Does fluid interaction affect regularity in the three-dimensional Keller-Segel system with saturated sensitivity?

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Abstract

A class of Keller-Segel-Stokes systems generalizing the prototype

$$\begin{cases} n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot \Big(n(n+1)^{-\alpha} \nabla c \Big), \\ c_t + u \cdot \nabla c &= \Delta c - c + n, \\ u_t + \nabla P &= \Delta u + n \nabla \phi + f(x,t), \qquad \nabla \cdot u = 0, \end{cases} \qquad (\star)$$

is considered in a bounded domain $\Omega \subset \mathbb{R}^3$, where ϕ and f are given sufficiently smooth functions such that f is bounded in $\Omega \times (0,\infty)$.

It is shown that under the condition that

$$\alpha > \frac{1}{3},$$

for all sufficiently regular initial data a corresponding Neumann-Neumann-Dirichlet initial-boundary value problem possesses a global bounded classical solution. This extends previous findings asserting a similar conclusion only under the stronger assumption $\alpha > \frac{1}{2}$.

In view of known results on the existence of exploding solutions when $\alpha < \frac{1}{3}$, this indicates that with regard to the occurrence of blow-up the criticality of the decay rate $\frac{1}{3}$, as previously found for the fluid-free counterpart of $(\star)$, remains essentially unaffected by fluid interaction of the type considered here.

Key words: chemotaxis, Stokes, boundedness, maximal Sobolev regularity
AMS Classification: 35B65 (primary); 35Q35, 35Q92, 92C17 (secondary)

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1 Introduction

One of the most characteristic mathematical features of the classical Keller-Segel system, in its simplest form given by

$$\begin{cases} n_t = \Delta n - \nabla \cdot (n \nabla c), \\ c_t = \Delta c - c + n, \end{cases} \tag{1.1}$$

consists in its ability to generate singular behavior by enforcing finite-time blow-up of some solutions in spatially two- or higher-dimensional situations ([15], [41]). Well-established as a model for the collective behavior in populations of cells chemotactically biased by a signal substance produced by themselves, (1.1) thus may well describe phenomena of spontaneous cell aggregation arising in various experimental contexts ([16]). In order to adequately describe chemotactic migration also in biological frameworks in which such an emergence of unbounded population densities seems unrealistic, considerable efforts have been undertaken since the introduction of (1.1) ([18]) to develop modified variants thereof in which the occurrence of explosions is a priori ruled out.

One frequently discussed and in its mathematical consequences quite comprehensively understood direction of refinement consists in assuming the cell motility to depend differently on the population density than supposed in (1.1), especially at large densities; this may lead to certain saturation effects in the cross-diffusion term, or to nonlinear diffusivities e.g. in the sense of a porous medium-type enhancement of diffusion at large densities, or to a combination of both (see e.g. the survey [16]). Focusing here on the former type of modification, as reflected in the variant

$$\begin{cases} n_t = \Delta n - \nabla \cdot (nS(n) \nabla c), \\ c_t = \Delta c - c + n, \end{cases} \tag{1.2}$$

of (1.1) with nonnegative $S(n)$ possibly becoming small at large values of n, we may interpret the corresponding literature as identifying the decay rate of the prototypical choice

$$S(n) = (n+1)^{-\frac{N-2}{N}}, \qquad n \geq 0, \tag{1.3}$$

as critical for the occurrence of blow-up in the spatially N-dimensional version of (1.2): Indeed, if $N \geq 2$ and $S \in C^2([0,\infty))$ is such that

$$S(n) \leq K_S (n+1)^{-\alpha} \qquad \text{for all } n \geq 0 \tag{1.4}$$

and some $K_S > 0$ and $\alpha > \frac{N-2}{N}$, then for all reasonably regular nonnegative initial data the no-flux initial-boundary value problem for (1.2) in smoothly bounded domains $\Omega \subset \mathbb{R}^N$ possesses a globally defined bounded classical solution ([17], [25]); on the other hand, if

$$S(n) \geq K'_S (n+1)^{-\alpha'} \qquad \text{for all } n \geq 0 \tag{1.5}$$

and some $K'_S > 0$ and $\alpha' < \frac{N-2}{N}$, then in each ball $\Omega \subset \mathbb{R}^N$ there exist solutions which become unbounded ([4], [38]).

It is the purpose of the present work to study the question how far this borderline role of the behavior (1.3) may be affected by interaction of cells with a liquid environment, where intending to incorporate an assumption underlying the model development in [30] we will suppose that this interaction occurs not only through transport but possibly also through a buoyancy-driven feedback of cells to the fluid velocity. Indeed, numerical evidence suggests that the combination of these mechanisms may at least enforce a delay in blow-up of some solutions to an accordingly modified two-dimensional variant of (1.1) ([22]). More drastically, a recent rigorous analytical result shows that even in absence of any influence of cells on the fluid motion, a purely transport-determined interplay in fact may fully suppress blow-up in the sense that for widely arbitrary fixed initial data one can construct a solenoidal fluid velocity field such that a corresponding initial value problem associated with an either two- or three-dimensional variant of (1.1) possesses globally bounded solutions ([19]).

With our focus slightly differing from that in the latter study, we will henceforth concentrate on the problem of deciding whether for some given sensitivity parameter function S, in the extension

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot\Big(nS(n)\nabla c\Big), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - c + n, & x\in\Omega,\ t>0,\\ u_t + \nabla P = \Delta u + n\nabla\phi + f(x,t), \qquad \nabla\cdot u = 0, & x\in\Omega,\ t>0,\\ \frac{\partial n}{\partial\nu}=0, \quad \frac{\partial c}{\partial\nu}=0, \quad u=0, & x\in\partial\Omega,\ t>0,\\ n(x,0)=n_0(x), \quad c(x,0)=c_0(x), \quad u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.6}$$

of the no-flux initial-boundary value problem for (1.2) it is *at all* possible to observe the occurrence of blow-up for *some* solution in presence of *some* suitably regular gravitational potential ϕ and external fluid force f in *some* bounded domain $\Omega\subset\mathbb{R}^N$.

Within this problem setting it then immediately becomes clear on letting $u_0\equiv 0$, $\phi\equiv 0$, $f\equiv 0$ and $\Omega := B_1(0)\subset\mathbb{R}^N$ that assuming (1.5) to be valid for some $K_S'>0$ and $\alpha'<\frac{N-2}{N}$ trivially remains sufficient for the existence of some exploding solutions in (1.6) as well. In the case $N=2$, this condition in fact appears to stay essentially optimal also for (1.6) in view of recent results asserting global existence of bounded classical solutions for all suitably regular initial data at least when $f\equiv 0$, thus ruling out any blow-up phenomenon ([35]), even in the more complicated case when the fluid flow is governed by an associated version of the full Navier-Stokes equations ([34]).

In the three-dimensional version of (1.6), the seemingly only available result on global existence and boundedness of classical solutions for arbitrarily large initial data relies on the requirement that (1.4) holds for some $K_S>0$ and $\alpha>\frac{1}{2}$ ([36]), thus leaving open the question how far the value $\frac{1}{3}$ accordingly appearing in (1.3) continues to play the role of a critical blow-up exponent for (1.6); after all, under the mere assumption that (1.4) be valid with some $K_S>0$ and $\alpha>\frac{1}{3}$, certain global generalized solutions could be constructed for the actually even more complex Keller-Segel-Navier-Stokes variant of (1.6) ([31], cf. also [21]), but unless in cases when suitable additional smallness conditions on the initial data are imposed ([20]) the knowledge on their boundedness features is yet quite poor.

Main results: Criticality of the decay exponent $\frac{1}{3}$. The main outcome of this study reveals that the validity of (1.8) with some $K_S>0$ and $\alpha>\frac{1}{3}$ is actually sufficient to exclude any singularity formation also in the full chemotaxis-Stokes system (1.6) unde reasonable assumptions on ϕ, f and the initial data, thereby indicating, in the sense specified above, that the possibility of observing blow-up in a suitable constellation remains unaffected by fluid interaction of the considered type.

To make this more precise, let us consider (1.6) in a bounded domain $\Omega\subset\mathbb{R}^3$, where for simplicity we shall assume that

$$\phi \in C^2(\overline{\Omega}) \qquad \text{and} \qquad f \in C^1(\overline{\Omega} \times [0,\infty);\mathbb{R}^3) \cap L^\infty(\Omega \times (0,\infty);\mathbb{R}^3), \tag{1.7}$$

and where we shall suppose throughout the sequel that $S \in C^2([0,\infty))$ satisfies

$$|S(n)| \le K_S(n+1)^{-\alpha} \qquad \text{for all } n \ge 0 \tag{1.8}$$

with some $\alpha > 0$ and $K_S > 0$. The initial data in (1.6) will be assumed to be such that

$$\begin{cases} n_0 \in C^0(\bar{\Omega}) \quad \text{with } n_0 \ge 0, \\ c_0 \in W^{1,\infty}(\Omega) \quad \text{with } c_0 \ge 0 \quad \text{and} \\ u_0 \in D(A^\beta) \quad \text{for some } \beta \in (\frac{3}{4},1), \end{cases} \tag{1.9}$$

where $A = -\mathcal{P}\Delta$ represents the Stokes operator in $L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega;\mathbb{R}^3) \mid \nabla \cdot \varphi = 0\}$, with its domain given by $D(A) := W^{2,2}(\Omega;\mathbb{R}^3) \cap W^{1,2}_0(\Omega;\mathbb{R}^3) \cap L^2_\sigma(\Omega)$, and with $\mathcal{P}$ denoting the Helmholtz projection from $L^2(\Omega;\mathbb{R}^3)$ into $L^2_\sigma(\Omega)$.

In this context, our main results read as follows.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, let ϕ and f satisfy (1.7), and let $S \in C^2([0,\infty))$ be such that (1.8) holds with some*

$$\alpha > \frac{1}{3}. \tag{1.10}$$

Then for all n_0, c_0 and u_0 fulfilling (1.9), the problem (1.6) possesses a global classical solution (n,c,u,P), uniquely determined by the inclusions

$$\begin{cases} n \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ c \in \bigcap_{p>3} C^0([0,\infty);W^{1,p}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ u \in C^0([0,\infty);D(A^\beta)) \cap C^{2,1}(\overline{\Omega} \times [0,\infty);\mathbb{R}^3), \\ P \in C^{1,0}(\overline{\Omega} \times (0,\infty)), \end{cases} \tag{1.11}$$

for which $n \ge 0$ and $c \ge 0$ in $\Omega \times (0,\infty)$. Moreover, given any $p > 1$ one can find $C > 0$ such that

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,p}(\Omega)} + \|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t \ge 0. \tag{1.12}$$

With regard to the question of global solvability by bounded functions for arbitrary coefficient functions ϕ and f and initial data, the problem of identifying a critical decay rate of S, up to evident remaining open topics arising when e.g. in (1.3) we precisely have equality, thereby seems comprehensively solved in the spatially three-dimensional case. In comparison to this, the picture seems much less complete in neighboring families of systems in which chemotactic cross-diffusion interacts with either alternative or further mechanisms. For instance, logistic-type growth restrictions, as modeled by additional summands of the form $\rho n - \mu n^2$ in the respective equation for n, have recently been shown to prevent blow-up in corresponding Keller-Segel-fluid variants of (1.6) if either $N = 2$ and $\mu > 0$ is arbitrary, even in the case when the fluid flow is governed by the full Navier-Stokes equations ([8], [29]), or $N = 3$ and $\mu > 0$ is suitably large ([28]). This generalizes previously known facts for the corresponding fluid-free Keller-Segel-growth system ([23], [39]), but due to the lack of any complementary result on blow-up e.g. for $N = 3$ and small $\mu > 0$, this only partially clarifies how far the potential to enforce explosions is influenced by fluid interaction in such circumstances. Similar observations concern related chemotaxis(-fluid) systems accounting for consumption, rather than production, of the chemical signal by the cells, in the most prototypical form requiring a replacement of the reaction term $-c + n$ with $-nc$ in the equation determining the evolution of c. Models of this form have been studied quite thoroughly in the literature, both with diffusion and cross-diffusion of the form in (1.6) ([6], [40], [3], [2]), and also with focus on blow-up-inhibiting effects of either nonlinear variants of cross-diffusion rates as in (1.2) ([32], [33]), or of porous medium-type diffusion ([5], [7], [27], [44], [37]). In fact, various sets of conditions could be identified as sufficient for global solvability in such systems within classes of bounded functions ([37], [44], [42], [26]), but due to missing examples of blow-up it seems widely unclear yet how far they are necessary therefor in the respective setting.

We remark that as a by-product, Theorem 1.1 also asserts global existence of bounded solutions to the corresponding Neumann initial-boundary value problem for the two-component chemotaxis-transport system

$$\begin{cases} n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot \Big(nS(n)\nabla c \Big), \qquad & x \in \Omega, \ t > 0, \\ c_t + u \cdot \nabla c &= \Delta c - c + n, & x \in \Omega, \ t > 0, \end{cases}$$

with any prescribed sufficiently smooth and bounded solenoidal fluid field u; in fact, this can readily be verified upon obvious choices of ϕ and f in Theorem 1.1.

Main ideas. In the literature on the fluid-free system (1.2), proofs for boundedness under the optimal version of (1.8) could be built on analyzing functionals of the form

$$y(t) := \int_\Omega n^p(\cdot, t) + \int_\Omega |\nabla c(\cdot, t)|^r, \qquad t > 0,$$

for suitably chosen $p > 1$ and $r > 1$ ([17], [25]). Indeed, it can be seen that in a correspondingly obtained ODE for y, by making use of (1.8) it becomes possible to control the respective crucial cross-diffusive contribution by means of appropriate interpolation in order to show that y satisfies an ODI of the form $y' + ay \le b$ with some $a > 0$ and $b > 0$. However, besides on mass conservation any such interpolation procedure appears to rely on uniform boundedness of c with respect to the norm in $L^q(\Omega)$ for q close to the largest value $\frac{N}{N-2}$ that can be expected for such a property in the heat equation $c_t = \Delta c - c + h$ in $\Omega \times (0, T)$ with h only known to belong to $L^\infty((0, T); L^1(\Omega))$.

Now in presence of an additional fluid interaction of the form in (1.6), it seems unclear whether this is sufficient to warrant that the latter basic integrability property of the signal c remains to be valid in the entire optimal range $1 \le q < 3 = \frac{N}{N-2}$; accordingly, pursuing strategies in the flavor of the above needs to cope with weaker a priori information on c which eventually requires stronger assumptions, such as e.g. in [36], where bounds for c in $L^\infty((0, T); L^2(\Omega))$, yet available in the whole regime $\alpha > \frac{1}{3}$, are used to finally derive boundedness under the suboptimal condition $\alpha > \frac{1}{2}$.

A major technical challenge will thus consist in developing an alternative approach capable of deriving boundedness of solutions in the optimal range of α but relying on basic regularity information on n, c and u not substantially going beyond that mentioned above. In the present work this will be achieved by a series of arguments which at their core are based on an analysis of the simple functional $z(t) := \int_\Omega n^p(\cdot,t)$, $t > 0$, for suitably large $p > 1$. In order to appropriately estimate the respective cross-diffusive summand arising in an associated ODE for z (cf. (6.8)), unlike in most previous related works we shall make essential use of maximal Sobolev regularity properties of the heat and the Stokes evolution equations to derive bounds for the divergence Δc of the cross-diffusive gradient which immediately arises herein (Lemma 5.1 and Lemma 5.3), and the velocity u to which the regularity of the latter is linked (Lemma 5.2). These estimates will be formulated in terms of the quantities given by

$$I_p(T) := \sup_{t\in[\tau,T-\tau]} \int_t^{t+\tau} \int_\Omega |\nabla n^{\frac{p}{2}}|^2$$

for suitable $\tau \in (0,1]$ and within suitable ranges of $T > 2\tau$, and a crucial observation will reveal by means of appropriate interpolation arguments (Lemma 3.1, Lemma 4.1 and Lemma 6.1) that when merely $\alpha > \frac{1}{3}$, for sufficiently large $p > 1$ these quantities will satisfy inequalities of the form $I_p(T) \le aI_p^\gamma(T) + b$ with some $a > 0, b > 0$ and $\gamma \in (0,1)$ conveniently independent of T (Lemma 6.2). The boundedness properties of $I_p(T)$ thereby implied will afterwards entail estimates for n with respect to the norm in $L^p(\Omega)$ for arbitrarily large $p > 1$ (Lemma 6.3) and thus, through subsequent applications of basically well-established methods, yield estimates sufficient for the derivation of Theorem 1.1 (Section 7). We emphasize that during our interpolation procedures we shall only rely on an easily obtained weak a priori boundedness information on (n,c,u) in the spaces $L^1(\Omega) \times L^1(\Omega) \times L^p(\Omega;\mathbb{R}^3)$ for arbitrary $p \in (1,3)$ (see Lemma 2.2 and Lemma 2.3).

2 Preliminaries

2.1 Local existence and basic solution properties

Let us first state a basic result on local existence and extensibilty that can be achieved by means of arguments well-known in the theory of chemotaxis and chemotaxis-fluid systems ([40], [17], [1]).

Lemma 2.1 *Let $\phi \in C^2(\overline{\Omega})$, $f \in C^1(\overline{\Omega} \times [0,\infty);\mathbb{R}^3)$ and $S \in C^2([0,\infty))$, and suppose that n_0, c_0 and u_0 comply with (1.9). Then there exist $T_{max} \in (0,\infty]$ and a uniquely determined quadruple (n,c,u,P) of functions*

$$\begin{cases} n \in C^0(\overline{\Omega} \times [0,T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{max})), \\ c \in \bigcap_{p>3} C^0([0,T_{max}); W^{1,p}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,T_{max})), \\ u \in C^0([0,T_{max}); D(A^\beta)) \cap C^{2,1}(\overline{\Omega} \times [0,T_{max}); \mathbb{R}^3), \\ P \in C^{1,0}(\overline{\Omega} \times (0,T_{max})), \end{cases} \tag{2.1}$$

which are such that $n \ge 0$ and $c \ge 0$ in $\Omega \times (0,T_{max})$, that (n,c,u,P) solves (1.6) in the classical sense in $\Omega \times (0,T_{max})$, and that

$$\text{if } T_{max} < \infty \quad \text{then} \quad \limsup_{t\nearrow T_{max}} \Big(\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,p}(\Omega)} + \|A^\beta u(\cdot,t)\|_{L^2(\Omega)} \Big) = \infty \quad \text{for all } p > 3. \tag{2.2}$$

The first two solution components can easily be seen to belong to $L^\infty((0,T_{max}); L^1(\Omega))$:

Lemma 2.2 *Under the assumptions of Lemma 2.1, the solution of (1.6) satisfies*

$$\int_\Omega n(\cdot,t) = \int_\Omega n_0 \qquad \text{for all } t \in [0,T_{max}) \tag{2.3}$$

and

$$\int_\Omega c(\cdot,t) \le \max\Big\{ \int_\Omega n_0\,,\,\int_\Omega c_0 \Big\} \qquad \text{for all } t \in [0,T_{max}). \tag{2.4}$$

Proof. We firstly obtain (2.3) as an immediate consequence of the fact that $\frac{d}{dt}\int_\Omega n = 0$ for all $t \in (0,T_{max})$ by (1.6). Thereafter, noting that thus $\frac{d}{dt}\int_\Omega c = -\int_\Omega c + \int_\Omega n = -\int_\Omega c + \int_\Omega n_0$ for all $t \in (0,T_{max})$, we may invoke an ODE comparison argument to readily verify (2.4). $\Box$

Under the boundedness assumption on f from Theorem 1.1, due to (2.3) also the fluid velocity enjoys a basic boundedness property. As precedent derivations of similar features in related systems apparently only address contexts without external source terms (see e.g. [36, Lemma 2.5]), let us include a short proof of this essentially well-known fact here for completeness.

Lemma 2.3 *If, beyond the assumptions of Lemma 2.1, f is bounded in $\Omega\times(0,\infty)$, then for each $p \in (1,3)$ there exists $C>0$ such that*

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le C \qquad \text{for all } t \in [0,T_{max}). \tag{2.5}$$

Proof. Since $p<3$ and hence $\frac{3}{2}-\frac{3}{2p}<1$, it is possible to fix $\gamma\in(0,1)$ such that $\gamma > \frac{3}{2}-\frac{3}{2p}$, which by a known embedding property ([43, Lemma 3.3]) ensures the existence of $C_1>0$ such that

$$\|A^{-\gamma}\mathcal{P}\varphi\|_{L^p(\Omega)} \le C_1\|\varphi\|_{L^1(\Omega)} \qquad \text{for all } \varphi \in C^1(\overline{\Omega};\mathbb{R}^3).$$

According to well-known smoothing properties of the Stokes semigroup ([24], [12]), on the basis of a variation-of-constants representation of u we thus infer that with some $C_2>0$ and $\lambda_1>0$ we have

$$\begin{aligned}
\|u(\cdot,t)\|_{L^p(\Omega)} &= \bigg\| e^{-tA}u_0 + \int_0^t A^\gamma e^{-(t-s)A}A^{-\gamma}\mathcal{P}\Big[n(\cdot,s)\nabla\phi + f(\cdot,s)\Big]ds\bigg\|_{L^p(\Omega)} ds \\
&\le C_2\|u_0\|_{L^p(\Omega)} + C_2\int_0^t (t-s)^{-\gamma}e^{-\lambda_1(t-s)}\Big\|A^{-\gamma}\mathcal{P}\Big[n(\cdot,s)\nabla\phi + f(\cdot,s)\Big]\Big\|_{L^1(\Omega)} ds \\
&\le C_2\|u_0\|_{L^p(\Omega)} + C_1C_2\int_0^t (t-s)^{-\gamma}e^{-\lambda_1(t-s)}\|n(\cdot,s)\nabla\phi + f(\cdot,s)\|_{L^1(\Omega)}
\end{aligned}$$

for all $t\in[0,T_{max})$. Since using (2.3) we obtain that

$$\|n(\cdot,s)\nabla\phi + f(\cdot,s)\|_{L^1(\Omega)} \le \|\nabla\phi\|_{L^\infty(\Omega)}\int_\Omega n_0 + |\Omega|\cdot\|f\|_{L^\infty(\Omega\times(0,\infty))} \qquad \text{for all } s\in(0,T_{max}),$$

and since the requirement that $\gamma<1$ implies that $\int_0^t (t-s)^{-\gamma}e^{-\lambda_1(t-s)}ds \le \int_0^\infty \sigma^{-\gamma}e^{-\lambda_1\sigma}d\sigma < \infty$ for all $t\ge 0$, this immediately yields (2.5). $\Box$

2.2 An ODE lemma

For later use in Lemma 4.1 and Lemma 6.2, let us provide an elementary statement on upper estimates in superlinearly dampened ordinary differential inequalities involving forcing terms only known to be bounded in average.

Lemma 2.4 *Let $t_\star \in \mathbb{R}$, $T > t_\star$ and $\tau \in (0, T - t_\star)$, and suppose that $y \in C^0([t_\star, T)) \cap C^1((t_\star, T))$, $g \in L^1((t_\star, T))$ and $h \in L^1((t_\star, T))$ are nonnegative and such that*

$$y'(t) + ay^\gamma(t) + g(t) \le h(t) \qquad \textit{for all } t \in (t_\star, T) \tag{2.6}$$

and

$$\int_t^{t+\tau} h(s)ds \le b \qquad \textit{for all } t \in [t_\star, T-\tau] \tag{2.7}$$

with some $a > 0, b > 0$ and $\gamma > 1$. Then

$$y(t) \le b + C \qquad \textit{for all } t \in [t_\star, T] \tag{2.8}$$

and

$$\int_t^{t+\tau} g(s)ds \le 2b + C \qquad \textit{for all } t \in [t_\star, T-\tau], \tag{2.9}$$

where

$$C := \max\left\{ y(t_\star)\, ,\, [(\gamma-1)a\tau]^{-\frac{1}{\gamma-1}} \right\}. \tag{2.10}$$

Proof. Abbreviating $C_1 := [(\gamma-1)a]^{-\frac{1}{\gamma-1}}$ and without loss of generality assuming that $t_\star = 0$, we first claim that then for any choice of $t_0 \in [0, T)$ we have

$$y(t) \le \overline{y}(t) := C_1 (t-t_0)^{-\frac{1}{\gamma-1}} + \int_{t_0}^t h(s)ds \qquad \text{for all } t \in (t_0, T]. \tag{2.11}$$

To verify this, we observe that

$$\begin{aligned}
\overline{y}'(t) + a\overline{y}^\gamma(t) - h(t) &= -\frac{C_1}{\gamma-1}(t-t_0)^{-\frac{\gamma}{\gamma-1}} + h(t) + a \cdot \left\{ C_1 (t-t_0)^{-\frac{1}{\gamma-1}} + \int_{t_0}^t h(s)ds \right\}^\gamma - h(t) \\
&\ge -\frac{C_1}{\gamma-1}(t-t_0)^{-\frac{\gamma}{\gamma-1}} + a \cdot \left\{ C_1 (t-t_0)^{-\frac{1}{\gamma-1}} \right\}^\gamma \\
&= 0 \qquad \text{for all } t \in (t_0, T),
\end{aligned}$$

because $\frac{C_1}{\gamma-1} = aC_1^\gamma$ according to our definition of C_1. Since y is bounded and $\overline{y}(t) \nearrow +\infty$ as $t \searrow t_0$, an ODE comparison argument on $[t_0 + \delta, T]$ with suitably small $\delta \in (0, T - t_0)$ therefore yields (2.11). Now for $t \ge \tau$, we may therein choose $t_0 := t - \tau$ to see that in view of (2.7) and (2.10),

$$y(t) \le C_1 \tau^{-\frac{1}{\gamma-1}} + \int_{t-\tau}^t h(s)ds \le C_1 \tau^{-\frac{1}{\gamma-1}} + b \le C + b \qquad \text{for all } t \in [\tau, T],$$

whereas for smaller t we simply neglect two nonnegative summands on the left of (2.6) to find upon integration that again due to (2.7),

$$y(t) \le y(0) + \int_0^t h(s)ds \le C + \int_0^\tau h(s)ds \le C + b \qquad \text{for all } t \in [0,\tau),$$

because h is nonnegative.
Having thereby established (2.8), by means of another integration in (2.6) we finally obtain that

$$\begin{aligned} \int_t^{t+\tau} g(s)ds &\le y(t) - y(t+\tau) + \int_t^{t+\tau} h(s)ds \\ &\le (b+C) + b \qquad \text{for all } t \in [0, T-\tau], \end{aligned}$$

and that thus also (2.9) is valid. □

3 A space-time regularity property of n implied by bounds for $\nabla n^{\frac{p}{2}}$

In order to simplify notation, throughout the remaining analysis we assume unless otherwise stated that ϕ, f, S and (n_0, c_0, u_0) are such that the hypotheses of Lemma 2.1 are satisfied, that moreover f is bounded, and that (1.8) holds with some $K_S > 0$ and $\alpha > 0$. We then let (n, c, u, P) and $T_{max} \in (0, \infty]$ be as provided by Lemma 2.1, and set

$$\tau := \min\Big\{1\,,\,\frac{1}{4}T_{max}\Big\}. \tag{3.1}$$

Now in the major part of our subsequent reasoning, a crucial role will be played by the quantities defined by

$$I_p(T) := \sup_{t\in[\tau,T-\tau]} \int_t^{t+\tau}\int_\Omega |\nabla n^{\frac{p}{2}}|^2 \qquad \text{for } T \in (2\tau, T_{max}) \quad \text{and } p > 1, \tag{3.2}$$

which contain the dissipated quantity appearing in a standard L^p testing procedure when applied to the first equation in (1.6). Our arguments to control the cross-diffusive contributions therein will be prepared by a series of bounds for n, c and u in terms of I_p, with the final ambition to estimate $I_p(T)$ by, essentially, a sublinear power thereof (Lemma 6.2).

Our first step in this direction, based on a simple interpolation argument involving (2.3), will frequently be applied in the following lemmata.

Lemma 3.1 *Let $p > 1$, and suppose that $\kappa > 1$ and $\lambda > 0$ are such that*

$$\kappa \le 3p \tag{3.3}$$

and

$$\frac{3(\kappa-1)}{(3p-1)\kappa} \cdot \lambda < 1. \tag{3.4}$$

Then there exists $C > 0$ such that for all $T \in (2\tau, T_{max})$ we have

$$\int_t^{t+\tau} \|n(\cdot,s)\|_{L^\kappa(\Omega)}^\lambda ds \le C + C I_p^{\frac{3(\kappa-1)\lambda}{(3p-1)\kappa}}(T) \qquad \text{for all } t \in [0, T-\tau]. \tag{3.5}$$

PROOF. Using that $\kappa \ge 1$ and $\kappa \le 3p$, we invoke the Gagliardo-Nirenberg inequality to fix $C_1 > 0$ such that

$$\|\varphi\|_{L^{\frac{2\kappa}{p}}(\Omega)}^{\frac{2\lambda}{p}} \le C_1 \|\nabla\varphi\|_{L^2(\Omega)}^{\frac{6(\kappa-1)\lambda}{(3p-1)\kappa}} \|\varphi\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(3p-\kappa)\lambda}{p(3p-1)\kappa}} + C_1 \|\varphi\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2\lambda}{p}} \qquad \text{for all } \varphi \in W^{1,2}(\Omega),$$

and apply this to $\varphi := n^{\frac{p}{2}}(\cdot, s)$ for $s \in (\tau, T_{max})$ to see upon a time integration that since $\|n^{\frac{p}{2}}(\cdot,s)\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{p}} = C_2 := \int_\Omega n_0$ for all $s \in (\tau, T_{max})$ by (2.3),

$$\begin{aligned} \int_t^{t+\tau} \|n(\cdot,s)\|_{L^\kappa(\Omega)}^\lambda ds &= \int_t^{t+\tau} \|n^{\frac{p}{2}}(\cdot,s)\|_{L^{\frac{2\kappa}{p}}(\Omega)}^{\frac{2\lambda}{p}} ds \\ &\le C_3 \int_t^{t+\tau} \|\nabla n^{\frac{p}{2}}(\cdot,s)\|_{L^2(\Omega)}^{\frac{6(\kappa-1)\lambda}{(3p-1)\kappa}} ds + C_4\tau \qquad \text{for all } t \in [\tau, T_{max}-\tau) \end{aligned} \tag{3.6}$$

with $C_3 := C_1 C_2^{\frac{(3p-\kappa)\lambda}{(3p-1)\kappa}}$ and $C_4 := C_1 C_2^\lambda$. Here thanks to the fact that $\frac{6(\kappa-1)\lambda}{(3p-1)\kappa} < 2$ according to (3.4), we may employ the Hölder inequality to obtain that

$$\int_t^{t+\tau} \|\nabla n^{\frac{p}{2}}(\cdot,s)\|_{L^2(\Omega)}^{\frac{6(\kappa+1)\lambda}{(3p-1)\kappa}} ds \le \left\{ \int_t^{t+\tau} \|\nabla n^{\frac{p}{2}}(\cdot,s)\|_{L^2(\Omega)}^2 ds \right\}^{\frac{3(\kappa-1)\lambda}{(3p-1)\kappa}} \cdot \tau^{1-\frac{3(\kappa-1)\lambda}{(3p-1)\kappa}} \qquad \text{for all } t \in [\tau, T_{max}-\tau).$$

Using that $\tau \le 1$, given $T \in (2\tau, T_{max})$ from (3.6) we thus conclude that due to the definition of I_p we have

$$\int_t^{t+\tau} \|n(\cdot,s)\|_{L^\kappa(\Omega)}^\lambda ds \le C_3 I_p^{\frac{3(\kappa-1)\lambda}{(3p-1)\kappa}}(T) + C_4 \qquad \text{for all } t \in [\tau, T-\tau],$$

which implies (3.5) due to the fact that n is bounded in $\Omega \times [0,\tau)$ by Lemma 2.1. $\Box$

4 An L^q bound for c in terms of $I_p(T)$

A first application of Lemma 3.1 yields the following L^q estimate for c in dependence on $I_p(T)$, provided that p is suitably large relative to q. Our derivation thereof is based on an L^q testing procedure for the second equation in (1.6) and thus, due to the solenoidality of the velocity field, does not rely on any explicit bound on u.

Lemma 4.1 *Let $p > 1$ and $q > 1$ be such that*

$$q < 3p. \tag{4.1}$$

Then there exists $C > 0$ such that for all $T \in (2\tau, T_{max})$,

$$\int_\Omega c^q(\cdot,t) \le C + C I_p^{\frac{q-1}{3p-1}}(T) \qquad \textit{for all } t \in [0,T]. \tag{4.2}$$

Proof. We use c^{q-1} as a test function for the second equation in (1.6) and note that $\nabla\cdot u\equiv 0$ to see by means of the Hölder inequality that

$$\begin{aligned}
\frac{1}{q}\frac{d}{dt}\int_\Omega c^q+(q-1)\int_\Omega c^{q-2}|\nabla c|^2+\int_\Omega c^q &= \int_\Omega nc^{q-1}=\int_\Omega n\cdot(c^{\frac{q}{2}})^{\frac{2(q-1)}{q}}\\
&\le \|n\|_{L^{\frac{3q}{2q+1}}(\Omega)}\|c^{\frac{q}{2}}\|_{L^6(\Omega)}^{\frac{2(q-1)}{q}} \qquad \text{for all } t\in(0,T_{max}).
\end{aligned}\tag{4.3}$$

Here employing the three-dimensional Sobolev inequality followed by Young's inequality we can find $C_1>0$ and $C_2>0$ such that

$$\begin{aligned}
\|n\|_{L^{\frac{3q}{2q+1}}(\Omega)}\|c^{\frac{q}{2}}\|_{L^6(\Omega)}^{\frac{2(q-1)}{q}} &\le C_1\|n\|_{L^{\frac{3q}{2q-1}}(\Omega)}\cdot\Big\{\|\nabla c^{\frac{q}{2}}\|_{L^2(\Omega)}^2+\|c^{\frac{q}{2}}\|_{L^2(\Omega)}^2\Big\}^{\frac{q-1}{q}}\\
&\le \frac{2(q-1)}{q^2}\cdot\Big\{\|\nabla c^{\frac{q}{2}}\|_{L^2(\Omega)}^2+\|c^{\frac{q}{2}}\|_{L^2(\Omega)}^2\Big\}+C_2\|n\|^q_{L^{\frac{3q}{2q+1}}(\Omega)}\\
&\le \frac{2(q-1)}{q^2}\int_\Omega|\nabla c^{\frac{q}{2}}|^2+\int_\Omega c^q+C_2\|n\|^q_{L^{\frac{3q}{2q+1}}(\Omega)} \qquad \text{for all } t\in(0,T_{max}),
\end{aligned}$$

because $\frac{2(q-1)}{q^2}\le 1$. Since $(q-1)\int_\Omega c^{q-2}|\nabla c|^2=\frac{4(q-1)}{q^2}\int_\Omega|\nabla c^{\frac{q}{2}}|^2$ for all $t\in(0,T_{max})$, from (4.3) we thus infer that

$$\frac{d}{dt}\int_\Omega c^q+C_3\int_\Omega|\nabla c^{\frac{q}{2}}|^2\le C_2q\|n\|^q_{L^{\frac{3q}{2q+1}}(\Omega)} \qquad \text{for all } t\in(0,T_{max}) \tag{4.4}$$

with $C_3:=\frac{2(q-1)}{q}>0$. Now combining the Gagliardo-Nirenberg inequality with the fact that $\|c^{\frac{q}{2}}(\cdot,t)\|^{\frac{2}{q}}_{L^{\frac{2}{q}}(\Omega)}\le\max\{\int_\Omega n_0,\int_\Omega c_0\}$ by (2.4), we can furthermore find $C_4>0$ and $C_5>0$ such that

$$\begin{aligned}
\Big\{\int_\Omega c^q\Big\}^{\frac{3q-1}{3(q-1)}} &= \|c^{\frac{q}{2}}\|_{L^2(\Omega)}^{\frac{2(3q-1)}{3(q-1)}}\\
&\le C_4\|\nabla c^{\frac{q}{2}}\|^2_{L^2(\Omega)}\|c^{\frac{q}{2}}\|^{\frac{4}{3(q-1)}}_{L^{\frac{2}{q}}(\Omega)}+C_4\|c^{\frac{q}{2}}\|^{\frac{2(3q-1)}{3(q-1)}}_{L^{\frac{2}{q}}(\Omega)}\\
&\le C_5\|\nabla c^{\frac{q}{2}}\|^2_{L^2(\Omega)}+C_5 \qquad \text{for all } t\in(0,T_{max})
\end{aligned}$$

and hence

$$\int_\Omega|\nabla c^{\frac{q}{2}}|^2\ge\frac{1}{C_5}\cdot\Big\{\int_\Omega c^q\Big\}^{\frac{3q-1}{3(q-1)}}-1 \qquad \text{for all } t\in(0,T_{max}).$$

Consequently, (4.4) can be turned into the inequality

$$\frac{d}{dt}\int_\Omega c^q+\frac{C_3}{C_5}\cdot\Big\{\int_\Omega c^q\Big\}^{\frac{3q-1}{3(q-1)}}\le C_2q\|n\|^q_{L^{\frac{3q}{2q+1}}(\Omega)}+C_3 \qquad \text{for all } t\in(0,T_{max}),$$

which by Lemma 2.4 implies that whenever $T \in (2\tau, T_{max})$,

$$\begin{aligned}
\int_\Omega c^q(\cdot,t) &\le \sup_{s\in[\tau,T-\tau]} \int_s^{s+\tau} \Big\{ C_2 q \|n(\cdot,\sigma)\|^q_{L^{\frac{3q}{2q+1}}(\Omega)} + C_3 \Big\} d\sigma + C_6 \\
&= C_2 q \sup_{s\in[\tau,T-\tau]} \int_s^{s+\tau} \|n(\cdot,\sigma)\|^q_{L^{\frac{3q}{2q+1}}(\Omega)} d\sigma + C_3\tau + C_6 \qquad \text{for all } t \in [\tau,T] \qquad (4.5)
\end{aligned}$$

with $C_6 := \max\Big\{ \int_\Omega c_0^q, [\frac{2}{3(q-1)} \cdot \frac{C_3}{C_5} \cdot \tau]^{-\frac{3(q-1)}{2}} \Big\}$.

In order to further estimate the right-hand side in (4.5) on the basis of Lemma 3.1, we observe that $\kappa := \frac{3q}{2q+1}$ and $\lambda := q$ satisfy

$$1 < \frac{2q+q}{2q+1} = \kappa < \frac{3q}{2q} = \frac{3}{2} < 3p$$

due to our assumptions that $q > 1$ and $p > 1$, and that our additional requirement (4.1) ensures that

$$\frac{3(\kappa-1)}{(3p-1)\kappa} \cdot \lambda = \frac{q-1}{3p-1} < 1.$$

Therefore, Lemma 3.1 indeed becomes applicable so as to yield $C_7 > 0$ such that

$$\int_s^{s+\tau} \|n(\cdot,\sigma\|^q_{L^{\frac{3q}{2q+1}}(\Omega)} d\sigma \le C_7 + C_7 I_p^{\frac{q-1}{3p-1}}(T) \qquad \text{for all } s \in [\tau, T-\tau],$$

whereupon (4.2) results from (4.5) and the boundedness of c in $\Omega \times [0,\tau)$ entailed by Lemma 2.1. □

5 Estimates for Δc in terms of $I_p(T)$ via maximal Sobolev regularity

Approaching the core of our analysis, our next goal consists in controlling the cross-diffusive gradient in (1.6) by quantities containing suitably small powers of $I_p(T)$ under appropriate further assumptions on α and the yet free parameter p. Here our first result will relate a second-order Sobolev norm of c to regularity properties of the three quantities n, c and u making up the inhomogeneity $h := n - u\cdot\nabla c$ in the heat equation $c_t = \Delta c - c + h$. This will be achieved through an argument based on a maximal Sobolev regularity feature of the latter, along with a suitable temporal regularization procedure which we prepare by fixing a nondecreasing $\zeta_0 \in C^\infty(\mathbb{R})$ such that $\zeta_0 \equiv 0$ in $(-\infty,-\tau]$ and $\zeta_0 \equiv 1$ in $[0,\infty)$, and defining a family of functions $(\zeta^{(t_0)})_{t_0\in\mathbb{R}}$ by letting

$$\zeta^{(t_0)}(t) := \zeta_0(t-t_0) \qquad \text{for } t_0 \in \mathbb{R} \text{ and } t \in \mathbb{R}. \qquad (5.1)$$

Our first step toward estimating Δc will now consist in the following inequality.

Lemma 5.1 *Let $q > \frac{3}{2}$ and $r > 1$. Then there exists $C > 0$ such that*

$$\begin{aligned}
\int_t^{t+\tau} \|c(\cdot,s)\|^r_{W^{2,\frac{3}{2}}(\Omega)} ds \le\ & C + C\int_{t-\tau}^{t+\tau} \|n(\cdot,s)\|^r_{L^{\frac{3}{2}}(\Omega)} ds \\
& + C\cdot \Big\{ \sup_{s\in[0,t+\tau]} \|c(\cdot,s)\|_{L^q(\Omega)} \Big\}^r \cdot \int_{t-\tau}^{t+\tau} \|u(\cdot,s)\|^{2r}_{L^{\frac{6q}{2q-3}}(\Omega)} ds \\
& + C\cdot \sup_{s\in[0,t+\tau]} \|c(\cdot,s)\|^r_{L^{\frac{3}{2}}(\Omega)} \qquad \textit{for all } t \in [\tau, T_{max}-\tau). \qquad (5.2)
\end{aligned}$$

Proof. We fix $t_0 \in [\tau, T_{max} - \tau)$, and with $\zeta := \zeta^{(t_0)}$ as defined in (5.1) we let

$$\widehat{c}(x,t) := \zeta(t)c(x,t), \qquad x \in \overline{\Omega},\ t \in [t_0 - \tau, T_{max}).$$

Then $\widehat{c}$ is a solution of

$$\begin{cases} \widehat{c}_t = \Delta\widehat{c} - \widehat{c} + \zeta n - \zeta u \cdot \nabla c + \zeta_t c, & x \in \Omega,\ t \in (t_0 - \tau, T_{max}), \\ \frac{\partial \widehat{c}}{\partial \nu} = 0, & x \in \partial\Omega,\ t \in (t_0 - \tau, T_{max}), \\ \widehat{c}(x, t_0 - \tau) = 0, & x \in \Omega, \end{cases}$$

so that known results on maximal Sobolev regularity in the Neumann problem for the heat equation ([13]) provide $C_1 > 0$ such that

$$\begin{aligned} \int_{t_0-\tau}^{t_0+\tau} \|\widehat{c}(\cdot,s)\|^r_{W^{2,\frac{3}{2}}(\Omega)} ds &\le C_1 \int_{t_0-\tau}^{t_0+\tau} \Big\| \zeta(s)n(\cdot,s) - \zeta(s)u(\cdot,s)\cdot\nabla c(\cdot,s) + \zeta_t(s)c(\cdot,s) \Big\|^r_{L^{\frac{3}{2}}(\Omega)} ds \\ &\le C_1 \int_{t_0-\tau}^{t_0+\tau} \|n(\cdot,s)\|^r_{L^{\frac{3}{2}}(\Omega)} ds + C_1 \int_{t_0-\tau}^{t_0+\tau} \Big\| \zeta(s)u(\cdot,s)\cdot\nabla c(\cdot,s) \Big\|^r_{L^{\frac{3}{2}}(\Omega)} ds \\ &\quad +2C_1C_2 \sup_{s\in[0,t_0+\tau]} \|c(\cdot,s)\|^r_{L^{\frac{3}{2}}(\Omega)} \end{aligned} \tag{5.3}$$

with $C_2 := \|(\zeta_0)_t\|^r_{L^\infty((-\tau,\tau))}$, because $|\zeta_0| \le 1$ and $\tau \le 1$. Moreover, using the Hölder inequality we see that

$$\begin{aligned} &\int_{t_0-\tau}^{t_0+\tau} \Big\| \zeta(s)u(\cdot,s)\cdot\nabla c(\cdot,s) \Big\|^r_{L^{\frac{3}{2}}(\Omega)} ds \\ &\qquad\le \int_{t_0-\tau}^{t_0+\tau} \zeta^r(s) \|u(\cdot,s)\|^r_{L^{\frac{6q}{2q-3}}(\Omega)} \|\nabla c(\cdot,s)\|^r_{L^{\frac{6q}{2q+3}}(\Omega)} ds \\ &\qquad\le \Big\{ \int_{t_0-\tau}^{t_0+\tau} \|u(\cdot,s)\|^{2r}_{L^{\frac{6q}{2q-3}}(\Omega)} ds \Big\}^{\frac{1}{2}} \cdot \Big\{ \int_{t_0-\tau}^{t_0+\tau} \zeta^{2r}(s) \|\nabla c(\cdot,s)\|^{2r}_{L^{\frac{6q}{2q+3}}(\Omega)} ds \Big\}^{\frac{1}{2}}, \end{aligned} \tag{5.4}$$

where according to the Gagliardo-Nirenberg inequality there exists $C_2 > 0$ such that

$$\begin{aligned} \zeta^{2r}(s)\|\nabla c(\cdot,s)\|^{2r}_{L^{\frac{6q}{2q+3}}(\Omega)} &\le C_2 \zeta^{2r}(s) \|c(\cdot,s)\|^r_{W^{2,\frac{3}{2}}(\Omega)} \|c(\cdot,s)\|^r_{L^q(\Omega)} \\ &\le C_2 \|\widehat{c}(\cdot,s)\|^r_{W^{2,\frac{3}{2}}(\Omega)} \cdot \Big\{ \sup_{\sigma\in[0,t_0+\tau]} \|c(\cdot,\sigma)\|_{L^q(\Omega)} \Big\}^r \quad \text{for all } s \in [t_0-\tau, t_0+\tau], \end{aligned}$$

again due to the fact that $|\zeta_0| \le 1$. Upon an application of Young's inequality, (5.4) therefore entails that

$$\begin{aligned} &C_1 \int_{t_0-\tau}^{t_0+\tau} \Big\| \zeta(s)u(\cdot,s)\cdot\nabla c(\cdot,s) \Big\|^r_{L^{\frac{3}{2}}(\Omega)} ds \\ &\le C_1\sqrt{C_2} \cdot \Big\{ \sup_{s\in[0,t_0+\tau]} \|c(\cdot,s)\|_{L^q(\Omega)} \Big\}^{\frac{r}{2}} \cdot \Big\{ \int_{t_0-\tau}^{t_0+\tau} \|u(\cdot,s)\|^{2r}_{L^{\frac{6q}{2q-3}}(\Omega)} ds \Big\}^{\frac{1}{2}} \cdot \Big\{ \int_{t_0-\tau}^{t_0+\tau} \|\widehat{c}(\cdot,s)\|^r_{W^{2,\frac{3}{2}}(\Omega)} ds \Big\}^{\frac{1}{2}} \\ &\le \frac{1}{2} \int_{t_0-\tau}^{t_0+\tau} \|\widehat{c}(\cdot,s)\|^r_{W^{2,\frac{3}{2}}(\Omega)} ds + \frac{C_1^2C_2}{2} \cdot \Big\{ \sup_{s\in[0,t_0+\tau]} \|c(\cdot,s)\|_{L^q(\Omega)} \Big\}^r \cdot \int_{t_0-\tau}^{t_0+\tau} \|u(\cdot,s)\|^{2r}_{L^{\frac{6q}{2q-3}}(\Omega)} ds. \end{aligned}$$

In view of the fact that $\zeta \equiv 1$ on $[t_0, t_0+\tau]$ and that hence $\widehat{c} \equiv c$ throughout $\Omega \times [t_0, t_0+\tau]$, together with (5.3) this establishes (5.2). $\Box$

The expressions on the right of (5.2) containing n and c can be estimated in terms of $I_p(T)$ by means of Lemma 3.1 and Lemma 4.1. In relating the remaining rightmost integral therein to $I_p(T)$ as well, we rely on a maximal Sobolev regularity property now of the Stokes evolution system to see that this indeed is possible when the summability power r in (5.2) is suitably small.

Lemma 5.2 *Let $p > 1, q > \frac{3}{2}$ and $r > 1$ be such that*

$$r < \frac{(3p-1)q}{3}. \tag{5.5}$$

Then one can find $C > 0$ with the property that for all $T \in (2\tau, T_{max})$,

$$\int_t^{t+\tau} \|u(\cdot,s)\|^{2r}_{L^{\frac{6q}{2q-3}}(\Omega)} ds \le C + C I_p^{\frac{3r}{(3p-1)q}}(T) \qquad \text{for all } t \in [0, T-\tau]. \tag{5.6}$$

Proof. As u is bounded in $\Omega \times [0,\tau)$ by Lemma 2.1, we only need to derive the claimed inequality in the time interval $[\tau, T-\tau]$ for arbitrary $T \in (2\tau, T_{max})$. To this end, fixing $t_0 \in [\tau, T-\tau]$ we once more take $\zeta := \zeta^{(t_0)}$ from (5.1) and let

$$\widehat{u}(x,t) := \zeta(t)u(x,t), \qquad x \in \overline{\Omega}, \ t \in [t_0-\tau, T_{max}),$$

so that

$$\begin{cases} \widehat{u}_t = \Delta\widehat{u} - \zeta\nabla P + \zeta n\nabla\phi + \zeta_t u + \zeta f, & x \in \Omega, \ t \in (t_0-\tau, T_{max}), \\ \widehat{u} = 0, & x \in \partial\Omega, \ t \in (t_0-\tau, T_{max}), \\ \widehat{u}(x, t_0-\tau) = 0, & x \in \Omega. \end{cases}$$

A maximal Sobolev regularity property of the Stokes evolution semigroup ([13]) thus yields $C_1 > 0$ such that

$$\begin{aligned} \int_{t_0-\tau}^{t_0+\tau} \|\widehat{u}(\cdot,s)\|^{2r}_{W^{2,\frac{2q}{2q-1}}(\Omega)} ds \ &\le \ C_1 \int_{t_0-\tau}^{t_0+\tau} \Big\| \zeta(s)n(\cdot,s)\nabla\phi + \zeta_t(s)u(\cdot,s) + \zeta(s)f(\cdot,s) \Big\|^{2r}_{L^{\frac{2q}{2q-1}}(\Omega)} ds \\ &\le \ C_2 \int_{t_0-\tau}^{t_0+\tau} \|n(\cdot,s)\|^{2r}_{L^{\frac{2q}{2q-1}}(\Omega)} ds + C_3 \int_{t_0-\tau}^{t_0+\tau} \|u(\cdot,s)\|^{2r}_{L^{\frac{2q}{2q-1}}(\Omega)} ds + C_4 \end{aligned} \tag{5.7}$$

with $C_2 := C_1\|\nabla\phi\|_{L^\infty(\Omega)}$, $C_3 := C_1\|(\zeta_0)_t\|_{L^\infty((-\tau,\tau))}$ and $C_4 := 2C_1|\Omega|^{\frac{r(2q-1)}{q}}\|f\|^{2r}_{L^\infty(\Omega\times(0,\infty))}$, because $|\zeta_0| \le 1$ and $(t_0+\tau)-(t_0-\tau) = 2\tau \le 2$.
To estimate the two integrals on the right-hand side herein, we write $\kappa := \frac{2q}{2q-1}$ and $\lambda := 2r$ and note that since $q > \frac{3}{2}$ we have

$$1 < \kappa < \frac{2\cdot\frac{3}{2}}{2\cdot\frac{3}{2}-1} = \frac{3}{2} < 3 < 3p, \tag{5.8}$$

and that thanks to (5.5) we moreover know that

$$\frac{3(\kappa-1)}{(3p-1)\kappa}\cdot\lambda = \frac{3r}{(3p-1)q} < 1. \tag{5.9}$$

From (5.8) we particularly see that Lemma 2.3 becomes applicable to show that there exists $C_4 > 0$ such that

$$\|u(\cdot,s)\|_{L^{\frac{2q}{2q-1}}(\Omega)} \le C_4 \qquad \text{for all } s \in (0,T_{max}), \tag{5.10}$$

and combining (5.8) with (5.9) we may invoke Lemma 3.1 to find $C_5 > 0$ fulfilling

$$\int_t^{t+\tau} \|n(\cdot,s)\|^{2r}_{L^{\frac{2q}{2q-1}}(\Omega)} ds \le C_5 + C_5 I_p^{\frac{3r}{(3p-1)q}}(T) \qquad \text{for all } t \in [0,T-\tau],$$

so that from (5.7) and (5.10) we thus infer that

$$\int_{t_0-\tau}^{t_0+\tau} \|\widehat{u}(\cdot,s)\|^{2r}_{W^{2,\frac{2q}{2q-1}}(\Omega)} ds \le 2C_2C_5 + 2C_2C_5 I_p^{\frac{3r}{(3p-1)q}}(T) + 2C_3C_4^{2r}\tau,$$

Since $\widehat{u} \equiv u$ in $\Omega \times [t_0, t_0+\tau]$ by (5.1), and since $W^{2,\frac{2q}{2q-1}}(\Omega) \hookrightarrow L^{\frac{6q}{2q-3}}(\Omega)$ in the present three-dimensional setting, this establishes (5.6) in the time interval $[\tau, T-\tau]$, as intended. $\square$

We can now formulate the main result of this section by combining Lemma 5.1 with Lemma 5.2, Lemma 4.1 and Lemma 3.1, where the latter turns out to be applicable here under a further smallness assumption on r.

Lemma 5.3 *Suppose that $p > 1, q > \frac{3}{2}$ and $r > 1$ are such that (4.1) and (5.5) hold as well as*

$$r < 3p-1. \tag{5.11}$$

Then there exists $C > 0$ such that whenever $T \in (2\tau, T_{max})$,

$$\int_t^{t+\tau} \|\Delta c(\cdot,s)\|^r_{L^{\frac{3}{2}}(\Omega)} ds \le C + C I_p^{\frac{r}{3p-1}}(T) + C I_p^{\frac{(q+2)r}{(3p-1)q}}(T) \qquad \textit{for all } t \in [\tau, T-\tau]. \tag{5.12}$$

Proof. Based on our assumptions that $q > \frac{3}{2}$ and that (4.1) and (5.5) hold, we first employ Lemma 4.1 and Lemma 5.2 to find positive constants C_1, C_2 and C_3 such that given $T \in (2\tau, T_{max})$ we know that

$$\|c(\cdot,s)\|^r_{L^q(\Omega)} \le C_1 + C_1 I_p^{\frac{(q-1)r}{(3p-1)q}}(T) \qquad \text{for all } s \in [0,T] \tag{5.13}$$

and

$$\|c(\cdot,s)\|^r_{L^{\frac{3}{2}}(\Omega)} \le C_2 + C_2 I_p^{\frac{r}{3(3p-1)}}(T) \qquad \text{for all } s \in [0,T] \tag{5.14}$$

as well as

$$\int_{t_0}^{t_0+\tau} \|u(\cdot,s)\|^{2r}_{L^{\frac{6q}{2q-3}}(\Omega)} ds \le C_3 + C_3 I_p^{\frac{3r}{(3p-1)q}}(T) \qquad \text{for all } t_0 \in [0,T-\tau]. \tag{5.15}$$

Moreover, writing $\kappa := \frac{3}{2}$ and $\lambda := r$ we obviously have $1 < \kappa < 3p$, whereas (5.11) guarantees that

$$\frac{3(\kappa-1)}{(3p-1)\kappa} \cdot \lambda < 1,$$

so that as a consequence of Lemma 3.1 we can pick $C_4 > 0$ satisfying

$$\int_{t_0}^{t_0+\tau} \|n(\cdot,s)\|_{L^{\frac{3}{2}}(\Omega)}^r ds \le C_4 + C_4 I_p^{\frac{r}{3p-1}}(T) \qquad \text{for all } t_0 \in [0, T-\tau]. \tag{5.16}$$

Now from Lemma 5.1 it follows that there exists $C_5 > 0$ such that

$$\begin{aligned}
\int_t^{t+\tau} \|\Delta c(\cdot,s)\|_{L^{\frac{3}{2}}(\Omega)}^r ds \\
&\le C_5 + C_5 \int_{t-\tau}^{t+\tau} \|n(\cdot,s)\|_{L^{\frac{3}{2}}(\Omega)}^r ds \\
&\quad + C_5 \cdot \Big\{ \sup_{s\in[0,t+\tau]} \|c(\cdot,s)\|_{L^q(\Omega)} \Big\}^r \cdot \int_{t-\tau}^{t+\tau} \|u(\cdot,s)\|_{L^{\frac{6q}{2q-3}}(\Omega)}^{2r} ds \\
&\quad + C_5 \sup_{s\in[0,t+\tau]} \|c(\cdot,s)\|_{L^{\frac{3}{2}}(\Omega)} \qquad \text{for all } t \in [\tau, T_{max}-\tau),
\end{aligned}$$

which in light of (5.13)-(5.16) particularly entails that

$$\begin{aligned}
\int_t^{t+\tau} \|\Delta c(\cdot,s)\|_{L^{\frac{3}{2}}(\Omega)}^r ds \\
&\le C_5 + C_5 \cdot \Big\{ 2C_4 + 2C_4 I_p^{\frac{r}{3p-1}}(T) \Big\} \\
&\quad + C_5 \cdot \Big\{ C_1 + C_1 I_p^{\frac{(q-1)r}{(3p-1)q}}(T) \Big\} \cdot \Big\{ 2C_3 + 2C_3 I_p^{\frac{3r}{(3p-1)q}}(T) \Big\} \\
&\quad + C_5 \cdot \Big\{ C_2 + C_2 I_p^{\frac{r}{3(3p-1)}}(T) \Big\} \qquad \text{for all } t \in [\tau, T-\tau].
\end{aligned}$$

As three applications of Young's inequality show that

$$\begin{aligned}
\Big\{ C_1 + C_1 I_p^{\frac{(q-1)r}{(3p-1)q}}(T) \Big\} \cdot \Big\{ 2C_3 + 2C_3 I_p^{\frac{3r}{(3p-1)q}}(T) \Big\} \\
&= 2C_1C_3 \cdot \Big\{ 1 + I_p^{\frac{(q-1)r}{(3p-1)q}}(T) + I_p^{\frac{3r}{(3p-1)q}}(T) + I_p^{\frac{(q+2)r}{(3p-1)q}}(T) \Big\} \\
&\le 2C_1C_3 \cdot \Big\{ 3 + 3 I_p^{\frac{(q+2)r}{(3p-1)q}}(T) \Big\}
\end{aligned}$$

and

$$C_2 I_p^{\frac{r}{3(3p-1)}}(T) \le C_2 I_p^{\frac{r}{3p-1}}(T) + C_2,$$

the derivation of (5.12) is complete. □

6 L^p bounds for n by closing the loop

Now controlling the cross-diffusive action in the announced testing procedure for n, to be detailed in Lemma 6.2, will amount to appropriately estimating $\int_\Omega n^{p-\alpha}|\Delta c|$. This can be achieved by means of Lemma 5.3 and, again, Lemma 3.1 if the exponent r in addition to the assumptions therein satisfies a further condition requiring r not to be too small:

Lemma 6.1 *Let $p>1, q>\frac{3}{2}$ and $r>1$ satisfy (4.1), (5.5) and (5.11) as well as*

$$p > \alpha + \frac{1}{3} \tag{6.1}$$

and

$$r > \frac{3p-1}{3\alpha}. \tag{6.2}$$

Then one can find $C>0$ such that for each $T \in (2\tau, T_{max})$,

$$\int_t^{t+\tau} \|n(\cdot,s)\|_{L^{3(p-\alpha)}(\Omega)}^{p-\alpha} \|\Delta c(\cdot,s)\|_{L^{\frac{3}{2}}(\Omega)} ds \le C + C I_p^{\frac{3(p-\alpha)}{3p-1}}(T) + C I_p^{\frac{3(p-\alpha)q+2}{(3p-1)q}}(T) \qquad \text{for all } t \in [\tau, T-\tau]. \tag{6.3}$$

Proof. By the Hölder inequality,

$$\begin{aligned} &\int_t^{t+\tau} \|n(\cdot,s)\|_{L^{3(p-\alpha)}(\Omega)}^{p-\alpha} \|\Delta c(\cdot,s)\|_{L^{\frac{3}{2}}(\Omega)} ds \\ &\quad \le \bigg\{ \int_t^{t+\tau} \|n(\cdot,s)\|_{L^{3(p-\alpha)}(\Omega)}^{\frac{(p-\alpha)r}{r-1}} ds \bigg\}^{1-\frac{1}{r}} \cdot \bigg\{ \int_t^{t+\tau} \|\Delta c(\cdot,s)\|_{L^{\frac{3}{2}}(\Omega)}^r \bigg\}^{\frac{1}{r}} \quad \text{for all } t \in [\tau, T_{max}-\tau). \end{aligned} \tag{6.4}$$

Here letting $\kappa := 3(p-\alpha)$ we trivially have $\kappa < 3p$, while (6.1) asserts that $\kappa > 1$. Furthermore, the hypothesis (6.2) guarantees that if we define $\lambda := \frac{(p-\alpha)r}{r-1}$, then

$$\frac{3(\kappa-1)}{(3p-1)\kappa} \cdot \lambda = \frac{3(p-\alpha)-1}{(3p-1)\cdot(1-\frac{1}{r})} < \frac{3(p-\alpha)-1}{(3p-1)\cdot(1-\frac{3\alpha}{3p-1})} = 1,$$

whence invoking Lemma 3.1 we can fix $C_1>0$ such that for all $T \in (2\tau, T_{max})$,

$$\begin{aligned} \bigg\{ \int_t^{t+\tau} \|n(\cdot,s)\|_{L^{3(p-\alpha)}(\Omega)}^{\frac{(p-\alpha)r}{r-1}} ds \bigg\}^{1-\frac{1}{r}} &\le C_1 + C_1 I_p^{\frac{3(p-\alpha)-1}{(3p-1)\cdot(1-\frac{1}{r})}\cdot(1-\frac{1}{r})}(T) \\ &= C_1 + C_1 I_p^{\frac{3(p-\alpha)-1}{3p-1}}(T) \qquad \text{for all } t \in [\tau, T-\tau]. \end{aligned} \tag{6.5}$$

Next, relying on (4.1), (5.5) and (5.11) we employ Lemma 5.3 to find $C_2>0$ with the property that for any such T,

$$\bigg\{ \int_t^{t+\tau} \|\Delta c(\cdot,s)\|_{L^{\frac{3}{2}}(\Omega)}^r ds \bigg\}^{\frac{1}{r}} \le C_2 + C_2 I_p^{\frac{1}{3p-1}}(T) + C_2 I_p^{\frac{q+2}{(3p-1)q}}(T) \qquad \text{for all } t \in [\tau, T-\tau].$$

In conjunction with (6.4) and (6.5), on three straightforward applications of Young's inequality this shows that

$$\begin{aligned} &\int_t^{t+\tau} \|n(\cdot,s)\|_{L^{3(p-\alpha)}(\Omega)}^{p-\alpha} \|\Delta c(\cdot,s)\|_{L^{\frac{3}{2}}(\Omega)} ds \\ &\le C_1C_2 \cdot \Big\{ 1 + I_p^{\frac{3(p-\alpha)-1}{3p-1}}(T) \Big\} \cdot \Big\{ 1 + I_p^{\frac{1}{3p-1}}(T) + I_p^{\frac{q+2}{(3p-1)q}}(T) \Big\} \\ &= C_1C_2 \cdot \Big\{ 1 + I_p^{\frac{1}{3p-1}}(T) + I_p^{\frac{q+2}{(3p-1)q}}(T) + I_p^{\frac{3(p-\alpha)-1}{3p-1}}(T) + I_p^{\frac{3(p-\alpha)}{3p-1}}(T) + I_p^{\frac{3(p-\alpha)q+2}{(3p-1)q}}(T) \Big\} \\ &\le C_1C_2 \cdot \Big\{ 4 + 3 I_p^{\frac{3(p-\alpha)}{3p-1}}(T) + 2 I_p^{\frac{3(p-\alpha)q+2}{(3p-1)q}}(T) \Big\} \end{aligned}$$

for all $t \in [\tau, T-\tau]$. $\square$

We are now prepared for closing our circle of arguments by an application of Lemma 2.4 to an ODI obtained on the basis of the announced L^p testing when combined with Lemma 6.1, provided that α satisfies the assumption from Theorem 1.1 and the exponent q originating from Lemma 4.1 is thereafter fixed appropriately large.

Lemma 6.2 *Suppose that $\alpha > \frac{1}{3}$, and let $p > 1, q > \frac{3}{2}$ and $r > 1$ be such that (4.1), (5.5), (5.11), (6.1) and (6.2) hold, and such that moreover*

$$q > \frac{2}{3\alpha - 1}. \tag{6.6}$$

Then there exists $C > 0$ such that

$$\int_\Omega n^p(\cdot, t) \le C \qquad \text{for all } t \in [0, T_{max}). \tag{6.7}$$

PROOF. We multiply the first equation in (1.6) by n^{p-1} to find using several integrations by parts that

$$\begin{aligned} \frac{d}{dt}\int_\Omega n^p + p(p-1)\int_\Omega n^{p-2}|\nabla n|^2 &= p(p-1)\int_\Omega n^{p-1}S(n)\nabla n \cdot \nabla c \\ &= p(p-1)\int_\Omega \nabla\Psi(n)\cdot\nabla c \\ &= -p(p-1)\int_\Omega \Psi(n)\Delta c \qquad \text{for all } t \in (0, T_{max}), \end{aligned} \tag{6.8}$$

where we have set

$$\Psi(\xi) := \int_0^\xi \sigma^{p-1}S(\sigma)d\sigma \qquad \text{for } \xi \ge 0.$$

Here thanks to (1.8), we can estimate

$$\begin{aligned} |\Psi(\xi)| &\le K_S \int_0^\xi \sigma^{p-1}(\sigma+1)^{-\alpha}d\sigma \\ &\le K_S \int_0^\xi \sigma^{p-1-\alpha}d\sigma \\ &= \frac{K_S}{p-\alpha}\xi^{p-\alpha} \qquad \text{for all } \xi \ge 0, \end{aligned}$$

so that by means of the Hölder inequality, on the right-hand side of (6.8) we obtain

$$\begin{aligned} -p(p-1)\int_\Omega \Psi(n)\Delta c &\le C_1 \int_\Omega n^{p-\alpha}|\Delta c| \\ &\le C_1 \|n\|^{p-\alpha}_{L^{3(p-\alpha)}(\Omega)} \|\Delta c\|_{L^{\frac{3}{2}}(\Omega)} \qquad \text{for all } t \in (0, T_{max}) \end{aligned}$$

with $C_1 := \frac{p(p-1)K_S}{p-\alpha}$. Apart from that, using the Gagliardo-Nirenberg inequality together with (2.3) we see that with some $C_2 > 0$ and $C_3 > 0$ we have

$$\left\{ \int_\Omega n^p \right\}^{\frac{3p-1}{3(p-1)}} = \|n^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{2(3p-1)}{3(p-1)}} \le C_2 \|\nabla n^{\frac{p}{2}}\|_{L^2(\Omega)}^2 \|n^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{4}{3(p-1)}} + C_2 \|n^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(3p-1)}{3(p-1)}} \le C_3 \|\nabla n^{\frac{p}{2}}\|_{L^2(\Omega)}^2 + C_3$$

for all $t \in (0, T_{max})$, and that abbreviating $C_4 := \frac{2(p-1)}{p}$ we thus can estimate

$$p(p-1) \int_\Omega n^{p-2} |\nabla n|^2 = 2C_4 \int_\Omega |\nabla n^{\frac{p}{2}}|^2 \ge \frac{C_4}{C_3} \cdot \left\{ \int_\Omega n^p \right\}^{\frac{3p-1}{3(p-1)}} - \frac{C_4}{C_3} + C_4 \int_\Omega |\nabla n^{\frac{p}{2}}|^2$$

for all $t \in (0, T_{max})$. From (6.8) we thus infer that

$$y(t) := \int_\Omega n^p(\cdot, t), \quad g(t) := C_4 \int_\Omega |\nabla n^{\frac{p}{2}}(\cdot, t)|^2 \quad \text{and}$$
$$h(t) := \frac{C_4}{C_3} + C_1 \|n(\cdot, t)\|_{L^{3(p-\alpha)}(\Omega)}^{p-\alpha} \|\Delta c(\cdot, t)\|_{L^{\frac{3}{2}}(\Omega)}, \qquad t \in [\tau, T_{max}),$$

satisfy

$$y'(t) + \frac{C_4}{C_3} y^{\frac{3p-1}{3(p-1)}}(t) + g(t) \le h(t) \qquad \text{for all } t \in (\tau, T_{max}),$$

where due to (4.1), (5.5), (5.11), (6.1) and (6.2) we may invoke Lemma 6.1 to find $C_5 > 0$ such that for all $T \in (2\tau, T_{max})$ we have

$$\int_t^{t+\tau} h(s) ds \le C_5 + C_5 I_p^{\frac{3(p-\alpha)}{3p-1}}(T) + C_5 I_p^{\frac{3(p-\alpha)q+2}{(3p-1)q}}(T) \qquad \text{for all } t \in [\tau, T-\tau]. \tag{6.9}$$

Therefore, Lemma 2.4 firstly states that if we let

$$C_6 := \max \left\{ \int_\Omega n^p(\cdot, \tau), \left[\frac{2}{3(p-1)} \cdot \frac{C_4}{C_3} \cdot \tau \right]^{-\frac{3(p-1)}{2}} \right\},$$

then

$$\int_t^{t+\tau} g(s) ds \le 2C_5 + 2C_5 I_p^{\frac{3(p-\alpha)}{3p-1}}(T) + 2C_5 I_p^{\frac{3(p-\alpha)q+2}{(3p-1)q}}(T) + C_6 \qquad \text{for all } t \in [\tau, T-\tau]$$

and hence, by definition of $I_p(T)$,

$$I_p(T) \le C_7 + C_7 I_p^{\frac{3(p-\alpha)}{3p-1}}(T) + C_7 I_p^{\frac{3(p-\alpha)q+2}{(3p-1)q}}(T) \qquad \text{for all } T \in (2\tau, T_{max} - \tau) \tag{6.10}$$

with $C_7 := \frac{2C_5 + C_6}{C_4}$. We can now rely on our assumptions that $\alpha > \frac{1}{3}$ and that (6.6) holds, which namely ensure that

$$\theta_1 := \frac{3(p-\alpha)}{3p-1} < \frac{3(p-\frac{1}{3})}{3p-1} = 1$$

and

$$\theta_2 := \frac{3(p-\alpha)q+2}{(3p-1)q} = \frac{3(p-\alpha)+\frac{2}{q}}{3p-1} < \frac{3(p-\alpha)+(3\alpha-1)}{3p-1} = 1,$$

respectively. Therefore, writing $\theta := \max\{\theta_1, \theta_1\} \in (0,1)$ and noting that

$$I_p(T) \le 2C_7 + 2C_7 I_p^\theta(T) \qquad \text{for all } T \in (2\tau, T_{max} - \tau)$$

by (6.10) and Young's inequality, we conclude by an elementary argument that

$$I_p(T) \le C_8 := \max\left\{1\,,\,(4C_7)^{\frac{1}{1-\theta}}\right\} \qquad \text{for all } T \in (2\tau, T_{max} - \tau).$$

In view of (6.9), this in turn implies that

$$\int_t^{t+\tau} h(s)ds \le C_9 := C_5 + C_5 C_8^{\theta_1} + C_5 C_8^{\theta_2} \qquad \text{for all } t \in [\tau, T_{max} - \tau),$$

whereupon Lemma 2.4 secondly guarantees that

$$y(t) \le C_9 + C_6 \qquad \text{for all } t \in [\tau, T_{max})$$

and thereby entails (6.7), again because n is bounded in $\Omega \times [0, \tau)$ by Lemma 2.1. $\square$

It remains to make sure that the above requirements on the auxiliary parameters q and r can indeed be fulfilled for arbitrarily large p to end up with the following.

Lemma 6.3 *Suppose that $\alpha > \frac{1}{3}$. Then given any $p > 1$, one can find $C > 0$ such that*

$$\int_\Omega n^p(\cdot, t) \le C \qquad \textit{for all } t \in [0, T_{max}). \tag{6.11}$$

Proof. As Ω is bounded, without loss of generality we may assume that p additionally satisfies

$$p > \max\left\{\frac{2}{3(3\alpha-1)}\,,\,\frac{1}{3\alpha}\,,\,\alpha + \frac{1}{3}\right\}. \tag{6.12}$$

We can then firstly pick $q > \frac{3}{2}$ such that

$$q < 3p \tag{6.13}$$

and

$$q > \frac{2}{3\alpha - 1} \tag{6.14}$$

as well as

$$q > \frac{1}{\alpha}, \tag{6.15}$$

where the latter ensures that

$$\frac{(3p-1)q}{3} > \frac{3p-1}{3\alpha}.$$

Since furthermore our hypotheses that $p > 1$ and $q > \frac{3}{2}$ warrant that

$$\frac{(3p-1)q}{3} > \frac{(3\cdot 1 - 1)\cdot \frac{3}{2}}{3} = 1$$

and that clearly also $3p - 1 > 1$, it is thereafter possible to choose $r > 1$ in such a way that

$$\frac{3p-1}{3\alpha} < r < \min\Big\{\frac{(3p-1)q}{3}\, ,\, 3p-1\Big\}. \tag{6.16}$$

Now from (6.13) and the third restriction in (6.12) it follows that (4.1) and (6.1) hold, whereas (6.16) guarantees validity of (5.5), (5.11) and (6.2). As moreover (6.6) is satisfied thanks to (6.14), Lemma 6.2 becomes applicable so as to assert the claimed boundedness property. $\Box$

7 Further regularity properties. Proof of Theorem 1.1

Higher integrability properties can now be derived by applying arguments which are essentially standard in the analysis of the heat and the Stokes equations. Firstly, the uniform boundedness of n with respect to the norm in $L^2(\Omega)$, together with our overall assumption that f be bounded, entails the following.

Lemma 7.1 *Let $\alpha > \frac{1}{3}$. Then there exists $C > 0$ such that*

$$\|A^\beta u(\cdot,t)\|_{L^2(\Omega)} \le C \qquad \textit{for all } t \in [0, T_{max}) \tag{7.1}$$

and

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t \in [0, T_{max}). \tag{7.2}$$

Proof. On the basis of a Duhamel formula associated with the Stokes subsystem of (1.6), by means of well-known smoothing properties of the Stokes semigroup ([24]) we see that with some $\lambda_1 > 0$ and $C_1 > 0$ we have

$$\begin{aligned}
\|A^\beta u(\cdot,t)\|_{L^2(\Omega)} &= \bigg\| e^{-tA}A^\beta u_0 + \int_0^t A^\beta e^{-(t-s)A}\mathcal{P}\Big[n(\cdot,s)\nabla\phi + f(\cdot,s)\Big]ds \bigg\|_{L^2(\Omega)} \\
&\le \|A^\beta u_0\|_{L^2(\Omega)} + C_1\int_0^t (t-s)^{-\beta}e^{-\lambda_1(t-s)}\Big\|\mathcal{P}\Big[n(\cdot,s)\nabla\phi + f(\cdot,s)\Big]\Big\|_{L^2(\Omega)}ds
\end{aligned}$$

for all $t \in [0, T_{max})$, because $\mathcal{P}$ acts as an orthogonal projection on $L^2(\Omega;\mathbb{R}^3)$ ([24]). Since Lemma 6.3 together with the boundedness of $\nabla\phi$ and f entails the existence of $C_2 > 0$ such that $\|n(\cdot,s)\nabla\phi + f(\cdot,s)\|_{L^2(\Omega)} \le C_2$ for all $s \in [0, T_{max})$, and since $C_3 := \int_0^\infty \sigma^{-\beta}e^{-\lambda_1\sigma}d\sigma$ is finite due to the fact that $\beta < 1$, this implies that

$$\|A^\beta u(\cdot,t)\|_{L^2(\Omega)} \le \|A^\beta u_0\|_{L^2(\Omega)} + C_1C_2C_3\|\nabla\phi\|_{L^\infty(\Omega)} \qquad \text{for all } t \in [0, T_{max})$$

and hence proves (7.1), for $u_0 \in D(A^\beta)$ by (1.9). As our assumption $\beta > \frac{3}{4}$ warrants that $D(A^\beta) \hookrightarrow L^\infty(\Omega;\mathbb{R}^3)$ ([11], [14]), this also entails (7.2). $\Box$

In conjunction again with Lemma 6.3, the latter entails a bound for c in the flavor needed for an application of Lemma 2.1 for the derivation of Theorem 1.1.

Lemma 7.2 *If $\alpha > \frac{1}{3}$, then for all $p > 1$ there exists $C > 0$ such that*

$$\|c(\cdot,t)\|_{W^{1,p}(\Omega)} \le C \qquad \textit{for all } t \in [0, T_{max}). \tag{7.3}$$

PROOF. We let B denote the realization of $-\Delta + \frac{1}{2}$ under homogeneous Neumann boundary conditions in $L^p(\Omega)$ and then obtain that B is sectorial with its spectrum contained in $[\frac{1}{2}, \infty)$, and that for arbitrary $\gamma \in (\frac{1}{2}, 1)$ the corresponding fractional power B^γ has its domain satisfy $D(B^\gamma) \hookrightarrow W^{1,p}(\Omega)$ ([14]), so that

$$\|\varphi\|_{W^{1,p}(\Omega)} \le C_1(\gamma)\|B^\gamma \varphi\|_{L^p(\Omega)} \qquad \text{for all } \varphi \in D(B^\gamma) \tag{7.4}$$

with some $C_1(\gamma) > 0$. Hencoforth fixing any $\gamma \in (\frac{1}{2}, 1)$ and $\gamma' \in (\frac{1}{2}, \gamma)$, by a well-known interpolation property ([9]) we can find $C_2 > 0$ such that

$$\|B^{\gamma'}\varphi\|_{L^p(\Omega)} \le C_2 \|B^\gamma \varphi\|^a_{L^p(\Omega)} \|\varphi\|^{1-a}_{L^p(\Omega)} \qquad \text{for all } \varphi \in D(B^\gamma) \tag{7.5}$$

with $a := \frac{\gamma'}{\gamma} \in (0,1)$, where according to the Gagliardo-Nirenberg inequality there exists $C_3 > 0$ fulfilling

$$\|\varphi\|^{1-a}_{L^p(\Omega)} \le C_3 \|\varphi\|^{(1-a)b}_{W^{1,p}(\Omega)} \|\varphi\|^{(1-a)(1-b)}_{L^1(\Omega)} \qquad \text{for all } \varphi \in W^{1,p}(\Omega)$$

with $b := \frac{3(p-1)}{4p-3} \in (0,1)$. Together with (7.4) and (7.5), this shows that if we let $d := a + (1-a)b \in (0,1)$ and $C_4 := C_1^{(1-a)b}(\gamma) C_2 C_3$, then

$$\|B^{\gamma'}\varphi\|_{L^p(\Omega)} \le C_4 \|B^\gamma \varphi\|^d_{L^p(\Omega)} \|\varphi\|^{1-d}_{L^1(\Omega)} \qquad \text{for all } \varphi \in D(B^\gamma). \tag{7.6}$$

Now since $c_t = -(B + \frac{1}{2})c + n - u \cdot \nabla c$ in $\Omega \times (0, T_{max})$ by (1.6), an associated variation-of-constants representation together with known regularization features of the corresponding analytic semigroup $(e^{-tB})_{t \ge 0}$ shows that there exists $C_5 > 0$ such that

$$\begin{aligned}
\|B^\gamma c(\cdot,t)\|_{L^p(\Omega)} &= \Bigg\| B^\gamma e^{-(t-\tau)(B+\frac{1}{2})} c(\cdot,\tau) + \int_\tau^t B^\gamma e^{-(t-s)(B+\frac{1}{2})} n(\cdot,s)\,ds \\
&\qquad - \int_\tau^t B^\gamma e^{-(t-s)(B+\frac{1}{2})} u(\cdot,s) \cdot \nabla c(\cdot,s)\,ds \Bigg\|_{L^p(\Omega)} \\
&\le C_5 \|B^\gamma c(\cdot,\tau)\|_{L^p(\Omega)} + C_5 \int_\tau^t (t-s)^{-\gamma} e^{-\frac{1}{2}(t-s)} \|n(\cdot,s)\|_{L^p(\Omega)}\,ds \\
&\qquad + C_5 \int_\tau^t (t-s)^{-\gamma} e^{-\frac{1}{2}(t-s)} \Big\| u(\cdot,s) \cdot \nabla c(\cdot,s) \Big\|_{L^p(\Omega)} ds \qquad \text{for all } t \in [\tau, T_{max}).
\end{aligned} \tag{7.7}$$

Here by Lemma 6.3 we can find $C_6 > 0$ such that

$$\|n(\cdot,s)\|_{L^p(\Omega)} \le C_6 \qquad \text{for all } s \in [\tau, T_{max}), \tag{7.8}$$

while Lemma 7.1 together with (7.4), (7.6) and (2.4) shows that with some $C_7 > 0$ we have

$$\Big\| u(\cdot,s) \cdot \nabla c(\cdot,s) \Big\|_{L^p(\Omega)} \le \|u(\cdot,s)\|_{L^\infty(\Omega)} \|\nabla c(\cdot,s)\|_{L^p(\Omega)}$$

$$
\begin{aligned}
&\le C_7\|\nabla c(\cdot,s)\|_{L^p(\Omega)} \\
&\le C_1(\gamma')C_7\|B^{\gamma'}c(\cdot,s)\|_{L^p(\Omega)} \\
&\le C_1(\gamma')C_4C_7\|B^{\gamma}c(\cdot,s)\|^d_{L^p(\Omega)}\|c(\cdot,s)\|^{1-d}_{L^1(\Omega)} \\
&\le C_8\|B^{\gamma}c(\cdot,s)\|^d_{L^p(\Omega)} \qquad \text{for all } s\in[\tau,T_{max})
\end{aligned}
$$

with $C_8 := C_1(\gamma')C_2C_5\cdot\max\Big\{\int_\Omega c_0,\int_\Omega n_0\Big\}$. Combining this with (7.8) and (7.7) and abbreviating $C_9 := \int_0^\infty \sigma^{-\gamma}e^{-\frac{\sigma}{2}}d\sigma<\infty$ as well as $M(T):=\sup_{t\in[\tau,T]}\|B^\gamma c(\cdot,t)\|_{L^p(\Omega)}$ for $T\in(\tau,T_{max})$, we obtain

$$
\|B^\gamma c(\cdot,t)\|_{L^p(\Omega)} \le C_5\|B^\gamma c(\cdot,\tau)\|_{L^p(\Omega)} + C_5C_6C_9 + C_5C_8C_9M^d(T) \qquad \text{for all } t\in[\tau,T]
$$

and hence

$$
M(T)\le C_{10}+C_{10}M^d(T) \qquad \text{for all } T\in[\tau,T_{max})
$$

with $C_{10} := \max\{C_5\|B^\gamma c(\cdot,\tau)\|_{L^p(\Omega)}+C_5C_6C_9, C_5C_8C_9\}$. As $d<1$, this entails that $M(T)\le \max\{1,(2C_{10})^{\frac{1}{1-d}}\}$ for all $T\in[\tau,T_{max})$, which in light of (7.4) establishes (7.3) due to the inclusion $c\in L^\infty((0,\tau);W^{1,p}(\Omega))$ asserted by Lemma 2.1. □

Finally, pointwise boundednes of n results from a standard argument contained in the literature.

Lemma 7.3 *Let $\alpha>\frac{1}{3}$. Then with some $C>0$ we have*

$$
\|n(\cdot,t)\|_{L^\infty(\Omega)}\le C \qquad \textit{for all } t\in[0,T_{max}). \tag{7.9}
$$

Proof. We write the first equation in (1.6) in the form $n_t=\Delta n+\nabla\cdot h(x,t)$ with $h:=-nS(n)\nabla c-nu$ and then obtain from (1.6) that $h\cdot\nu=0$ on $\partial\Omega\times(0,T_{max})$, whereas (1.8), Lemma 6.3, Lemma 7.2 and Lemma 7.1 entail that $h\in L^\infty((0,T_{max});L^p(\Omega;\mathbb{R}^3))$ for each $p>1$. Since moreover $n\in L^\infty((0,T_{max});L^p(\Omega))$ for any such p, (7.9) can e.g. be derived by a Moser-type iterative argument; for a statement precisely covering the present situation we refer to [25, Lemma A.1]. □

Thanks to the extensibility criterion (2.2), the derivation of our main results thus consists in a mere collection of the latter three lemmata.

Proof of Theorem 1.1. In view of Lemma 2.1, the boundedness properties obtained Lemma 7.3, Lemma 7.2 and Lemma 7.1 assert both global extensibility and the claimed regularity features of the local-in-time solution from Lemma 2.1, as well as the temporally uniform estimate in (1.12). □

Acknowledgement. The author acknowledges support of the *Deutsche Forschungsgemeinschaft* in the context of the project *Analysis of chemotactic cross-diffusion in complex frameworks*, and he is grateful to Yulan Wang for numerous helpful remarks on this manuscript.

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