Global weak solutions to a strongly degenerate haptotaxis model

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Abstract

We consider a one-dimensional version of a model obtained in [9] and describing the anisotropic spread of tumor cells in a tissue network. The model consists of a reaction-diffusion-taxis equation for the density of tumor cells coupled with an ODE for the density of tissue fibers and allows for strong degeneracy both in the diffusion and the haptotaxis terms. In this setting we prove the global existence of weak solutions to an associated no-flux initial-boundary value problem. Numerical simulations are performed in order to illustrate the model behavior.

Key words: haptotaxis; degenerate diffusion; global existence
MSC: 35K65, 35K51, 35K57 (primary); 35D30, 35K55, 92C17, 35Q30, 35Q92 (secondary)

1 Introduction

Models with degenerate diffusion in the context of taxis equations have received increased interest during the last decade. They describe the dynamics of a cell population in response to a chemoattractant [7, 14, 20], moving up the gradient of an insoluble signal (haptotaxis) [24], or performing both chemo- and haptotaxis [15, 19, 22].

In this work we consider a reaction-diffusion-transport-haptotaxis model which is inspired by the effective equations obtained in [9] via parabolic scaling upon starting from a multiscale model for glioma invasion in the anisotropic brain tissue and relying on the setting introduced in [8]. More precisely, the following PDE-ODE model was considered for the density function $p(t,x,v,y)$ of glioma cells depending on time t, position $x \in \mathbb{R}^n$, velocity $v \in V := s\mathbb{S}^{n-1}$, and density $y \in Y := (0, R_0)$ of cell surface receptors[1] bound to tissue fibers, and for the subcellular dynamics simplified to mass action kinetics of the mentioned receptor binding:

$$\partial_t p + \nabla_x \cdot (vp) + \nabla_y \cdot (G(y,w)p) = \mathcal{L}[\lambda]p + \mathcal{P}(p) \tag{1.1}$$

$$\dot{y} = G(y,w). \tag{1.2}$$

Thereby, $w(x)$ represents the (macroscopic) volume fraction of tissue, the turning operator $\mathcal{L}[\lambda]p := -\lambda(y)p + \int_V \lambda(y)K(x,v,v')p(v')dv'$ describes the reorientation of cells due to contact guidance by

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[1]R_0 denotes the total amount of receptors, assumed to be constant

tissue, and the term $\mathcal{P}(p) := \mu(x, \bar{p}, v) \int_Y \chi(x, y, y')p(t, x, v, y')w(x)dy'$ models proliferation subsequent to cell-tissue interactions. The function $\lambda(y)$ denotes the cell reorientation rate, $K(x, v, v')$ is the turning kernel depending on the directional distribution $q(x, v)$ of tissue fibers (obtained from diffusion tensor imaging data), μ represents the proliferation rate depending on the macroscopic cell density $\bar{p} = \int_V \int_Y p(t, x, v, y)dydv$, and χ is a kernel characterizing the transition from the state y to the state y' during a proliferative action.

An appropriate parabolic scaling led to the macroscopic equation for (an approximation of) the tumor cell density:

$$\partial_t u - \nabla\nabla : (\mathbb{D}_T u) + \nabla \cdot (a(w)\mathbb{D}_T \nabla w\ u) = w\mu(x, u)u, \tag{1.3}$$

where $a(w)$ is a function containing both macroscopic and subcellular level information,
$\mathbb{D}_T = \text{const} \int_V qv \otimes vdv$ is the tumor diffusion tensor encrypting the medical data about the structure of brain tissue, and

$$\nabla\nabla : (\mathbb{D}_T u) = \nabla \cdot (\mathbb{D}_T(x)\nabla u) + \nabla \cdot (\zeta(x)u) \tag{1.4}$$

with the drift velocity $\zeta(x) = \text{const} \int_V v \otimes v\nabla qdv$. For more details and the precise definitions we refer to [9].

Equation (1.3) is of the reaction-diffusion-transport-(hapto)taxis type and characterizes the evolution of the tumor cell density for a known underlying structure of brain tissue; in practice, the functions q and w are assessed at a certain time point t from medical data. This facilitates both its mathematical analysis and efficient numerical handling, however in fact the tumor evolution in a patient also induces dynamical changes in the tissue such as e.g. depletion or remodeling, which play an essential role in the disease development, see e.g. [2, 17] and the references therein. Therefore, a further equation is needed to describe these tissue modifications under the influence of tumor cells. Although in practice it is not feasible from the viewpoint of medical imaging to assess the tissue structure dynamically, by way of model-based predictions relying on such PDE-ODE coupled systems it is possible to use a sequence of just a few images in order to obtain via numerical simulations a good approximation of the dynamics over the whole timespan of interest.

Another issue is related to possible (local) degeneracies of the tumor diffusion tensor $\mathbb{D}_T(x)$, which is particularly relevant e.g. when modeling resected or irradiated regions of the tumor, where the tissue has been depleted as well. In the respective domains, this indeed reduces the otherwise diffusion-dominated PDE (1.3) to a hyperbolic transport equation with nonlinear source term. The mathematically quite delicate features of such strongly degenerate systems become manifest already in the case when any taxis or source terms are absent, that is, when $a \equiv 0$ and $\mu \equiv 0$ in (1.3). Indeed, in [11] the linear scalar parabolic equation

$$\partial_t u = (d_1(y)u)_{xx} + (d_2(y)u)_{yy}, \qquad (x, y) \in \Omega = (0, L_x) \times (0, L_y),\ t > 0, \tag{1.5}$$

has been studied, motivated among others by a monoscale model for anisotropic glioma spread in [16], and it was shown there that if the functions d_1 and d_2 are smooth and nonnegative and such that d_1 is strictly positive but d_2 vanishes precisely in some subinterval $[a, b]$ of $(0, L_y)$, then solutions to an associated no-flux initial-boundary value problem asymptotically approach a singular state reflecting concentration of mass within the degeneracy region $[0, L_x] \times [a, b]$ and extinction outside.

In this paper we intend to provide a first step toward a mathematical understanding of corresponding systems when beyond such strongly degenerate diffusion processes, further crucial mechanisms and especially nonlinear haptotaxis are involved. In order to concentrate on essential aspects of such types of interplay within the framework of a model that captures the essential properties but beyond that remains as simple as possible, we may restrict to the spatially one-dimensional case, in which the tumor diffusion tensor $\mathbb{D}_T$ in (1.3) actually reduces to a scalar function. In the context of a simple evolution law for the haptotactic attractant, particularly neglecting remodeling mechanisms, this leads to coupled parabolic-ODE systems of the form

$$\begin{cases} u_t = \big(d(x)u\big)_{xx} - \big(d(x)u\psi(v)v_x\big)_x, \\ v_t = -uh(v), \end{cases} \tag{1.6}$$

with given nonnegative functions d, ψ and h.

Although in our current 1D setting (1.6) the model in [9] loses most of its anisotropy relevance, some of it is retained in the space-dependent diffusion and haptotactic sensitivity coefficients. Likewise, the multiscality considered in [9] and leading to a haptotactic coefficient depending on the subcellular dynamics can still be partially retained in this model, in spite of the modified transport term, in which the drift velocity has now a simpler form, yet depending on $d(x)$. The very presence of the haptotaxis term is a consequence of taking the receptor binding dynamics into account when describing the evolution of the cell density function on the mesoscopic level and scaling up to the macroscopic one. Hence, essential features of the model obtained in [9] are preserved even in this simplified, dimension-reduced setting.

Other related models featuring degenerate diffusion in the context of haptotaxis were proposed and investigated in [23, 24]. The kind of degeneracy considered there is, however, different from the one in this and previous models, as it affects both the diffusion and the haptotaxis coefficients, thereby allowing the diffusion to degenerate due to one or both solution components (tumor cell density and tissue density). The model in [23] involves two subpopulations of tumor cells, differentiating between moving and proliferating [2] ones, but allowing mutual transitions. Unlike the present model, in [23, 24] there is (apart from the taxis) no other transport term.

Problem setup and main result. In order to make the essential mathematical aspects of (1.6) more transparent, let us write (1.6) in a form involving a constant haptotacitc sensitivity, which according to the simple ODE structure of the second equation therein can readily be achieved on substituting $w = \Psi(v)$ with $\Psi(v) := \int_0^v \psi(\sigma)d\sigma$, $v \geq 0$. Accordingly, in an open bounded interval $\Omega \subset \mathbb{R}$ we will henceforth consider the initial-boundary value problem

$$\begin{cases} u_t = (d(x)u)_{xx} - (d(x)uw_x)_x + uf(x,u,w), & x \in \Omega,\ t > 0, \\ w_t = -ug(w), & x \in \Omega,\ t > 0, \\ (d(x)u)_x - d(x)uw_x = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.7}$$

with given parameter functions $d : \bar{\Omega} \to [0,\infty)$, $f : \bar{\Omega} \times [0,\infty)^2 \to \mathbb{R}$ and $g : [0,\infty) \to [0,\infty)$ satisfying

$$\sqrt{d} \in W^{1,\infty}(\Omega), \quad f \in C^1(\bar{\Omega} \times [0,\infty)^2) \quad \text{and} \quad g \in C^1([0,\infty)), \tag{1.8}$$

[2] whence in virtue of the go-or-grow dichotomy assumed to be non-motile

and with prescribed initial data u_0 and w_0 which are such that

$$\begin{cases} 0 \le u_0 \in C^0(\bar\Omega) \text{ satisfies } u_0 \not\equiv 0 \quad \text{and} \\ 0 \le w_0 \in W^{1,2}(\Omega) \text{ has the property that } \int_\Omega \frac{w_{0x}^2}{g(w_0)} < \infty. \end{cases} \tag{1.9}$$

As for the parameter functions in (1.7), throughout our analysis we shall furthermore assume that

$$f(x,u,w) \le \rho(w) \quad \text{for all } (x,u,w) \in \bar\Omega \times [0,\infty)^2 \quad \text{with some nondecreasing } \rho : [0,\infty) \to [0,\infty), \tag{1.10}$$

and that there exists $\delta > 0$ such that writing

$$M := \|w_0\|_{L^\infty(\Omega)} + \delta, \tag{1.11}$$

we have

$$g(0) = 0, \qquad g(w) > 0 \quad \text{for all } w \in (0,M] \qquad \text{and} \qquad g'(w) > 0 \quad \text{for all } w \in [0,M] \tag{1.12}$$

as well as

$$\liminf_{w \searrow 0} \frac{g'(w)}{g(w)} > 0, \tag{1.13}$$

whence in particular there exist $\Gamma > 0$ and $\gamma > 0$ fulfilling

$$g(w) \le \Gamma w \quad \text{for all } w \in [0,M] \tag{1.14}$$

and

$$\frac{g'(w)}{g(w)} \ge \gamma \qquad \text{for all } w \in (0,M]. \tag{1.15}$$

Beyond the analytically simplest case obtained on letting

$$g(w) = w, \qquad w \ge 0,$$

this inter alia includes more general choices such as

$$g(w) = w(1-w), \qquad w \ge 0,$$

upon which via the substitution $w = \frac{v}{1+v}$, on the set of solutions fulfilling $v < 1$ the system (1.7) becomes formally equivalent to a corresponding initial-boundary value problem for the special version

$$\begin{cases} u_t = \big(d(x)u\big)_{xx} - \Big(\frac{d(x)u}{(1+v)^2} v_x\Big)_x, \\ v_t = -uv, \end{cases} \tag{1.16}$$

of (1.6), as proposed in [24] for modeling tumor invasion in a tissue network, thereby paying increased attention to the form of the haptotaxis coefficient. Specifically, the latter accounts for microscopic cell-tissue interactions, which –besides having a haptotaxis term at all– retains a supplementary trace of multiscality in our macroscopic model, although in a rather indirect way, as we do not explicitly couple some ODE for receptor binding kinetics to the two PDEs for u and v. The presence of $d(x)$ in both diffusion/transport and haptotaxis coefficients is motivated by the deduction in [9].

The main results of our analysis indicate that even in this general setting, thus allowing for virtually arbitrary strength of degeneracies in diffusion, haptotactic cross-diffusion does not result in a finite-time collapse of solutions into e.g. persistent Dirac-type singularities. More precisely, let us introduce the following solution concept to pursued below, in which we use the abbreviation $\{d>0\} := \big\{x \in \bar\Omega \,\big|\, d(x)>0\big\}$ which along with a corresponding definition of $\{d=0\}$ will frequently be used throughout the sequel.

Definition 1.1 *A pair (u,w) of nonnegative functions*

$$\begin{cases} u \in L^1_{loc}(\bar\Omega\times[0,\infty)), \\ w \in L^\infty_{loc}(\bar\Omega\times[0,\infty)) \cap L^1_{loc}([0,\infty);W^{1,1}(\{d>0\})) \end{cases} \tag{1.17}$$

satisfying

$$uf(\cdot,u,w) \in L^1_{loc}(\bar\Omega\times[0,\infty)) \qquad \textit{and} \qquad ug(w) \in L^1_{loc}(\bar\Omega\times[0,\infty)) \tag{1.18}$$

as well as

$$duw_x \in L^1_{loc}([0,\infty);L^1(\{d>0\})) \tag{1.19}$$

will be called a global weak solution *of (1.7) if*

$$-\int_0^\infty\int_\Omega u\varphi_t - \int_\Omega u_0\varphi(\cdot,0) = \int_0^\infty\int_{\{d>0\}} du\varphi_{xx} + \int_0^\infty\int_{\{d>0\}} duw_x\varphi_x + \int_0^\infty\int_\Omega uf(\cdot,u,w)\varphi \tag{1.20}$$

for all $\varphi\in C_0^\infty(\bar\Omega\times[0,\infty))$ such that $\varphi_x=0$ on $\partial\Omega\times(0,\infty)$ and

$$\int_0^\infty\int_\Omega w\varphi_t + \int_\Omega w_0\varphi(\cdot,0) = \int_0^\infty\int_\Omega ug(w)\varphi \tag{1.21}$$

for all $\varphi\in C_0^\infty(\Omega\times[0,\infty))$.

Within this framework, a global solution of (1.7) can always be constructed:

Theorem 1.2 *Suppose that $\Omega\subset\mathbb{R}$ is a bounded interval, and that u_0, w_0, d, f and g satisfy (1.9), (1.8), (1.10) and (1.12). Then (1.7) possesses at least one global weak solution in the sense specified in Definition 1.1 below.*

This paper is organized as follows: In Section 2 we introduce a regularized version of the degenerate problem, for which some useful properties are obtained. Section 3 is concerned with studying an entropy functional which allows to deduce a quasi-dissipative property of the regularized system, inter alia asserting global existence of its solution. Some precompactness and regularity properties of terms involved in that system follow in Sections 4 and 5, respectively, succeeded in Section 6 by regularity features of corresponding time derivatives. Sections 7 and 8 provide convergence properties of the approximate solution in the region with no degeneracy; further properties of the respective limits are obtained in Section 9. Finally, Section 10 concludes the existence proof for the strongly degenerate problem (1.7).

2 Regularized problems and their basic properties

In order to prepare the construction of an appropriate family of non-degenerate approximations of (1.7), according to the nonnegativity of d and the inclusion $\sqrt{d} \in W^{1,\infty}(\Omega)$ we may first choose $(d_\varepsilon)_{\varepsilon\in(0,1)} \subset C^3(\bar\Omega)$ in such a way that $d_{\varepsilon x} = 0$ on $\partial\Omega$ and that with some $K_1 > 0$, for each $\varepsilon \in (0,1)$ we have

$$\sqrt{\varepsilon} \le d_\varepsilon(x) \le \|d\|_{L^\infty(\Omega)} + 1 \qquad \text{for all } x \in \bar\Omega, \tag{2.1}$$

as well as

$$\frac{d_{\varepsilon x}^2(x)}{d_\varepsilon(x)} \le K_1 \qquad \text{for all } x \in \bar\Omega, \tag{2.2}$$

and such that moreover

$$d_\varepsilon \to d \quad \text{in } L^\infty(\Omega) \qquad \text{as } \varepsilon \searrow 0 \tag{2.3}$$

and

$$d_{\varepsilon x} \to d_x \quad \text{a.e. in } \Omega \qquad \text{as } \varepsilon \searrow 0. \tag{2.4}$$

We next note that according to (1.12) it is possible to fix $\varepsilon_0 \in (0,1)$ such that $g(M) > \varepsilon_0$, whereupon with δ as introduced in the course of the definition (1.11) of M, for each $\varepsilon \in (0,\varepsilon_0)$ we can choose $\delta_\varepsilon \in (0,\delta^2)$ such that

$$g(w) \ge \varepsilon \quad \text{for all } w \in [\delta_\varepsilon, M], \tag{2.5}$$

and such that moreover $\delta_\varepsilon \to 0$ as $\varepsilon \searrow 0$. It is then easy to see that one can find $(\eta_\varepsilon)_{\varepsilon\in(0,\varepsilon_0)} \subset (0,1)$ with the two properties that

$$\eta_\varepsilon \ln \frac{1}{\sqrt{\delta_\varepsilon}} \to +\infty \qquad \text{as } \varepsilon \searrow 0, \tag{2.6}$$

and that

$$\eta_\varepsilon \to 0 \qquad \text{as } \varepsilon \searrow 0; \tag{2.7}$$

indeed, it can readily be checked that this can be achieved on choosing

$$\eta_\varepsilon := \frac{\ln\ln \frac{A}{\sqrt{\delta_\varepsilon}}}{\ln \frac{A}{\sqrt{\delta_\varepsilon}}}, \qquad \varepsilon \in (0,\varepsilon_0),$$

with some suitably large $A > 0$. For $\varepsilon \in (0,\varepsilon_0)$, we then let

$$w_{0\varepsilon}(x) := w_0(x) + \sqrt{\delta_\varepsilon}, \qquad x \in \bar\Omega, \tag{2.8}$$

and consider the regularized variant of (1.7) given by

$$\begin{cases} u_{\varepsilon t} = (d_\varepsilon u_\varepsilon)_{xx} - \Big(d_\varepsilon \frac{u_\varepsilon}{(1+\eta_\varepsilon u_\varepsilon)^2} w_{\varepsilon x}\Big)_x + u_\varepsilon f(x,u_\varepsilon,w_\varepsilon), & x \in \Omega,\ t>0, \\ w_{\varepsilon t} = \varepsilon\Big(\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x - \frac{u_\varepsilon}{1+\eta_\varepsilon u_\varepsilon} g(w_\varepsilon), & x \in \Omega,\ t>0, \\ u_{\varepsilon x} = w_{\varepsilon x} = 0, & x \in \partial\Omega,\ t>0, \\ u_\varepsilon(x,0) = u_0(x), \qquad w_\varepsilon(x,0) = w_{0\varepsilon}(x), & x \in \Omega, \end{cases} \tag{2.9}$$

Due to the additionally introduced artificial diffusion in the equation for w_ε each of these problems can be viewed as a variant of the well-studied Keller-Segel chemotaxis system; in fact, as can be seen by straightforward adaptation of arguments well-established in the analysis of chemotaxis problems ([1], [12], [21]), all these problems allow for local-in-time classical solutions which enjoy a favorable extensibility criterion:

Lemma 2.1 *For each $\varepsilon \in (0, \varepsilon_0)$, there exist $T_{max,\varepsilon} \in (0, \infty]$ and nonnegative functions*

$$\begin{cases} u_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{max,\varepsilon})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max,\varepsilon})), \\ w_\varepsilon \in C^0([0, T_{max,\varepsilon}); W^{1,2}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max,\varepsilon})), \end{cases}$$

which solve (2.9) in the classical sense in $\Omega \times (0, T_{max,\varepsilon})$, and which are such that

$$\text{if } T_{max,\varepsilon} < \infty, \text{ then } \limsup_{t \nearrow T_{max,\varepsilon}} \left(\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|w_\varepsilon(\cdot, t)\|_{W^{1,2}(\Omega)} + \left\| \frac{1}{g(w_\varepsilon(\cdot, t))} \right\|_{L^\infty(\Omega)} \right) = \infty. \tag{2.10}$$

Let us first collect some basic properties of these solutions. We first assert some useful pointwise upper and lower bounds for w_ε.

Lemma 2.2 *Let $\varepsilon \in (0, \varepsilon_0)$. Then*

$$w_\varepsilon(x, t) \le M \qquad \text{for all } x \in \Omega \text{ and } t \in (0, T_{max,\varepsilon}) \tag{2.11}$$

and

$$w_\varepsilon(x, t) \ge \sqrt{\delta_\varepsilon} e^{-\frac{\Gamma}{\eta_\varepsilon} t} \qquad \text{for all } x \in \Omega \text{ and } t \in (0, T_{max,\varepsilon}), \tag{2.12}$$

where $\Gamma > 0$ is as in (1.14).

Proof. Since according to our choices of δ_ε and M we have

$$w_\varepsilon(x, 0) = w_0(x) + \sqrt{\delta_\varepsilon} \le \|w_0\|_{L^\infty(\Omega)} + \delta = M \qquad \text{for all } x \in \Omega,$$

the inequality in (2.11) immediately results from the maximum principle applied to the second equation in (2.9). As a consequence thereof, in view of (1.12) we know that $g'(w_\varepsilon) \ge 0$ in $\Omega \times (0, T_{max,\varepsilon})$, whence

$$\frac{u_\varepsilon}{1 + \eta_\varepsilon u_\varepsilon} g(w_\varepsilon) \le \frac{1}{\eta_\varepsilon} g(w_\varepsilon) \qquad \text{in } \Omega \times (0, T_{max,\varepsilon}),$$

so that using (2.9) and (1.14) we see that

$$w_{\varepsilon t} \ge \left(\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \right)_x - \frac{\Gamma}{\eta_\varepsilon} w_\varepsilon \qquad \text{in } \Omega \times (0, T_{max,\varepsilon}).$$

Since

$$\underline{w}(x, t) := \sqrt{\delta_\varepsilon} e^{-\frac{\Gamma}{\eta_\varepsilon} t}, \qquad x \in \bar{\Omega}, \ t \ge 0,$$

satisfies

$$\underline{w}_t - \left(\frac{\underline{w}_x}{\sqrt{g(w_\varepsilon)}} \right)_x + \frac{\Gamma}{\eta_\varepsilon} \underline{w} = 0 \qquad \text{in } \Omega \times (0, \infty)$$

and $\frac{\partial w}{\partial \nu} = 0$ on $\partial\Omega \times (0,\infty)$ as well as

$$\underline{w}(x,0) = \sqrt{\delta_\varepsilon} \le w_\varepsilon(x,0) \qquad \text{for all } x \in \Omega$$

by (2.8), the comparison principle therefeore ensures that $w_\varepsilon \ge \underline{w}$ in $\Omega \times (0, T_{max,\varepsilon})$ and that thus also (2.12) is valid. $\Box$

Using the latter along with (1.10), we easily obtain the following information on the evolution of $\int_\Omega u_\varepsilon$.

Lemma 2.3 *With M as defined in (1.11) and ρ taken from (1.10), we have*

$$\frac{d}{dt}\int_\Omega u_\varepsilon \le \rho(M)\int_\Omega u_\varepsilon \qquad \textit{for all } t \in (0, T_{max,\varepsilon}) \tag{2.13}$$

and

$$\int_\Omega u_\varepsilon(\cdot,t) \le \left\{ \int_\Omega u_0 \right\} \cdot e^{\rho(M)t} \qquad \textit{for all } t \in (0, T_{max,\varepsilon}). \tag{2.14}$$

Proof. We integrate the first equation in (2.9) and use (1.10) together with (2.11) to find that

$$\frac{d}{dt}\int_\Omega u_\varepsilon = \int_\Omega u_\varepsilon f(x,u_\varepsilon,w_\varepsilon) \le \int_\Omega u_\varepsilon \rho(w_\varepsilon) \le \rho(M)\int_\Omega u_\varepsilon \qquad \text{for all } t \in (0, T_{max,\varepsilon}),$$

and that hence (2.13) holds, from which in turn (2.14) results upon integration in time. $\Box$

3 Implications of an entropy-like structure

Now the core of our approach consists in the detection of a favorable quasi-dissipative property of the system (2.9) which can be revealed by following the well-established strategy of considering the time evolution of a functional that combines a logarithmic entropy of the cell distribution with a properly chosen summand annihilating the correspondingly obtained cross-diffusive interaction integral. In order to clarify which precise form the latter takes in the context of the approximate problems (2.9), let us begin by separately tracking the logarithmic entropy.

Lemma 3.1 *Let ρ, M and K_1 be as introduced in (1.10), (1.11) and (2.2). Then for each $\varepsilon \in (0,\varepsilon_0)$ we have*

$$\begin{aligned} \frac{d}{dt}\int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{1}{2}\int_\Omega d_\varepsilon \frac{u_{\varepsilon x}^2}{u_\varepsilon} \quad &\le \quad \int_\Omega d_\varepsilon \frac{u_{\varepsilon x}}{(1+\eta_\varepsilon u_\varepsilon)^2} w_{\varepsilon x} + \Big(\rho(M) + \frac{K_1}{2}\Big) \cdot \left\{ \int_\Omega u_0 \right\} \cdot e^{\rho(M)t} \\ &\quad + \int_\Omega u_\varepsilon \ln u_\varepsilon \cdot f(x,u_\varepsilon,w_\varepsilon) \qquad \textit{for all } t \in (0, T_{max,\varepsilon}). \end{aligned} \tag{3.1}$$

Proof. Since u_ε is positive in $\bar\Omega \times (0, T_{max,\varepsilon})$ by the strong maximum principle, we may multiply the first equation in (2.9) by $\ln u_\varepsilon$ and integrate by parts to see that

$$\begin{aligned} \frac{d}{dt}\int_\Omega u_\varepsilon \ln u_\varepsilon &= \int_\Omega u_{\varepsilon t} \ln u_\varepsilon + \frac{d}{dt}\int_\Omega u_\varepsilon \\ &= -\int_\Omega (d_\varepsilon u_\varepsilon)_x \cdot \frac{u_{\varepsilon x}}{u_\varepsilon} + \int_\Omega d_\varepsilon \frac{u_{\varepsilon x}}{(1+\eta_\varepsilon u_\varepsilon)^2} w_{\varepsilon x} + \int_\Omega u_\varepsilon \ln u_\varepsilon \cdot f(x,u_\varepsilon,w_\varepsilon) \\ &\quad + \frac{d}{dt}\int_\Omega u_\varepsilon \qquad \text{for all } t \in (0, T_{max,\varepsilon}). \end{aligned} \tag{3.2}$$

Here by Lemma 2.3 we have

$$\frac{d}{dt}\int_\Omega u_\varepsilon \le \rho(M)\int_\Omega u_\varepsilon \qquad \text{for all } t\in(0,T_{max,\varepsilon}), \tag{3.3}$$

and using Young's inequality and (2.2) we obtain

$$\begin{aligned}
-\int_\Omega (d_\varepsilon u_\varepsilon)_x \cdot \frac{u_{\varepsilon x}}{u_\varepsilon} &= -\int_\Omega d_\varepsilon \frac{u_{\varepsilon x}^2}{u_\varepsilon} - d_{\varepsilon x} u_{\varepsilon x} \\
&\le -\frac{1}{2}\int_\Omega d_\varepsilon \frac{u_{\varepsilon x}^2}{u_\varepsilon} + \frac{1}{2}\int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon} u_\varepsilon \\
&\le -\frac{1}{2}\int_\Omega d_\varepsilon \frac{u_{\varepsilon x}^2}{u_\varepsilon} + \frac{K_1}{2}\int_\Omega u_\varepsilon \qquad \text{for all } t\in(0,T_{max,\varepsilon}).
\end{aligned} \tag{3.4}$$

Since

$$\Big(\rho(M)+\frac{K_1}{2}\Big)\int_\Omega u_\varepsilon \le \Big(\rho(M)+\frac{K_1}{2}\Big)\cdot\Big\{\int_\Omega u_0\Big\}\cdot e^{\rho(M)t} \qquad \text{for all } t\in(0,T_{max,\varepsilon})$$

by Lemma 2.3, combining (3.2)-(3.4) thus yields (3.1). $\square$

Thanks to a favorable exact relationship between the approximate signal absorption rate $0 \le u \mapsto \frac{u}{1+\eta_\varepsilon u}$ and the tactic sensitivity $0 \le \frac{1}{(1+\eta_\varepsilon u)^2}$ in (2.9), an exact compensation of the first summand on the right of (3.1) can be achieved on complementing the above by the following.

Lemma 3.2 *With K_1 as in (2.2), we have*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{g(w_\varepsilon)} + \frac{\varepsilon}{2}\int_\Omega d_\varepsilon\cdot\frac{1}{\sqrt{g(w_\varepsilon)}}\cdot\Big(\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x^2 + \frac{1}{2}\int_\Omega d_\varepsilon\cdot\frac{u_\varepsilon}{1+\eta_\varepsilon u_\varepsilon}\cdot\frac{g'(w_\varepsilon)}{g(w_\varepsilon)} w_{\varepsilon x}^2 \\
&\qquad \le -\int_\Omega d_\varepsilon \frac{u_{\varepsilon x}}{(1+\eta_\varepsilon u_\varepsilon)^2} w_{\varepsilon x} + \frac{\varepsilon K_1}{2}\int_\Omega \frac{w_{\varepsilon x}^2}{\sqrt{g(w_\varepsilon)}^3} \qquad \textit{for all } t\in(0,T_{max,\varepsilon})
\end{aligned} \tag{3.5}$$

whenever $\varepsilon\in(0,\varepsilon_0)$.

PROOF. Using that $g(w_\varepsilon)$ is positive in $\bar\Omega\times[0,T_{max,\varepsilon})$ due to Lemma 2.2 and (1.12), on the basis of the second equation in (2.9) we compute

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{g(w_\varepsilon)} &= 2\int_\Omega d_\varepsilon \frac{1}{g(w_\varepsilon)} w_{\varepsilon x} w_{\varepsilon xt} - \int_\Omega d_\varepsilon \frac{g'(w_\varepsilon)}{g^2(w_\varepsilon)} w_{\varepsilon x}^2 w_{\varepsilon t} \\
&= 2\int_\Omega d_\varepsilon \frac{1}{g(w_\varepsilon)} w_{\varepsilon x} \cdot \Big\{ \varepsilon \Big(\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \Big)_{xx} - \Big(\frac{u_\varepsilon}{1+\eta_\varepsilon u_\varepsilon} g(w_\varepsilon) \Big)_x \Big\} \\
&\quad - \int_\Omega d_\varepsilon \frac{g'(w_\varepsilon)}{g^2(w_\varepsilon)} w_{\varepsilon x}^2 \cdot \Big\{ \varepsilon \Big(\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \Big)_x - \frac{u_\varepsilon}{1+\eta_\varepsilon u_\varepsilon} g(w_\varepsilon) \Big\} \\
&= -2\varepsilon \int_\Omega \Big(d_\varepsilon \frac{1}{g(w_\varepsilon)} w_{\varepsilon x} \Big)_x \cdot \Big(\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \Big)_x \\
&\quad -2\int_\Omega d_\varepsilon \frac{1}{g(w_\varepsilon)} w_{\varepsilon x} \cdot \Big(\frac{u_\varepsilon}{1+\eta_\varepsilon u_\varepsilon} g(w_\varepsilon) \Big)_x \\
&\quad -\varepsilon \int_\Omega d_\varepsilon \frac{g'(w_\varepsilon)}{g^2(w_\varepsilon)} w_{\varepsilon x}^2 \cdot \Big(\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \Big)_x \\
&\quad + \int_\Omega d_\varepsilon \frac{u_\varepsilon}{1+\eta_\varepsilon u_\varepsilon} \frac{g'(w_\varepsilon)}{g(w_\varepsilon)} w_{\varepsilon x}^2 \qquad \text{for all } t \in (0, T_{max,\varepsilon}). \qquad (3.6)
\end{aligned}
$$

Here we expand

$$
\begin{aligned}
-2\int_\Omega d_\varepsilon \frac{1}{g(w_\varepsilon)} w_{\varepsilon x} \cdot \Big(\frac{u_\varepsilon}{1+\eta_\varepsilon u_\varepsilon} g(w_\varepsilon) \Big)_x &= -2\int_\Omega d_\varepsilon \frac{u_{\varepsilon x}}{(1+\eta_\varepsilon u_\varepsilon)^2} w_{\varepsilon x} \\
&\quad -2\int_\Omega d_\varepsilon \frac{u_\varepsilon}{1+\eta_\varepsilon u_\varepsilon} \frac{g'(w_\varepsilon)}{g(w_\varepsilon)} w_{\varepsilon x}^2 \qquad \text{for all } t \in (0, T_{max,\varepsilon}),
\end{aligned}
$$

so that

$$
\begin{aligned}
&-2\int_\Omega d_\varepsilon \frac{1}{g(w_\varepsilon)} w_{\varepsilon x} \cdot \Big(\frac{u_\varepsilon}{1+\eta_\varepsilon u_\varepsilon} g(w_\varepsilon) \Big)_x + \int_\Omega d_\varepsilon \frac{u_\varepsilon}{1+\eta_\varepsilon u_\varepsilon} \frac{g'(w_\varepsilon)}{g(w_\varepsilon)} w_{\varepsilon x}^2 \\
&\qquad = -2\int_\Omega d_\varepsilon \frac{u_{\varepsilon x}}{(1+\eta_\varepsilon u_\varepsilon)^2} w_{\varepsilon x} - \int_\Omega d_\varepsilon \frac{u_\varepsilon}{1+\eta_\varepsilon u_\varepsilon} \frac{g'(w_\varepsilon)}{g(w_\varepsilon)} w_{\varepsilon x}^2 \qquad \text{for all } t \in (0, T_{max,\varepsilon}). \qquad (3.7)
\end{aligned}
$$

We next use the identity

$$
w_{\varepsilon xx} = \sqrt{g(w_\varepsilon)} \cdot \Big(\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \Big)_x + \frac{g'(w_\varepsilon)}{2g(w_\varepsilon)} w_{\varepsilon x}^2 \qquad \text{in } \Omega \times (0, T_{max,\varepsilon}) \qquad (3.8)
$$

to rewrite

$$
\begin{aligned}
\Big(d_\varepsilon \frac{1}{g(w_\varepsilon)} w_{\varepsilon x} \Big)_x &= d_\varepsilon \frac{1}{g(w_\varepsilon)} w_{\varepsilon xx} - d_\varepsilon \frac{g'(w_\varepsilon)}{g^2(w_\varepsilon)} w_{\varepsilon x}^2 + d_{\varepsilon x} \frac{1}{g(w_\varepsilon)} w_{\varepsilon x} \\
&= d_\varepsilon \frac{1}{\sqrt{g(w_\varepsilon)}} \Big(\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \Big)_x - \frac{1}{2} d_\varepsilon \frac{g'(w_\varepsilon)}{g^2(w_\varepsilon)} w_{\varepsilon x}^2 + d_{\varepsilon x} \frac{1}{g(w_\varepsilon)} w_{\varepsilon x} \qquad \text{in } \Omega \times (0, T_{max,\varepsilon}),
\end{aligned}
$$

so that on the right-hand side of (3.6) we can employ Young's inequality to see that

$$
\begin{aligned}
-2\varepsilon \int_\Omega \Big(d_\varepsilon \frac{1}{g(w_\varepsilon)} w_{\varepsilon x}\Big)_x \cdot \Big(\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x - \varepsilon \int_\Omega d_\varepsilon \frac{g'(w_\varepsilon)}{g^2(w_\varepsilon)} w_{\varepsilon x}^2 \cdot \Big(\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x \\
= \quad -2\varepsilon \int_\Omega d_\varepsilon \frac{1}{\sqrt{g(w_\varepsilon)}} \Big(\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x^2 - 2\varepsilon \int_\Omega d_{\varepsilon x} \frac{1}{g(w_\varepsilon)} w_{\varepsilon x} \cdot \Big(\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x \\
\le \quad -\varepsilon \int_\Omega d_\varepsilon \frac{1}{\sqrt{g(w_\varepsilon)}} \Big(\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x^2 + \varepsilon \int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon} \frac{1}{\sqrt{g(w_\varepsilon)}} w_{\varepsilon x}^2 \\
\le \quad -\varepsilon \int_\Omega d_\varepsilon \frac{1}{\sqrt{g(w_\varepsilon)}} \Big(\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x^2 + \varepsilon K_1 \int_\Omega \frac{1}{\sqrt{g(w_\varepsilon)}} w_{\varepsilon x}^2 \qquad \text{for all } t \in (0, T_{max,\varepsilon}),
\end{aligned}
$$

again due to (2.2). In conjunction with (3.7) and (3.6) this yields (3.5). □

In fact, combining the latter two lemmata yields a quasi-entropy inequality, the essential implications of which can be summarized as follows.

Lemma 3.3 *Let $T > 0$. Then there exist $\varepsilon_\star(T) \in (0, \varepsilon_0)$ and $C(T) > 0$ such that for any choice of $\varepsilon \in (0, \varepsilon_\star(T))$, the solution of (2.9) satisfies*

$$
\int_{\{u_\varepsilon(\cdot,t)\ge 1\}} u_\varepsilon(\cdot,t) \ln u_\varepsilon(\cdot,t) \le C(T) \qquad \textit{for all } t \in (0, \widehat{T}_\varepsilon) \tag{3.9}
$$

and

$$
\int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2(\cdot,t)}{g(w_\varepsilon(\cdot,t))} \le C(T) \qquad \textit{for all } t \in (0, \widehat{T}_\varepsilon), \tag{3.10}
$$

and such that moreover

$$
\int_0^{\widehat{T}_\varepsilon} \int_\Omega d_\varepsilon \frac{u_{\varepsilon x}^2}{u_\varepsilon} \le C(T) \tag{3.11}
$$

and

$$
\int_0^{\widehat{T}_\varepsilon} \int_{\{u_\varepsilon(\cdot,t)\ge 1\}} u_\varepsilon \ln u_\varepsilon \cdot f_-(\cdot, u_\varepsilon, w_\varepsilon) \le C(T) \tag{3.12}
$$

as well as

$$
\int_0^{\widehat{T}_\varepsilon} \int_\Omega d_\varepsilon \frac{u_\varepsilon}{1+\eta_\varepsilon u_\varepsilon} \frac{g'(w_\varepsilon)}{g(w_\varepsilon)} w_{\varepsilon x}^2 \le C(T), \tag{3.13}
$$

where with $T_{max,\varepsilon}$ as in Lemma 2.1 we have set $\widehat{T}_\varepsilon := \min\{T, T_{max,\varepsilon}\}$.

Proof. We add the inequalities provided by Lemma 3.1 and Lemma 3.2 to see on dropping a nonnegative summand on the right that for all $\varepsilon \in (0, \varepsilon_0)$ we have

$$
\begin{aligned}
\frac{d}{dt}\Big\{ \int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{1}{2} \int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{g(w_\varepsilon)} \Big\} + \frac{1}{2} \int_\Omega d_\varepsilon \frac{u_{\varepsilon x}^2}{u_\varepsilon} + \frac{1}{2} \int_\Omega d_\varepsilon \frac{u_\varepsilon}{1+\eta_\varepsilon u_\varepsilon} \frac{g'(w_\varepsilon)}{g(w_\varepsilon)} w_{\varepsilon x}^2 \\
\le \quad c_1 + \int_\Omega u_\varepsilon \ln u_\varepsilon \cdot f(x, u_\varepsilon, w_\varepsilon) + \frac{\varepsilon K_1}{2} \int_\Omega \frac{w_{\varepsilon x}^2}{\sqrt{g(w_\varepsilon)}^3} \qquad \text{for all } t \in (0, T_{max,\varepsilon})
\end{aligned} \tag{3.14}
$$

with $c_1 \equiv c_1(T) := \Big(\rho(M) + \frac{K_1}{2}\Big) \cdot \Big\{ \int_\Omega u_0 \Big\} \cdot e^{\rho(M)T}$. Here we split $f = f_+ - f_-$ and

$$\begin{aligned}
\int_\Omega u_\varepsilon \ln u_\varepsilon \cdot f(x,u_\varepsilon,w_\varepsilon) &= \int_{\{u_\varepsilon<1\}} u_\varepsilon \ln u_\varepsilon \cdot f_+(x,u_\varepsilon,w_\varepsilon) - \int_{\{u_\varepsilon<1\}} u_\varepsilon \ln u_\varepsilon \cdot f_-(x,u_\varepsilon,w_\varepsilon) \\
&\quad + \int_{\{u_\varepsilon\ge1\}} u_\varepsilon \ln u_\varepsilon \cdot f_+(x,u_\varepsilon,w_\varepsilon) - \int_{\{u_\varepsilon\ge1\}} u_\varepsilon \ln u_\varepsilon \cdot f_-(x,u_\varepsilon,w_\varepsilon) \qquad (3.15)
\end{aligned}$$

for $t \in (0, T_{max,\varepsilon})$, where clearly

$$\int_{\{u_\varepsilon<1\}} u_\varepsilon \ln u_\varepsilon \cdot f_+(x,u_\varepsilon,w_\varepsilon) \le 0 \qquad \text{for all } t \in (0, T_{max,\varepsilon}), \tag{3.16}$$

and where using that

$$\xi \ln \xi \ge -\frac{1}{e} \qquad \text{for all } \xi > 0, \tag{3.17}$$

we see that

$$-\int_{\{u_\varepsilon<1\}} u_\varepsilon \ln u_\varepsilon \cdot f_-(x,u_\varepsilon,w_\varepsilon) \le \frac{|\Omega|}{e} c_2 \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \tag{3.18}$$

with

$$c_2 := \max_{(x,u,w)\in\bar\Omega\times[0,1]\times[0,M]} f_-(x,u,w)$$

being finite by continuity of f. Since

$$f_+(\cdot, u_\varepsilon, w_\varepsilon) \le \rho(M) \qquad \text{in } \Omega \times (0, T_{max,\varepsilon}) \tag{3.19}$$

by (1.10) and Lemma 2.2, again relying on (3.17) we see that

$$\begin{aligned}
\int_{\{u_\varepsilon\ge1\}} u_\varepsilon \ln u_\varepsilon \cdot f_+(x,u_\varepsilon,w_\varepsilon) &\le \rho(M) \int_{\{u_\varepsilon\ge1\}} u_\varepsilon \ln u_\varepsilon \\
&= \rho(M) \cdot \Big\{ \int_\Omega u_\varepsilon \ln u_\varepsilon - \int_{\{u_\varepsilon<1\}} u_\varepsilon \ln u_\varepsilon \Big\} \\
&\le \rho(M) \int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{\rho(M)|\Omega|}{e} \qquad \text{for all } t \in (0, T_{max,\varepsilon}),
\end{aligned}$$

so that from (3.15), (3.16) and (3.18) we infer that

$$\int_\Omega u_\varepsilon \ln u_\varepsilon \cdot f(x,u_\varepsilon,w_\varepsilon) \le \rho(M) \int_\Omega u_\varepsilon \ln u_\varepsilon - \int_{\{u_\varepsilon\ge1\}} u_\varepsilon \ln u_\varepsilon \cdot f_-(x,u_\varepsilon,w_\varepsilon) + c_3 \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \tag{3.20}$$

with $c_3 := \frac{|\Omega|}{e} \cdot (c_2 + \rho(M))$.
Next, the rightmost summand in (3.14) can be estimated using Lemma 2.2 along with the defining properties of $(d_\varepsilon)_{\varepsilon\in(0,\varepsilon_0)}$ and $(\eta_\varepsilon)_{\varepsilon\in(0,\varepsilon_0)}$: Indeed, given $T > 0$ we may use (2.6) to fix $\varepsilon_\star = \varepsilon_\star(T) \in (0, \varepsilon_0)$ small enough such that with Γ as in (1.14) we have

$$T \le \frac{\eta_\varepsilon}{\Gamma} \cdot \ln \frac{1}{\sqrt{\delta_\varepsilon}} \qquad \text{for all } \varepsilon \in (0, \varepsilon_\star),$$

which implies that for any such ε,

$$e^{-\frac{\Gamma}{\eta_\varepsilon}t} \ge e^{-\ln\frac{1}{\sqrt{\delta_\varepsilon}}} = \sqrt{\delta_\varepsilon} \qquad \text{for all } t \in (0,T)$$

and hence, by Lemma 2.2,

$$M \ge w_\varepsilon(x,t) \ge \sqrt{\delta_\varepsilon} \cdot e^{-\frac{\Gamma}{\eta_\varepsilon}t} \ge \delta_\varepsilon \qquad \text{for all } x \in \Omega \text{ and } t \in (0,\widehat{T}_\varepsilon).$$

Therefore, (2.5) applies so as to ensure that

$$g(w_\varepsilon) \ge \varepsilon \qquad \text{for all } x \in \Omega \text{ and } t \in (0,\widehat{T}_\varepsilon),$$

so that the term in question satisfies

$$\begin{aligned} \frac{\varepsilon K_1}{2}\int_\Omega \frac{w_{\varepsilon x}^2}{\sqrt{g(w_\varepsilon)}^3} &\le \frac{\sqrt{\varepsilon}K_1}{2}\int_\Omega \frac{w_{\varepsilon x}^2}{g(w_\varepsilon)} \\ &\le \frac{K_1}{2}\int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{g(w_\varepsilon)} \qquad \text{for all } t \in (0,\widehat{T}_\varepsilon), \end{aligned} \tag{3.21}$$

because $d_\varepsilon \ge \sqrt{\varepsilon}$ in Ω by (2.1). Together with (3.20) and (3.14), this shows that

$$y_\varepsilon(t) := \int_\Omega u_\varepsilon(\cdot,t)\ln u_\varepsilon(\cdot,t) + \frac{1}{2}\int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2(\cdot,t)}{g(w_\varepsilon(\cdot,t))}, \qquad t \in [0,T_{max,\varepsilon}),$$

and

$$\begin{aligned} h_\varepsilon(t) &:= \frac{1}{2}\int_\Omega d_\varepsilon \frac{u_{\varepsilon x}^2(\cdot,t)}{u_\varepsilon(\cdot,t)} + \frac{1}{2}\int_\Omega d_\varepsilon \frac{u_\varepsilon(\cdot,t)}{1+\eta_\varepsilon u_\varepsilon(\cdot,t)} \frac{g'(w_\varepsilon(\cdot,t))}{g(w_\varepsilon(\cdot,t))} w_{\varepsilon x}^2(\cdot,t) \\ &\quad + \int_{\{u_\varepsilon(\cdot,t)\ge 1\}} u_\varepsilon(\cdot,t)\ln u_\varepsilon(\cdot,t) \cdot f_-(\cdot,u_\varepsilon(\cdot,t),w_\varepsilon(\cdot,t)), \qquad t \in (0,T_{max,\varepsilon}), \end{aligned}$$

have the property that

$$\begin{aligned} y_\varepsilon'(t) + h_\varepsilon(t) &\le c_1 + c_3 + \rho(M)\int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{K_1}{2}\int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{g(w_\varepsilon)} \\ &= c_1 + c_3 + \rho(M)\cdot\Big\{y_\varepsilon(t) - \frac{1}{2}\int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{g(w_\varepsilon)}\Big\} + K_1 \cdot \Big\{y_\varepsilon(t) - \int_\Omega u_\varepsilon \ln u_\varepsilon\Big\} \\ &\le c_4 + c_5 y_\varepsilon(t) \qquad \text{for all } t \in (0,\widehat{T}_\varepsilon) \end{aligned} \tag{3.22}$$

with $c_4 := c_1 + c_3 + \frac{K_1|\Omega|}{e}$ and $c_5 := \rho(M) + K_1$, where we again have used (3.17).
Now by nonnegativity of h_ε and (2.1), an integration of (3.22) firstly yields

$$\begin{aligned} y_\varepsilon(t) &\le y_\varepsilon(0)e^{c_5 t} + \frac{c_4}{c_5}e^{c_5 t}\Big(1 - e^{-c_5 t}\Big) \\ &\le c_6 := \Big\{\int_\Omega u_0 \ln u_0 + \frac{1}{2}\Big(\|d\|_{L^\infty(\Omega)} + 1\Big)\cdot \sup_{\varepsilon\in(0,\varepsilon_0)}\int_\Omega \frac{w_{0x}^2}{g(w_0+\delta_\varepsilon)} + \frac{c_4}{c_5}\Big\}\cdot e^{c_5 T} \end{aligned} \tag{3.23}$$

for all $t \in [0, \widehat{T}_\varepsilon)$ and $\varepsilon \in (0, \varepsilon_\star)$, where we note that c_6 is finite according to (1.9), because $\delta_\varepsilon \to 0$ as $\varepsilon \searrow 0$, and because due to the fact that $g' \geq 0$ on $[0, M]$, as asserted by (1.12), Beppo Levi's theorem warrants that as $k \to \infty$ we have

$$\int_\Omega \frac{w_{0x}^2}{g(w_0 + \frac{1}{k})} \nearrow \int_\Omega \frac{w_{0x}^2}{g(w_0)} < \infty.$$

Once more in view of (3.17), this entails both (3.9) and (3.10) with some suitably large $C(T) > 0$, whereas another integration of (3.22), this time making use of (3.23), shows that

$$\begin{aligned} \int_0^{\widehat{T}_\varepsilon} h_\varepsilon(t)dt &\leq y_\varepsilon(0) - y_\varepsilon(\widehat{T}_\varepsilon) + c_4\widehat{T}_\varepsilon + c_5 \int_0^{\widehat{T}_\varepsilon} y_\varepsilon(t)dt \\ &\leq c_7 := c_6 + \frac{|\Omega|}{e} + c_4 T + c_5 c_6 T, \end{aligned} \tag{3.24}$$

and that hence (3.10)-(3.13) hold with some possibly enlarged $C(T)$. $\square$

Due to the boundedness property (2.11) of w_ε, (1.12) and (1.15), from (3.10) and (3.13) we particularly obtain corresponding estimates for integrals no longer containing $\frac{1}{g(w_\varepsilon)}$ and $\frac{g'(w_\varepsilon)}{g(w_\varepsilon)}$.

Corollary 3.4 *Suppose that $T > 0$, and let $\varepsilon_\star(T) \in (0, \varepsilon_0)$ be as given by Lemma 3.3. Then there exists $C(T) > 0$ with the property that for all $\varepsilon \in (0, \varepsilon_\star(T))$,*

$$\int_\Omega d_\varepsilon w_{\varepsilon x}^2(\cdot, t) \leq C(T) \qquad \textit{for all } t \in (0, \widehat{T}_\varepsilon) \tag{3.25}$$

and

$$\int_0^{\widehat{T}_\varepsilon} \int_\Omega d_\varepsilon \frac{u_\varepsilon}{1 + \eta_\varepsilon u_\varepsilon} w_{\varepsilon x}^2 \leq C(T), \tag{3.26}$$

where again $\widehat{T}_\varepsilon := \min\{T, T_{max,\varepsilon}\}$.

Proof. Since $g(w_\varepsilon) \leq g(M)$ in $\Omega \times (0, T_{max,\varepsilon})$ by Lemma 2.2 and (1.12), we immediately obtain (3.25) from (3.10). As furthermore (1.15) warrants that

$$\frac{g'(w_\varepsilon)}{g(w_\varepsilon)} \geq \gamma > 0 \qquad \text{in } \Omega \times (0, T_{max,\varepsilon})$$

by Lemma 2.2, we also infer from (3.13) that (3.26) is valid with some adequately large $C(T) > 0$. $\square$

As one consequence of (3.25) when combined with the boundedness of $\frac{u_\varepsilon}{(1+\eta_\varepsilon u_\varepsilon)^2}$ and the uniform positivity of d_ε, we can infer that in fact our approximate solutions cannot blow up in finite time:

Lemma 3.5 *For each $\varepsilon \in (0, \varepsilon_0)$, the solution of (2.9) is global in time; that is, in Lemma 2.1 we have $T_{max,\varepsilon} = \infty$.*

Proof. Assuming on the contrary that $T_{max,\varepsilon}$ be finite, combining (2.12) with (1.12) we first obtain that then there would exist $c_1 > 0$ such that

$$\frac{1}{g(w_\varepsilon)} \leq c_1 \qquad \text{in } \Omega \times (0, T_{max,\varepsilon}). \tag{3.27}$$

Moreover, as $d_\varepsilon > 0$ in $\bar\Omega$ by (2.1), Corollary 3.4 and Lemma 2.2 would yield $c_2 > 0$ fulfilling

$$\|w_\varepsilon(\cdot,t)\|_{W^{1,2}(\Omega)} \le c_2 \qquad \text{for all } t \in (0, T_{max,\varepsilon}). \tag{3.28}$$

In particular, the latter along with (2.1) and the fact that $\frac{\xi}{(1+\eta_\varepsilon \xi)^2} \le \frac{1}{4\eta_\varepsilon}$ for all $\xi \ge 0$ ensures that the cross-diffusive flux in the first equation in (2.9) satisfies

$$\Big\| d_\varepsilon \frac{u_\varepsilon(\cdot,t)}{(1+\eta_\varepsilon u_\varepsilon(\cdot,t))^2} w_{\varepsilon x}(\cdot,t)\Big\|_{L^2(\Omega)} \le \Big(\|d\|_{L^\infty(\Omega)} + 1\Big) \cdot \frac{1}{4\eta_\varepsilon} \cdot c_2 \qquad \text{for all } t \in (0, T_{max,\varepsilon}).$$

Since furthermore, by (1.10) and again Lemma 2.2,

$$f(\cdot, u_\varepsilon, w_\varepsilon) \le \rho(M) \qquad \text{in } \Omega \times (0, T_{max,\varepsilon}),$$

a standard argument based on the smoothing properties of the non-degenerate linear semigroup $(e^{t(d_\varepsilon \cdot)_{xx}})_{t\ge0}$ (cf. e.g. the reasoning in [3, Lemma 3.2]) provides $c_3 > 0$ such that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le c_3 \qquad \text{for all } t \in (0, T_{max,\varepsilon}).$$

In view of the extensibility criterion (2.10), together with (3.27) and (3.28) this shows that our assumption $T_{max,\varepsilon} < \infty$ was absurd. $\square$

4 Weak precompactness properties of $u_\varepsilon f(x, u_\varepsilon, w_\varepsilon)$ and $u_\varepsilon g(w_\varepsilon)$ in L^1

In appropriately passing to the limit in the zero-order integrals appearing in the respective weak formulations of (2.9), we shall make essential use of two compactness properties of the solutions thereof which appear to go beyond trivial implications of the bounds provided by Lemma 3.3. As a preparation for our arguments in this respect, let us state the following observation on a lower bound for all possible values of $u \ge 0$ at which $u \cdot f_-(x,u,w)$ may become large for some $x \in \bar\Omega$ and $w \in [0, M]$. This will be used in Lemma 4.2 to assert that for arbitrarily large $\kappa > 0$ one can pick $N > 0$ in such a way that whenever $u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon) \ge N$, we know that $u_\varepsilon \ge \kappa$.

Lemma 4.1 *With $M > 0$ as in (1.11), let*

$$\mathcal{S}(N) := \Big\{ u \ge 0 \ \Big| \ u \cdot f_-(x,u,w) \ge N \textit{ for some } x \in \bar\Omega \textit{ and } w \in [0, M] \Big\} \tag{4.1}$$

and

$$\kappa(N) := \begin{cases} \inf \mathcal{S}(N) & \textit{if } \mathcal{S}(N) \ne \emptyset, \\ +\infty & \textit{else} \end{cases} \tag{4.2}$$

for $N \in \mathbb{N}$. Then

$$\limsup_{N\to\infty} \kappa(N) = +\infty. \tag{4.3}$$

Proof. If (4.3) was false, then there would exist $N_0 \in \mathbb{N}$ such that for all $N \ge N_0$ we would have $\mathcal{S}(N) \ne \emptyset$ and $\kappa(N) < c_1$ with some $c_1 > 0$. By definition of $\mathcal{S}(N)$ and $\kappa(N)$, this would mean that we could find $(x_N)_{N\ge N_0} \subset \bar\Omega$, $(u_N)_{N\ge N_0} \subset [0,\infty)$ and $(w_N)_{N \ge N_0} \subset [0, M]$ fulfilling

$$u_N \cdot f_-(x_N, u_N, w_N) \ge N \qquad \text{for all } N \ge N_0 \tag{4.4}$$

and

$$u_N \le c_1 \qquad \text{for all } N \ge N_0,$$

where passing to a subsequence if necessary we may assume that as $N \to \infty$ we have $x_N \to x_\infty$, $u_N \to u_\infty$ and $w_N \to w_\infty$ with some $x_\infty \in \bar{\Omega}$, $u_\infty \in [0, c_1]$ and $w_\infty \in [0, M]$. By continuity of f_-, however, this would imply that

$$u_N \cdot f_-(x_N, u_N, w_N) \to u_\infty \cdot f_-(x_\infty, u_\infty, w_\infty) \qquad \text{as } N \to \infty$$

and thereby contradict (4.4). □

Making use of the latter, by means of the Dunford-Pettis theorem we can now establish suitable compactness properties of the rightmost summands in the first two equations in (2.9).

Lemma 4.2 *Let $T > 0$. Then with $\varepsilon_\star(T) \in (0, \varepsilon_0)$ as in Lemma 3.3,*

$$\Big(u_\varepsilon f(\cdot, u_\varepsilon, w_\varepsilon)\Big)_{\varepsilon \in (0, \varepsilon_\star(T))} \quad \textit{is relatively compact with respect to the weak topology in } L^1(\Omega \times (0,T)), \tag{4.5}$$

and moreover

$$\Big(u_\varepsilon g(w_\varepsilon)\Big)_{\varepsilon \in (0, \varepsilon_\star(T))} \quad \textit{is relatively compact with respect to the weak topology in } L^1(\Omega \times (0,T)). \tag{4.6}$$

Proof. According to Lemma 3.3, we can fix positive constants c_1 and c_2 such that

$$\int_{\{u_\varepsilon(\cdot,t) \ge 1\}} u_\varepsilon(\cdot,t) \ln u_\varepsilon(\cdot,t) \le c_1 \qquad \text{for all } t \in (0,T) \tag{4.7}$$

and

$$\int_0^T \int_{\{u_\varepsilon(\cdot,t) \ge 1\}} u_\varepsilon \ln u_\varepsilon \cdot f_-(\cdot, u_\varepsilon, w_\varepsilon) \le c_2 \tag{4.8}$$

whenever $\varepsilon \in (0, \varepsilon_\star(T))$. Aiming at an application of the Dunford-Pettis theorem, given $\mu > 0$ we first fix an integer $N \ge 1$ suitably large such that

$$\frac{c_1 \rho(M) T}{\ln N} < \frac{\mu}{4} \tag{4.9}$$

and

$$\frac{c_1 g(M) T}{\ln N} < \frac{\mu}{2}, \tag{4.10}$$

and such that with $\kappa(N)$ as defined in Lemma 4.1 we have $\kappa(N) > 1$ and

$$\frac{c_2}{\ln \kappa(N)} < \frac{\mu}{4}, \tag{4.11}$$

where the latter is possible due to the outcome of Lemma 4.1. Thereafter, we choose $\iota > 0$ small enough fulfilling

$$\rho(M) N \iota < \frac{\mu}{4} \tag{4.12}$$

and

$$N\iota < \frac{\mu}{4} \tag{4.13}$$

as well as

$$g(M)N\iota < \frac{\mu}{2}, \tag{4.14}$$

and fix an arbitrary measurable set $E \subset \Omega \times (0,T)$ satisfying $|E| < \iota$. Then decomposing

$$\int\int_E \Big| u_\varepsilon f(\cdot, u_\varepsilon, w_\varepsilon) \Big| = \int\int_E u_\varepsilon f_+(\cdot, u_\varepsilon, w_\varepsilon) + \int\int_I u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon), \tag{4.15}$$

by combining (4.7) with Lemma 2.2, (1.10) and (4.12) we can estimate

$$\begin{aligned}
\int_E u_\varepsilon f_+(\cdot, u_\varepsilon, w_\varepsilon) &= \int\int_{E\cap\{u_\varepsilon<N\}} u_\varepsilon f_+(\cdot, u_\varepsilon, w_\varepsilon) + \int_{E\cap\{u_\varepsilon\ge N\}} u_\varepsilon f_+(\cdot, u_\varepsilon, w_\varepsilon) \\
&\le \rho(M) \int_{E\cap\{u_\varepsilon<N\}} u_\varepsilon + \frac{\rho(M)}{\ln N} \int\int_{E\cap\{u_\varepsilon\ge N\}} u_\varepsilon \ln u_\varepsilon \\
&\le \rho(M)\cdot N|E| + \frac{\rho(M)}{\ln N} \int_0^T \int_{\{u_\varepsilon(\cdot,t)\ge 1\}} u_\varepsilon \ln u_\varepsilon \\
&\le \rho(M)N\iota + \frac{\rho(M)}{\ln N}\cdot c_1 T \\
&< \frac{\mu}{4} + \frac{\mu}{4} = \frac{\mu}{2} \qquad \text{for all } \varepsilon \in (0, \varepsilon_\star(T)).
\end{aligned} \tag{4.16}$$

Likewise, relying on (4.13) we see that

$$\begin{aligned}
\int\int_E u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon) &= \int\int_{E\cap\{u_\varepsilon f_-(\cdot,u_\varepsilon,w_\varepsilon)<N\}} u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon) + \int\int_{E\cap\{u_\varepsilon f_-(\cdot,u_\varepsilon,w_\varepsilon)\ge N\}} u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon) \\
&\le N|E| + \int\int_{E\cap\{u_\varepsilon f_-(\cdot,u_\varepsilon,w_\varepsilon)\ge N\}} u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon) \\
&< \frac{\mu}{4} + \int\int_{E\cap\{u_\varepsilon f_-(\cdot,u_\varepsilon,w_\varepsilon)\ge N\}} u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon) \qquad \text{for all } \varepsilon \in (0, \varepsilon_\star(T)),
\end{aligned} \tag{4.17}$$

and in order to appropriately control the last summand herein we recall the definition (4.2) of $\kappa(N)$ to observe that whenever $u_\varepsilon(x,t)f_-(x, u_\varepsilon(x,t), w_\varepsilon(x,t)) \ge N$ for some $\varepsilon \in (0, \varepsilon_0)$, $x \in \bar\Omega$ and $t \ge 0$, we necessarily must have $u_\varepsilon(x,t) \ge \kappa(N)$. Consequently, $E \cap \{u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon) \ge N\} \subset E \cap \{u_\varepsilon \ge \kappa(N)\}$, so that (4.8) and (4.11) become applicable so as to guarantee that

$$\begin{aligned}
\int\int_{E\cap\{u_\varepsilon f_-(\cdot,u_\varepsilon,w_\varepsilon)\ge N\}} u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon) &\le \int\int_{E\cap\{u_\varepsilon\ge\kappa(N)\}} u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon) \\
&\le \frac{1}{\ln\kappa(N)} \int\int_{E\cap\{u_\varepsilon\ge\kappa(N)\}} u_\varepsilon \ln u_\varepsilon \cdot f_-(\cdot, u_\varepsilon, w_\varepsilon) \\
&\le \frac{1}{\ln\kappa(N)} \cdot c_2 \\
&< \frac{\mu}{4} \qquad \text{for all } \varepsilon \in (0, \varepsilon_\star(T)),
\end{aligned}$$

which along with (4.15), (4.16) and (4.17) shows that for any such E we have

$$\int\int_E \Big| u_\varepsilon f(\cdot, u_\varepsilon, w_\varepsilon)\Big| < \mu \qquad \text{for all } \varepsilon \in (0, \varepsilon_\star(T)). \tag{4.18}$$

Similarly, using Lemma 2.2 together with (1.12) we obtain

$$\begin{aligned}
\int\int_E \Big| u_\varepsilon g(w_\varepsilon)\Big| &= \int\int_{E\cap\{u_\varepsilon<N\}} u_\varepsilon g(w_\varepsilon) + \int\int_{E\cap\{u_\varepsilon\ge N\}} u_\varepsilon g(w_\varepsilon) \\
&\le g(M)\int\int_{E\cap\{u_\varepsilon<N\}} u_\varepsilon + g(M)\int\int_{E\cap\{u_\varepsilon\ge N\}} u_\varepsilon \\
&\le g(M)\cdot N|E| + \frac{g(M)}{\ln N}\int_0^T\int_{\{u_\varepsilon(\cdot,t)\ge 1\}} u_\varepsilon \\
&\le g(M)N\iota + \frac{g(M)}{\ln N}\cdot c_1 T \\
&< \frac{\mu}{2} + \frac{\mu}{2} = \mu \qquad \text{for all } \varepsilon \in (0, \varepsilon_\star(T)),
\end{aligned}$$

because of (4.10) and (4.14). By means of the Dunford-Pettis compactness criterion, from this we infer that (4.6) holds, and that (4.5) is a consequence of (4.18). $\square$

5 Regularity properties of $\sqrt{d_\varepsilon}u_\varepsilon$

In order to further prepare our limit procedure, especially with regard to pointwise convergence of u_ε and of convergence in the cross-diffusive flux term $d_\varepsilon \frac{u_\varepsilon}{(1+\eta_\varepsilon u_\varepsilon)^2} w_{\varepsilon x}$ in (2.9), we next plan to combine the weak compactness feature of the part $\sqrt{d_\varepsilon}w_{\varepsilon x}$ thereof, as naturally implied by Corollary 3.4, by an appropriate result on convergence in the complementary factor $\sqrt{d_\varepsilon}\frac{u_\varepsilon}{(1+\eta_\varepsilon u_\varepsilon)^2}$ in a strong L^2 topology. To achieve this in Lemma 8.1 by using underway an argument based on the Aubin-Lions lemma, let us suitably interpolate between the inequalities in (2.14) and (3.11) to derive the following spatio-temporal estimates for the quantity $\sqrt{d_\varepsilon}u_\varepsilon$ forming the core of the factor in question.

Lemma 5.1 *Let $T > 0$ and $\varepsilon_\star(T) \in (0, \varepsilon_0)$ be as in Lemma 3.3. Then there exists $C(T) > 0$ such that for all $\varepsilon \in (0, \varepsilon_\star(T))$,*

$$\int_0^T \Big\|(\sqrt{d_\varepsilon}u_\varepsilon(\cdot,t))_x\Big\|^2_{L^1(\Omega)} dt \le C(T) \tag{5.1}$$

and

$$\int_0^T \Big\|\sqrt{d_\varepsilon}u_\varepsilon(\cdot,t)\Big\|^2_{L^\infty(\Omega)} dt \le C(T) \tag{5.2}$$

as well as

$$\int_0^T\int_\Omega \sqrt{d_\varepsilon}^3 u_\varepsilon^3 \le C(T). \tag{5.3}$$

Proof. According to Lemma 2.3 and Lemma 3.3, there exist $c_1 = c_1(T) > 0$ and $c_2 = c_2(T) > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ we have

$$\int_\Omega u_\varepsilon \le c_1 \qquad \text{for all } t \in (0, T), \tag{5.4}$$

and that

$$\int_0^T \int_\Omega d_\varepsilon \frac{u_{\varepsilon x}^2}{u_\varepsilon} \le c_2 \tag{5.5}$$

whenever $\varepsilon \in (0, \varepsilon_\star(T))$. Since

$$\begin{aligned}
\Big|(\sqrt{d_\varepsilon} u_\varepsilon)_x\Big| &= \Big|\sqrt{d_\varepsilon} u_{\varepsilon x} + \frac{d_{\varepsilon x}}{2\sqrt{d_\varepsilon}} u_\varepsilon\Big| \\
&\le \sqrt{d_\varepsilon}|u_{\varepsilon x}| + \frac{\sqrt{K_1}}{2} u_\varepsilon \qquad \text{in } \Omega \times (0,\infty)
\end{aligned}$$

due to (2.2), by the Cauchy-Schwarz inequality these estimates imply that

$$\begin{aligned}
\int_0^T \Big\|(\sqrt{d_\varepsilon} u_\varepsilon(\cdot,t))_x\Big\|_{L^1(\Omega)}^2 dt &\le \int_0^T \bigg\{ \int_\Omega \sqrt{d_\varepsilon}|u_{\varepsilon x}| \bigg\}^2 + \frac{\sqrt{K_1}}{2} \int_0^T \bigg\{ \int_\Omega u_\varepsilon \bigg\}^2 \\
&\le \int_0^T \bigg\{ \int_\Omega d_\varepsilon \frac{u_{\varepsilon x}^2}{u_\varepsilon} \bigg\} \cdot \bigg\{ \int_\Omega u_\varepsilon \bigg\} + \frac{\sqrt{K_1}}{2} \int_0^T \bigg\{ \int_\Omega u_\varepsilon \bigg\}^2 \\
&\le c_3 \equiv c_3(T) := c_1 c_2 + \frac{\sqrt{K_1}}{2} c_1^2 T \qquad \text{for all } \varepsilon \in (0, \varepsilon_\star(T))
\end{aligned}$$

and thereby proves (5.1), whereupon (5.2) follows from Lemma 2.3 and the fact that $W^{1,1}(\Omega) \hookrightarrow L^\infty(\Omega)$. As the Gagliardo-Nirenberg inequality provides $c_4 > 0$ such that

$$\|\varphi\|_{L^3(\Omega)}^3 \le c_4 \|\varphi_x\|_{L^1(\Omega)}^2 \|\varphi\|_{L^1(\Omega)} + c_4 \|\varphi\|_{L^1(\Omega)}^3 \qquad \text{for all } \varphi \in W^{1,1}(\Omega),$$

in view of (2.1) this furthermore entails that with $c_5 := \sqrt{\|d\|_{L^\infty(\Omega)} + 1}$ we have

$$\begin{aligned}
\int_0^T \int_\Omega \sqrt{d_\varepsilon}^3 u_\varepsilon^3 &= \int_0^T \Big\|\sqrt{d_\varepsilon} u_\varepsilon(\cdot,t)\Big\|_{L^3(\Omega)}^3 dt \\
&\le c_4 \int_0^T \Big\|(\sqrt{d_\varepsilon} u_\varepsilon(\cdot,t))_x\Big\|_{L^1(\Omega)}^2 \cdot \Big\|\sqrt{d_\varepsilon} u_\varepsilon(\cdot,t)\Big\|_{L^1(\Omega)} dt + c_4 \int_0^T \Big\|\sqrt{d_\varepsilon} u_\varepsilon(\cdot,t)\Big\|_{L^1(\Omega)}^3 dt \\
&\le c_4 c_5 \big\|u_\varepsilon\big\|_{L^\infty((0,T);L^1(\Omega))} \int_0^T \Big\|(\sqrt{d_\varepsilon} u_\varepsilon(\cdot,t))_x\Big\|_{L^1(\Omega)}^2 dt + c_4 c_5^3 \int_0^T \Big\|u_\varepsilon(\cdot,t)\Big\|_{L^1(\Omega)}^3 dt \\
&\le c_4 c_5 c_1 c_3 + c_4 c_5^3 c_1^3 T \qquad \text{for all } \varepsilon \in (0, \varepsilon_\star(T)),
\end{aligned}$$

and that thus also (5.3) holds. □

6 Regularity in time

As a final preparation for our first subsequence extraction procedure, we combine our previously gained estimates to obtain some regularity properties involving time derivatives of the solution components u_ε and w_ε.

Lemma 6.1 *Let $T>0$ and $\varepsilon_\star(T)\in(0,\varepsilon_0)$ be as in Lemma 3.3. Then there exists $C(T)>0$ such that*

$$\int_0^T \Big\|\partial_t\sqrt{d_\varepsilon(u_\varepsilon(\cdot,t)+1)}\Big\|_{(W^{1,3}(\Omega))^\star}dt\le C(T) \qquad \textit{for all } \varepsilon\in(0,\varepsilon_\star(T)). \tag{6.1}$$

Proof. For fixed $t>0$ and $\psi\in C^1(\bar\Omega)$, from the first equation in (2.9) we obtain that

$$\begin{aligned}
\int_\Omega \partial_t\sqrt{d_\varepsilon(u_\varepsilon(\cdot,t)+1)}\psi &= -\frac12\int_\Omega\Big(\sqrt{d_\varepsilon}\frac{1}{\sqrt{u_\varepsilon+1}}\psi\Big)_x\cdot(d_\varepsilon u_\varepsilon)_x\\
&\quad+\frac12\int_\Omega\Big(\sqrt{d_\varepsilon}\frac{1}{\sqrt{u_\varepsilon+1}}\psi\Big)_x\cdot d_\varepsilon\frac{u_\varepsilon}{(1+\eta_\varepsilon u_\varepsilon)^2}w_{\varepsilon x}\\
&\quad+\frac12\int_\Omega\sqrt{d_\varepsilon}\frac{1}{\sqrt{u_\varepsilon+1}}u_\varepsilon f(\cdot,u_\varepsilon,w_\varepsilon)\psi\\
&= \frac14\int_\Omega\sqrt{d_\varepsilon}^3\frac{1}{\sqrt{u_\varepsilon+1}^3}u_{\varepsilon x}^2\psi+\frac14\int_\Omega\sqrt{d_\varepsilon}d_{\varepsilon x}\frac{u_\varepsilon}{\sqrt{u_\varepsilon+1}^3}u_{\varepsilon x}\psi\\
&\quad-\frac14\int_\Omega\sqrt{d_\varepsilon}d_{\varepsilon x}\frac{1}{\sqrt{u_\varepsilon+1}}u_{\varepsilon x}\psi-\frac14\int_\Omega\frac{d_{\varepsilon x}^2}{\sqrt{d_\varepsilon}}\frac{u_\varepsilon}{\sqrt{u_\varepsilon+1}}\psi\\
&\quad-\frac12\int_\Omega\sqrt{d_\varepsilon}^3\frac{1}{\sqrt{u_\varepsilon+1}}u_{\varepsilon x}\psi_x-\frac12\int_\Omega\sqrt{d_\varepsilon}d_{\varepsilon x}\frac{u_\varepsilon}{\sqrt{u_\varepsilon+1}}\psi_x\\
&\quad-\frac14\int_\Omega\sqrt{d_\varepsilon}^3\frac{u_\varepsilon}{\sqrt{u_\varepsilon+1}^3(1+\eta_\varepsilon u_\varepsilon)^2}u_{\varepsilon x}w_{\varepsilon x}\psi\\
&\quad+\frac14\int_\Omega\sqrt{d_\varepsilon}d_{\varepsilon x}\frac{u_\varepsilon}{\sqrt{u_\varepsilon+1}(1+\eta_\varepsilon u_\varepsilon)^2}w_{\varepsilon x}\psi\\
&\quad+\frac12\int_\Omega\sqrt{d_\varepsilon}^3\frac{u_\varepsilon}{\sqrt{u_\varepsilon+1}(1+\eta_\varepsilon u_\varepsilon)^2}w_{\varepsilon x}\psi_x\\
&\quad+\frac12\int_\Omega\sqrt{d_\varepsilon}\frac{u_\varepsilon}{\sqrt{u_\varepsilon+1}}f(\cdot,u_\varepsilon,w_\varepsilon)\psi \qquad \text{for all } \varepsilon\in(0,\varepsilon_0).
\end{aligned}\tag{6.2}$$

Here using (2.1) and Young's inequality, we see that

$$\begin{aligned}
\left|\frac14\int_\Omega\sqrt{d_\varepsilon}^3\frac{1}{\sqrt{u_\varepsilon+1}^3}u_{\varepsilon x}^2\psi\right| &\le \frac{\sqrt{c_1}}{4}\Big\{\int_\Omega d_\varepsilon\frac{u_{\varepsilon x}^2}{u_\varepsilon}\Big\}^{\frac12}\|\psi\|_{L^\infty(\Omega)}\\
&\le \frac{\sqrt{c_1}}{8}\Big\{\int_\Omega d_\varepsilon\frac{u_{\varepsilon x}^2}{u_\varepsilon}+1\Big\}\|\psi\|_{L^\infty(\Omega)}
\end{aligned}\tag{6.3}$$

with $c_1:=\|d\|_{L^\infty(\Omega)}+1$, and similarly,

$$\begin{aligned}
\left|\frac14\int_\Omega\sqrt{d_\varepsilon}d_{\varepsilon x}\frac{u_\varepsilon}{\sqrt{u_\varepsilon+1}^3}u_{\varepsilon x}\psi\right| &\le \frac14\Big\{\int_\Omega d_\varepsilon\frac{u_{\varepsilon x}^2}{u_\varepsilon}\Big\}^{\frac12}\cdot\Big\{\int_\Omega d_{\varepsilon x}^2\frac{u_\varepsilon^3}{(u_\varepsilon+1)^3}\psi^2\Big\}^{\frac12}\\
&\le \frac{\sqrt{K_1c_1}}{4}\Big\{\int_\Omega d_\varepsilon\frac{u_{\varepsilon x}^2}{u_\varepsilon}\Big\}^{\frac12}\|\psi\|_{L^2(\Omega)}\\
&\le \frac{\sqrt{K_1c_1}}{8}\Big\{\int_\Omega d_\varepsilon\frac{u_{\varepsilon x}^2}{u_\varepsilon}+1\Big\}\|\psi\|_{L^2(\Omega)},
\end{aligned}\tag{6.4}$$

because

$$d_{\varepsilon x}^2 \le K_1 d_\varepsilon \le K_1 c_1 \qquad \text{in } \Omega \tag{6.5}$$

thanks to (2.2) and (2.1). Next, by the Hölder inequality and again due to (6.5), (2.2), (2.1) and Young's inequality,

$$\begin{aligned}
\left| -\frac{1}{4}\int_\Omega \sqrt{d_\varepsilon} d_{\varepsilon x} \frac{1}{\sqrt{u_\varepsilon+1}} u_{\varepsilon x}\psi \right| &\le \frac{1}{4}\left\{ \int_\Omega d_\varepsilon \frac{u_{\varepsilon x}^2}{u_\varepsilon} \right\}^{\frac{1}{2}} \cdot \left\{ \int_\Omega d_{\varepsilon x}^2 \frac{u_\varepsilon}{u_\varepsilon+1}\psi^2 \right\}^{\frac{1}{2}} \\
&\le \frac{\sqrt{K_1 c_1}}{4} \left\{ \int_\Omega d_\varepsilon \frac{u_{\varepsilon x}^2}{u_\varepsilon} \right\}^{\frac{1}{2}} \|\psi\|_{L^2(\Omega)} \\
&\le \frac{\sqrt{K_1 c_1}}{8} \left\{ \int_\Omega d_\varepsilon \frac{u_{\varepsilon x}^2}{u_\varepsilon} + 1 \right\} \|\psi\|_{L^2(\Omega)}
\end{aligned} \tag{6.6}$$

and

$$\begin{aligned}
\left| -\frac{1}{4}\int_\Omega \frac{d_{\varepsilon x}^2}{\sqrt{d_\varepsilon}} \frac{u_\varepsilon}{\sqrt{u_\varepsilon+1}}\psi \right| &\le \frac{1}{4}\left\{ \int_\Omega \sqrt{d_\varepsilon}^3 u_\varepsilon^3 \right\}^{\frac{1}{6}} \cdot \left\{ \int_\Omega d_\varepsilon^{-\frac{9}{10}} |d_{\varepsilon x}|^{\frac{12}{5}} \Big(\frac{u_\varepsilon}{u_\varepsilon+1}\Big)^{\frac{3}{5}} |\psi|^{\frac{6}{5}} \right\}^{\frac{5}{6}} \\
&\le \frac{K_1}{4}\left\{ \int_\Omega \sqrt{d_\varepsilon}^3 u_\varepsilon^3 \right\}^{\frac{1}{6}} \cdot \left\{ \int_\Omega d_\varepsilon^{\frac{3}{10}} |\psi|^{\frac{6}{5}} \right\}^{\frac{5}{6}} \\
&\le \frac{K_1 c_1^{\frac{1}{4}}}{4}\left\{ \int_\Omega \sqrt{d_\varepsilon}^3 u_\varepsilon^3 \right\}^{\frac{1}{6}} \|\psi\|_{L^{\frac{6}{5}}(\Omega)} \\
&\le \frac{K_1 c_1^{\frac{1}{4}}}{24}\left\{ \int_\Omega \sqrt{d_\varepsilon}^3 u_\varepsilon^3 + 1 \right\} \|\psi\|_{L^{\frac{6}{5}}(\Omega)}
\end{aligned} \tag{6.7}$$

as well as

$$\begin{aligned}
\left| -\frac{1}{2}\int_\Omega \sqrt{d_\varepsilon}^3 \frac{1}{\sqrt{u_\varepsilon+1}} u_{\varepsilon x}\psi_x \right| &\le \frac{1}{2}\left\{ \int_\Omega d_\varepsilon \frac{u_{\varepsilon x}^2}{u_\varepsilon} \right\}^{\frac{1}{2}} \cdot \left\{ \int_\Omega d_\varepsilon^2 \frac{u_\varepsilon}{u_\varepsilon+1}\psi_x^2 \right\}^{\frac{1}{2}} \\
&\le \frac{c_1}{2} \left\{ \int_\Omega d_\varepsilon \frac{u_{\varepsilon x}^2}{u_\varepsilon} \right\}^{\frac{1}{2}} \|\psi_x\|_{L^2(\Omega)} \\
&\le \frac{c_1}{4} \left\{ \int_\Omega d_\varepsilon \frac{u_{\varepsilon x}^2}{u_\varepsilon} + 1 \right\} \|\psi_x\|_{L^2(\Omega)}
\end{aligned} \tag{6.8}$$

and

$$\begin{aligned}
\left| -\frac{1}{2}\int_\Omega \sqrt{d_\varepsilon} d_{\varepsilon x} \frac{u_\varepsilon}{\sqrt{u_\varepsilon+1}}\psi_x \right| &\le \frac{1}{2}\left\{ \int_\Omega \sqrt{d_\varepsilon}^3 u_\varepsilon^3 \right\}^{\frac{1}{6}} \cdot \left\{ d_\varepsilon^{\frac{3}{10}} |d_{\varepsilon x}|^{\frac{6}{5}} \Big(\frac{u_\varepsilon}{u_\varepsilon+1}\Big)^{\frac{3}{5}} |\psi_x|^{\frac{6}{5}} \right\}^{\frac{5}{6}} \\
&\le \frac{\sqrt{K_1}}{2}\left\{ \int_\Omega \sqrt{d_\varepsilon}^3 u_\varepsilon^3 \right\}^{\frac{1}{6}} \cdot \left\{ d_\varepsilon^{\frac{9}{10}} |\psi_x|^{\frac{6}{5}} \right\}^{\frac{5}{6}} \\
&\le \frac{\sqrt{K_1} c_1^{\frac{3}{4}}}{2}\left\{ \int_\Omega \sqrt{d_\varepsilon}^3 u_\varepsilon^3 \right\}^{\frac{1}{6}} \|\psi_x\|_{L^{\frac{6}{5}}(\Omega)} \\
&\le \frac{\sqrt{K_1} c_1^{\frac{3}{4}}}{2}\left\{ \int_\Omega \sqrt{d_\varepsilon}^3 u_\varepsilon^3 + 1 \right\} \|\psi_x\|_{L^{\frac{6}{5}}(\Omega)}.
\end{aligned} \tag{6.9}$$

Likewise, the integrals in (6.2) stemming from the cross-diffusive interaction can be estimated according to

$$\begin{aligned}
\left| -\frac{1}{4}\int_\Omega \sqrt{d_\varepsilon}^3 \frac{u_\varepsilon}{\sqrt{u_\varepsilon+1}^3(1+\eta_\varepsilon u_\varepsilon)^2} u_{\varepsilon x} w_{\varepsilon x}\psi \right| &\le \frac{1}{4}\left\{\int_\Omega d_\varepsilon \frac{u_{\varepsilon x}^2}{u_\varepsilon}\right\}^{\frac{1}{2}} \cdot \left\{\int_\Omega d_\varepsilon^2 \frac{u_\varepsilon^3}{(u_\varepsilon+1)^3} w_{\varepsilon x}^2\right\}^{\frac{1}{2}} \|\psi\|_{L^\infty(\Omega)} \\
&\le \frac{\sqrt{c_1}}{4}\left\{\int_\Omega d_\varepsilon \frac{u_{\varepsilon x}^2}{u_\varepsilon}\right\}^{\frac{1}{2}} \cdot \left\{\int_\Omega d_\varepsilon w_{\varepsilon x}^2\right\}^{\frac{1}{2}} \|\psi\|_{L^\infty(\Omega)} \\
&\le \frac{\sqrt{c_1}}{8}\left\{\int_\Omega d_\varepsilon \frac{u_{\varepsilon x}^2}{u_\varepsilon} + \int_\Omega d_\varepsilon w_{\varepsilon x}^2\right\} \|\psi\|_{L^\infty(\Omega)} \qquad (6.10)
\end{aligned}$$

and

$$\begin{aligned}
\left| \frac{1}{4}\int_\Omega \sqrt{d_\varepsilon} d_{\varepsilon x} \frac{u_\varepsilon}{\sqrt{u_\varepsilon+1}(1+\eta_\varepsilon u_\varepsilon)^2} w_{\varepsilon x}\psi \right| &\le \frac{1}{4}\left\{\int_\Omega d_\varepsilon w_{\varepsilon x}^2\right\}^{\frac{1}{2}} \cdot \left\{\int_\Omega d_{\varepsilon x}^2 \frac{u_\varepsilon^2}{u_\varepsilon+1}\psi^2\right\}^{\frac{1}{2}} \\
&\le \frac{\sqrt{K_1 c_1}}{4}\left\{\int_\Omega d_\varepsilon w_{\varepsilon x}^2\right\}^{\frac{1}{2}} \cdot \left\{\int_\Omega u_\varepsilon\right\}^{\frac{1}{2}} \|\psi\|_{L^2(\Omega)} \\
&\le \frac{\sqrt{K_1 c_1}}{8}\left\{\int_\Omega d_\varepsilon w_{\varepsilon x}^2 + \int_\Omega u_\varepsilon\right\} \|\psi\|_{L^2(\Omega)} \qquad (6.11)
\end{aligned}$$

as well as

$$\begin{aligned}
\left| \frac{1}{2}\int_\Omega \sqrt{d_\varepsilon}^3 \frac{u_\varepsilon}{\sqrt{u_\varepsilon+1}(1+\eta_\varepsilon u_\varepsilon)^2} w_{\varepsilon x}\psi_x \right| &\le \frac{1}{2}\left\{\int_\Omega d_\varepsilon w_{\varepsilon x}^2\right\}^{\frac{1}{2}} \cdot \left\{\int_\Omega d_\varepsilon^2 \frac{u_\varepsilon^2}{u_\varepsilon+1}\psi_x^2\right\}^{\frac{1}{2}} \\
&\le \frac{c_1^{\frac{3}{4}}}{2}\left\{\int_\Omega d_\varepsilon w_{\varepsilon x}^2\right\}^{\frac{1}{2}} \cdot \left\{\int_\Omega \sqrt{d_\varepsilon} u_\varepsilon \psi_x^2\right\}^{\frac{1}{2}} \\
&\le \frac{c_1^{\frac{3}{4}}}{2}\left\{\int_\Omega d_\varepsilon w_{\varepsilon x}^2\right\}^{\frac{1}{2}} \cdot \left\{\int_\Omega \sqrt{d_\varepsilon}^3 u_\varepsilon^3\right\}^{\frac{1}{6}} \|\psi_x\|_{L^3(\Omega)} \\
&\le \frac{c_1^{\frac{3}{4}}}{4}\left\{\int_\Omega d_\varepsilon w_{\varepsilon x}^2 + \int_\Omega \sqrt{d_\varepsilon}^3 u_\varepsilon^3 + 1\right\} \|\psi_x\|_{L^3(\Omega)}. \qquad (6.12)
\end{aligned}$$

Since finally

$$\left| \frac{1}{2}\int_\Omega \sqrt{d_\varepsilon} \frac{u_\varepsilon}{\sqrt{u_\varepsilon+1}} f(\cdot,u_\varepsilon,w_\varepsilon)\psi \right| \le \frac{c_1}{2}\left\{\int_\Omega u_\varepsilon |f(\cdot,u_\varepsilon,w_\varepsilon)|\right\} \|\psi\|_{L^\infty(\Omega)}, \qquad (6.13)$$

and since in the present one-dimensional setting we have $W^{1,3}(\Omega) \subset W^{1,\frac{6}{5}}(\Omega) \hookrightarrow L^\infty(\Omega) \subset L^3(\Omega) \subset L^2(\Omega) \subset L^{\frac{6}{5}}(\Omega)$, in view of the estimates implied by Lemma 3.3, Corollary 3.4, Lemma 5.1, Lemma 2.3 and Lemma 4.2 we only need to collect (6.3), (6.4) and (6.6)-(6.13) to derive (6.1) from (6.2). $\Box$

It may be not surprising that our derivation of a corresponding property of w_ε is much less involved:

Lemma 6.2 *Let $T > 0$, and let $\varepsilon_\star(T) \in (0, \varepsilon_0)$ be as in Lemma 3.3. Then one can find $C(T) > 0$ such that*

$$\int_0^T \left\| \partial_t\Big(\sqrt{d_\varepsilon} w_\varepsilon(\cdot,t)\Big) \right\|_{(W^{1,2}(\Omega))^\star}^3 dt \le C(T) \qquad \textit{for all } \varepsilon \in (0, \varepsilon_\star(T)). \qquad (6.14)$$

Proof. For arbitrary $\psi \in C^1(\bar\Omega)$, the second equation in (2.9) shows that

$$\begin{aligned}
\int_\Omega \partial_t\Big(\sqrt{d_\varepsilon} w_\varepsilon(\cdot,t)\Big)\psi &= -\varepsilon\int_\Omega \frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\big(\sqrt{d_\varepsilon}\psi\big)_x - \int_\Omega \sqrt{d_\varepsilon}\frac{u_\varepsilon}{1+\eta_\varepsilon u_\varepsilon} g(w_\varepsilon)\psi \\
&= -\frac{\varepsilon}{2}\int_\Omega \frac{d_{\varepsilon x}}{\sqrt{d_\varepsilon}}\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\psi - \varepsilon\int_\Omega \sqrt{d_\varepsilon}\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\psi_x \\
&\quad - \int_\Omega \sqrt{d_\varepsilon}\frac{u_\varepsilon}{1+\eta_\varepsilon u_\varepsilon} g(w_\varepsilon)\psi \qquad (6.15)
\end{aligned}$$

for all $\varepsilon \in (0,\varepsilon_0)$, where by the Cauchy-Schwarz inequality and (2.2),

$$\begin{aligned}
\left| -\frac{\varepsilon}{2}\int_\Omega \frac{d_{\varepsilon x}}{\sqrt{d_\varepsilon}}\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\psi\right| &\le \frac{\varepsilon}{2}\bigg\{\int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{g(w_\varepsilon)}\bigg\}^{\frac12}\cdot\bigg\{\int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon^2}\psi^2\bigg\}^{\frac12} \\
&\le \frac{\sqrt{K_1}\varepsilon^{\frac34}}{2}\bigg\{\int_\Omega d_\varepsilon\frac{w_{\varepsilon x}^2}{g(w_\varepsilon)}\bigg\}^{\frac12}\|\psi\|_{L^2(\Omega)}, \qquad (6.16)
\end{aligned}$$

because $\frac{1}{d_\varepsilon} \le \frac{1}{\sqrt{\varepsilon}}$ in Ω according to (2.1). Furthermore,

$$\left| -\varepsilon\int_\Omega \sqrt{d_\varepsilon}\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\psi_x\right| \le \varepsilon\bigg\{\int_\Omega d_\varepsilon\frac{w_{\varepsilon x}^2}{g(w_\varepsilon)}\bigg\}^{\frac12}\|\psi_x\|_{L^2(\Omega)}, \qquad (6.17)$$

whereas again invoking Lemma 2.2 along with (1.12) we see that

$$\left| -\int_\Omega \sqrt{d_\varepsilon}\frac{u_\varepsilon}{1+\eta_\varepsilon u_\varepsilon} g(w_\varepsilon)\psi\right| \le g(M)\int_\Omega \sqrt{d_\varepsilon} u_\varepsilon\psi \le g(M)\bigg\{\int_\Omega \sqrt{d_\varepsilon}^3 u_\varepsilon^3\bigg\}^{\frac13}\|\psi\|_{L^{\frac32}(\Omega)} \qquad (6.18)$$

for all $\varepsilon \in (0,\varepsilon_0)$. Thus, since $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega) \subset L^{\frac32}(\Omega)$, and due to Lemma 3.3 and Lemma 5.1, we obtain that for any $T>0$,

$$\sup_{\varepsilon\in(0,\varepsilon_\star(T))}\int_0^T\bigg\{\int_\Omega d_\varepsilon(x)\frac{w_{\varepsilon x}^2(x,t)}{g(w_\varepsilon(x,t))}dx\bigg\}^{\frac32}dt \le T\cdot\bigg\{\sup_{\varepsilon\in(0,\varepsilon_\star(T))}\sup_{t\in(0,T)}\int_\Omega d_\varepsilon(x)\frac{w_{\varepsilon x}^2(x,t)}{g(w_\varepsilon(x,t))}dx\bigg\}^{\frac32} < \infty$$

and

$$\sup_{\varepsilon\in(0,\varepsilon_\star(T))}\int_0^T\int_\Omega \sqrt{d_\varepsilon(x)}^3 u_\varepsilon^3(x,t)dxdt < \infty,$$

it follows from (6.16), (6.17), and (6.18) that (6.15) entails (6.14). $\Box$

7 Construction of limit functions in $\{d>0\}$

We are now prepared for the construction of a limit function inside the positivity set of d through a straightforward extraction process based on straighforward compactness arguments. We remark that at this stage, besides the weighted functions $\sqrt{d_\varepsilon}w_\varepsilon$, our reasoning yet involves the quantities $\sqrt{d_\varepsilon(u_\varepsilon+1)}$, rather than those addressed in Lemma 5.1.

Lemma 7.1 *There exist a sequence $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,\varepsilon_0)$ and nonnegative functions $\widetilde{u}$ and $\widetilde{w}$ defined in $\{d>0\}\times(0,\infty)$ such that $\varepsilon_k \searrow 0$ as $k\to\infty$ and*

$$u_\varepsilon \to \widetilde{u} \qquad \textit{a.e. in } \{d>0\}\times(0,\infty), \tag{7.1}$$

$$u_\varepsilon \rightharpoonup \widetilde{u} \qquad \textit{in } L^1_{loc}([0,\infty);L^1(\{d>0\})), \tag{7.2}$$

$$\sqrt{d_\varepsilon(u_\varepsilon+1)} \to \sqrt{d(\widetilde{u}+1)} \qquad \textit{in } L^2_{loc}([0,\infty);L^2(\{d>0\})), \tag{7.3}$$

$$w_\varepsilon \to \widetilde{w} \qquad \textit{a.e. in } \{d>0\}\times(0,\infty) \qquad \textit{and} \tag{7.4}$$

$$\sqrt{d_\varepsilon}w_\varepsilon \rightharpoonup \sqrt{d}\widetilde{w} \qquad \textit{in } L^2_{loc}([0,\infty);W^{1,2}(\{d>0\})) \tag{7.5}$$

as $\varepsilon=\varepsilon_k\searrow 0$.

Proof. Since given $T>0$ we can use (2.2) to estimate

$$\begin{aligned}
\int_0^T\int_\Omega \Big|(\sqrt{d_\varepsilon(u_\varepsilon+1)})_x\Big|^2 &= \frac14\int_0^T\int_\Omega \Big|\sqrt{d_\varepsilon}\frac{u_{\varepsilon x}}{\sqrt{u_\varepsilon+1}} + \frac{d_{\varepsilon x}}{\sqrt{d_\varepsilon}}\sqrt{u_\varepsilon+1}\Big|^2 \\
&\le \frac12\int_0^T\int_\Omega d_\varepsilon\frac{u_{\varepsilon x}^2}{u_\varepsilon+1} + \frac12\int_0^T\int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon}(u_\varepsilon+1) \\
&\le \frac12\int_0^T\int_\Omega d_\varepsilon\frac{u_{\varepsilon x}^2}{u_\varepsilon+1} + \frac{K_1}{2}\int_0^T\int_\Omega (u_\varepsilon+1)
\end{aligned}$$

for all $\varepsilon\in(0,\varepsilon_0)$, it follows from Lemma 3.3 and Lemma 2.3 that with $\varepsilon_\star(T)$ as introduced there,

$$\Big(\sqrt{d_\varepsilon(u_\varepsilon+1)}\Big)_{\varepsilon\in(0,\varepsilon_\star(T))} \qquad \text{is bounded in } L^2((0,T);W^{1,2}(\Omega)).$$

Therefore, in view of Lemma 6.1 the Aubin-Lions lemma asserts that for any such T,

$$\Big(\sqrt{d_\varepsilon(u_\varepsilon+1)}\Big)_{\varepsilon\in(0,\varepsilon_\star(T))} \qquad \text{is relatively compact in } L^2((0,T);L^2(\Omega)),$$

from which it follows by a standard argument that for a suitable sequence $(\varepsilon_k)_{k\in\mathbb{N}}\subset(0,\varepsilon_0)$ and some $z\in L^2_{loc}([0,\infty);L^2(\Omega))$ we have $\varepsilon_k\searrow 0$ as $k\to\infty$ and

$$\sqrt{d_\varepsilon(u_\varepsilon+1)} \to z \qquad \text{in } L^2_{loc}([0,\infty);L^2(\Omega)) \tag{7.6}$$

and

$$\sqrt{d_\varepsilon(u_\varepsilon+1)} \to z \qquad \text{a.e. in } \Omega\times(0,\infty) \tag{7.7}$$

as $\varepsilon=\varepsilon_k\searrow 0$. Since $d_\varepsilon\to d$ a.e. in Ω as $\varepsilon\searrow 0$ by (2.3), this means that if we let $\widetilde{u}(x,t) := \frac{z^2(x,t)}{d(x)} - 1$ for $x\in\{d>0\}$ and $t>0$, then (7.7) and (7.6) imply (7.1) and (7.3), whereupon (7.1) a posteriori also shows that $\widetilde{u}$ must be nonnegative.
We next make use of the estimate (3.9) from Lemma 3.3 to infer that for $T>0$ and $\varepsilon_\star(T)$ as above,

$$\Big(u_\varepsilon \ln u_\varepsilon\Big)_{\varepsilon\in(0,\varepsilon_\star(T))} \qquad \text{is bounded in } L^1(\Omega\times(0,T)),$$

so that the Dunford-Pettis theorem guarantees that $(u_\varepsilon)_{\varepsilon\in(0,\varepsilon_\star(T))}$ is relatively compact with respect to the weak topology in $L^1(\Omega\times(0,T))$, and that hence (7.2) can be achieved on extracting a subsequence of $(\varepsilon_k)_{k\in\mathbb{N}}$ if necessary.

As for the second solution component, we first use (2.2) to see that

$$\begin{aligned}\int_\Omega \Big|(\sqrt{d_\varepsilon} w_\varepsilon)_x\Big|^2 &= \int_\Omega \Big|\sqrt{d_\varepsilon} w_{\varepsilon x} + \frac{d_{\varepsilon x}}{2\sqrt{d_\varepsilon}} w_\varepsilon\Big|^2 \\ &\le 2\int_\Omega d_\varepsilon w_{\varepsilon x}^2 + \frac{1}{2}\int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon} w_\varepsilon^2 \\ &\le 2\int_\Omega d_\varepsilon w_{\varepsilon x}^2 + \frac{K_1}{2}\int_\Omega w_\varepsilon^2 \qquad \text{for all } t>0,\end{aligned}$$

so that for $T>0$ and $\varepsilon_\star(T)$ as before, Corollary 3.4 and Lemma 2.2 warrant that

$$\Big(\sqrt{d_\varepsilon} w_\varepsilon\Big)_{\varepsilon\in(0,\varepsilon_\star(T))} \quad \text{is bounded in } L^\infty((0,T);W^{1,2}(\Omega)). \tag{7.8}$$

Thus,

$$\Big(\sqrt{d_\varepsilon} w_\varepsilon\Big)_{\varepsilon\in(0,\varepsilon_\star(T))} \quad \text{is relatively compact with respect to the weak topology in } L^2((0,T);W^{1,2}(\Omega)), \tag{7.9}$$

whereas (7.8) in conjunction with Lemma 6.2 and the Aubin-Lions lemma ensures that

$$\Big(\sqrt{d_\varepsilon} w_\varepsilon\Big)_{\varepsilon\in(0,\varepsilon_\star(T))} \quad \text{is relatively compact in } L^2(\Omega\times(0,T)).$$

Consequently, arguing as above we conclude upon passing to a further subsequence if necessary that both (7.4) and (7.5) hold with some $\widetilde{w} : \{d>0\}\times(0,\infty)\to[0,\infty)$. $\square$

In dealing with the taxis term in (2.9), the following consequence of (7.5) will turn out to be more convenient.

Corollary 7.2 *With $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,\varepsilon_0)$ and $\widetilde{w}$ as in Lemma 7.1, we have*

$$\sqrt{d_\varepsilon} w_{\varepsilon x} \rightharpoonup \sqrt{d}\widetilde{w}_x \qquad \text{in } L^2_{loc}([0,\infty);L^2(\{d>0\})) \tag{7.10}$$

as $\varepsilon=\varepsilon_k \searrow 0$.

PROOF. We rewrite

$$\sqrt{d_\varepsilon} w_{\varepsilon x} = (\sqrt{d_\varepsilon} w_\varepsilon)_x - \frac{d_{\varepsilon x}}{2\sqrt{d_\varepsilon}} w_\varepsilon \tag{7.11}$$

and note that in view of the dominated convergence theorem, combining (2.4), (2.3) and (7.4) with (2.2) and Lemma 2.2 shows that for any $T>0$ and $\varphi\in L^2(\{d>0\}\times(0,T))$ we obtain

$$\int_0^T \int_\Omega \int_{\{d>0\}} \frac{d_{\varepsilon x}}{2\sqrt{d_\varepsilon}} w_\varepsilon \varphi \to \int_0^T \int_{\{d>0\}} \frac{d_x}{2\sqrt{d}} \widetilde{w}\varphi$$

and hence

$$\frac{d_{\varepsilon x}}{2\sqrt{d_\varepsilon}} w_\varepsilon \rightharpoonup \frac{d_x}{2\sqrt{d}}\widetilde{w} \qquad \text{in } L^2_{loc}([0,\infty);L^2(\{d>0\}))$$

as $\varepsilon=\varepsilon_k\searrow 0$. Therefore, (7.10) results from (7.11) on using (7.5). $\square$

8 Further convergence and integrability properties

Let us now make use of the pointwise convergence property (3.9) from Lemma 3.3 to accomplish our previously formulated goal concerning strong L^2 convergence of $\sqrt{d_\varepsilon}\frac{u_\varepsilon}{(1+\eta_\varepsilon u_\varepsilon)^2}$. Indeed, through an argument based on Egorov's theorem this will result from the fact that Lemma 5.1 implies bounds for this quantity in Lebesgue spaces involving superquadratic integrability.

Lemma 8.1 *Let $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,\varepsilon_0)$ be as provided by Lemma 7.1. Then*

$$\sqrt{d_\varepsilon}\frac{u_\varepsilon}{(1+\eta_\varepsilon u_\varepsilon)^2} \to \sqrt{d}\widetilde{u} \qquad \text{in } L^2_{loc}([0,\infty);L^2(\{d>0\})) \tag{8.1}$$

as $\varepsilon=\varepsilon_k \searrow 0$.

Proof. As a consequence of Lemma 5.1, given $T>0$ we can find $c_1=c_1(T)>0$ such that with $\varepsilon_\star(T)$ as in Lemma 3.3,

$$\int_0^T\int_\Omega \sqrt{d_\varepsilon}^3 u_\varepsilon^3 \le c_1 \qquad \text{for all } \varepsilon\in(0,\varepsilon_\star(T)). \tag{8.2}$$

Since $\eta_\varepsilon>0$ for all $\varepsilon\in(0,\varepsilon_0)$, this implies that for

$$z_\varepsilon := \sqrt{d_\varepsilon}\frac{u_\varepsilon}{(1+\eta_\varepsilon u_\varepsilon)^2}, \qquad \varepsilon\in(0,\varepsilon_\star(T)),$$

we have

$$\int_0^T\int_\Omega z_\varepsilon^3 \le c_1 \qquad \text{for all } \varepsilon\in(0,\varepsilon_\star(T)). \tag{8.3}$$

Since $\eta_\varepsilon\to 0$ as $\varepsilon\searrow 0$ by (2.7), from Lemma 7.1 we moreover know that

$$z_\varepsilon \to \sqrt{d}\widetilde{u} \quad \text{a.e. in } \{d>0\}\times(0,\infty) \qquad \text{as } \varepsilon=\varepsilon_k\searrow 0. \tag{8.4}$$

Therefore, according to a standard argument involving Egorov's theorem it particularly follows from (8.3) that

$$z_\varepsilon \rightharpoonup \sqrt{d}\widetilde{u} \quad \text{in } L^2(\{d>0\}\times(0,T)) \qquad \text{as } \varepsilon=\varepsilon_k\searrow 0,$$

so that it remains to show that

$$\limsup_{\varepsilon=\varepsilon_k\searrow 0}\int_0^T\int_{\{d>0\}} z_\varepsilon^2 \le \int_0^T\int_{\{d>0\}} d\widetilde{u}^2. \tag{8.5}$$

To this end, supposing on the contrary that for some $c_2>\int_0^T\int_{\{d>0\}} du^2$ and some subsequence $(\varepsilon_{k_j})_{j\in\mathbb{N}}$ of $(\varepsilon_k)_{k\in\mathbb{N}}$ we had

$$\int_0^T\int_{\{d>0\}} z_\varepsilon^2 \to c_2 \qquad \text{as } \varepsilon=\varepsilon_{k_j}\searrow 0, \tag{8.6}$$

once more by means of (8.3) we could extract a further subsequence, again denoted by $(\varepsilon_{k_j})_{j\in\mathbb{N}}$ here fore convenience, along which for some $\widehat{z} \in L^{\frac{3}{2}}(\{d>0\}\times(0,T))$ we would have

$$z_\varepsilon^2 \rightharpoonup \widehat{z} \quad \text{in } L^{\frac{3}{2}}(\{d>0\}\times(0,T)) \qquad \text{as } \varepsilon=\varepsilon_{k_j}\searrow 0.$$

Since (8.4) warrants that $z_\varepsilon^2 \to du^2$ a.e. in $\{d>0\}\times(0,\infty)$ as $\varepsilon=\varepsilon_k\searrow 0$, again by Egorov's theorem this would imply that actually

$$z_\varepsilon^2 \rightharpoonup d\widetilde{u}^2 \quad \text{in } L^{\frac{3}{2}}(\{d>0\}\times(0,T)) \qquad \text{as } \varepsilon=\varepsilon_{k_j}\searrow 0,$$

so that since the boundedness of $\{d>0\}\times(0,T)$ allows for choosing nontrivial constants as test functions here, we would conclude that we would conclude that

$$\int_0^T \int_{\{d>0\}} z_\varepsilon^2 \to \int_0^T \int_{\{d>0\}} du^2 \qquad \text{as } \varepsilon=\varepsilon_{k_j}\searrow 0.$$

This contradiction to (8.6) shows that in fact (8.5) must hold, whence the proof becomes complete. □

A further property of the limit couple $(\widetilde{u},\widetilde{w})$, quite plausible in view of Corollary 3.4, can also be justified on the basis of Egorov's theorem.

Lemma 8.2 *Suppose that $\widetilde{u}$ and $\widetilde{w}$ are as constructed in Lemma 7.1. Then for all $T>0$,*

$$\int_0^T \int_{\{d>0\}} d\widetilde{u}\widetilde{w}_x^2 < \infty. \tag{8.7}$$

Proof. According to Lemma 7.1, (2.7), and Corollary 7.2, with $(\varepsilon_k)_{k\in\mathbb{N}}\subset(0,\varepsilon_0)$ as in Lemma 7.1 we have

$$\sqrt{\frac{u_\varepsilon}{1+\eta_\varepsilon u_\varepsilon}} \to \sqrt{\widetilde{u}} \qquad \text{a.e. in } \{d>0\}\times(0,T) \tag{8.8}$$

and

$$\sqrt{d_\varepsilon}w_{\varepsilon x} \rightharpoonup \sqrt{d}\widetilde{w}_x \qquad \text{in } L^2(\{d>0\}\times(0,T)) \tag{8.9}$$

as $\varepsilon=\varepsilon_k\searrow 0$. Next, Corollary 3.4 entails that with $\varepsilon_\star(T)\in(0,\varepsilon_0)$ taken from Lemma 3.3, the family $\left(\sqrt{d_\varepsilon\frac{u_\varepsilon}{1+\eta_\varepsilon u_\varepsilon}}w_{\varepsilon x}\right)_{\varepsilon\in(0,\varepsilon_\star(T))}$ is bounded in $L^2(\{d>0\}\times(0,T))$, so that we can find

$$z\in L^2(\{d>0\}\times(0,T)) \tag{8.10}$$

and a subsequence $(\varepsilon_{k_j})_{j\in\mathbb{N}}$ of $(\varepsilon_k)_{k\in\mathbb{N}}$ in such a way that

$$\sqrt{\frac{u_\varepsilon}{1+\eta_\varepsilon u_\varepsilon}}\cdot\left(\sqrt{d_\varepsilon}w_{\varepsilon x}\right) \equiv \sqrt{d_\varepsilon\frac{u_\varepsilon}{1+\eta_\varepsilon u_\varepsilon}}w_{\varepsilon x} \rightharpoonup z \qquad \text{in } L^2(\{d>0\}\times(0,T))$$

as $\varepsilon=\varepsilon_{k_j}\searrow 0$. Here a known consequence of Egorov's theorem ([24, Lemma A.1]) asserts that due to (8.8) and (8.9) we may identify

$$z=\sqrt{\widetilde{u}}\cdot\left(\sqrt{d}\widetilde{w}_x\right) \equiv \sqrt{d\widetilde{u}}\widetilde{w}_x \qquad \text{a.e. in } \{d>0\}\times(0,T),$$

so that (8.7) results from (8.10). □

9 Solution properties of $\widetilde{u}$ and $\widetilde{w}$

We are now ready to make sure that $(\widetilde{u}, \widetilde{w})$ indeed solves (1.7) when restricted to $\{d > 0\}$ in the following sense.

Lemma 9.1 *Let $\widetilde{u}$ and $\widetilde{w}$ be as obtained in Lemma 7.1.*

i) If $\varphi \in C_0^\infty(\bar{\Omega} \times [0, \infty))$ is such that $\varphi_x = 0$ on $\partial\Omega \times (0, \infty)$ and additionally

$$\operatorname{supp}\varphi \subset \{d > 0\} \times [0, \infty), \tag{9.1}$$

then

$$-\int_0^\infty \int_{\{d>0\}} \widetilde{u}\varphi_t - \int_{\{d>0\}} u_0 \varphi(\cdot, 0) = \int_0^\infty \int_{\{d>0\}} d\widetilde{u}\varphi_{xx} + \int_0^\infty \int_{\{d>0\}} d\widetilde{u}\widetilde{w}_x\varphi_x + \int_0^\infty \int_{\{d>0\}} \widetilde{u} f(\cdot, \widetilde{u}, \widetilde{w})\varphi. \tag{9.2}$$

ii) For all $\varphi \in C_0^\infty(\Omega \times [0, \infty))$ fulfilling (9.1), we have

$$\int_0^\infty \int_{\{d>0\}} \widetilde{w}\varphi_t + \int_{\{d>0\}} w_0 \varphi(\cdot, 0) = \int_0^\infty \widetilde{u} g(\widetilde{w})\varphi. \tag{9.3}$$

Proof. On testing the first equation in (2.9) by φ we see that

$$-\int_0^\infty \int_\Omega u_\varepsilon \varphi_t - \int_\Omega u_0\varphi(\cdot, 0) = \int_0^\infty \int_\Omega d_\varepsilon u_\varepsilon \varphi_{xx} + \int_0^\infty \int_\Omega d_\varepsilon \frac{u_\varepsilon}{(1 + \eta_\varepsilon u_\varepsilon)^2} w_{\varepsilon x}\varphi_x + \int_0^\infty \int_\Omega u_\varepsilon f(\cdot, u_\varepsilon, w_\varepsilon)\varphi \qquad \text{for all } \varepsilon \in (0, \varepsilon_0), \tag{9.4}$$

where since $u_\varepsilon \rightharpoonup \widetilde{u}$ in $L^1_{loc}([0, \infty); L^1(\{d > 0\}))$ as $\varepsilon = \varepsilon_k \searrow 0$ by Lemma 7.1, according to (9.1) we have

$$-\int_0^\infty \int_\Omega u_\varepsilon \varphi_t \to -\int_0^\infty \int_{\{d>0\}} \widetilde{u}\varphi_t$$

and

$$\int_0^\infty \int_\Omega d_\varepsilon u_\varepsilon \varphi_{xx} \to \int_0^\infty \int_{\{d>0\}} d\widetilde{u}\varphi_{xx}$$

as $\varepsilon = \varepsilon_k \searrow 0$, because $d_\varepsilon \to d$ in $L^\infty(\Omega)$ as $\varepsilon \searrow 0$ due to (2.3).
Next, since Lemma 7.1 warrants that also $u_\varepsilon \to \widetilde{u}$ and $w_\varepsilon \to \widetilde{w}$ a.e. in $\{d > 0\} \times (0, \infty)$ as $\varepsilon = \varepsilon_k \searrow 0$, it follows from Lemma 4.2 and a standard argument, again involving Egorov's theorem, that

$$u_\varepsilon f(\cdot, u_\varepsilon, w_\varepsilon) \rightharpoonup \widetilde{u} f(\cdot, \widetilde{u}, \widetilde{w}) \qquad \text{in } L^1_{loc}([0, \infty); L^1(\{d > 0\})),$$

and that hence

$$\int_0^\infty \int_\Omega u_\varepsilon f(x, u_\varepsilon, w_\varepsilon) \to \int_0^\infty \int_{\{d>0\}} \widetilde{u} f(x, \widetilde{u}, \widetilde{w})$$

as $\varepsilon = \varepsilon_k \searrow 0$.

Finally, from Corollary 7.2 we know that

$$\sqrt{d_\varepsilon} w_{\varepsilon x} \rightharpoonup \sqrt{d}\widetilde{w}_x \qquad \text{in } L^2_{loc}([0,\infty); L^2(\{d>0\}))$$

as $\varepsilon = \varepsilon_k \searrow 0$, which combined with the strong convergence property of $\sqrt{d_\varepsilon}\frac{u_\varepsilon}{(1+\eta_\varepsilon u_\varepsilon)^2}$ in $L^2_{loc}([0,\infty); L^2(\{d > 0\}))$ asserted by Lemma 8.1 ensures that

$$\int_0^\infty \int_\Omega d_\varepsilon \frac{u_\varepsilon}{(1+\eta_\varepsilon u_\varepsilon)^2} w_{\varepsilon x}\varphi_x = \int_0^\infty \int_\Omega \Big(\sqrt{d_\varepsilon}\frac{u_\varepsilon}{(1+\eta_\varepsilon u_\varepsilon)^2}\Big) \cdot \Big(\sqrt{d_\varepsilon} w_{\varepsilon x}\Big)\varphi_x \to \int_0^\infty \int_{\{d>0\}} d\widetilde{u}\widetilde{w}_x\varphi_x$$

as $\varepsilon = \varepsilon_k \searrow 0$. Therefore, (9.2) is a consequence of (9.4).

To verify (9.3), given $\varphi \in C_0^\infty(\Omega \times [0,\infty))$ fulfilling (9.1) we obtain from (2.9) that

$$\int_0^\infty \int_\Omega w_\varepsilon \varphi_t + \int_\Omega w_{0\varepsilon}\varphi(\cdot,0) = -\varepsilon \int_0^\infty \int_\Omega \frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\varphi_x - \int_0^\infty \int_\Omega \frac{u_\varepsilon}{1+\eta_\varepsilon u_\varepsilon} g(w_\varepsilon)\varphi \tag{9.5}$$

for all $\varepsilon \in (0,\varepsilon_0)$. Here by Lemma 7.1, Lemma 2.2 and the dominated convergence theorem,

$$\int_0^\infty \int_\Omega w_\varepsilon\varphi_t \to \int_0^\infty \int_{\{d>0\}} \widetilde{w}\varphi_t \tag{9.6}$$

as $\varepsilon = \varepsilon_k \searrow 0$, whereas (2.8) trivially ensures that

$$\int_\Omega w_{0\varepsilon}\varphi(\cdot,0) \to \int_{\{d>0\}} w_0\varphi(\cdot,0) \tag{9.7}$$

as $\varepsilon = \varepsilon_k \searrow 0$. Moreover, combining Lemma 4.2 with the pointwise convergence properties in (7.1) and (7.4) we easily infer that

$$u_\varepsilon g(w_\varepsilon) \rightharpoonup \widetilde{u} g(\widetilde{w}) \qquad \text{in } L^1_{loc}([0,\infty); L^1(\{d>0\}))$$

and thus, also relying on (2.7) and again on Lemma 7.1, we obtain

$$\int_0^\infty \int_\Omega \frac{u_\varepsilon}{1+\eta_\varepsilon u_\varepsilon} g(w_\varepsilon)\varphi \to \int_0^\infty \int_{\{d>0\}} \widetilde{u} g(\widetilde{w})\varphi \tag{9.8}$$

as $\varepsilon = \varepsilon_k \searrow 0$. Finally, once more relying on the fact that $d_\varepsilon \ge \sqrt{\varepsilon}$ by (2.1), we see by using the Cauchy-Schwarz inequality that if $T > 0$ is large enough such that $\varphi \equiv 0$ in $\Omega \times (T,\infty)$, then

$$\begin{aligned} \left|\varepsilon \int_0^\infty \int_\Omega \frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\varphi_x\right| &\le \varepsilon \int_0^\infty \Big\{\int_\Omega d_\varepsilon \frac{w^2_{\varepsilon x}}{g(w_\varepsilon)}\Big\}^{\frac12} \cdot \Big\{\int_\Omega \frac{\varphi_x^2(\cdot,t)}{d_\varepsilon}\Big\}^{\frac12} dt \\ &\le \varepsilon^{\frac34} \sup_{t\in(0,T)} \Big\{\int_\Omega d_\varepsilon \frac{w^2_{\varepsilon x}(\cdot,t)}{g(w_\varepsilon(\cdot,t))}\Big\}^{\frac12} \cdot \int_0^\infty \Big\{\int_\Omega \varphi_x^2(\cdot,t)\Big\}^{\frac12} dt \end{aligned}$$

for all $\varepsilon \in (0,\varepsilon_0)$. Therefore, since with $\varepsilon_\star(T) \in (0,\varepsilon_0)$ as given by Lemma 3.3 we know from (3.10) that

$$\sup_{\varepsilon\in(0,\varepsilon_\star(T))} \sup_{t\in(0,T)} \int_\Omega d_\varepsilon \frac{w^2_{\varepsilon x}(\cdot,t)}{g(w_\varepsilon(\cdot,t))} < \infty,$$

it follows that

$$\varepsilon \int_0^\infty \int_\Omega \frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \varphi_x \to 0$$

as $\varepsilon = \varepsilon_k \searrow 0$. In combination with (9.6)-(9.8), this shows that (9.5) implies (9.3). $\Box$

10 Proof of Theorem 1.2

We can finally extend the above spatially local solution in an evident manner so as to become a global weak solution in the flavor of Definition 1.1. In the verification of the desired solution property near the boundary of $\{d > 0\}$ we shall make use of the following consequence of the inclusion $\sqrt{d} \in W^{1,\infty}(\Omega)$.

Lemma 10.1 *Let $x \in \bar{\Omega}$. Then*

$$d(x) \le \frac{K_1}{4} \Big\{ \mathrm{dist}\,(x, \{d = 0\}) \Big\}^2. \tag{10.1}$$

Proof. We only need to consider the case when $d(x) > 0$, in which by the closedness of $\{d = 0\}$ we can pick $x_0 \in \bar{\Omega}$ such that $d(x_0) = 0$ and $|x - x_0| = \mathrm{dist}\,(x, \{d = 0\}) > 0$. Thanks to (2.2), we then have

$$\sqrt{d(x)} = \int_{x_0}^x (\sqrt{d})_y dy = \frac{1}{2} \int_{x_0}^x \frac{d_x(y)}{\sqrt{d(y)}} dy \le \frac{\sqrt{K_1}}{2} |x - x_0| = \frac{\sqrt{K_1}}{2} \cdot \mathrm{dist}\,(x, \{d = 0\}),$$

from which (10.1) follows. $\Box$

By means of an appropriate cut-off procedure we can thereby proceed to show that the natural extension of $(\widetilde{u}, \widetilde{w})$, consisting of a solution to the ODE system formally associated with (1.7) in $\{d = 0\}$ indeed solves (1.7) in the desired sense.

Proof of Theorem 1.2. We let $\widetilde{u}$ and $\widetilde{w}$ denote the functions defined on $\{d > 0\} \times (0, \infty)$ in Lemma 7.1, and for fixed $x \in \{d = 0\}$ we let $(\widehat{u}(x, \cdot), \widehat{w}(x, \cdot)) \in (C^1([0, \infty)))^2$ be the solution of the initial-value problem

$$\begin{cases} \widehat{u}_t = \widehat{u} f(x, \widehat{u}, \widehat{w}), & t > 0, \\ \widehat{w}_t = -\widehat{u} g(\widehat{w}), & t > 0, \\ \widehat{u}(x, 0) = u_0(x), & \widehat{w}(x, 0) = w_0(x). \end{cases} \tag{10.2}$$

Indeed, it follows from (1.8), (1.10) and (1.12) that for any such x this ODE problem possesses a globally defined solution fulfilling

$$0 \le \widehat{w}(x, t) \le M \qquad \text{for all } t > 0 \tag{10.3}$$

and

$$0 \le \widehat{u}(x, t) \le u_0(x) e^{\rho(M)t} \qquad \text{for all } t > 0, \tag{10.4}$$

and since u_0 and w_0 are continuous in $\bar{\Omega}$ by (1.9), standard ODE theory warrants that both $\widehat{u}$ and $\widehat{w}$ are continuous in $\{d = 0\} \times [0, \infty)$. Therefore,

$$(u, w)(x, t) := \begin{cases} (\widetilde{u}, \widetilde{w})(x, t), & x \in \{d > 0\},\ t > 0, \\ (\widehat{u}, \widehat{w})(x, t), & x \in \{d = 0\},\ t > 0, \end{cases} \tag{10.5}$$

defines a pair of nonnegative measurable functions on all of $\Omega \times (0,\infty)$ which thanks to Lemma 7.1, Lemma 4.2, Lemma 2.2, (1.12), (10.3) and (10.4) satisfy (1.17) and (1.18), and for which Lemma 8.2 in particular entails that also (1.19) holds.

In order to verify (1.20), we first make use of the fact that by continuity of d the set $\{d>0\}$ is relatively open in $\bar\Omega$, and hence consists of countably many connected components; that is, there exist an index set $I \subset \mathbb{N}$ and intervals $P_i \subset \bar\Omega$, $i \in I$, such that $\{d>0\} = \bigcup_{i\in I} P_i$ and $P_i \cap P_j = \emptyset$ for $i,j \in I$ with $i \neq j$. Now for each $i \in I$, there exist $a_i \in \bar\Omega$ and $b_i \in \bar\Omega$ such that $(a_i,b_i) \subset P_i \subset [a_i,b_i]$, where $a_i \in P_i$ if and only if $a_i \in \partial\Omega$ and $b_i \in P_i$ if and only if $b_i \in \partial\Omega$. For fixed $\delta \in (0,1)$, defining $\delta_i := 2^{-i}\delta$, $i \in I$, it is then possible to pick $(\zeta_\delta^{(i)})_{i\in I} \subset C^\infty(\bar\Omega)$ such that for all $i \in I$ we have $0 \le \zeta_\delta^{(i)} \le 1$ in $\bar\Omega$, $\zeta_\delta^{(i)}(x) = 1$ whenever $x \in P_i$ is such that $\operatorname{dist}(x,\partial P_i) \ge \delta_i$, $\zeta_\delta^{(i)} \equiv 0$ in $\bar\Omega \setminus P_i$, and

$$|\zeta_{\delta x}^{(i)}| \le \frac{2}{\delta_i} \qquad \text{in } \bar\Omega \tag{10.6}$$

as well as

$$|\zeta_{\delta xx}^{(i)}| \le \frac{16}{\delta_i^2} \qquad \text{in } \bar\Omega, \tag{10.7}$$

where in the exceptional case $a_i \in \partial\Omega$ we can additionally achieve that $\zeta_\delta^{(i)} \equiv 1$ holds even throughout $[a_i, b_i - \delta_i]$, and where, similarly, in the case $b_i \in \partial\Omega$ we require that $\zeta_\delta^{(i)} \equiv 1$ in $[a_i+\delta_i, b_i]$.
Now given $\varphi \in C_0^\infty(\bar\Omega \times [0,\infty))$ satisfying $\varphi_x = 0$ on $\partial\Omega \times (0,\infty)$, from (10.2) and (10.4) we immediately see that

$$-\int_0^\infty \int_{\{d=0\}} u\varphi_t - \int_{\{d=0\}} u_0\varphi(\cdot,0) = \int_0^\infty \int_{\{d=0\}} uf(x,u,w)\varphi. \tag{10.8}$$

Moreover, Lemma 9.1 guarantees that if we let

$$\zeta_\delta := \sum_{i\in I} \zeta_\delta^{(i)}, \qquad \delta \in (0,1),$$

then since $\operatorname{supp}(\zeta_\delta \cdot \varphi) \subset \{d>0\} \times [0,\infty)$, we have

$$\begin{aligned}
&-\int_0^\infty \int_{\{d>0\}} \zeta_\delta u\varphi_t - \int_{\{d>0\}} \zeta_\delta u_0 \varphi(\cdot,0) \\
&\quad= \int_0^\infty \int_{\{d>0\}} du \cdot (\zeta_\delta\varphi)_{xx} + \int_0^\infty \int_{\{d>0\}} duw_x \cdot (\zeta_\delta\varphi)_x \\
&\qquad + \int_0^\infty \int_{\{d>0\}} \zeta_\delta uf(\cdot,u,w)\varphi \\
&\quad= \int_0^\infty \int_{\{d>0\}} \zeta_\delta du\varphi_{xx} + 2\int_0^\infty \int_{\{d>0\}} \zeta_{\delta x} du\varphi_x + \int_0^\infty \int_{\{d>0\}} \zeta_{\delta xx} du\varphi \\
&\qquad + \int_0^\infty \int_{\{d>0\}} \zeta_\delta duw_x\varphi_x + \int_0^\infty \int_{\{d>0\}} \zeta_{\delta x} duw_x\varphi \\
&\qquad + \int_0^\infty \int_{\{d>0\}} \zeta_\delta uf(\cdot,u,w)\varphi \qquad \text{for all } \delta \in (0,1).
\end{aligned} \tag{10.9}$$

Here we may use that $0 \le \zeta_\delta \le 1$ and that as $\delta \searrow 0$ we have $\zeta_\delta \to 1$ a.e. in $\{d > 0\}$ to infer from the dominated convergence theorem that

$$- \int_0^\infty \int_{\{d>0\}} \zeta_\delta u \varphi_t \to - \int_0^\infty \int_{\{d>0\}} u \varphi_t \tag{10.10}$$

and

$$- \int_{\{d>0\}} \zeta_\delta u_0 \varphi(\cdot, 0) \to - \int_{\{d>0\}} u_0 \varphi(\cdot, 0) \tag{10.11}$$

as well as

$$\int_0^\infty \int_{\{d>0\}} \zeta_\delta d u \varphi_{xx} \to \int_0^\infty \int_{\{d>0\}} d u \varphi_{xx} \tag{10.12}$$

and

$$\int_0^\infty \int_{\{d>0\}} \zeta_\delta d u w_x \varphi_x \to \int_0^\infty \int_{\{d>0\}} d u w_x \varphi_x \tag{10.13}$$

and

$$\int_0^\infty \int_{\{d>0\}} \zeta_\delta u f(\cdot, u, w) \varphi \to \int_0^\infty \int_{\{d>0\}} u f(\cdot, u, w) \varphi \tag{10.14}$$

as $\delta \searrow 0$. In order to estimate the integrals on the right of (10.9) which contain derivatives of ζ_δ, let us first observe that as a consequence of (10.6), (10.7) and Lemma 10.1 we know that whenever $x \in \bar{\Omega}$ is such that $\zeta_{\delta x}(x) \ne 0$, for some $i \in I$ we have $x \in P_i$ and $\operatorname{dist}(x, \{d = 0\}) \le \delta_i$ and hence

$$d(x) \zeta_{\delta x}^2(x) \le \frac{K_1 \delta_i^2}{4} \cdot \Big(\frac{2}{\delta_i}\Big)^2 = K_1 \tag{10.15}$$

as well as

$$d(x) \cdot |\zeta_{\delta xx}(x)| \le \frac{K_1 \delta_i^2}{4} \cdot \frac{16}{\delta_i^2} = 4K_1. \tag{10.16}$$

Furthermore, again by mutual disjointness of the P_i,

$$\Big| \operatorname{supp} \zeta_{\delta x} \Big| \le \sum_{i \in I} 2 \cdot \delta_i = \sum_{i \in I} 2 \cdot (2^{-i} \delta) \le 2\delta \qquad \text{for all } \delta \in (0, 1),$$

so that since we know from Lemma 5.1, Lemma 8.2 and Fatou's lemma that with $T > 0$ taken large enough fulfilling $\varphi \equiv 0$ in $\Omega \times (T, \infty)$ we have

$$\sqrt{d}^3 u^3 \in L^1(\{d > 0\} \times (0, T))$$

and

$$d u w_x^2 \in L^1(\{d > 0\} \times (0, T)),$$

from the dominated convergence theorem it follows that

$$\int_0^T \int_{\operatorname{supp} \zeta_{\delta x}} \sqrt{d}^3 u^3 \to 0 \tag{10.17}$$

and

$$\int_0^T \int_{\operatorname{supp}\zeta_{\delta x}} duw_x^2 \to 0 \tag{10.18}$$

as $\delta \searrow 0$, whereas combining (10.16) with the dominated convergence theorem shows that also

$$\int_{\operatorname{supp}\zeta_{\delta x}} |\zeta_{\delta xx}| \cdot d \to 0 \tag{10.19}$$

as $\delta \searrow 0$.
Thus, using the Hölder inequality along with (10.15) and (10.17) we obtain that

$$\begin{aligned}
\left|2\int_0^\infty \int_{\{d>0\}} \zeta_{\delta x} du\varphi_x\right| &\leq 2\|\varphi_x\|_{L^\infty(\Omega\times(0,\infty))} \left\{\int_0^T \int_{\operatorname{supp}\zeta_{\delta x}} \sqrt{d}^3 u^3\right\}^{\frac{1}{3}} \cdot \left\{\int_0^T \int_{\operatorname{supp}\zeta_{\delta x}} |\zeta_{\delta x}|^{\frac{3}{2}} d^{\frac{3}{4}}\right\}^{\frac{2}{3}} \\
&\leq 2\sqrt{K_1} T^{\frac{2}{3}} \|\varphi_x\|_{L^\infty(\Omega\times(0,\infty))} \left\{\int_0^T \int_{\operatorname{supp}\zeta_{\delta x}} \sqrt{d}^3 u^3\right\}^{\frac{1}{3}} \\
&\to 0
\end{aligned} \tag{10.20}$$

as $\delta \searrow 0$, while from (10.19) we infer that

$$\begin{aligned}
\left|\int_0^\infty \int_{\{d>0\}} \zeta_{\delta xx} du\varphi\right| &\leq \|\varphi\|_{L^\infty(\Omega\times(0,\infty))} \int_0^T \left\|\sqrt{d}u(\cdot,t)\right\|_{L^\infty(\{d>0\})} \int_{\operatorname{supp}\zeta_{\delta x}} |\zeta_{\delta xx}| \cdot d\ dt \\
&\leq \sqrt{T}\|\varphi\|_{L^\infty(\Omega\times(0,\infty))} \left\{\int_0^T \left\|\sqrt{d}u(\cdot,t)\right\|_{L^\infty(\{d>0\})} dt\right\}^{\frac{1}{2}} \cdot \int_{\operatorname{supp}\zeta_{\delta x}} |\zeta_{\delta xx}| \cdot d \\
&\to 0
\end{aligned} \tag{10.21}$$

as $\delta \searrow 0$, because Lemma 5.1 together with Fatou's lemma warrants that

$$\int_0^T \left\|\sqrt{d}u(\cdot,t)\right\|^2_{L^\infty(\{d>0\})} dt \leq \liminf_{\varepsilon=\varepsilon_k\searrow 0} \int_0^T \left\|\sqrt{d_\varepsilon}u_\varepsilon(\cdot,t)\right\|^2_{L^\infty(\Omega)} dt < \infty.$$

Since finally (10.18) along with (10.15) ensures that also

$$\begin{aligned}
\left|\int_0^\infty \int \{d>0\}\zeta_{\delta x} duw_x\varphi\right| &\leq \|\varphi\|_{L^\infty(\Omega\times(0,\infty))} \left\{\int_0^T \int_{\operatorname{supp}\zeta_{\delta x}} duw_x^2\right\}^{\frac{1}{2}} \cdot \left\{\int_0^T \int_\Omega \zeta_{\delta x}^2 d\right\}^{\frac{1}{2}} \\
&\leq \sqrt{K_1 T}\|\varphi\|_{L^\infty(\Omega\times(0,\infty))} \left\{\int_0^T \int_{\operatorname{supp}\zeta_{\delta x}} duw_x^2\right\}^{\frac{1}{2}} \\
&\to 0
\end{aligned}$$

as $\delta \searrow 0$, from (10.9)-(10.14), (10.20) and (10.21) we conclude that

$$\begin{aligned}
-\int_0^\infty \int_{\{d>0\}} u\varphi_t - \int_{\{d>0\}} u_0\varphi(\cdot,0) = \int_0^\infty \int_{\{d>0\}} du\varphi_{xx} + \int_0^\infty \int_{\{d>0\}} duw_x\varphi_x \\
+\int_0^\infty \int_{\{d>0\}} uf(\cdot,u,w)\varphi,
\end{aligned}$$

which in combination with (10.8) shows that indeed (1.20) is valid for any such φ.

The derivation of (1.21) is much less involved: Given $\varphi \in C_0^\infty(\Omega \times [0,\infty))$, from (10.5) and (10.2) we first obtain that

$$\int_0^\infty \int_{\{d=0\}} w\varphi_t + \int_{\{d=0\}} w_0\varphi(\cdot,0) = \int_0^\infty \int_{\{d=0\}} ug(w)\varphi, \tag{10.22}$$

whereas with $(\zeta_\delta)_{\delta\in(0,1)}$ as introduced above we obtain from Lemma 9.1 that

$$\int_0^\infty \int_\Omega \zeta_\delta w\varphi_t + \int_\Omega \zeta_\delta w_0\varphi(\cdot,0) = \int_0^\infty \int_\Omega \zeta_\delta ug(w)\varphi \tag{10.23}$$

for all $\delta \in (0,1)$. Using that w and $ug(w)$ belong to $L^1_{loc}([0,\infty); L^1(\{d>0\}))$ by Lemma 2.2, Lemma 4.2 and Lemma 7.1, we may again employ the dominated convergence theorem here to see that in the limit $\delta \searrow 0$, (10.23) implies that

$$\int_0^\infty \int_{\{d>0\}} w\varphi_t + \int_{\{d>0\}} w_0\varphi(\cdot,0) = \int_0^\infty \int_{\{d>0\}} ug(w)\varphi,$$

and that thus in view of (10.22) also (1.21) holds. $\square$

11 Discussion

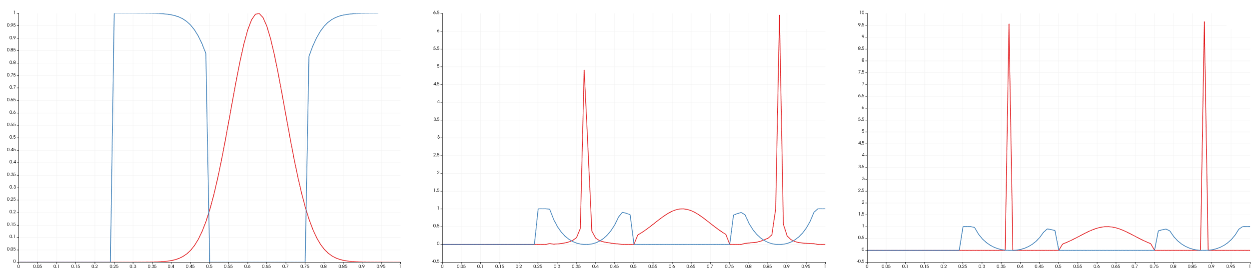
In this paper we considered a model for tumor cell migration in an anisotropic environment. More precisely, this is a 1D version of a model deduced in [9] to characterize glioma invasion. Here we explicitly allow for strongly degenerate diffusion and the setting also involves haptotaxis and a drift term coming from the non-Fickian transport obtained via parabolic scaling in [9]. We proved the global existence of weak solutions, the boundedness and the uniqueness issues remaining open.

In order to illustrate the solution behavior some numerical experiments have been performed upon using a finite volume method. It is well known that there are many serious numerical issues related to handling chemotaxis and haptotaxis systems, even in the case with linear diffusion for the cell density, as in such settings the solutions can infer large aggregations and even exhibit blow-up. We are not aware of any well investigated numerical methods dealing with strongly degenerate diffusion and haptotaxis, thus had to rely on methods which have been developed for other purposes in order to get a glimpse into the solution behavior in this framework. Figure 1 shows the densities of cancer cells and of tissue at several time points; the simulations confirm the expected qualitative behavior and, moreover, hint on possible blowups in finite time, which in this work we have neither been able to prove nor to rule out. Thereby, we considered for $x \in [0,1]$ a diffusivity function

$$d(x) = (\sin(4\pi(x-0.25)))^{2\alpha}\mathbb{1}_{[0.25,0.5]} + (\sin(4\pi(x-0.75)))^{2\alpha}\mathbb{1}_{[0.75,1.0]}$$

with $\alpha > 0$ and started with initial conditions like shown in Figure 1a; they describe a tumor at time $t=0$ situated in a tissue region which has already been degraded by the cancer cells, most of the latter still being located in the immediate neighbourhood of the tumor. Then the cells spread gradually into the available tissue, eventually aggregating at the sites with highest difference between diffusivities, see Figures 1b 1c. Our results actually hint on the solution blowing up at those sites, see Figure 1c.

Figure 1: Simulation results. Red line: cancer cell density, blue line: tissue density; $\alpha = 0.14$.



(a) Initial condition. (b) Simulation at time $t = 80$ (c) Simulation at time $t = 200$

This behavior corresponds to previous observations, in particular in the context of glioma migration, which is heavily relying on the patient-specific brain tissue structure in which the tumor cells are invading. Indeed, glioma cells have been observed to accumulate along interfaces between white and gray matter [5]. This might be a consequence of the very different diffusivities in the two regions: the white matter is much softer and allows for cell motility (along white matter tracts) which is 5-25 times higher than in gray matter, see [18, 13, 4]. According to the discussion in [4], such behavior can be recovered, however, only if the equations involve myopic transport (i.e., the cells sense only locally their neighbourhood, which is described by a term of the form $\nabla\nabla : (D(x)u))$ instead of the classical Fickian diffusion, whence supplementing the diffusion with a transport term. The model we presented and analyzed here features this myopic transport, but moreover also accounts for haptotaxis, which is crucial for cells invading a tissue, [6] the more so for glioma cells following white matter tracts, [10, 8]. Most of the previous models addressing the issue of glioma invasion (including the already mentioned ones [18, 13, 4] and [16]) did not involve haptotaxis; as mentioned in Section 1 our macroscopic model is obtained from a micro-meso setting and the haptotaxis term is a consequence of the multiscality. Our numerical experiments suggest strong cell aggregation at sites with high diffusivity situated in the surroundings of regions with much lower diffusivity, whereby the latter may include areas with sparse tissue. Given the heterogeneity of brain tissue it is possible to have a more or less frequent alternation of such regions. This then leads to a rather fractal pattern for the tumor spread, featuring an alternation of large cancer cell accumulations with smaller tumor cell groups, possibly separated by hypocellular zones, occasionally exhibiting sharp transitions -as shown by our simulations. This behavior is in line with the highly infiltrative character of glioma spread.
Both the mathematical analysis and the numerical handling of a model featuring haptotaxis, degenerate diffusion, and myopic transport are much more challenging than for the prevalent systems with (nonlinear) diffusion and taxis. Not only are such models more realistic, but they also offer a research field for several interesting questions, both from the mathematical and the biological point of view.

Aknowledgement

The authors thank Alexander Hunt (TU Kaiserslautern) for doing the numerical simulations.

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