Global classical solvability and stabilization in a two-dimensional chemotaxis-Navier-Stokes system modeling coral fertilization

Elio Espejo*
Faculty of Mathematics, Kyushu University,
Fukuoka 819-0395, Japan

Michael Winkler#
Institut für Mathematik, Universität Paderborn,
33098 Paderborn, Germany

Abstract

The interplay of chemotaxis, convection and reaction terms is studied in the particular framework of a refined model for coral broadcast spawning, consisting of three equations describing the population densities of unfertilized sperms and eggs and the concentration of a chemical released by the latter, coupled to the incompressible Navier-Stokes equations.

Under mild assumptions on the initial data, global existence of classical solutions to an associated initial-boundary value problem in bounded planar domains is established. Moreover, all these solutions are shown to approach a spatially homogeneous equilibrium in the large time limit.

Key words: chemotaxis; Navier-Stokes; global existence; boundedness, asymptotic behavior
MSC 2010: 35B65, 35B40 (primary); 35K55, 92C17, 35Q30, 35Q92 (secondary)

*espejo@math.kyushu-u.ac.jp
#michael.winkler@math.uni-paderborn.de

1 Introduction

Chemotactic contributions to cell migration may have substantial implications on the collective behavior in cell populations. Especially in situations when the respective signal is actively produced by the participating cells, even drastic effects such as spontaneous aggregation in small spatial regions are known to occur in numerous experimental settings (cf. the survey [14]), and various types of mathematical results on unboundedness phenomena confirm the appropriateness of Keller-Segel-type systems for the qualitatively correct modeling of such processes ([13], [22], [37], [39]). Beyond this, also in some contexts of significantly weaker mechanisms of self-enhancement, such as e.g. in cases when the signal is either produced through more indirect processes ([25]) or even consumed by the cells ([7], [30]), experiments indicate a relevant influence of chemotaxis, but apparently only few rigorous mathematical results on nontrivial behavior in presence of such types of interplay are available (see [29] and also the survey [1]).

An important effect of chemotaxis on fertilization processes, exemplified by focusing on coral fertilization, has been studied in [15] and [16] with regard to the particular question to which extent chemotactically directed motion of spermatozoids toward eggs may benefit the overall collective behavior. In the framework of a two-component model of the form

$$\begin{cases} \rho_t + u \cdot \nabla\rho = \Delta\rho - \chi\nabla\cdot(\rho\nabla c) - \rho^q, \\ 0 = \Delta c + \rho, \end{cases} \tag{1.1}$$

for the unknown concentration c of a signal released by the eggs and the density ρ of the population of both sperms and eggs, with a given smooth solenoidal fluid vector field u, it has inter alia been shown there that in an associated Cauchy problem in the whole plane $\mathbb{R}^2$, in the case $q > 2$ of supercritical reaction the large time limit of the total population size $\int_{\mathbb{R}^2} \rho$ indeed becomes arbitrarily small with increasing χ, thus indicating enhancement of fertilization in presence of large taxis terms ([15]), whereas in the critical case $q = 2$ a corresponding weaker but yet relevant effect within finite time intervals is detected ([16]).

The above approach implicitly relies on the simplifying hypothesis that unlike the cells which are transported through the fluid, the chemical remains unaffected by the fluid motion. A refinement in this direction has been achieved in [5], where in the framework of the chemotaxis-(Navier-)Stokes model

$$\begin{cases} \rho_t + u \cdot \nabla\rho = \Delta\rho - \chi\nabla\cdot(\rho\nabla c) - \mu\rho^2, \\ c_t + u\cdot\nabla c = \Delta c - c + \rho, \\ u_t + \kappa(u\cdot\nabla)u = \Delta u - \nabla P + \rho\nabla\phi, \qquad \nabla\cdot u = 0, \end{cases} \tag{1.2}$$

also buoyancy effects of the cells on the fluid velocity u, mediated through a given gravitational potential ϕ, are accounted for. Here in comparison to (1.1), treating the fluid field as a further unknown gives rise to evident additional mathematical challenges already at the level of basic questions from existence and regularity theory, but at least some numerical evidence reports significant effects of such fluid interaction on the possibile occurrence of explosions in the case $\mu = 0$ ([21]; cf. also [6]). Precedents in the analytical literature on chemotaxis-fluid systems have essentially concentrated on systems involving signal consumption mechanisms. Indeed, such supplementary dissipative features, mathematically reflected in replacing the second identity in (1.2) by equations of the form $c_t+u\cdot\nabla c = \Delta c - nc$, have been shown to allow for the construction of global solutions to associated initial value problems in N-dimensional physical domains even when $\mu = 0$ and the full Navier-Stokes case $\kappa = 1$ is considered, in the framework of smooth solutions when either $N = 2$ ([36]) or $N = 3$ and the initial data are suitably small ([9], [17], [2]), or in classes of weak solutions in widely arbitrary three-dimensional situations ([40]); some related results and partial extensions to models involving nonlinear diffusion can be found in [20], [9], [19], [4] and [41], for instance. In presence of absorptive effects of this type, even quite comprehensive results on the large time behavior of solutions are available, in particular including statements on stabilization of each individual solution toward spatially homogeneous steady states in the large time limit ([38], [43], [42]; cf. also [3] for some decay results addressing Cauchy problems in $\mathbb{R}^2$ and $\mathbb{R}^3$, and [18] for asymptotics in a system involving logistic cell kinetics).

In the full system (1.2) involving signal production through cells and thereby retaining fundamental properties of the original Keller-Segel chemotaxis system, in light of known blow-up results for the latter it seems necessary to rely on the alternative quadratic growth-inhibiting mechanism included when explicitly requiring that the parameter μ therein be positive, which in the respective fluid-free chemotaxis system indeed suppresses explosions in two-dimensional cases ([23]) and also in higher-dimensional situations when in addition μ is large enough ([34]). In a first analytical approach, this assumption could be identified as sufficient to enforce global existence of weak solutions actually also in corresponding planar Cauchy problems for (1.2) when $\kappa = 0$ ([5]). Recent works have shown that in fact global bounded classical solutions can be constructed in this two-dimensional situation, even in the case $\kappa = 1$, and that moreover all these solutions asymptotically decay toward the trivial equilibrium of (1.2) ([28]); a similar conclusion is available for the three-dimensional version of (1.2) when $\kappa = 0$ and $\mu > 0$ is suitably large ([27]). In absence of such logistic dampening mechanisms, global bounded solutions emanating from large initial data have only been found in modified variants of (1.2) involving certain saturation effects in the chemotactic sensitivity at large densities ([32], [33], [31]).

Main results. It is the objective of the present work to analyze a further refinement of the model (1.2) which explicitly distinguishes between sperms and eggs, thus unlike (1.1) and (1.2) not presupposing the density of both subpopulations to precisely coincide at each point. In particular, it thereby becomes possible to account for the plausible hypothesis that only the dynamics of spermatozoids will be affected by chemotaxis, whereas the evolution of the egg population is merely determined by diffusion, fluid transport and degradation upon contact with sperms. Hence splitting $\rho = n + m$ with n and m denoting the population densities of unfertilized sperms and eggs, respectively, in a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary we will subsequently consider the four-component chemotaxis-Navier-Stokes system

$$\begin{cases} n_t + u\cdot\nabla n &= \Delta n - \nabla\cdot(n\nabla c) - nm, & & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c &= \Delta c - c + m, & & x\in\Omega,\ t>0,\\ m_t + u\cdot\nabla m &= \Delta m - nm, & & x\in\Omega,\ t>0,\\ u_t + (u\cdot\nabla)u &= \Delta u - \nabla P + (n+m)\nabla\phi, & \nabla\cdot u = 0, & x\in\Omega,\ t>0. \end{cases} \tag{1.3}$$

In comparison to (1.2), it is still assumed here that the signal is produced by cells, whereas the dampening effect of the genuinely quadratic death term in (1.2) has been replaced by the apparently weaker absorptive summand $-nm$ in the first equation of (1.3), reflecting a respective dependence of sperm population degradation on encounters with unfertilized eggs. Our goal will be to make sure that despite this, in the considered two-dimensional context the regularizing action of the diffusion processes in (1.3), in interaction with the additional absorptive mechanism expressed in the third equation in (1.3), is strong enough to overbalance the destabilizing effects of both chemotacitc cross-diffusion and also of convection. In order to complete the description of the specific setting within which this problem will be addressed, let us consider (1.3) under the boundary conditions

$$\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = \frac{\partial m}{\partial \nu} = 0 \quad \text{and} \quad u = 0, \qquad x \in \partial\Omega,\ t > 0, \tag{1.4}$$

and the initial conditions

$$n(x,0) = n_0(x), \quad c(x,0) = c_0(x), \quad m(x,0) = m_0(x) \quad \text{and} \quad u(x,0) = u_0(x), \qquad x \in \Omega. \tag{1.5}$$

Throughout the sequel, we shall assume that herein $\phi \in W^{2,\infty}(\Omega)$, and that

$$\begin{cases} n_0 \in C^0(\bar\Omega) \quad \text{is nonnegative with } n_0 \not\equiv 0, \\ c_0 \in W^{1,\infty}(\Omega) \quad \text{is nonnegative,} \\ m_0 \in C^0(\bar\Omega) \quad \text{is nonnegative, and} \\ u_0 \in D(A^\alpha) \quad \text{with some } \alpha \in (\frac{1}{2}, 1), \end{cases} \tag{1.6}$$

where $A := -\mathcal{P}\Delta$ denotes the realization of the Stokes operator in $L^2(\Omega; R^2)$, defined on its domain $D(A) := W^{2,2}(\Omega;\mathbb{R}^2) \cap W_0^{1,2}(\Omega;\mathbb{R}^2) \cap L^2_\sigma(\Omega;\mathbb{R}^2)$ with $L^2_\sigma(\Omega;\mathbb{R}^2) := \{\varphi \in L^2(\Omega;\mathbb{R}^2) \mid \nabla \cdot u = 0\}$, and with $\mathcal{P}$ representing the Helmholtz projection of $L^2(\Omega;\mathbb{R}^2)$ onto $L^2_\sigma(\Omega;\mathbb{R}^2)$.

In this framework, the considered initial-boundary value problem indeed admits globally defined classical solutions:

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and let $\phi \in W^{1,\infty}(\Omega)$. Then for all (n_0, c_0, m_0, u_0) satisfying (1.6), the problem (1.3)-(1.5) possesses a globally defined classical solution (n, c, m, u, P) which for each $p > 2$ is uniquely determined by the requirements*

$$\begin{cases} n \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)), \\ c \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)) \cap L^\infty((0,\infty); W^{1,p}(\Omega)), \\ m \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)), \\ u \in C^0(\bar\Omega \times [0,\infty);\mathbb{R}^2) \cap C^{2,1}(\bar\Omega \times (0,\infty);\mathbb{R}^2) \cap L^\infty((0,\infty); D(A^\alpha)), \\ P \in C^{1,0}(\bar\Omega \times (0,\infty)), \end{cases} \tag{1.7}$$

and for which n, c and m are nonnegative in $\Omega \times (0,\infty)$. Moreover, this solution is bounded in the sense that given any $p > 2$ one can find $C = C(p) > 0$ fulfilling

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,p}(\Omega)} + \|m(\cdot,t)\|_{L^\infty(\Omega)} + \|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} \le C \qquad \textit{for all } t > 0. \tag{1.8}$$

Moreover, the large time behavior of all these solutions can be described quite comprehensively as follows.

Theorem 1.2 *Under the assumptions of Theorem 1.1, the global classical solution (n, c, m, u, P) of (1.3)-(1.5) satisfies*

$$n(\cdot,t) \to \left\{ \fint_\Omega n_0 - \fint_\Omega m_0 \right\}_+ \quad \textit{in } L^\infty(\Omega) \qquad \textit{as } t \to \infty \tag{1.9}$$

and

$$c(\cdot,t) \to \Big\{ \fint_\Omega m_0 - \fint_\Omega n_0 \Big\}_+ \quad \textit{in } L^\infty(\Omega) \qquad \textit{as } t\to\infty \tag{1.10}$$

as well as

$$m(\cdot,t) \to \Big\{ \fint_\Omega m_0 - \fint_\Omega n_0 \Big\}_+ \quad \textit{in } L^\infty(\Omega) \qquad \textit{as } t\to\infty \tag{1.11}$$

and

$$u(\cdot,t) \to 0 \quad \textit{in } L^\infty(\Omega;\mathbb{R}^2) \qquad \textit{as } t\to\infty. \tag{1.12}$$

Main ideas. A key role in our existence analysis is played by the observation that for appropriate positive constants a and b, the functional

$$\mathcal{F}(t) := \int_\Omega n(\cdot,t)\ln n(\cdot,t) + a\int_\Omega |\nabla c(\cdot,t)|^2 + b\int_\Omega |u(\cdot,t)|^2$$

possesses a favorable entropy-like property, where remarkably the respective choices of a and b will depend on the particular choice of the initial data (Section 3.1). The basic boundedness properties thereby implied will thereafter be seen in Sections 3.2-3.5 to entail further regularity estimates which will turn out to be sufficient for global extensibility of local-in-time solutions in Section 3.6.

According to additional temporally uniform regularity and compactness properties obtained as further consequences, in Section 4 we shall see turn the basic relaxation properties expressed in the inequalities

$$\int_0^\infty\int_\Omega nm < \infty \qquad \text{and} \qquad \int_0^\infty\int_\Omega |\nabla m|^2 < \infty$$

into the uniform stabilization statements formulated in Theorem 1.2.

2 Local existence and basic properties

The following statement asserting local existence and uniqueness of classical solutions as well as a convenient extensibility can be obtained by a straightforward adaptation of well-established arguments based on the contraction mapping principle (see e.g. [36, Lemma 2.1]).

Lemma 2.1 *Assume that $\Omega\subset\mathbb{R}^2$ is a bounded domain with smooth boundary, that $\phi\in W^{1,\infty}(\Omega)$, and that n_0, c_0, m_0 and u_0 satisfy (1.6). Then there exist $T_{max}\in(0,\infty]$ and a classical solution of (1.3)-(1.5) in $\Omega\times(0,T_{max})$, for any $p>2$ uniquely determined by the inclusions*

$$\begin{cases} n\in C^0(\bar\Omega\times[0,T_{max}))\cap C^{2,1}(\bar\Omega\times(0,T_{max})),\\ c\in C^0(\bar\Omega\times[0,T_{max}))\cap C^{2,1}(\bar\Omega\times(0,T_{max}))\cap L^\infty_{loc}([0,T_{max});W^{1,p}(\Omega)),\\ m\in C^0(\bar\Omega\times[0,T_{max}))\cap C^{2,1}(\bar\Omega\times(0,T_{max})),\\ u\in C^0(\bar\Omega\times[0,T_{max});\mathbb{R}^2)\cap C^{2,1}(\bar\Omega\times(0,T_{max});\mathbb{R}^2)\cap L^\infty_{loc}(([0,T_{max});D(A^\alpha)),\\ P\in C^{1,0}(\bar\Omega\times(0,T_{max})), \end{cases} \tag{2.1}$$

such that n, c and m are nonnegative in $\Omega\times(0,T_{max})$, and such that

$$\textit{if } T_{max}<\infty, \textit{ then } \limsup_{t\nearrow T_{max}} \Big\{ \|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,p}(\Omega)} + \|m(\cdot,t)\|_{L^\infty(\Omega)} + \|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} \Big\} = \infty \tag{2.2}$$

for each $p>2$.

Throughout the sequel, we shall tacitly suppose that the assumptions of Lemma 2.1 are satisfied, and we let (n, c, m, u, P) denote the corresponding local solution of (1.3)-(1.5), according to Lemma 2.1 maximally extended so as to exist on $\Omega \times (0, T_{max})$ with $T_{max} \in (0, \infty]$ fulfilling (2.2).

Some basic but important properties of this solution can directly be obtained by several standard arguments, in summary leading to the following.

Lemma 2.2 *The solution of (1.3)-(1.5) satisfies*

$$\frac{d}{dt}\int_\Omega n(\cdot,t) \le 0 \quad \textit{and} \quad \frac{d}{dt}\int_\Omega m(\cdot,t) \le 0 \qquad \textit{for all } t \in (0, T_{max}) \tag{2.3}$$

as well as

$$\int_\Omega n(\cdot,t) \le \int_\Omega n_0 \quad \textit{and} \quad \int_\Omega m(\cdot,t) \le \int_\Omega m_0 \qquad \textit{for all } t \in (0, T_{max}) \tag{2.4}$$

and

$$\int_\Omega n(\cdot,t) - \int_\Omega m(\cdot,t) = \int_\Omega n_0 - \int_\Omega m_0 \qquad \textit{for all } t \in (0, T_{max}), \tag{2.5}$$

and we have

$$\|m(\cdot,t)\|_{L^\infty(\Omega)} \le \|m_0\|_{L^\infty(\Omega)} \qquad \textit{for all } t \in (0, T_{max}) \tag{2.6}$$

and

$$\|c(\cdot,t)\|_{L^\infty(\Omega)} \le \max\Big\{\|c_0\|_{L^\infty(\Omega)}\,,\, \|m_0\|_{L^\infty(\Omega)}\Big\} \qquad \textit{for all } t \in (0, T_{max}). \tag{2.7}$$

Moreover,

$$\int_0^t\int_\Omega nm \le \min\Big\{\int_\Omega n_0\,,\, \int_\Omega m_0\Big\} \qquad \textit{for all } t \in (0, T_{max}) \tag{2.8}$$

and

$$\int_0^t\int_\Omega |\nabla m|^2 \le \frac{1}{2}\int_\Omega m_0^2 \qquad \textit{for all } t \in (0, T_{max}). \tag{2.9}$$

Proof. On integrating the first and the third equation in (1.3) over Ω and using that $\nabla \cdot u = 0$, we obtain the identities

$$\frac{d}{dt}\int_\Omega n = -\int_\Omega nm \quad \text{and} \quad \frac{d}{dt}\int_\Omega m = -\int_\Omega nm \qquad \text{for all } t \in (0, T_{max}), \tag{2.10}$$

which directly imply both (2.3) and (2.4) as well as (2.5). Furthermore, a time integration of the first identity in (2.10) shows that

$$\int_0^t\int_\Omega nm = \int_\Omega n_0 - \int_\Omega n(\cdot,t) \le \int_\Omega n_0 \qquad \text{for all } t \in (0, T_{max}),$$

which entails (2.8) together with the observation that, similarly, $\int_0^t\int_\Omega nm \le \int_\Omega m_0$ for all $t \in (0, T_{max})$. Next, (2.6) is an immediate consequence of the maximum principle applied to the third equation in (1.3), whereupon it follows that $\hat{c}(x,t) := \max\{\|c_0\|_{L^\infty(\Omega)}\,,\, \|m_0\|_{L^\infty(\Omega)}\}$, $(x,t) \in \bar{\Omega} \times [0,\infty)$, satisfies

$$\hat{c}_t - \Delta\hat{c} + \hat{c} - m(x,t) + u(x,t)\nabla\hat{c} = \hat{c} - m(x,t) \ge 0 \qquad \text{in } \Omega \times (0, T_{max}),$$

so that also (2.7) becomes a consequence of a parabolic comparison argument.
Finally, testing the third equation in (1.3) by m we see that

$$\frac{1}{2}\int_\Omega m^2(\cdot,t) + \int_0^t\int_\Omega |\nabla m|^2 = \frac{1}{2}\int_\Omega m_0^2 \qquad \text{for all } t\in(0,T_{max}),$$

and that hence also (2.9) is valid. $\square$

3 Global existence. Proof of Theorem 1.1

3.1 Construction of an entropy-like functional

In order to achieve a priori information beyond that from Lemma 2.2, in this section we shall concentrate on the detection of further global dissipative properties of (1.3). Indeed, in Lemma 3.4 we shall see that if in dependence on the initial data we choose the constants $a>0$ and $b>0$ appropriately, then

$$\mathcal{F}(t) := \int_\Omega n(\cdot,t)\ln n(\cdot,t) + a\int_\Omega |\nabla c(\cdot,t)|^2 + b\int_\Omega |u(\cdot,t)|^2, \qquad t\in(0,T_{max}), \tag{3.1}$$

acts as a quasi-entropy functional along the trajectory associated with the particular solution under consideration. The regularity properties implied by a corresponding entropy-dissipation inequality will constitute key ingredients for the verification that the solution from Lemma 2.1 in fact exists globally in time.

An important role in our first two steps toward this, to be achieved in Lemma 3.1 and Lemma 3.2, will be played by the interpolation inequality

$$\|\nabla\varphi\|_{L^4(\Omega)} \le C\cdot\Big\{\|\Delta\varphi\|_{L^2(\Omega)}^{\frac{1}{2}}\|\varphi\|_{L^\infty(\Omega)}^{\frac{1}{2}} + \|\varphi\|_{L^\infty(\Omega)}\Big\}, \tag{3.2}$$

which thanks to a combination of the Gagliardo-Nirenberg inequality with standard elliptic regularity theory can readily be seen to hold for some $C>0$ and any $\varphi\in C^2(\bar\Omega)$ satisfying $\frac{\partial\varphi}{\partial\nu}=0$ on $\partial\Omega$.
Indeed, an argument based on (3.2) allows for a favorable control of the cross-diffusive interaction in (1.3) during a standard testing procedure applied to the first equation therein.

Lemma 3.1 *There exists $C>0$ such that*

$$\frac{d}{dt}\int_\Omega n\ln n + \frac{1}{2}\int_\Omega \frac{|\nabla n|^2}{n} \le C\cdot\Big\{\int_\Omega |\Delta c|^2 + 1\Big\} \qquad \textit{for all } t\in(0,T_{max}). \tag{3.3}$$

Proof. Noting that n is strictly positive in $\bar\Omega\times(0,T_{max})$ by the strong maximum principle, we test the first equation in (1.3) by $\ln n$ to compute

$$\frac{d}{dt}\int_\Omega n\ln n + \int_\Omega \frac{|\nabla n|^2}{n} = \int_\Omega \nabla n\cdot\nabla c - \int_\Omega n\ln n\cdot m \qquad \text{for all } t\in(0,T_{max}), \tag{3.4}$$

where since $\xi\ln\xi \ge -\frac{1}{e}$ for all $\xi>0$, by (2.4) we can estimate

$$-\int_\Omega n\ln n\cdot m \le \frac{1}{e}\int_\Omega m \le \frac{1}{e}\int_\Omega m_0 \qquad \text{for all } t\in(0,T_{max}). \tag{3.5}$$

Moreover, due to Young's inequality and the Cauchy-Schwarz inequality we have

$$\begin{aligned}\int_\Omega \nabla n\cdot\nabla c &\le \frac14\int_\Omega\frac{|\nabla n|^2}{n}+\int_\Omega n|\nabla c|^2\\ &\le \frac14\int_\Omega\frac{|\nabla n|^2}{n}+\Big\{\int_\Omega n^2\Big\}^{\frac12}\cdot\Big\{|\nabla c|^4\Big\}^{\frac12}\qquad\text{for all } t\in(0,T_{max}),\end{aligned}\tag{3.6}$$

where an application of the Gagliardo-Nirenberg inequality provides $C_1>0$ such that

$$\begin{aligned}\Big\{\int_\Omega n^2\Big\}^{\frac12} &= \|\sqrt{n}\|^2_{L^4(\Omega)}\\ &\le C_1\|\nabla\sqrt{n}\|_{L^2(\Omega)}\|\sqrt{n}\|_{L^2(\Omega)}+C_1\|\sqrt{n}\|^2_{L^2(\Omega)}\qquad\text{for all } t\in(0,T_{max}).\end{aligned}$$

As $\|\sqrt{n}\|^2_{L^2(\Omega)}=\int_\Omega n\le\int_\Omega n_0$ for all $t\in(0,T_{max})$ by (2.4), this entails the existence of $C_2>0$ such that

$$\Big\{\int_\Omega n^2\Big\}^{\frac12}\le C_2\cdot\Big\{\Big\{\int_\Omega\frac{|\nabla n|^2}{n}\Big\}^{\frac12}+1\Big\}\qquad\text{for all } t\in(0,T_{max}),$$

whence invoking Young's inequality we infer from (3.6) that

$$\begin{aligned}\int_\Omega\nabla n\cdot\nabla c &\le \frac14\int_\Omega\frac{|\nabla n|^2}{n}+C_2\Big\{\Big\{\int_\Omega\frac{|\nabla n|^2}{n}\Big\}^{\frac12}+1\Big\}\cdot\Big\{\int_\Omega|\nabla c|^4\Big\}^{\frac12}\\ &\le \frac14\int_\Omega\frac{|\nabla n|^2}{n}+\frac18\Big\{\Big\{\int_\Omega\frac{|\nabla n|^2}{n}\Big\}^{\frac12}+1\Big\}^2+2C_2^2\int_\Omega|\nabla c|^4\\ &\le \frac14\int_\Omega\frac{|\nabla n|^2}{n}+\frac14\int_\Omega\frac{|\nabla n|^2}{n}+\frac14+2C_2^2\int_\Omega|\nabla c|^4\\ &= \frac12\int_\Omega\frac{|\nabla n|^2}{n}+2C_2^2\int_\Omega|\nabla c|^4+\frac14\qquad\text{for all } t\in(0,T_{max}).\end{aligned}$$

As (3.2) and (2.7) yield $C_3>0$ and $C_4>0$ such that

$$\begin{aligned}2C_2^2\int_\Omega|\nabla c|^4 &\le C_3\|\Delta c\|^2_{L^2(\Omega)}\|c\|^2_{L^\infty(\Omega)}+C_3\|c\|^4_{L^\infty(\Omega)}\\ &\le C_4\int_\Omega|\Delta c|^2+C_4\qquad\text{for all } t\in(0,T_{max}),\end{aligned}$$

combining this with (3.5) and (3.4) readily establishes (3.3). □

In order to estimate the integral on the right of (3.3), we perform another straightforward testing procedure to the second equation in (1.3). As this evidently needs to be concerned with the time evolution of a functional containing spatial derivatives of c, unlike the above situation the result will now involve the fluid velocity, where thanks to (3.2), however, this dependence will take the following favorable form.

Lemma 3.2 *There exists $C > 0$ such that*

$$\frac{d}{dt}\int_\Omega |\nabla c|^2 + \int_\Omega |\Delta c|^2 + 2\int_\Omega |\nabla c|^2 \le C\cdot\Big\{\int_\Omega |\nabla u|^2 + 1\Big\} \qquad \textit{for all } t\in(0,T_{max}). \tag{3.7}$$

Proof. We multiply the second equation in (1.3) by $-\Delta c$ and integrate by parts to see that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c|^2 + \int_\Omega |\Delta c|^2 + \int_\Omega |\nabla c|^2 = -\int_\Omega m\Delta c + \int_\Omega (u\cdot\nabla c)\Delta c \qquad \text{for all } t\in(0,T_{max}), \tag{3.8}$$

where by Young's inequality and (2.6),

$$\begin{aligned} -\int_\Omega m\Delta c &\le \frac{1}{4}\int_\Omega |\Delta c|^2 + \int_\Omega m^2 \\ &\le \frac{1}{4}\int_\Omega |\Delta c|^2 + |\Omega|\cdot\|m_0\|^2_{L^\infty(\Omega)} \qquad \text{for all } t\in(0,T_{max}). \end{aligned} \tag{3.9}$$

In the last summand in (3.8), we once more integrate by parts to find that

$$\int_\Omega (u\cdot\nabla c)\Delta c = -\int_\Omega \nabla c\cdot\nabla(u\cdot\nabla c) = -\int_\Omega \nabla c\cdot(\nabla u\cdot\nabla c) - \int_\Omega \nabla c\cdot(D^2c\cdot u) \qquad \text{for all } t\in(0,T_{max}).$$

Since u is solenoidal and vanishes on $\partial\Omega$, herein we have

$$-\int_\Omega \nabla c\cdot(D^2c\cdot u) = -\frac{1}{2}\int_\Omega u\cdot\nabla|\nabla c|^2 = 0 \qquad \text{for all } t\in(0,T_{max}),$$

so that

$$\begin{aligned} \int_\Omega (u\cdot\nabla c)\Delta c &= \int_\Omega \nabla c\cdot(\nabla u\cdot\nabla c) \\ &\le \Big\{\int_\Omega |\nabla c|^4\Big\}^{\frac{1}{2}}\cdot\Big\{\int_\Omega |\nabla u|^2\Big\}^{\frac{1}{2}} \qquad \text{for all } t\in(0,T_{max}) \end{aligned} \tag{3.10}$$

by the Cauchy-Schwarz inequality. Now thanks to (3.2) we can find $C_1 > 0$ fulfilling

$$\Big\{\int_\Omega |\nabla c|^4\Big\}^{\frac{1}{2}} \le C_1\|\Delta c\|_{L^2(\Omega)}\|c\|_{L^\infty(\Omega)} + C_1\|c\|^4_{L^\infty(\Omega)} \qquad \text{for all } t\in(0,T_{max}),$$

whence in view of (2.7) there exists $C_2 > 0$ such that

$$\Big\{\int_\Omega |\nabla c|^4\Big\}^{\frac{1}{2}} \le C_2\cdot\Big\{\Big\{\int_\Omega |\Delta c|^2\Big\}^{\frac{1}{2}} + 1\Big\} \qquad \text{for all } t\in(0,T_{max}).$$

Again by Young's inequality, we thus infer from (3.10) that

$$\begin{aligned} \int_\Omega (u\cdot\nabla c)\Delta c &\le C_2\cdot\Big\{\Big\{\int_\Omega |\Delta c|^2\Big\}^{\frac{1}{2}} + 1\Big\}\cdot\Big\{\int_\Omega |\nabla u|^2\Big\}^{\frac{1}{2}} \\ &\le \frac{1}{8}\cdot\Big\{\Big\{\int_\Omega |\Delta c|^2\Big\}^{\frac{1}{2}} + 1\Big\}^2 + 2C_2^2\int_\Omega |\nabla u|^2 \\ &\le \frac{1}{4}\int_\Omega |\Delta c|^2 + \frac{1}{4} + 2C_2^2\int_\Omega |\nabla u|^2 \qquad \text{for all } t\in(0,T_{max}), \end{aligned}$$

and hence (3.7) becomes a consequence of (3.8) and (3.9). $\square$

Finally, the Dirichlet integral on the right of (3.7) appears as the dissipation rate in the standard energy inequality associated with the Navier-Stokes subsystem of (1.3). More precisely, in the present context with coupling to the quantity n, we have the following.

Lemma 3.3 *For any $p > 1$ one can find $C(p) > 0$ such that*

$$\frac{d}{dt}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 \le C(p) \cdot \Big\{ \|n\|^2_{L^p(\Omega)} + 1 \Big\} \qquad \textit{for all } t \in (0, T_{max}). \tag{3.11}$$

PROOF. Taking u as a test function in the fourth equation in (1.3) and using the Hölder inequality, writing $C_1 := \|\nabla\phi\|_{L^\infty(\Omega)}$ we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 = \int_\Omega (n+m)u\cdot\nabla\phi \le C_1\|n+m\|_{L^p(\Omega)}\|u\|_{L^{\frac{p}{p-1}}(\Omega)} \qquad \text{for all } t \in (0, T_{max}), \tag{3.12}$$

where since $W_0^{1,2}(\Omega) \hookrightarrow L^{\frac{p}{p-1}}(\Omega)$, according to a corresponding Poincaré-Sobolev inequality we can find $C_2 > 0$ such that

$$\|u\|_{L^{\frac{p}{p-1}}(\Omega)} \le C_2\|\nabla u\|_{L^2(\Omega)} \qquad \text{for all } t \in (0, T_{max}).$$

By Young's inequality, we can thus estimate

$$\begin{aligned} C_1\|n+m\|_{L^p(\Omega)}\|u\|_{L^{\frac{p}{p-1}}(\Omega)} &\le C_1C_2\|n+m\|_{L^p(\Omega)}\|\nabla u\|_{L^2(\Omega)} \\ &\le \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{C_1^2C_2^2}{2}\|n+m\|^2_{L^p(\Omega)} \\ &\le \frac{1}{2}\int_\Omega |\nabla u|^2 + C_1^2C_2^2 \cdot \Big\{ \|n\|^2_{L^p(\Omega)} + \|m\|^2_{L^p(\Omega)} \Big\} \qquad \text{for all } t \in (0, T_{max}), \end{aligned}$$

so that since (2.6) warrants that

$$\|m\|^2_{L^p(\Omega)} \le |\Omega|^{\frac{2}{p}}\|m_0\|^2_{L^\infty(\Omega)} \qquad \text{for all } t \in (0, T_{max}),$$

we conclude from (3.12) that (3.11) holds. $\square$

Now taking an appropriate linear combination of the three inequalities provided by Lemma 3.1, Lemma 3.2 and Lemma 3.3, we can verify that indeed a functional of the form in (3.1) satisfies a favorable ordinary differential inequality. As a particular outcome of the latter, we obtain the following spatial and spatio-temporal regularity properties beyond those from Lemma 2.2.

Lemma 3.4 *There exists $C > 0$ with the property that*

$$\int_\Omega |\nabla c(\cdot,t)|^2 \le C \qquad \textit{for all } t \in (0, T_{max}) \tag{3.13}$$

and

$$\int_\Omega |u(\cdot,t)|^2 \le C \qquad \textit{for all } t \in (0, T_{max}) \tag{3.14}$$

as well as

$$\int_t^{t+\tau}\int_\Omega \frac{|\nabla n|^2}{n} \le C \qquad \textit{for all } t\in(0,T_{max}-\tau) \tag{3.15}$$

and

$$\int_t^{t+\tau}\int_\Omega |\nabla u|^2 \le C \qquad \textit{for all } t\in(0,T_{max}-\tau), \tag{3.16}$$

where $\tau := \min\{1,\frac{1}{2}T_{max}\}$.

PROOF. We choose any $p\in(1,2)$ and apply Lemma 3.1, Lemma 3.2 and Lemma 3.3 to find positive constants C_1, C_2 and C_3 such that

$$\frac{d}{dt}\int_\Omega n\ln n + \frac{1}{2}\int_\Omega \frac{|\nabla n|^2}{n} \le C_1\int_\Omega |\Delta c|^2 + C_1 \qquad \text{for all } t\in(0,T_{max}) \tag{3.17}$$

and

$$\frac{d}{dt}\int_\Omega |\nabla c|^2 + \int_\Omega |\Delta c|^2 + 2\int_\Omega |\nabla c|^2 \le C_2\int_\Omega |\nabla u|^2 + C_2 \qquad \text{for all } t\in(0,T_{max}) \tag{3.18}$$

as well as

$$\frac{d}{dt}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 \le C_3\|n\|^2_{L^p(\Omega)} + C_3 \qquad \text{for all } t\in(0,T_{max}). \tag{3.19}$$

We now fix $a>0$ large enough such that

$$a \ge 2C_1$$

and thereafter pick some large $b>0$ fulfilling

$$b \ge 2C_2 a,$$

and we then combine (3.17)-(3.19) to infer that these choices ensure that

$$\begin{aligned}
\frac{d}{dt}\Big\{\int_\Omega n\ln n + a\int_\Omega |\nabla c|^2 + b\int_\Omega |u|^2\Big\} &+ \int_\Omega n\ln n + 2a\int_\Omega |\nabla c|^2 \\
&+\frac{1}{2}\int_\Omega \frac{|\nabla n|^2}{n} + \frac{a}{2}\int_\Omega |\Delta c|^2 + \frac{b}{2}\int_\Omega |\nabla u|^2 \\
\le\ & \int_\Omega n\ln n - \frac{a}{2}\int_\Omega |\Delta c|^2 - \frac{b}{2}\int_\Omega |\nabla u|^2 \\
&+\Big\{C_1\int_\Omega |\Delta c|^2 + C_1\Big\} + \Big\{C_2 a\int_\Omega |\nabla u|^2 + C_2 a\Big\} + \Big\{C_3 b\|n\|^2_{L^p(\Omega)} + C_3 b\Big\} \\
\le\ & C_1 + C_2 a + C_3 b + C_3 b\|n\|^2_{L^p(\Omega)} + \int_\Omega n\ln n \qquad \text{for all } t\in(0,T_{max}).
\end{aligned} \tag{3.20}$$

Here using the validity of the inequality $\xi\ln\xi \le \frac{1}{(p-1)e}\xi^p$ for all $\xi>0$, by Young's inequality we see that since $p<2$ we have

$$\begin{aligned}
\int_\Omega n\ln n &\le \frac{1}{(p-1)e}\|n\|^p_{L^p(\Omega)} \\
&\le \frac{1}{(p-1)e}\cdot\Big\{\|n\|^2_{L^p(\Omega)} + 1\Big\} \qquad \text{for all } t\in(0,T_{max}),
\end{aligned}$$

and by means of the Gagliardo-Nirenberg inequality and (2.4) we can find $C_4 > 0$ and $C_5 > 0$ such that

$$\begin{aligned}
\|n\|_{L^p(\Omega)}^2 &= \|\sqrt{n}\|_{L^{2p}(\Omega)}^4 \\
&\le C_4 \|\nabla\sqrt{n}\|_{L^2(\Omega)}^{\frac{4(p-1)}{p}} \|\sqrt{n}\|_{L^2(\Omega)}^{\frac{4}{p}} + \|\sqrt{n}\|_{L^2(\Omega)}^4 \\
&\le C_5 \cdot \Big\{ \int_\Omega \frac{|\nabla n|^2}{n} \Big\}^{\frac{2(p-1)}{p}} + C_5 \qquad \text{for all } t \in (0, T_{max}).
\end{aligned}$$

As our restriction $p < 2$ warrants that $\frac{2(p-1)}{p} < 1$, we may again invoke Young's inequality to obtain $C_6 > 0$ such that

$$C_1 + C_2 a + C_3 b + C_3 b \|n\|_{L^p(\Omega)}^2 + \int_\Omega n \ln n \le \frac{1}{4} \int_\Omega \frac{|\nabla n|^2}{n} + C_6 \qquad \text{for all } t \in (0, T_{max}). \tag{3.21}$$

Since finally the Poincaré inequality provides $C_7 > 0$ such that

$$\int_\Omega |u|^2 \le C_7 \int_\Omega |\nabla u|^2 \qquad \text{for all } t \in (0, T_{max}),$$

from (3.20) and (3.21) we infer that

$$y(t) := \int_\Omega n(\cdot,t) \ln n(\cdot,t) + a \int_\Omega |\nabla c(\cdot,t)|^2 + b \int_\Omega |u(\cdot,t)|^2, \qquad t \in (0, T_{max}),$$

and

$$g(t) := \frac{1}{4} \int_\Omega \frac{|\nabla n(\cdot,t)|^2}{n(\cdot,t)} + \frac{a}{2} \int_\Omega |\Delta c(\cdot,t)|^2 + \frac{b}{4} \int_\Omega |\nabla u(\cdot,t)|^2, \qquad t \in (0, T_{max}),$$

satisfy

$$y'(t) + \int_\Omega n \ln n + 2a \int_\Omega |\nabla c|^2 + \frac{b}{4C_7} \int_\Omega |u|^2 + g(t) \le C_6 \qquad \text{for all } t \in (0, T_{max}),$$

and that hence with $C_8 := \min \Big\{ 1\,,\, 2a\,,\, \frac{b}{4C_7} \Big\}$ we have

$$y'(t) + C_8 y(t) + g(t) \le C_9 := C_6 + \frac{(1 - C_8)|\Omega|}{e} \qquad \text{for all } t \in (0, T_{max}), \tag{3.22}$$

because once more due to the elementary inequality

$$\xi \ln \xi \ge -\frac{1}{e} \qquad \text{for all } \xi > 0 \tag{3.23}$$

we can estimate

$$\begin{aligned}
\int_\Omega n \ln n &= C_8 \int_\Omega n \ln n + (1 - C_8) \int_\Omega n \ln n \\
&\ge C_8 \int_\Omega n \ln n - \frac{(1 - C_8)|\Omega|}{e} \qquad \text{for all } t \in (0, T_{max}).
\end{aligned}$$

Now since g is nonnegative, (3.22) firstly implies that

$$y(t) \le C_{10} := \max\left\{ \|y\|_{L^\infty((0,\tau))}\,,\, \frac{C_9}{C_8} \right\} \qquad \text{for all } t \in (0, T_{max}), \tag{3.24}$$

where we note that C_{10} is finite thanks to the regularity properties of n, c and u asserted by Lemma 2.1. Thereafter, we secondly infer from (3.22) on integration that

$$\begin{aligned} \int_t^{t+\tau} g(s)ds &\le y(t+\tau) - y(t) - c_8 \int_t^{t+\tau} y(s)ds + C_9\tau \\ &\le C_{10} + \frac{|\Omega|}{e} + \frac{C_8|\Omega|\tau}{e} + C_9\tau \qquad \text{for all } t \in (0, T_{max} - \tau), \end{aligned} \tag{3.25}$$

because again by (3.23) we see that

$$y(t) \ge \int_\Omega n \ln n \ge -\frac{|\Omega|}{e} \qquad \text{for all } t \in (0, T_{max}).$$

In view of the latter inequality herein, (3.24) readily implies both (3.13) and (3.14), whereas (3.15) and (3.16) directly result from (3.25). □

3.2 Boundedness of u in $L^p(\Omega)$ for any finite p

In order to prepare subsequent arguments ensuring further regularity properties of c, n and m, in this section we shall derive temporally uniform estimates on u in $L^p(\Omega)$ for arbitrary $p \in (1,\infty)$. To initiate this, let us state an immediate consequence of (3.15) when combined with (2.4).

Corollary 3.5 *There exists $C > 0$ such that with $\tau := \min\{1, \frac{1}{2}T_{max}\}$ we have*

$$\int_t^{t+\tau} \int_\Omega n^2 \le C \qquad \textit{for all } t \in (0, T_{max} - \tau). \tag{3.26}$$

PROOF. As the Gagliardo-Nirenberg inequality yields $C_1 > 0$ such that

$$\int_\Omega n^2 = \|\sqrt{n}\|^4_{L^4(\Omega)} \le C_1 \|\nabla\sqrt{n}\|^2_{L^2(\Omega)} \|\sqrt{n}\|^2_{L^2(\Omega)} + C_1\|\sqrt{n}\|^2_{L^2(\Omega)} \qquad \text{for all } t \in (0, T_{max}),$$

in view of (2.4) this immediately results from (3.15). □

Thereby providing a spatio-temporal L^2 control over the forcing term in the Navier-Stokes system in (1.3), by means of an adaptation of essentially well-established arguments ([26], [36, p.340]) the latter entails boundedness of the Dirichlet integral of u.

Lemma 3.6 *There exists $C > 0$ such that*

$$\int_\Omega |\nabla u(\cdot,t)|^2 \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{3.27}$$

Proof. We apply the Helmholtz projection $\mathcal{P}$ to the fourth equation in (1.3) and test the resulting identity by Au to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |A^{\frac{1}{2}}u|^2+\int_\Omega |Au|^2 = -\int_\Omega \mathcal{P}[(u\cdot\nabla)u]\cdot Au+\int_\Omega \mathcal{P}[(n+m)\nabla\phi]\cdot Au \qquad \text{for all } t\in(0,T_{max}), \quad (3.28)$$

where by Young's inequality and (2.6),

$$\begin{aligned}\int_\Omega \mathcal{P}[(n+m)\nabla\phi]\cdot Au &\le \frac{1}{2}\int_\Omega |Au|^2+\frac{1}{2}\int_\Omega \Big|\mathcal{P}[(n+m)\nabla\phi]\Big|^2 \\ &\le \frac{1}{2}\int_\Omega |Au|^2+\frac{1}{2}\int_\Omega |(n+m)\nabla\phi|^2 \\ &\le \frac{1}{2}\int_\Omega |Au|^2+\frac{C_1}{2}\int_\Omega (n+m)^2 \\ &\le \frac{1}{2}\int_\Omega |Au|^2+C_1\int_\Omega n^2+C_1\int_\Omega m^2 \\ &\le \frac{1}{2}\int_\Omega |Au|^2+C_1\int_\Omega n^2+C_2 \qquad \text{for all } t\in(0,T_{max}) \end{aligned} \qquad (3.29)$$

with $C_1 := \|\nabla\phi\|^2_{L^\infty(\Omega)}$ and $C_2 := C_1|\Omega|\|m_0\|^2_{L^\infty(\Omega)}$, because $\mathcal{P}$ acts as an orthogonal projection on $L^2(\Omega;\mathbb{R}^2)$. For the same reason, by means of the Hölder inequality we can estimate

$$\begin{aligned}-\int_\Omega \mathcal{P}[(u\cdot\nabla)u]\cdot Au &\le \Big\|\mathcal{P}[(u\cdot\nabla)u]\Big\|_{L^2(\Omega)}\|Au\|_{L^2(\Omega)} \\ &\le \|(u\cdot\nabla)u\|_{L^2(\Omega)}\|Au\|_{L^2(\Omega)} \\ &\le \|u\|_{L^4(\Omega)}\|\nabla u\|_{L^4(\Omega)}\|Au\|_{L^2(\Omega)} \qquad \text{for all } t\in(0,T_{max}).\end{aligned}$$

Here twice invoking the Gagliardo-Nirenberg inequality and recalling that in $D(A)$, $\|\nabla(\cdot)\|_{L^2(\Omega)}$ and $\|A(\cdot)\|_{L^2(\Omega)}$ define norms equivalent to $\|\cdot\|_{W^{1,2}(\Omega)}$ and $\|\cdot\|_{W^{2,2}(\Omega)}$, respectively, in light of (3.14) we hence obtain positive constants C_3, C_4 and C_5 such that

$$\begin{aligned}-\int_\Omega \mathcal{P}[(u\cdot\nabla)u]\cdot Au &\le \Big\{C_3\|\nabla u\|^{\frac{1}{2}}_{L^2(\Omega)}\|u\|^{\frac{1}{2}}_{L^2(\Omega)}\Big\}\cdot\Big\{C_4\|Au\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla u\|^{\frac{1}{2}}_{L^2(\Omega)}\Big\}\cdot\|Au\|_{L^2(\Omega)} \\ &\le C_5\|\nabla u\|_{L^2(\Omega)}\|Au\|^{\frac{3}{2}}_{L^2(\Omega)} \\ &\le \frac{1}{2}\int_\Omega |Au|^2+\frac{27C_5^4}{32}\Big\{\int_\Omega |\nabla u|^2\Big\}^2 \qquad \text{for all } t\in(0,T_{max}),\end{aligned}$$

where in the last step we have also made use of Young's inequality. Since $\int_\Omega |A^{\frac{1}{2}}u|^2=\int_\Omega |\nabla u|^2$ for all $t\in(0,T_{max})$, from (3.28) and (3.29) we thus infer that

$$\frac{d}{dt}\int_\Omega |\nabla u|^2 \le \frac{27C_5^4}{16}\Big\{\int_\Omega |\nabla u|^2\Big\}^2+2C_1\int_\Omega n^2+2C_2 \qquad \text{for all } t\in(0,T_{max}),$$

meaning that $y(t) := \int_\Omega |\nabla u(\cdot,t)|^2$, $g(t) := \frac{27C_5^4}{16}\int_\Omega |\nabla u(\cdot,t)|^2$ and $h(t) := 2C_1\int_\Omega n^2(\cdot,t)+2C_2$, $t\in(0,T_{max})$, satisfy

$$y'(t)\le g(t)y(t)+h(t) \qquad \text{for all } t\in(0,T_{max}). \qquad (3.30)$$

Now again abbreviating $\tau := \min\{1, \frac{1}{2}T_{max}\}$, from Lemma 3.4 and Corollary 3.5 we obtain $C_6 > 0$ and $C_7 > 0$ such that

$$\int_t^{t+\tau} \int_\Omega |\nabla u|^2 \le C_6 \qquad \text{for all } t \in (0, T_{max} - \tau) \tag{3.31}$$

and

$$\int_t^{t+\tau} \int_\Omega n^2 \le C_7 \qquad \text{for all } t \in (0, T_{max} - \tau),$$

which since $\tau \le 1$ firstly implies that

$$\int_t^{t+\tau} g(s)ds \le \frac{27C_5^4 C_6}{16} \quad \text{and} \quad \int_t^{t+\tau} h(s)ds \le 2C_1C_7 + 2C_2 \qquad \text{for all } t \in (0, T_{max} - \tau),$$

and where secondly (3.31) moreover warrants that for each $t \in (\tau, T_{max})$ we can find $t_\star(t) \in (t - \tau, t)$ such that

$$y(t_\star(t)) \le \frac{C_6}{\tau}.$$

Therefore, by an ODE comparison argument we conclude from (3.30) that

$$\begin{aligned}
y(t) &\le y(t_\star(t)) \cdot e^{\int_{t_\star(t)}^t g(s)ds} + \int_{t_\star(t)}^t e^{\int_s^t g(\sigma)d\sigma} \cdot h(s)ds \\
&\le \frac{C_6}{\tau} \cdot e^{\frac{27C_5^4C_6}{16}} + e^{\frac{27C_5^4C_6}{16}} \int_{t_\star(t)}^t h(s)ds \\
&\le \frac{C_6}{\tau} \cdot e^{\frac{27C_5^4C_6}{16}} + e^{\frac{27C_5^4C_6}{16}} \cdot (2C_1C_7 + 2C_2) \qquad \text{for all } t \in (\tau, T_{max}),
\end{aligned}$$

because $t - t_\star(t) < \tau$. As y is bounded in $(0, \tau]$ according to Lemma 2.1, this verifies (3.27). $\Box$

Thanks to a corresponding Sobolev embedding, we thus arrive at our main goal of this section.

Corollary 3.7 *Let $p \in (1, \infty)$. Then there exists $C(p) > 0$ such that*

$$\|u(\cdot, t)\|_{L^p(\Omega)} \le C(p) \qquad \textit{for all } t \in (0, T_{max}). \tag{3.32}$$

PROOF. As u belongs to $L^\infty((0, T_{max}); W_0^{1,2}(\Omega; \mathbb{R}^2))$ due to Lemma 3.6, this evidently results from the continuity of the embedding $W_0^{1,2}(\Omega; \mathbb{R}^2) \hookrightarrow L^p(\Omega; \mathbb{R}^2)$ for any such p. $\Box$

3.3 Boundedness of ∇c in $L^p(\Omega)$ for any finite p

We shall next make use of the above information (3.32) to make sure that beyond the result from Lemma 3.4, ∇c remains bounded actually in any space $L^p(\Omega)$ with finite $p < \infty$.

Lemma 3.8 *For any $p \in (2, \infty)$, there exists $C(p) > 0$ such that*

$$\|\nabla c(\cdot, t)\|_{L^p(\Omega)} \le C(p) \qquad \textit{for all } t \in (0, T_{max}). \tag{3.33}$$

PROOF. We again let $\tau := \min\{1, \frac{1}{2}T_{max}\}$, and given $p \in (2,\infty)$ and $T \in (\tau, T_{max})$ we introduce the finite number

$$M(T) := \sup_{t\in(\tau,T)} \|\nabla c(\cdot,t)\|_{L^p(\Omega)}.$$

In order to control $M(T)$ appropriately, we employ a variation-of-constants representation of c involving the Neumann heat semigroup $(e^{\sigma\Delta})_{\sigma\geq 0}$ in Ω, to see that for each fixed $t \in (\tau, T)$ we have

$$c(\cdot,t) = e^{\tau(\Delta-1)}c(\cdot,t-\tau) + \int_{t-\tau}^t e^{(t-s)(\Delta-1)}m(\cdot,s)ds - \int_{t-\tau}^t e^{(t-s)(\Delta-1)}[u(\cdot,s)\cdot\nabla c(\cdot,s)]ds,$$

so that estimating $e^{-(t-s)} \leq 1$ for $s \in [t-\tau, t]$ we obtain

$$\begin{aligned}
\|\nabla c(\cdot,t)\|_{L^p(\Omega)} &\leq \left\|\nabla e^{\tau\Delta}c(\cdot,t-\tau)\right\|_{L^p(\Omega)} + \int_{t-\tau}^t \left\|\nabla e^{(t-s)\Delta}m(\cdot,s)\right\|_{L^p(\Omega)}ds \\
&\quad + \int_{t-\tau}^t \left\|\nabla e^{(t-s)\Delta}[u(\cdot,s)\cdot\nabla c(\cdot,s)]\right\|_{L^p(\Omega)}ds \qquad \text{for all } t \in (\tau, T_{max}). \qquad (3.34)
\end{aligned}$$

Here combining known smoothing properties of the Neumann heat semigroup ([35, Lemma 1.3]) with (2.7) and (2.6) yields positive constants C_1, C_2 and C_3 such that

$$\begin{aligned}
\left\|\nabla e^{(t-s)\Delta}c(\cdot,t-\tau)\right\|_{L^p(\Omega)} &\leq C_1\tau^{-\frac{1}{2}}\|c(\cdot,t-\tau)\|_{L^p(\Omega)} \\
&\leq C_1\tau^{-\frac{1}{2}}|\Omega|^{\frac{1}{p}}\|c(\cdot,t-\tau)\|_{L^\infty(\Omega)} \\
&\leq C_2 \qquad \text{for all } t \in (\tau, T_{max}) \qquad (3.35)
\end{aligned}$$

and

$$\begin{aligned}
\int_{t-\tau}^t \left\|\nabla e^{(t-s)\Delta}m(\cdot,s)\right\|_{L^p(\Omega)}ds &\leq C_3\int_{t-\tau}^t (t-s)^{-\frac{1}{2}}\|m(\cdot,s)\|_{L^p(\Omega)}ds \\
&\leq C_3|\Omega|^{\frac{1}{p}}\|m_0\|_{L^\infty(\Omega)}\int_{t-\tau}^t (t-s)^{-\frac{1}{2}}ds \\
&= 2C_3|\Omega|^{\frac{1}{p}}\|m_0\|_{L^\infty(\Omega)}\tau^{\frac{1}{2}} \qquad \text{for all } t \in (\tau, T_{max}). \qquad (3.36)
\end{aligned}$$

As for the rightmost summand in (3.34), we fix any $q \in (2,p)$ such that $q > \frac{2p}{p+2}$, and then pick $\theta > 1$ such that still $q\theta < p$, to firstly find $C_4 > 0$ such that

$$\begin{aligned}
\int_{t-\tau}^t \left\|\nabla e^{(t-s)\Delta}[u(\cdot,s)\cdot\nabla c(\cdot,s)]\right\|_{L^p(\Omega)}ds &\leq C_4\int_{t-\tau}^t (t-s)^{-\frac{1}{2}-(\frac{1}{q}-\frac{1}{p})}\|u(\cdot,s)\cdot\nabla c(\cdot,s)\|_{L^q(\Omega)}ds \\
&\qquad \text{for all } t \in (\tau, T_{max}), \qquad (3.37)
\end{aligned}$$

and to secondly twice apply the Hölder inequality in estimating

$$\begin{aligned}
\|u(\cdot,s)\cdot\nabla c(\cdot,s)\|_{L^q(\Omega)} &\leq \|u(\cdot,s)\|_{L^{\frac{q\theta}{q-1}}(\Omega)}\|\nabla c(\cdot,s)\|_{L^{q\theta}(\Omega)} \\
&\leq \|u(\cdot,s)\|_{L^{\frac{q\theta}{q-1}}(\Omega)}\|\nabla c(\cdot,s)\|^a_{L^p(\Omega)}\|\nabla c(\cdot,s)\|^{1-a}_{L^2(\Omega)} \qquad \text{for all } s \in (0, T_{max})
\end{aligned}$$

with

$$a := \frac{\frac{1}{2} - \frac{1}{q\theta}}{\frac{1}{2} - \frac{1}{p}} \in (0,1).$$

In view of Corollary 3.7, Lemma 3.4 and our definition of $M(T)$, we thus infer the existence of $C_5 > 0$ fulfilling

$$\|u(\cdot,s)\cdot\nabla c(\cdot,s)\|_{L^q(\Omega)} \le C_5 M^a(T) \qquad \text{for all } s \in (\tau,T),$$

so that

$$\|u(\cdot,s)\cdot\nabla c(\cdot,s)\|_{L^q(\Omega)} \le C_5 M^a(T) + C_6 \qquad \text{for all } s \in (0,T)$$

with $C_6 := \|u\cdot\nabla c\|_{L^\infty((0,\tau);L^q(\Omega))}$ being finite due to Lemma 2.1. As our restriction $q > \frac{2p}{p+2}$ guarantees that $\beta := \frac{1}{2} + (\frac{1}{q} - \frac{1}{p}) < 1$, (3.37) therefore entails that

$$\begin{aligned}
\int_{t-\tau}^{t} \Big\|\nabla e^{(t-s)\Delta}[u(\cdot,s)\cdot\nabla c(\cdot,s)]\Big\|_{L^p(\Omega)} ds &\le C_4(C_5M^a(T)+C_6)\int_{t-\tau}^{t}(t-s)^{-\beta}ds \\
&= \frac{C_4(C_5M^a(T)+C_6)\tau^{1-\beta}}{1-\beta} \qquad \text{for all } t \in (\tau,T),
\end{aligned}$$

which in conjunction with (3.34), (3.35) and (3.36) implies that with some $C_7 > 0$ we have

$$\|\nabla c(\cdot,t)\|_{L^p(\Omega)} \le C_7 M^a(T) + C_7 \qquad \text{for all } t \in (\tau,T).$$

Since this entails that

$$M(T) \le C_7 M^a(T) + C_7 \qquad \text{for all } T \in (\tau, T_{max})$$

and that hence

$$M(T) \le \max\Big\{1\,,\,(2C_7)^{\frac{1}{1-a}}\Big\} \qquad \text{for all } T \in (\tau, T_{max}),$$

and since ∇c belongs to $L^\infty((0,\tau);L^p(\Omega))$ by Lemma 2.1, this completes the proof. □

3.4 Boundedness of n

Now having appropriate control over both the cross-diffusive flux as well as the convective velocity in the first equation of (1.3), by adapting a known argument (see e.g. [1, Lemma 3.2] or also [27, Lemma 4.2]) we can derive boundedness of n in $\Omega\times(0,T_{max})$.

Lemma 3.9 *There exists $C > 0$ with the property that*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t \in (0,T_{max}). \tag{3.38}$$

PROOF. Again with $\tau := \min\{1, \frac{1}{2}T_{max}\}$, we let

$$M(T) := \sup_{t\in(\tau,T)} \|n(\cdot,t)\|_{L^\infty(\Omega)}, \qquad T \in (\tau, T_{max}),$$

and for $t \in (\tau, T_{max})$ we represent n according to

$$n(\cdot,t) = e^{\tau\Delta}n(\cdot,t-\tau) - \int_{t-\tau}^t e^{(t-s)\Delta}\nabla\cdot\Big(n(\cdot,s)g(\cdot,s)\Big)ds - \int_{t-\tau}^t e^{(t-s)\Delta}[n(\cdot,s)m(\cdot,s)]ds,$$

where

$$g(x,s) := \nabla c(x,s) + u(x,s), \qquad x \in \Omega,\ \ s \in (0,T_{max}).$$

As the last summand in (3.39) is nonpositive by the maximum principle, we can thus estimate

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \le \|e^{\tau\Delta}n(\cdot,t-\tau)\|_{L^\infty(\Omega)} + \int_{t-\tau}^t \Big\|e^{(t-s)\Delta}\nabla\cdot\Big(n(\cdot,s)g(\cdot,s)\Big)\Big\|_{L^\infty(\Omega)}ds \tag{3.39}$$

for all $t \in (\tau, T_{max})$, where known regularization properties of $(e^{\sigma\Delta})_{\sigma\ge0}$ ([35, Lemma 1.3]) in conjunction with (2.4) provide positive constants C_1, C_2 and C_3 such that

$$\|e^{\tau\Delta}n(\cdot,t-\tau)\|_{L^\infty(\Omega)} \le C_1\tau^{-1}\|n(\cdot,t-\tau)\|_{L^1(\Omega)} \le C_2 \qquad \text{for all } t \in (\tau,T_{max}) \tag{3.40}$$

and

$$\int_{t-\tau}^t \Big\|e^{(t-s)\Delta}\nabla\cdot\Big(n(\cdot,s)g(\cdot,s)\Big)\Big\|_{L^\infty(\Omega)}ds \le C_3\int_{t-\tau}^t (t-s)^{-\frac{3}{4}}\|n(\cdot,s)g(\cdot,s)\|_{L^4(\Omega)}ds \quad \text{for all } t \in (\tau,T_{max}). \tag{3.41}$$

Here by the Hölder inequality we have

$$\begin{aligned}\|n(\cdot,s)g(\cdot,s)\|_{L^4(\Omega)} &\le \|n(\cdot,s)\|_{L^8(\Omega)}\|g(\cdot,s)\|_{L^8(\Omega)} \\ &\le \|n(\cdot,s)\|_{L^\infty(\Omega)}^{\frac{7}{8}}\|n(\cdot,s)\|_{L^1(\Omega)}^{\frac{1}{8}}\|g(\cdot,s)\|_{L^8(\Omega)} \qquad \text{for all } s \in (0,T_{max}),\end{aligned}$$

so that since g lies in $L^\infty((0,T_{max});L^8(\Omega))$ due to Corollary 3.7 and Lemma 3.8, in view of (2.4) we can find $C_4 > 0$ such that

$$\begin{aligned}\|n(\cdot,s)g(\cdot,s)\|_{L^4(\Omega)} &\le C_4\|n(\cdot,s)\|_{L^\infty(\Omega)}^{\frac{7}{8}} \\ &\le C_4M^{\frac{7}{8}}(T) \qquad \text{for all } s \in (\tau,T)\end{aligned}$$

and that hence

$$\|n(\cdot,s)g(\cdot,s)\|_{L^4(\Omega)} \le C_4M^{\frac{7}{8}}(T) + C_5 \qquad \text{for all } s \in (0,T)$$

with $C_5 := \|ng\|_{L^\infty((0,\tau);L^4(\Omega))}$. Consequently, (3.41) implies that

$$\int_{t-\tau}^t \Big\|e^{(t-s)\Delta}\nabla\cdot\Big(n(\cdot,s)g(\cdot,s)\Big)\Big\|_{L^\infty(\Omega)}ds \le 4C_3\cdot\Big(C_4M^{\frac{7}{8}}(T) + C_5\Big)\cdot\tau^{\frac{1}{4}} \qquad \text{for all } t \in (\tau,T),$$

whence from (3.39) and (3.40) we conclude that with some $C_6 > 0$ we have

$$M(T) \le C_6 M^{\frac{7}{8}}(T) + C_6 \qquad \text{for all } T \in (\tau, T_{max})$$

and therefore

$$M(T) \le \max\Big\{1\,,\,(2C_6)^8\Big\} \qquad \text{for all } T \in (\tau, T_{max}).$$

By boundedness of n in $\Omega \times (0,\tau]$, this establishes (3.38). □

3.5 Boundedness of $A^\alpha u$ in $L^2(\Omega)$

As a final preparation for our proof of global solvability, following a straightforward argument ([36, p.340], [28, Lemma 3.11]) we turn the above information on boundedness of n into the following regularity property of the fluid velocity.

Lemma 3.10 *With $\alpha \in (\frac{1}{2}, 1)$ taken from (1.6), we can find a constant $C > 0$ such that*

$$\|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{3.42}$$

Proof. We represent u according to

$$u(\cdot,t) = e^{-\tau A} u(\cdot, t-\tau) - \int_{t-\tau}^t e^{-(t-s)A}\mathcal{P}\Big[(u(\cdot,s)\cdot\nabla)u(\cdot,s)\Big]ds + \int_{t-\tau}^t e^{-(t-s)A}\mathcal{P}[n(\cdot,s)\nabla\phi]ds$$

for $t \in (\tau, T_{max})$, where once more $\tau := \min\{1, \frac{1}{2}T_{max}\}$. Hence,

$$\begin{aligned} \|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} \le\ & \Big\|A^\alpha e^{-\tau A} u(\cdot,t-\tau)\Big\|_{L^2(\Omega)} \\ & + \int_{t-\tau}^t \Big\|A^\alpha e^{-(t-s)A}\mathcal{P}\Big[(u\cdot,s)\cdot\nabla)u(\cdot,s)\Big]\Big\|_{L^2(\Omega)} ds \\ & + \int_{t-\tau}^t \Big\|A^\alpha e^{-(t-s)A}\mathcal{P}\Big[(n(\cdot,s)+m(\cdot,s))\nabla\phi\Big]\Big\|_{L^2(\Omega)} ds \end{aligned} \tag{3.43}$$

for all $t \in (\tau, T_{max})$, and using well-known smoothing estimates for the Stokes semigroup $(e^{-\sigma A})_{\sigma\ge 0}$ ([11, p.201]), thanks to Lemma 3.4, Lemma 3.9 and (2.6) we obtain positive constants C_1, C_2, C_3 and C_4 such that

$$\Big\|A^\alpha e^{-\tau A} u(\cdot,t-\tau)\Big\|_{L^2(\Omega)} \le C_1\tau^{-\alpha}\|u(\cdot,t-\tau)\|_{L^2(\Omega)} \le C_2 \qquad \text{for all } t\in(\tau,T_{max}), \tag{3.44}$$

and such that

$$\begin{aligned} & \int_{t-\tau}^t \Big\|A^\alpha e^{-(t-s)A}\mathcal{P}\Big[(n(\cdot,s)+m(\cdot,s))\nabla\phi\Big]\Big\|_{L^2(\Omega)} ds \\ & \qquad \le\ C_3\int_{t-\tau}^t (t-s)^{-\alpha}\Big\|\mathcal{P}\Big[(n(\cdot,s)+m(\cdot,s))\nabla\phi\Big]\Big\|_{L^2(\Omega)} ds \end{aligned}$$

$$
\begin{aligned}
&\leq C_3 \int_{t-\tau}^{t} (t-s)^{-\alpha} \Big\| (n(\cdot,s)+m(\cdot,s))\nabla\phi \Big\|_{L^2(\Omega)} ds \\
&\leq C_4 \int_{t-\tau}^{t} (t-s)^{-\alpha} ds \\
&= \frac{C_4 \tau^{1-\alpha}}{1-\alpha} \qquad \text{for all } t\in(\tau,T_{max}),
\end{aligned} \tag{3.45}
$$

again because of the projection properties of $\mathcal{P}$ and the boundedness of $\nabla\phi$ in Ω, and because $\alpha<1$. In order to treat the second summand on the right of (3.43) suitably, we once more use that $\alpha<1$ in fixing $p\in(1,2)$ conveniently close to 2 such that $p>\frac{2}{3-2\alpha}$, and employ a corresponding L^p-L^2 estimate for the Stokes semigroup ([11, p.201]) as we as a known result on boundedness of the Helmholtz projection in $L^p(\Omega)$ ([10]) to find $C_5>0$ and $C_6>0$ satisfying

$$
\begin{aligned}
&\int_{t-\tau}^{t} \Big\| A^\alpha e^{-(t-s)A}\mathcal{P}\Big[(u(\cdot,s)\cdot\nabla)u(\cdot,s)\Big]\Big\|_{L^2(\Omega)} ds \\
&\qquad\leq C_5 \int_{t-\tau}^{t} (t-s)^{-\alpha-(\frac{1}{p}-\frac{1}{2})} \Big\|\mathcal{P}\Big[(u(\cdot,s)\cdot\nabla)u(\cdot,s)\Big]\Big\|_{L^p(\Omega)} ds \\
&\qquad\leq C_6 \int_{t-\tau}^{t} (t-s)^{-\alpha-(\frac{1}{p}-\frac{1}{2})} \Big\|(u(\cdot,s)\cdot\nabla)u(\cdot,s)\Big\|_{L^p(\Omega)} ds \qquad \text{for all } t\in(\tau,T_{max}).
\end{aligned} \tag{3.46}
$$

Here since $p<2$, we may apply the Hölder inequality along with Lemma 3.6 and Corollary 3.7 to infer that with some $C_7>0$ we have

$$
\Big\|(u(\cdot,s)\cdot\nabla)u(\cdot,s)\Big\|_{L^p(\Omega)} \|u(\cdot,s)\|_{L^{\frac{2p}{2-p}}(\Omega)} \|\nabla u(\cdot,s)\|_{L^2(\Omega)} \leq C_7 \qquad \text{for all } s\in(0,T_{max}),
$$

so that since our restriction $p>\frac{2}{3-2\alpha}$ ensures $\beta:=\alpha+(\frac{1}{p}-\frac{1}{2})$ satisfies $\beta<1$, from (3.46) we infer that

$$
\begin{aligned}
\int_{t-\tau}^{t} \Big\| A^\alpha e^{-(t-s)A}\mathcal{P}\Big[(u(\cdot,s)\cdot\nabla)u(\cdot,s)\Big]\Big\|_{L^2(\Omega)} ds &\leq C_6C_7 \int_{t-\tau}^{t}(t-s)^{-\beta}ds \\
&= \frac{C_6C_7\tau^{1-\beta}}{1-\beta} \qquad \text{for all } t\in(\tau,T_{max}).
\end{aligned} \tag{3.47}
$$

In summary, (3.43), (3.44), (3.45) and (3.47) show that

$$
\|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} \leq C_2 + \frac{C_4\tau^{1-\alpha}}{1-\alpha} + \frac{C_6C_7\tau^{1-\beta}}{1-\beta} \qquad \text{for all } t\in(\tau,T_{max})
$$

and thereby prove (3.42), because $\sup_{t\in(0,\tau)} \|A^\alpha u(\cdot,t)\|_{L^2(\Omega)}$ is finite according to Lemma 2.1. □

3.6 Proof of Theorem 1.1

We are now prepared for the derivation of our main result on global existence in (1.3).

Proof of Theorem 1.1. Due to the extensibility criterion (2.2) in Lemma 2.1, together with (2.7) and (2.6) the estimates gained in Lemma 3.9, Lemma 3.8 and Lemma 3.10 firstly show that in Lemma 2.1 we actually have $T_{max}=\infty$, meaning that (n,c,m,u,P) exists globally in time, and that hence, secondly, (1.8) holds. □

4 Large time behavior. Proof of Theorem 1.2

4.1 Stabilization of n and m in spatial average

We next address the problem of determining the large time behavior of the solution just constructed, and the fundament of our arguments in this direction will be formed by the basic relaxation properties expressed in (2.8) and (2.9), which we briefly repeat here for emphasis.

Corollary 4.1 *The global solution of (1.3)-(1.5) satisfies*

$$\int_0^\infty \int_\Omega nm < \infty \tag{4.1}$$

and

$$\int_0^\infty \int_\Omega |\nabla m|^2 < \infty. \tag{4.2}$$

Proof. These properties are immediate consequences of (2.8) and (2.9). $\square$

Indeed, in conjunction with the global boundedness of n these inequalities imply stabilization of the spatial averages of both n and m toward their expected limits.

Lemma 4.2 *We have*

$$\int_\Omega n(\cdot,t) \to \Big\{ \int_\Omega n_0 - \int_\Omega m_0 \Big\}_+ \qquad \textit{as } t\to\infty \tag{4.3}$$

and

$$\int_\Omega m(\cdot,t) \to \Big\{ \int_\Omega m_0 - \int_\Omega n_0 \Big\}_+ \qquad \textit{as } t\to\infty. \tag{4.4}$$

Proof. Pursuing a strategy demonstrated in [38, Lemma 4.2], we start by noting that as a first consequence of Corollary 4.1 we know that

$$\int_{t-1}^t \int_\Omega nm \to 0 \qquad \text{as } t\to\infty, \tag{4.5}$$

where we rewrite the term on the left according to

$$\begin{aligned}
\int_{t-1}^t \int_\Omega nm &= \int_{t-1}^t \int_\Omega n(x,s)\Big(m(x,s)-\overline{m(\cdot,s)}\Big)dxds + \int_{t-1}^t \overline{m(\cdot,s)} \int_\Omega n(x,s)dxds \\
&= \int_{t-1}^t \int_\Omega n(x,s)\Big(m(x,s)-\overline{m(\cdot,s)}\Big)dxds \\
&\quad + \frac{1}{|\Omega|}\int_{t-1}^t \Big\{\int_\Omega m(x,s)dx\Big\}\cdot\Big\{\int_\Omega n(x,s)dx\Big\}ds, \qquad t>1.
\end{aligned} \tag{4.6}$$

Here we may use the Poincaré inequality along with the boundedness of n in $\Omega\times(0,\infty)$ to see that with some $C_1>0$ we have

$$\begin{aligned}
\bigg| \int_{t-1}^t \int_\Omega n(x,s)\Big(m(x,s)-\overline{m(\cdot,s)}\Big)dxds \bigg| &\le \int_{t-1}^t \|n(\cdot,s)\|_{L^2(\Omega)} \|m(\cdot,s)-\overline{m(\cdot,s)}\|_{L^2(\Omega)}ds \\
&\le C_1 \int_{t-1}^t \|\nabla m(\cdot,s)\|_{L^2(\Omega)}ds \qquad \text{for all } t>1,
\end{aligned}$$

so that since due to the Cauchy-Schwarz inequality, Corollary 4.1 moreover asserts that

$$\int_{t-1}^{t} \|\nabla m(\cdot,s)\|_{L^2(\Omega)} ds \le \left\{ \int_{t-1}^{t} \|\nabla m(\cdot,s)\|_{L^2(\Omega)}^2 ds \right\}^{\frac{1}{2}} \to 0 \qquad \text{as } t \to \infty,$$

combining (4.5) with (4.6) we infer that

$$\int_{t-1}^{t} \left\{ \int_\Omega m(x,s)dx \right\} \cdot \left\{ \int_\Omega n(x,s)dx \right\} ds \to 0 \qquad \text{as } t \to \infty. \tag{4.7}$$

Now in the case when $\int_\Omega n_0 \ge \int_\Omega m_0$, (2.5) warrants that also $\int_\Omega n(x,s)dx \ge \int_\Omega m(x,s)dx$ for all $s > 0$, so that (4.7) shows that then

$$\int_{t-1}^{t} \left\{ \int_\Omega m(x,s)dx \right\}^2 ds \to 0 \qquad \text{as } t \to \infty.$$

Since

$$\int_\Omega m(x,s)dx \ge \int_\Omega m(x,t)dx \qquad \text{for all } s \in (0,t)$$

by (2.3), this implies that

$$\int_\Omega m(x,t)dx \le \left\{ \int_{t-1}^{t} \left\{ \int_\Omega m(x,s)dx \right\}^2 ds \right\}^{\frac{1}{2}} \to 0 \qquad \text{as } t \to \infty$$

and thus proves that in this case indeed (4.4) is valid, whereupon thanks to (2.5) we moreover obtain

$$\int_\Omega n(x,s)dx \to \int_\Omega n_0 - \int_\Omega m_0 \qquad \text{as } t \to \infty$$

and hence conclude that also (4.3) holds.
By quite a similar argument, it can be seen that (4.7) entails both (4.3) and (4.4) also when $\int_\Omega n_0 > \int_\Omega m_0$. □

4.2 Hölder continuity properties

In order to derive further information from both Corollary 4.1 and Lemma 4.2, it will be convenient to know that our solution enjoys certain Hölder regularity properties. A first result on this topic is actually a simple by-product of Lemma 3.10.

Lemma 4.3 *There exist $\theta \in (0,1)$ and $C > 0$ with the property that*

$$\|u(\cdot,t)\|_{C^\theta(\bar\Omega)} \le C \qquad \textit{for all } t > 0. \tag{4.8}$$

PROOF. This is an immediate consequence of Lemma 3.10, because our overall assumption that $\alpha > \frac{1}{2}$ guarantees that $D(A^\alpha) \hookrightarrow C^\theta(\bar\Omega)$ for each $\theta \in (0, 2\alpha - 1)$ ([10], [12]). □

Next, standard parabolic regularity theory asserts that the boundedness properties collected above are sufficient for the following Hölder estimates, where especially for the component m the additional time Hölder regularity property will be essential to our subsequent arguments (see Lemma 4.5).

Lemma 4.4 *There exist $\theta \in (0,1)$ and $C > 0$ such that*

$$\|n\|_{C^{\theta,\frac{\theta}{2}}(\bar\Omega\times[t,t+1])} \le C \qquad \text{for all } t > 1 \tag{4.9}$$

and

$$\|c\|_{C^{\theta,\frac{\theta}{2}}(\bar\Omega\times[t,t+1])} \le C \qquad \text{for all } t > 1 \tag{4.10}$$

as well as

$$\|m\|_{C^{\theta,\frac{\theta}{2}}(\bar\Omega\times[t,t+1])} \le C \qquad \text{for all } t > 1. \tag{4.11}$$

Proof. We once more make use of the identity $\nabla \cdot u \equiv 0$ in rewriting the first equation in (1.3) according to

$$n_t = \Delta n + \nabla \cdot a_1(x,t) + b_1(x,t), \qquad x \in \Omega,\ t > 0,$$

with

$$a_1(x,t) := -n(x,t)\nabla c(x,t) - n(x,t)u(x,t) \quad \text{and} \quad b_1(x,t) := -n(x,t)m(x,t), \qquad x \in \Omega,\ t > 0,$$

Since according to (2.6), Lemma 3.9, Lemma 4.3 and Lemma 3.8 we know that b_1 is bounded in $\Omega \times (0,\infty)$ and a_1 belongs to $L^\infty((0,\infty);L^p(\Omega))$ for each finite $p > 1$, (4.9) therefore becomes a consequence of a standard result on parabolic Hölder regularity in parabolic equations ([24, Theorem 1.3]).
Similarly, (4.10) follows from the fact that in

$$c_t = \Delta c + \nabla \cdot a_2(x,t) + b_2(x,t), \qquad x \in \Omega,\ t > 0,$$

the functions given by

$$a_2(x,t) := -c(x,t)u(x,t) \quad \text{and} \quad b_2(x,t) := -c(x,t) + m(x,t), \qquad x \in \Omega,\ t > 0,$$

are both bounded in $\Omega \times (0,\infty)$ due to (2.7), (2.6) and Lemma 4.3.
Finally, observing that

$$m_t = \Delta m + \nabla \cdot a_3(x,t) + b_3(x,t), \qquad x \in \Omega,\ t > 0,$$

with

$$a_3(x,t) := -m(x,t)u(x,t) \quad \text{and} \quad b_3(x,t) := -n(x,t)m(x,t), \qquad x \in \Omega,\ t > 0,$$

we obtain the estimate (4.11) from the boundedness of a_3 and b_3 in $\Omega \times (0,\infty)$, as asserted by (2.6), Lemma 4.3 and Lemma 3.9. □

4.3 Uniform stabilization of m

Using the Hölder bound (4.11) provided by Lemma 4.4, we can now turn the weak homogenization property implied by (2.9) together with the convergence information from Lemma 4.2 on the respective average into uniform convergence of m in the sense claimed in Theorem 1.2.

Lemma 4.5 *We have*

$$m(\cdot,t) \to \left\{ \fint_\Omega m_0 - \fint_\Omega n_0 \right\}_+ \qquad in\ L^\infty(\Omega) \quad as\ t \to \infty. \tag{4.12}$$

Proof. Abbreviating $m_\infty := \left\{ \fint_\Omega m_0 - \fint_\Omega n_0 \right\}_+$, we see that if (4.12) was false, then there would exist $\varepsilon > 0$, $(t_k)_{k\in\mathbb{N}} \subset (1,\infty)$ and $(x_k)_{k\in\mathbb{N}} \subset \Omega$ such that $t_k \to \infty$ as $k \to \infty$ and

$$|m(x_k,t_k) - m_\infty| \ge \varepsilon \qquad \text{for all } k \in \mathbb{N},$$

so that for

$$z_k(x,s) := m(x, t_k + s), \qquad x \in \bar\Omega,\ s \in [0,1],\ k \in \mathbb{N},$$

we would have

$$|z_k(x_k,0) - m_\infty| \ge \varepsilon \qquad \text{for all } k \in \mathbb{N}. \tag{4.13}$$

Now from Lemma 4.4 we know that there exists $\theta \in (0,1)$ such that

$$(z_k)_{k\in\mathbb{N}} \text{ is bounded in } C^{\theta,\frac{\theta}{2}}(\bar\Omega \times [0,1]), \tag{4.14}$$

which in particular entails equicontinuity of $(z_k)_{k\in\mathbb{N}}$ and thus, by (4.13), ensures the existence of $\delta \in (0,1)$ and $r > 0$ such that

$$|z_k(x,s) - m_\infty| \ge \frac{\varepsilon}{2} \qquad \text{for all } x \in B_r(x_k), \text{ each } s \in (0,\delta) \text{ and any } k \in \mathbb{N}. \tag{4.15}$$

We next apply Lemma 4.2 which combined with Lemma 2.2 makes sure that $\int_\Omega m(\cdot,t) \searrow m_\infty|\Omega|$ as $t \to \infty$, implying that for all $s \in (0,1)$ we have

$$\int_\Omega z_k(\cdot,s) \searrow m_\infty|\Omega| \qquad \text{as } k \to \infty$$

and that hence, by Dini's theorem,

$$\sup_{s\in(0,1)} \left| \overline{z_k(\cdot,s)} - m_\infty \right| \to 0 \qquad \text{as } k \to \infty. \tag{4.16}$$

As a final ingredient to our argument, let us once more invoke Corollary 4.1 which entails that since $t_k \to \infty$ as $k \to \infty$ we have

$$\int_0^t \int_\Omega |\nabla z_k(x,s)|^2 dxds = \int_{t_k}^{t_k+1} \int_\Omega |\nabla m(x,t)|^2 dxdt \to 0 \qquad \text{as } k \to \infty. \tag{4.17}$$

Thus, fixing a Poincaré constant $C_1 > 0$ such that

$$\int_\Omega |\varphi - \overline{\varphi}|^2 \le C_1 \int_\Omega |\nabla\varphi|^2 \qquad \text{for all } \varphi \in W^{1,2}(\Omega),$$

from (4.17) we obtain that

$$\int_0^1 \int_\Omega \Big| z_k(x,s) - \overline{z_k(\cdot,s)} \Big|^2 dxds \to 0 \qquad \text{as } k \to \infty,$$

and that therefore

$$\begin{aligned}
\int_0^1 \int_\Omega \Big| z_k(x,s) - m_\infty \Big|^2 dxds &\le 2\int_0^1 \int_\Omega \Big| z_k(x,s) - \overline{z_k(\cdot,s)} \Big|^2 dxds + 2\int_0^1 \int_\Omega \Big| \overline{z_k(\cdot,s)} - m_\infty \Big|^2 dxds \\
&\le 2\int_0^1 \int_\Omega \Big| z_k(x,s) - \overline{z_k(\cdot,s)} \Big|^2 dxds + 2|\Omega| \cdot \sup_{s\in(0,1)} \Big| \overline{z_k(\cdot,s)} - m_\infty \Big|^2 \\
&\to 0 \qquad \text{as } k \to \infty
\end{aligned} \tag{4.18}$$

according to (4.16). By (4.15), however, we have

$$\begin{aligned}
\int_0^1 \int_\Omega \Big| z_k(x,s) - m_\infty \Big|^2 dxds &\ge \int_0^\delta \int_{B_r(x_k)\cap\Omega} \Big| z_k(x,s) - m_\infty \Big|^2 dxds \\
&\ge \frac{\varepsilon^2}{4} \cdot \delta \cdot |B_r(x_k)\cap\Omega| \qquad \text{for all } k \in \mathbb{N},
\end{aligned}$$

which contradicts (4.18), because the smoothness of $\partial\Omega$ ensures that $\inf_{k\in\mathbb{N}} |B_r(x_k)\cap\Omega|$ must be positive. □

4.4 Stabilization of n, c and u in $L^2(\Omega)$

For proving uniform stabilization of the components n, c and u, in view of corresponding precompactness features in $L^\infty(\Omega)$ asserted by Lemma 4.3 and Lemma 4.4 when combined with the Arzelá-Ascoli theorem it will be sufficient to derive the respective convergence properties with respect to the norm in $L^2(\Omega)$. This will be achieved separately for each of these components by means of some further testing procedures, where in drawing conclusions from the resultng ordinary differential inequalities we shall twice make use of the following elementary lemma guaranteeing decay in absorptive linear ODEs with inhomogeneities decaying in a certain average sense.

Lemma 4.6 *Let $y \in C^1([0,\infty))$ and $g \in C^0([0,\infty))$ be nonnegative functions satisfying*

$$y'(t) + \lambda y(t) \le g(t) \qquad \textit{for all } t > 0 \tag{4.19}$$

with some $\lambda > 0$. Then if

$$\int_t^{t+1} g(s)ds \to 0 \qquad \textit{as } t \to \infty, \tag{4.20}$$

we have

$$y(t) \to 0 \qquad \textit{as } t \to \infty. \tag{4.21}$$

PROOF. We abbreviate $B := \|g\|_{L^\infty((0,\infty))}$, and given $\varepsilon > 0$ we first fix an integer k large enough such that

$$\frac{B}{\lambda} \cdot e^{-\lambda k} < \frac{\varepsilon}{3}, \tag{4.22}$$

thereafter pick $\delta > 0$ suitably small fulfilling

$$k\delta < \frac{\varepsilon}{3} \tag{4.23}$$

and finally use (4.10) in choosing $t_0 > k$ large satisfying

$$y(0)e^{-\lambda t_0} < \frac{\varepsilon}{3} \qquad \text{and} \qquad \int_t^{t+1} g(s)ds < \delta \quad \text{for all } t \geq t_0 - k. \tag{4.24}$$

Then since by a comparison argument we have

$$\begin{aligned} y(t) &\leq y(0)e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} g(s)ds \\ &\leq y(0)e^{-\lambda t} + \int_0^{t-k} e^{-\lambda(t-s)} g(s)ds + \sum_{j=0}^{k-1} \int_{t-k+j}^{t-k+j+1} e^{-\lambda(t-s)} g(s)ds \qquad \text{for all } t > k, \end{aligned}$$

it follows from (4.24), (4.22) and (4.23) that for each $t > t_0$ we can estimate

$$\begin{aligned} y(t) &< \frac{\varepsilon}{3} + \int_0^{t-k} e^{-\lambda(t-s)} \cdot B ds + \sum_{j=0}^{k-1} \int_{t-k+j}^{t-k+j+1} g(s)ds \\ &= \frac{\varepsilon}{3} + \frac{B}{\lambda} \cdot \Big(e^{-\lambda k} - e^{-\lambda t} \Big) + \sum_{j=0}^{k-1} \int_{t-k+j}^{t-k+j+1} g(s)ds \\ &< \frac{\varepsilon}{3} + \frac{B}{\lambda} \cdot e^{-\lambda 1} + k\delta \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}. \end{aligned}$$

As y is nonnegative, this establishes (4.21). □

As a first application thereof, we can derive the following stabilization property of c, along with an additional decay information on ∇c which will be used in Lemma 4.8 below.

Lemma 4.7 *We have*

$$c(\cdot,t) \to \Big\{ \fint_\Omega m_0 - \fint_\Omega n_0 \Big\}_+ \qquad \text{in } L^2(\Omega) \quad \text{as } t \to \infty \tag{4.25}$$

and

$$\int_t^{t+1} \int_\Omega |\nabla c|^2 \to 0 \qquad \text{as } t \to \infty. \tag{4.26}$$

PROOF. Once more writing $m_\infty := \left\{ \fint_\Omega m_0 - \fint_\Omega n_0 \right\}_+$, we test the second equation in (1.3) against $c - m_\infty$ to find using Young's inequality that

$$\begin{aligned}
\frac{2}{2}\frac{d}{dt}\int_\Omega (c-m_\infty)^2 + \int_\Omega |\nabla c|^2 &= \int_\Omega (c-m_\infty)(-c+m) \\
&= -\int_\Omega (c-m_\infty)^2 + \int_\Omega (c-m_\infty)(m-m_\infty) \\
&\le -\frac{1}{2}\int_\Omega (c-m_\infty)^2 + \frac{1}{2}\int_\Omega (m-m_\infty)^2 \qquad \text{for all } t>0.
\end{aligned}$$

Therefore, $y(t) := \int_\Omega (c(\cdot,t)-m_\infty)^2$, $t \ge 0$, as well as $g(t) := 2\int_\Omega |\nabla c(\cdot,t)|^2$ and $h(t) := \int_\Omega (m(\cdot,t) - m_\infty)^2$, $t>0$, satisfy

$$y'(t) + y(t) + g(t) \le h(t) \qquad \text{for all } t>0, \tag{4.27}$$

so that since from Lemma 4.5 we know that

$$h(t) \to 0 \qquad \text{as } t \to \infty, \tag{4.28}$$

thanks to Lemma 4.6 and the nonnegativity of y and g this firstly implies that

$$y(t) \to 0 \qquad \text{as } t \to \infty \tag{4.29}$$

and that hence (4.25) holds. Secondly, an integration of (4.27) thereupon shows that as a consequence of (4.29) and (4.28) we have

$$\int_t^{t+1} g(s)ds \le y(t) + \int_t^{t+1} h(s)ds \to 0 \qquad \text{as } t \to \infty,$$

thus verifying (4.26). $\square$

Again making use of Lemma 4.6, thanks to the decay properties of $\int_\Omega nm$ and $\int_\Omega |\nabla c|^2$ provided by Corollary 4.1 and Lemma 4.7 we can now obtain convergence with respect to the norm in $L^2(\Omega)$ also of the crucial quantity n.

Lemma 4.8 *The first solution component satisfies*

$$n(\cdot,t) \to \left\{ \fint_\Omega n_0 - \fint_\Omega m_0 \right\}_+ \qquad \text{in } L^2(\Omega) \quad \text{as } t \to \infty. \tag{4.30}$$

PROOF. We use the first equation in (1.3) to compute

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega \Big(n(\cdot,t) - \overline{n(\cdot,t)}\Big)^2 &= \int_\Omega (n-\overline{n}) \cdot \Big(n_t - (\overline{n})_t\Big) \\
&= \int_\Omega (n-\overline{n}) \cdot \Big(n_t - (\overline{n})_t\Big) \\
&= \int_\Omega (n-\overline{n}) \cdot \Big(\Delta n - \nabla\cdot(n\nabla c) - nm - u\cdot\nabla n + \overline{nm}\Big) \\
&= -\int_\Omega |\nabla n|^2 + \int_\Omega n\nabla n\cdot\nabla c - \int_\Omega (n-\overline{n})\cdot nm + \int_\Omega (n-\overline{n})\cdot\overline{nm} \qquad (4.31)
\end{aligned}$$

for all $t>0$, because $\int_\Omega (n-\overline{n})=0$ and, again, because $\nabla\cdot u\equiv 0$. Here, using Young's inequality and the boundedness of n we can estimate

$$\begin{aligned}\int_\Omega n\nabla n\cdot\nabla c &\le \frac12\int_\Omega |\nabla n|^2+\frac12\int_\Omega n^2|\nabla c|^2\\ &\le \frac12\int_\Omega |\nabla n|^2+\frac{C_1^2}{2}\int_\Omega|\nabla c|^2 \qquad \text{for all } t>0\end{aligned}$$

with $C_1:=\|n\|_{L^\infty(\Omega\times(0,\infty))}$, whereas by nonnegativity of n and m, according to (2.4) we have

$$-\int_\Omega (n-\overline{n})\cdot nm\le \overline{n}\cdot\int_\Omega nm\le C_2\int_\Omega nm \qquad \text{for all } t>0$$

and similarly

$$\int_\Omega (n-\overline{n})\cdot\overline{nm}\le\int_\Omega n\cdot\overline{nm}\le C_2\int_\Omega nm \qquad \text{for all } t>0,$$

where $C_2:=\fint_\Omega n_0$. As the Poincaré inequality yields $C_3>0$ such that

$$\int_\Omega |\nabla n|^2\ge C_3\int_\Omega (n-\overline{n})^2 \qquad \text{for all } t>0,$$

from (4.31) we thus infer that

$$\frac{d}{dt}\int_\Omega (n-\overline{n})^2-C_3\int_\Omega (n-\overline{n})^2\le C_1^2\int_\Omega|\nabla c|^2+4C_2\int_\Omega nm \qquad \text{for all } t>0.$$

Since

$$\int_t^{t+1}\int_\Omega |\nabla c|^2\to 0 \qquad \text{as } t\to\infty$$

by Lemma 4.7, and since

$$\int_t^{t+1}\int_\Omega nm\to 0 \qquad \text{as } t\to\infty$$

due to Corollary 4.1, employing Lemma 4.6 we conclude that

$$\int_\Omega \Big(n(\cdot,t)-\overline{n(\cdot,t)}\Big)^2\to 0 \qquad \text{as } t\to\infty. \tag{4.32}$$

We now recall that according to Lemma 4.2 we have

$$\overline{n(\cdot,t)}\to n_\infty:=\Big\{\fint_\Omega n_0-\fint_\Omega m_0\Big\}_+ \qquad \text{as } t\to\infty,$$

which together with (4.32) implies that

$$\begin{aligned}\int_\Omega \Big(n(\cdot,t)-n_\infty\Big)^2 &\le 2\int_\Omega \Big(n(\cdot,t)-\overline{n(\cdot,t)}\Big)^2 + 2\int_\Omega \Big(\overline{n(\cdot,t)}-n_\infty\Big)^2 \\ &= 2\int_\Omega \Big(n(\cdot,t)-\overline{n(\cdot,t)}\Big)^2 + 2\Big(\overline{n(\cdot,t)}-n_\infty\Big)^2|\Omega| \\ &\to 0 \qquad \text{as } t\to\infty\end{aligned}$$

and thereby entails (4.31). □

Finally, with the above decay information on the source term in the Navier-Stokes system in (1.3) at hand, again by utilizing the corresponding energy inequality it is not difficult to derive temporal decay of u in $L^2(\Omega)$.

Lemma 4.9 *We have*

$$u(\cdot,t)\to 0 \qquad \textit{in } L^2(\Omega) \quad \textit{as } t\to\infty. \tag{4.33}$$

Proof. Again abbreviating $n_\infty := \Big\{ ⨍_\Omega n_0 - ⨍_\Omega m_0\Big\}_+$ and $m_\infty := \Big\{ ⨍_\Omega m_0 - ⨍_\Omega n_0\Big\}_+$, from the fourth equation in (1.3) we obtain the associated Navier-Stokes energy inequality in the form

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 &= \int_\Omega (n+m)u\cdot\nabla\phi \\ &= \int_\Omega (n-n_\infty)u\cdot\nabla\phi + \int_\Omega (m-m_\infty)u\cdot\nabla\phi \qquad \text{for all } t>0,\end{aligned} \tag{4.34}$$

because $\int_\Omega u\cdot\nabla\phi$ due to the solenoidality of u. Here by the Poincaré inequality, we can find $C_1>0$ such that

$$\int_\Omega |\nabla u|^2 \ge C_1\int_\Omega |u|^2 \qquad \text{for all } t>0,$$

and thereafter employing Young's ineqiality we see that

$$\begin{aligned}\int_\Omega (n-n_\infty)u\cdot\nabla\phi &\le \frac{C_1}{4}\int_\Omega |u|^2 + \frac{1}{C_1}\int_\Omega (n-n_\infty)^2|\nabla\phi|^2 \\ &\le \frac{C_1}{4}\int_\Omega |u|^2 + C_2\int_\Omega (n-n_\infty)^2 \qquad \text{for all } t>0,\end{aligned}$$

and that similarly

$$\int_\Omega (m-m_\infty)u\cdot\nabla\phi \le \frac{C_1}{4}\int_\Omega |u|^2 + C_2\int_\Omega (m-m_\infty)^2 \qquad \text{for all } t>0$$

with $C_2 := \frac{\|\nabla\phi\|^2_{L^\infty(\Omega)}}{C_1}$. In consequence, (4.34) implies that

$$\frac{d}{dt}\int_\Omega |u|^2 + C_1\int_\Omega |u|^2 \le 2C_2\int_\Omega (n-n_\infty)^2 + 2C_2\int_\Omega (m-m_\infty)^2 \qquad \text{for all } t>0,$$

so that since from Lemma 4.8 and Lemma 4.5 we know that

$$\int_\Omega \Big(n(\cdot,t)-n_\infty\Big)^2 \to 0 \quad \text{and} \int_\Omega \Big(m(\cdot,t)-m_\infty\Big)^2 \to 0 \qquad \text{as } t\to\infty,$$

Lemma 4.6 entails (4.33). □

4.5 Stabilization of n, c and u in $L^\infty(\Omega)$. Proof of Theorem 1.2

We now only need to collect the above convergence properties to complete the derivation of our main result on stabilization by means of a standard argument based on an Ehrling-type lemma.

Proof of Theorem 1.2. The claim (1.11) concerning m has precisely been derived in Lemma 4.5 already. The convergence properties in (1.9), (1.10) and (1.12), however, follow from the respective stabilization statements in $L^2(\Omega)$, as asserted by Lemma 4.8, Lemma 4.7 and Lemma 4.9 when combined with the compactness features provided by Lemma 4.4 and Lemma 4.3. In fact, after applying Lemma 4.4 to pick $\theta \in (0,1)$ and $C_1 > 0$ such that with $n_\infty := \Big\{ \fint_\Omega n_0 - \fint_\Omega m_0 \Big\}_+$ we have

$$\|n(\cdot,t)\|_{C^\theta(\bar\Omega)} \le C_1 \qquad \text{for all } t > 1, \tag{4.35}$$

given $\varepsilon > 0$ we may use the compactness of the first of the embeddings $C^\theta(\bar\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \hookrightarrow L^2(\Omega)$ to fix, through an associated Ehrling lemma, a constant $C_2 > 0$ such that

$$\|\varphi\|_{L^\infty(\Omega)} \le \frac{\varepsilon}{2C_1}\|\varphi\|_{C^\theta(\bar\Omega)} + C_2\|\varphi\|_{L^2(\Omega)} \qquad \text{for all } \varphi \in C^\theta(\bar\Omega). \tag{4.36}$$

Now since $n(\cdot,t) \to n_\infty$ in $L^2(\Omega)$ as $t \to \infty$ by Lemma 4.8, we may choose $t_0 > 1$ large enough such that

$$\|n(\cdot,t) - n_\infty\|_{L^2(\Omega)} < \frac{\varepsilon}{2C_2} \qquad \text{for all } t > t_0.$$

Combined with (4.36) and (4.35), this shows that in fact

$$\begin{aligned}
\|n(\cdot,t) - n_\infty\|_{L^\infty(\Omega)} &\le \frac{\varepsilon}{2C_1}\|n(\cdot,t) - n_\infty\|_{C^\theta(\bar\Omega)} + C_2\|n(\cdot,t) - n_\infty\|_{L^2(\Omega)} \\
&< \frac{\varepsilon}{2C_1}\cdot C_1 + C_2 \cdot \frac{\varepsilon}{2C_2} \\
&= \varepsilon \qquad \text{for all } t > t_0,
\end{aligned}$$

which implies (1.9), for $\varepsilon > 0$ was arbitrary. Likewise, (1.10) follows from Lemma 4.7 combined with Lemma 4.4, whereas Lemma 4.9 in conjunction with Lemma 4.3 entails (1.12). □

Acknowledgement. M. Winkler acknowledges support of the *Deutsche Forschungsgemeinschaft* in the context of the project *Analysis of chemotactic cross-diffusion in complex frameworks.*
E. Espejo acknowledges support of the Patrimonio Autónomo Fondo Nacional de Financiamiento Para la Ciencia, la Tecnología y la Innovación, Francisco José de Caldas and Universidad Nacional de Colombia (UN).

References

[1] Bellomo, N., Bellouquid, A., Tao, Y., Winkler, M.: *Toward a Mathematical Theory of Keller-Segel Models of Pattern Formation in Biological Tissues.* Math. Mod. Meth. Appl. Sci. **25**, 1663-1763 (2015)

[2] Cao, X., Lankeit, J.: *Global classical small-data solutions for a three-dimensional chemotaxis Navier-Stokes system involving matrix-valued sensitivities.* Calc. Var. Part. Differential Eq. **55**, Paper No. 107, 39 pp (2016)

[3] Chae, M., Kang, K., Lee, J.: *Global existence and temporal decay in Keller-Segel models coupled to fluid equations.* Comm. Part. Differ. Eq. **39**, 1205-1235 (2014)

[4] Chertock, A., Fellner, K., Kurganov, A., Lorz, A., Markowich, P.A.: *Sinking, merging and stationary plumes in a coupled chemotaxis-fluid model: a high-resolution numerical approach.* J. Fluid Mech. **694**, 155-190 (2012)

[5] Espejo, E.E., Suzuki, T.: *Reaction terms avoiding aggregation in slow fluids.* Nonlin. Anal. Real World Appl. **21**, 110-126 (2015)

[6] Espejo, E.E., Suzuki, T.: *Reaction enhancement by chemotaxis.* Nonlin. Anal. Real World Appl. **35**, 102-131 (2017)

[7] Dombrowski, C., Cisneros, L., Chatkaew, S., Goldstein, R.E., Kessler, J.O.: *Self-concentration and large-scale coherence in bacterial dynamics.* Phys. Rev. Lett. **93**, 098103-1-4 (2004)

[8] DiFrancesco, M., Lorz, A., Markowich, P.A.: *Chemotaxis-fluid coupled model for swimming bacteria with nonlinear diffusion: global existence and asymptotic behavior.* Discr. Cont. Dyn. Syst. A **28**, 1437-1453 (2010)

[9] Duan, R.J., Lorz, A., Markowich, P.A.: *Global solutions to the coupled chemotaxis-fluid equations.* Comm. Part. Differ. Eq. **35**, 1635-1673 (2010)

[10] Giga, Y.: *The Stokes operator in L_r spaces.* Proc. Japan Acad. S. **2**, 85-89 (1981)

[11] Giga, Y.: *Solutions for Semilinear Parabolic Equations in L_p and Regularity of Weak Solutions of the Navier-Stokes System.* J. Differential Eq. **61**, 186-212 (1986)

[12] Henry, D.: *Geometric theory of semilinear parabolic equations.* Lecture Notes in Mathematics. 840. Springer, Berlin-Heidelberg-New York, 1981

[13] Herrero, M.A., Velázquez, J.J.L.: *A blow-up mechanism for a chemotaxis model.* Ann. Scuola Normale Superiore Pisa **24**, 633-683 (1997)

[14] Hillen, T., Painter, K.J.: *A user's guide to PDE models for chemotaxis.* J. Math. Biol. **58**, 183-217 (2009)

[15] Kiselev, A., Ryzhik, L.: *Biomixing by chemotaxis and enhancement of biological reactions.* Comm. Part. Differ. Eq. **37** (1-3), 298-318 (2012)

[16] Kiselev, A., Ryzhik, L.: *Biomixing by chemotaxis and efficiency of biological reactions: the critical reaction case.* J. Math. Phys. **53** (11), 115609, 9 p. (2012)

[17] Kozono, H., Miura, M., Sugiyama, Y.: *Existence and uniqueness theorem on mild solutions to the Keller-Segel system coupled with the Navier-Stokes fluid.* J. Funct. Anal. **270**, 1663-1683 (2016)

[18] Lankeit, J.: *Long-term behaviour in a chemotaxis-fluid system with logistic source.* Math. Mod. Meth. Appl. Sci. **26**, 2071-2109 (2016)

[19] Liu, J.-G., Lorz, A.: *A Coupled Chemotaxis-Fluid Model: Global Existence.* Ann. Inst. Henri Poincaré, Anal. Non Linéaire **28** (5), 643-652 (2011)

[20] Lorz, A.: *Coupled chemotaxis fluid model.* Math. Mod. Meth. Appl. Sci. **20**, 987-1004 (2010)

[21] Lorz, A.: *Coupled Keller-Segel-Stokes model: global existence for small initial data and blow-up delay.* Comm. Math. Sci. **10**, 555-574 (2012)

[22] Nagai, T.: *Blowup of Nonradial Solutions to Parabolic-Elliptic Systems Modeling Chemotaxis in Two-Dimensional Domains.* J. Inequal. Appl. **6**, 37-55 (2001)

[23] Osaki, K., Tsujikawa, T., Yagi, A., Mimura, M.: *Exponential attractor for a chemotaxis-growth system of equations.* Nonlin. Anal. Theory Meth. Appl. **51**, 119-144 (2002)

[24] Porzio, M.M., Vespri, V.: *Holder Estimates for Local Solutions of Some Doubly Nonlinear Degenerate Parabolic Equations.* J. Differential Equations **103**, 146-178 (1993)

[25] Strohm, S., Tyson, R.C., Powell, J.A.: *Pattern formation in a model for mountain pine beetle dispersal: linking model predictions to data.* Bull. Math. Biol. **75**, 1778-1797 (2013)

[26] Sohr, H.: *The Navier-Stokes Equations. An Elementary Functional Analytic Approach.* Birkhäuser, Basel, 2001

[27] Tao, Y., Winkler, M.: *Boundedness and decay enforced by quadratic degradation in a three-dimensional chemotaxis-fluid system.* Z. Angew. Math. Phys. **66**, 2555-2573 (2015)

[28] Tao, Y., Winkler, M.: *Blow-up prevention by quadratic degradation in a two-dimensional Keller-Segel-Navier-Stokes system.* Z. Angew. Math. Phys. **67**, Art. 138, 23 pp (2016)

[29] Tao, Y., Winkler, M.: *Critical mass for infinite-time aggregation in a chemotaxis model with indirect signal production.* J. Eur. Math. Soc., to appear

[30] Tuval, I., Cisneros, L., Dombrowski, C., Wolgemuth, C.W., Kessler, J.O., Goldstein, R.E.: *Bacterial swimming and oxygen transport near contact lines.* Proc. Nat. Acad. Sci. USA **102**, 2277-2282 (2005)

[31] Wang, Y., Winkler, M., Xiang, Z.: *Global classical solutions in a two-dimensional chemotaxis-Navier-Stokes system with subcritical sensitivity.* Ann. Scu. Norm. Sup. Pisa Cl. Sci., to appear

[32] Wang, Y., Xiang, Z.: *Global existence and boundedness in a Keller-Segel-Stokes system involving a tensor-valued sensitivity with saturation.* J. Differential Eq. **259**, 7578-7609 (2015)

[33] Wang, Y., Xiang, Z.: *Global existence and boundedness in a Keller-Segel-Stokes system involving a tensor-valued sensitivity with saturation: The 3D case.* J. Differ. Eq. **261**, 4944-4973 (2016)

[34] Winkler, M.: *Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source.* Comm. Part. Differ. Eq. **35**, 1516-1537 (2010)

[35] Winkler, M.: *Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model.* J. Differential Equations **248**, 2889-2905 (2010)

[36] Winkler, M.: *Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops.* Comm. Part. Differ. Eq. **37**, 319351 (2012)

[37] Winkler, M: *Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system.* Journal de Mathématiques Pures et Appliquées **100**, 748-767 (2013), `arXiv:1112.4156v1`

[38] Winkler, M.: *Stabilization in a two-dimensional chemotaxis-Navier-Stokes system.* Arch. Ration. Mech. Anal. **211** (2), 455-487 (2014)

[39] Winkler, M.: *How far can chemotactic cross-diffusion enforce exceeding carrying capacities?* J. Nonlinear Sci. **24**, 809-855 (2014)

[40] Winkler, M.: *Global weak solutions in a three-dimensional chemotaxis-Navier-Stokes system.* Ann. Inst. Henri Poincaré, Anal. Non Linéaire **33**, 1329-1352 (2016)

[41] Winkler, M.: *Boundedness and large time behavior in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion and general sensitivity.* Calc. Var. Part. Differ. Eq. **54**, 3789-3828 (2015)

[42] Winkler, M.: *How far do oxytaxis-driven forces influence regularity in the Navier-Stokes system?* Trans. Amer. Math. Soc. **369**, 3067-3125 (2017)

[43] Zhang, Q., Li, Y.: *Convergence rates of solutions for a two-dimensional chemotaxis-Navier-Stokes system.* Discr. Cont. Dyn. Syst. B **20**, 2751-2759 (2015)