One-dimensional super-fast diffusion: Persistence vs. extinction revisited. Extinction at spatial infinity

Michael Winkler*
Institut für Mathematik, Universität Paderborn,
33098 Paderborn, Germany

Abstract

We consider positive classical solutions of

$$v_t = (v^{m-1} v_x)_x, \qquad x \in \mathbb{R},\ t > 0, \qquad (\star)$$

in the super-fast diffusion range $m < -1$. Our main interest is in smooth positive initial data $v_0 = v(\cdot, 0)$ which decay as $x \to +\infty$, but which are possibly unbounded as $x \to -\infty$, having in mind monotonically decreasing data as prototypes.

It is firstly proved that if v_0 decays sufficiently fast only in one direction by satisfying

$$v_0(x) \le c x^{-\beta} \qquad \text{for all } x > 0 \quad \text{with some} \quad \beta > \tfrac{2}{1-m}$$

and some $c > 0$, then the so-called proper solution of $(\star)$ vanishes identically in $\mathbb{R} \times (0, \infty)$, and accordingly no positive classical solution exists in any time interval in this case. Complemented by some sufficient criteria for solutions to remain positive either locally or globally in time, this condition for instantaneous extinction is shown to be optimal at least with respect to algebraic decay of the initial data. This partially extends some known nonexistence results for $(\star)$ (Daskalopoulos and Del Pino, *Arch. Rat. Mech. Anal.* **137** (1997)) in that it does not require any knowledge on the behavior of $v_0(x)$ for $x < 0$.

Next focusing on the phenomenon of extinction in finite time, we show that in this respect a mass influx from $x = -\infty$ can interact with mass loss at $x = +\infty$ in a nontrivial manner. Namely, we shall detect examples of monotone initial data, with critical decay as $x \to +\infty$ and exponential growth as $x \to -\infty$, that lead to solutions of $(\star)$ which become extinct at a finite positive time, but which have empty extinction sets. This is in sharp contrast to known extinction mechanisms which are such that the corresponding extinction sets coincide with all of $\mathbb{R}$.

Key words: fast diffusion, extinction, extinction set, traveling wave
AMS Classification: 35K65, 35K55, 35B40 (secondary)

*michael.winkler@math.uni-paderborn.de

1 Introduction

In the modeling of nonlinear diffusion processes, an outstanding role is played by the prototypical equation

$$v_t = (v^{m-1} v_x)_x, \qquad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

and its n-dimensional analogue, where m is a real parameter and the unknown $v = v(x,t)$ is supposed to be nonnegative. If $m = 1$, this becomes the usual linear heat equation, whereas choosing $m > 1$ leads to the porous medium equation that was thoroughly explored in the 1970s and 1980s (see [2] and, for more recent developments, [13] and [32]). The study of the regime $m < 1$ began slightly later, and this range is often referred to as the fast diffusion range, because in this case the diffsusivity v^{m-1} in (1.1) increases at small densities and even becomes singular in the limit $v \searrow 0$ (cf. [32] and [13] for comprehensive surveys).

As one particular consequence of sufficiently singular diffusion, it is known that if $m \le -1$ then the phenomenon of extinction may occur if the initial data decay fast enough as $|x| \to \infty$. For $m < -1$, an instructive example for such an effect is provided by the family $(v_T)_{T>0}$ of explicitly given functions defined by

$$v_T(x,t) := K\Big(\frac{T-t}{x^2}\Big)^{\frac{1}{1-m}}, \qquad x \in \mathbb{R},\ t \in (0,T), \qquad \text{with } K := \Big(\frac{2(-m-1)}{1-m}\Big)^{\frac{1}{1-m}}, \tag{1.2}$$

which classically solve (1.1) in $(\mathbb{R} \setminus \{0\}) \times (0,T)$ and tend to zero everywhere as $t \nearrow T$. Along with an associated family of slightly less explicit nonsingular relatives sharing the same extinction behavior (cf. (5.1) below), all these functions decay like $|x|^{-\frac{2}{1-m}}$ as $|x| \to \infty$ for all $t \in [0,T)$. Accordingly, a straightforward analysis based on comparison arguments shows that for any bounded initial data which lie below some multiple of $|x|^{-\frac{2}{1-m}}$ for all $x \in \mathbb{R}$, the corresponding solution of (1.1) becomes extinct after a finite time ([32, Chapter 9]). Even more drastically, this extinction time is actually zero, and thus no resonable solution exists at all, whenever $m < -1$ and $v(x,0) \le c(1+|x|)^{-\frac{2}{1-m}-\varepsilon}$ holds for all $x \in \mathbb{R}$ and some $c > 0$ and $\varepsilon > 0$ ([11]).

Let us mention that both these effects can also be found in the higher-dimensional version of (1.1) for appropriately small m; in fact, in space dimension $n \ge 1$ and for respectively subcritical m considerable knowledge has been gained concerning the set of initial data which either lead to extinction after a positive time, or to instantaneous extinction, that is, to local nonexistence (see [9], [28], [11], [12], [32, Chapter 5, Chapter 9] and also [15] for extinction in slightly more general models). Recent studies have moreover revealed quite a large variety of quantitative results on how the respective extinction mechanisms depend on the spatial decay of the initial data in fast diffusion equations involving such subcritical m, with a particularly rich variety of facets appearing for intial data decaying at the critical rate $|x|^{-\frac{2}{1-m}}$ at first order ([14], [6], [7], [21], [17], [18], [19], [20]), but also some non-standard types of behavior for data with faster spatial decay ([24], [21], [27]).

When looking for explanations for the above phenomena of mass loss, one is led to the interpretation that if the initial distribution decays rapidly enough in space then the influence of fast diffusion, present wherever the solution is small, is strong enough to quickly transport as much mass as available away from each compact region towards infinity. The objective of the present work is to address the question how this mass transport may be affected when spatial decay of the data is prescribed *only* *in one direction*, say, as $x \to +\infty$ in the one-dimensional equation (1.1), possibly accompanied by unboundedness in the other. Thus having in mind as prototypes monotone initial data decreasing to zero as $x \to +\infty$, we particularly ask whether one can observe a significant interplay between the effects of mass loss towards $x = +\infty$ and influx coming from the left.

Main results I: Instantaneous extinction enforced by decay in one direction. Let us clarify the framework within which we can formulate the first of our main results. We shall assume throughout that the initial data v_0 are positive on $\mathbb{R}$ and, for simplicity in presentation, belong to $C^3(\mathbb{R})$, and hence subsequently we will consider the Cauchy problem

$$\begin{cases} v_t = (v^{m-1} v_x)_x, & x \in \mathbb{R},\ t > 0, \\ v(x,0) = v_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.3}$$

Since in view of well-known results ([16]) we neither expect uniqueness, nor classical solvability for all such data, in the present setting we find it adequate to adopt from [22] the notion of a *proper solution*, which one may think of as the largest function, defined on $\mathbb{R} \times [0,\infty)$, that can be approximated from above by a sequence $(v_k)_{k\in\mathbb{N}}$ of positive classical solutions of (1.1) with initial data converging to v_0 as $k \to \infty$ (cf. Definition 3.3 for a precise definition of proper solutions). This guarantees that for any choice of initial data with the mentioned properties, we will always be able to refer to some globally defined object that deserves being called a solution of (1.3), regardless of any positivity properties of this function, for instance.

Our first results concern the phenomenon of instantaneous extinction, and in this respect they essentially rediscover criticality ([11]) of the decay in (1.2) even when present only as $x \to +\infty$.

Theorem 1.1 *Let $m < -1$, and suppose that $v_0 \in C^3(\mathbb{R})$ is positive and such that*

$$v_0(x) \le c x^{-\beta} \qquad \text{for all } x > 0 \tag{1.4}$$

with some $\beta > \frac{2}{1-m}$ and $c > 0$. Then the proper solution v of (1.3) satisfies $v \equiv 0$ in $\mathbb{R} \times (0,\infty)$. In particular, in this case the problem (1.3) does not possess a positive classical solution in $\mathbb{R} \times (0,T)$ for any $T > 0$.

To underline the optimality of the above condition $\beta > \frac{2}{1-m}$, in part i) of the following proposition we include a corresponding statement on local existence for initial data lying above a multiple of $|x|^{-\frac{2}{1-m}}$ for all large values of $|x|$. Whereas this is a basically well-known consequence of a comparison procedure involving separated solutions (cf. Section 4.2), the second part on global existence under a slightly stronger condition seems to be new in this context.

Proposition 1.2 *Let $m < -1$, and assume that $v_0 \in C^3(\mathbb{R})$ is positive.*
i) If there exists $c > 0$ such that

$$v_0(x) \ge c(1+|x|)^{-\frac{2}{1-m}} \qquad \text{for all } x \in \mathbb{R},$$

then there exists $T > 0$ with the property that the proper solution v is a positive classical solution of (1.3) in $\mathbb{R} \times (0,T)$.
ii) In the case when moreover

$$|x|^{\frac{2}{1-m}} v_0(x) \to \infty \qquad \text{as } |x| \to \infty,$$

the same is true for $T = \infty$.

Theorem 1.1 especially means that if for large *positive* x, $v_0(x)$ lies significantly below all the singular solutions in (1.2), then even arbitrary growth of v_0 as $x \to -\infty$ is not sufficient to prevent instantaneous extinction: The strong loss of mass due to super-fast diffusion in one spatial direction cannot be compensated by any diffusion-driven influx. Moreover, since v_0 can be described at will for $x < 0$ in Theorem 1.1, in this case we conclude that even arbitrarily high amounts of mass can immediately be transported from any bounded region towards $x = +\infty$. An interesting problem left open here consists in determining to which extent parallels to this can be found in higher-dimensional nonlinear diffusion flows. For a related study on issues of global existence and blow-up in dependence on the asymptotics of initial data in the porous medium equation on Riemannian manifolds, we refer to [23]; certain integral conditions for the occurrence of instantaneous extinction in nonlinear fractional diffusion processes have recently been derived in [8].

Main results II: Extinction at spatial infinity. We next investigate in more detail the behavior of some solutions evolving from initial data decaying critically as $x \to +\infty$, and we are thereby led to studying the phenomenon of extinction in a finite but positive time. As to this, (1.2) provides an example in which decay of the data as both $x \to +\infty$ and $x \to -\infty$ entails that extinction will occur at a rate determined by $(T-t)^{\frac{1}{1-m}}$, and that it does so *at each point* $x \in \mathbb{R}$. Consequently, the *extinction set*

$$\mathcal{E} \;:=\; \Big\{ x \in \mathbb{R} \;\Big|\; \exists (x_k, t_k)_{k\in\mathbb{N}} \subset \mathbb{R} \times (0,T) \text{ such that } x_k \to x, t_k \to T \text{ and } v(x_k,t_k) \to 0 \text{ as } k \to \infty \Big\} \tag{1.5}$$

of these solutions is the whole line, $\mathcal{E} = \mathbb{R}$, and to the best of our knowledge this is the only possibility ever detected in comparable situations (see the discussion in [32, Section 7] and in particular [32, Theorem 7.6], for instance).

In contrast to this, we shall reveal a particular mechanism in which mass transport from $x = -\infty$ interacts with mass transport to $x = +\infty$ in a nontrivial manner, and thereby essentially influences the asymptotics near extinction. Namely, we shall see that whenever $m < 0$, the problem (1.3) possesses classical solutions v that are positive on $\mathbb{R} \times (0,T)$ and become extinct at some finite positive time T, but which have *empty extinction sets*:

Theorem 1.3 *Let $m < -1$. Then for each $k > 0$ and all $T > 0$ there exists a positive classical solution v of (1.3) which can be written in the form*

$$v(x,t) = (T-t)^{\frac{1}{1-m}} \cdot F_k\Big(x + k \cdot \ln(T-t)\Big), \qquad x \in \mathbb{R},\ t \in [0,T), \tag{1.6}$$

where $F_k \in C^\infty(\mathbb{R})$ is a decreasing positive solution of

$$(F^{m-1}F')' = -\frac{1}{1-m}F - kF' \qquad \textit{on } \mathbb{R}. \tag{1.7}$$

The initial data $v_0 := v(\cdot,0)$ are decreasing on $\mathbb{R}$ with an asymptotic behavior described by

$$c_0 e^{-\frac{1}{(1-m)k}x} \le v_0(x) \le c_1 e^{-\frac{1}{(1-m)k}\cdot x} \qquad \textit{for all } x \le 0$$

and

$$d_0 x^{-\frac{2}{1-m}} \le v_0(x) \le d_1 x^{-\frac{2}{1-m}} \quad \textit{for all } x \ge 1,$$

and the solution satisfies

$$v(x,t) \ge c e^{-\frac{1}{(1-m)k}\cdot x} \qquad \textit{for all } x \in \mathbb{R} \textit{ and } t \in (0,T) \tag{1.8}$$

as well as

$$\limsup_{t \nearrow T} v(x,t) \le C e^{-\frac{1}{(1-m)k}\cdot x} \qquad \textit{for all } x \in \mathbb{R} \tag{1.9}$$

with some positive constants c_0, c_1, d_0, d_1, c and C.
Moreover, v undergoes an extinction at $t = T$ in the sense that for any $\tilde{T} > T$, there is no positive classical solution of (1.3) in $\mathbb{R}\times(0,\tilde{T})$ which coincides with v in $\mathbb{R}\times(0,T)$. In particular, the extinction set of v is empty, $\mathcal{E} = \emptyset$.

This curious effect, sometimes also referred to as *extinction at spatial infinity* according to the fact that in the situation addressed in (1.5) it requires that $x_k \to \infty$, may be regarded as reflecting the possibility of a certain compensation of mass loss near $x = +\infty$ by an appropriate amount of mass influx from $x = -\infty$, leading to a rather precise balance of these processes until some time at which the wave mechanism in (1.6) has established the state characterized by (1.8) and (1.9), and beyond which no extension will thus be possible due to the exponential decay described in (1.9). It would of course be interesting to see whether this phenomenon is stable in any sense, and if it can be observed for other than the above choices of the initial data. Since we do not pursue the issue of uniqueness of classical solutions of (1.3) here, we do not even know whether all solutions evolving from the initial data in Theorem 1.3 have empty extinction sets. We also have to leave open whether the solutions in Theorem 1.3 are maximal and thus coincide with the proper solutions discussed before. Another natural question not addressed here concerns the precise extinction profile of the solutions found above, which amounts to studying the asymptotics of the solutions of (1.7) as $\xi \to -\infty$ in more detail.

Consequences implied by the Bäcklund transform. We finally remark that by means of the Bäcklund transform, from Theorem 1.3 we equivalently obtain a result on finite-time blow-up with empty blow-up set, that is, on blow-up at spatial infinity, in one-dimensional porous medium equations for monotone initial data satisfying a critical growth condition: Namely, taking v as in Theorem 1.3 and setting

$$z(y,t) := \frac{1}{v(x,t)}, \quad \text{where } y(x,t) := \varphi(t) + \int_0^x v(\xi,t)d\xi \quad \text{with } \varphi(t) := \frac{1}{m}\int_0^t (v^m)_x(0,s)ds \tag{1.10}$$

for $x \in \mathbb{R}$ and $t \ge 0$, we see that $y(\cdot,t)$ is a diffeomorphism from $\mathbb{R}$ onto itself, and can easily verify that z satisfies the porous medium equation $z_t = \frac{1}{|m|}(z^{|m|})_{yy}$ on $\mathbb{R}\times(0,T)$ ([31]). Moreover, from the properties of v_0 it can be computed that $z_0 := z(\cdot,0)$ increases on $\mathbb{R}$ with

$$\frac{\tilde{c}_0}{|y|} \le z_0(y) \le \frac{\tilde{c}_1}{|y|} \qquad \text{for all } y \le -1$$

and

$$\tilde{d}_0 y^{\frac{2}{|m|-1}} \le z_0(y) \le \tilde{d}_1 y^{\frac{2}{|m|-1}} \qquad \text{for all } y \ge 1.$$

The latter growth condition is precisely critical in respect of local existence of solutions for this porous medium equation ([3], [5]), so that it is not surprising that z blows up in a finite positive time. However, we are not aware of any result asserting emptyness of a corresponding blow-up set in related situations.

Let us mention that some of our results also apply to different ranges of m that are not in the main focus of this paper. In order not to distract from our main purpose, we will not state them here but rather refer to Remarks 1, 2 and 4 below.

The paper is organized as follows. After introducing a convenient transformation in Section 2, in Section 3 we shall describe the procedure along which we will construct proper solutions. Sections 4.1 and 4.2 will the be devoted to the proofs of Theorem 1.1 and Proposition 1.2, respectively, while Theorem 1.3 will be established in Section 5.

2 Transformation to a degenerate parabolic problem

We find it more convenient for our analysis to reformulate (1.3) by introducing the new unknown

$$u(x,t) := v^m(x,t), \qquad x \in \mathbb{R},\ t > 0, \tag{2.1}$$

which should satisfy the degenerate parabolic problem

$$\begin{cases} u_t = u^p u_{xx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.2}$$

where $p := \frac{1-m}{-m}$ satisfies

$$1 < p < 2$$

when $m < -1$, the borderline case $m = -1$ corresponding to $p = 2$. According to our assumptions on v_0, the initial data $u_0 := v_0^m \in C^3(\mathbb{R})$ are positive on $\mathbb{R}$.
One technical advantage in addressing (2.2) rather than (1.3) will consist of the fact that in this new formulation, minimal solutions can be approximated by solutions of asscociated problems in bounded intervals with prescribed Dirichlet data zero on the lateral boundary.
All of our results shall be derived in terms of the variable u first, and then be translated to the original coordinates. As a natural by-product, we thus obtain some statements for (2.2) concerning immediate blow-up, and on finite-time blow-up with empty blow-up sets.

3 Proper solutions and approximation by bounded solutions

To begin with, we describe one possible approximation procedure by which we will obtain an object that deserves being called a solution of (2.2) even when the value $+\infty$ is allowed to be attained. Although most parts of the reasoning are standard, we attempt to be as concise as possible here, because in some of the rigorous arguments in Section 4 we shall essentially rely on some properties of our approximate solutions, and accordingly we will frequently refer to some particular stages of the construction process (cf. Lemma 4.2, for instance).

In a first step we attempt to approximate (2.2) by problems having bounded solutions. To this end, we consider solutions $u = u_M$ of

$$\begin{cases} u_t = u^p u_{xx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_{0M}(x), & x \in \mathbb{R}, \end{cases} \tag{3.1}$$

where $M > 0$. Here, we choose the initial data u_{0M} to be smooth bounded functions satisfying $0 \le u_{0M} \le M$ on $\mathbb{R}$ and converging to u_0 as $M \to \infty$: To achieve this, we fix a nondecreasing $\rho \in C^3([0,\infty))$ such that $\rho(s) = s$ if $s \le \frac{1}{4}$, $\rho(s) = 1$ if $s \ge 1$ and $s\rho'(s) \le \rho(s)$ for $s \ge 0$, and given $M > 0$ we let $\rho_M(s) := M \cdot \rho(\frac{s}{M})$, $s \ge 0$. Then

$$u_{0M}(x) := \rho_M(u_0(x)), \qquad x \in \mathbb{R}, \tag{3.2}$$

belongs to $C^3(\mathbb{R})$ with $0 \le u_{0M} \le M$, and u_{0M} coincides with u_0 wherever $u_0(x) \le \frac{M}{4}$. Moreover, for all $x \in \mathbb{R}$ we have $u_{0M}(x) \nearrow u_0(x)$ as $M \to \infty$, because $s\rho'(s) \le \rho(s)$ for $s \ge 0$ implies that ρ_M is nondecreasing with M.

To act in bounded intervals rather than on the whole real line, we shall next approximate u_M by solutions $u = u_{MR}^{(x_0)}$ in $B_R(x_0) \times (0,\infty)$, $x_0 \in \mathbb{R}, R > 0$, of

$$\begin{cases} u_t = u^p u_{xx}, & x \in B_R(x_0),\ t > 0, \\ u(x_0 \pm R, t) = 0, & t > 0, \\ u(x,0) = u_{0MR}^{(x_0)}(x), & x \in B_R(x_0), \end{cases} \tag{3.3}$$

with data $u_{0MR}^{(x_0)}$ constructed in such a way that

$$u_{0MR}^{(x_0)} = u_{0MRx}^{(x_0)} = u_{0MRxx}^{(x_0)} = 0 \qquad \text{for } x = x_0 \pm R, \tag{3.4}$$

and that in addition positivity of u_0 on $\mathbb{R}$ implies positivity of $u_{0MR}^{(x_0)}$ inside $B_R(x_0)$. To this end, we take a nonincreasing function $\chi \in C^3([0,1])$ such that $\chi \equiv 1$ in $[0, \frac{1}{2}]$, $\chi > 0$ in $[0,1)$, and such that $\chi(1) = \chi_x(1) = \chi_{xx}(1) = 0$. For $x_0 \in \mathbb{R}$ and $R > 0$ we let $\chi_R^{(x_0)}(x) := \chi(\frac{|x-x_0|}{R})$, $x \in \bar{B}_R(x_0)$. Then

$$u_{0MR}^{(x_0)}(x) := u_{0M}(x) \cdot \chi_R^{(x_0)}(x), \qquad x \in \bar{B}_R(x_0), \tag{3.5}$$

belongs to $C^3(\bar{B}_R(x_0))$ and satisfies $u_{0MR}^{(x_0)} \nearrow u_{0MR}$ as $R \to \infty$ for each $x_0 \in \mathbb{R}$, and it can easily be checked that our assumptions on χ guarantee that (3.4) holds.
We remark here that although for most of our purposes it would be sufficient to fix $x_0 = 0$ throughout, in Lemma 4.2 below we shall need to pick different x_0 in order to prove that spatial monotonicity of the initial data is inherited by solutions of (3.1).

Finally, following a standard way to achieve a non-degenerate parabolic regularization, for $\varepsilon \in (0,1)$ we consider solutions $u = u_{MR\varepsilon}^{(x_0)}$ of

$$\begin{cases} u_t = u^p u_{xx}, & x \in B_R(x_0),\ t > 0, \\ u(x_0 \pm R, t) = \varepsilon, & t > 0, \\ u(x,0) = u_{0MR\varepsilon}^{(x_0)}(x), & x \in B_R(x_0), \end{cases} \tag{3.6}$$

where

$$u^{(x_0)}_{0MR\varepsilon}(x) := u^{(x_0)}_{0MR}(x) + \varepsilon, \qquad x \in \bar{B}_R(x_0).$$

Then also $u^{(x_0)}_{0MR\varepsilon}$ belongs to $C^3(\bar{B}_R(x_0))$ and clearly satisfies $u^{(x_0)}_{0MR\varepsilon} \ge \varepsilon$ and $u^{(x_0)}_{0MR\varepsilon} \searrow u^{(x_0)}_{0MR}$ in $\bar{B}_R(x_0)$ as $\varepsilon \searrow 0$, whereas (3.4) guarantees

$$u^{(x_0)}_{0MR\varepsilon}(x_0 \pm R) = \varepsilon \qquad \text{as well as} \qquad u^{(x_0)}_{0MR\varepsilon xx}(x_0 \pm R) = 0. \tag{3.7}$$

This will entail convenient regularity properties for solutions of (3.6) that will be useful in Lemma 4.3 below.

The solvability of (3.6) and of (3.3) can be asserted by standard arguments, which we shortly repeat for the sake of completeness in the following lemma. Here and in the sequel, we shall frequently apply parabolic comparison arguments to sub- and supersolutions of suitable boundary value problems associated with (2.2). However, known results on nonuniqueness in (2.2) for $p > 2$ ([16]) and even for (weak solutions of) the corresponding Dirichlet problem in bounded domains when $p = 1$ ([26]) underline the need for being cautious in such reasonings in presence of degenerate diffusion. In [33], the reader can find a corresponding version of the comparison principle that is adequate for all of our arguments in this direction.

Lemma 3.1 *Suppose that $u_0 \in C^3(\mathbb{R})$ satisfies $u_0 > 0$ on $\mathbb{R}$. Then for all $x_0 \in \mathbb{R}, M > 0, R > 0$ and $\varepsilon \in (0,1)$, the problem (3.6) has a global positive classical solution $u^{(x_0)}_{MR\varepsilon} \in C^{2,1}(\bar{B}_R(x_0) \times [0,\infty))$. As $\varepsilon \searrow 0$, we have $u^{(x_0)}_{MR\varepsilon} \searrow u^{(x_0)}_{MR}$ in $\mathbb{R} \times [0,\infty)$ and $u^{(x_0)}_{MR\varepsilon} \to u^{(x_0)}_{MR}$ in $C^0_{loc}(\bar{B}_R(x_0) \times [0,\infty)) \cap C^{2,1}_{loc}(B_R(x_0) \times (0,\infty))$, where $u^{(x_0)}_{MR}$ is the unique positive global classical solution of (3.3).*

PROOF. For convenience we drop the superscript (x_0). In view of the comparison principle we have the a priori estimates $\varepsilon \le u_{MR\varepsilon} \le M + \varepsilon$ for $t > 0$ and $x \in B_R(x_0)$, so that in fact (3.6) is non-degenerate and hence admits a global classical solution $u_{MR\varepsilon}$ which even belongs to $C^{2,1}(\bar{B}_R(x_0) \times [0,\infty))$, because (3.7) implies that the compatibility conditions for (3.6) up to first order are fulfilled ([25]). Moreover, using that u_0 is positive on $\mathbb{R}$ and that hence u_{0MR} is positive inside $B_R(x_0)$, it can readily be verified that for any choice of $R' \in (0,R)$ one can find $c_{MR'} > 0$ such that $u_{0MR}(x) \ge \tilde{c}_{MR'}\Theta_{R'}(x)$ holds for all $x \in B_{R'}(x_0)$ with $\Theta_{R'}(x) \equiv \Theta^{(x_0)}_{R'}(x) := \cos\frac{\pi(x-x_0)}{2R'}$. Since the separated function

$$\underline{u}(x,t) := y(t) \cdot \Theta_{R'}(x), \qquad x \in \bar{B}_{R'}(x_0),\ t \ge 0,$$

with

$$y(t) := \Big(\tilde{c}^p_{MR'} + \Big(\frac{\pi}{2R'}\Big)^2 pt\Big)^{-\frac{1}{p}}, \qquad t \ge 0,$$

can readily be seen to satisfy $\underline{u}_t \le \underline{u}^p \underline{u}_{xx}$ in $B_{R'}(x_0) \times (0,\infty)$, another comparison thus yields the two-sided ε-independent estimate

$$y(t) \cdot \Theta_{R'}(x) \le u_{MR\varepsilon}(x,t) \le M + 1 \qquad \text{for all } x \in B_{R'}(x_0) \text{ and } t \ge 0. \tag{3.8}$$

As $R' \in (0,R)$ was arbitrary here, this allows for invoking parabolic regularity theory to derive uniform estimates for $u_{MR\varepsilon}$ in $C^{\theta,\frac{\theta}{2}}_{loc}(B_R(x_0)\times[0,\infty))$ and in $C^{2,1}_{loc}(B_R(x_0)\times(0,\infty))$ for some $\theta>0$ ([25]). Along with the evident ordering property of $(u_{MR\varepsilon})_{\varepsilon\in(0,1)}$, this implies that as $\varepsilon\searrow 0$, we have $u_{MR\varepsilon}\to u_{MR}$ in $C^0_{loc}(B_R(x_0)\times[0,\infty))\cap C^{2,1}_{loc}(B_R(x_0)\times(0,\infty))$ for some nonnegative function u_{MR} defined on $\bar B_R\times[0,\infty)$ which is positive and continuous in $B_R\times[0,\infty)$ and satisfies the initial value problem in (3.3) classically, and which moreover is upper semicontinuous in $\bar B_R(x_0)\times[0,\infty)$. Since clearly u_{MR} vanishes on $\partial B_R(x_0)$, this implies that actually u_{MR} belongs to $C^0(\bar B_R(x_0)\times[0,\infty))$ with $u_{MR}|_{\partial B_R(x_0)}=0$ and that hence $u_{MR\varepsilon}\to u_{MR}$ even in $C^0_{loc}(\bar B_R(x_0)\times[0,\infty))$ as $\varepsilon\searrow 0$ according to Dini's theorem (cf. also [34, Lemma 1.2] for a precedent variant of this argument).
To see its uniqueness along the procedure presented in [33], we only need to make sure that whenever $T>0$, any positive classical solution v of (3.3) in $\mathbb{R}\times(0,T)$ coincides with the limit $u:=u^{(0)}_{MR}$ that we have just found upon the particular choice $x_0:=0$. Indeed, since $u_{MR\varepsilon}\ge v$ by classical comparison, we already know that $u\ge v$ and thus $H(u)\ge H(v)$, where $H(s):=\int_1^s\sigma^{-p}d\sigma$, $s>0$. On the other hand, for any $r\in(0,R)$, with $\Theta_r=\Theta^{(0)}_r$ as introduced above we have

$$\begin{aligned}\frac{d}{dt}\int_{B_r(0)}\Big(H(u(x,t))-H(v(x,t))\Big)\cdot\Theta_r(x)dx &= \int_{B_r(0)}(u_{xx}-v_{xx})\cdot\Theta_r\\ &= \int_{B_r(0)}(u-v)\cdot\Theta_{rxx}\\ &\quad -\Big(u(r,t)-v(r,t)\Big)\cdot\Theta_{rx}(r)\\ &\quad +\Big(u(-r,t)-v(-r,t)\Big)\cdot\Theta_{rx}(-r)\end{aligned}$$

for all $t>0$. As $\Theta_{rxx}\le 0$ in $B_r(0)$ and $\pm\Theta_{rx}(\pm r)=-\frac{\pi}{2r}<0$, using $0\le v\le u$ we infer that

$$\frac{d}{dt}\int_{B_r(0)}\Big(H(u(x,t))-H(v(x,t))\Big)\cdot\Theta_r(x)dx\le\frac{\pi}{r}\cdot\delta(r,T)\qquad\text{for all } t\in(0,T),$$

where $\delta(r,T):=\sup_{t\in(0,T)}u(\pm r,t)\to 0$ as $r\nearrow R$ by continuity of u and the fact that u vanishes on $\partial B_R(0)$. In view of the fact that $H(u(\cdot,0))-H(v(\cdot,0))\le 0$ in $B_R(0)$, integrating with respect to $t\in(0,T)$ and then taking $r\nearrow R$ thus yields $H(u)\le H(v)$ in $B_R(0)\times(0,T)$ by Fatou's lemma. This implies $u\le v$ and hence $u\equiv v$. □

Another limit process enables us to establish the existence of a minimal solution to (3.1).

Lemma 3.2 *Let $M>0$. Then (3.1) possesses a minimal positive classical solution u_M which is global in time and satisfies $u_M\le M$ in $\mathbb{R}\times(0,\infty)$. For arbitrary $x_0\in\mathbb{R}$, this solution can be obtained as the limit in $C^0_{loc}(\mathbb{R}\times[0,\infty))\cap C^{2,1}_{loc}(\mathbb{R}\times(0,\infty))$ of the solutions $u^{(x_0)}_{MR}$ of (3.3) as $R\nearrow\infty$.*

Proof. From the properties of $(\chi_R)_{R>0}$ it is easy to see using the comparison principle that for each $x_0\in\mathbb{R}$, $u^{(x_0)}_{MR}$ is nondecreasing with respect to R and hence monotonically approaches some limit $u^{(x_0)}_M$ from below. Since $u^{(x_0)}_{MR}\le M$ for all R by the maximum principle, parabolic regularity theory ensures that actually $u^{(x_0)}_{MR}\to u^{(x_0)}_M$ takes place in the asserted topology. To see that $u^{(x_0)}_M$ is minimal, we only need to observe that any positive classical solution $\tilde u$ of (3.1) satisfies $\tilde u\ge u^{(x_0)}_{MR}$ in $B_R(x_0)\times(0,\infty)$

for all $x_0 \in \mathbb{R}$ and $R > 0$ by another comparison argument, and hence we have $\tilde{u} \ge u_M^{(x_0)}$. This entails that in fact all the $u_M^{(x_0)}$, $x_0 \in \mathbb{R}$, coincide, as desired. $\Box$

Once more thanks to the comparison principle, the family $(u_M)_{M>0}$ of minimal positive classical solutions of (3.1) is ordered, so that

$$u(x,t) := \lim_{M\to\infty} u_M(x,t), \qquad x \in \mathbb{R},\ t > 0, \tag{3.9}$$

provides a well-defined function with values in $(0, +\infty]$. Without any further investigation of its particular solution properties in respect of the problem (2.2) directly, we adopt a notion from [22] in giving the following definition.

Definition 3.3 *Let $u_0 \in C^3(\mathbb{R})$ be positive. Then the function u defined through (3.9) will be called the* proper solution *of (2.2).*
Given a positive $v_0 \in C^3(\mathbb{R})$, the proper solution v of (1.3) is defined as the $\frac{1}{m}$-th power of the proper solution u of (2.2) with data $u_0 := v_0^m$, $v := u^{\frac{1}{m}}$, where we set $v(x,t) := 0$ if $u(x,t) = +\infty$.

Repeating the comparison argument from Lemma 3.2, we immediately obtain the following extremality feature of proper solutions.

Corollary 3.4 *Let $u_0 \in C^3(\mathbb{R})$ be positive. Assume that $T > 0$, and that $\tilde{u}$ is a positive classical solution of (2.2) in $\mathbb{R} \times (0,T)$ with initial data u_0. Then the proper solution u of (2.2) satisfies $u \le \tilde{u}$ in $\mathbb{R} \times (0,T)$.*
Correspondingly, if $\tilde{v}$ is a positive classical solution of (1.3) in $\mathbb{R}\times(0,T)$ then $\tilde{v} \le v$ holds in $\mathbb{R}\times(0,T)$ with the proper solution v of (1.3).

4 Occurrence vs. absence of immediate blow-up in the transformed problem

In this part our goal is to prove Theorem 1.1. In view of (2.1, this amounts to providing respective conditions on the growth of $u_0(x)$ as $x \to +\infty$ that either enforce or rule out immediate blow-up of the proper solution u of (2.2). Here we observe that since $p = \frac{1-m}{-m}$, the decay claimed to be critical in Theorem 1.1 corresponds to a growth rate of $u_0(x)$ as $x \to +\infty$ determined by $x^{\frac{2}{p}}$.

4.1 Immediate blow-up for data with supercritical growth as $x \to +\infty$. Proof of Theorem 1.1

Let us first address Theorem 1.1, that is, we wish to assert immediate blow-up for u if $u_0(x) \ge cx^\alpha$ for all $x > 0$ with some $\alpha > \frac{2}{p}$ and $c > 0$. The main difficulty to be overcome here stems from the fact that we claim to require no information about the behavior of u_0 for $x < 0$ other than just positivity. An essential preliminary step towards the desired result consists of giving a first a priori bound from below for solutions on the half-line emanating from initial data with supercritical growth as $x \to +\infty$.

Lemma 4.1 *Suppose that $p \in (0,2)$, and that $u_0 \in C^3(\mathbb{R})$ is positive and such that*

$$u_0(x) \ge c_0 x^\alpha \qquad \textit{for all } x > 0 \tag{4.1}$$

with some $c_0 > 0$ and $\alpha > \frac{2}{p}$. Then there exists $c > 0$ with the property that for the proper solution u of (2.2) we have

$$u(x,t) \ge cx^\alpha \qquad \text{for all } x > 0 \text{ and } t > 0. \tag{4.2}$$

PROOF. For $\delta > 0$ and $R > 0$, we let $w_{\delta R}$ be the solution of

$$\begin{cases} -w_{\delta Rxx} = \delta w_{\delta R}^{1-p}, & x \in B_R(0), \\ w_{\delta R}(\pm R) = 0. \end{cases} \tag{4.3}$$

Then by a first integral procedure we obtain

$$\frac{1}{2} w_{\delta Rx}^2(x) = \frac{\delta}{2-p} \Big(A_{\delta R}^{2-p} - w_{\delta R}^{2-p}(x) \Big), \qquad x \in B_R(0), \tag{4.4}$$

where $A_{\delta R} := w_{\delta R}(0)$. Upon another integration, we find that

$$\int_{\frac{w_{\delta R}(x)}{A_{\delta R}}}^1 \frac{d\sigma}{\sqrt{1-\sigma^{2-p}}} = \sqrt{\frac{2\delta}{2-p}} \cdot A_{\delta R}^{-\frac{p}{2}} \cdot |x|, \qquad x \in \bar{B}_R(0),$$

which evaluated at $x = R$ allows us to determine $A_{\delta R}$ according to

$$A_{\delta R} = \left(\frac{\sqrt{\frac{2\delta}{2-p}} R}{I} \right)^{\frac{2}{p}} \qquad \text{with} \qquad I := \int_0^1 \frac{d\sigma}{\sqrt{1-\sigma^{2-p}}}. \tag{4.5}$$

Given $x_0 > 0$, our goal is to compare u_M for suitably large $M > 0$ to

$$\underline{u}_\delta(x,t) := y_\delta(t) \cdot w_{\delta R_\delta}(x - x_0 - R_\delta), \qquad x_0 \le x \le x_0 + 2R_\delta, \ t \ge 0,$$

for $\delta > 0$ and some appropriate $R_\delta > 0$, where

$$\begin{cases} y_\delta'(t) = -\delta y_\delta^{p+1}(t), & t > 0, \\ y_\delta(0) = 1, \end{cases} \tag{4.6}$$

that is, $y_\delta(t) := (1 + p\delta t)^{-\frac{1}{p}}$. From (4.3) and (4.6) we see that for any choice of δ and R_δ,

$$\underline{u}_{\delta t} - \underline{u}_\delta^p \underline{u}_{\delta xx} = (y_\delta' + \delta y_\delta^{p+1}) \cdot w_{\delta R_\delta} = 0 \qquad \text{for all } x \in (x_0, x_0 + 2R_\delta) \text{ and } t > 0.$$

Since $\underline{u}_\delta = 0$ for $x = x_0$ and for $x = x_0 + 2R_\delta$, the comparison principle will tell us that $u_M \ge \underline{u}_\delta$ in $(x_0, x_0 + 2R_\delta) \times (0, \infty)$ whenever δ and R_δ are such that $u_M \ge \underline{u}_\delta$ holds initially. In view of (4.1), however, this is true if

$$w_{\delta R_\delta}(x - x_0 - R_\delta) \le c_0 x^\alpha \qquad \text{for all } x \in (x_0, x_0 + 2R_\delta), \tag{4.7}$$

and if moreover

$$\frac{M}{4} \ge c_0 \cdot (x_0 + 2R_\delta)^\alpha, \tag{4.8}$$

because then according to our construction of u_{0M} and the monotonicity of the function ρ_M appearing therein,

$$u_{0M}(x) = \rho_M(u_0(x)) \geq \rho_M(c_0 x^\alpha) = c_0 x^\alpha \qquad \text{for all } x \in (x_0, x_0 + 2R_\delta)$$

due to the fact that then $c_0 x^\alpha \leq \frac{M}{4}$ for any such x by (4.8).
We now claim that (4.7) holds if, given $\delta > 0$, we let

$$R_\delta := c_1 \delta^{-\frac{1}{2-p}} \cdot x_0^{\frac{p(\alpha-1)}{2-p}} \tag{4.9}$$

with

$$c_1 := \Big(\frac{2-p}{2}\Big)^{\frac{1}{2-p}} \cdot I \cdot c_2^{-\frac{p(\alpha-1)}{2-p}}, \qquad c_2 := (\alpha^{-\frac{1}{\alpha-1}} - \alpha^{-\frac{\alpha}{\alpha-1}}) \cdot c_0^{-\frac{1}{\alpha-1}},$$

where we observe that $c_2 > 0$ because $\alpha > \frac{2}{p}$ entails $\alpha > 1$.
To see that (4.9) implies (4.7), we note that by (4.4),

$$\begin{aligned} w_{\delta R_\delta}(x - x_0 - R_\delta) &\leq w_{\delta R_\delta}(-R_\delta) + w_{\delta R_\delta x}(-R_\delta) \cdot (x - x_0) \\ &= \sqrt{\frac{2\delta}{2-p} A_{\delta R_\delta}^{2-p}} \cdot (x - x_0) \qquad \text{for all } x \in (x_0, x_0 + 2R_\delta), \end{aligned}$$

so that (4.7) holds if

$$\varphi(x) := c_0 x^\alpha - B \cdot (x - x_0), \qquad x > x_0,$$

is nonnegative on (x_0, ∞), where we have set $B := \sqrt{\frac{2\delta}{2-p} A_{\delta R_\delta}^{2-p}}$. Computing $\varphi'(x) = \alpha c x^{\alpha-1} - B$ for $x > x_0$, we see that unless φ is strictly increasing, φ attains its minimum at $x_m = (\frac{B}{\alpha c_0})^{\frac{1}{\alpha-1}}$ with

$$\begin{aligned} \inf_{x \in (x_0, \infty)} \varphi(x) = \varphi(x_m) &= c_0 \Big(\frac{B}{\alpha c_0}\Big)^{\frac{\alpha}{\alpha-1}} - B\Big(\frac{B}{\alpha c_0}\Big)^{\frac{1}{\alpha-1}} + Bx_0 \\ &= -c_2 B^{\frac{\alpha}{\alpha-1}} + Bx_0. \end{aligned}$$

We therefore obtain $\varphi \geq 0$ on (x_0, ∞) provided that $Bx_0 \geq c_2 B^{\frac{\alpha}{\alpha-1}}$, that is, if $B \leq (\frac{x_0}{c_2})^{\alpha-1}$. Rewritten in terms of $A_{\delta R_\delta}$ and thus, via (4.5), of R_δ, this becomes

$$R_\delta \leq \sqrt{\frac{2-p}{2\delta}} \cdot I \cdot \Big\{\frac{2-p}{2\delta} \cdot \Big(\frac{x_0}{c_2}\Big)^{2(\alpha-1)}\Big\}^{\frac{p}{2(2-p)}}$$

and is thus asserted by (4.9).

Consequently, for such R_δ by comparison we conclude that for any $M > 0$ fulfilling (4.8) we have $u_M \geq \underline{u}_\delta$ in $(x_0, x_0 + 2R_\delta) \times (0, \infty)$ and thus, since $u \geq u_M$,

$$u(x,t) \geq (1 + p\delta t)^{-\frac{1}{p}} \cdot w_{\delta R_\delta}(x - x_0 - R_\delta) \qquad \text{for all } x \in (x_0, x_0 + 2R_\delta) \text{ and } t > 0. \tag{4.10}$$

In order to derive (4.2) from this, let us make sure that there exist $c_3 > 0$ independent of x_0 and $\delta_0(x_0) > 0$ such that

$$w_{\delta R_\delta}(x_0 - R_\delta) \ge c_3 x_0^\alpha \qquad \text{whenever } \delta \in (0, \delta_0(x_0)). \tag{4.11}$$

Indeed, let

$$\delta_0(x_0) := c_1^{2-p} x_0^{p\alpha-2}, \qquad x_0 > 0.$$

Then for $\delta < \delta_0(x_0)$ we have

$$\begin{aligned} \frac{x_0}{R_\delta} &= \frac{1}{c_1} \cdot \delta^{\frac{1}{2-p}} \cdot x_0^{1-\frac{p(\alpha-1)}{2-p}} \\ &< \frac{1}{c_1} \cdot c_1 x_0^{p\alpha-2} \cdot x_0^{1-\frac{p(\alpha-1)}{2-p}} \\ &= 1, \end{aligned}$$

and hence $x_0 - R_\delta < 0$. Since $w_{\delta R_\delta}$ increases on $(-R_\delta, 0)$, by (4.4) this means that if $w_{\delta R_\delta}(x_0 - R_\delta) \le \frac{1}{2} A_{\delta R_\delta}$ then

$$w_{\delta R_\delta x}(x) \ge \sqrt{\frac{2\delta}{2-p} \cdot (1 - 2^{p-2}) \cdot A_{\delta R_\delta}^{2-p}} \qquad \text{for all } x \in (-R_\delta, x_0 - R_\delta).$$

Expressing $A_{\delta R_\delta}$ via (4.5) and (4.9) according to

$$A_{\delta R_\delta} = c_4 \delta^{-\frac{1}{2-p}} \cdot x_0^{\frac{2(\alpha-1)}{2-p}}$$

with c_4 depending on p and α only, in this case we obtain

$$w_{\delta R_\delta x} \ge c_5 x_0^{\alpha-1} \qquad \text{on } (-R_\delta, x_0 - R_\delta)$$

for some $c_5 > 0$ and therefore upon integration

$$w_{\delta R_\delta}(x_0 - R_\delta) \ge c_5 x_0^\alpha \qquad \text{if } w_{\delta R_\delta}(x_0 - R_\delta) \le \frac{1}{2} A_{\delta R_\delta}. \tag{4.12}$$

On the other hand, if $w_{\delta R_\delta}(x_0 - R_\delta) > \frac{1}{2} A_{\delta R_\delta}$ then we directly find

$$\begin{aligned} w_{\delta R_\delta}(x_0 - R_\delta) &> \frac{1}{2} A_{\delta R_\delta} \\ &= \frac{c_4}{2} \delta^{-\frac{1}{2-p}} \cdot x_0^{\frac{2(\alpha-1)}{2-p}} \\ &> \frac{c_4}{2c_1} x_0^{-\frac{p\alpha-2}{2-p}} \cdot x_0^{\frac{2(\alpha-1)}{2-p}} \\ &= \frac{c_4}{2c_1} x_0^\alpha \qquad \text{if } w_{\delta R_\delta}(x_0 - R_\delta) > \frac{1}{2} A_{\delta R_\delta} \end{aligned}$$

when $\delta < \delta_0(x_0)$, which together with (4.12) proves (4.11). Hence, (4.10) tells us that

$$u(2x_0,t) \ge c_3(1+p\delta t)^{-\frac{1}{p}} x_0^\alpha \qquad \text{for all } t>0$$

whenever $\delta < \delta_0(x_0)$, which upon letting $\delta \searrow 0$ yields (4.2) in view of the fact that $x_0 > 0$ was arbitrary. □

Another preparation addresses the question whether spatial monotonicity of u_0 is inherited by the solutions of (3.1), and thus of (2.2). This issue appears to be obvious at first glance, but since we do not see how to apply a corresponding comparison argument to the Cauchy problem (3.1) directly, it seems that a rigorous argument should rather refer to the boundary value problems (3.3). However, since in the latter we prescribe zero Dirichlet data, none of the approximations $u_{MR}^{(x_0)}$ of u_M can be nondecreasing in space (unless $u_{MR}^{(x_0)} \equiv 0$). Fortunately, their limit will nevertheless have this property whenever u_0 is nondereasing, and the proof of this implication is the only place in this work where we substantially need the freedom to choose the point x_0 in the approximations (3.3) and (3.6) arbitrarily in $\mathbb{R}$.

Lemma 4.2 *Suppose that $u_{0x} \ge 0$ on $\mathbb{R}$. Then for all $M>0$ we have*

$$u_{Mx} \ge 0 \qquad \text{in } \mathbb{R}\times(0,\infty).$$

Proof. We fix $x_0 \in \mathbb{R}$ and claim that for all $t>0$ we have

$$u_M(x_0+h,t) \ge u_M(x_0-h,t) \qquad \text{for all } h>0, \tag{4.13}$$

which evidently will imply $u_{Mx}(x_0,t) \ge 0$ for all $t>0$. To see (4.13), for arbitrary $R>h$ and $\varepsilon \in (0,1)$ we let

$$z(y,t) := u_{MR\varepsilon}^{(x_0)}(x_0+y,t) - u_{MR\varepsilon}^{(x_0)}(x_0-y,t), \qquad y\in[0,R],\ t\ge 0,$$

and observe that

$$z(0,t)=0 \qquad \text{and} \qquad z(R,t)=0 \qquad \text{for all } t\ge 0. \tag{4.14}$$

Moreover, from the assumption $u_{0x}\ge 0$ it immediately follows that $u_{0Mx}\ge 0$, which shows that

$$\begin{aligned} z(y,0) &= \Big(\chi\Big(\frac{|y|}{R}\Big)\cdot u_{0M}(x_0+y)+\varepsilon\Big) - \Big(\chi\Big(\frac{|y|}{R}\Big)\cdot u_{0M}(x_0-y)+\varepsilon\Big) \\ &= \chi\Big(\frac{|y|}{R}\Big)\cdot\Big(u_{0M}(x_0+y)-u_{0M}(x_0-y)\Big) \\ &\ge 0 \qquad \text{for all } y\in[0,R]. \end{aligned} \tag{4.15}$$

Next, a straightforward computation reveals that

$$z_t = a(y,t)z_{yy} + b(y,t)z, \qquad y\in(0,R),\ t>0, \tag{4.16}$$

holds with

$$a(y,t) := \Big(u_{MR\varepsilon}^{(x_0)}(x_0+y,t)\Big)^p \qquad \text{and}$$

$$b(y,t) := u_{MR\varepsilon xx}^{(x_0)}(x_0-y,t)\cdot\int_0^1 p\cdot\Big\{s\cdot u_{MR\varepsilon}^{(x_0)}(x_0+y,t)+(1-s)\cdot u_{MR\varepsilon}^{(x_0)}(x_0-y,t)\Big\}^{p-1} ds.$$

Since $\varepsilon \le u^{(x_0)}_{MR\varepsilon} \le M$ and $u^{(x_0)}_{MR\varepsilon}$ belongs to $C^{2,1}(\bar{B}_R(x_0)\times[0,\infty))$, for each fixed ε the functions a and b are continuous on $\bar{B}_R(x_0)\times[0,\infty)$ with $a \ge \varepsilon^p$. Therefore the comparison principle can be applied to guarantee that as a consequence of (4.14)-(4.16) we have $z \ge 0$ in $(0,R)\times(0,\infty)$. Taking $\varepsilon \searrow 0$ and then $R \to \infty$ now easily yields (4.13). $\Box$

As a last preliminary, we recall the following well-known semi-convexity estimate (cf. also [32], [1]).

Lemma 4.3 *Let $p > 0$. Then for all $M > 0$ we have*

$$\frac{u_{Mt}}{u_M} \ge -\frac{1}{pt} \qquad in\ \mathbb{R}\times(0,\infty). \tag{4.17}$$

PROOF. For $R > 0$ and $\varepsilon \in (0,1)$, we let $z := \frac{u^{(0)}_{MR\varepsilon t}}{u^{(0)}_{MR\varepsilon}}$. Then it follows from the regularity properties of $u^{(0)}_{MR\varepsilon}$ that z is continuous in $\bar{B}_R(0)\times[0,\infty)$ and solves

$$z_t = a(x,t)z_{xx} + b(x,t)z_x + pz^2 \qquad \text{in } B_R(0)\times(0,\infty)$$

with $a := (u^{(0)}_{MR\varepsilon})^p$ and $b := 2(u^{(0)}_{MR\varepsilon})^{p-1}\cdot u^{(0)}_{MR\varepsilon x}$. Hence, by comparison we find that z lies above $\underline{z}(x,t) := -\frac{1}{p(t+\tau)}$ in $B_R(0)\times(0,\infty)$ for all sufficiently small $\tau > 0$, because $\underline{z}_t - a\underline{z}_{xx} - b\underline{z}_x - p\underline{z}^2 = 0$, $\underline{z}(\pm R,t) < 0 = z(\pm R,t)$ for $t > 0$ and $\underline{z}(x,0) = -\frac{1}{p\tau} < z(x,0)$ for all $x \in B_R(0)$ if τ is small enough. Taking $\tau \searrow 0$, then $\varepsilon \searrow 0$ and finally $R \to \infty$ shows that (4.17) is true. $\Box$

We are now in the position to prove immediate blow-up of the proper solution of (2.2) under the assumption of supercritical growth of u_0 as $x \to +\infty$. Our proof will be based on a contradictory argument, combining comparison techniques with an analysis of certain weighted integral norms of inverse powers of the solution.

Lemma 4.4 *Let $p \in (0,2)$ and assume that $u_0 \in C^3(\mathbb{R})$ is positive on $\mathbb{R}$ and fulfills*

$$u_0(x) \ge cx^\alpha \qquad for\ all\ x > 0 \tag{4.18}$$

with some $c > 0$ and $\alpha > \frac{2}{p}$. Then the proper solution u of (2.2) satisfies

$$u(x,t) = +\infty \qquad for\ all\ x \in \mathbb{R}\ and\ t > 0. \tag{4.19}$$

In particular, in this case the problem (2.2) does not possess a positive classical solution in $\mathbb{R}\times(0,T)$ for any $T > 0$.

PROOF. Since $u_{0\star}(x) := \frac{1}{2}\min_{y\ge x} u_0(y)$, $x \in \mathbb{R}$, defines a continuous nondecreasing positive function on $\mathbb{R}$ fulfillig $u_{0\star} < u_0$ on $\mathbb{R}$, on performing a straightforward mollifying procedure to the latter it is possible to find $\underline{u_0} \in C^3(\mathbb{R})$ fulfilling (4.18) such that $0 < \underline{u_0} < u_0$ as well as $(\underline{u_0})_x \ge 0$ on $\mathbb{R}$. Applying a comparison argument to the approximate problems (3.6), we may therefore assume that $u_{0x} \ge 0$ on $\mathbb{R}$ and hence obtain from Lemma 4.2 that $u_{Mx} \ge 0$ in $\mathbb{R}\times(0,\infty)$ for all $M > 0$. Supposing (4.19) to be false, we could then find $x_0 \in \mathbb{R}, t_0 > 0$ and $c_1 > 0$ such that $u_M(x_0,t_0) \le c_1$ and hence

$$u_M(x,t_0) \le c_1 \qquad \text{for all } x \in (-\infty,x_0]$$

and each $M>0$. Integrating the inequality $(\ln u_M)_t \ge (\ln t^{-\frac{1}{p}})_t$ asserted by Lemma 4.3 shows that

$$u_M(x,t) \le u_M(x,t_0)\cdot\Big(\frac{t_0}{t}\Big)^{\frac{1}{p}} \qquad \text{for all } x\in(-\infty,x_0] \text{ and } t\in(0,t_0]$$

and thus

$$u_M(x,t) \le c_2 := 2^{\frac{1}{p}} c_1 \qquad \text{for all } x\in(-\infty,x_0] \text{ and } t\in\Big[\frac{t_0}{2},t_0\Big]. \tag{4.20}$$

On the other hand, our assumptions on u_0 guarantee that

$$u_0(x) \ge c_3\cdot(x-x_0+3)^\alpha \qquad \text{for all } x>x_0-2$$

holds with some $c_3>0$, whence an application of Lemma 4.1 yields

$$u(x,t)\ge c_4\cdot(x-x_0+3)^\alpha =: w(x,t) \qquad \text{for all } x>x_0-3 \text{ and } t>0 \tag{4.21}$$

with a certain $c_4>0$. In view of the monotone convergence $u_M \nearrow u$, Dini's theorem applied to $(u_M-u)_+$ enables us to find $\overline{M}>0$ and $c_5>0$ such that whenever $M\ge\overline{M}$, we have

$$u_M(x,t)\ge c_5 \qquad \text{for all } x\in[x_0-2,x_0] \text{ and } t\in[0,t_0]. \tag{4.22}$$

According to (4.20) and (4.22), parabolic regularity theory ([25]) ensures that for some $c_6>0$ and any such M we have

$$u_{Mx}(x_0-1,t)\le c_6 \qquad \text{for all } t\in\Big[\frac{t_0}{2},t_0\Big]. \tag{4.23}$$

In order to show that actually (4.20) is absurd, we note that since $p\in(0,2)$ it is possible to pick a positive number β satisfying

$$p-1<\beta<\frac{p}{2} \tag{4.24}$$

and then let

$$\kappa := 1-\frac{2\beta}{p}>0.$$

With $A:=2-x_0$, multiplying (3.1) by $(A+x)^{-\kappa}\cdot u_M^{-\beta-1}$ and integrating by parts with respect to $x\in(x_0-1,a)$ for $a>x_0+1$ then yields

$$\begin{aligned}
\frac{1}{\beta}\frac{d}{dt}\int_{x_0-1}^a (A+x)^{-\kappa}\cdot u_M^{-\beta}(x,t)dx &= -\int_{x_0-1}^a (A+x)^{-\kappa}\cdot\cdot u_M^{p-\beta-1}u_{Mxx}dx \\
&= -(\beta-p+1)\int_{x_0-1}^a (A+x)^{-\kappa}\cdot u_M^{p-\beta-2}u_{Mx}^2dx \\
&\quad -\kappa\int_{x_0-1}^a (A+x)^{-\kappa-1}\cdot u_M^{p-\beta-1}u_{Mx}dx \\
&\quad -(A+a)^{-\kappa}\cdot u_M^{p-\beta-1}(a,t)\cdot u_{Mx}(a,t) \\
&\quad +u_M^{p-\beta-1}(x_0-1,t)\cdot u_{Mx}(x_0-1,t) \\
&=: I_1+I_2+I_3+I_4
\end{aligned} \tag{4.25}$$

for $t \in (\frac{t_0}{2}, t_0)$. Here, $I_1 \le 0$ by (4.24) and $I_3 \le 0$ due to the fact that $u_{Mx} \ge 0$. Moreover, (4.22) and (4.23) ensure that

$$I_4 \le c_7$$

with a constant c_7 depending on x_0 but not on M. As to I_2, we once more integrate by parts to see that

$$\begin{aligned} I_2 &= -\frac{\kappa}{p-\beta} \int_{x_0-1}^a (A+x)^{-\kappa-1} \cdot (u_M^{p-\beta})_x dx \\ &= -\frac{\kappa(\kappa+1)}{p-\beta} \int_{x_0-1}^a (A+x)^{-\kappa-2} \cdot u_M^{p-\beta} dx \\ &\quad -\frac{\kappa}{p-\beta} \cdot (A+a)^{-\kappa-1} \cdot u_M^{p-\beta}(a,t) \\ &\quad +\frac{\kappa}{p-\beta} \cdot u_M^{p-\beta}(x_0-1,t) \\ &=: I_{21} + I_{22} + I_{23}. \end{aligned}$$

Since (4.24) entails that $p-\beta > 0$, we clearly have $I_{22} \le 0$ and, by (4.20),

$$I_{23} \le c_8$$

whenever $t \in (\frac{t_0}{2}, t_0)$, with $c_8 > 0$ independent of M. Altogether, (4.25) leads to the inequality

$$\frac{1}{\beta}\frac{d}{dt} \int_{x_0-1}^a (A+x)^{-\kappa} \cdot u_M^{-\beta}(x,t)dx \le -\frac{\kappa(\kappa+1)}{p-\beta} \int_{x_0-1}^a (A+x)^{-\kappa-2} \cdot u_M^{p-\beta}(x,t)dx + c_7 + c_8 \tag{4.26}$$

for all $t \in (\frac{t_0}{2}, t_0)$. Now by the Hölder inequality, choosing $q := \frac{p}{p-\beta} > 1$ we find

$$\begin{aligned} \int_{x_0-1}^a \frac{1}{A+x} dx &= \int_{x_0-1}^a \Big((A+x)^{-\kappa} \cdot u_M^{-\beta}\Big)^{\frac{1}{q}} \cdot (A+x)^{\frac{\kappa}{q}-1} \cdot u_M^{\frac{\beta}{q}} dx \\ &\le \Big(\int_{x_0-1}^a (A+x)^{-\kappa} \cdot u_M^{-\beta} dx\Big)^{\frac{1}{q}} \cdot \Big(\int_{x_0-1}^a (A+x)^{(\frac{\kappa}{q}-1)\cdot\frac{q}{q-1}} \cdot u_M^{\frac{\beta}{q-1}} dx\Big)^{\frac{q-1}{q}} \\ &\le \Big(\int_{x_0-1}^a (A+x)^{-\kappa} \cdot u_M^{-\beta} dx\Big)^{\frac{1}{q}} \cdot \Big(\int_{x_0-1}^a (A+x)^{-\kappa-2} \cdot u_M^{p-\beta} dx\Big)^{\frac{q-1}{q}}, \end{aligned}$$

which gives

$$\int_{x_0-1}^a (A+x)^{-\kappa-2} \cdot u_M^{p-\beta} dx \ge \Big(\ln(A+a)\Big)^{\frac{p}{\beta}} \cdot \Big(\int_{x_0-1}^a (A+x)^{-\kappa} \cdot u_M^{-\beta} dx\Big)^{-\frac{p-\beta}{\beta}}.$$

Accordingly, (4.26) shows that $y(t) := \int_{x_0-1}^a (A+x)^{-\kappa} \cdot u_M^{-\beta}(x,t)dx$, $t \in [\frac{t_0}{2}, t_0]$, satisfies

$$y'(t) \le -c_9 \cdot K(a) \cdot y^{-\delta}(t) + c_{10} \qquad \text{for all } t \in \Big(\frac{t_0}{2}, t_0\Big) \tag{4.27}$$

with $K(a) := (\ln(A+a))^{\frac{p}{\beta}}$, $\delta := \frac{p-\beta}{\beta}$ and positive constants c_9 and c_{10} possibly depending on t_0 and x_0 but neither on M nor on a.
In order to control the value of y at $t = \frac{t_0}{2}$, we recall (4.21) in estimating

$$\begin{aligned} \int_{x_0-1}^{a} (A+x)^{-\kappa} \cdot u^{-\beta}\Big(x, \frac{t_0}{2}\Big) dx &\le c_4^{-\beta} \int_{x_0-1}^{a} (A+x)^{-\kappa} \cdot (x - x_0 + 3)^{-\alpha\beta} dx \\ &\le c_4^{-\beta} \int_{x_0-1}^{\infty} (A+x)^{-\kappa-\alpha\beta} dx \\ &=: c_{11} \end{aligned}$$

and observe that c_{11} is finite, because $\alpha > \frac{2}{p}$ implies

$$\kappa + \alpha\beta = 1 - \frac{2\beta}{p} + \alpha\beta = 1 + \beta \cdot \Big(\alpha - \frac{2}{p}\Big) > 1.$$

Consequently, for all $a > x_0 - 1$ we can pick some $M_a \ge \overline{M}$ such that whenever $M \ge M_a$ we have

$$y\Big(\frac{t_0}{2}\Big) \le 2c_{11}. \tag{4.28}$$

Now since $K(a) \to +\infty$ as $a \to \infty$, we can choose some a large enough fulfilling

$$K(a) \ge \frac{2c_{10} \cdot (2c_{11})^{\delta}}{c_9} \qquad \text{and} \qquad K(a) > \frac{4 \cdot (2c_{11})^{\delta+1}}{(\delta+1) c_9 t_0}$$

and thus easily obtain that for any $M \ge M_a$,

$$-c_9 K(a) \cdot y^{-\delta}\Big(\frac{t_0}{2}\Big) + c_{10} \le -\frac{1}{2} c_9 K(a) \cdot y^{-\delta}\Big(\frac{t_0}{2}\Big) \ < 0. \tag{4.29}$$

Thus, a standard ODE argument ensures that for such a and M, y decreases on $[\frac{t_0}{2}, t_0]$ and moreover satisfies

$$y'(t) \le -\frac{1}{2} c_9 K(a) \cdot y^{-\delta}(t) \qquad \text{for all } t \in \Big(\frac{t_0}{2}, t_0\Big),$$

so that an explicit integration yields

$$y^{\delta+1}(t) \le y^{\delta+1}\Big(\frac{t_0}{2}\Big) - \frac{(\delta+1) c_9 K(a)}{2} \cdot \Big(t - \frac{t_0}{2}\Big) \qquad \text{for all } t \in \Big[\frac{t_0}{2}, t_0\Big].$$

Using the second inequality in (4.29), from (4.28) we arrive at the absurd conclusion

$$y(t_0) < 0$$

and thereby infer that our assumption that $u(x_0, t_0)$ be bounded must have been impossible. This establishes (4.19) and thereby completes the proof in view of Corollary 3.4. □

The above statement precisely establishes the statement in Theorem 1.1:

PROOF of Theorem 1.1. In view of (2.1), this part is an immediate consequence of Lemma 4.4 applied to $p = \frac{1-m}{-m} \in (1,2)$. □

Remark 1. We note that Lemma 4.4 actually addresses the full range $p \in (0,2)$, and so one may consider its respective conclusions in the cases $p \in (0,1)$ and $p = 1$ not referred to in Theorem 1.1.
i) Since $p < 1$ if and only if $m = \frac{1}{1-p} > 1$, Lemma 4.4 says that for any $m > 1$ the corresponding proper solution v of the porous medium equation (1.1) satisfies $v \equiv +\infty$ in $\mathbb{R} \times (0,\infty)$ whenever the initial data v_0 satisfy $v_0(x) \ge c|x|^{\frac{2}{m-1}+\delta}$ for all $x > 0$ and some $c > 0$ and $\delta > 0$. However, this is essentially no novelty, because it is known ([10]) that any continuous weak solution v of (1.1) in $\mathbb{R} \times (0,T)$, $T > 0$, must satisfy $v(x,t) \le C(t)(1+|x|)^{\frac{2}{m-1}}$ for all $x \in \mathbb{R}$ and $t \in (0,T)$ with some $C(t) > 0$.
ii) Accordingly, in the borderline case $p = 1$ Lemma 4.4 says that in the equation $v_t = (e^v v_x)_x$, the proper solution evolving from v_0 will immediately blow up everywhere if $v_0(x) \ge (2+\delta)\ln x - C$ for $x > 0$ with positive δ and C.

4.2 Local and global finiteness for slowly growing data

In contrast to the above situation, when u_0 grows at most as fast as a multiple of $x^{\frac{2}{p}}$ as $x \to +\infty$, we will be able to compare our solution from above by certain – accordingly transformed – relatives of (1.2) which are regular in the sense that they are everywhere positive. By quite an elementary ODE analysis, it can be seen that these exhibit a similar growth:

Lemma 4.5 *Let $p \in (0,2)$, and let h denote the solution of the initial value problem*

$$\begin{cases} h'' = h^{1-p}, & y > 0, \\ h(0) = 1, \quad h'(0) = 0. \end{cases} \tag{4.30}$$

Then h exists globally with $h' > 0$ on $(0,\infty)$, and there exist positive constants k_0 and k_1 such that

$$k_0(1+y)^{\frac{2}{p}} \le h(y) \le k_1(1+y)^{\frac{2}{p}} \qquad \text{for all } y \ge 0. \tag{4.31}$$

PROOF. Since $p > 0$, it can easily be checked that h exists in the whole interval $(0,\infty)$ and satisfies $h'' > 0$ and $h' > 0$ throughout $(0,\infty)$. Thus, a multiplication of $h'' = h^{1-p}$ by h' upon integration yields

$$h' = \sqrt{\frac{2}{2-p}} \cdot \sqrt{h^{2-p} - 1}, \qquad y > 0. \tag{4.32}$$

On the one hand, this shows that $h' \le \sqrt{\frac{2}{2-p}} h^{\frac{2-p}{2}}$ on $(0,\infty)$, whence the right inequality in (4.31) results from another integration.
To prove the left one, we let $y_0 := \sup\{y > 0 \mid h(y) \le 2\}$ and observe that y_0 must be finite since evidently h cannot be bounded. For $y \ge y_0$, however, (4.32) entails that

$$h' \ge \sqrt{\frac{2}{2-p}} \cdot \sqrt{h^{2-p} - \Big(\frac{h}{2}\Big)^{2-p}} = c_1 h^{\frac{2-p}{2}}, \qquad y > y_0,$$

with $c_1 := \sqrt{\frac{2}{2-p}} \cdot \sqrt{1-2^{p-2}} > 0$. Accordingly,

$$h(y) \ge \Big(2^{\frac{p}{2}} + \frac{pc_1}{2}(y-y_0)\Big)^{\frac{2}{p}} \qquad \text{for all } y \ge y_0,$$

which in view of the monotonicity of h completes the proof of (4.31). $\square$

Now a straightforward comparison allows us to derive boundedness of $u(x,t)$ for each fixed $x \in \mathbb{R}$ and small $t > 0$, provided that u_0 does not grow faster than a multiple of $|x|^{\frac{2}{p}}$ as $x \to \pm\infty$.

Lemma 4.6 *Let $p \in (0,2)$, and suppose that $u_0 \in C^3(\mathbb{R})$ is positive and such that*

$$u_0(x) \le c(1+|x|)^{\frac{2}{p}} \qquad \textit{for all } x \in \mathbb{R} \tag{4.33}$$

with some $c > 0$. Then there exists $T > 0$ such that the proper solution of (2.2) satisfies

$$u(x,t) < \infty \qquad \textit{for all } x \in \mathbb{R} \textit{ and } t \in (0,T), \tag{4.34}$$

and u is a classical solution of (2.2) in $\mathbb{R} \times (0,T)$.

Proof. With c as in (4.33) and k_0 taken from Lemma 4.5, we let $y = y(t)$ denote the solution of

$$\begin{cases} y' = y^{p+1}, & t > 0, \\ y(0) = \frac{c}{k_0}, \end{cases}$$

that is, we set

$$y(t) := \Big\{ \Big(\frac{k_0}{c}\Big)^p - pt \Big\}^{-\frac{1}{p}}, \qquad t \in [0,T),$$

where $T := \frac{k_0^p}{pc^p}$. Then

$$\overline{u}(x,t) := y(t) \cdot h(|x|), \qquad x \in \mathbb{R}, \ t \in [0,T),$$

with h as in Lemma 4.5, satisfies

$$\overline{u}(x,0) = \frac{c}{k_0} \cdot h(|x|) \ge \frac{c}{k_0} \cdot k_0 (1+|x|)^{\frac{2}{p}} \ge u_0(x) \qquad \text{for all } x \in \mathbb{R}$$

by (4.31), which means that $\overline{u}(\cdot,0) \ge u_{0MR}^{(0)}$ on $B_R(0)$ for all $M > 0$ and $R > 0$, because $u_{0MR}^{(0)} \le u_0$ in $B_R(0)$ by (3.5) and (3.2). Since moreover

$$\overline{u}_t - \overline{u}^p \overline{u}_{xx} = y' \cdot h(|x|) - y^{p+1} \cdot h^p(|x|) \cdot h''(|x|) = (y' - y^{p+1}) \cdot h(|x|) = 0 \qquad \text{in } \mathbb{R} \times (0,T),$$

we may invoke the comparison principle to obtain that

$$u_{MR}^{(0)} \le \overline{u} \qquad \text{in } B_R(0) \times (0,T) \tag{4.35}$$

for all $M > 0$ and $R > 0$. Upon taking $R \to \infty$ and then $M \to \infty$, this easily yields $u \le \overline{u}$ in $\mathbb{R} \times (0,\infty)$ and thereby proves (4.34). According to parabolic regularity theory, (4.35) in conjunction with the monotonicity properties of $(u_{MR}^{(0)})_{R>0}$ and $(u_M^{(0)})_{M>0}$ also implies that u indeed classically solves (2.2) in $\mathbb{R} \times (0,T)$. $\square$

Under a slightly stronger growth restriction we obtain finiteness of u globally rather than locally in time.

Lemma 4.7 *Suppose that $p \in (0,2)$, and that $u_0 \in C^3(\mathbb{R})$ is a positive function fulfilling*

$$|x|^{-\frac{2}{p}} \cdot u_0(x) \to 0 \qquad \text{as } |x| \to \infty. \tag{4.36}$$

Then the proper solution u of (2.2) satisfies

$$u(x,t) < \infty \qquad \text{for all } x \in \mathbb{R} \text{ and } t > 0, \tag{4.37}$$

and u is a classical solution of (2.2) in $\mathbb{R} \times (0,\infty)$.

PROOF. We fix $T > 0$ and pick $\delta < 0$ small such that

$$\delta < \frac{1}{pT}. \tag{4.38}$$

According to (4.36), it is then possible to fix $R_0 > 0$ large such that

$$u_0(x) \le k_0 \delta^{\frac{1}{p}} \cdot |x|^{\frac{2}{p}} \qquad \text{for all } x \in \mathbb{R} \setminus B_{R_0}(0) \tag{4.39}$$

holds with k_0 as in Lemma 4.5. We finally choose $A > 0$ large fulfilling

$$u_0(x) \le A \qquad \text{for all } x \in B_{R_0}(0) \tag{4.40}$$

and define

$$\overline{u}(x,t) := y(t) \cdot w(x), \qquad x \in \mathbb{R},\ t \in [0,T],$$

where

$$y(t) := (1 - p\delta t)^{-\frac{1}{p}}, \qquad t \in [0,T],$$

and

$$w(x) := A \cdot h\Big(\delta^{\frac{1}{2}} A^{-\frac{p}{2}} \cdot |x|\Big), \qquad x \in \mathbb{R},$$

with h as provided by Lemma 4.5. Here we observe that indeed y is well-defined and bounded on $[0,T]$ because of (4.38), and it can easily be checked that $y' = \delta y^{p+1}$ on $(0,T)$. Since moreover

$$\begin{aligned} w_{xx}(x) &= \delta A^{1-p} \cdot h''\Big(\delta^{\frac{1}{2}} A^{-\frac{p}{2}} \cdot |x|\Big) \\ &= \delta A^{1-p} \cdot h^{1-p}\Big(\delta^{\frac{1}{2}} A^{-\frac{p}{2}} \cdot |x|\Big) \\ &= \delta A^{1-p} \cdot \Big(\frac{w(x)}{A}\Big)^{1-p} \\ &= \delta w^{1-p}(x) \qquad \text{for all } x \in \mathbb{R} \end{aligned}$$

by (4.30), we obtain that $\overline{u}_t = \overline{u}^p \overline{u}_{xx}$ in $\mathbb{R} \times (0,T)$. Furthermore, at $t = 0$, for large $|x|$ we have

$$\begin{aligned} \overline{u}(x,0) &= w(x) \\ &\ge A \cdot k_0 \Big(1 + \delta^{\frac{1}{2}} A^{-\frac{p}{2}} \cdot |x|\Big)^{\frac{2}{p}} \\ &\ge k_0 \delta^{\frac{1}{p}} \cdot |x|^{\frac{2}{p}} \\ &\ge u_0(x) \qquad \text{for all } x \in \mathbb{R} \setminus B_{R_0}(0) \end{aligned}$$

thanks to (4.31) and (4.39), whereas near the origin we use (4.40) in estimating

$$\overline{u}(x,0) \geq A \geq u_0(x) \qquad \text{for all } B_{R_0}(0),$$

because the monotonicity of h ensures that $h \geq 1$ on $(0,\infty)$. Having thereby asserted that $\overline{u}$ dominates u_0 at $t=0$, by comparison we infer that $\overline{u} \geq u_{MR}^{(0)}$ in $B_R \times (0,T)$ for all $R>0$ and $M>0$. This yields $\overline{u} \geq u$ in $\mathbb{R} \times (0,T)$ and thus proves (4.37), because $T>0$ was arbitrary. The pointwise solution properties of u follow as in Lemma 4.6. □

Remark 2. In the next section (cf. (5.1) and Theorem 5.2) we shall see several examples which indicate that in presence of critical growth of u_0, blow-up in a finite but positive time may occur. This will justify the separate consideration of Lemma 4.6 and Lemma 4.7.
An open problem is to classify the set of initial data leading to finite-time blow-up in (2.2) in the above sense. We conjecture, but cannot prove here, that in fact $u(\cdot,T) \equiv +\infty$ holds for some $T>0$ whenever $x^{-\frac{2}{p}}u_0(x) \to L > 0$ as $x \to +\infty$ for some $L>0$ and $u_0(x)$ does not decay too fast as $x \to -\infty$.

We proceed to prove Proposition 1.2.

Proof of Proposition 1.2. The assertion on local-in-time positivity is provided by Lemma 4.6, whereas translating Lemma 4.7 to the original variables yields the statement on global positivity. □

Remark 3. i) Again, in the porous medium case $p \in (0,1)$, the results given in Lemma 4.6 and Lemma 4.7 are well-known ([5]).
ii) When $p=1$, Lemma 4.6 guarantees local existence of a classical solution for $v_t = (e^v v_x)_x$, provided that the data satisfy $v_0(x) \leq 2\ln(1+|x|) + C$ for some $C>0$ and all $x \in \mathbb{R}$, while Lemma 4.7 asserts global solvability under the additional requirement that $v_0(x) - 2\ln|x| \to -\infty$ as $|x| \to \infty$.

5 Accelerating waves blowing up at spatial inifinity. Proof of Theorem 1.3

Our goal in this section is to investigate how mass transport occurs when the initial data u_0 in (2.2) exhibit *critical* growth as $x \to +\infty$. One rather explicit example can be constructed by separation of variables, which leads to the family $(u_{aT})_{a>0,T>0}$ of solutions of (2.2) given by

$$u_{aT}(x,t) = (pa^2)^{-\frac{1}{p}}(T-t)^{-\frac{1}{p}}z(ax), \qquad x \in \mathbb{R}, t \in [0,T), \tag{5.1}$$

where

$$z_{yy} = z^{1-p}, \quad y \in \mathbb{R} \qquad \text{with} \qquad z_y(0) = 0 \text{ and } z(0) = 1,$$

that is, $z(y)$ is implicitly defined by

$$\int_1^{z(y)} \frac{ds}{\sqrt{s^{2-p}-1}} = \sqrt{\frac{2}{2-p}}|y|, \qquad y \in \mathbb{R}.$$

In order to establish a link to previous material, let us observe that setting $v = u^{\frac{1}{m}}$ yields a corresponding family of separated smooth solutions of (1.1), and in the limit $a \to \infty$ we rediscover the fully explicit example in (1.2).

Clearly, u_{aT} blows up everywhere in $\mathbb{R}$ at time $t = T$, with its blow-up set $\mathcal{B}(u_{aT}) := \{x \in \mathbb{R} \mid \exists (x_k, t_k)_{k\in\mathbb{N}} \subset \mathbb{R} \times (0,T)$ such that $x_k \to x, t_k \to T$ and $u_{aT}(x_k,t_k) \to +\infty$ as $k \to \infty\}$ coinciding with all of $\mathbb{R}$.

We shall see that the latter need no longer be true when critical growth of the initial data is merely prescribed as $x \to +\infty$, possibly complemented by boundedness, or even some decay, as $x \to -\infty$. We shall obtain some examples of solutions u emanating from such initial data, and one particular conclusion will be that the solution still may blow up, but with empty blow-up set $\mathcal{B}(u) = \emptyset$. In fact, in Theorem 5.2 below for any $\gamma > 0$ we shall find some u blowing up at time $t = T$ but satisfying an estimate of the form $u(x,t) \le ce^{\gamma x}$ for all $x \in \mathbb{R}$ and $t \in (0,T)$ and some $c > 0$.

Our approach will be based on wave-like solutions of the form

$$u(x,t) = (T-t)^{-\frac{1}{p}} \cdot f\Big(x + k \cdot \ln(T-t)\Big), \qquad x \in \mathbb{R},\ t \in [0,T), \tag{5.2}$$

having their speed increasing with time and blowing up at time T, where $k > 0$ is a parameter. In fact, for $p > 2$ such solutions have been found to exist and to be extensible to all of $\mathbb{R} \times \mathbb{R}$ so as to become a homclinic orbit of (2.2) connecting the trivial equilibrium to itself ([35]). Inserting this ansatz into (2.2) suggests to solve

$$f^p(\xi) f''(\xi) = \frac{1}{p} f(\xi) - k f'(\xi), \qquad \xi \in \mathbb{R}, \tag{5.3}$$

and we shall see that this problem has positive classical solutions actually for any value of $k > 0$ and $p > 0$.

Lemma 5.1 *Let $p > 0$. Then for all $k > 0$, (5.3) possesses a positive solution $f = f_k \in C^\infty(\mathbb{R})$. This solution increases on $\mathbb{R}$ and satisfies*

$$c_0 e^{\frac{1}{pk}\xi} \le f(\xi) \le c_1 e^{\frac{1}{pk}\xi} \qquad \text{for all } \xi \le 0, \tag{5.4}$$

and

$$\begin{cases} d_0 \xi^{\frac{2}{p}} \le f(\xi) \le d_1 \xi^{\frac{2}{p}} \quad \text{for all } \xi \ge 2 & \text{if } p < 2, \\ d_0 \xi \sqrt{\ln \xi} \le f(\xi) \le \xi \sqrt{\ln \xi} \quad \text{for all } \xi \ge 2 & \text{if } p = 2, \\ d_0 \xi \le f(\xi) \le d_1 \xi \quad \text{for all } \xi \ge 2 & \text{if } p > 2 \end{cases} \tag{5.5}$$

with certain positive constants c_0, c_1, d_0 and d_1.

Proof. We normalize (5.3) by setting

$$f(\xi) = (pk^2)^{\frac{1}{p}} \cdot g\Big(\frac{1}{pk} \cdot \xi\Big), \qquad \xi \in \mathbb{R}, \tag{5.6}$$

and are thereby lead to studying

$$g^p(\sigma) g''(\sigma) = g(\sigma) - g'(\sigma), \qquad \sigma \in \mathbb{R}. \tag{5.7}$$

In order to solve the latter first on $(-\infty, 0]$, we will rely on a comparison argument anticipating a behavior of the form described in (5.4), suggesting that $\frac{g''(\sigma)}{g(\sigma)} \simeq 1$ as $\sigma \to -\infty$. Indeed, we recall from [35, Lemma 2.1] that for $a_+ := 1, a_- := \frac{1}{2}, s_+ := \frac{1}{2}$ and all sufficiently small $s_- \in (0, s_+)$, the solutions of the Bernoulli-type problem

$$\begin{cases} g'_\pm(\sigma) = g_\pm(\sigma) - a_\pm g_\pm^{p+1}(\sigma), & \sigma < 0, \\ g_\pm(0) = s_\pm, \end{cases} \tag{5.8}$$

can be seen to satisfy $g_- < g_+$ on $(-\infty, 0]$ and

$$-g_-^p g_-'' - g_-' + g_- < 0 < -g_+^p g_+'' - g_+' + g_+ \qquad \text{on } (-\infty, 0].$$

Using standard elliptic existence results ([30]), we thus infer that on the left half-line, (5.7) admits a solution $g_1 \in C^\infty((-\infty, 0])$ which fulfills $g_- \le g_1 \le g_+$ on $(-\infty, 0]$. Since (5.8) can explicitly be solved, we can therefore easily verify that with certain constants $c_2 > 0$ and $c_3 > 0$,

$$c_2 e^\sigma \le g_1(\sigma) \le c_3 e^\sigma \qquad \text{for all } \sigma \in (-\infty, 0] \tag{5.9}$$

holds for this solution, and that accordingly $g_1' > 0$ on $(-\infty, 0]$, because at each point where g_1 is positive and $g_1' = 0$ we must have $g_1'' > 0$ by (5.7).
In order to extend g_1 to a solution of (5.7) on all of $\mathbb{R}$, we let $A := g_1(0) > 0$ and $B := g_1'(0) > 0$ and consider the solution g of the initial-value problem

$$\begin{cases} g^p(\sigma) g''(\sigma) = g(\sigma) - g'(\sigma), & \sigma \in J, \\ g(0) = A, \quad g'(0) = B, \end{cases} \tag{5.10}$$

defined on its maximal existence interval $J \subset \mathbb{R}$ such that $0 \in J$. Then clearly, by a uniqueness argument, $(-\infty, 0] \subset J$ and $g \equiv g_1$ on $(-\infty, 0]$. Moreover, the fact that $B > 0$ ensures that $g' > 0$ on J and hence $g \ge A$ on $J \cap (0, \infty)$. Thus, $-A^{-p} g' \le g'' \le A^{-p} g$ on $J \cap (0, \infty)$, which implies that in fact $J = \mathbb{R}$.
Now in the case $p < 2$, multiplying the inequality $g'' = g^{1-p} - g^{-p} g' \le g^{1-p}$ by $g' > 0$ shows that

$$\frac{1}{2} g'^2(\sigma) - \frac{1}{2} B^2 \le \frac{1}{2-p} (g^{2-p}(\sigma) - A^{2-p}) \qquad \text{for all } \sigma > 0,$$

and hence we have

$$g'(\sigma) \le \sqrt{\frac{2}{2-p} g^{2-p}(\sigma) + B^2} \le c_4 \sqrt{g^{2-p}(\sigma)} \qquad \text{for all } \sigma > 0 \tag{5.11}$$

with $c_4 := \sqrt{\frac{2}{2-p} + A^{p-2} B^2}$. Upon another integration, this gives

$$g(\sigma) \le \Big(\frac{p c_4}{2} \sigma + A^{\frac{p}{2}} \Big)^{\frac{2}{p}} \qquad \text{for all } \sigma > 0. \tag{5.12}$$

In particular, the latter entails that for some large $\sigma_0 > 0$ we must have $g(\sigma_0) \ge g'(\sigma_0)$, because the contrary assumption that $g' > g$ be valid throughout $(0, \infty)$ would evidently contradict (5.12). Since $\psi := g - g'$ satisfies

$$\psi' = g' - \frac{g - g'}{g^p} = g - \psi - \frac{\psi}{g^p} > -\Big(1 + \frac{1}{g^p}\Big) \cdot \psi \qquad \text{on } (\sigma_0, \infty),$$

by comparison we conclude that $\psi \ge 0$ and hence $g'' \ge 0$ on $[\sigma_0, \infty)$. Using that $g'(\sigma_0) > 0$, we thus obtain that $g(\sigma) \to +\infty$ as $\sigma \to +\infty$, so that for some $\sigma_1 > 0$ we have

$$c_4 g^{-\frac{p}{2}}(\sigma) \le \frac{1}{2} \qquad \text{for all } \sigma \ge \sigma_1.$$

Consequently, by (5.11) and (5.10) we see that

$$g^p(\sigma) g''(\sigma) \ge g(\sigma) \cdot \Big(1 - c_4 g^{-\frac{p}{2}}(\sigma)\Big) \ge \frac{1}{2} g(\sigma) \qquad \text{for all } \sigma \ge \sigma_1,$$

which upon multiplication by $\frac{g'(\sigma)}{g^p(\sigma)}$ yields

$$\frac{1}{2} g'^2(\sigma) - \frac{1}{2} g'^2(\sigma_1) \ge \frac{1}{2-p} \Big(g^{2-p}(\sigma) - g^{2-p}(\sigma_1)\Big) \qquad \text{for all } \sigma \ge \sigma_1$$

and therefore

$$g'(\sigma) \ge \sqrt{\frac{2}{2-p}\Big(g^{2-p}(\sigma) - g^{2-p}(\sigma_1)\Big)} \ge c_5 \sqrt{g^{2-p}(\sigma)} \qquad \text{for all } \sigma \ge \sigma_1 + 1$$

is valid with $c_5 := \sqrt{\frac{2}{2-p}\Big(1 - \frac{g^{2-p}(\sigma_1)}{g^{2-p}(\sigma_1+1)}\Big)} > 0$. Integrating this, we obtain

$$g(\sigma) \ge \Big(\frac{pc_5}{2}(\sigma - \sigma_1 - 1) + g^{\frac{p}{2}}(\sigma_1 + 1)\Big)^{\frac{2}{p}} \qquad \text{for all } \sigma \ge \sigma_1 + 1. \tag{5.13}$$

We finally define f via (5.6) and collect (5.9), (5.12) and (5.13) to easily end up with (5.4) and (5.5) in the case $p < 2$.

If $p = 2$ we proceed similarly and instead of (5.11) now obtain

$$g'(\sigma) \le c_6 \sqrt{\ln \frac{g(\sigma)}{A}} \qquad \text{for all } \sigma > 0 \tag{5.14}$$

with some $c_6 > 0$. In order to convert this into an estimate from above for g, we fix any $\sigma_2 > 1$ and then pick $c_7 > c_6$ large enough fulfilling

$$c_7 \sigma_2 \sqrt{\ln \sigma_2} \ge g(\sigma_2) \tag{5.15}$$

and

$$(c_7^2 - c_6^2) \ln \sigma \ge c_6^2 \ln \frac{c_7}{A} + \frac{c_6^2}{2} \ln \ln \sigma \qquad \text{for all } \sigma > \sigma_2. \tag{5.16}$$

Then

$$\overline{g}(\sigma) := c_7 \sigma \sqrt{\ln \sigma}, \qquad \sigma \ge \sigma_2,$$

satisfies $\overline{g}(\sigma_2) \ge g(\sigma_2)$ by (5.15), and since $\overline{g}'(\sigma) \ge c_7 \sqrt{\ln \sigma} > 0$, we have

$$\begin{aligned} \overline{g}'^2(\sigma) - \Big(c_6 \sqrt{\ln \frac{\overline{g}(\sigma)}{A}}\Big)^2 &\ge c_7^2 \ln \sigma - c_6^2 \ln \frac{c_7}{A} - c_6^2 \ln \sigma - \frac{c_6^2}{2} \ln \ln \sigma \\ &\ge 0 \qquad \text{for all } \sigma \ge \sigma_2 \end{aligned}$$

in view of (5.16). Thus, by comparison we obtain $g \le \overline{g}$ in $[\sigma_2, \infty)$ and thereby verify the right inequality in (5.5). As before, this entails that $g''(\sigma) \ge 0$ for all suitably large σ, so that with some $c_8 \in (0,1)$ we have

$$g(\sigma) \ge c_8\sigma \qquad \text{for all } \sigma \ge 0. \tag{5.17}$$

In particular, this again implies that $g(\sigma) \to +\infty$ as $\sigma \to +\infty$, whence using (5.14) we infer that $g^2(\sigma)g''(\sigma) = g(\sigma) - g'(\sigma) \ge \frac{1}{2}g(\sigma)$ for all large σ. By integration, we therefore derive the inequality

$$g'(\sigma) \ge \sqrt{\ln \frac{g(\sigma)}{\tilde{A}}} \qquad \text{for all } \sigma > \sigma_3$$

with some $\sigma_3 > 0$ and $\tilde{A} := g(\sigma_3)$. Let us now fix a large number $\sigma_4 > \sigma_3$ fulfilling

$$\sigma_4 \ge e \qquad \text{and} \qquad (1 - c_8^2)\ln\sigma_4 \ge -\ln\frac{c_8}{\tilde{A}} + \frac{5}{4}c_8^2 \tag{5.18}$$

and set

$$\underline{g}(\sigma) := c_8\sigma\sqrt{\ln\frac{e\sigma}{\sigma_4}}, \qquad \sigma \ge \sigma_4.$$

Then $\underline{g}(\sigma_4) = c_8\sigma_4 \le g(\sigma_4)$ thanks to (5.17), and (5.18) implies that $\ln\frac{e}{\sigma_4} \le 0$. As clearly $\ln\ln\frac{e\sigma}{\sigma_4} \ge 0$ for all $\sigma > \sigma_4$, we accordingly obtain that

$$\begin{aligned}
\underline{g}'^2(\sigma) - \left(\sqrt{\ln\frac{\underline{g}(\sigma)}{\tilde{A}}}\right)^2 &= c_8^2\Big(\ln\sigma + \ln\frac{e}{\sigma_4} + 1 + \frac{1}{4\ln\frac{e\sigma}{\sigma_4}}\Big) - \ln\frac{c_8}{\tilde{A}} - \ln\sigma - \frac{1}{2}\ln\ln\frac{e\sigma}{\tilde{A}} \\
&\le -(1-c_8^2)\ln\sigma + c_8^2 + \frac{c_8^2}{4} - \ln\frac{c_8}{\tilde{A}} \\
&\le 0 \qquad \text{for all } \sigma > \sigma_4
\end{aligned}$$

by the second requirement in (5.18), so that by comparison we conclude that $g \ge \underline{g}$ in $[\sigma_4, \infty)$, from which the left inequality in (5.5) easily follows in the case $p = 2$.

Finally, in the case $p > 2$ the statement of the lemma is precisely covered by [35, Lemma 2.1, Lemma 2.2]. □

As an immediate consequence for (2.2), we obtain the following.

Theorem 5.2 *Let $p > 0$. Then for each $k > 0$ and all $T > 0$ there exists a positive classical solution $u \in C^\infty(\mathbb{R} \times [0,T))$ of (2.2) of the form (5.2), where $f = f_k \in C^\infty(\mathbb{R})$ is the increasing positive solution of (5.3) provided by Lemma 5.1.*
In particular, for the initial data $u_0 := u(\cdot, 0)$ we have $u_{0x} > 0$ on $\mathbb{R}$,

$$c_0 e^{\frac{1}{pk}x} \le u_0(x) \le c_1 e^{\frac{1}{pk}x} \qquad \textit{for all } x \le 0 \tag{5.19}$$

and

$$\begin{cases} d_0 x^{\frac{2}{p}} \le u_0(x) \le d_1 x^{\frac{2}{p}} \quad \textit{for all } x \ge 2 & \textit{if } p < 2, \\ d_0 x\sqrt{\ln x} \le u_0(x) \le d_1 x\sqrt{\ln x} \quad \textit{for all } x \ge 2 & \textit{if } p = 2, \\ d_0 x \le u_0(x) \le d_1 x \quad \textit{for all } x \ge 2 & \textit{if } p > 2, \end{cases} \tag{5.20}$$

and the solution satisfies

$$u(x,t) \le Ce^{\frac{1}{pk}x} \qquad \text{for all } x \in \mathbb{R} \text{ and } t \in (0,T) \tag{5.21}$$

as well as

$$\liminf_{t \nearrow T} u(x,t) \ge c\, e^{\frac{1}{pk}x} \qquad \text{for all } x \in \mathbb{R}. \tag{5.22}$$

with some positive constants c_0, c_1, d_0, d_1, c and C.

Moreover, if $p < 2$ then u blows up at $t = T$ in the sense that for any $\tilde{T} > T$, there is no positive classical solution of (2.2) in $\mathbb{R} \times (0, \tilde{T})$ which coincides with u in $\mathbb{R} \times (0,T)$.
On the other hand, if $p > 2$ then u can be continued for all times so as to exist as a classical solution of (2.2) in $\mathbb{R} \times [0,\infty)$.

Proof. The solution properties of u and the estimates (5.19) and (5.20) are immediate consequences of the properties of f_k. Furthermore, the right inequalities in (5.4) and (5.5) imply that actually $f_k(\xi) \le Ce^{\frac{1}{pk}\xi}$ holds for all $\xi \in \mathbb{R}$ with some $C > 0$, so that (5.2) shows that

$$u(x,t) \ \le \ (T-t)^{-\frac{1}{p}} \cdot Ce^{\frac{1}{pk}(x+k\cdot\ln(T-t))} = Ce^{\frac{1}{pk}x} \qquad \text{for all } x \in \mathbb{R} \text{ and } t \in (0,T).$$

Similarly, since for each fixed $x \in \mathbb{R}$ we have $x + k \cdot \ln(T-t) \le 0$ for all $t \in (t_0(x), T)$ with some $t_0(x)$ sufficiently close to T, from the left inequality in (5.4) we obtain that

$$u(x,t) \ge (T-t)^{-\frac{1}{p}} \cdot c_0 e^{\frac{1}{pk}(x+k\cdot\ln(T-t))} = c_0 e^{\frac{1}{pk}x}$$

holds for all $t \in (t_0(x), T)$ and some $c > 0$, which yields (5.22).
Let us now assume that $p < 2$, and that there exist $\tilde{T} > T$ and a positive classical solution $\tilde{u}$ of (2.2) on $\mathbb{R} \times (0,\tilde{T})$ fulfilling $\tilde{u} \equiv u$ in $\mathbb{R} \times (0,T)$. Since $\tilde{u}$ is continuous, by (5.22) we then would have $\tilde{u}(x,T) \ge ce^{\frac{1}{pk}x}$ for all $x \in \mathbb{R}$, and hence an application of Corollary 3.4 along with Lemma 4.4 would assert that $\tilde{u}(x,t) = +\infty$ for all $x \in \mathbb{R}$ and $t \in (T, \tilde{T})$, which is absurd. Therefore, u indeed ceases to exist at time T.
Finally, if $p > 2$ then due to (5.21), (5.22) and parabolic regularity theory, for each $x \in \mathbb{R}$ the limit $u(x,T) := \lim_{t \nearrow T} u(x,t)$ exists and defines a smooth positive function on $\mathbb{R}$ which is dominated by some multiple of $e^{\frac{1}{pk}x}$ throughout $\mathbb{R}$. According to known results ([29], [35]), (2.2) therefore possesses a classical solution $\tilde{u}$ in $\mathbb{R} \times [T,\infty)$ with $\tilde{u}(\cdot,T) = u(\cdot,T)$, which glued together with u evidently yields a global extension of u, as desired. ☐

Proof of Theorem 1.3. We only need to consider $p \in (1,2)$ in the above theorem and transform (5.2) and (5.3) to the original variables. ☐

Remark 4. Theorem 5.2 again goes slightly beyond our original purpose in that it applies to any $p > 0$. Via (2.1), as a by-product this provides examples of mass transport for the porous medium equation (1.1) when $p \in (0,1)$, for the equation $v_t = (e^v v_x)_x$ when $p = 1$ and for the super-fast diffusion equations (1.1) in the intermediate regime $m \in [-1,0)$ when $p \ge 2$.

Acknowledgment. The author would like to sincerely thank the anonymous reviewer for numerous detailed remarks and suggestions which substantially supported improvement of this manuscript.

References

[1] Aronson, D.G.: *Regularity properties of flows through porous media: the interface.* Arch. Rat. Mech. Anal. **37**, 1-10 (1970).

[2] Aronson, D.G.: *The porous medium equation.* Nonlinear diffusion problems, Lect. 2nd 1985 Sess. C.I.M.E.. Montecatini Terme/Italy 1985, Lect. Notes Math. **1224**, 1-46 (1986)

[3] Aronson, D.G., Caffarelli, L.A.: *The initial trace of a solution of the porous medium equation.* Trans. Amer. Math. Soc. **280**, 351-366 (1983)

[4] Barenblatt, G.I.: *On some unsteady motions of a liquid or a gas in a porous medium (in Russian).* Prikl. Mat. Mekh. **16**, 67-78 (1952)

[5] Bénilan, Ph., Crandall, M.G., Pierre, M.: *Solutions of the porous medium equation in $\mathbb{R}^N$ under optimal conditions on initial values.* Indiana Univ. Math. J. **33**, 51-87 (1984)

[6] Blanchet, A., Bonforte, M., Dolbeault, J., Grillo, G., Vázquez, J.L.: *Asymptotics of the fast diffusion equation via entropy estimates.* Arch. Rat. Mech. Anal. **191**, 347-385 (2009)

[7] Bonforte, M., Dolbeault, J., Grillo, G., Vázquez, J.L.: *Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities.* Proc. Nat. Acad. Sciences **107**, 16459-16464 (2010)

[8] Bonforte, M., Segatti, A., Vázquez, J.L.: *Non-existence and instantaneous extinction of solutions for singular nonlinear fractional diffusion equations.* Calc. Var. Part. Differential Eq. **55**: 68, 23 pp. (2016)

[9] Brézis, H., Friedman, A.: *Nonlinear parabolic equations involving measures as initial conditions.* J. Math. Pures Appl. **62**, 73-97 (1983)

[10] Dahlberg, B.E.J., Kenig, C.E.: *Nonnegative solutions of the porous medium equation.* Comm. Part. Differ. Eq. **9**, 409-437 (1984)

[11] Daskalopoulos, P., del Pino, M.: *On nonlinear parabolic equations of very fast diffusion.* Arch. Rational Mech. Anal. **137** (4), 363-380 (1997)

[12] Daskalopoulos, P., del Pino, M.: *On the Cauchy problem for $u_t = \Delta \log u$ in higher dimensions.* Math. Ann. **313** (2), 189-206 (1999)

[13] Daskalopoulos, P., Kenig, C.E.: Degenerate diffusions. Initial value problems and local regularity theory. EMS Tracts in Mathematics, European Mathematical Society, Zürich, 2007

[14] daskalopoulos, P., Sesum, N.: *On the extinction profile of solutions to fast diffusion.* J. Reine Angew. Math. **622**, 95-119 (2008)

[15] Diaz, G., Diaz, J.I.: *Finite extinction time for a class of non-linear parabolic equations.* Comm. Part. Diff. Eq. **4**, 1213-1231 (1979)

[16] Esteban, J.R., Rodriguez, A., Vázquez, J.L.: *A nonlinear heat equation with singular diffusivity.* Comm. Part. Diff. Eq. **13**, 985-1039 (1988)

[17] Fila, M., King, J.R., Winkler, M.: *Rate of convergence to Barenblatt profiles for the fast diffusion equation with a critical exponent.* J. London Math. Soc. **90**, 167-183 (2014)

[18] Fila, M., Vázquez, J.L., Winkler, M.: *A continuum of extinction rates for the fast diffusion equation.* Comm. Pure Appl. Anal. **10** 1129-1147 (2011)

[19] Fila, M., Vázquez, J.L., Winkler, M., Yanagida, E.: *Rate of convergence to Barenblatt profiles for the fast diffusion equation.* Arch. Rat. Mech. Anal. **204** 599-625 (2012)

[20] Fila, M., Winkler, M.: *Optimal rates of convergence to the singular Barenblatt profile for the fast diffusion equation.* Proc. Roy. Soc. Edinburgh Sect. A **146**, 309-324 (2016)

[21] Galaktionov, V.A., Peletier, L.A.: *Asymptotic behaviour near finite-time extinction for the fast diffusion equation.* Arch. Rat. Mech. Anal. **139**, 83-98 (1997)

[22] Galaktionov, V.A., Vázquez, J.L.: *Continuation of blowup solutions of nonlinear heat equations in several space dimensions.* Comm. Pure Appl. Math. **50** (1), 1-67 (1997)

[23] Grillo, G., Muratori, M., Punzo, F.: *The porous medium equation with large initial data on negatively curved Riemannian manifolds.* arXiv: https://arxiv.org/abs/1609.06498

[24] King, J. R.: *Self-similar behaviour for the equation of fast nonlinear diffusion.* Phil. Trans. Roy. Soc. London A **343**, 337-375 (1993)

[25] Ladyzenskaja, O.A., Solonnikov, V.A., Ural'ceva, N.N.: Linear and Quasi-linear Equations of Parabolic Type. AMS, Providence, 1968

[26] Luckhaus, S., Dal Passo, R.: *A Degenerate Diffusion Problem Not in Divergence Form.* J. Differential Eq. **69**, 1-14 (1987)

[27] Peletier, M.A., Zhang, H.: *Self-similar solutions of a fast diffusion equation that do not conserve mass.* Diff. Int. Eq. **8**, 2045-2064 (1995)

[28] Pierre, M.: Nonlinear fast diffusion with measures as data. In: Nonlinear parabolic equations: qualitative properties of solutions (Rome, 1985). Pitman Res. Notes Math. Ser. **149**, Longman Sci. Tech., Harlow, 179-188 (1987)

[29] Rodriguez, A., Vázquez, J.L.: *A well-posed problem in singular Fickian diffusion.* Arch. Rat. Mech. Anal. **110** (2), 141-163 (1990)

[30] Tomi, F.: *Über semilineare elliptische Differentialgleichungen zweiter Ordnung.* Math. Z. **111**, 350-366 (1969)

[31] Vázquez, J.L.: *Darcy'c Law and the theory of shrinking solutions of fast diffusion equations.* SIAM J. Math. Anal. **35** (4), 1005-10284 (2004)

[32] Vázquez, J.L.: Smoothing and Decay Estimates for Nonlinear Diffusion Equations. Equations of Porous Medium Type. Oxford Lecture Series in Mathematics and Its Applications **33**, Oxford University Press, 2006

[33] Wiegner, M.: *A Degenerate Diffusion Equation with a Nonlinear Source Term.* Nonlin. Anal. TMA **28**, 1977-1995 (1997)

[34] Winkler, M.: *Conservation of boundary decay and nonconvergent bounded gradients in degenerate diffusion problems.* Adv. Differential Eq. **13**, 27-54 (2008)

[35] Winkler, M.: *Spatially monotone homoclinic orbits in nonlinear parabolic equations of superfast diffusion type.* Math. Ann. **355** (2), 519-549 (2013)