Singular structure formation in a degenerate haptotaxis model involving myopic diffusion

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Abstract

We consider the system

$$\begin{cases} u_t = \big(d(x)u\big)_{xx} - \big(d(x)uw_x\big)_x, \\ w_t = -ug(w), \end{cases} \tag{0.1}$$

which arises as a simple model for haptotactic migration in heterogeneous environments, such as typically occurring in the invasive dynamics of glioma. A particular focus is on situations when the diffusion herein is degenerate in the sense that the zero set of d is not empty.

It is shown that if such possibly present degeneracies are sufficiently mild in the sense that

$$\int_\Omega \frac{1}{d} < \infty, \tag{0.2}$$

then under appropriate assumptions on the initial data a corresponding initial-boundary value problem for (0.1), posed under no-flux boundary conditions in a bounded open interval $\Omega \subset \mathbb{R}$, possesses at least one globally defined generalized solution.

Moreover, despite such degeneracies the myopic diffusion mechanism in (0.1) is seen to asymptotically determine the solution behavior in the sense that for some constant $\mu_\infty > 0$, the obtained solution satisfies

$$u(\cdot,t) \rightharpoonup \frac{\mu_\infty}{d} \quad \text{in } L^1(\Omega) \qquad \text{and} \qquad w(\cdot,t) \to 0 \quad \text{in } L^\infty(\Omega) \qquad \text{as } t \to \infty, \tag{0.3}$$

and that hence in the degenerate case the solution component u stabilizes toward a state involving infinite densities, which is in good accordance with experimentally observed phenomena of cell aggregation.

Finally, under slightly stronger hypotheses inter alia requiring that $\frac{1}{d}$ belong to $L\log L(\Omega)$, a substantial effect of diffusion is shown to appear already immediately by proving that for a.e. $t > 0$, the quantity $\ln(du(\cdot,t))$ is bounded in Ω. In degenerate situations, this particularly implies that the blow-up phenomena expressed in (0.3) in fact occur instantaneously.

Keywords: haptotaxis; degenerate diffusion; global existence; large time behavior; blow-up
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1 Introduction

In the theoretical description of collective cell behavior at macroscopic scales, taxis mechanisms have been playing an increasingly substantial role ([20]). In the past two decades, an accordingly growing literature on mathematical analysis of such processes has brought about quite a thorough knowledge of various classes of corresponding PDE models, containing cross-diffusive parabolic equations as their most characteristic ingredient, especially in situations when the attractive signal is a chemical and hence diffusible (see [2] for a recent survey). Unlike such chemotaxis systems, considerably less understood seem so-called haptotaxis systems which substantially differ from the former in that they address cases of non-diffusible cues, as naturally involved when tumors invade healthy tissue. Moreover, virtually all analytical studies on taxis systems assume that random movement of cells is of Fickian diffusion type, either linear or nonlinear, with few exceptions considering fractional diffusion chemotaxis models ([6], [7]). Recent modeling approaches, however, indicate that in situations of significantly heterogeneous environments, adequate macroscopic limits of random walks based on individually local sensing rather lead to certain non-Fickian diffusion operators ([4], [14]).

The main focus of the present work is on the question how far the latter concept, in the literature also referred to as *myopic diffusion* ([4]), can rigorously be proved appropriate for the description of spontaneous structure generation in the context of simple haptotaxis systems in heterogeneous environments. We thereby intend to provide some analytical evidence for heuristic reasonings ([4]) suggesting that in contrast to those based on Fickian diffusion, this modeling framework may indeed much more accurately describe the emergence of neighborhood-adapted structures in such populations of myopic individuals, with aggregation phenomena of glioma near thin interfaces between white and grey matter in mouse brains forming a corresponding experimental observation of particular importance ([8]).

To this end, we will consider a particular version of an evolution system recently proposed as a model for the description of glioma spread in heterogeneous tissue ([14]), for mathematical purposes simplified in that any proliferation effects are neglected and that the spatial setting is assumed to be one-dimensional. Specifically, we shall be concerned with the initial-boundary value problem

$$\begin{cases} u_t = (d(x)u)_{xx} - (d(x)uw_x)_x, & x \in \Omega,\ t > 0, \\ w_t = -ug(w), & x \in \Omega,\ t > 0, \\ (d(x)u)_x - d(x)uw_x = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

for the unknown cell density $u = u(x,t)$ and the density $w = w(x,t)$ of tissue fibers acting as a haptotactic cue in a bounded open interval $\Omega \subset \mathbb{R}$, with given nonnegative functions d, u_0 and w_0 on $\overline{\Omega}$ and g generalizing the prototypical choice $g(s) = s$, $s \geq 0$, in a sense to be specified in (1.7) and (1.8) below. The formal parabolic limit procedure performed in [14, Section 3.1], adequately accounting for the influence of the underlying tissue structure on tumor cell movement, led to the above concrete form of the macroscopic equations featuring myopic diffusion and haptotaxis. Both these types of terms in their respective coefficient functions, involve the so-called tumor diffusion tensor explicitly deduced e.g. in [14, Formula (3.11)]. In the latter reference, the distribution of the tissue density is assessed from medical data and plays the role of an input to the equation for the space-time evolution of the tumor cell population. When the tissue dynamics is taken into account, as done through the second equation in (1.4), then the mathematical analysis of the resulting system becomes challenging, the more so in situations with possibly degenerate diffusion, which can indeed occur during the migration of glioma through the tissue, either when the latter is locally too dense and isotropic, thus impairing the spread of cells which have to overcome it, or too sparse, which in turn is hindering the spread of cells, as they have to rely on it both for migration and proliferation. In this work we therefore concentrate on the one-dimensional version of the system obtained in [14], which correspondingly uses the same motility coefficient function $d = d(x)$ in both the diffusive and the advective terms in (1.4) and allow this function to degenerate.

We moreover note that as can readily be verified on substituting $w = \Psi(v) := \int_0^v \psi(\xi)d\xi$ and $g(w) = \Psi^{-1}(w) \cdot \psi(\Psi^{-1}(w))$, for arbitrary smooth positive $\psi : [0,\infty) \to \mathbb{R}$ this thereby implicitly includes solutions with sufficiently small component v of the respective initial-boundary value problem for

$$\begin{cases} u_t = (d(x)u)_{xx} - (d(x)u\psi(v)v_x)_x, & x \in \Omega,\ t > 0, \\ v_t = -uv, & x \in \Omega,\ t > 0, \end{cases}$$

where the choice $\psi(v) = \frac{1}{(1+v)^2}$ corresponds to the particular tumor invasion model recently analyzed in [51].

In the case $d \equiv 1$ representing spatially homogeneous conditions for both diffusion and cross-diffusion, (1.4) reduces to the apparently simplest reasonable model for haptotactic interaction ([26]), containing the essential aspects of several more complex systems that have been discussed in the modeling literature ([31], [9], [10]; cf. also [3]) and also analyzed analytically. Beyond statements on global existence in various functional frameworks (see [45], [13] and [28] for some classical and e.g. [37] as well as [34] for more recent examples) and scattered results on boundedness ([29], [42], [18]), however, even in this non-degenerate and homogeneous setting a detailed description of further qualitative facets such as the large time behavior could be established only in very particular cases up to now; moreover, apparently all available results in this direction are either restricted to solutions suitably close to equilibria ([16], [18]), or to situations when a strongly dissipative action of additional logistic-type cell kinetic terms can be shown to dominate on large time scales ([27], [46], [43], [42], [21]), meaning that in the latter cases solutions exclusively stabilize toward spatially homogeneous and hence unstructured equilibria. This lack of rigorous knowledge in situations of expectedly more colorful solution behavior may be viewed as reflecting the circumstance that unless suitably compensated by further mechanisms, tactic cross-diffusion of the form in (1.4) may substantially affect the regularity of solutions and hence obstruct mathematical analysis at various stages. This strongly destabilizing potential is well-known from various findings detecting unboundedness phenomena especially in self-reinforced taxis models, even in apparently more regular settings determined by cross-diffusive interaction with a diffusible quantity such as in the classical Keller-Segel chemotaxis system and derivates thereof ([19], [30], [48], [24]), already in some spatially one-dimensional scenarios ([23], [49]), but also in some models for tactic migration toward non-diffusible attractants ([25], [32]).

Main results. In the presently considered context of the model (1.4), our analysis will reveal that under appropriate assumptions inter alia requiring mildness of possible degeneracies in diffusion, such types of taxis-driven collapse do not occur, but that the solution behavior is rather essentially prearranged by the environmental conditions. Indeed, our main results will show that for a large class of initial data, certain global generalized solutions can be constructed which in the large time limit approach a positive multiple of the reciprocal myopic diffusion coefficient $\frac{1}{d}$ in their first component, as predicted in [4]; in particular, this reflects asymptotic aggregation of cells in regions where d is small, in presence of zeros of d even in the mathematically extreme sense of stabilization toward a singular state. Beyond this asymptotic statement, we will identify a solution property that indicates a certain predominance of the diffusion process in (1.4) already at intermediate and even small time scales: Namely, we shall see that under slightly stronger assumptions, for a.e. $t > 0$ the quantity $du(\cdot, t)$ is bounded from above and below in Ω by positive constants only depending on t. This firstly ensures local boundedness of $u(\cdot, t)$ inside the positivity set of d and hence rules out any significant taxis-forced aggregation; secondly, and more drastically, however, this implies that singularities near points of degenerate diffusion, according to the above arising at least in the long-term limit, in fact emerge instantaneously.

In order to formulate these results more precisely, let us specify the framework to be considered henceforth by assuming $d \in C^0(\overline{\Omega})$ to be nonnegative and such that

$$d \in C^1(\{d > 0\}), \tag{1.5}$$

as well as

$$\int_\Omega \frac{1}{d} < \infty, \tag{1.6}$$

where $\{d > 0\} := \{x \in \overline{\Omega} \mid d(x) > 0\}$, with this and similar notation frequently being used throughout the sequel without further explicit definition. We observe that (1.6) in particular requires the set of all zeros of d to be a null set of points, thus inter alia excluding situations when diffusion may become degenerate throughout entire subintervals of Ω. In application contexts, this corresponds to limiting situations of small interfacial layers of inhibited diffusion, such as typically occurring in the mentioned framework of glioma spread addressed in [4]. Mathematically, it may be noted that at least formally, (1.4) would predict temporal constancy of u inside the interior of such degeneracy regions; a partial rigorous justification thereof has recently been achieved in [35]. Thinking of the particular problem setting of glioma invasion, let us recall that the tumor diffusion tensor obtained during the macroscopic scaling process in [14] is proportional to the water diffusion tensor assessed by diffusion tensor imaging (cf. also [11] and [36] for independently obtained similar links); accordingly, in the one-dimensional framework at hand the corresponding scalar coefficient function d is also supposed to be tightly related to the diffusivity of water molecules. Thereby, sharp intersections of the one-dimensional diffusion direction of water molecules by tissue fibers which are very thin, single objects, at those sites lead to essentially single-point degeneracies in the diffusion of water molecules and, the more so, of tumor cells. In addition, (1.6) implicitly requires that d grows suitably fast near its zeros, in the prototypical case when $d(x) = |x - x_0|^\theta$ for all $x \in \overline{\Omega}$, some $x_0 \in \overline{\Omega}$ and some $\theta > 0$ reducing to the hypothesis that $\theta < 1$. Biologically, this corresponds to situations in which the diffusivity undergoes a rapid enhancement in the immediate proximity of the sites of degeneration, e.g., where it was 'blocked' by the fibers ([14]); further indications for the occurrence of such sudden increases in diffusivity at interfaces is provided by experimental evidence reporting that the diffusivity in white brain matter is much higher than in grey matter and leads to differences in cell motility 5-25 times higher in white than in grey matter (see e.g. [4] and the references therein). Besides their biological plausibility, these assumptions will also serve technical purposes that will become evident in the discussion below, e.g. around the formulation of Theorem 1.2.

As for the signal absorption coefficient function in (1.4), we shall suppose that $g \in C^2([0,\infty))$ is such that $g(0)=0$ and that with some positive constants $\underline{\gamma}$ and $\overline{\gamma}$ we have

$$\underline{\gamma} \le g'(s) \le \overline{\gamma} \qquad \text{for all } s \ge 0 \tag{1.7}$$

and hence also

$$\underline{\gamma} s \le g(s) \le \overline{\gamma} s \qquad \text{for all } s \ge 0, \tag{1.8}$$

and the intial data are required to be such that

$$\begin{cases} 0 \le u_0 \in C^0(\overline{\Omega}) \text{ satisfies } u_0 \not\equiv 0, \quad \text{and that} \\ 0 \le w_0 \in C^0(\overline{\Omega}) \ \text{ is such that } \sqrt{w_0} \in W^{1,2}(\Omega). \end{cases} \tag{1.9}$$

Within this setting, the first of our main results establishes global existence of a solution to (1.4) under an appropriate additional condition requiring a certain smallness property of w_0 near zeros of d. We emphasize already here that due to our mild assumptions on d, in view of the statement on instantaneous blow-up formulated in Theorem 1.3 we can in general not expect boundedness of the first solution component with respect to the norm in $L^p(\Omega)$ for any $p>1$, not even locally in time, so that our notion of solution needs to be adequately adapted to this circumstance. After all, our analysis will reveal that it is not necessary to resort to concepts involving measure-valued solutions, but that it is rather possible to construct solutions with their first component belonging to the space $C^0_w([0,\infty);L^1(\Omega))$ of $L^1(\Omega)$-valued functions defined on $[0,\infty)$ which are continuous with respect to the weak topology in $L^1(\Omega)$.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}$ be a bounded interval, and suppose that $d \in C^0(\overline{\Omega})$ is nonnegative and such that (1.5) and (1.6) hold. Moreover, let $g \in C^2([0,\infty))$ be such that $g(0)=0$ and that (1.7) is valid with some $\underline{\gamma}>0$ and $\overline{\gamma}>0$. Then for all initial data u_0 and w_0 which satisfy (1.9) and which are such that furthermore*

$$\int_\Omega \frac{d_x^2}{d} w_0 < \infty, \tag{1.10}$$

there exists at least one pair (u,w) of nonnegative functions

$$\begin{cases} u \in C^0_w([0,\infty);L^1(\Omega)) \cap L^\infty((0,\infty);L^1(\Omega)), \\ w \in C^0(\overline{\Omega}\times[0,\infty)) \cap L^\infty(\Omega\times(0,\infty)) \cap L^1_{loc}([0,\infty);W^{1,1}(\Omega)), \end{cases} \tag{1.11}$$

which form a global weak solution of (1.4) in the sense of Definition 2.1, and for which we have

$$\int_\Omega u(\cdot,t) = \int_\Omega u_0 \qquad \textit{for all } t>0. \tag{1.12}$$

Next, our main result concerning qualitative behavior in (1.4) asserts that in the large time limit, each of these solutions approaches a steady state of (1.4). Here since the nonnegative equilibria of (1.4) are precisely the pairs $(\frac{\mu}{d},0)$ with $\mu \ge 0$, in light of the mass conservation property (1.12) this a posteriori underlines the crucial role of our overall integrability assumption (1.6) for this central result.

Theorem 1.2 *Suppose that the assumptions of Theorem 1.1 are fulfilled. Then the global generalized solution (u, w) of (1.4) obtained in Theorem 1.1 satisfies*

$$u(\cdot, t) \rightharpoonup \frac{\mu_\infty}{d} \quad \text{in } L^1(\Omega) \qquad \text{as } t \to \infty \tag{1.13}$$

and

$$w(\cdot, t) \to 0 \quad \text{in } L^\infty(\Omega) \qquad \text{as } t \to \infty \tag{1.14}$$

with the positive number

$$\mu_\infty := \frac{\int_\Omega u_0}{\int_\Omega \frac{1}{d}}. \tag{1.15}$$

We note that in presence of zeros of d, (1.13) actually asserts that the quantity u undergoes a certain blow-up phenomenon at least in the large time limit. We finally make sure that this explosion actually occurs immediately and persistently, provided that diffusion is slightly less degenerate than admitted in Theorem 1.1, and that $\frac{w_0}{d}$ is bounded. In fact, the following states that under these hypotheses, the regularizing action of diffusion is strong enough, both relatively to haptotaxis and absolutely, so as to allow for the conclusion that, at least in an appropriate weakened form, the quantity du enjoys properties of instantaneous positivity and boundedness well-known for solutions of the heat equation.

Theorem 1.3 *Assume that in addition to the hypotheses of Theorem 1.1,*

$$\int_\Omega \frac{1}{d} \ln \frac{1}{d} < \infty \tag{1.16}$$

and

$$\frac{w_0}{d} \in L^\infty(\Omega). \tag{1.17}$$

Then the global generalized solution (u, w) of (1.4) from Theorem 1.1 has the property that

$$\int_\tau^T \left\| \ln \Big(du(\cdot, t) \Big) \right\|_{L^\infty(\Omega)}^3 dt < \infty \qquad \text{for all } T > 0 \text{ and } \tau \in (0, T). \tag{1.18}$$

In particular, for a.e. $t > 0$ there exist $C_1(t) > 0$ and $C_2(t) > 0$ such that

$$\frac{C_1(t)}{d(x)} \le u(x, t) \le \frac{C_2(t)}{d(x)} \qquad \text{for a.e. } x \in \Omega, \tag{1.19}$$

and if $\frac{1}{d} \notin L^\infty(\Omega)$, then

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty \qquad \text{for a.e. } t > 0. \tag{1.20}$$

The above results seem to go beyond previous knowledge even in cases when haptotactic interaction is neglected e.g. by formally setting $w \equiv 0$ in (1.4). In the non-degenerate version of the correspondingly obtained linear diffusion problem, that is, when $d > 0$ in $\overline{\Omega}$, global existence of classical solutions, smoothly approaching the steady state in (1.13), can readily be established by standard methods. As for degenerate limit cases thereof, a result on global existence of certain very weak solutions, as well as on their stabilization toward an associated singular equilibrium, can be found in [22]. A very early caveat indicating criticality of the assumption (1.6) goes back to [15], where it is shown that if the diffusion degeneracy is slightly stronger in that $d(x) = x$ in $\Omega = (0,\infty)$, then prescribing boundary conditions at $x = 0$ in the resulting simple equation $u_t = (xu)_{xx}$ is meaningless in the sense that solutions to the initial-value problem therefor are uniquely determined already by their prescribed (reasonably regular) initial data.

Main ideas. Our analysis is rooted in the observation that in the context of non-degenerate and suitably regular diffusion, a supposedly given smooth solution to (1.4) satisfies the energy inequality

$$\frac{d}{dt}\bigg\{\int_\Omega u\ln(du) + \frac{1}{2}\int_\Omega d\frac{w_x^2}{g(w)} + \frac{\overline{\gamma}}{\underline{\gamma}^2}\int_\Omega \frac{d_x^2}{d}w\bigg\} + \int_\Omega \frac{(du)_x^2}{du} + \frac{\underline{\gamma}}{4\overline{\gamma}}\int_\Omega du\frac{w_x^2}{w} \le 0 \tag{1.21}$$

where our hypothesis that $\int_\Omega \frac{1}{d}$ be finite warrants that the Lyapunov functional therein is bounded from below (cf. Lemma 3.5). Thus generalizing the corresponding identity for the special case $d \equiv const.$, as already observed in [13] and frequently adapted to various related cases involving spatially homogeneous diffusion (cf. [34] for a recent even quite complex example), (1.21) contains in its dissipated part, as a main novel ingredient, the fraction $\frac{d_x^2}{d}$ which our assumption (1.6) enforces to have infinite integral around each zero of d (see Lemma 2.3). Mainly due to this circumstance, considerable efforts will be undertaken in Section 2 to carefully design a sequence of regularized problems, indexed by a small positive parameter ε, that will involve nondegenerate diffusion in the respective first equation as well as a parabolic approximation of the second equation in (1.4), and at the core of which the construction of suitable approximations d_ε and $w_{0\varepsilon}$ to d and w_0, respectively, is guided by the intention to remain basically consistent with the structure expressed in (1.21). In Section 3 this will enable us to obtain an approximate counterpart of (1.21) and derive correspondingly implied a priori estimates for the respective solutions $(u_\varepsilon, w_\varepsilon)$ in the central Lemma 3.5, inter alia containing a regularized variant of the global dissipation property

$$\int_0^\infty \int_\Omega \frac{(du)_x^2}{du} < \infty \tag{1.22}$$

formally resulting from (1.21). By means of standard testing procedures, in Section 4 these will be seen to entail further regularity properties, now possibly ε-dependent, which enable us to extend each of these approximate solutions so as to exist globally.

Beyond some local-in-time estimates for $u_{\varepsilon x}$ and $w_{\varepsilon t}$, Section 5.1 will thereafter reveal two key regularity features, namely firstly uniform integrability of u_ε and of $w_{\varepsilon x}$ with respect to both the time variable and the approximation parameter (Lemma 5.1 and Lemma 5.2), and secondly an approximate analogue of the relaxation property

$$\int_0^\infty \|u_t(\cdot,t)\|^2_{W^{1,\infty}(\Omega))^\star} dt < \infty \tag{1.23}$$

formally implied by (1.21) (Lemma 5.4). Along with a crucial strong L^2 compactness property of the first factor $\sqrt{d_\varepsilon}u_\varepsilon$ in the corresponding cross-diffusive flux (Lemma 6.3 and Lemma 6.4), these will allow for constructing a solution to (1.4) through an appropriate extraction procedure based on straightforward compactness arguments (Section 6), and thus for proving Theorem 1.1 (Section 7).

Section 8 will then be devoted to the derivation of the stabilization results in Theorem 1.2, where first concentrating on the solution component u we will make essential use of the weak decay information implicitly contained in (1.22) and (1.23), as well as a now evident equi-integrability feature of $(u(\cdot,t))_{t>0}$ (Sections 8.1-8.3). Thereafter, the fact that thus u approaches a positive limit will be combined with the equicontinuity of $(w(\cdot,t))_{t>0}$, as implied by the above, to verify that the decreasing quantity $\|w(\cdot,t)\|_{L^\infty(\Omega)}$ must actually decay (Section 8.4).

Finally, Section 9 provides a proof of Theorem 1.3, with a key step consisting in deriving an estimate of the form

$$\int_0^T \int_\Omega w_{\varepsilon x}^2 \le C(T), \qquad T > 0, \tag{1.24}$$

(Lemma 9.3), used to control the right-hand side in the regularized analogue of

$$\frac{d}{dt} \int_\Omega \frac{1}{d} \ln u \ge \frac{1}{2} \int_\Omega \frac{(du)_x^2}{(du)^2} - \frac{1}{2} \int_\Omega w_x^2 \tag{1.25}$$

adequately (Lemma 9.8). For smooth solutions, (1.24) would trivially result as a by-product of (1.21) due to the evident fact that as a consequence of (1.4) and the assumptions in Theorem 1.3, $\frac{d}{w}$ would have a positive lower bound, and hence would $\frac{d}{g(w)}$ by (1.8). Due to positivity of w_ε enforced by artificial diffusion, however, a corresponding upper bound for $\frac{w_\varepsilon}{d_\varepsilon}$ seems available only in certain L^p spaces, with the integrability power p herein fortunately increasing with decreasing ε, however (Lemma 9.2). Therefore, (1.24) can be obtained by means of a subtle interpolation argument (Lemma 9.3) involving an additional regularity information on $w_{\varepsilon x}$ which stems from the artificially introduced dissipation and is thus of higher order, but singular with respect to ε (Lemma 9.1).

Before going into details, let us remark that due to the delicate coupling of diffusion and haptotactic cross-diffusion in (1.4), in the general framework determined by our conditions and especially by (1.6) we do not expect solutions to possess spatially global regularity properties substantially beyond those obtained by our analysis, as already discussed above in the context of Theorem 1.1. An interesting question going beyond the scope of the present work consists in describing possible further regularity aspects inside the positivity region of d where in the purely diffusive case when $w \equiv 0$, standard parabolic theory essentially provides smoothness up to an extent determined by the smoothness of d and u_0. After all, a subsequent study in this direction will inter alia show that imposing the slightly stronger assumption $\int_\Omega \frac{1}{d^2} < \infty$ on the behavior of d near its zeros ensures that the quantity du remains bounded in $L^p(\Omega)$ for any $p \in (1,\infty)$, that locally in $\{d>0\} \times (0,\infty)$ the function u itself is even Hölder continuous, and that the convergence in (1.13) in fact is locally uniform in $\{d>0\}$ ([33]).

2 Approximation of (1.4) by a family of regularized problems

2.1 A weak solution concept

To begin with, let us specify our generalized solution concept in order to substantiate the goal to be pursued in the context of our existence analysis.

Definition 2.1 *A pair (u,w) of nonnegative functions*

$$\begin{cases} u \in L^1_{loc}(\overline{\Omega} \times [0,\infty)), \\ w \in L^\infty_{loc}(\overline{\Omega} \times [0,\infty)) \cap L^1_{loc}([0,\infty); W^{1,1}(\Omega)) \end{cases} \tag{2.1}$$

satisfying

$$duw_x \in L^1_{loc}(\overline{\Omega} \times [0,\infty)) \tag{2.2}$$

will be called a global weak solution *of (1.4) if*

$$-\int_0^\infty \int_\Omega u\varphi_t - \int_\Omega u_0\varphi(\cdot,0) = \int_0^\infty \int_\Omega du\varphi_{xx} + \int_0^\infty \int_\Omega duw_x\varphi_x \tag{2.3}$$

for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$ *such that* $\varphi_x = 0$ *on* $\partial\Omega \times (0,\infty)$ *and*

$$\int_0^\infty \int_\Omega w\varphi_t + \int_\Omega w_0\varphi(\cdot,0) = \int_0^\infty \int_\Omega ug(w)\varphi \tag{2.4}$$

for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$.

2.2 Construction of energy-compatible sequences approximating d and w_0

A natural first step in the construction of globally defined functions solving (1.4) in the above sense consists in considering appropriately regularized problems. In order to allow for classical solvability, the latter should in particular involving non-degenerate diffusion in the respective cruicial first equation; as smooth solvability furthermore seems to require second-order spatial differentiability of the haptoattractant therein, apart from that a certain smoothness-enforcing regularization in the second equation appears to be in order. In the context of the questions addressed here, however, nearby approaches based e.g. on straightforward introduction of artificial non-degenerate diffusion in both sub-problems of (1.4) apparently need to face two essential challenges: Firstly, our assumption (1.6) of suitably weak degeneracy implicitly forces d to be non-smooth near possible zeros; in particular, the function d_x appearing as a coefficient in the divergence-like reformulation of the diffusion operator $(du)_{xx} = (du_x + d_xu)_x$ need not belong to any of the spaces $L^p(\Omega)$ for $p > 1$; accordingly, for guaranteeing the existence of suitably smooth solutions to our regularized problems it seems adequate to approximate d by appropriate functions each of which, beyond being strictly positive, is sufficiently regular. Secondly, and more drastically, in view of our goal to exploit the energy structure (1.21) formally associated with (1.4), unlike in situations when only global solvability is strived for ([35]) our design of regularization will be restricted to approximate problems which are essentially consistent with this structure. Here, in view of a considerably strong singularity of $\frac{d_x^2}{d}$ necessarily appearing near any zero of d (Lemma 2.3), a particularly crucial role will be played by the last integral $\int_\Omega \frac{d_x^2}{d}w$ arising in the Lyapunov functional in (1.21), especially at the initial time where it seems far from obvious how far our mere assumptions in (1.9) and (1.10) may warrant boundedness of the respective expression when d is replaced by approximate variants; accordingly, our regularization procedure will moreover include a suitable modification of w_0 near zeros of d.

In order to adequately cope with both these challenges, in this section we describe a possible construction of a sequence of approximate versions of (1.4), indexed by a small parameter $\varepsilon \in (0,1)$ which will eventually be restricted so as to run along an appropriately chosen decreasing sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ (see Lemma 2.6). In order to avoid abundant technicalities at this stage, we postpone details of the corresponding analysis to an appendix below.

As a first step within our procedure, we will make sure that d can monotonically be approximated by a family of smooth positive functions d_ε with convenient further properties.

Lemma 2.2 *Suppose that $d \in C^0(\overline{\Omega})$ is such that (1.5) holds. Then there exists a family $(d_\varepsilon)_{\varepsilon\in(0,1)} \subset C^\infty(\overline{\Omega})$ with the properties that as $\varepsilon \searrow 0$ we have*

$$d_\varepsilon \to d \qquad \text{in } L^\infty(\Omega) \tag{2.5}$$

and

$$d_{\varepsilon x} \to d_x \qquad \text{in } L^\infty_{loc}(\{d>0\}\cap\Omega) \text{ and in } L^p_{loc}(\{d>0\}) \text{ for all } p\in[1,\infty), \tag{2.6}$$

that

$$d_\varepsilon \le d_{\varepsilon'} \quad \text{in } \Omega \text{ whenever } 0<\varepsilon\le\varepsilon'<1, \tag{2.7}$$

that for all $\varepsilon \in (0,1)$ we have $d_\varepsilon > 0$ in $\overline{\Omega}$,

$$d_{\varepsilon x} = 0 \qquad \text{on } \partial\Omega \tag{2.8}$$

and

$$d_\varepsilon \le \|d\|_{L^\infty(\Omega)} + 1 \qquad \text{in } \Omega, \tag{2.9}$$

and such that

$$\varepsilon^2 \int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon^3} \le 1 \tag{2.10}$$

and

$$\sqrt{\varepsilon} \int_\Omega \frac{d_{\varepsilon x}^4}{d_\varepsilon^2} \le 1 \tag{2.11}$$

as well as

$$\varepsilon^{\frac14} \cdot \frac{1}{\inf_{x\in\Omega} d_\varepsilon(x)} \le 1 \tag{2.12}$$

and

$$\varepsilon^{\frac14} \cdot \Big\| \frac{d_{\varepsilon x}}{d_\varepsilon} \Big\|_{L^\infty(\Omega)} \le 1 \tag{2.13}$$

for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$.

In view of (2.5) and (2.6), taking $\varepsilon \searrow 0$ in the expression $\int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon} w_0$ will not go along with any difficulty in the special case when w_0 has compact support in $\{d > 0\}$. That it is reasonable to use such functions for the approximation of a general w_0, beyond the required regularity assumptions merely satisfying (1.10), is indicated by the observation to be made in Lemma 2.4, which itself is prepared by the following implication of our assumptions on d.

Lemma 2.3 *Let $d \in C^0(\overline{\Omega})$ be nonnegative and such that (1.5) and (1.6) are satisfied. Then for any set $\Omega_0 \subset \overline{\Omega}$ which is relatively open in $\overline{\Omega}$ and such that $\Omega_0 \cap \{d=0\} \ne \emptyset$, we have*

$$\int_{\Omega_0} \frac{d_x^2}{d} = \infty. \tag{2.14}$$

We can thereby easily assert that any w_0 compatible with the hypotheses of Theorem 1.1 indeed must vanish at each zero of d.

Lemma 2.4 *Let $d \in C^0(\overline{\Omega})$ be nonnegative and such that (1.5) and (1.6) are valid, and suppose that $w_0 \in C^0(\overline{\Omega})$ is a nonnegative function fulfilling (1.10). Then*

$$w_0(x) = 0 \qquad \textit{for all } x \in \{d = 0\}. \tag{2.15}$$

We shall next use the above fact together with our overall regularity assumption that $\sqrt{w_0}$ belongs to $W^{1,2}(\Omega)$ to construct a monotone sequence of approximations to w_0 which are all compactly supported in $\{d > 0\}$, and which moreover are compatible with the energy functional in (1.21) in the sense that not only the third but also the second intergal therein remains bounded along this sequence.

Lemma 2.5 *Assume that the nonnegative function $d \in C^0(\overline{\Omega})$ satisfies (1.5) and (1.6), and that w_0 complies with (1.9) and (1.10). Then there exists $(w_{0j})_{j\in\mathbb{N}} \subset L^\infty(\Omega)$ such that for all $j \in \mathbb{N}$ we have $w_{0j} \ge 0$ in Ω and $\sqrt{w_{0j}} \in W^{1,2}(\Omega)$ as well as*

$$\operatorname{supp} w_{0j} \subset \{d > 0\}, \tag{2.16}$$

and such that

$$w_{0j} \nearrow w_0 \quad \textit{in } \Omega \qquad \textit{as } j \to \infty \tag{2.17}$$

and

$$\sup_{j\in\mathbb{N}} \int_\Omega d \frac{w_{0jx}^2}{w_{0j}} < \infty. \tag{2.18}$$

We finally combine the outcomes of Lemma 2.2 and Lemma 2.5 to select a suitable decreasing sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ along which the interplay of the correspondingly defined function d_{ε_j} with a slightly shifted variant of w_{0j} is favorable with regard to both relevant integrals appearing in the Lyapunov functional in (1.21).

Lemma 2.6 *Let $d \in C^0(\overline{\Omega})$ be nonnegative and such that (1.5) and (1.6) hold, and let w_0 satisfy (1.9) and (1.10). Then there exists $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$, and such that for $(d_{\varepsilon_j})_{j\in\mathbb{N}}$ as determined by Lemma 2.2, and for*

$$w_{0\varepsilon}(x) := w_{0j}(x) + \varepsilon^{\frac{1}{4}}, \qquad x \in \overline{\Omega}, \quad \varepsilon = \varepsilon_j, \quad j \in \mathbb{N}, \tag{2.19}$$

with $(w_{0j})_{j\in\mathbb{N}}$ taken from Lemma 2.5, we can find $C > 0$ such that

$$\int_\Omega d_\varepsilon \frac{w_{0\varepsilon x}^2}{w_{0\varepsilon}} \le C \qquad \textit{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}} \tag{2.20}$$

and

$$\int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon} w_{0\varepsilon} \le C \qquad \textit{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}. \tag{2.21}$$

2.3 Regularized problems: local existence and extensibility

Upon the choices specified in Lemma 2.6, for $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ we henceforth consider the approximate variants of (1.4) given by

$$\begin{cases} u_{\varepsilon t} = (d_\varepsilon u_\varepsilon)_{xx} - (d_\varepsilon u_\varepsilon w_{\varepsilon x})_x, & x \in \Omega,\ t > 0, \\ w_{\varepsilon t} = \varepsilon\Big(d_\varepsilon \frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x - u_\varepsilon g(w_\varepsilon), & x \in \Omega,\ t > 0, \\ u_{\varepsilon x} = w_{\varepsilon x} = 0, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x,0) = u_0(x), \qquad w_\varepsilon(x,0) = w_{0\varepsilon}(x), & x \in \Omega, \end{cases} \tag{2.22}$$

which are all solvable at least locally in time, and for which a convenient criterion for extensibility can be obtained:

Lemma 2.7 *For each $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, there exist $T_{max,\varepsilon} \in (0,\infty]$ and functions*

$$\begin{cases} u_\varepsilon \in C^0(\overline{\Omega} \times [0,T_{max,\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{max,\varepsilon})), \\ w_\varepsilon \in C^0([0,T_{max,\varepsilon}); W^{1,2}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,T_{max,\varepsilon})), \end{cases} \tag{2.23}$$

for which we have $u_\varepsilon > 0$ in $\overline{\Omega} \times (0,T_{max,\varepsilon})$ and $w_\varepsilon > 0$ in $\overline{\Omega} \times [0,T_{max,\varepsilon})$, which solve (2.22) in the classical sense in $\Omega \times (0,T_{max,\varepsilon})$, and which are such that

$$\textit{if } T_{max,\varepsilon} < \infty, \textit{ then } \limsup_{t\nearrow T_{max,\varepsilon}} \left\{ \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|w_\varepsilon(\cdot,t)\|_{W^{1,2}(\Omega)} + \left\|\frac{1}{w_\varepsilon(\cdot,t)}\right\|_{L^\infty(\Omega)} \right\} = \infty. \tag{2.24}$$

PROOF. In light of the positivity of both d_ε and $w_{0\varepsilon}$ in $\overline{\Omega}$, as asserted by Lemma 2.2 and Lemma 2.6, this can be seen on adapting well-established arguments from the analysis of chemotaxis problems and of parabolic problems involving nonlinear degenerate diffusion ([39], [1], [47]) to the present context. $\Box$

The following two properties of these solutions are almost trivial but important.

Lemma 2.8 *Let $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$. Then*

$$\int_\Omega u_\varepsilon(\cdot,t) = \int_\Omega u_0 \qquad \textit{for all } t \in (0,T_{max,\varepsilon}) \tag{2.25}$$

and

$$w_\varepsilon(x,t) \le M := \|w_0\|_{L^\infty(\Omega)} + 1 \qquad \textit{for all } x \in \Omega \textit{ and } t \in (0,T_{max,\varepsilon}), \tag{2.26}$$

and furthermore we have

$$\int_\Omega w_\varepsilon(\cdot,t) \le \int_\Omega w_\varepsilon(\cdot,t_0) \le \int_\Omega w_{0\varepsilon} \qquad \textit{whenever } 0 < t_0 < t < T_{max,\varepsilon} \tag{2.27}$$

as well as

$$\int_0^t \int_\Omega u_\varepsilon w_\varepsilon \le \frac{1}{\underline{\gamma}} \int_\Omega w_{0\varepsilon} \qquad \textit{for all } t \in (0,T_{max,\varepsilon}). \tag{2.28}$$

PROOF. The identity (2.25) immediately results on integration of the first equation in (2.22) over $\Omega \times (0,t)$. For the derivation of (2.26), we only need to observe that by the maximum principle,

$$w_\varepsilon \le \|w_{0\varepsilon}\|_{L^\infty(\Omega)} \qquad \text{in } \Omega \times (0, T_{max,\varepsilon}),$$

and that herein by definition (2.19) of $w_{0\varepsilon}$, due to the fact that $\varepsilon_j \le 1$ for all $j \in \mathbb{N}$ we have

$$w_{0\varepsilon_j} = w_{0j} + \varepsilon_j^{\frac{1}{4}} \le w_0 + 1 \le M \quad \text{in } \Omega \qquad \text{for all } j \in \mathbb{N},$$

because $w_{0j} \le w_0$ in Ω for all $j \in \mathbb{N}$ by Lemma 2.5.
Finally, since from the second equation in (2.22) we obtain

$$\frac{d}{dt}\int_\Omega w_\varepsilon = -\int_\Omega u_\varepsilon g(w_\varepsilon) \le -\underline{\gamma}\int_\Omega u_\varepsilon w_\varepsilon \qquad \text{for all } t \in (0, T_{max,\varepsilon}),$$

after an integration in time we readily infer that also (2.27) and (2.28) hold. □

3 An approximate energy inequality

In order to derive some fundamental a priori information beyond that from Lemma 2.8, we shall next make use of our particular construction of the functions d_ε and $w_{0\varepsilon}$ to establish an approximate version of the energy inequality (1.21). This will be achieved in Lemma 3.4, and thereafter further exploited in Lemma 3.5, on the basis of three testing procedures performed in Lemma 3.1, Lemma 3.2 and Lemma 3.3.

We first consider the part containing the logarithmic entropy functional.

Lemma 3.1 *For all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ and arbitrary $\delta > 0$,*

$$\frac{d}{dt}\int_\Omega u_\varepsilon \ln(d_\varepsilon u_\varepsilon) + \int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{d_\varepsilon u_\varepsilon} \le \int_\Omega d_\varepsilon u_{\varepsilon x} w_{\varepsilon x} + \delta \int_\Omega d_\varepsilon u_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} + \frac{1}{4\delta}\int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon} u_\varepsilon w_\varepsilon \tag{3.1}$$

for all $t \in (0, T_{max,\varepsilon})$.

PROOF. We multiply the first equation in (2.22) by the function $\ln(d_\varepsilon u_\varepsilon)$ which by Lemma 2.2 and the strong maximum principle is positive in $\overline{\Omega} \times (0, T_{max,\varepsilon})$. On integrating by parts and using (2.25) we thereby obtain the identity

$$\begin{aligned} \frac{d}{dt}\int_\Omega u_\varepsilon \ln(d_\varepsilon u_\varepsilon) + \int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{d_\varepsilon u_\varepsilon} &= \int_\Omega (d_\varepsilon u_\varepsilon)_x w_{\varepsilon x} \\ &= \int_\Omega d_\varepsilon u_{\varepsilon x} w_{\varepsilon x} + \int_\Omega d_{\varepsilon x} u_\varepsilon w_{\varepsilon x} \qquad \text{for all } t \in (0, T_{max,\varepsilon}), \end{aligned}$$

in which by Young's inequality, for each $\delta > 0$ we have

$$\int_\Omega d_{\varepsilon x} u_\varepsilon w_{\varepsilon x} \le \delta \int_\Omega d_\varepsilon u_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} + \frac{1}{4\delta}\int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon} u_\varepsilon w_\varepsilon \qquad \text{for all } t \in (0, T_{max,\varepsilon}),$$

so that (3.1) directly follows. $\square$

As already observed in [13] and essentially used in numerous further precedent works on haptotaxis systems (see e.g. [28], [41]), the interaction term in (3.1) containing the gradients of both the population density and the attractant, precisely appears during an appropriate testing process applied to the second equation in (2.22). Thanks to the dissipative character of the signal consumption mechanism in (2.22), this furthermore provides an absorptive term that can be used to compensate the second summand on the right of (3.1). The next lemma will moreover reveal the fortunate circumstance that the particular diffusive regularization chosen in the second equation in (2.22) is in favorable accordance with these stuctural properties.

Lemma 3.2 *Let $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$. Then with $\underline{\gamma}$ and $\overline{\gamma}$ taken from (1.7), we have*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{g(w_\varepsilon)} + \frac{\underline{\gamma}}{2\overline{\gamma}}\int_\Omega d_\varepsilon u_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} + \varepsilon\int_\Omega \frac{1}{\sqrt{g(w_\varepsilon)}}\Big(d_\varepsilon \frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x^2 \le -\int_\Omega d_\varepsilon u_{\varepsilon x} w_{\varepsilon x} \tag{3.2}$$

for all $t \in (0, T_{max,\varepsilon})$.

Proof. Using that $w_\varepsilon > 0$ in $\overline{\Omega}\times[0,T_{max,\varepsilon})$ by Lemma 2.7, and that hence (1.8) warrants that also $g(w_\varepsilon)$ is positive in $\overline{\Omega}\times[0,T_{max,\varepsilon})$, on the basis of the second equation in (2.22) and an integration by parts we compute

$$\begin{aligned}
\frac{d}{dt}\int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{g(w_\varepsilon)} &= 2\int_\Omega d_\varepsilon \frac{w_{\varepsilon x}}{g(w_\varepsilon)}\cdot\Big\{\varepsilon\Big(d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_{xx} - \Big(u_\varepsilon g(w_\varepsilon)\Big)_x\Big\} \\
&\quad -\int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{g^2(w_\varepsilon)} g'(w_\varepsilon)\cdot\Big\{\varepsilon\Big(d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x - u_\varepsilon g(w_\varepsilon)\Big\} \\
&= -2\varepsilon\int_\Omega \Big(d_\varepsilon\frac{w_{\varepsilon x}}{g(w_\varepsilon)}\Big)_x\cdot\Big(d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x \\
&\quad -2\int_\Omega d_\varepsilon u_{\varepsilon x} w_{\varepsilon x} - 2\int_\Omega d_\varepsilon u_\varepsilon g'(w_\varepsilon)\frac{w_{\varepsilon x}^2}{g(w_\varepsilon)} \\
&\quad -\varepsilon\int_\Omega d_\varepsilon g'(w_\varepsilon)\frac{w_{\varepsilon x}^2}{g^2(w_\varepsilon)}\cdot\Big(d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x \\
&\quad +\int_\Omega d_\varepsilon u_\varepsilon g'(w_\varepsilon)\frac{w_{\varepsilon x}^2}{g(w_\varepsilon)} \\
&= -2\varepsilon\int_\Omega \frac{1}{\sqrt{g(w_\varepsilon)}}\cdot\Big(d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x^2 \\
&\quad -2\int_\Omega d_\varepsilon u_{\varepsilon x} w_{\varepsilon x} \\
&\quad -\int_\Omega d_\varepsilon u_\varepsilon g'(w_\varepsilon)\frac{w_{\varepsilon x}^2}{g(w_\varepsilon)} \qquad \text{for all } t\in(0,T_{max,\varepsilon}),
\end{aligned} \tag{3.3}$$

where we have used the pointwise identity

$$2\Big(d_\varepsilon\frac{w_{\varepsilon x}}{g(w_\varepsilon)}\Big)_x + d_\varepsilon g'(w_\varepsilon)\frac{w_{\varepsilon x}^2}{g^2(w_\varepsilon)}$$

$$
\begin{aligned}
&= 2\Big(\frac{1}{\sqrt{g(w_\varepsilon)}}\cdot d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x + d_\varepsilon g'(w_\varepsilon)\frac{w_{\varepsilon x}^2}{g^2(w_\varepsilon)} \\
&= -\frac{g'(w_\varepsilon)w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}^3}\cdot d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} + 2\cdot\frac{1}{\sqrt{g(w_\varepsilon)}}\cdot\Big(d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x + d_\varepsilon g'(w_\varepsilon)\frac{w_{\varepsilon x}^2}{g^2(w_\varepsilon)} \\
&= 2\frac{1}{\sqrt{g(w_\varepsilon)}}\cdot\Big(d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x \qquad \text{in } \Omega\times(0,T_{max,\varepsilon}).
\end{aligned}
$$

Since (1.7) and (1.8) entail that

$$
g'(w_\varepsilon)\ge\underline{\gamma} \quad\text{and}\quad g(w_\varepsilon)\le\overline{\gamma}w_\varepsilon \qquad \text{in } \Omega\times(0,T_{max,\varepsilon}),
$$

from (3.3) we obtain (3.2). $\square$

Finally, in order to absorb the rightmost summand in (3.1) appropriately, we shall add a suitable multiple of the inequality contained in the following.

Lemma 3.3 *Let $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}$ and $\delta>0$. Then for all $t\in(0,T_{max,\varepsilon})$,*

$$
\frac{d}{dt}\int_\Omega\frac{d_{\varepsilon x}^2}{d_\varepsilon}w_\varepsilon + \underline{\gamma}\int_\Omega\frac{d_{\varepsilon x}^2}{d_\varepsilon}u_\varepsilon w_\varepsilon \le \delta\varepsilon\int_\Omega\frac{1}{\sqrt{g(w_\varepsilon)}}\cdot\Big(d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x^2 + \frac{\sqrt{\overline{\gamma}M\varepsilon}}{4\delta}, \tag{3.4}
$$

where $\underline{\gamma},\overline{\gamma}$ and M are as in (1.7) and (2.26), respectively.

Proof. By means of (2.22), for $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}$ we calculate

$$
\frac{d}{dt}\int_\Omega\frac{d_{\varepsilon x}^2}{d_\varepsilon}w_\varepsilon = \varepsilon\int_\Omega\frac{d_{\varepsilon x}^2}{d_\varepsilon}\cdot\Big(d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x - \int_\Omega\frac{d_{\varepsilon x}^2}{d_\varepsilon}u_\varepsilon g(w_\varepsilon) \qquad \text{for all } t\in(0,T_{max,\varepsilon}), \tag{3.5}
$$

where thanks to (1.8),

$$
-\int_\Omega\frac{d_{\varepsilon x}^2}{d_\varepsilon}u_\varepsilon g(w_\varepsilon) \le -\underline{\gamma}\int_\Omega\frac{d_{\varepsilon x}^2}{d_\varepsilon}u_\varepsilon w_\varepsilon \qquad \text{for all } t\in(0,T_{max,\varepsilon}). \tag{3.6}
$$

In order to estimate the first term on the right of (3.5), we first invoke Young's inequality to see that for each $\delta>0$ we have

$$
\varepsilon\int_\Omega\frac{d_{\varepsilon x}^2}{d_\varepsilon}\cdot\Big(d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x \le \delta\varepsilon\int_\Omega\frac{1}{\sqrt{g(w_\varepsilon)}}\cdot\Big(d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x^2 + \frac{\varepsilon}{4\delta}\int_\Omega\frac{d_{\varepsilon x}^4}{d_\varepsilon^2}\sqrt{g(w_\varepsilon)} \qquad \text{for all } t\in(0,T_{max,\varepsilon}), \tag{3.7}
$$

and here in the rightmost summand we recall (1.8) and (2.26) to find that

$$
\sqrt{g(w_\varepsilon)}\le\sqrt{\overline{\gamma}M} \qquad \text{in } \Omega\times(0,T_{max,\varepsilon}).
$$

Since in Lemma 2.2 we have asserted that

$$
\int_\Omega\frac{d_{\varepsilon x}^4}{d_\varepsilon^2}\le\frac{1}{\sqrt{\varepsilon}} \qquad \text{for all } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}},
$$

this entails that

$$\frac{\varepsilon}{4\delta}\int_\Omega \frac{d_{\varepsilon x}^4}{d_\varepsilon^2}\sqrt{g(w_\varepsilon)} \le \frac{\sqrt{\overline{\gamma}}M\varepsilon}{4\delta} \qquad \text{for all } t\in(0,T_{max,\varepsilon}),$$

so that combining (3.6) and (3.7) with (3.5) yields (3.4). □

In summary, on adequately joining the above three lemmata we obtain the desired approximate analogue of the energy inequality (1.21).

Lemma 3.4 *Let $\underline{\gamma},\overline{\gamma}$ and M denote the constants from (1.7) and (2.26). Then whenever $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}$,*

$$\begin{aligned} &\frac{d}{dt}\Big\{\int_\Omega u_\varepsilon\ln(d_\varepsilon u_\varepsilon) + \frac12\int_\Omega d_\varepsilon\frac{w_{\varepsilon x}^2}{g(w_\varepsilon)} + \frac{\overline{\gamma}}{\underline{\gamma}^2}\int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon}w_\varepsilon\Big\} \\ &\qquad\qquad + \int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{d_\varepsilon u_\varepsilon} + \frac{\underline{\gamma}}{4\overline{\gamma}}\int_\Omega d_\varepsilon u_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} + \frac{\varepsilon}{2}\int_\Omega \frac{1}{\sqrt{g(w_\varepsilon)}}\cdot\Big(d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x^2 \\ &\le \frac{\sqrt{\overline{\gamma}^5 M\varepsilon}}{2\underline{\gamma}^4} \qquad \textit{for all } t\in(0,T_{max,\varepsilon}). \end{aligned} \tag{3.8}$$

Proof. We choose the free parameters δ in Lemma 3.1 and Lemma 3.3 to equal $\frac{\underline{\gamma}}{4\overline{\gamma}}$ and $\frac{\underline{\gamma}^2}{2\overline{\gamma}}$, respectively, to see on linearly combining (3.1), (3.2) and (3.4) that for all $t\in(0,T_{max,\varepsilon})$,

$$\begin{aligned} &\frac{d}{dt}\Big\{\int_\Omega u_\varepsilon\ln(d_\varepsilon u_\varepsilon) + \frac12\int_\Omega d_\varepsilon\frac{w_{\varepsilon x}^2}{g(w_\varepsilon)} + \frac{\overline{\gamma}}{\underline{\gamma}^2}\int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon}w_\varepsilon\Big\} + \int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{d_\varepsilon u_\varepsilon} + \frac{\underline{\gamma}}{2\overline{\gamma}}\int_\Omega d_\varepsilon u_\varepsilon\frac{w_{\varepsilon x}^2}{w_\varepsilon} \\ &\qquad\qquad +\varepsilon\int_\Omega \frac{1}{\sqrt{g(w_\varepsilon)}}\cdot\Big(d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x^2 + \frac{\overline{\gamma}}{\underline{\gamma}^2}\cdot\underline{\gamma}\int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon}u_\varepsilon w_\varepsilon \\ &\le \int_\Omega d_\varepsilon u_{\varepsilon x}w_{\varepsilon x} + \frac{\underline{\gamma}}{4\overline{\gamma}}\int_\Omega d_\varepsilon u_\varepsilon\frac{w_{\varepsilon x}^2}{w_\varepsilon} + \frac{\overline{\gamma}}{\underline{\gamma}}\int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon}u_\varepsilon w_\varepsilon \\ &\qquad -\int_\Omega d_\varepsilon u_{\varepsilon x}w_{\varepsilon x} \\ &\qquad +\frac{\overline{\gamma}}{\underline{\gamma}^2}\cdot\frac{\underline{\gamma}^2}{2\overline{\gamma}}\cdot\varepsilon\int_\Omega \frac{1}{\sqrt{g(w_\varepsilon)}}\cdot\Big(d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x^2 + \frac{\overline{\gamma}}{\underline{\gamma}^2}\cdot\frac{\sqrt{\overline{\gamma}}M\varepsilon}{4\frac{\underline{\gamma}^2}{2\overline{\gamma}}}, \end{aligned}$$

which can readily be simplified so as to yield (3.8). □

We now use Lemma 2.6 to make sure that the respective energy values at the initial time are bounded from above uniformly with respect to $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}$. Therefore, an integration of (3.8) yields the following.

Lemma 3.5 *There exists $C>0$ with the property that whenever $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}$, we have*

$$\int_\Omega u_\varepsilon(\cdot,t)\Big|\ln(d_\varepsilon u_\varepsilon(\cdot,t))\Big| \le C\cdot(1+\sqrt{\varepsilon}t) \qquad \textit{for all } t\in(0,T_{max,\varepsilon}) \tag{3.9}$$

and

$$\int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2(\cdot,t)}{w_\varepsilon(\cdot,t)} \le C\cdot(1+\sqrt{\varepsilon}t) \qquad \textit{for all } t\in(0,T_{max,\varepsilon}) \tag{3.10}$$

as well as

$$\int_0^t \int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{d_\varepsilon u_\varepsilon} \le C\cdot(1+\sqrt{\varepsilon}t) \qquad \textit{for all } t\in(0,T_{max,\varepsilon}) \tag{3.11}$$

and

$$\int_0^t \int_\Omega d_\varepsilon u_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} \le C\cdot(1+\sqrt{\varepsilon}t) \qquad \textit{for all } t\in(0,T_{max,\varepsilon}) \tag{3.12}$$

and

$$\varepsilon\int_0^t \int_\Omega \frac{1}{\sqrt{g(w_\varepsilon)}}\cdot\Big(d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x^2 \le C\cdot(1+\sqrt{\varepsilon}t) \qquad \textit{for all } t\in(0,T_{max,\varepsilon}). \tag{3.13}$$

PROOF. For $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}$ we obtain from Lemma 3.4 that if we let $\underline{\gamma}>0, \overline{\gamma}>0$ and $M>0$ be as specified in (1.7) and (2.26), then

$$y_\varepsilon(t) := \int_\Omega u_\varepsilon(\cdot,t)\ln(d_\varepsilon u_\varepsilon(\cdot,t)) + \frac12\int_\Omega d_\varepsilon\frac{w_{\varepsilon x}^2(\cdot,t)}{g(w_\varepsilon(\cdot,t))} + \frac{\overline{\gamma}}{\underline{\gamma}^2}\int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon} w_\varepsilon(\cdot,t), \qquad t\in[0,T_{max,\varepsilon}),$$

and

$$\begin{aligned} h_\varepsilon(t) \;:=\; & \int_\Omega \frac{(d_\varepsilon u_\varepsilon(\cdot,t))_x^2}{d_\varepsilon u_\varepsilon(\cdot,t)} + \frac{\underline{\gamma}}{4\overline{\gamma}}\int_\Omega d_\varepsilon u_\varepsilon(\cdot,t)\frac{w_{\varepsilon x}^2(\cdot,t)}{w_\varepsilon(\cdot,t)} \\ & + \frac{\varepsilon}{2}\int_\Omega \frac{1}{\sqrt{g(w_\varepsilon(\cdot,t))}}\cdot\Big(d_\varepsilon\frac{w_{\varepsilon x}(\cdot,t)}{\sqrt{g(w_\varepsilon(\cdot,t))}}\Big)_x^2, \qquad t\in(0,T_{max,\varepsilon}), \end{aligned}$$

satisfy

$$y_\varepsilon'(t) + h_\varepsilon(t) \le c_1\sqrt{\varepsilon} \qquad \text{for all } t\in(0,T_{max,\varepsilon}), \tag{3.14}$$

where $c_1 := \frac{\sqrt{\overline{\gamma}^5 M}}{2\underline{\gamma}^4}$. To conclude (3.9)-(3.12) from this, we observe that at the initial time we can use (2.9) to estimate

$$\int_\Omega u_0\ln(d_\varepsilon u_0) \le c_2 := \|u_0\|_{L^\infty(\Omega)}\ln\Big\{(\|d\|_{L^\infty(\Omega)}+1)\|u_0\|_{L^\infty(\Omega)}\Big\} \qquad \text{for all } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}, \tag{3.15}$$

whereas Lemma 2.6 ensures the existence of $c_3>0$ and $c_4>0$ such that

$$\int_\Omega d_\varepsilon\frac{w_{0\varepsilon x}^2}{w_{0\varepsilon}} \le c_3 \qquad \text{for all } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}} \tag{3.16}$$

and

$$\int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon} w_{0\varepsilon} \le c_4 \qquad \text{for all } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}. \tag{3.17}$$

Since $\frac{w_{0\varepsilon x}^2}{g(w_{0\varepsilon})} \le \frac{1}{\underline{\gamma}}\frac{w_{0\varepsilon x}^2}{w_{0\varepsilon}}$ in Ω by (1.8), (3.15)-(3.17) show that

$$y_\varepsilon(0) \le c_5 := c_2 + \frac{c_3}{2\underline{\gamma}} + \frac{c_4\overline{\gamma}}{\underline{\gamma}^2} \qquad \text{for all } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}},$$

by (3.14) implying that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,

$$y_\varepsilon(t) + \int_0^t h_\varepsilon(s)ds \le c_5 + c_1\sqrt{\varepsilon}t \qquad \text{for all } t \in (0, T_{max,\varepsilon}). \tag{3.18}$$

Here we note that since $\xi \ln \xi \ge -\frac{1}{e}$ for all $\xi > 0$ and $d_\varepsilon \ge d$ in Ω for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ thanks to Lemma 2.2, with $c_6 := \frac{1}{e}\int_\Omega \frac{1}{d}$ being finite according to (1.6), we have

$$\begin{aligned}
-\int_{\{d_\varepsilon u_\varepsilon<1\}} u_\varepsilon \ln(d_\varepsilon u_\varepsilon) &= -\int_{\{d_\varepsilon u_\varepsilon<1\}} \frac{1}{d_\varepsilon} \cdot \Big\{ d_\varepsilon u_\varepsilon \cdot \ln(d_\varepsilon u_\varepsilon)\Big\} \\
&\le \frac{1}{e}\int_{\{d_\varepsilon u_\varepsilon<1\}} \frac{1}{d_\varepsilon} \\
&\le c_6 \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}} \text{ and } t \in (0, T_{max,\varepsilon}),
\end{aligned}$$

which along with (1.8) in particular entails that

$$\begin{aligned}
y_\varepsilon(t) &\ge \int_\Omega u_\varepsilon \ln(d_\varepsilon u_\varepsilon) + \frac{1}{2\overline{\gamma}}\int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} \\
&= \int_\Omega u_\varepsilon \Big| \ln(d_\varepsilon u_\varepsilon)\Big| + 2\int_{\{d_\varepsilon u_\varepsilon<1\}} u_\varepsilon \ln(d_\varepsilon u_\varepsilon) + \frac{1}{2\overline{\gamma}}\int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} \\
&\ge \int_\Omega u_\varepsilon \Big| \ln(d_\varepsilon u_\varepsilon)\Big| - 2c_6 + \frac{1}{2\overline{\gamma}}\int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}} \text{ and } t \in (0, T_{max,\varepsilon}).
\end{aligned}$$

Therefore, (3.18) implies that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ we have

$$\int_\Omega u_\varepsilon \Big| \ln(d_\varepsilon u_\varepsilon)\Big| + \frac{1}{2\overline{\gamma}}\int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} + \int_0^t h_\varepsilon(s)ds \le 2c_6 + c_5 + c_1\sqrt{\varepsilon}t \qquad \text{for all } t \in (0, T_{max,\varepsilon}),$$

which in view of the definition of h_ε yields all claimed inequalities. □

4 Global existence in the approximate problems

With the above information at hand, we can now make sure that in fact all our approximate solutions are global in time. To achieve this in Lemma 4.5 on the basis of the extensibility criterion in Lemma 2.7, for each individual $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ we will derive further estimates which may depend on ε. We begin with a pointwise lower estimate for w_ε that we obtain by a comparison argument combined with Lemma 3.5, and that will be used in Lemma 4.2.

Lemma 4.1 *Assume that $T_{max,\varepsilon} < \infty$ for some $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$. Then there exists $C(\varepsilon) > 0$ such that*

$$w_\varepsilon(x,t) \ge C(\varepsilon) \qquad \textit{for all } x \in \Omega \textit{ and } t \in (0, T_{max,\varepsilon}). \tag{4.1}$$

PROOF. We first observe that under the current hypothesis, Lemma 3.5 says that

$$\int_0^{T_{max,\varepsilon}} \int_\Omega \Big| \Big(\sqrt{d_\varepsilon u_\varepsilon}\Big)_x\Big|^2 < \infty, \tag{4.2}$$

and we claim that along with (2.25) this provides sufficient regularity information on the absorption coefficient function u_ε in the second equation in (2.22) to rule out finite-time formation of zeros of w_ε in the sense of (4.1). To verify this, we use the continuity of the embedding $W^{1,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ as well as (2.25) and (2.9) to fix $c_1 > 0$ and $c_2 > 0$ fulfilling

$$\begin{aligned}
\int_0^{T_{max,\varepsilon}} \Big\| \sqrt{d_\varepsilon u_\varepsilon(\cdot,t)} \Big\|_{L^\infty(\Omega)}^2 dt &\le c_1 \int_0^{T_{max,\varepsilon}} \bigg\{ \Big\| \Big(\sqrt{d_\varepsilon u_\varepsilon(\cdot,t)} \Big)_x \Big\|_{L^2(\Omega)}^2 + \Big\| \sqrt{d_\varepsilon u_\varepsilon(\cdot,t)} \Big\|_{L^2(\Omega)}^2 \bigg\} dt \\
&\le c_2 \int_0^{T_{max,\varepsilon}} \bigg\{ \Big\| \Big(\sqrt{d_\varepsilon u_\varepsilon(\cdot,t)} \Big)_x \Big\|_{L^2(\Omega)}^2 + 1 \bigg\} dt,
\end{aligned}$$

and to see that thus (4.2) entails that

$$\int_0^{T_{max,\varepsilon}} \Big\| \sqrt{d_\varepsilon u_\varepsilon(\cdot,t)} \Big\|_{L^\infty(\Omega)}^2 dt < \infty.$$

hence, by positivity of d_ε in $\overline{\Omega}$, also the number

$$c_3 := \int_0^{T_{max,\varepsilon}} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} dt$$

is finite, which in particular implies that the solution $y \in C^1([0, T_{max,\varepsilon}))$ of the initial-value problem

$$\begin{cases} y'(t) = -\overline{\gamma} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \cdot y(t), \qquad t \in (0, T_{max,\varepsilon}), \\ y(0) = \varepsilon^{\frac{1}{4}}, \end{cases} \tag{4.3}$$

satisfies

$$y(t) = \varepsilon^{\frac{1}{4}} e^{-\overline{\gamma} \int_0^t \|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)} ds} \ge c_4 := \varepsilon^{\frac{1}{4}} e^{-\overline{\gamma} c_3} \qquad \text{for all } t \in (0, T_{max,\varepsilon}). \tag{4.4}$$

It can now readily be verified that $\overline{\Omega} \times [0, T_{max,\varepsilon}) \ni (x,t) \mapsto y(t)$ is a classical subsolution to the initial-boundary problem solved by w_ε in $\Omega \times (0, T_{max,\varepsilon})$, so that by (4.4), $w_\varepsilon(x,t) \ge c_4$ for all $x \in \Omega$ and $t \in (0, T_{max,\varepsilon})$, which yields (4.1). □

This lower bound enables us to suitably estimate singular denominators appearing in the following lemma which, apart from that and the positivity of d_ε, again only relies on Lemma 3.5 only.

Lemma 4.2 *Let $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, and suppose that $T_{max,\varepsilon} < \infty$. Then there exists $C(\varepsilon) > 0$ such that*

$$\int_\Omega w_{\varepsilon x}^4(\cdot,t) \le C(\varepsilon) \qquad \textit{for all } t \in (0, T_{max,\varepsilon}). \tag{4.5}$$

PROOF. According to Lemma 3.5 and Lemma 4.1, our hypothesis that $T_{max,\varepsilon} < \infty$ again warrants that

$$\int_0^{T_{max,\varepsilon}} \int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{d_\varepsilon u_\varepsilon} < \infty, \tag{4.6}$$

and that moreover with some $c_1 > 0$ and $c_2 > 0$ we have

$$\int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} \le c_1 \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \tag{4.7}$$

as well as

$$w_\varepsilon \geq c_2 \qquad \text{in } \Omega \times (0, T_{max,\varepsilon}). \tag{4.8}$$

Since combining the Gagliardo-Nirenberg inequality with (2.25) yields positive constants c_3 and c_4 such that

$$\begin{aligned}
\int_0^{T_{max,\varepsilon}} \int_\Omega (d_\varepsilon u_\varepsilon)^3 &= \int_0^{T_{max,\varepsilon}} \left\| \sqrt{d_\varepsilon u_\varepsilon(\cdot,t)} \right\|_{L^6(\Omega)}^6 dt \\
&\leq c_3 \int_0^{T_{max,\varepsilon}} \left\{ \left\| \left(\sqrt{d_\varepsilon u_\varepsilon(\cdot,t)} \right)_x \right\|_{L^2(\Omega)}^2 \left\| \sqrt{d_\varepsilon u_\varepsilon(\cdot,t)} \right\|_{L^2(\Omega)}^4 + \left\| \sqrt{d_\varepsilon u_\varepsilon(\cdot,t)} \right\|_{L^2(\Omega)}^6 \right\} dt \\
&\leq c_4 \int_0^{T_{max,\varepsilon}} \left\{ \left\| \left(\sqrt{d_\varepsilon u_\varepsilon(\cdot,t)} \right)_x \right\|_{L^2(\Omega)}^2 + 1 \right\} dt,
\end{aligned}$$

it follows from (4.6) that also

$$\int_0^{T_{max,\varepsilon}} \int_\Omega (d_\varepsilon u_\varepsilon)^3 < \infty.$$

As

$$c_5 := \inf_{x\in\Omega} d_\varepsilon(x) \tag{4.9}$$

is positive thanks to Lemma 2.2, this implies that

$$c_6 := \int_0^{T_{max,\varepsilon}} \int_\Omega u_\varepsilon^3 \tag{4.10}$$

is finite, whereas (4.7) and (2.26) show that with some $c_7 > 0$ we have

$$\int_\Omega w_{\varepsilon x}^2 \leq c_7 \qquad \text{for all } t \in (0, T_{max,\varepsilon}). \tag{4.11}$$

We now use the second equation in (2.22) to compute

$$\begin{aligned}
\frac{1}{4}\frac{d}{dt}\int_\Omega w_{\varepsilon x}^4 &= -3\int_\Omega w_{\varepsilon x}^2 w_{\varepsilon xx} w_{\varepsilon t} \\
&= -3\varepsilon \int_\Omega w_{\varepsilon x}^2 w_{\varepsilon xx} \cdot \left(d_\varepsilon \frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \right)_x + 3\int_\Omega \int_\Omega u_\varepsilon g(w_\varepsilon) w_{\varepsilon x}^2 w_{\varepsilon xx} \\
&= -3\varepsilon \int_\Omega d_\varepsilon \frac{1}{\sqrt{g(w_\varepsilon)}} w_{\varepsilon x}^2 w_{\varepsilon xx}^2 + \frac{3}{2}\varepsilon \int_\Omega d_\varepsilon \frac{g'(w_\varepsilon)}{\sqrt{g(w_\varepsilon)}^3} w_{\varepsilon x}^4 w_{\varepsilon xx} \\
&\quad -3\varepsilon \int_\Omega d_{\varepsilon x} \frac{1}{\sqrt{g(w_\varepsilon)}} w_{\varepsilon x}^3 w_{\varepsilon xx} + 3\int_\Omega u_\varepsilon g(w_\varepsilon) w_{\varepsilon x}^2 w_{\varepsilon xx} \qquad \text{for all } t \in (0, T_{max,\varepsilon}),
\end{aligned} \tag{4.12}$$

where by Young's inequality, (4.8), (4.9), (1.7), (1.8) and (2.26), for all $t \in (0, T_{max,\varepsilon})$ we can estimate

$$\begin{aligned}
\frac{3}{2}\varepsilon \int_\Omega d_\varepsilon \frac{g'(w_\varepsilon)}{\sqrt{g(w_\varepsilon)}^3} w_{\varepsilon x}^4 w_{\varepsilon xx} &\leq \varepsilon \int_\Omega d_\varepsilon \frac{1}{\sqrt{g(w_\varepsilon)}} w_{\varepsilon x}^2 w_{\varepsilon xx}^2 + \frac{9}{16}\varepsilon \int_\Omega d_\varepsilon \frac{g'^2(w_\varepsilon)}{\sqrt{g(w_\varepsilon)}^5} w_{\varepsilon x}^6 \\
&\leq \varepsilon \int_\Omega d_\varepsilon \frac{1}{\sqrt{g(w_\varepsilon)}} w_{\varepsilon x}^2 w_{\varepsilon xx}^2 + c_8 \int_\Omega w_{\varepsilon x}^6
\end{aligned} \tag{4.13}$$

and

$$
\begin{aligned}
-3\varepsilon \int_\Omega d_{\varepsilon x} \frac{1}{\sqrt{g(w_\varepsilon)}} w_{\varepsilon x}^3 w_{\varepsilon xx} &\le \varepsilon \int_\Omega d_\varepsilon \frac{1}{\sqrt{g(w_\varepsilon)}} w_{\varepsilon x}^2 w_{\varepsilon xx}^2 + \frac{9}{4}\varepsilon \int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon} \frac{1}{\sqrt{g(w_\varepsilon)}} w_{\varepsilon x}^4 \\
&\le \varepsilon \int_\Omega d_\varepsilon \frac{1}{\sqrt{g(w_\varepsilon)}} w_{\varepsilon x}^2 w_{\varepsilon xx}^2 + c_9 \int_\Omega w_{\varepsilon x}^4 \\
&\le \varepsilon \int_\Omega d_\varepsilon \frac{1}{\sqrt{g(w_\varepsilon)}} w_{\varepsilon x}^2 w_{\varepsilon xx}^2 + c_9 \int_\Omega w_{\varepsilon x}^6 + c_9 |\Omega| \qquad (4.14)
\end{aligned}
$$

as well as

$$
\begin{aligned}
3\int_\Omega u_\varepsilon g(w_\varepsilon) w_{\varepsilon x}^2 w_{\varepsilon xx} &\le \frac{\varepsilon}{2} \int_\Omega d_\varepsilon \frac{1}{\sqrt{g(w_\varepsilon)}} w_{\varepsilon x}^2 w_{\varepsilon xx}^2 + \frac{9}{2\varepsilon} \int_\Omega \frac{1}{d_\varepsilon} u_\varepsilon^2 \sqrt{g(w_\varepsilon)}^5 w_{\varepsilon x}^2 \\
&\le \frac{\varepsilon}{2} \int_\Omega d_\varepsilon \frac{1}{\sqrt{g(w_\varepsilon)}} w_{\varepsilon x}^2 w_{\varepsilon xx}^2 + c_{10} \int_\Omega u_\varepsilon^2 w_{\varepsilon x}^2 \\
&\le \frac{\varepsilon}{2} \int_\Omega d_\varepsilon \frac{1}{\sqrt{g(w_\varepsilon)}} w_{\varepsilon x}^2 w_{\varepsilon xx}^2 + c_{10} \int_\Omega u_\varepsilon^3 + c_{10} \int_\Omega w_{\varepsilon x}^6 \qquad (4.15)
\end{aligned}
$$

with $c_8 := \frac{9\varepsilon\overline{\gamma}^2 \|d_\varepsilon\|_{L^\infty(\Omega)}}{16\sqrt{\underline{\gamma} c_2}^5}$, $c_9 := \frac{9\varepsilon \|d_{\varepsilon x}\|_{L^\infty(\Omega)}^2}{4c_5\sqrt{\underline{\gamma} c_2}}$ and $c_{10} := \frac{9\sqrt{\overline{\gamma} M}^5}{2\varepsilon c_5}$. Since (4.9), (1.8) and (2.26) moreover entail that writing $c_{11} := \frac{\varepsilon c_5}{2\sqrt{\overline{\gamma} M}}$ we obtain

$$
\frac{\varepsilon}{2} \int_\Omega d_\varepsilon \frac{1}{\sqrt{g(w_\varepsilon)}} w_{\varepsilon x}^2 w_{\varepsilon xx}^2 \ge c_{11} \int_\Omega w_{\varepsilon x}^2 w_{\varepsilon xx}^2 = \frac{c_{11}}{4} \int_\Omega (w_{\varepsilon x}^2)_x^2 \qquad \text{for all } t \in (0, T_{max,\varepsilon}),
$$

and since the Gagliardo-Nirenberg inequality and Young's inequality together with (4.11) show that with some positive constants c_{12}, c_{13} and c_{14} we have

$$
\begin{aligned}
(c_8 + c_9 + c_{10}) \int_\Omega w_{\varepsilon x}^6 &= (c_8 + c_9 + c_{10}) \|w_{\varepsilon x}^2\|_{L^3(\Omega)}^3 \\
&\le c_{12} \left\| (w_{\varepsilon x}^2)_x \right\|_{L^2(\Omega)}^{\frac{4}{3}} \|w_{\varepsilon x}^2\|_{L^1(\Omega)}^{\frac{5}{3}} + c_{12} \|w_{\varepsilon x}^2\|_{L^1(\Omega)}^3 \\
&\le c_{13} \left\| (w_{\varepsilon x}^2)_x \right\|_{L^2(\Omega)}^{\frac{4}{3}} + c_{13} \\
&\le \frac{c_{11}}{4} \int_\Omega (w_{\varepsilon x}^2)_x^2 + c_{14} \qquad \text{for all } t \in (0, T_{max,\varepsilon}),
\end{aligned}
$$

on combining (4.12)-(4.15) we therefore see that

$$
\frac{1}{4} \frac{d}{dt} \int_\Omega w_{\varepsilon x}^4 \le c_9 |\Omega| + c_{10} \int_\Omega u_\varepsilon^3 + c_{14} \qquad \text{for all } t \in (0, T_{max,\varepsilon}),
$$

and that hence

$$
\int_\Omega w_{\varepsilon x}^4(\cdot, t) \le \int_\Omega w_{0\varepsilon x}^4 + 4c_{10}c_6 + 4(c_9|\Omega| + c_{14}) \cdot t \qquad \text{for all } t \in (0, T_{max,\varepsilon})
$$

because of (4.10). Again since $T_{max,\varepsilon}$ was assumed to be finite, this entails (4.5). $\Box$

The above regularity information on the haptotactic gradient is now sufficient to warrant an ε-dependent bound for u_ε in $L^p(\Omega)$ for arbitrarily large p.

Lemma 4.3 *Assume that $T_{max,\varepsilon} < \infty$ for some $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$. Then for all $p \in (1,\infty)$ there exists $C(\varepsilon,p) > 0$ such that*

$$\int_\Omega u_\varepsilon^p(\cdot,t) \le C(\varepsilon,p) \qquad \textit{for all } t \in (0,T_{max,\varepsilon}). \tag{4.16}$$

PROOF. We test the first equation in (2.22) against u_ε^{p-1} and use Young's inequality to see that

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u_\varepsilon^p &= \int_\Omega u_\varepsilon^{p-1}\cdot\Big\{(d_\varepsilon u_\varepsilon)_x - d_\varepsilon u_\varepsilon w_{\varepsilon x}\Big\}_x \\
&= -(p-1)\int_\Omega d_\varepsilon u_\varepsilon^{p-2}u_{\varepsilon x}^2 - (p-1)\int_\Omega d_{\varepsilon x}u_\varepsilon^{p-1}u_{\varepsilon x} + (p-1)\int_\Omega d_\varepsilon u_\varepsilon^{p-1}u_{\varepsilon x}w_{\varepsilon x} \\
&\le -\frac{p-1}{2}\int_\Omega d_\varepsilon u_\varepsilon^{p-2}u_{\varepsilon x}^2 + (p-1)\int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon}u_\varepsilon^p + (p-1)\int_\Omega d_\varepsilon u_\varepsilon^p w_{\varepsilon x}^2 \qquad \text{for all } t\in(0,T_{max,\varepsilon}),
\end{aligned}$$

so that since d_ε is smooth and positive throughout $\overline{\Omega}$, we can find $c_1 > 0$ and $c_2 > 0$ fulfilling

$$\frac{d}{dt}\int_\Omega u_\varepsilon^p + c_1\int_\Omega (u_\varepsilon^{\frac{p}{2}})_x^2 \le c_2\int_\Omega u_\varepsilon^p + c_2\int_\Omega u_\varepsilon^p w_{\varepsilon x}^2 \qquad \text{for all } t\in(0,T_{max,\varepsilon}). \tag{4.17}$$

Here using the Cauchy-Schwarz inequality, the Gagliardo-Nirenberg inequality and Young's inequality along with (2.25) and the estimate from Lemma 4.2, we obtain positive constants c_3, c_4, c_5 and c_6 such that

$$\begin{aligned}
c_2\int_\Omega u_\varepsilon^p w_{\varepsilon x}^2 &\le c_2\cdot\Big\{\int_\Omega u_\varepsilon^{2p}\Big\}^{\frac{1}{2}}\cdot\Big\{\int_\Omega w_{\varepsilon x}^4\Big\}^{\frac{1}{2}} \\
&\le c_3\|u_\varepsilon^{\frac{p}{2}}\|_{L^4(\Omega)}^2 \\
&\le c_4\|(u_\varepsilon^{\frac{p}{2}})_x\|_{L^2(\Omega)}^{\frac{2p-1}{p+1}}\|u_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{3}{p+1}} + c_4\|u_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^2 \\
&\le c_5\|(u_\varepsilon^{\frac{p}{2}})_x\|_{L^2(\Omega)}^{\frac{2p-1}{p+1}} + c_5 \\
&\le c_1\int_\Omega (u_\varepsilon^{\frac{p}{2}})_x^2 + c_6 \qquad \text{for all } t\in(0,T_{max,\varepsilon}),
\end{aligned}$$

because $\frac{2p-1}{p+1} < 2$. Therefore, (4.17) entails that

$$\frac{d}{dt}\int_\Omega u_\varepsilon^p \le c_2\int_\Omega u_\varepsilon^p + c_6 \qquad \text{for all } t\in(0,T_{max,\varepsilon})$$

and thus, upon integration, that

$$\int_\Omega u_\varepsilon^p \le \Big\{\int_\Omega u_0^p\Big\}\cdot e^{c_2 t} + \frac{c_6}{c_2}(e^{c_2 t}-1) \qquad \text{for all } t\in(0,T_{max,\varepsilon}),$$

which implies (4.16). $\Box$

By means of a standard result based on a Moser-type iteration, along with Lemma 4.2 this readily yields boundedness of u_ε whenever $T_{max,\varepsilon} < \infty$.

Lemma 4.4 *If $T_{max,\varepsilon} < \infty$ for some $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, then there exists $C(\varepsilon) > 0$ such that*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C(\varepsilon) \qquad \textit{for all } t \in (0, T_{max,\varepsilon}). \tag{4.18}$$

PROOF. We rewrite the first equation in (2.22) in the form

$$u_{\varepsilon t} = (d_\varepsilon u_{\varepsilon x})_x + h_{\varepsilon x}, \qquad x \in \Omega,\ t \in (0, T_{max,\varepsilon}),$$

with

$$h_\varepsilon(x,t) := d_{\varepsilon x}(x) u_\varepsilon(x,t) - d_\varepsilon(x) u_\varepsilon(x,t) w_{\varepsilon x}(x,t), \qquad x \in \Omega,\ t \in (0, T_{max,\varepsilon}),$$

and note that for any fixed $q \in (3,4)$, by the Hölder inequality we have

$$\begin{aligned}
\int_\Omega |h_\varepsilon(\cdot,t)|^q &\le 2^{q-1} \int_\Omega |d_\varepsilon|^q u_\varepsilon^q + 2^{q-1} \int_\Omega d_\varepsilon^q u_\varepsilon^q |w_{\varepsilon x}|^q \\
&\le 2^{q-1} |d_{\varepsilon x}\|_{L^\infty(\Omega)}^q \int_\Omega u_\varepsilon^q + 2^{q-1} \|d_\varepsilon\|_{L^\infty(\Omega)}^q \cdot \left\{ \int_\Omega u_\varepsilon^{\frac{4q}{4-q}} \right\}^{\frac{4-q}{4}} \cdot \left\{ \int_\Omega w_{\varepsilon x}^4 \right\}^{\frac{q}{4}}
\end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$. As Lemma 4.2 and Lemma 4.3 guarantee that

$$\sup_{t\in(0,T_{max,\varepsilon})} \int_\Omega w_{\varepsilon x}^4(\cdot,t) < \infty \qquad \text{and} \qquad \sup_{t\in(0,T_{max,\varepsilon})} \int_\Omega u_\varepsilon^p(\cdot,t) < \infty \quad \text{for all } p \in (1,\infty), \tag{4.19}$$

this implies that h_ε belongs to $L^\infty((0,T_{max,\varepsilon}); L^q(\Omega))$, so that using that $q > 3$ we may apply a known Moser-type result on boundedness in scalar parabolic equations ([40, Lemma A.1]) to see that along with the identities

$$u_{\varepsilon x} = 0 \quad \text{and} \quad h_\varepsilon = 0 \qquad \text{on } \partial\Omega \times (0, T_{max,\varepsilon}),$$

the latter being asserted by the fact that $d_{\varepsilon x} = 0$ on $\partial\Omega$ by (2.8), the second property in (4.19) is sufficient to warrant (4.18). $\Box$

In conclusion, finite-time blow-up cannot occur in any of the approximate problems.

Lemma 4.5 *For all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, the solution of (2.22) is global in time.*

PROOF. In view of the extensibility criterion (2.24), we only need to collect the outcomes of Lemma 4.1, Lemma 4.2 and Lemma 4.4. $\Box$

5 Further ε-independent regularity properties of (2.22)

5.1 Equi-integrability properties

Now a key to both our existence proof and our stabilization results consists in the observation that due to Lemma 3.5, and again due to the assumed integrability of $\frac{1}{d}$, the solution component u_ε enjoys a certain doubly uniform integrability property. In order to prepare this and also our subsequent analysis, let us introduce

$$\omega_d(\delta) := \sup\left\{ \int_E \frac{1}{d} \;\middle|\; E \subset \Omega \text{ is measurable with } |E| \le \delta \right\} \qquad \text{for } \delta > 0, \tag{5.1}$$

and observe that then our integrability assumption (1.6) warrants that

$$\omega_d(\delta) \to 0 \qquad \text{as } \delta \searrow 0. \tag{5.2}$$

Along with Lemma 3.5, this will entail the following.

Lemma 5.1 *For all $\eta > 0$ there exists $\delta > 0$ such that whenever $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,*

$$\int_E u_\varepsilon(\cdot,t) \le \eta \qquad \text{for all } t \in \Big(0, \frac{1}{\sqrt{\varepsilon}}\Big) \quad \text{and any measurable } E \subset \Omega \text{ such that } |E| \le \delta. \tag{5.3}$$

Proof. According to Lemma 3.5 we can fix $c_1 > 0$ such that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ we have

$$\int_\Omega u_\varepsilon \Big| \ln(d_\varepsilon u_\varepsilon) \Big| \le c_1 \cdot (1 + \sqrt{\varepsilon} t) \qquad \text{for all } t > 0, ,$$

whence in particular

$$\int_\Omega u_\varepsilon \Big| \ln(d_\varepsilon u_\varepsilon) \Big| \le 2c_1 \qquad \text{for all } t \in \Big(0, \frac{1}{\sqrt{\varepsilon}}\Big). \tag{5.4}$$

We then let $\eta > 0$ be given and pick $L > 1$ large enough fulfilling $\frac{2c_1}{\ln L} \le \frac{\eta}{2}$, and thereafter make use of (5.2) in choosing some $\delta > 0$ such that $\omega_d(\delta) \le \frac{\eta}{2L}$. Then for any measurable $E \subset \Omega$ with $|E| \le \delta$ and each $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, again using the fact that $d_\varepsilon \ge d$ we find that

$$\begin{aligned}
\int_E u_\varepsilon &= \int_{E\cap\{d_\varepsilon u_\varepsilon > L\}} u_\varepsilon + \int_{E\cap\{d_\varepsilon u_\varepsilon \le L\}} u_\varepsilon \\
&\le \int_{E\cap\{d_\varepsilon u_\varepsilon > L\}} u_\varepsilon \cdot \frac{\ln(d_\varepsilon u_\varepsilon)}{\ln L} + \int_{E\cap\{d_\varepsilon u_\varepsilon \le L\}} \frac{L}{d_\varepsilon} \\
&\le \frac{1}{\ln L} \cdot \int_{E\cap\{d_\varepsilon u_\varepsilon > L\}} u_\varepsilon \Big| \ln(d_\varepsilon u_\varepsilon) \Big| + L \int_{E\cap\{d_\varepsilon u_\varepsilon \le L\}} \frac{1}{d_\varepsilon} \\
&\le \frac{1}{\ln L} \cdot \int_\Omega u_\varepsilon \Big| \ln(d_\varepsilon u_\varepsilon) \Big| + L \int_E \frac{1}{d_\varepsilon} \\
&\le \frac{1}{\ln L} \cdot 2c_1 + L\omega_d(\delta) \\
&\le \frac{\eta}{2} + \frac{\eta}{2} = \eta \qquad \text{for all } t \in \Big(0, \frac{1}{\sqrt{\varepsilon}}\Big),
\end{aligned}$$

as claimed. $\Box$

Likewise, the weighted L^2 estimate for $w_{\varepsilon x}$ in Lemma 3.5 can be turned into a corresponding equi-integrability statement for $w_{\varepsilon x}$, and apart from that it implies an additional boundedness property of w_ε in a space compactly embedded into $C^0(\overline{\Omega})$.

Lemma 5.2 *For all $\eta > 0$ there exists $\delta > 0$ with the property that for arbitrary $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,*

$$\int_E |w_{\varepsilon x}(\cdot,t)| \le \eta \qquad \textit{for all } t \in \Big(0, \frac{1}{\sqrt{\varepsilon}}\Big) \quad \textit{whenever } E \subset \Omega \textit{ is measurable with } |E| \le \delta. \tag{5.5}$$

Moreover, there exists $C > 0$ such that for arbitrary $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,

$$\|w_\varepsilon(\cdot,t)\|_Y \le C \qquad \textit{for all } t \in \Big(0, \frac{1}{\sqrt{\varepsilon}}\Big), \tag{5.6}$$

where the Banach space Y is defined by

$$Y := \left\{ \varphi \in C^0(\overline{\Omega}) \;\middle|\; \|\varphi\|_Y := \|\varphi\|_{L^\infty(\Omega)} + \sup_{x,y\in\Omega,\ x\ne y} \frac{|\varphi(x)-\varphi(y)|}{\sqrt{\omega_d(|x-y|)}} < \infty \right\} \tag{5.7}$$

with ω_d as in (5.1).

Proof. From Lemma 3.5 and (2.26) we obtain $c_1 > 0$ such that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,

$$\int_\Omega d_\varepsilon w_{\varepsilon x}^2 \le c_1 \cdot (1 + \sqrt{\varepsilon} t) \qquad \text{for all } t > 0$$

and hence

$$\int_\Omega d_\varepsilon w_{\varepsilon x}^2 \le 2c_1 \qquad \text{for all } t \in \Big(0, \frac{1}{\sqrt{\varepsilon}}\Big). \tag{5.8}$$

Therefore, an application of the Cauchy-Schwarz inequality shows that for arbitrary measurable $F \subset \Omega$ we can estimate

$$\int_F |w_{\varepsilon x}| \le \Big\{ \int_F d_\varepsilon w_{\varepsilon x}^2 \Big\}^{\frac12} \cdot \Big\{ \int_F \frac{1}{d_\varepsilon} \Big\}^{\frac12} \le \sqrt{2c_1}\sqrt{\omega_d(|F|)} \qquad \text{for all } t \in \Big(0, \frac{1}{\sqrt{\varepsilon}}\Big). \tag{5.9}$$

In particular, if given $\eta > 0$ we let $\delta > 0$ be such that $\omega_d(\delta) \le \frac{\eta^2}{2c_1}$, then for each measurable $E \subset \Omega$ fulfilling $|E| \le \delta$ we conclude from (5.9) that

$$\int_E |w_{\varepsilon x}| \le \sqrt{2c_1}\sqrt{\omega_d(\delta)} \le \eta \qquad \text{for all } t \in \Big(0, \frac{1}{\sqrt{\varepsilon}}\Big)$$

and that thus (5.5) holds. Furthermore, for $x \in \Omega$ and $y \in \Omega$ with $y < x$, a second application of (5.9), now to $F := (y,x)$, shows that

$$|w_\varepsilon(x,t) - w_\varepsilon(y,t)| = \left| \int_y^x w_{\varepsilon x}(z,t)dz \right| \le \int_y^x |w_{\varepsilon x}(z,t)|dz \le \sqrt{2c_1}\sqrt{\omega_d(|x-y|)} \quad \text{for all } t \in \Big(0, \frac{1}{\sqrt{\varepsilon}}\Big),$$

which together with a similar estimate in the case $y > x$ establishes (5.6). $\Box$

5.2 A local estimate for $u_{\varepsilon x}$

In order to ultimately achieve pointwise convergence of u_ε along a subsequence of $(\varepsilon_j)_{j\in\mathbb{N}}$ through a compactness argument based on the Aubin-Lions lemma in Lemma 6.1, let us combine the weighted estimate for $(\sqrt{d_\varepsilon u_\varepsilon})_x$ from Lemma 3.5 with (2.25) and the boundedness properties of d_ε inside $\{d>0\}$ to derive the following local but unweighted integral estimate for $u_{\varepsilon x}$ itself.

Lemma 5.3 *Let $K \subset \{d>0\}\cap\Omega$ be compact. Then there exists $C(K)>0$ such that for all $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}$,*

$$\int_0^T \left\{ \int_K |u_{\varepsilon x}(x,t)|dx \right\}^2 dt \le C(K)\cdot(1+T) \qquad \textit{for all } T>0. \tag{5.10}$$

Proof. According to Lemma 2.2, our assumption on K ensures that with some $c_1>0$ we have

$$d_\varepsilon \ge c_1 \quad \text{in } K \qquad \text{for all } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}, \tag{5.11}$$

and that moreover $d_{\varepsilon x}\to d_x$ in $L^\infty(K)$, whence there exists $c_2>0$ fulfilling

$$|d_{\varepsilon x}| \le c_2 \quad \text{in } K \qquad \text{for all } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}. \tag{5.12}$$

We now make use of the fact that Lemma 3.5 yields $c_3>0$ satisfying

$$\int_0^T\int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{d_\varepsilon u_\varepsilon} \le c_3\cdot(1+T) \qquad \text{for all } T>0 \text{ and } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}},$$

which namely implies that for any such ε we have

$$\begin{aligned} c_3\cdot(1+T) &\ge \int_0^T\int_K \frac{(d_\varepsilon u_{\varepsilon x}+d_{\varepsilon x}u_\varepsilon)^2}{d_\varepsilon u_\varepsilon} \\ &\ge \int_0^T\int_K \frac{\frac{1}{2}d_\varepsilon^2 u_{\varepsilon x}^2 - d_{\varepsilon x}^2 u_\varepsilon^2}{d_\varepsilon u_\varepsilon} \\ &= \frac{1}{2}\int_0^T\int_K d_\varepsilon \frac{u_{\varepsilon x}^2}{u_\varepsilon} - \int_0^T\int_K \frac{d_{\varepsilon x}^2}{d_\varepsilon} u_\varepsilon \qquad \text{for all } T>0, \end{aligned}$$

because $(\xi+\eta)^2 \ge \frac{1}{2}\xi^2-\eta^2$ for all $\xi\in\mathbb{R}$ and $\eta\in\mathbb{R}$. In view of (5.11), (5.12) and (2.25), this shows that

$$\begin{aligned} \frac{c_1}{2}\int_0^T\int_K \frac{u_{\varepsilon x}^2}{u_\varepsilon} &\le \frac{1}{2}\int_0^T\int_K d_\varepsilon\frac{u_{\varepsilon x}^2}{u_\varepsilon} \\ &\le \int_0^T\int_K \frac{d_{\varepsilon x}^2}{d_\varepsilon}u_\varepsilon + c_3\cdot(1+T) \\ &\le \frac{c_2^2}{c_1}\int_0^T\int_\Omega u_\varepsilon + c_3\cdot(1+T) \\ &= \frac{c_2^2}{c_1}T\int_\Omega u_0 + c_3\cdot(1+T) \qquad \text{for all } T>0, \end{aligned}$$

which readily implies (5.10) upon the observation that

$$\begin{aligned}
\int_0^t \Big\{ \int_K |u_{\varepsilon x}| \Big\}^2 &\le \int_0^T \Big\{ \int_K \frac{u_{\varepsilon x}^2}{u_\varepsilon} \Big\} \cdot \Big\{ \int_K u_\varepsilon \Big\} \\
&\le \Big\{ \int_0^T \int_K \frac{u_{\varepsilon x}^2}{u_\varepsilon} \Big\} \cdot \Big\{ \int_\Omega u_0 \Big\} \qquad \text{for all } T > 0
\end{aligned}$$

according to the Cauchy-Schwarz inequality and (2.25). $\Box$

5.3 Time regularity

As a final preparation for our subsequence extraction, let us derive some regularity features of the respective time derivatives. The first of these, again resulting from Lemma 3.5, is actually asymptotically independent of the length of the time interval appearing therein, and hence can serve below as a first information on decay of temporal oscillations.

Lemma 5.4 *There exists $C > 0$ such that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,*

$$\int_0^T \|u_{\varepsilon t}(\cdot,t)\|^2_{(W^{1,\infty}(\Omega))^\star} dt \le C \cdot (1 + \sqrt{\varepsilon} T) \qquad \textit{for all } T > 0. \tag{5.13}$$

PROOF. We fix $t > 0$ and $\psi \in W^{1,\infty}(\Omega)$ such that $\|\psi\|_{W^{1,\infty}(\Omega)} \le 1$, and then obtain on testing the first equation in (2.22) by ψ and using the Cauchy-Schwarz inequality and (2.9) as well as (2.25) and (2.26) that

$$\begin{aligned}
\Big| \int_\Omega u_{\varepsilon t}(\cdot,t)\psi \Big| &= \Big| - \int_\Omega (d_\varepsilon u_\varepsilon)_x \psi_x + \int_\Omega d_\varepsilon u_\varepsilon w_{\varepsilon x} \psi_x \Big| \\
&\le \int_\Omega \Big| (d_\varepsilon u_\varepsilon)_x \Big| + \int_\Omega d_\varepsilon u_\varepsilon |w_{\varepsilon x}| \\
&\le \Big\{ \int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{d_\varepsilon u_\varepsilon} \Big\}^{\frac{1}{2}} \cdot \Big\{ \int_\Omega d_\varepsilon u_\varepsilon \Big\}^{\frac{1}{2}} + \Big\{ \int_\Omega d_\varepsilon u_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} \Big\}^{\frac{1}{2}} \cdot \Big\{ \int_\Omega d_\varepsilon u_\varepsilon w_\varepsilon \Big\}^{\frac{1}{2}} \\
&\le \Big\{ \int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{d_\varepsilon u_\varepsilon} \Big\}^{\frac{1}{2}} \cdot (\|d\|_{L^\infty(\Omega)} + 1)^{\frac{1}{2}} \cdot \Big\{ \int_\Omega u_0 \Big\}^{\frac{1}{2}} \\
&\quad + \Big\{ \int_\Omega d_\varepsilon u_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} \Big\}^{\frac{1}{2}} \cdot (\|d\|_{L^\infty(\Omega)} + 1)^{\frac{1}{2}} \cdot \Big\{ \int_\Omega u_0 \Big\}^{\frac{1}{2}} \cdot \sqrt{M}
\end{aligned}$$

for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$. Writing $c_1 := (\|d\|_{L^\infty(\Omega)} + 1) \cdot \int_\Omega u_0$, we thus infer that for any such ε,

$$\|u_{\varepsilon t}(\cdot,t)\|^2_{(W^{1,\infty}(\Omega))^\star} \le 2c_1 \int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{d_\varepsilon u_\varepsilon} + 2c_1 M \int_\Omega d_\varepsilon u_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} \qquad \text{for all } t > 0,$$

which in view of Lemma 3.5 implies (5.13) on integration in time. $\Box$

Next, the estimates from Lemma 3.5 imply the following temporally local estimate for $w_{\varepsilon t}$ in a straightforward manner.

Lemma 5.5 *Let $T > 0$. Then there exists $C(T) > 0$ such that*

$$\int_0^T \|w_{\varepsilon t}(\cdot,t)\|_{L^1(\Omega)}^2 dt \le C(T) \qquad \textit{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}. \tag{5.14}$$

PROOF. By directly using the second equation in (2.22) we can estimate

$$\left\{ \int_\Omega |w_{\varepsilon t}| \right\}^2 \le 2 \cdot \left\{ \varepsilon \cdot \int_\Omega \left| \left(d_\varepsilon \frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \right)_x \right| \right\}^2 + 2 \cdot \left\{ \int_\Omega u_\varepsilon g(w_\varepsilon) \right\}^2 \qquad \text{for all } t > 0, \tag{5.15}$$

where due to the Cauchy-Schwarz inequality, (1.8) and (2.26),

$$\begin{aligned} \left\{ \varepsilon \cdot \int_\Omega \left| \left(d_\varepsilon \frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \right)_x \right| \right\}^2 &\le \varepsilon^2 \cdot \left\{ \int_\Omega \frac{1}{\sqrt{g(w_\varepsilon)}} \cdot \left(d_\varepsilon \frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \right)_x^2 \right\} \cdot \left\{ \int_\Omega \sqrt{g(w_\varepsilon)} \right\} \\ &\le \sqrt{\overline{\gamma} M} |\Omega| \cdot \varepsilon \int_\Omega \frac{1}{\sqrt{g(w_\varepsilon)}} \cdot \left(d_\varepsilon \frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \right)_x^2 \qquad \text{for all } t > 0, \end{aligned}$$

because $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$. As (1.8) and (2.26) together with (2.25) assert that

$$\left\{ \int_\Omega u_\varepsilon g(w_\varepsilon) \right\}^2 \le \overline{\gamma}^2 M^2 \cdot \left\{ \int_\Omega u_\varepsilon \right\}^2 = \overline{\gamma}^2 M^2 \cdot \left\{ \int_\Omega u_0 \right\}^2 \qquad \text{for all } t > 0,$$

in view of Lemma 3.5 we thus obtain (5.14) from (5.15). □

6 Global existence in the degenerate problem

6.1 Construction of limit functions

By means of a straightforward extraction procedure based on our estimates collected so far as well as standard compactness arguments, we can now construct a limit object that will finally turn out to solve (1.4) in the considered generalized sense.

Lemma 6.1 *There exist a subsequence $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ of $(\varepsilon_j)_{j\in\mathbb{N}}$ and nonnegative functions*

$$\begin{cases} u \in L^1_{loc}(\overline{\Omega} \times [0,\infty)) \cap C^0([0,\infty); (W^{1,\infty}(\Omega))^\star) \qquad \textit{and} \\ w \in L^\infty(\Omega \times (0,\infty)) \cap C^0([0,\infty); L^1(\Omega)) \cap L^1_{loc}([0,\infty); W^{1,1}(\Omega)) \end{cases} \tag{6.1}$$

such that

$$\begin{aligned} u_\varepsilon &\to u && \textit{a.e. in } \Omega \times (0,\infty), && (6.2) \\ u_\varepsilon &\to u && \textit{in } L^1_{loc}(\overline{\Omega} \times [0,\infty)), && (6.3) \\ u_\varepsilon &\to u && \textit{in } C^0_{loc}([0,\infty); (W^{1,\infty}(\Omega))^\star), && (6.4) \\ w_\varepsilon &\to w && \textit{in } C^0_{loc}(\overline{\Omega} \times [0,\infty)), && (6.5) \\ w_{\varepsilon x} &\rightharpoonup w_x && \textit{in } L^1_{loc}(\overline{\Omega} \times [0,\infty)) \qquad \textit{and} && (6.6) \\ \sqrt{d_\varepsilon} w_{\varepsilon x} &\rightharpoonup \sqrt{d} w_x && \textit{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)) && (6.7) \end{aligned}$$

as $\varepsilon = \varepsilon_{j_k} \searrow 0$.

Proof. We first combine Lemma 5.3 with (2.25) to see that for any open $\Omega_0 \subset \Omega$ satisfying $\overline{\Omega}_0 \subset \{d > 0\} \cap \Omega$,

$$(u_{\varepsilon_j})_{j\in\mathbb{N}} \quad \text{is bounded in } L^2((0,T); W^{1,1}(\Omega_0)) \text{ for all } T > 0,$$

whereas Lemma 5.4, asserting that

$$(u_{\varepsilon_j t})_{j\in\mathbb{N}} \quad \text{is bounded in } L^2\Big((0,T); (W^{1,\infty}(\Omega))^\star\Big) \text{ for all } T > 0, \tag{6.8}$$

entails that

$$(u_{\varepsilon_j t})_{j\in\mathbb{N}} \quad \text{is bounded in } L^2\Big((0,T); (W_0^{1,\infty}(\Omega_0))^\star\Big) \text{ for all } T > 0$$

due to the observation that the trivial extension ψ of any $\psi_0 \in W_0^{1,\infty}(\Omega_0)$ to all of Ω satisfies $\psi \in W^{1,\infty}(\Omega)$ with $\|\psi\|_{W^{1,\infty}(\Omega)} \le \|\psi_0\|_{W_0^{1,\infty}(\Omega_0)}$. For any such Ω_0, in view of the compactness of the first of the embeddings $W^{1,1}(\Omega_0) \hookrightarrow L^2(\Omega) \hookrightarrow (W_0^{1,\infty}(\Omega_0))^\star$ the Aubin-Lions lemma ([44]) thus guarantees that

$$(u_{\varepsilon_j})_{j\in\mathbb{N}} \quad \text{is relatively compact in } L^2(\Omega_0 \times (0,T)),$$

so that since d is continuous in $\overline{\Omega}$, and since our assumption that $\frac{1}{d} \in L^1(\Omega)$ especially ensures that $d > 0$ a.e. in Ω, by means of a straightforward successive extraction procedure we obtain a decreasing subsequence $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ of $(\varepsilon_j)_{j\in\mathbb{N}}$ and a nonnegative measurable function $u : \Omega \times (0,\infty) \to \mathbb{R}$ such that (6.2) holds. As from Lemma 5.1 we particularly know that

$$(u_{\varepsilon_j})_{j\in\mathbb{N}} \quad \text{is equi-integrable in } \Omega \times (0,T) \text{ for all } T > 0,$$

due to (6.2) we may invoke the Vitali convergence theorem to see that also (6.3) holds along this sequence. Moreover, combining (6.8) with the fact that

$$(u_{\varepsilon_j})_{j\in\mathbb{N}} \quad \text{is bounded in } L^\infty((0,T); L^1(\Omega)) \text{ for all } T > 0$$

due to (2.25), we may make use of the compactness of the embedding $L^1(\Omega) \hookrightarrow (W^{1,\infty}(\Omega))^\star$ in employing the Arzelà-Ascoli theorem to conclude that

$$(u_{\varepsilon_j})_{j\in\mathbb{N}} \quad \text{is relatively compact in } C^0\Big([0,T]; (W^{1,\infty}(\Omega))^\star\Big) \text{ for all } T > 0,$$

and that hence on modification of u on a null set of times we can also achieve (6.4).

As for the second solution component, we first note that as a consequence of Lemma 5.2, with Y as introduced in (5.7) we have that

$$(w_{\varepsilon_j})_{j\in\mathbb{N}} \quad \text{is bounded in } L^\infty((0,T); Y) \quad \text{for all } T > 0,$$

so that since due to Lemma 5.5,

$$(w_{\varepsilon_j t})_{j\in\mathbb{N}} \quad \text{is bounded in } L^2((0,T); L^1(\Omega)) \quad \text{for all } T > 0,$$

and since Y is compactly emdedded into $C^0(\overline{\Omega})$ according to the Arzelà-Acsoli theorem, another application of an Aubin-Lions lemma shows that

$$(w_{\varepsilon_j})_{j\in\mathbb{N}} \quad \text{is relatively compact in } C^0([0,T];C^0(\overline{\Omega})) \quad \text{for all } T>0. \tag{6.9}$$

As combining Lemma 5.2 with the Dunford-Pettis theorem apart from that warrants that

$$(w_{\varepsilon_j x})_{j\in\mathbb{N}} \quad \text{is relatively compact with respect to the weak topology in } L^1(\Omega\times(0,T))$$

for all $T>0$, we may assume on passing to a further subsequence if necessary that also (6.5) and (6.6) hold, and since furthermore Lemma 3.5 implies that

$$\Big(\sqrt{d_{\varepsilon_j}}w_{\varepsilon_j x}\Big)_{j\in\mathbb{N}} \quad \text{is bounded in } L^\infty((0,T);L^2(\Omega)) \quad \text{for all } T>0,$$

upon a final extraction process we can also achieve (6.7). □

6.2 Strong convergence of $\sqrt{d_\varepsilon}u_\varepsilon$ in L^2

In view of (6.7), for appropriate passing to the limit in the regularized counterpart of the haptotactic integral in (2.3) it seems in order to assert strong convergence of the expression $\sqrt{d_\varepsilon}u_\varepsilon$ with respect to the norm in $L^2(\Omega\times(0,T))$ for fixed $T>0$. In achieving this on the basis of the Vitali convergence theorem, we will make use of the following generalization of the Gagliardo-Nirenberg inequality that can be obtained by straighforward adaptation of the argument in [5] (cf. also [41, Lemma A.5]).

Lemma 6.2 *There exists $C>0$ such that for any choice of $\eta\in(0,1)$ one can find $C(\eta)>0$ with the property that*

$$\|\varphi\|_{L^\infty(\Omega)}^4 \le \eta\|\varphi_x\|_{L^2(\Omega)}^2\cdot\Big\||\varphi|\ln|\varphi|^{\frac{1}{2}}\Big\|_{L^2(\Omega)}^2 + C\|\varphi\|_{L^2(\Omega)}^4 + C(\eta) \qquad \text{for all } \varphi\in W^{1,2}(\Omega). \tag{6.10}$$

We can thereby once more exploit the estimates for u_ε from Lemma 3.5 to infer the following spatio-temporal equi-integrability property of $d_\varepsilon u_\varepsilon^2$.

Lemma 6.3 *Let $T>0$. Then for all $\eta>0$ one can find $\delta>0$ such that for any choice of $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}$,*

$$\int\int_E d_\varepsilon u_\varepsilon^2 \le \eta \qquad \text{for all measurable } E\subset\Omega\times(0,T) \text{ fulfilling } |E|\le\delta. \tag{6.11}$$

Proof. In conclusion of Lemma 3.5, we can fix $c_1>0$ and $c_2>0$ such that for all $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}$ we have

$$\int_\Omega u_\varepsilon\Big|\ln(d_\varepsilon u_\varepsilon)\Big| \le c_1 \qquad \text{for all } t\in(0,T) \tag{6.12}$$

and

$$\int_0^T\int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{d_\varepsilon u_\varepsilon} \le c_2. \tag{6.13}$$

Then for arbitrary $\eta > 0$, applying Lemma 6.2 and using that $c_3 := \int_\Omega \frac{1}{d}$ is finite, we may pick $c_4 > 0$ such that

$$\|\varphi\|_{L^\infty(\Omega)}^4 \le \frac{4\eta}{c_1 c_2 c_3 (\|d\|_{L^\infty(\Omega)}+1)} \|\varphi_x\|_{L^2(\Omega)}^2 \Big\|\varphi \ln^{\frac{1}{2}}|\varphi|\Big\|_{L^2(\Omega)}^2 + c_4 \|\varphi\|_{L^2(\Omega)}^4 + c_4 \qquad \text{for all } \varphi \in W^{1,2}(\Omega) \tag{6.14}$$

and abbreviate $c_5 := c_4(\|d\|_{L^\infty\Omega)}+1)^2 \Big\{ \int_\Omega u_0 \Big\}^2 + c_4$. Now once more since $\frac{1}{d} \in L^1(\Omega)$ and hence also $\frac{1}{d} \in L^1(\Omega \times (0,T))$, we can find $\delta > 0$ such that

$$\int\int_E \frac{1}{d} \le \frac{\eta}{2c_5} \qquad \text{for each measurable } E \subset \Omega \times (0,T) \text{ satisfying } |E| \le \delta. \tag{6.15}$$

In order to derive (6.11) from this, we observe that by (6.14),

$$\begin{aligned} d_\varepsilon u_\varepsilon^2 &= \frac{\sqrt{d_\varepsilon u_\varepsilon}^4}{d_\varepsilon} \\ &\le \frac{4\eta}{c_1 c_2 c_3 (\|d\|_{L^\infty(\Omega)}+1) d_\varepsilon} \Big\|(\sqrt{d_\varepsilon u_\varepsilon})_x\Big\|_{L^2(\Omega)}^2 \Big\|\sqrt{d_\varepsilon u_\varepsilon} \cdot \sqrt{|\ln \sqrt{d_\varepsilon u_\varepsilon}|}\Big\|_{L^2(\Omega)}^2 + \frac{c_4}{d_\varepsilon} \|\sqrt{d_\varepsilon u_\varepsilon}\|_{L^2(\Omega)}^4 + \frac{c_4}{d_\varepsilon} \\ &= \frac{\eta}{2 c_1 c_2 c_3 (\|d\|_{L^\infty(\Omega)}+1) d_\varepsilon} \cdot \Big\{ \int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{d_\varepsilon u_\varepsilon} \Big\} \cdot \Big\{ \int_\Omega d_\varepsilon u_\varepsilon \Big| \ln(d_\varepsilon u_\varepsilon) \Big| \Big\} \\ &\quad + \frac{c_4}{d_\varepsilon} \cdot \Big\{ \int_\Omega d_\varepsilon u_\varepsilon \Big\}^2 + \frac{c_4}{d_\varepsilon} \qquad \text{for all } x \in \Omega \text{ and } t > 0, \end{aligned} \tag{6.16}$$

where according to (6.12), (2.9) and (2.25),

$$\int_\Omega d_\varepsilon u_\varepsilon \Big| \ln(d_\varepsilon u_\varepsilon) \Big| \le (\|d\|_{L^\infty(\Omega)}+1) \cdot \int_\Omega u_\varepsilon \Big| \ln(d_\varepsilon u_\varepsilon) \Big| \le (\|d\|_{L^\infty(\Omega)}+1) \cdot c_1 \qquad \text{for all } \in (0,T)$$

and

$$\frac{c_4}{d_\varepsilon} \cdot \Big\{ \int_\Omega d_\varepsilon u_\varepsilon \Big\}^2 + \frac{c_4}{d_\varepsilon} \le \frac{c_4}{d_\varepsilon} \cdot (\|d\|_{L^\infty(\Omega)}+1)^2 \cdot \Big\{ \int_\Omega u_0 \Big\}^2 + \frac{c_4}{d_\varepsilon} = \frac{c_5}{d_\varepsilon} \qquad \text{for all } x \in \Omega \text{ and } t > 0.$$

Therefore, given any measurable $E \subset (0,T)$ with $|E| \le \delta$, we infer on integrating (6.16) that due to (6.13) and (6.15), indeed we have

$$\begin{aligned} \int\int_E d_\varepsilon u_\varepsilon^2 &\le \frac{\eta}{2 c_2 c_3} \cdot \int\int_E \frac{1}{d_\varepsilon} \cdot \Big\{ \int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{d_\varepsilon u_\varepsilon} \Big\} + c_5 \int\int_E \frac{1}{d_\varepsilon} \\ &\le \frac{\eta}{2 c_2 c_3} \cdot \int_0^T \Big\{ \int_\Omega \frac{1}{d} \Big\} \cdot \Big\{ \frac{(d_\varepsilon u_\varepsilon)_x^2}{d_\varepsilon u_\varepsilon} \Big\} + c_5 \int\int_E \frac{1}{d} \\ &\le \frac{\eta}{2} + \frac{\eta}{2} = \eta \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}, \end{aligned}$$

again because $d_\varepsilon \ge d$. □

In consequence, the Vitali convergence theorem entails the desired strong convergence feature.

Lemma 6.4 *With $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ taken from Lemma 6.1, we have*

$$\sqrt{d_\varepsilon}u_\varepsilon \to \sqrt{d}u \quad \text{in } L^2_{loc}(\overline{\Omega}\times[0,\infty)) \qquad \text{as } \varepsilon=\varepsilon_{j_k}\searrow 0.$$

PROOF. In view of the Vitali convergence theorem, this is a direct consequence of Lemma 6.3 when combined with the fact that due to Lemma 2.2 and Lemma 6.1 we have $\sqrt{d_\varepsilon}u_\varepsilon \to \sqrt{d}u$ a.e. in $\Omega\times(0,\infty)$ as $\varepsilon=\varepsilon_{j_k}\searrow 0$. □

6.3 Global existence in (1.4)

We are now prepared for appropriate limit procedures in each of the integrals related to (2.3) and (2.4).

Lemma 6.5 *The pair (u,w) obtained in Lemma 6.1 is a global generalized solution of (1.4) in the sense of Definition 2.1.*

PROOF. The regularity properties in (2.1) are implied by (6.1), whereas if we take $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ as provided by Lemma 6.1, then the strong L^2 convergence property of $(\sqrt{d_{\varepsilon_{j_k}}}u_{\varepsilon_{j_k}})_{k\in\mathbb{N}}$ asserted by Lemma 6.4 along with the weak L^2 approximation feature of $(\sqrt{d_{\varepsilon_{j_k}}}w_{\varepsilon_{j_k}x})_{k\in\mathbb{N}}$ gained in Lemma 6.1 warrants that

$$d_\varepsilon u_\varepsilon w_{\varepsilon x} = (\sqrt{d_\varepsilon}u_\varepsilon)\cdot(\sqrt{d_\varepsilon}w_{\varepsilon x}) \rightharpoonup d u w_x \qquad \text{in } L^1_{loc}(\overline{\Omega}\times[0,\infty)) \tag{6.17}$$

as $\varepsilon=\varepsilon_{j_k}\searrow 0$, and that hence also (2.2) holds.
The verification of the integral identity (2.3) is now straightofrward: Fixing an arbitrary $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$ such that $\varphi_x=0$ on $\partial\Omega\times(0,\infty)$, we obtain from the first equation in (2.22) that for each $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}$,

$$-\int_0^\infty\int_\Omega u_\varepsilon\varphi_t - \int_\Omega u_0\varphi(\cdot,0) = \int_0^\infty\int_\Omega d_\varepsilon u_\varepsilon\varphi_{xx} + \int_0^\infty\int_\Omega d_\varepsilon u_\varepsilon w_{\varepsilon x}\varphi_x, \tag{6.18}$$

where (6.17) ensures that

$$\int_0^\infty\int_\Omega d_\varepsilon u_\varepsilon w_{\varepsilon x}\varphi_x \to \int_0^\infty\int_\Omega d u w_x\varphi_x \qquad \text{as } \varepsilon=\varepsilon_{j_k}\searrow 0,$$

whereas using that $d_\varepsilon\to d$ in $L^\infty(\Omega)$ as $\varepsilon=\varepsilon_j\searrow 0$ we infer from (6.3) that

$$-\int_0^\infty\int_\Omega u_\varepsilon\varphi_t \to -\int_0^\infty\int_\Omega u\varphi_t \quad \text{and} \quad -\int_0^\infty\int_\Omega d_\varepsilon u_\varepsilon\varphi_{xx} \to \int_0^\infty\int_\Omega d u\varphi_{xx} \qquad \text{as } \varepsilon=\varepsilon_{j_k}\searrow 0,$$

so that (6.18) entails (2.3).
Likewise, for fixed $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$ the second equation in (2.22) yields

$$-\int_0^\infty\int_\Omega w_\varepsilon\varphi_t - \int_\Omega w_{0\varepsilon}\varphi(\cdot,0) = -\varepsilon\int_0^\infty\int_\Omega d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\varphi_x - \int_0^\infty\int_\Omega u_\varepsilon g(w_\varepsilon)\varphi \tag{6.19}$$

for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, where according to our construction of $(w_{0\varepsilon_j})_{j\in\mathbb{N}}$ in Lemma 2.6 and Lemma 2.5 we know that

$$\begin{aligned} -\int_\Omega w_{0\varepsilon_{j_k}}\varphi(\cdot,0) &= -\int_\Omega w_{0j_k}\varphi(\cdot,0) - \varepsilon_{j_k}^{\frac{1}{4}}\int_\Omega \varphi(\cdot,0) \\ &\to -\int_\Omega w_0\varphi(\cdot,0) \qquad \text{as } k\to\infty, \end{aligned}$$

and where according to the uniform convergence statement in (6.5), the L^1 approximation property (6.3) and the continuity of g on $[0,\infty)$,

$$-\int_0^\infty\int_\Omega w_\varepsilon\varphi_t \to -\int_0^\infty\int_\Omega w\varphi_t \qquad \text{as } \varepsilon=\varepsilon_{j_k}\searrow 0$$

and

$$-\int_0^\infty\int_\Omega u_\varepsilon g(w_\varepsilon)\varphi \to -\int_0^\infty\int_\Omega ug(w)\varphi \qquad \text{as } \varepsilon=\varepsilon_{j_k}\searrow 0.$$

Since the Cauchy-Schwarz inequality together with (1.8) implies that

$$\begin{aligned} \left| -\varepsilon\int_0^\infty\int_\Omega d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\varphi_x\right| &\le \varepsilon\cdot\left\{\int_0^\infty\int_\Omega \frac{1}{\sqrt{g(w_\varepsilon)}}\cdot\Big(d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x^2\right\}^{\frac{1}{2}}\cdot\left\{\int_0^\infty\int_\Omega \sqrt{g(w_\varepsilon)}\varphi^2\right\}^{\frac{1}{2}} \\ &\le \sqrt{\varepsilon}\cdot\left\{\sqrt{\varepsilon}\int_0^\infty\int_\Omega \frac{1}{\sqrt{g(w_\varepsilon)}}\cdot\Big(d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x^2\right\}^{\frac{1}{2}}\cdot(\overline{\gamma}M)^{\frac{1}{4}}\cdot\left\{\int_0^\infty\int_\Omega\varphi^2\right\}^{\frac{1}{2}} \end{aligned}$$

for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, and that hence

$$-\varepsilon\int_0^\infty\int_\Omega d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\varphi_x \to 0 \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0$$

thanks to Lemma 3.5, on taking $\varepsilon=\varepsilon_{j_k}\searrow 0$ in (6.19) we also obtain (2.4). $\square$

7 Further regularity properties of (u,w). Proof of Theorem 1.1

In order to complete the proof of Theorem 1.1, but also to further prepare our subsequent asymptotic analysis, let us use the equi-integrability and equicontinuity properties contained in Section 5.1 to firstly derive corresponding conclusions for the respective limit functions, and to secondly assert the continuity and mass conservation properties claimed in Theorem 1.1.

Lemma 7.1 *Let $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ be as in Lemma 6.1. The solution component u belongs to $C^0_w([0,\infty);L^1(\Omega))$, and with $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ taken from Lemma 6.1, we have*

$$u_\varepsilon(\cdot,t) \rightharpoonup u(\cdot,t) \quad \text{in } L^1(\Omega) \qquad \text{as } \varepsilon=\varepsilon_{j_k}\searrow 0. \tag{7.1}$$

Moreover,

$$(u(\cdot,t))_{t>0} \quad \text{is equi-integrable over } \Omega \tag{7.2}$$

and

$$(w(\cdot,t))_{t>0} \quad \text{is equicontinuous in } \overline{\Omega}. \tag{7.3}$$

PROOF. Once more using that with Y as in (5.7), the family $(w_{\varepsilon_j})_{j\in\mathbb{N}}$ is bounded in $L^\infty((0,\infty);Y)$, we directly see from (6.5) that $(w(\cdot,t))_{t>0}$ is bounded in Y and hence equicontinuous in $\overline{\Omega}$ according to (5.7).

Next, fixing an arbitrary $t>0$ we know from (6.4) that $u_\varepsilon(\cdot,t)\to u(\cdot,t)$ in $(W^{1,\infty}(\Omega))^\star$ as $\varepsilon=\varepsilon_{j_k}\searrow 0$, whereas Lemma 5.1 shows that $(u_{\varepsilon_j}(\cdot,t))_{j\in\mathbb{N}}$ is relatively compact with respect to the weak topology in $L^1(\Omega)$ due to the Dunford-Pettis theorem. Combining these two properties implies that any accumulation point of $(u_{\varepsilon_{j_k}}(\cdot,t))_{k\in\mathbb{N}}$ in the weak topology of $L^1(\Omega)$ must coincide with $u(\cdot,t)$, hence implying that $u(\cdot,t)\in L^1(\Omega)$ and $u_\varepsilon(\cdot,t)\rightharpoonup u(\cdot,t)$ in $L^1(\Omega)$ along the entire sequence $\varepsilon=\varepsilon_{j_k}\searrow 0$. Having thus verified (7.1), in view of the fact that this entails $\int_E u_\varepsilon(\cdot,t)\to\int_E u(\cdot,t)$ as $\varepsilon=\varepsilon_{j_k}\searrow 0$ for each measurable $E\subset\Omega$, we immediately also obtain (7.2) as a consequence of Lemma 5.1. Finally, the inclusion $u\in C^0_w([0,\infty);L^1(\Omega))$ can be seen by quite a similar argument: Given $t_0\ge 0$ and $(t_k)_{k\in\mathbb{N}}\subset(0,\infty)$ such that $t_k\to t_0$ as $k\to\infty$, again relying on (6.4) we note that $u(\cdot,t_k)\to u(\cdot,t)$ in $(W^{1,\infty}(\Omega))^\star$ as $k\to\infty$, whereas (7.2) in conjunction with the Dunford-Pettis theorem warrants that $(u(\cdot,t_k))_{k\in\mathbb{N}}$ is relatively compact with respect to the weak topology in $L^1(\Omega)$. As thus $u(\cdot,t)$ is the only cluster point of $(u(\cdot,t_k))_{k\in\mathbb{N}}$ in the latter space, we infer that indeed $u(\cdot,t_k)\rightharpoonup u(\cdot,t_0)$ in $L^1(\Omega)$ as $k\to\infty$. $\square$

Thus particularly knowing that not only $w(\cdot,t)$ but also $u(\cdot,t)$ is a well-defined element of $L^1(\Omega)$ for all $t>0$, we can proceed to formulate corresponding dissipation and conservation properties in this space, both of which being of great importance for our stabilization proof below.

Lemma 7.2 *We have*

$$\int_\Omega u(\cdot,t)=\int_\Omega u_0 \qquad \textit{for all } t>0 \tag{7.4}$$

and

$$\|w(\cdot,t)\|_{L^1(\Omega)}\le\|w(\cdot,t_0)\|_{L^1(\Omega)}\le\|w_0\|_{L^1(\Omega)} \qquad \textit{for all } t_0>0 \textit{ and any } t>t_0 \tag{7.5}$$

as well as

$$\int_0^\infty\int_\Omega uw\le\frac{1}{\underline{\gamma}}\int_\Omega w_0. \tag{7.6}$$

PROOF. The conservation property (7.4) is an immediate consequence of (2.25) and Lemma 7.1. For the derivation of (7.5) and (7.6) we integrate the second equation in (2.22) to see that

$$\frac{d}{dt}\int_\Omega w_\varepsilon=-\int_\Omega u_\varepsilon g(w_\varepsilon) \qquad \text{for all } t>0, \tag{7.7}$$

whence in particular

$$\int_\Omega w_\varepsilon(\cdot,t)\le\int_\Omega w_\varepsilon(\cdot,t_0)\le\int_\Omega w_{0\varepsilon} \qquad \text{for all } t_0>0 \text{ and any } t\in(t_0,\infty). \tag{7.8}$$

Recalling that by Lemma 2.6 and Lemma 2.5,

$$\int_\Omega w_{0\varepsilon_j}=\int_\Omega w_{0j}+\varepsilon_j^{\frac14}|\Omega|\to\int_\Omega w_0 \qquad \text{as } j\to\infty, \tag{7.9}$$

in view of (6.5) we thus obtain (7.5) from (7.8).
Finally, further integration of (7.7) shows that due to (1.8),

$$\underline{\gamma}\int_0^t\int_\Omega u_\varepsilon w_\varepsilon \le \int_0^t\int_\Omega u_\varepsilon g(w_\varepsilon) = \int_\Omega w_{0\varepsilon} - \int_\Omega w_\varepsilon(\cdot,t) \le \int_\Omega w_{0\varepsilon} \qquad \text{for all } t>0,$$

so that (7.9) along with (6.2) and (6.5) establishes (7.6) by means of Fatou's lemma. $\Box$

The proof of our main result on global existence, regularity and mass conservation is thereby complete:

Proof of Theorem 1.1. In Lemma 6.5 we have seen that (u,w) is a global generalized solution of (1.4) in the desired sense. The additional boundedness and continuity properties in (1.11) as well as the mass conservation law (1.12) readily result from Lemma 7.1, Lemma 6.1 and Lemma 7.2. $\Box$

8 Stabilization. Proof of Theorem 1.2

We next intend to properly exploit the global dissipative properties expressed in Lemma 3.5, Lemma 5.4 and Lemma 7.2 so as to derive the convergence results claimed in Theorem 1.2. We will first concentrate on the respective statement concerning u and thereafter consider the decay of the component w.

8.1 An averaged stabilization property of u

Let us first state a conseqence of Lemma 5.4 for the limit u in a form which does no longer involve time derivatives but rather concentrates on the quantity u itself, but which still reflects an appropriate relaxation property in the large time limit. The argument underlying the following lemma was kindly pointed out to us by one of the reviewers.

Lemma 8.1 *For each $\varphi \in L^\infty(\Omega)$, we have*

$$\sup_{\tau\in[0,1]}\left|\int_\Omega \Big(u(\cdot,t+\tau)-u(\cdot,t)\Big)\cdot\varphi\right| \to 0 \qquad \text{as } t\to\infty. \tag{8.1}$$

Proof. Given $\eta>0$, thanks to the equi-integrability property (7.2) we can fix $\delta>0$ such that whenever $E\subset\Omega$ is measurable with $|E|\le\delta$, we have

$$2\cdot\Big(2\|\varphi\|_{L^\infty(\Omega)}+1\Big)\int_E u(\cdot,t) \le \frac{\eta}{4} \qquad \text{for all } t>0. \tag{8.2}$$

Next, employing a standard regularization procedure we can find $(\varphi_k)_{k\in\mathbb{N}} \subset X := W^{1,\infty}(\Omega)$ such that

$$\|\varphi_k\|_{L^\infty(\Omega)} \le \|\varphi\|_{L^\infty(\Omega)}+1 \quad \text{for all } k\in\mathbb{N} \qquad \text{and} \qquad \varphi_k\to\varphi \quad \text{a.e. in } \Omega \text{ as } k\to\infty. \tag{8.3}$$

Due to Egorov's theorem, the latter approximation property in particular enables us to pick $k_0\in\mathbb{N}$ and a measurable $E\subset\Omega$ such that $|E|\le\delta$ and

$$2\cdot\left\{\int_\Omega u_0\right\}\cdot\|\varphi-\varphi_{k_0}\|_{L^\infty(\Omega\setminus E)} \le \frac{\eta}{4}. \tag{8.4}$$

Finally, Lemma 5.4 asserts the existence of $c_1 > 0$ such that

$$\int_0^T \|u_{\varepsilon t}(\cdot,t)\|_{X^\star}^2 dt \le c_1 \cdot (1+\sqrt{\varepsilon}T), \qquad \text{for all } T>0 \text{ and any } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}},$$

from which it readily follows by means of Lemma 6.1 and a lower semicontinuity argument that $\int_0^\infty \|u_t(\cdot,t)\|_{X^\star}^2 dt < \infty$ and that hence we can choose $t_0 > 0$ large enough fulfilling

$$\|\varphi_{k_0}\|_X \cdot \int_{t_0}^\infty \|u_t(\cdot,t)\|_{X^\star}^2 dt \le \frac{\eta}{2}. \tag{8.5}$$

Now decomposing the expression under consideration according to

$$\int_\Omega \Big(u(\cdot,t+\tau)-u(\cdot,t)\Big)\cdot\varphi = \int_\Omega \Big(u(\cdot,t+\tau)-u(\cdot,t)\Big)\cdot\varphi_{k_0} + \int_\Omega \Big(u(\cdot,t+\tau)-u(\cdot,t)\Big)\cdot(\varphi-\varphi_{k_0})$$

for $t>0$ and $\tau\in[0,1]$, by using the Cauchy-Schwarz inequality and (8.5) we may estimate

$$\begin{aligned}
\left|\int_\Omega \Big(u(\cdot,t+\tau)-u(\cdot,t)\Big)\cdot\varphi_{k_0}\right| &= \left|\int_t^{t+\tau}\langle u_t(\cdot,s),\varphi_{k_0}\rangle ds\right| \\
&\le \|\varphi_{k_0}\|_X \int_t^{t+\tau} \|u_t(\cdot,s)\|_{X^\star} ds \\
&\le \|\varphi_{k_0}\|_X \int_t^{\infty} \|u_t(\cdot,s)\|_{X^\star}^2 ds \\
&\le \frac{\eta}{2} \qquad \text{for all } t\ge t_0 \text{ and any } \tau\in[0,1],
\end{aligned}$$

where $\langle\cdot,\cdot,\rangle$ denotes the duality pairing between $X^\star$ and X. Since furthermore from (7.4), (8.3), (8.4) and (8.2) we know that

$$\begin{aligned}
\left|\int_\Omega \Big(u(\cdot,t+\tau)-u(\cdot,t)\Big)\cdot(\varphi-\varphi_{k_0})\right| &\le \left\{\int_{\Omega\setminus E}\Big(u(\cdot,t+\tau)-u(\cdot,t)\Big)\right\}\cdot\|\varphi-\varphi_{k_0}\|_{L^\infty(\Omega\setminus E)} \\
&\quad +\|\varphi-\varphi_{k_0}\|_{L^\infty(E)}\cdot\left\{\int_E u(\cdot,t+\tau)+\int_E u(\cdot,t)\right\} \\
&\le 2\cdot\left\{\int_\Omega u_0\right\}\cdot\|\varphi-\varphi_{k_0}\|_{L^\infty(\Omega\setminus E)} \\
&\quad +\Big(2\|\varphi\|_{L^\infty(\Omega)}+1\Big)\cdot\left\{\int_E u(\cdot,t+\tau)+\int_E u(\cdot,t)\right\} \\
&\le \frac{\eta}{4}+\frac{\eta}{4}=\frac{\eta}{2} \qquad \text{for all } t>0 \text{ and each } \tau\in[0,1],
\end{aligned}$$

we thus infer that

$$\left|\int_\Omega \Big(u(\cdot,t+\tau)-u(\cdot,t)\Big)\cdot\varphi\right| \le \eta \qquad \text{for all } t\ge t_0 \text{ and } \tau\in[0,1],$$

as intended. $\square$

8.2 Decaying deviation of du from its spatial average

Next aiming at a direct exploitation of (3.11), in view of the fact that through a Poincaré inequality the spatial gradients appearing therein control appropriate L^p norms of deviations from respective spatial means, let us briefly address the spatial averages relevant to our approach in the following.

Lemma 8.2 *The function μ defined on $[0,\infty)$ by letting*

$$\mu(t) := \frac{1}{|\Omega|}\int_\Omega d(x)u(x,t)dx, \qquad t>0, \tag{8.6}$$

is bounded and continuous on $[0,\infty)$, and with $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ as provided by Lemma 6.1 we have

$$\mu_\varepsilon(t) \to \mu(t) \quad \textit{for all } t>0 \qquad \textit{as } \varepsilon=\varepsilon_{j_k}\searrow 0, \tag{8.7}$$

where we have set

$$\mu_\varepsilon(t) := \frac{1}{|\Omega|}\int_\Omega d_\varepsilon(x)u_\varepsilon(x,t)dx, \qquad t\ge 0,\ \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}.$$

Moreover,

$$\sup_{\tau\in[0,1]}\Big|\mu(t+\tau)-\mu(t)\Big| \to 0 \qquad \textit{as } t\to\infty. \tag{8.8}$$

PROOF. As d is bounded, the continuity of μ is an immediate consequence of Lemma 7.1, whereas its boundedness is evident from (7.4). The approximation property (8.7) results upon observing that Lemma 7.1 asserts that as $\varepsilon=\varepsilon_{j_k}\searrow 0$, for all $t>0$ we have $u_\varepsilon(\cdot,t)\rightharpoonup u(\cdot,t)$ in $L^1(\Omega)$ and hence also $d_\varepsilon u_\varepsilon(\cdot,t)\to du(\cdot,t)$ in $L^1(\Omega)$ due to the fact that $d_\varepsilon\to d$ in $L^\infty(\Omega)$ by Lemma 2.2. Finally, (8.8) directly results on applying Lemma 8.1 to $\varphi:=d$. □

In terms of the function μ thus defined, (3.11) implies the following.

Lemma 8.3 *With μ as defined in (8.6), we have*

$$\int_0^\infty \|du(\cdot,t)-\mu(t)\|^2_{L^1(\Omega)}dt<\infty. \tag{8.9}$$

PROOF. According to a Poincaré inequality we can find $c_1>0$ such that

$$\|\varphi-\overline{\varphi}\|_{L^1(\Omega)} \le c_1\|\varphi_x\|_{L^1(\Omega)} \qquad \text{for all } \varphi\in W^{1,1}(\Omega),$$

so that for arbitrary $T>0$ we may once more combine the Cauchy-Schwarz inequality with (2.9) and (2.25) to see that with μ_ε as introduced in Lemma 8.2 we have

$$\begin{aligned}
\int_0^T \|d_\varepsilon u_\varepsilon(\cdot,t)-\mu_\varepsilon(t)\|^2_{L^1(\Omega)}dt &\le c_1^2\int_0^T\bigg\{\int_\Omega\Big|(d_\varepsilon u_\varepsilon(\cdot,t))_x\Big|\bigg\}^2dt \\
&\le c_1^2\int_0^T\bigg\{\int_\Omega\frac{(d_\varepsilon u_\varepsilon)_x^2}{d_\varepsilon u_\varepsilon}\bigg\}\cdot\bigg\{\int_\Omega d_\varepsilon u_\varepsilon\bigg\} \\
&\le c_2\int_0^T\int_\Omega\frac{(d_\varepsilon u_\varepsilon)_x^2}{d_\varepsilon u_\varepsilon} \qquad \text{for all } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}
\end{aligned}$$

with $c_2 := c_1^2(\|d\|_{L^\infty(\Omega)}+1)\int_\Omega u_0$. Since Lemma 3.5 provides $c_3>0$ such that

$$\int_0^T \int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{d_\varepsilon u_\varepsilon} \le c_3\cdot(1+\sqrt{\varepsilon}T) \qquad \text{for all } T>0 \text{ and } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}},$$

from this we infer that for all $T>0$,

$$\int_0^T \|d_\varepsilon u_\varepsilon(\cdot,t)-\mu_\varepsilon(t)\|_{L^1(\Omega)}^2 dt \le 2c_2c_3 \qquad \text{whenever } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}} \text{ is such that } \varepsilon\le\frac{1}{T^2}. \tag{8.10}$$

We now use that as a particular consequence of Lemma 6.1 we have $d_\varepsilon u_\varepsilon\to du$ in $L^2_{loc}([0,\infty);L^1(\Omega))$ as $\varepsilon=\varepsilon_{j_k}\searrow 0$, which together with Lemma 8.2 guarantees that for all $T>0$ and any $\tau\in(0,T)$,

$$\int_\tau^T \|d_\varepsilon u_\varepsilon(\cdot,t)-\mu_\varepsilon(t)\|_{L^1(\Omega)}^2 dt \to \int_\tau^T \|du(\cdot,t)-\mu(t)\|_{L^1(\Omega)}^2 dt \qquad \text{as } \varepsilon=\varepsilon_{j_k}\searrow 0.$$

Therefore, (8.10) implies that

$$\int_\tau^T \|du(\cdot,t)-\mu(t)\|_{L^1(\Omega)}^2 dt \le 2c_2c_3 \qquad \text{for all } T>0 \text{ and } \tau\in(0,T),$$

from which (8.9) results on taking $\tau\searrow 0$ and $T\to\infty$. $\square$

Once again relying on Lemma 8.1, we thereby indeed arrive at the main result of this section.

Lemma 8.4 *With μ as defined in (8.6), we have*

$$du(\cdot,t)-\mu(t)\rightharpoonup 0 \quad \textit{in } L^1(\Omega) \qquad \textit{as } t\to\infty. \tag{8.11}$$

Proof. We fix $\varphi\in L^\infty(\Omega)$ and $\eta>0$ and then obtain from Lemma 8.1 that there exists $t_1>0$ such that

$$\sup_{\tau\in[0,1]}\left|\int_\Omega d\varphi\cdot\Big(u(\cdot,t+\tau)-u(\cdot,t)\Big)\right| \le \frac{\eta}{3} \qquad \text{for all } t\ge t_1, \tag{8.12}$$

whereas (8.8) says that with some $t_2\ge t_1$ we have

$$\|\varphi\|_{L^1(\Omega)}\cdot\sup_{\tau\in[0,1]}\Big|\mu(t+\tau)-\mu(t)\Big| \le \frac{\eta}{3} \qquad \text{for all } t\ge t_2, \tag{8.13}$$

and finally invoking Lemma 8.3 we can pick $t_0\ge t_2$ satisfying

$$\|\varphi\|_{L^\infty(\Omega)}\cdot\int_{t_0}^\infty \|du(\cdot,t)-\mu(t)\|_{L^1(\Omega)}^2 dt \le \frac{\eta}{3}. \tag{8.14}$$

We now write

$$\begin{aligned}
\int_\Omega \Big(du(\cdot,t)-\mu(t)\Big)\cdot\varphi &= \int_0^1\int_\Omega \Big(d(x)u(x,t)-\mu(t)\Big)\cdot\varphi(x)dxd\tau \\
&= \int_0^1\int_\Omega d(x)\Big(u(x,t)-u(x,t+\tau)\Big)\cdot\varphi(x)dxd\tau \\
&\quad+\int_0^1\int_\Omega \Big(d(x)u(x,t+\tau)-\mu(t+\tau)\Big)\cdot\varphi(x)dxd\tau \\
&\quad+\int_0^t \Big(\mu(t+\tau)-\mu(t)\Big)\cdot\int_\Omega\varphi(x)dxd\tau \qquad \text{for } t>0,
\end{aligned} \tag{8.15}$$

and use (8.12) to see that herein for all $t \geq t_0 \geq t_1$,

$$\left| \int_0^1 \int_\Omega d(x)\Big(u(x,t) - u(x,t+\tau)\Big) \cdot \varphi(x)dxd\tau \right| \leq \sup_{\tau\in[0,1]} \left| \int_\Omega d\varphi \cdot \Big(u(\cdot,t+\tau) - u(\cdot,t)\Big) \right| \leq \frac{\eta}{3}.$$

Moreover, (8.13) entails that

$$\left| \int_0^t \Big(\mu(t+\tau) - \mu(t)\Big) \cdot \int_\Omega \varphi(x)dxd\tau \right| \leq \|\varphi\|_{L^1(\Omega)} \cdot \sup_{\tau\in[0,1]} \Big|\mu(t+\tau) - \mu(t)\Big| \leq \frac{\eta}{3} \qquad \text{for all } t \geq t_0 \geq t_2,$$

while combining the Cauchy-Schwarz inequality with (8.14) shows that

$$\begin{aligned}
\left| \int_0^1 \int_\Omega \Big(d(x)u(x,t+\tau) - \mu(t+\tau)\Big) \cdot \varphi(x)dxd\tau \right| &\leq \|\varphi\|_{L^\infty(\Omega)} \cdot \int_0^1 \|du(\cdot,t+\tau) - \mu(t+\tau)\|_{L^1(\Omega)}d\tau \\
&= \|\varphi\|_{L^\infty(\Omega)} \cdot \int_t^{t+1} \|du(\cdot,s) - \mu(s)\|_{L^1(\Omega)}ds \\
&\leq \|\varphi\|_{L^\infty(\Omega)} \cdot \int_t^{\infty} \|du(\cdot,s) - \mu(s)\|_{L^1(\Omega)}^2 ds \\
&\leq \frac{\eta}{3} \qquad \text{for all } t \geq t_0.
\end{aligned}$$

In summary, (8.15) implies that

$$\left| \int_\Omega \Big(du(\cdot,t) - \mu(t)\Big) \cdot \varphi \right| \leq \eta \qquad \text{for all } t \geq t_0$$

and thereby yields (8.11). $\square$

8.3 Weak L^1 convergence of u

We are now in the position to address the claimed convergence statement concerning the quantity u itself. As a last preparation, let us use Lemma 8.4 and again the uniform integrability of $(u(\cdot,t))_{t>0}$ to derive the following.

Lemma 8.5 *Let μ be as in (8.6). Then for each $\varphi \in L^\infty(\Omega)$,*

$$\int_\Omega u(\cdot,t)\varphi - \mu(t)\int_\Omega \frac{\varphi}{d} \to 0 \qquad \textit{as } t \to \infty. \tag{8.16}$$

Proof. Observing that $|\{d \leq \nu\}| \to 0$ as $\nu \searrow 0$, for fixed $\varphi \in L^\infty(\Omega)$ and $\eta > 0$ we first employ (7.2) to pick $\nu > 0$ small enough such that

$$\|\varphi\|_{L^\infty(\Omega)} \cdot \int_{\{d\leq\nu\}} u(\cdot,t) \leq \frac{\eta}{3} \qquad \text{for all } t > 0 \tag{8.17}$$

and such that moreover

$$c_1\|\varphi\|_{L^\infty(\Omega)} \cdot \int_{\{d\leq\nu\}} \frac{1}{d} \leq \frac{\eta}{3}, \tag{8.18}$$

with $c_1 := \sup_{t>0} \mu(t)$ being finite according to Lemma 8.2. As $\frac{\varphi}{d} \cdot \chi_{\{d>\nu\}}$ belongs to $L^\infty(\Omega)$, we may now rely on Lemma 8.4 in choosing $t_0 > 0$ suitably large such that

$$\left| \int_{\{d>\nu\}} \Big(du(\cdot,t) - \mu(t) \Big) \cdot \frac{\varphi}{d} \right| \le \frac{\eta}{3} \qquad \text{for all } t \ge t_0. \tag{8.19}$$

Then in the identity

$$\int_\Omega u(\cdot,t)\varphi - \mu(t) \int_\Omega \frac{\varphi}{d} = \int_{\{d\le\nu\}} u(\cdot,t)\varphi + \int_{\{d>\nu\}} \Big(du(\cdot,t) - \mu(t) \Big) \cdot \frac{\varphi}{d} - \mu(t) \int_{\{d\le\nu\}} \frac{\varphi}{d}, \quad t > 0, \tag{8.20}$$

we may use (8.17) to estimate

$$\left| \int_{\{d\le\nu\}} u(\cdot,t)\varphi \right| \le \|\varphi\|_{L^\infty(\Omega)} \cdot \int_{\{d\le\nu\}} u(\cdot,t) \le \frac{\eta}{3} \qquad \text{for all } t > 0,$$

and apply (8.18) to see that

$$\left| - \mu(t) \cdot \int_{\{d\le\nu\}} \frac{\varphi}{d} \right| \le \mu(t) \|\varphi\|_{L^\infty(\Omega)} \cdot \int_{\{d\le\nu\}} \frac{1}{d} \le \frac{\eta}{3} \qquad \text{for all } t > 0.$$

In view of (8.19), from (8.20) we thus infer that

$$\left| \int_\Omega u(\cdot,t)\varphi - \mu(t) \cdot \int_\Omega \frac{\varphi}{d} \right| \le \eta \qquad \text{for all } t \ge t_0$$

and conclude. $\Box$

Two applications thereof now yield the claimed stabilization property of u.

Lemma 8.6 *Let μ_∞ be as specified in Theorem 1.2. Then*

$$u(\cdot,t) \rightharpoonup \frac{\mu_\infty}{d} \quad \textit{in } L^1(\Omega) \qquad \textit{as } t \to \infty. \tag{8.21}$$

PROOF. A first application of Lemma 8.5 shows that due to (7.4), with μ as in (8.6) we have

$$\int_\Omega u_0 - \mu(t) \cdot \int_\Omega \frac{1}{d} = \int_\Omega u(\cdot,t) - \mu(t) \cdot \int_\Omega \frac{1}{d} \to 0 \qquad \text{as } t \to \infty$$

and that hence by definition of μ_∞,

$$\mu(t) \to \mu_\infty \qquad \text{as } t \to \infty.$$

Therefore, once more employing Lemma 8.5, now with arbitrary $\varphi \in L^\infty(\Omega)$, shows that for any such φ,

$$\int_\Omega u(\cdot,t)\varphi = \left\{ \int_\Omega u(\cdot,t)\varphi - \mu(t) \cdot \int_\Omega \frac{\varphi}{d} \right\} + \mu(t) \cdot \int_\Omega \frac{\varphi}{d} \to \mu_\infty \int_\Omega \frac{\varphi}{d} \qquad \text{as } t \to \infty,$$

and that thus (8.21) holds. $\Box$

8.4 Uniform decay of w. Proof of Theorem 1.2

Now since we already know that $u(\cdot,t)$ stabilizes with respect to the weak topology in $L^1(\Omega)$ to a positive limit function as $t \to \infty$, thanks to the equicontinuity feature of w expressed in Lemma 7.1 the integral decay property (7.6) can be used to derive the following.

Lemma 8.7 *We have*

$$\int_t^{t+1} \int_\Omega w \to 0 \qquad \text{as } t \to \infty. \tag{8.22}$$

Proof. Given $\eta > 0$, relying on the fact that $u_0 \not\equiv 0$ and that hence the number μ_∞ in Theorem 1.2 is positive, we can fix $\delta > 0$ small enough such that with $c_1 := \frac{\mu_\infty}{\|d\|_{L^\infty(\Omega)}}$ we have

$$\delta \cdot \int_\Omega u_0 \le \frac{c_1 \eta}{6}. \tag{8.23}$$

We thereafter once again make use of Lemma 7.1 which in conjunction with the Arzelà-Ascoli theorem ensures that the set $(w(\cdot,t))_{t>0}$ is relatively compact in $C^0(\bar\Omega)$, implying that there exist $k_0 \in \mathbb{N}$ and $(w_k)_{k\in\{1,\dots,k_0\}} \subset C^0(\bar\Omega)$ with the property that for all $t > 0$ one can choose $k(t) \in \{1, ..., k_0\}$ fulfilling

$$\|w(\cdot,t) - w_{k(t)}\|_{L^\infty(\Omega)} \le \delta. \tag{8.24}$$

Since $\{1, ..., k_0\}$ is finite, thanks to the fact that $u(\cdot,t) \rightharpoonup \frac{\mu_\infty}{d}$ in $L^1(\Omega)$ as $t \to \infty$, as asserted by Lemma 8.6, it is then possible to pick $t_1 > 0$ such that

$$\left| \int_\Omega u(\cdot,t) w_k - \int_\Omega \frac{\mu_\infty}{d} w_k \right| \le \frac{c_1\eta}{6} \qquad \text{for all } k \in \{1, ..., k_0\} \text{ and each } t > t_1. \tag{8.25}$$

Finally, the integrability property (7.6) enables us to find $t_0 > t_1$ such that

$$\int_{t_0}^\infty \int_\Omega uw \le \frac{c_1\eta}{6}, \tag{8.26}$$

and we claim that these choices guarantee that

$$\int_t^{t+1} \int_\Omega w \le \eta \qquad \text{for all } t > t_0. \tag{8.27}$$

To verify this, we split

$$\begin{aligned}
\int_t^{t+1} \int_\Omega u(x,s) w(x,s)\,dxds &= \int_t^{t+1} \int_\Omega \frac{\mu_\infty}{d(x)} w(x,s)\,dxds \\
&\quad + \int_t^{t+1} \int_\Omega u(x,s) \cdot \Big\{ w(x,s) - w_{k(s)}(x) \Big\} dxds \\
&\quad + \int_t^{t+1} \Big\{ \int_\Omega u(x,s) w_{k(s)}(x)\,dx - \int_\Omega \frac{\mu_\infty}{d(x)} w_{k(s)}(x)\,dx \Big\} ds \\
&\quad + \int_t^{t+1} \int_\Omega \frac{\mu_\infty}{d(x)} \cdot \Big\{ w_{k(s)}(x) - w(x,s) \Big\} dxds \qquad \text{for } t > 0
\end{aligned} \tag{8.28}$$

and use (7.4) together with (8.24) and (8.23) to see that

$$\begin{aligned}
\left| \int_t^{t+1} \int_\Omega u(x,s) \cdot \Big\{ w(x,s) - w_{k(s)}(x) \Big\} dx ds \right| &\le \int_t^{t+1} \bigg\{ \int_\Omega u(\cdot,s) \bigg\} \cdot \|w(\cdot,s) - w_{k(s)}\|_{L^\infty(\Omega)} ds \\
&= \bigg\{ \int_\Omega u_0 \bigg\} \cdot \int_t^{t+1} \|w(\cdot,s) - w_{k(s)}\|_{L^\infty(\Omega)} ds \\
&\le \bigg\{ \int_\Omega u_0 \bigg\} \cdot \delta \\
&\le \frac{c_1\eta}{6} \qquad \text{for all } t>0,
\end{aligned}$$

and that, similarly, by definition of μ_∞ we have

$$\begin{aligned}
\left| \int_t^{t+1} \int_\Omega \frac{\mu_\infty}{d(x)} \cdot \Big\{ w_{k(s)}(x) - w(x,s) \Big\} dx ds \right| &\le \mu_\infty \cdot \bigg\{ \int_\Omega \frac{1}{d} \bigg\} \cdot \int_t^{t+1} \|w_{k(s)} - w(\cdot,s)\|_{L^\infty(\Omega)} ds \\
&\le \mu_\infty \cdot \bigg\{ \int_\Omega \frac{1}{d} \bigg\} \cdot \delta \\
&\le \frac{c_1\eta}{6} \qquad \text{for all } t>0.
\end{aligned}$$

As moreover (8.25) along with our restriction $t_0 > t_1$ ensures that

$$\begin{aligned}
\left| \int_t^{t+1} \bigg\{ \int_\Omega u(x,s) w_{k(s)}(x) dx - \int_\Omega \frac{\mu_\infty}{d(x)} w_{k(s)}(x) dx \bigg\} ds \right| &\le \int_t^{t+1} \max_{k\in\{1,\dots,k_0\}} \left| \int_\Omega u(\cdot,s) w_k - \int_\Omega \frac{\mu_\infty}{d} w_k \right| ds \\
&\le \frac{c_1\eta}{6} \qquad \text{for all } t>t_0,
\end{aligned}$$

from (8.28) we altogether obtain that

$$\int_t^{t+1} \int_\Omega uw \ge \int_t^{t+1} \int_\Omega \frac{\mu_\infty}{d} w - \frac{c_1\eta}{6} - \frac{c_1\eta}{6} - \frac{c_1\eta}{6} = \int_t^{t+1} \int_\Omega \frac{\mu_\infty}{d} w - \frac{c_1\eta}{2} \qquad \text{for all } t>t_0.$$

Since apart from that

$$\int_\Omega \frac{\mu_\infty}{d} w \ge \frac{\mu_\infty}{\|d\|_{L^\infty(\Omega)}} \int_\Omega w = c_1 \int_\Omega w \qquad \text{for all } t>0,$$

combined with (8.26) this shows that

$$c_1 \int_t^{t+1} \int_\Omega w \le \int_t^{t+1} \int_\Omega uw + \frac{c_1\eta}{2} \le \frac{c_1\eta}{2} + \frac{c_1\eta}{2} = c_1\eta \qquad \text{for all } t>t_0$$

and thereby establishes (8.27). □

Together with the monotonicity information (7.5), this entails decay of $w(\cdot,t)$ with respect to the norm in $L^1(\Omega)$.

Lemma 8.8 *We have*

$$\|w(\cdot,t)\|_{L^1(\Omega)} \to 0 \qquad \textit{as } t\to\infty. \tag{8.29}$$

PROOF. Since from (7.5) we know that

$$\|w(\cdot,t)\|_{L^1(\Omega)} \le \int_{t-1}^t \|w(\cdot,s)\|_{L^1(\Omega)} ds \qquad \text{for all } t \ge 1,$$

this is a direct consequence of Lemma 8.7. $\square$

Again by Lemma 7.1, the topological information herein can be improved.

Lemma 8.9 *We have*

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } t \to \infty. \tag{8.30}$$

PROOF. If this was false, there would exist $(t_k)_{k\in\mathbb{N}} \subset (0,\infty)$, $(x_k)_{k\in\mathbb{N}} \subset \bar{\Omega}$ and $c_1 > 0$ such that

$$w(x_k,t_k) \ge c_1 \qquad \text{for all } k \in \mathbb{N},$$

which due to the equicontinuity of $(w(\cdot,t_k))_{k\in\mathbb{N}}$ asserted by Lemma 7.1 would entail that with some $\delta > 0$ we would have

$$w(x,t_k) \ge \frac{c_1}{2} \qquad \text{for all } x \in (x_k-\delta, x_k+\delta) \cap \Omega \text{ and each } k \in \mathbb{N}.$$

This, however, would be incompatible with the outcome of Lemma 8.8. $\square$

We have thereby actually already completed the derivation of our main results concerning the large time behavior in (1.4).

PROOF of Theorem 1.2. We only need to combine Lemma 8.6 with Lemma 8.9. $\square$

9 Instantaneous blow-up. Proof of Theorem 1.3

Finally concerned with the verification of Theorem 1.3, we will pursue a strategy based on the additional dissipative structure expressed in the identity

$$\frac{d}{dt}\int_\Omega \frac{1}{d}\ln u = \int_\Omega \frac{(du)_x^2}{(du)^2} - \int_\Omega \frac{(du)_x}{du} w_x \tag{9.1}$$

formally associated with (1.4). In order to appropriately cope with the latter summand herein, even at the level of approximate solutions the preparation of a spatio-temporal estimate for w_x seems in order. In the limit problem (1.4), this could formally be obtained in a trivial manner under our assumption that $\frac{w_0}{d}$ and hence $\frac{w}{d}$ be bounded, together with the boundedness of $\int_\Omega d\frac{w_x^2}{w}$ implied by (1.21). At the level of approximate solutions, however, in view of diffusion-induced positivity of w_ε considerable additional efforts seem necessary to guarantee appropriate boundedness properties of $\frac{w_\varepsilon}{d_\varepsilon}$. Our approach toward this will therefore be restricted to the derivation of corresponding L^p bounds for large but finite p only (Lemma 9.2), thereby requiring to involve additional higher-order regularity features of $w_{\varepsilon x}$, possibly depending on ε in a singular manner (Lemma 9.1), to achieve the desired L^2 estimate through an interpolation argument (Lemma 9.3). Thereafter, on the basis of a regularized counterpart of (9.1) we will see in Section 9.2 that our hypothesis that $\int_\Omega \frac{1}{d}\ln\frac{1}{d}$ be finite, by guaranteeing boundedness of the functional $\int_\Omega \frac{1}{d}\ln u$ in (9.1) from above (Lemma 9.6), allows for deducing space-time L^2 bounds on $(\ln(d_\varepsilon u_\varepsilon))_x$ (Lemma 9.8) and hence for deriving Theorem 1.3.

9.1 An L^2 estimate for $w_{\varepsilon x}$ implied by boundedness of $\frac{w_0}{d}$

Let us first interpolate between two regularity estimates for $w_{\varepsilon x}$ from Lemma 3.5 to achieve the following bound involving a high integrability power but a singular dependence on ε.

Lemma 9.1 *There exists $C > 0$ with the property that for any choice of $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,*

$$\int_0^T \left\| d_\varepsilon \frac{w_{\varepsilon x}(\cdot,t)}{\sqrt{w_\varepsilon(\cdot,t)}} \right\|_{L^\infty(\Omega)}^4 dt \le \frac{C}{\varepsilon} \cdot (1+T) \cdot (1+\sqrt{\varepsilon}T)^2 \qquad \textit{for all } T > 0. \tag{9.2}$$

Proof. From Lemma 3.5 we obtain $c_1 > 0$ and $c_2 > 0$ such that

$$\varepsilon \int_0^T \int_\Omega \frac{1}{\sqrt{g(w_\varepsilon)}} \cdot \Big(d_\varepsilon \frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \Big)_x^2 \le c_1 \cdot (1+\sqrt{\varepsilon}T) \qquad \text{for all } T > 0$$

and

$$\int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} \le c_2 \cdot (1+\sqrt{\varepsilon}t) \qquad \text{for all } t > 0,$$

which in view of (1.8), (2.26) and (2.9) entails that

$$\begin{aligned} \varepsilon \int_0^T \int_\Omega \Big(d_\varepsilon \frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \Big)_x^2 &\le \sqrt{\gamma M} \cdot \varepsilon \int_0^T \int_\Omega \frac{1}{\sqrt{g(w_\varepsilon)}} \cdot \Big(d_\varepsilon \frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \Big)_x^2 \\ &\le c_3 \cdot (1+\sqrt{\varepsilon}T) \qquad \text{for all } T > 0 \end{aligned} \tag{9.3}$$

and

$$\begin{aligned} \int_\Omega \Big(d_\varepsilon \frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \Big)^2 &= \int_\Omega \frac{d_\varepsilon w_\varepsilon}{g(w_\varepsilon)} \cdot d_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} \\ &\le \frac{\|d\|_{L^\infty(\Omega)}+1}{\underline{\gamma}} \int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} \\ &\le c_4 \cdot (1+\sqrt{\varepsilon}t) \qquad \text{for all } t > 0 \end{aligned} \tag{9.4}$$

with obvious choices of $c_3 > 0$ and $c_4 > 0$. Now since the Gagliardo-Nirenberg inequality provides $c_5 > 0$ fulfilling

$$\|\varphi\|_{L^\infty(\Omega)}^4 \le c_5 \|\varphi_x\|_{L^2(\Omega)}^2 \|\varphi\|_{L^2(\Omega)}^2 + c_5 \|\varphi\|_{L^2(\Omega)}^4 \qquad \text{for all } \varphi \in W^{1,2}(\Omega),$$

combining (9.3) with (9.4) we infer that

$$\begin{aligned} \int_0^T \left\| d_\varepsilon \frac{w_{\varepsilon x}(\cdot,t)}{\sqrt{g(w_\varepsilon(\cdot,t))}} \right\|_{L^\infty(\Omega)}^4 dt &\le c_5 \int_0^T \left\| \Big(d_\varepsilon \frac{w_{\varepsilon x}(\cdot,t))}{\sqrt{g(w_\varepsilon(\cdot,t))}} \Big)_x \right\|_{L^2(\Omega)}^2 \left\| d_\varepsilon \frac{w_{\varepsilon x}(\cdot,t)}{\sqrt{g(w_\varepsilon(\cdot,t))}} \right\|_{L^2(\Omega)}^2 dt \\ &\quad + c_5 \int_0^T \left\| d_\varepsilon \frac{w_{\varepsilon x}(\cdot,t)}{\sqrt{g(w_\varepsilon(\cdot,t))}} \right\|_{L^2(\Omega)}^4 dt \\ &\le c_5 \cdot \frac{c_3}{\varepsilon}(1+\sqrt{\varepsilon}T) \cdot c_4(1+\sqrt{\varepsilon}T) + c_5 \cdot c_4^2 (1+\sqrt{\varepsilon}T)^2 T \qquad \text{for all } T > 0, \end{aligned}$$

which readily implies (9.2) due to the fact that

$$\frac{1}{\sqrt{g(w_\varepsilon)}} \ge \frac{1}{\sqrt{\gamma}\sqrt{w_\varepsilon}} \qquad \text{in } \Omega \times (0,\infty)$$

by (1.8). $\square$

Next, and independently from essentially all our previous analysis, a testing procedure applied to the second equation in (2.22) yields the following weighted L^p estimate for $\frac{w_\varepsilon}{d_\varepsilon}$ for asymptotically large but yet finite p.

Lemma 9.2 *Assume that $\frac{w_0}{d} \in L^\infty(\Omega)$. Then there exists $C>0$ such that whenever $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,*

$$\int_\Omega \frac{w_\varepsilon^p(\cdot,t)}{d_\varepsilon^{p+1}} \le \Big(C\cdot(1+t)\Big)^{2p} \qquad \textit{for all } t>0 \textit{ and any } p \in \Big[2, \frac{2}{\sqrt{\varepsilon}}\Big]. \tag{9.5}$$

Proof. We integrate by parts in the second equation in (2.22) and use the nonnegativity of $u_\varepsilon g(w_\varepsilon)$ as well as Young's inequality to obtain

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega \frac{w_\varepsilon^p}{d_\varepsilon^{p+1}} &= \int_\Omega \frac{w_\varepsilon^{p-1}}{d_\varepsilon^{p+1}} w_{\varepsilon t} \\
&\le \varepsilon\int_\Omega \frac{w_\varepsilon^{p-1}}{d_\varepsilon^{p+1}}\cdot\Big(d_\varepsilon \frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x \\
&= -(p-1)\varepsilon\int_\Omega \frac{1}{d_\varepsilon^p}\cdot\frac{w_\varepsilon^{p-2}}{\sqrt{g(w_\varepsilon)}} w_{\varepsilon x}^2 + (p+1)\varepsilon\int_\Omega \frac{d_{\varepsilon x}}{d_\varepsilon^{p+1}}\cdot\frac{w_\varepsilon^{p-1}}{\sqrt{g(w_\varepsilon)}} w_{\varepsilon x} \\
&\le \frac{(p+1)^2\varepsilon}{4(p-1)}\int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon^{p+2}}\cdot\frac{w_\varepsilon^p}{\sqrt{g(w_\varepsilon)}} \qquad \text{for all } t>0.
\end{aligned} \tag{9.6}$$

Here since according to (2.13) we have

$$\frac{d_{\varepsilon x}^2}{d_\varepsilon^2} \le \frac{1}{\sqrt{\varepsilon}} \qquad \text{in } \Omega,$$

due to (1.8) and our restriction $p \ge 2$ we can estimate

$$\begin{aligned}
\frac{(p+1)^2\varepsilon}{4(p-1)}\int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon^{p+2}}\cdot\frac{w_\varepsilon^p}{\sqrt{g(w_\varepsilon)}} &\le \frac{(2p)^2\varepsilon}{4\cdot\frac{p}{2}\cdot\sqrt{\gamma}}\cdot\int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon^{p+2}}\cdot w_\varepsilon^{p-\frac{1}{2}} \\
&= \frac{2p\varepsilon}{\sqrt{\gamma}}\cdot\int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon^2}\cdot\frac{w_\varepsilon^{\frac{2p-1}{2}}}{d_\varepsilon^p} \\
&\le \frac{2p\sqrt{\varepsilon}}{\sqrt{\gamma}}\cdot\int_\Omega \frac{w_\varepsilon^{\frac{2p-1}{2}}}{d_\varepsilon^p} \qquad \text{for all } t>0.
\end{aligned} \tag{9.7}$$

Since from (2.9) we know that

$$\frac{w_\varepsilon^{\frac{2p-1}{2}}}{d_\varepsilon^p} = d_\varepsilon^{\frac{p-1}{2p}}\cdot\Big(\frac{w_\varepsilon^p}{d_\varepsilon^{p+1}}\Big)^{\frac{2p-1}{2p}}$$

$$
\begin{aligned}
&\le (\|d\|_{L^\infty(\Omega)}+1)^{\frac{p-1}{2p}} \cdot \Big(\frac{w_\varepsilon^p}{d_\varepsilon^{p+1}}\Big)^{\frac{2p-1}{2p}} \\
&\le c_1 \Big(\frac{w_\varepsilon^p}{d_\varepsilon^{p+1}}\Big)^{\frac{2p-1}{2p}} \qquad \text{in } \Omega\times(0,\infty)
\end{aligned}
$$

with $c_1 := (\|d\|_{L^\infty(\Omega)}+1)^{\frac{1}{2}}$, in view of the Hölder inequality we see that (9.7) entails the inequality

$$
\begin{aligned}
\frac{(p+1)^2\varepsilon}{4(p-1)} \int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon^{p+2}} \cdot \frac{w_\varepsilon^p}{\sqrt{g(w_\varepsilon)}} &\le \frac{2c_1 p\sqrt{\varepsilon}}{\sqrt{\underline{\gamma}}} \int_\Omega \Big(\frac{w_\varepsilon^p}{d_\varepsilon^{p+1}}\Big)^{\frac{2p-1}{2p}} \\
&\le \frac{2c_1 p\sqrt{\varepsilon}}{\sqrt{\underline{\gamma}}} |\Omega|^{\frac{1}{2p}} \cdot \Big\{ \int_\Omega \frac{w_\varepsilon^p}{d_\varepsilon^{p+1}} \Big\}^{\frac{2p-1}{2p}} \\
&\le c_2 p\sqrt{\varepsilon} \cdot \Big\{ \int_\Omega \frac{w_\varepsilon^p}{d_\varepsilon^{p+1}} \Big\}^{\frac{2p-1}{2p}} \qquad \text{for all } t>0,
\end{aligned}
$$

where $c_2 := \frac{2c_1}{\sqrt{\underline{\gamma}}} \cdot \max\{|\Omega|^{\frac{1}{4}}, 1\}$. Therefore, (9.6) shows that

$$
y_\varepsilon(t) := \int_\Omega \frac{w_\varepsilon^p(\cdot,t)}{d_\varepsilon^{p+1}}, \qquad t\ge 0,
$$

satisfies

$$
y_\varepsilon'(t) \le c_2 p^2 \sqrt{\varepsilon} y_\varepsilon^{\frac{2p-1}{2p}}(t) \qquad \text{for all } t>0,
$$

which on integration implies that

$$
y_\varepsilon(t) \le \Big\{ y_\varepsilon^{\frac{1}{2p}}(0) + \frac{c_2}{2} p\sqrt{\varepsilon} t \Big\}^{2p} \qquad \text{for all } t>0,
$$

that is,

$$
\int_\Omega \frac{w_\varepsilon^p}{d_\varepsilon^{p+1}} \le \bigg\{ \Big\{ \int_\Omega \frac{w_{0\varepsilon}^p}{d_\varepsilon^{p+1}} \Big\}^{\frac{1}{2p}} + \frac{c_2}{2} p\sqrt{\varepsilon} t \bigg\}^{2p} \qquad \text{for all } t>0. \tag{9.8}
$$

Here thanks to the fact that from (2.12) we know that

$$
\frac{\varepsilon^{\frac{1}{4}}}{d_\varepsilon} \le 1 \qquad \text{in } \Omega,
$$

according to (2.7) and the definition of $w_{0\varepsilon}$ in Lemma 2.6 we can use Lemma 2.5 as well as our assumption that $\frac{w_0}{d}$ be bounded in Ω to see that writing $c_3 := \max\Big\{1\,,\,(\int_\Omega \frac{1}{d})^{\frac{1}{4}}\Big\}$ we have

$$
\begin{aligned}
\Big\{ \int_\Omega \frac{w_{0\varepsilon}^p}{d_\varepsilon^{p+1}} \Big\}^{\frac{1}{2p}} &\le \Big\| \frac{w_{0\varepsilon}}{d_\varepsilon} \Big\|_{L^\infty(\Omega)}^{\frac{1}{2}} \cdot \Big\{ \int_\Omega \frac{1}{d_\varepsilon} \Big\}^{\frac{1}{2p}} \\
&\le c_3 \Big\| \frac{w_{0\varepsilon}}{d_\varepsilon} \Big\|_{L^\infty(\Omega)}^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\le c_3 \Big\| \frac{w_0 + \varepsilon^{\frac14}}{d_\varepsilon} \Big\|_{L^\infty(\Omega)}^{\frac12} \\
&\le c_3 \Big\| \frac{w_0}{d_\varepsilon} \Big\|_{L^\infty(\Omega)}^{\frac12} + c_3 \Big\| \frac{\varepsilon^{\frac14}}{d_\varepsilon} \Big\|_{L^\infty(\Omega)}^{\frac12} \\
&\le c_4 := c_3 \Big\| \frac{w_0}{d} \Big\|_{L^\infty(\Omega)}^{\frac12} + c_3 \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}},
\end{aligned}
$$

because $w_{0\varepsilon} \le w_0 + \varepsilon^{\frac14}$ for any such ε. As moreover $p\sqrt{\varepsilon} \le 2$ due to our hypothesis, from (9.8) we thus infer that

$$
\int_\Omega \frac{w_\varepsilon^p}{d_\varepsilon^{p+1}} \le \Big\{ c_4 + c_2 t \Big\}^{2p} \qquad \text{for all } t > 0
$$

and that hence (9.5) holds with $C := \max\{c_2, c_4\}$. □

Fortunately, the largest admissible p in (9.5) is such that in the course of an interpolation argument, an estimation of the L^2 norm in question only involves powers of the inequality in (9.2) which are such that the singular dependence on ε therein disappears in the limit $\varepsilon = \varepsilon_j \searrow 0$.

Lemma 9.3 *Assume that $\frac{w_0}{d} \in L^\infty(\Omega)$. Then for all $T > 0$ one can find $C(T) > 0$ such that*

$$
\int_0^T \int_\Omega w_{\varepsilon x}^2 \le C(T) \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}.
$$

Proof. Let us first apply Lemma 3.5, Lemma 9.1 and Lemma 9.2 to fix constants $c_1 \ge 1, c_2 > 0$ and $c_3 > 0$ such that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ we have

$$
\int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} \le c_1 \cdot (1+t) \qquad \text{for all } t > 0 \tag{9.9}
$$

and

$$
\int_0^T \Big\| d_\varepsilon \frac{w_{\varepsilon x}(\cdot,t)}{\sqrt{w_\varepsilon(\cdot,t)}} \Big\|_{L^\infty(\Omega)}^4 dt \le \frac{c_2}{\varepsilon} \cdot (1+T)^3 \qquad \text{for all } T > 0 \tag{9.10}
$$

as well as

$$
\int_\Omega \frac{w_\varepsilon^{\frac{2}{\sqrt{\varepsilon}}}}{d_\varepsilon^{\frac{2}{\sqrt{\varepsilon}}+1}} \le \Big(c_3 \cdot (1+t) \Big)^{\frac{4}{\sqrt{\varepsilon}}} \qquad \text{for all } t > 0. \tag{9.11}
$$

Then invoking the Hölder inequality we see that

$$
\begin{aligned}
\int_\Omega w_{\varepsilon x}^2 &= \int_\Omega \Big| d_\varepsilon \frac{w_{\varepsilon x}}{\sqrt{w_\varepsilon}} \Big|^{\sqrt{\varepsilon}} \cdot \Big| d_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} \Big|^{\frac{2-\sqrt{\varepsilon}}{2}} \cdot \frac{w_\varepsilon}{d_\varepsilon^{1+\frac{\sqrt{\varepsilon}}{2}}} \\
&\le \Big\| d_\varepsilon \frac{w_{\varepsilon x}}{\sqrt{w_\varepsilon}} \Big\|_{L^\infty(\Omega)}^{\sqrt{\varepsilon}} \cdot \Big\{ \int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} \Big\}^{\frac{2-\sqrt{\varepsilon}}{2}} \cdot \Big\{ \int_\Omega \frac{w_\varepsilon^{\frac{2}{\sqrt{\varepsilon}}}}{d_\varepsilon^{\frac{2}{\sqrt{\varepsilon}}+1}} \Big\}^{\frac{\sqrt{\varepsilon}}{2}} \\
&\le c_1^{\frac{2-\sqrt{\varepsilon}}{2}} (1+t)^{\frac{2-\sqrt{\varepsilon}}{2}} \cdot \Big(c_3 \cdot (1+t) \Big)^2 \cdot \Big\| d_\varepsilon \frac{w_{\varepsilon x}}{\sqrt{w_\varepsilon}} \Big\|_{L^\infty(\Omega)}^{\sqrt{\varepsilon}} \\
&\le c_1 c_3^2 (1+t)^3 \Big\| d_\varepsilon \frac{w_{\varepsilon x}}{\sqrt{w_\varepsilon}} \Big\|_{L^\infty(\Omega)}^{\sqrt{\varepsilon}} \qquad \text{for all } t > 0,
\end{aligned}
$$

because $c_1 \ge 1$. In order to make use of (9.10) here, we integrate with respect to the time variable and once more employ the Hölder inequality to find that again since $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$,

$$\begin{aligned}
\int_0^T \int_\Omega w_{\varepsilon x}^2 &\le c_1 c_3^2 \int_0^T (1+t)^3 \bigg\| d_\varepsilon \frac{w_{\varepsilon x}(\cdot,t)}{\sqrt{w_\varepsilon(\cdot,t)}} \bigg\|_{L^\infty(\Omega)}^{\sqrt{\varepsilon}} dt \\
&\le c_1 c_3^2 (1+T)^3 \int_0^T \bigg\| d_\varepsilon \frac{w_{\varepsilon x}(\cdot,t)}{\sqrt{w_\varepsilon(\cdot,t)}} \bigg\|_{L^\infty(\Omega)}^{\sqrt{\varepsilon}} dt \\
&\le c_1 c_3^2 (1+T)^3 \cdot \bigg\{ \int_0^T \bigg\| d_\varepsilon \frac{w_{\varepsilon x}(\cdot,t)}{\sqrt{w_\varepsilon(\cdot,t)}} \bigg\|_{L^\infty(\Omega)}^4 dt \bigg\}^{\frac{\sqrt{\varepsilon}}{4}} \cdot T^{\frac{4-\sqrt{\varepsilon}}{4}} \\
&\le c_1 c_3^2 (1+T)^3 \cdot \Big\{ \frac{c_2}{\varepsilon} \cdot (1+T)^3 \Big\}^{\frac{\sqrt{\varepsilon}}{4}} \cdot T^{\frac{4-\sqrt{\varepsilon}}{4}} \\
&\le c_1 c_2^{\frac{1}{4}} c_3^2 (1+T)^4 \varepsilon^{-\frac{\sqrt{\varepsilon}}{4}} \qquad \text{for all } T > 0.
\end{aligned}$$

Consequently, the proof can be completed by the observation that

$$\varepsilon^{-\frac{\sqrt{\varepsilon}}{4}} = e^{-\frac{\sqrt{\varepsilon}}{2} \ln \sqrt{\varepsilon}} \le e^{\frac{1}{2e}} \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$$

due to the fact that $\xi \ln \xi > -\frac{1}{e}$ for all $\xi > 0$. □

9.2 A bound for $|\ln(d_\varepsilon u_\varepsilon)|$. Proof of Theorem 1.3

In order to prepare our estimates for the absolute value of $\ln(d_\varepsilon u_\varepsilon)$, let us first make sure that this quantity cannot attain large negative values throughout Ω.

Lemma 9.4 *There exists $C > 0$ such that for each $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,*

$$\int_\Omega d_\varepsilon u_\varepsilon(\cdot,t) \ge C \qquad \textit{for all } t \in \Big(0, \frac{1}{\sqrt{\varepsilon}}\Big). \tag{9.12}$$

Proof. From Lemma 5.1 we obtain $\delta > 0$ such that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,

$$\int_E u_\varepsilon \le \frac{1}{2} \int_\Omega u_0 \quad \text{for all } t \in \Big(0, \frac{1}{\sqrt{\varepsilon}}\Big) \text{ and each measurable } E \subset \Omega \text{ such that } |E| \le \delta,$$

so that since by integrability of $\frac{1}{d}$ we can find $\nu > 0$ fulfilling $|\{d \le \nu\}| \le \delta$, we infer that

$$\int_{\{d\le\nu\}} u_\varepsilon \le \frac{1}{2} \int_\Omega u_0 \qquad \text{for all } t \in \Big(0, \frac{1}{\sqrt{\varepsilon}}\Big).$$

Again using that $d_\varepsilon \ge d$ and (7.4), we obtain that indeed

$$\int_\Omega d_\varepsilon u_\varepsilon \ge \int_\Omega d u_\varepsilon \ge \nu \cdot \int_{\{d\ge\nu\}} u_\varepsilon = \nu \cdot \bigg\{ \int_\Omega u_\varepsilon - \int_{\{d\le\nu\}} u_\varepsilon \bigg\} \ge \frac{\nu}{2} \int_\Omega u_\varepsilon = \frac{\nu}{2} \int_\Omega u_0$$

for any such ε and t. □

In view of (2.25), this entails an upper bound for the spatial minimum of $|\ln(d_\varepsilon u_\varepsilon)|$.

Lemma 9.5 *Let $T > 0$. Then there exists $C > 1$ with the property that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ and any $t \in (0,T)$ one can find $x_0(t,\varepsilon) \in \overline{\Omega}$ such that*

$$\frac{1}{C} \le d_\varepsilon(x_0(t,\varepsilon))u_\varepsilon(x_0(t,\varepsilon),t) \le C. \tag{9.13}$$

PROOF. Since Lemma 9.4 along with (2.25) and (2.9) ensures the existence of $c_1 > 0$ and $c_2 > 0$ such that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ we have

$$c_1 \le \int_\Omega d_\varepsilon u_\varepsilon \le c_2 \qquad \text{for all } t \in (0,T),$$

by a mean-value theorem we can pick some $x_0 = x_0(t,\varepsilon) \in \overline{\Omega}$ fulfilling

$$d_\varepsilon(x_0)u_\varepsilon(x_0,t) = \frac{1}{|\Omega|}\int_\Omega d_\varepsilon u_\varepsilon(\cdot,t) \in \Big[\frac{c_1}{|\Omega|}, \frac{c_2}{|\Omega|}\Big],$$

so that the claim results on taking $C > 1$ suitably large. □

Now a straightforward application of Young's inequality yields the following inequality which inter alia entails an upper bound for the functional on the right of (9.1) at the approximate level.

Lemma 9.6 *Suppose that*

$$\int_\Omega \frac{1}{d}\ln\frac{1}{d} < \infty. \tag{9.14}$$

Then there exists $C > 0$ such that for each $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,

$$\int_{\{u_\varepsilon(\cdot,t)\ge 1\}} \frac{1}{d_\varepsilon}\ln u_\varepsilon(\cdot,t) \le C \qquad \text{for all } t > 0. \tag{9.15}$$

PROOF. As $\xi\eta \le \xi\ln\xi + e^{\eta-1}$ for all $\xi > 0$ and $\eta \in \mathbb{R}$, we may use (2.25) to see that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,

$$\begin{aligned}\int_{\{u_\varepsilon(\cdot,t)\ge 1\}} \frac{1}{d_\varepsilon}\ln u_\varepsilon(\cdot,t) &\le \int_\Omega \frac{1}{d_\varepsilon}\ln\frac{1}{d_\varepsilon} + \frac{1}{e}\int_\Omega u_\varepsilon(\cdot,t) \\ &= \int_\Omega \frac{1}{d_\varepsilon}\ln\frac{1}{d_\varepsilon} + \frac{1}{e}\int_\Omega u_0 \qquad \text{for all } t > 0.\end{aligned} \tag{9.16}$$

Since by monotonicity of $(1,\infty) \ni \xi \mapsto \xi\ln\xi$ and Lemma 2.2 we have

$$\int_\Omega \frac{1}{d_\varepsilon}\ln\frac{1}{d_\varepsilon} \le \int_{\{d_\varepsilon<1\}} \frac{1}{d_\varepsilon}\ln\frac{1}{d_\varepsilon} \le c_1 := \int_{\{d<1\}} \frac{1}{d}\ln\frac{1}{d} \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}},$$

and since c_1 is finite according to our assumption (9.14) and the boundedness of d, this already yields (9.15). □

In exploiting the regularized variant of (9.1), we shall moreover make use of the following elementary lemma concerned with an ODE comparison.

Lemma 9.7 *Let $T > 0$, and suppose that $y \in C^0([0,T)) \cap C^1((0,T))$ is such that*

$$y'(t) + ay_+^2(t) \le h(t) \qquad \text{for all } t \in (0,T) \tag{9.17}$$

with some $a > 0$ and some nonnegative $h \in L^1((0,T)) \cap C^0((0,T))$. Then

$$y(t) \le \frac{1}{at} + \int_0^t h(s)ds \qquad \text{for all } t \in (0,T). \tag{9.18}$$

PROOF. Since the expression on the right-hand side of (9.18) defines a supersolution of the problem in (9.17) which diverges to $+\infty$ as $t \searrow 0$, this readily results from an ODE comparison argument. $\Box$

We are now prepared for our analysis of the quasi-dissipative structure suggested by (9.1), relying on the assumption that $\frac{1}{d}$ belong to $L \log L(\Omega)$ through Lemma 9.6.

Lemma 9.8 *Suppose that*

$$\int_\Omega \frac{1}{d} \ln \frac{1}{d} < \infty \tag{9.19}$$

and that $\frac{w_0}{d} \in L^\infty(\Omega)$. Then for all $T > 0$ and any $\tau \in (0,T)$ there exists $C(T,\tau) > 0$ such that

$$\int_\Omega \frac{1}{d_\varepsilon} \ln u_\varepsilon(\cdot,t) \ge -C(T,\tau) \qquad \text{for all } t \in (\tau,T) \tag{9.20}$$

and

$$\int_\tau^T \int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{(d_\varepsilon u_\varepsilon)^2} \le C(T,\tau) \tag{9.21}$$

whenever $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$.

PROOF. We multiply the first equation in (2.22) by $\frac{1}{d_\varepsilon u_\varepsilon}$ and integrate by parts over Ω to see that

$$\begin{aligned} \frac{d}{dt}\int_\Omega \frac{1}{d_\varepsilon} \ln u_\varepsilon &= \int_\Omega \frac{1}{d_\varepsilon u_\varepsilon} u_{\varepsilon t} \\ &= \int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{(d_\varepsilon u_\varepsilon)^2} - \int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x}{d_\varepsilon u_\varepsilon} w_{\varepsilon x} \qquad \text{for all } t > 0, \end{aligned}$$

where by Young's inequality,

$$\int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x}{d_\varepsilon u_\varepsilon} w_{\varepsilon x} \le \frac{1}{2}\int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{(d_\varepsilon u_\varepsilon)^2} + \frac{1}{2}\int_\Omega w_{\varepsilon x}^2 \qquad \text{for all } t > 0,$$

so that

$$z_\varepsilon(t) := -\int_\Omega \frac{1}{d_\varepsilon} \ln u_\varepsilon(\cdot,t), \qquad t \ge 0,$$

satisfies

$$z_\varepsilon'(t) + \frac{1}{2}\int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{(d_\varepsilon u_\varepsilon)^2} \le \frac{1}{2}\int_\Omega w_{\varepsilon x}^2 \qquad \text{for all } t > 0. \tag{9.22}$$

Now given $T > 0$, we apply Lemma 9.5 to gain $c_1 > 1$ such that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ and each $t \in (0,T)$ we can pick $x_0 = x_0(t,\varepsilon) \in \overline{\Omega}$ such that

$$\frac{1}{c_1} \le d_\varepsilon(x_0)u_\varepsilon(x_0,t) \le c_1,$$

which implies that

$$\Big| \ln \Big(d_\varepsilon(x_0)u_\varepsilon(x_0,t)\Big)\Big| \le c_2 := \ln c_1.$$

Since by means of the Cauchy-Schwarz inequality we obtain that

$$\Big| \ln \Big(d_\varepsilon(x)u_\varepsilon(x,t)\Big) - \ln \Big(d_\varepsilon(x_0)u_\varepsilon(x_0,t)\Big)\Big| \le \sqrt{|\Omega|} \cdot \bigg\{ \int_\Omega \bigg| \bigg(\ln \Big(d_\varepsilon u_\varepsilon(\cdot,t)\Big)\bigg)_x \bigg|^2 \bigg\}^{\frac{1}{2}} \qquad \text{for all } x \in \Omega,$$

this entails that

$$\Big| \ln \Big(d_\varepsilon(x)u_\varepsilon(x,t)\Big)\Big| \le c_2 + \sqrt{|\Omega|} \cdot \bigg\{ \int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{(d_\varepsilon u_\varepsilon)^2} \bigg\}^{\frac{1}{2}} \qquad \text{for all } x \in \Omega \text{ and } t \in (0,T)$$

and that hence

$$\begin{aligned} \frac{1}{4}\int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{(d_\varepsilon u_\varepsilon)^2} &\ge \frac{1}{4|\Omega|} \cdot \Big\{ \Big\| \ln(d_\varepsilon u_\varepsilon)\Big\|_{L^\infty(\Omega)} - c_2 \Big\}_+^2 \\ &\ge \frac{1}{8|\Omega|} \Big\| \ln(d_\varepsilon u_\varepsilon)\Big\|_{L^\infty(\Omega)}^2 - c_3 \qquad \text{for all } t \in (0,T) \end{aligned} \tag{9.23}$$

with $c_3 := \frac{c_2^2}{4|\Omega|}$, because $(\xi - \eta)_+^2 \ge \frac{1}{2}\xi^2 - \eta^2$ for all $\xi \ge 0$ and $\eta \ge 0$. Now since again using that $\xi \ln \xi \ge -\frac{1}{e}$ for all $\xi > 0$ we can estimate

$$\begin{aligned} z_\varepsilon(t) &= -\int_\Omega \frac{1}{d_\varepsilon} \ln(d_\varepsilon u_\varepsilon) + \int_\Omega \frac{1}{d_\varepsilon} \ln d_\varepsilon \\ &\le -\int_\Omega \frac{1}{d_\varepsilon} \ln(d_\varepsilon u_\varepsilon) + \frac{|\Omega|}{e} \qquad \text{for all } t > 0, \end{aligned}$$

and since Lemma 2.2 warrants that

$$\begin{aligned} -\int_\Omega \frac{1}{d_\varepsilon} \ln(d_\varepsilon u_\varepsilon) &\le \Big\| \ln(d_\varepsilon u_\varepsilon)\Big\|_{L^\infty(\Omega)} \cdot \int_\Omega \frac{1}{d_\varepsilon} \\ &\le c_4 \Big\| \ln(d_\varepsilon u_\varepsilon)\Big\|_{L^\infty(\Omega)} \qquad \text{for all } t > 0 \end{aligned}$$

with $c_4 := \int_\Omega \frac{1}{d} < \infty$, from (9.23) we thus infer that

$$\frac{1}{4}\int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{(d_\varepsilon u_\varepsilon)^2} \ge \frac{1}{8c_4^2|\Omega|} \cdot \Big\{ z_\varepsilon(t) - \frac{|\Omega|}{e} \Big\}_+^2 - c_3 \qquad \text{for all } t \in (0,T).$$

Consequently, writing $c_5 := \frac{1}{8c_4^2|\Omega|}$ we see that (9.22) entails the inequality

$$z_\varepsilon'(t) + c_5 \cdot \Big\{ z_\varepsilon(t) - \frac{|\Omega|}{e} \Big\}_+^2 + \frac{1}{4} \int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{(d_\varepsilon u_\varepsilon)^2} \le \frac{1}{2} \int_\Omega w_{\varepsilon x}^2(\cdot,t) + c_3 \qquad \text{for all } t \in (0,T), \tag{9.24}$$

from which in view of Lemma 9.7 we firstly conclude that

$$z_\varepsilon(t) - \frac{|\Omega|}{e} \le \frac{1}{c_5 t} + \frac{1}{2} \int_0^t \int_\Omega w_{\varepsilon x}^2 + c_3 t \qquad \text{for all } t \in (0,T).$$

Since Lemma 9.3 provides $c_6 > 0$ such that

$$\int_0^T \int_\Omega w_{\varepsilon x}^2 \le c_6 \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}, \tag{9.25}$$

for arbitrary $\tau \in (0,T)$ and each $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ this entails the one-sided inequality

$$z_\varepsilon(t) \le c_7 := \frac{|\Omega|}{e} + \frac{1}{c_5\tau} + \frac{c_6}{2} + c_3 T \qquad \text{for all } t \in [\tau,T), \tag{9.26}$$

thus particularly establishing (9.20).

In order to achieve a corresponding upper bound, we now make use of our assumption (9.19), which allows us to invoke Lemma 9.6 to find $c_8 > 0$ fulfilling

$$-z_\varepsilon(t) \le \int_{\{u_\varepsilon(\cdot,t)\ge 1\}} \frac{1}{d_\varepsilon} \ln u_\varepsilon(\cdot,t) \le c_8 \qquad \text{for all } t > 0. \tag{9.27}$$

Therefore, namely, on integrating (9.24) and relying on (9.26) and again (9.25) we see that

$$\begin{aligned}
\frac{1}{4} \int_\tau^T \int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{(d_\varepsilon u_\varepsilon)^2} &\le z_\varepsilon(\tau) - z_\varepsilon(T) + \frac{1}{2} \int_\tau^T \int_\Omega w_{\varepsilon x}^2 + c_3(T-\tau) \\
&\le c_7 + c_8 + \frac{c_6}{2} + c_3(T-\tau) \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}},
\end{aligned}$$

and that thus also (9.21) is valid. $\square$

In order to turn this into a two-sided estimate for the quantity $\ln(d_\varepsilon u_\varepsilon)$ itself, we once more rely on Lemma 9.6 to assert a spatial L^1 bound therefor.

Lemma 9.9 *Assume that $\int_\Omega \frac{1}{d} \ln \frac{1}{d} < \infty$ and $\frac{w_0}{d} \in L^\infty(\Omega)$. Then for all $T > 0$ and $\tau \in (0,T)$ there exists $C(T,\tau) > 0$ such that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ we have*

$$\int_\Omega \Big| \ln \Big(d_\varepsilon u_\varepsilon(\cdot,t) \Big) \Big| \le C(T,\tau) \qquad \textit{for all } t \in (\tau,T). \tag{9.28}$$

Proof. In the inequality

$$\int_\Omega \Big| \ln(d_\varepsilon u_\varepsilon) \Big| \le \int_\Omega |\ln d_\varepsilon| + \int_\Omega |\ln u_\varepsilon|, \qquad t > 0, \tag{9.29}$$

we may first use that the validity of $\ln\xi \le \xi$ for all $\xi > 0$ entails that $|\ln\xi| \le \xi + \frac{1}{\xi}$ for all $\xi > 0$, so that according to Lemma 2.2, writing $c_1 := \|d\|_{L^\infty(\Omega)} + 1$ we have

$$\int_\Omega |\ln d_\varepsilon| \le \int_\Omega d_\varepsilon + \int_\Omega \frac{1}{d_\varepsilon} \le c_1|\Omega| + \int_\Omega \frac{1}{d} < \infty. \tag{9.30}$$

Likewise, in

$$\int_\Omega |\ln u_\varepsilon| = \int_{\{u_\varepsilon \ge 1\}} \ln u_\varepsilon - \int_{\{u_\varepsilon < 1\}} \ln u_\varepsilon, \qquad t > 0, \tag{9.31}$$

we have

$$\int_{\{u_\varepsilon \ge 1\}} \ln u_\varepsilon \le \int_{\{u_\varepsilon \ge 1\}} u_\varepsilon \le \int_\Omega u_\varepsilon = \int_\Omega u_0 \qquad \text{for all } t > 0 \tag{9.32}$$

by (2.25), whereas

$$\begin{aligned} -\int_{\{u_\varepsilon<1\}} \ln u_\varepsilon &= -\int_{\{u_\varepsilon<1\}} d_\varepsilon \cdot \frac{1}{d_\varepsilon} \ln u_\varepsilon \\ &\le -c_1 \int_{\{u_\varepsilon<1\}} \frac{1}{d_\varepsilon} \ln u_\varepsilon \\ &= -c_1 \int_\Omega \frac{1}{d_\varepsilon} \ln u_\varepsilon + c_1 \int_{\{u_\varepsilon \ge 1\}} \frac{1}{d_\varepsilon} \ln u_\varepsilon \qquad \text{for all } t > 0. \end{aligned} \tag{9.33}$$

Since Lemma 9.6 provides $c_2 > 0$ such that

$$\int_{\{u_\varepsilon \ge 1\}} \frac{1}{d_\varepsilon} \ln u_\varepsilon \le c_2 \qquad \text{for all } t > 0,$$

and since Lemma 9.8 says that given any $T > 0$ and $\tau \in (0,T)$ we can find $c_3(T,\tau) > 0$ fulfilling

$$\int_\Omega \frac{1}{d_\varepsilon} \ln u_\varepsilon \ge -c_3(T,\tau) \qquad \text{for all } t \in (\tau,T),$$

from (9.31), (9.32) and (9.33) we conclude that

$$\left| \int_\Omega \ln u_\varepsilon \right| \le \int_\Omega u_0 + c_1 c_3(T,\tau) + c_1 c_2 \qquad \text{for all } t \in (\tau,T),$$

which together with (9.29) and (9.30) verifies (9.28). □

Now by interpolation, the latter in conjunction with Lemma 9.8 entails (1.18).

Lemma 9.10 *Assume that $\int_\Omega \frac{1}{d} \ln \frac{1}{d} < \infty$ and $\frac{w_0}{d} \in L^\infty(\Omega)$. Then*

$$\int_\tau^T \left\| \ln\Big(du(\cdot,t)\Big) \right\|_{L^\infty(\Omega)}^3 dt < \infty \qquad \textit{for all } T > 0 \textit{ and } \tau \in (0,T). \tag{9.34}$$

PROOF. Given $T>0$ and $\tau\in(0,T)$, from Lemma 9.8 and Lemma 9.9 we obtain $c_1>0$ and $c_2>0$ such that for all $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}$,

$$\int_\tau^T\int_\Omega \Big(\ln(d_\varepsilon u_\varepsilon)\Big)_x^2 \le c_1 \tag{9.35}$$

and

$$\int_\Omega \Big|\ln(d_\varepsilon u_\varepsilon)\Big| \le c_2 \qquad \text{for all } t\in(\tau,T). \tag{9.36}$$

As a Gagliardo-Nirenberg inequality says that with some $c_3>0$ we have

$$\|\varphi\|_{L^\infty(\Omega)}^3 \le c_3\|\varphi_x\|_{L^2(\Omega)}^2\|\varphi\|_{L^1(\Omega)} + c_3\|\varphi\|_{L^1(\Omega)}^3 \qquad \text{for all } \varphi\in W^{1,2}(\Omega),$$

from this we infer that

$$\begin{aligned}\int_\tau^T \Big\|\ln\Big(d_\varepsilon u_\varepsilon(\cdot,t)\Big)\Big\|_{L^\infty(\Omega)}^3 dt &\le c_3\int_\tau^T \Big\|\Big(\ln\Big(d_\varepsilon u_\varepsilon(\cdot,t)\Big)\Big)_x\Big\|_{L^2(\Omega)}^2\Big\|\ln\Big(d_\varepsilon u_\varepsilon(\cdot,t)\Big)\Big\|_{L^1(\Omega)} dt \\ &\quad + c_3\int_\tau^T \Big\|\ln\Big(d_\varepsilon u_\varepsilon(\cdot,t)\Big)\Big\|_{L^1(\Omega)}^3 dt \\ &\le c_4 := c_1c_2c_3 + c_2^3c_3T \end{aligned}\tag{9.37}$$

for all $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}$. Now since Lemma 6.1 along with Lemma 2.2 warrants that with $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ as introduced in Lemma 6.1, for a.e. $t>0$ we have $d_\varepsilon u_\varepsilon(\cdot,t)\to du(\cdot,t)$ a.e. in Ω and hence

$$\Big\|\ln\Big(du(\cdot,t)\Big)\Big\|_{L^\infty(\Omega)} \le \liminf_{\varepsilon=\varepsilon_{j_k}\searrow 0}\Big\|\ln\Big(d_\varepsilon u_\varepsilon(\cdot,t)\Big)\Big\|_{L^\infty(\Omega)} \qquad \text{for a.e. } t>0,$$

using Fatou's lemma we thus obtain from (9.37) that

$$\begin{aligned}\int_\tau^T \Big\|\ln\Big(du(\cdot,t)\Big)\Big\|_{L^\infty(\Omega)}^3 dt &\le \int_\tau^T \liminf_{\varepsilon=\varepsilon_{j_k}\searrow 0}\Big\|\ln\Big(d_\varepsilon u_\varepsilon(\cdot,t)\Big)\Big\|_{L^\infty(\Omega)}^3 dt \\ &\le \liminf_{\varepsilon=\varepsilon_{j_k}\searrow 0}\int_\tau^T \Big\|\ln\Big(d_\varepsilon u_\varepsilon(\cdot,t)\Big)\Big\|_{L^\infty(\Omega)}^3 dt \\ &\le c_4\end{aligned}$$

and conclude. □

We thereby readily arrive at our main result on diffusive effects at intermediate time scales.

PROOF of Theorem 1.3. The integrability property (1.18) has precisely been asserted by Lemma 9.10. As a consequence, we may choose a null set $N_0\subset(0,\infty)$ such that $\ln(du(\cdot,t))\in L^\infty(\Omega)$ for all $t\in(0,\infty)\setminus N_0$, whence if for such t we abbreviate $c_1(t):=\|\ln(du(\cdot,t))\|_{L^\infty(\Omega)}$, then

$$-c_1(t)\le \ln\Big(d(x)u(x,t)\Big)\le c_1(t) \qquad \text{for a.e. } x\in\Omega,$$

that is,

$$\frac{e^{-c_1(t)}}{d(x)} \le u(x,t) \le \frac{e^{c_t(t)}}{d(x)} \qquad \text{for a.e. } x\in\Omega$$

whenever $t\in(0,\infty)\setminus N_0$. This yields (1.19), whereupon (1.20) becomes obvious. □

10 Appendix

This appendix is devoted to the details of the approximation procedures underlying Section 2.2. Let us first construct a family of smooth positive approximations to d with the properties listed in Lemma 2.2.

Proof of Lemma 2.2. Without loss of generality we may assume that $\Omega = (-R, R)$ with some $R > 0$, and fix a sequence $(K_j)_{j\in\mathbb{N}}$ of compact subsets of $\{d > 0\}$ such that $K_j \subset K_{j+1}$ for all $j \in \mathbb{N}$ and $\bigcup_{j\in\mathbb{N}} K_j = \{d > 0\}$, whence for $\widetilde{K}_j := K_j \cap [-R + \frac{1}{j}, R - \frac{1}{j}]$, $j \in \mathbb{N}$, we have $\widetilde{K}_j \subset \widetilde{K}_{j+1}$ for all $j \in \mathbb{N}$ and $\bigcup_{j\in\mathbb{N}} \widetilde{K}_j = \{d > 0\} \cap \Omega$. We first observe that then by continuity of d in $\overline{\Omega}$ and of d_x in $\{d > 0\}$, for each $\delta \in (0, 1)$,

$$\psi_\delta(x) := \begin{cases} d(-R), & x \le -\frac{R}{1+\delta}, \\ d\Big((1+\delta)x\Big), & -\frac{R}{1+\delta} \le x \le \frac{R}{1+\delta}, \\ d(R), & x \ge \frac{R}{1+\delta}, \end{cases} \tag{10.1}$$

defines a function $\psi_\delta \in C^0(\mathbb{R})$ fulfilling $\psi_\delta \in C^1(\{\psi_\delta > 0\} \cap (-\frac{R}{1+\delta}, \frac{R}{1+\delta})) \cap W^{1,\infty}_{loc}(\{\psi_\delta > 0\})$, for which $\psi_\delta \to d$ in $L^\infty(\Omega)$ and $\psi_{\delta x} \to d_x$ in $L^\infty_{loc}(\{d > 0\} \cap \Omega)$ and in $L^p_{loc}(\{d > 0\})$ for all $p \in [1, \infty)$ as $\delta \searrow 0$, so that for each $j \in \mathbb{N}$ we can pick $\delta_j \in (0, 1)$ such that $\widetilde{\varphi}_j(x) := \psi_{\delta_j}(x)$, $x \in \overline{\Omega}$, satisfies

$$\|\widetilde{\varphi}_j - d\|_{L^\infty(\Omega)} \le \frac{1}{2\cdot 3^j}, \qquad \|\widetilde{\varphi}_{jx} - d_x\|_{L^\infty(\widetilde{K}_j)} \le \frac{1}{2j} \qquad \text{and} \qquad \|\widetilde{\varphi}_{jx} - d_x\|_{L^j(K_j)} \le \frac{1}{2j} \tag{10.2}$$

Next, for $\eta \in (0, 1)$ letting $\rho_\eta \in C_0^\infty(\mathbb{R})$ denote an arbitrary mollifier having the properties that $\operatorname{supp}\rho_\eta \subset [-\eta, \eta]$ and $\int_{\mathbb{R}} \rho_\eta = 1$, we immediately see that if $\eta < \frac{\delta_j R}{1+\delta_j}$, then $\rho_\eta \star \widetilde{\varphi}_j \equiv \widetilde{\varphi}_j \equiv d(-R)$ in $(-\infty, -\frac{R}{1+\delta_j} - \eta)$ and $\rho_\eta \star \widetilde{\varphi}_j \equiv \widetilde{\varphi}_j \equiv d(R)$ in $(\frac{R}{1+\delta_j} + \eta, \infty)$ and hence, in particular, $(\rho_\eta \star \widetilde{\varphi}_j)_x = 0$ on $\partial\Omega$ for any such η. Since standard arguments ([17]) moreover show that $\rho_\eta \star \widetilde{\varphi}_j \to \widetilde{\varphi}_j$ in $L^\infty(\Omega)$ as well as $(\rho_\eta \star \widetilde{\varphi}_j)_x \to \widetilde{\varphi}_{jx}$ in $L^\infty(\widetilde{K}_j)$ and in $L^p(K_j)$ for all $p \in [1, \infty)$ as $\eta \searrow 0$, it follows that for any $j \in \mathbb{N}$ we may fix $\eta_j \in (0, 1)$ suitably small such that for $\widehat{\varphi}_j := \rho_{\eta_j} \star \widetilde{\varphi}_j$ we have $\widehat{\varphi}_{jx} = 0$ on $\partial\Omega$ as well as

$$\|\widehat{\varphi}_j - \widetilde{\varphi}_j\|_{L^\infty(\Omega)} \le \frac{1}{2\cdot 3^j}, \qquad \|\widehat{\varphi}_{jx} - \widetilde{\varphi}_{jx}\|_{L^\infty(\widetilde{K}_j)} \le \frac{1}{2j} \qquad \text{and} \qquad \|\widehat{\varphi}_{jx} - \widetilde{\varphi}_{jx}\|_{L^j(K_j)} \le \frac{1}{2j} \tag{10.3}$$

Writing $\varphi_j := \widehat{\varphi}_j + \frac{2}{3^j}$, $j \in \mathbb{N}$, we thus obtain $(\varphi_j)_{j\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ such that $\varphi_{jx} \equiv \widehat{\varphi}_{jx}$ in $\overline{\Omega}$ and thus still

$$\varphi_{jx} = 0 \quad \text{on } \partial\Omega \qquad \text{for all } j \in \mathbb{N}, \tag{10.4}$$

that, by (10.2) and (10.3),

$$\|\varphi_{jx} - d_x\|_{L^\infty(\widetilde{K}_j)} \le \|\widehat{\varphi}_{jx} - \widetilde{\varphi}_{jx}\|_{L^\infty(\widetilde{K}_j)} + \|\widetilde{\varphi}_{jx} - d_x\|_{L^\infty(\widetilde{K}_j)} \le \frac{1}{2j} + \frac{1}{2j} = \frac{1}{j} \qquad \text{for all } j \in \mathbb{N} \tag{10.5}$$

and similarly

$$\|\varphi_{jx} - d_x\|_{L^j(K_j)} \le \|\widehat{\varphi}_{jx} - \widetilde{\varphi}_{jx}\|_{L^j(K_j)} + \|\widetilde{\varphi}_{jx} - d_x\|_{L^j(K_j)} \le \frac{1}{2j} + \frac{1}{2j} = \frac{1}{j} \qquad \text{for all } j \in \mathbb{N}, \tag{10.6}$$

and that moreover

$$d + \frac{1}{3^j} \le \varphi_j \le d + \frac{3}{3^j} \quad \text{in } \Omega \qquad \text{for all } j \in \mathbb{N}, \tag{10.7}$$

which in particular ensures that

$$\varphi_{j+1} \le \varphi_j \quad \text{in } \Omega \qquad \text{for all } j \in \mathbb{N}. \tag{10.8}$$

Now in order to construct $(d_\varepsilon)_{\varepsilon\in(0,1)}$, we recursively define $(\varepsilon_j)_{j\in\mathbb{N}_0} \subset [0,1]$ by letting $\varepsilon_0 := 1$ and

$$\varepsilon_j := \min\left\{ \frac{\varepsilon_{j-1}}{2}, 3^{-4j}, \left\{ \int_\Omega \frac{\varphi_{jx}^2}{\varphi_j^3} \right\}^{-\frac{1}{2}}, \left\{ \int_\Omega \frac{\varphi_{jx}^4}{\varphi_j^2} \right\}^{-2}, \left\| \frac{\varphi_{jx}}{\varphi_j} \right\|_{L^\infty(\Omega)}^{-4} \right\}, \qquad j \ge 1, \tag{10.9}$$

and observe that this especially guarantees that $(\varepsilon_j)_{j\in\mathbb{N}_0}$ is strictly decreasing, and that for each $j \in \mathbb{N}$ we have $\varepsilon_j > 0$ due to (10.7) and the inclusion $\varphi_j \in C^1(\overline{\Omega})$. As a consequence, introducing

$$d_\varepsilon := \varphi_j \qquad \text{whenever } \varepsilon \in (\varepsilon_{j+1}, \varepsilon_j] \text{ for some } j \in \mathbb{N}_0$$

indeed yields a well-defined family $(d_\varepsilon)_{\varepsilon\in(0,1)} \subset C^\infty(\overline{\Omega})$ which thanks to (10.7), (10.5), (10.6), (10.8), (10.4) and the monotonicity of $(\varepsilon_j)_{j\in\mathbb{N}}$ satisfies (2.5), (2.9), (2.6), (2.7) and (2.8), and for which due to the second restriction expressed in (10.9) we know from the left inequality in (10.7) that for all $j \in \mathbb{N}_0$,

$$d_\varepsilon \ge d + \frac{1}{3^j} \ge \frac{1}{3^j} \ge \varepsilon_j^{\frac{1}{4}} \ge \varepsilon^{\frac{1}{4}} > 0 \quad \text{in } \overline{\Omega} \qquad \text{for all } \varepsilon \in (\varepsilon_{j+1}, \varepsilon_j].$$

Furthermore, the third, fourth and fifth requirements in (10.9) warrant that for any $j \in \mathbb{N}_0$ and each $\varepsilon \in (\varepsilon_{j+1}, \varepsilon_j]$ we have

$$\varepsilon^2 \int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon^3} = \varepsilon^2 \int_\Omega \frac{\varphi_{jx}^2}{\varphi_j^3} \le \varepsilon_j^2 \int_\Omega \frac{\varphi_{jx}^2}{\varphi_j^3} \le 1$$

and, similarly,

$$\sqrt{\varepsilon} \int_\Omega \frac{d_{\varepsilon x}^4}{d_\varepsilon^2} \le \sqrt{\varepsilon_j} \int_\Omega \frac{\varphi_{jx}^4}{\varphi_j^2} \le 1$$

as well as

$$\varepsilon^{\frac{1}{4}} \left\| \frac{d_{\varepsilon x}}{d_\varepsilon} \right\|_{L^\infty(\Omega)} \le \varepsilon_j^{\frac{1}{4}} \left\| \frac{\varphi_{jx}}{\varphi_j} \right\|_{L^\infty(\Omega)} \le 1,$$

and that thus also (2.10), (2.11) and (2.13) are valid. $\square$

We next verify that our assumptions on d and w_0 indeed entail the consequences specified in Lemma 2.3 and Lemma 2.4.

Proof of Lemma 2.3. Assuming on the contrary that $c_1 := \int_{\Omega_0} \frac{d_x^2}{d}$ be finite, by hypothesis we can find $x_0 \in \Omega_0$ and $\delta > 0$ such that $d(x_0) = 0$ and either $(x_0, x_0 + \delta) \subset \Omega_0$ or $(x_0 - \delta, x_0) \subset \Omega_0$, and concentrating on the former case we know from the continuity of d that for each $x_1 \in \Omega_1 := \{x \in$

$(x_0, x_0+\delta) \mid d(x) > 0\}$, the point $\tilde{x}_0 := \max\{x \in [x_0, x_1] \mid d(x) = 0\}$ belongs to $[x_0, x_1)$. As d is positive and hence continuously differentiable on $(\tilde{x}_0, x_1]$, using elementary calculus we can estimate

$$\begin{aligned}
\sqrt{d(x_1)} &= \sqrt{d(\tilde{x}_0)} + \int_{\tilde{x}_0}^{x_1} (\sqrt{d})_x(y)dy \\
&= \frac{1}{2}\int_{\tilde{x}_0}^{x_1} \frac{d_x(y)}{\sqrt{d(y)}}dy \\
&\le \frac{1}{2}\left\{\int_{\tilde{x}_0}^{x_1} \frac{d_x^2(y)}{d(y)}dy\right\}^{\frac{1}{2}} \cdot \sqrt{x_1 - \tilde{x}_0} \\
&\le \frac{\sqrt{c_1}}{2}\sqrt{x_1 - \tilde{x}_0} \\
&\le \frac{\sqrt{c_1}}{2}\sqrt{x_1 - x_0}.
\end{aligned}$$

Since $x_1 \in \Omega_1$ was arbitrary and $(x_0, x_0+\delta) \setminus \Omega_1 \subset \{d = 0\}$ is a null set by (1.6), this entails that

$$\int_\Omega \frac{1}{d} \ge \int_{\Omega_1} \frac{1}{d} \ge \int_{\Omega_1} \frac{4}{c_1(x-x_0)}dx = \int_{x_0}^{x_0+\delta} \frac{4}{c_1(x-x_0)}dx = \infty,$$

which in turn is incompatible with (1.6) and thereby establishes the claim. □

PROOF of Lemma 2.4. Let us assume for contradiction that there exists $x_0 \in \overline{\Omega}$ such that $d(x_0) = 0$ but $w_0(x_0) > 0$. Then by continuity of w_0 we can find $\delta > 0$ and an interval $\Omega_0 \subset \overline{\Omega}$, relatively open in $\overline{\Omega}$, such that $w_0 \ge \delta$ throughout Ω_0. As $d > 0$ a.e. in Ω as a consequence of (1.6), using Lemma 2.3 we therefore obtain

$$\int_{\Omega_0} \frac{d_x^2}{d} w_0 \ge \delta \int_{\Omega_0} \frac{d_x^2}{d} = \infty,$$

which contradicts (1.10). □

We are now in the position to provide an approximation of w_0 in the flavor of Lemma 2.5.

PROOF of Lemma 2.5. Without loss of generality we may assume that $\{d = 0\}$ is not empty. Then since d is continuous in $\overline{\Omega}$, there exist a countable set $I \subset \mathbb{N}$ and a family $(J_i)_{i\in I}$ of relatively open proper subintervals J_i of $\overline{\Omega}$ such that $J_i \cap J_j = \emptyset$ if $i \in I$ and $j \in I$ are such that $i \ne j$, and that $\bigcup_{i\in I} J_i = \{d > 0\}$. Accordingly, for each $i \in I$ there exist $a_i \in \overline{\Omega}$ and $b_i \in \overline{\Omega}$ such that $(a_i, b_i) \subset J_i \subset [a_i, b_i]$, where $a_i \in J_i$ (resp., $b_i \in J_i$) if and only if $a_i \in \partial\Omega$ (resp., $b_i \in \partial\Omega$).
Now for fixed $i \in I$, in the case $a_i \notin J_i$ we know from the defining properties of J_i that $d(a_i) = 0$, whence again by continuity of d we have $\|d\|_{L^\infty((a_i, a_i+\delta))} \to 0$ as $\delta \searrow 0$; likewise, if $b_i \notin J_i$ then $\|d\|_{L^\infty((b_i-\delta, b_i))} \to 0$ as $\delta \searrow 0$. Therefore, we can recursively define $(\delta_j^{(i)})_{j\in\mathbb{N}} \subset (0,1)$ such that

$$\delta_j^{(i)} < \frac{b_i - a_i}{4} \qquad \text{for all } j \in \mathbb{N} \tag{10.10}$$

and

$$\delta_{j+1}^{(i)} \le \min\left\{\delta_j^{(i)}, \frac{1}{j}\right\} \qquad \text{for all } j \in \mathbb{N}, \tag{10.11}$$

and such that if $a_i \notin J_i$, then

$$\|d\|_{L^\infty((a_i,a_i+2\delta_j^{(i)}))} \le \frac{1}{2^i} \qquad \text{for all } j \in \mathbb{N}, \tag{10.12}$$

and that if $b_i \notin J_i$ then

$$\|d\|_{L^\infty((b_i-2\delta_j^{(i)},b_i))} \le \frac{1}{2^i} \qquad \text{for all } j \in \mathbb{N}. \tag{10.13}$$

For $i \in I$ and $j \in \mathbb{N}$, we then introduce the piecewise linear functions $\zeta_j^{(i)} \in W^{1,\infty}(\Omega)$ by letting

$$\zeta_j^{(i)} := \begin{cases} 0 & \text{if } x \le a_i + \delta_j^{(i)}, \\ \frac{x-a_i-\delta_j^{(i)}}{\delta_j^{(i)}} & \text{if } a_i + \delta_j^{(i)} < x < a_i + 2\delta_j^{(i)}, \\ 1 & \text{if } a_i + 2\delta_j^{(i)} \le x \le b_i - 2\delta_j^{(i)}, \\ \frac{b_i-\delta_j^{(i)}-x}{\delta_j^{(i)}} & \text{if } b_i - 2\delta_j^{(i)} < x < b_i - \delta_j^{(i)}, \\ 0 & \text{if } x \ge b_i - \delta_j^{(i)} \end{cases} \tag{10.14}$$

whenever $J_i = (a_i, b_i)$ and

$$\zeta_j^{(i)} := \begin{cases} 1 & \text{if } x \le b_i - 2\delta_j^{(i)}, \\ \frac{b_i-\delta_j^{(i)}-x}{\delta_j^{(i)}} & \text{if } b_i - 2\delta_j^{(i)} < x < b_i - \delta_j^{(i)}, \\ 0 & \text{if } x \ge b_i - \delta_j^{(i)} \end{cases} \tag{10.15}$$

in the case $J_i = [a_i, b_i)$ and

$$\zeta_j^{(i)} := \begin{cases} 0 & \text{if } x \le a_i + \delta_j^{(i)}, \\ \frac{x-a_i-\delta_j^{(i)}}{\delta_j^{(i)}} & \text{if } a_i + \delta_j^{(i)} < x < a_i + 2\delta_j^{(i)}, \\ 1 & \text{if } x \ge a_i + 2\delta_j^{(i)} \end{cases} \tag{10.16}$$

when $J_i = (a_i, b_i]$, and for $j \in \mathbb{N}$ we let

$$\zeta_j(x) := \sum_{i\in I, i\le j} \zeta_j^{(i)}(x), \qquad x \in \overline{\Omega}, \tag{10.17}$$

as well as

$$w_{0j}(x) := \zeta_j^2(x) w_0(x), \qquad x \in \overline{\Omega}. \tag{10.18}$$

Then since (10.11) in particular asserts that $\delta_j^{(i)} \searrow 0$ as $j \to \infty$ for each $i \in I$, from the definition of ζ_j it follows that

$$0 \le \zeta_j(x) \le \zeta_{j+1}(x) \qquad \text{for all } x \in \Omega \text{ and } j \in \mathbb{N} \tag{10.19}$$

and

$$\zeta_j(x) \nearrow 1 \quad \text{as } j \to \infty \qquad \text{for all } x \in \{d > 0\}, \tag{10.20}$$

implying that $0 \le w_{0j} \le w_{0,j+1}$ in Ω for all $j \in \mathbb{N}$, and that both (2.16) and (2.17) hold. Moreover, it is clear from (10.18) and the inclusion $w_0 \in W^{1,2}(\Omega)$ implied by our assumptions on w_0 that $w_{0j} \in W^{1,2}(\Omega)$ with

$$\begin{aligned} w_{0jx}^2 &= (\zeta_j^2 w_{0x} + 2\zeta_j\zeta_{jx}w_0)^2 \\ &\le 2\zeta_j^4 w_{0x}^2 + 8\zeta_j^2\zeta_{jx}^2 w_0^2 \qquad \text{a.e. in } \Omega, \end{aligned}$$

so that

$$\frac{w_{0jx}^2}{w_{0j}} \le 2\zeta_j^2 \frac{w_{0x}^2}{w_0} + 8\zeta_{jx}^2 w_0 \qquad \text{a.e. in } \Omega. \tag{10.21}$$

Since $\frac{w_{0x}^2}{w_0} \in L^1(\Omega)$ by hypothesis, this firstly implies that for each fixed $j \in \mathbb{N}$ we have $\frac{w_{0jx}^2}{w_{0j}} \in L^1(\Omega)$ and hence $\sqrt{w_{0j}} \in W^{1,2}(\Omega)$, and according to (10.19) and (10.17), from (10.21) we furthermore obtain that

$$\begin{aligned} \int_\Omega d\frac{w_{0jx}^2}{w_{0j}} &\le 2\int_\Omega d\zeta_j^2\frac{w_{0x}^2}{w_0} + 8\int_\Omega d\zeta_{jx}^2 w_0 \\ &\le 2\int_\Omega d\frac{w_{0x}^2}{w_0} + 8\sum_{i\in I}\int_\Omega d(\zeta_{jx}^{(i)})^2 w_0 \qquad \text{for all } j \in \mathbb{N}. \end{aligned} \tag{10.22}$$

In order to estimate the rightmost summand herein, we first note that according to our choice of $(J_i)_{i\in I}$, for all $i \in I$ we have

$$d(a_i) = 0 \text{ whenever } a_i \notin J_i \qquad \text{and} \qquad d(b_i) = 0 \text{ when } b_i \notin J_i,$$

and that thus, as a consequence of (1.6) and (1.10) when combined with Lemma 2.4,

$$w_0(a_i) = 0 \text{ if } a_i \notin J_i \qquad \text{and} \qquad w_0(b_i) = 0 \text{ if } b_i \notin J_i.$$

Again since $\sqrt{w_0} \in W^{1,2}(\Omega)$, by means of the Cauchy-Schwarz inequality this implies that writing $c_1 := \int_\Omega (\sqrt{w_0})_x^2$ we have

$$w_0(x) \le c_1|x - a_i| \quad \text{for all } x \in \Omega \qquad \text{if } a_i \notin J_i$$

and

$$w_0(x) \le c_1|x - b_i| \quad \text{for all } x \in \Omega \qquad \text{if } b_i \notin J_i,$$

so that whenever $i \in I$ is such that $J_i = (a_i, b_i)$, in view of (10.14) we can use (10.12) and (10.13) to estimate

$$\begin{aligned} \int_\Omega d(\zeta_{jx}^{(i)})^2 w_0 &= \frac{1}{(\delta_j^{(i)})^2}\int_{a_i+\delta_j^{(i)}}^{a_i+2\delta_j^{(i)}} dw_0 + \frac{1}{(\delta_j^{(i)})^2}\int_{b_i-2\delta_j^{(i)}}^{b_i-\delta_j^{(i)}} dw_0 \\ &\le \frac{1}{(\delta_j^{(i)})^2}\cdot \delta_j^{(i)}\|d\|_{L^\infty((a_i,a_i+2\delta_j^{(i)}))}\|w_0\|_{L^\infty((a_i,a_i+2\delta_j^{(i)}))} \end{aligned}$$

$$
\begin{aligned}
& \frac{1}{(\delta_j^{(i)})^2} \cdot \delta_j^{(i)} \|d\|_{L^\infty((b_i-2\delta_j^{(i)},b_i))} \|w_0\|_{L^\infty((b_i-2\delta_j^{(i)},b_i))} \\
\le \;& 2c_1 \|d\|_{L^\infty((a_i,a_i+2\delta_j^{(i)}))} + 2c_1 \|d\|_{L^\infty((b_i-2\delta_j^{(i)},b_i))} \\
\le \;& 2c_1 \cdot \frac{1}{2^i} + 2c_1 \cdot \frac{1}{2^i} \\
= \;& \frac{4c_1}{2^i}.
\end{aligned}
$$

Along with a similar reasoning in the cases $J_i = [a_i, b_i)$ and $J_i = (a_i, b_i]$, this allows us to conclude that

$$
8 \sum_{i\in I} \int_\Omega d(\zeta_{jx}^{(i)})^2 w_0 \le 32c_1 \sum_{i\in I} \frac{1}{2^i} \le 32c_1 \sum_{i=1}^\infty \frac{1}{2^i} < \infty \qquad \text{for all } j \in \mathbb{N},
$$

because $I \subset \mathbb{N}$. In light of our assumption (1.9), from (10.22) we thus obtain (2.18). $\square$

Our final selection of the sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$, as used throughout our analysis, can be accomplished as follows.

Proof of Lemma 2.6. For fixed $j \in \mathbb{N}$ we estimate

$$
\int_\Omega d_\varepsilon \frac{(w_{0j} + \varepsilon^{\frac{1}{4}})_x^2}{w_{0j} + \varepsilon^{\frac{1}{4}}} = \int_\Omega d_\varepsilon \frac{w_{0jx}^2}{w_{0j} + \varepsilon^{\frac{1}{4}}} \le \int_\Omega d_\varepsilon \frac{w_{0jx}^2}{w_{0j}} \qquad \text{for all } \varepsilon \in (0,1), \tag{10.23}
$$

where using the inclusion $\sqrt{w_{0j}} \in W^{1,2}(\Omega)$, as asserted by Lemma 2.5, along with the monotonicity of the convergence $d_\varepsilon \to d$, as obtained in Lemma 2.2, we see that

$$
\int_\Omega d_\varepsilon \frac{w_{0jx}^2}{w_{0j}} \to \int_\Omega d \frac{w_{0jx}^2}{w_{0j}} \qquad \text{as } \varepsilon \searrow 0.
$$

As $c_1 := \sup_{j\in\mathbb{N}} \int_\Omega d \frac{w_{0jx}^2}{w_{0j}}$ is finite thanks to Lemma 2.5, from this and (10.23) we infer that for all $j \in \mathbb{N}$ we can fix $\varepsilon^{(1)}(j) \in (0,1)$ such that

$$
\int_\Omega d_\varepsilon \frac{(w_{0j} + \varepsilon^{\frac{1}{4}})_x^2}{w_{0j} + \varepsilon^{\frac{1}{4}}} \le c_1 + 1 \qquad \text{for all } \varepsilon \in (0, \varepsilon^{(1)}(j)]. \tag{10.24}
$$

Next, for arbitrary $j \in \mathbb{N}$ and $\varepsilon \in (0,1)$ we trivially split

$$
\int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon}(w_{0j} + \varepsilon^{\frac{1}{4}}) = \int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon} w_{0j} + \varepsilon^{\frac{1}{4}} \int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon} \tag{10.25}
$$

and note that here due to the Cauchy-Schwarz inequality, the boundedness property (2.11) derived in Lemma 2.2 ensures that

$$
\varepsilon^{\frac{1}{4}} \int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon} \le \varepsilon^{\frac{1}{4}} |\Omega|^{\frac{1}{2}} \bigg\{ \int_\Omega \frac{d_{\varepsilon x}^4}{d_\varepsilon^2} \bigg\}^{\frac{1}{2}} \le |\Omega|^{\frac{1}{2}} \qquad \text{for all } \varepsilon \in (0,1). \tag{10.26}
$$

Now since $K := \operatorname{supp} w_{0j}$ is a compact subset of $\{d > 0\}$ by Lemma 2.5, and since according to Lemma 2.2 we have $d_\varepsilon \to d$ in $L^\infty(\Omega)$ and $d_{\varepsilon x} \to d_x$ in $L^2_{loc}(\{d > 0\})$ and hence $\frac{d_{\varepsilon x}^2}{d_\varepsilon} \to \frac{d_x^2}{d}$ in $L^1(K)$ as $\varepsilon \searrow 0$, it follows that for any individual $j \in \mathbb{N}$,

$$\int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon} w_{0j} \to \int_\Omega \frac{d_x^2}{d} w_{0j} \qquad \text{as } \varepsilon \searrow 0.$$

Since $w_{0j} \le w_0$ by (2.17) and thus

$$\int_\Omega \frac{d_x^2}{d} w_{0j} \le c_2 := \int_\Omega \frac{d_x^2}{d} w_0 \qquad \text{for all } j \in \mathbb{N}$$

with c_2 being finite thanks to our assumptions on w_0, we thus conclude that for any $j \in \mathbb{N}$ we can pick $\varepsilon^{(2)}(j) \in (0,1)$ fulfilling

$$\int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon} w_{0j} \le c_2 + 1 \qquad \text{for all } \varepsilon \in (0, \varepsilon^{(2)}(j)],$$

which together with (10.25) and (10.26) entails that

$$\int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon} (w_{0j} + \varepsilon^{\frac{1}{4}}) \le c_2 + 1 + |\Omega|^{\frac{1}{2}} \qquad \text{for all } \varepsilon \in (0, \varepsilon^{(2)}(j)].$$

In conjunction with (10.24), this shows that if we pick any $\varepsilon_0 \in (0,1)$ and recursively define a nonincreasing sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ by letting

$$\varepsilon_j := \min\Big\{\varepsilon_{j-1}\,,\, \frac{1}{j}\,,\, \varepsilon^{(1)}(j)\,,\, \varepsilon^{(2)}(j)\Big\}, \qquad j \in \mathbb{N},$$

then $(w_{0\varepsilon_j})_{j\in\mathbb{N}}$ as given by (2.19) indeed satisfies (2.20) and (2.21). □

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