# The small-convection limit in a two-dimensional chemotaxis-Navier-Stokes system


Yulan Wang*
School of Science, Xihua University, Chengdu 610039, China

Michael Winkler#
Institut für Mathematatik, Universität Paderborn,
33098 Paderborn, Germany

Zhaoyin Xiang‡
School of Mathematical Sciences, University of Electronic Science and Technology of China,
Chengdu 611731, China



**Abstract**

This paper deals with an initial-boundary value problem for the chemotaxis-(Navier-)Stokes system

$$\begin{cases} n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot (n \nabla c), & x \in \Omega,\ t > 0, \\ c_t + u \cdot \nabla c &= \Delta c - nc, & x \in \Omega,\ t > 0, \\ u_t + \kappa (u \cdot \nabla) u &= \Delta u - \nabla P + n \nabla \phi, & x \in \Omega,\ t > 0, \\ \nabla \cdot u = 0, & & x \in \Omega,\ t > 0, \end{cases}$$

in a bounded convex domain $\Omega \subset \mathbb{R}^2$ with smooth boundary, with $\kappa \in \mathbb{R}$ and a given smooth potential $\phi : \Omega \to \mathbb{R}$.

It is known that for each $\kappa \in \mathbb{R}$ and all sufficiently smooth initial data this problem possesses a unique global classical solution $(n^{(\kappa)}, c^{(\kappa)}, u^{(\kappa)})$. The present work asserts that these solutions stabilize to $(n^{(0)}, c^{(0)}, u^{(0)})$ uniformly with respect to the time variable. More precisely, it is shown that there exist $\mu > 0$ and $C > 0$ such that whenever $\kappa \in (-1, 1)$,

$$\left\| n^{(\kappa)}(\cdot,t) - n^{(0)}(\cdot,t) \right\|_{L^\infty(\Omega)} + \left\| c^{(\kappa)}(\cdot,t) - c^{(0)}(\cdot,t) \right\|_{L^\infty(\Omega)} + \left\| u^{(\kappa)}(\cdot,t) - u^{(0)}(\cdot,t) \right\|_{L^\infty(\Omega)} \le C|\kappa| e^{-\mu t}$$

for all $t > 0$.

This result thereby provides an example for a rigorous quantification of stability properties in the Stokes limit process, as frequently considered in the literature on chemotaxis-fluid systems in application contexts involving low Reynolds numbers.



**Key words:** chemotaxis, Navier-Stokes, small-convection limit, exponential stabilization
**AMS Classification:** 35B40 (primary); 35K55, 92C17, 35Q30, 35Q92 (secondary)



*wangyulan-math@163.com
#michael.winkler@math.uni-paderborn.de
‡zxiang@uestc.edu.cn

# 1 Introduction

The interaction of chemotaxis and diffusion of nutrients in bacterial suspensions can produce a variety of structures with locally high concentrations of cells, including phyllotactic patterns, filaments, and concentrations in fabricated microstructures. To explore a class of situations in which actually concentrating hydrodynamic flows may arise in such circumstances, Goldstein et al. conducted a detailed experimental study of the collective behavior in populations of swimming bacteria of the species *Bacillus subtilis* when suspended in a sessile drop of water, as a striking result revealing spontaneous emergence of structures such as the formation of plume-like structures and large-scale convection patterns ([32]). As a model for the theoretical description of such processes, coupled chemotaxis-fluid systems of the form

$$\begin{cases} n_t + u\cdot\nabla n = \nabla\cdot(D(n)\nabla n) - \nabla\cdot(n\chi(n,c)\nabla c), \\ c_t + u\cdot\nabla c = \Delta c - nf(c), \\ u_t + \kappa(u\cdot\nabla)u + \nabla P = \varsigma\Delta u + n\nabla\phi, \\ \nabla\cdot u = 0 \end{cases} \tag{1.1}$$

are proposed in [32] as extensions of the classical Keller-Segel-type chemotaxis models to such situations of nontrivial interaction of chemotactically migrating cells with liquid environments. Here, the parameters $\kappa$ and $\varsigma$ are related to the strength of nonlinear fluid convection and the viscosity of the incompressible fluid, represented through its velocity $u$ and the associated pressure $P$, and where $n$ and $c$, respectively, denote the population density of cells and the concentration of the oxygen by which they are attracted and which they consume upon contact. The corresponding cell mobility $D(n)$, the chemotactic sensitivity function $\chi(c)$, and the per-capita oxygen consumption rate $f(c)$ are given scalar functions, where prototypical choices are given by

$$D \equiv const., \qquad \chi \equiv const. \qquad \text{and} \qquad f(c) = c, \quad c \ge 0. \tag{1.2}$$

**Chemotaxis-Navier-Stokes vs. chemotaxis-Stokes systems.** Mathematically analyzing models of the above form needs to adequately cope, inter alia, with the evident challenges related to the corresponding chemotaxis system and the equations from fluid mechanics, both contained in (1.1) as subsystems. Indeed, even the former seems well-understood only in the two-dimensional setting where global bounded classical solutions to an associated Neumann initial-boundary value problem are known to exist for widely arbitrary initial data ([28]); in the three-dimensional analogue, such solutions could be found under a smallness assumption on the initial data $\|c(\cdot,0)\|_{L^\infty(\Omega)}$ ([27]), whereas for large data up to now only global weak solutions could be constructed which, after all, become eventually smooth and classical ([28]).
As for the Navier-Stokes subsystem of (1.1) related to the choices $n = c \equiv 0$ and $\kappa = \varsigma = 1$, despite tremendous efforts throughout the past decades a comprehensive theory of global well-posedness in frameworks of smooth solutions e.g. to an associated Dirichlet problem in bounded domains seems available also only in the two-dimensional case, while in three-dimensional scenarios there apparently still remains a gap in knowledge, between Leray's old result on globally existing weak solutions on the one hand, and various statements on unique local-in-time regular solutions under diverse assumptions on the other ([19], [12], [25]). In sharp contrast to this, the knowledge on the corresponding linear Stokes evolution system, as obtained on letting $\kappa = 0$ and thus neglecting the nonlinear convective

term $(u\cdot\nabla)u$, is substantially more complete, yielding global smooth solutions without any restrictions on the spatial dimension ([25]).

Accordingly, several works on coupled systems of the form (1.1) concentrate on the respective chemotaxis-Stokes variant, relying on experimental observations on average swimming speeds of bacteria in water which suggest the relevant Reynolds number $\mathcal{R}e = \frac{\kappa}{\varsigma}$ to be of order $\mathcal{R}e \approx 10^{-5}$. Indeed, this simplification has turned out to allow for significantly more extensive results on existence and also on qualitative properties of solutions, especially in three-dimensional situations. For the initial-boundary value problem for (1.1) in smoothly bounded convex domains $\Omega \subset \mathbb{R}^3$ with $\frac{\partial n}{\partial\nu} = \frac{\partial c}{\partial\nu} = 0$ and $u = 0$ on $\partial\Omega$, for instance, only global weak solutions are known to exist for $\kappa = \varsigma = 1$ and $\chi$ and $f$ as in (1.2) in the linear case $D \equiv 1$ ([39]), but also in presence of nonlinear diffusion of the form $D(n) = n^{m-1}$, $n > 0$, for $m \ge \frac{2}{3}$ ([46]); in the case $D \equiv 1$ these solutions become smooth and classical after a certain time $T > 0$ ([40]), but possibly irregular behavior prior to this relaxation time can not yet be excluded, not even in cases of large $m$ in which cell diffusion is substantially enhanced at large population densities. Contrary to this, for the corresponding chemotaxis-Stokes variant with $D(n) = n^{m-1}, n > 0$, it could be shown that global weak solutions, yet known to exist whenever $m \ge 1$ ([8], [37]), are locally bounded in $\bar\Omega \times [0,\infty)$ if $m > \frac{8}{7}$ ([30]) and even globally bounded if $m > \frac{7}{6}$ ([41]). In full chemotaxis-Navier-Stokes systems, further boundedness properties could only be established upon imposing appropriate smallness conditions on the initial data ([7], [5], [18], [4], [44], [26]).
We remark here that essential use of corresponding Stokes simplifications has also been made in several earlier works on models of type (1.1) ([7], [6], [21], [33], [34], [35], [29]).

**Quantifying the Stokes approximation error. Main results.** In light of the above, it seems natural to ask to which extent the Stokes approximation affects the solution behavior in systems of type (1.1). Indeed, in view of results on continuous parameter dependence known from other contexts it appears to be not very daring to conjecture that in the limit $\kappa \to 0$, solutions to (1.1) will approach a solution of the corresponding chemotaxis-Stokes problem at least locally with respect to the time variable. The purpose of the present work is to perform a detailed quantitative analysis of the respective error, and to thereby show that actually the relaxation properties induced by the dissipative mechanisms of diffusion and, especially, of signal consumption in (1.1) are strong enough so as to allow for a uniform and global-in-time control of this error in terms of the parameter $\kappa$. In order to focus on this aspect, we concentrate on a concrete situation in which all parameter functions in (1.1) are specified through respective prototypical choices, and in which both the corresponding Stokes and the Navier-Stokes version are essentially well-understood. Accordingly, for $\kappa \in \mathbb{R}$ we shall subsequently consider

$$\begin{cases} n_t + u\cdot\nabla n &= \Delta n - \nabla\cdot(n\nabla c), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c &= \Delta c - nc, & x\in\Omega,\ t>0,\\ u_t + \kappa(u\cdot\nabla)u &= \Delta u - \nabla P + n\nabla\phi, & x\in\Omega,\ t>0,\\ \nabla\cdot u = 0, & & x\in\Omega,\ t>0, \end{cases} \tag{1.3}$$

under the boundary conditions

$$\frac{\partial n}{\partial\nu} = 0, \quad \frac{\partial c}{\partial\nu} = 0 \quad \text{and} \quad u = 0, \qquad x\in\partial\Omega,\ t>0, \tag{1.4}$$

and the initial conditions

$$n(x,0) = n_0(x), \quad c(x,0) = c_0(x) \quad \text{and} \quad u(x,0) = u_0(x), \qquad x\in\Omega, \tag{1.5}$$

in a bounded convex domain $\Omega \subset \mathbb{R}^2$ with smooth boundary. Here for simplicity we shall assume that $\phi \in W^{2,\infty}(\Omega)$, and that the initial data are such that

$$\begin{cases} n_0 \in C^0(\bar{\Omega}) \quad \text{is nonnegative with } n_0 \not\equiv 0, \\ c_0 \in W^{1,\infty}(\Omega) \quad \text{is nonnegative, and} \\ u_0 \in D(A^\alpha) \quad \text{for some } \alpha \in (\frac{1}{2}, 1), \end{cases} \tag{1.6}$$

where $A := -\mathcal{P}\Delta$ denotes the realization of the Stokes operator in $L^2(\Omega; R^2)$, defined on its domain $D(A) := W^{2,2}(\Omega;\mathbb{R}^2) \cap W_0^{1,2}(\Omega;\mathbb{R}^2) \cap L^2_\sigma(\Omega)$ with $L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega;\mathbb{R}^2) \mid \nabla \cdot u = 0\}$, and with $\mathcal{P}$ representing the Helmholtz projection of $L^2(\Omega;\mathbb{R}^2)$ onto $L^2_\sigma(\Omega)$.

Indeed, it is known that in this framework, (1.3)-(1.5) possesses a uniquely determined global classical solution ([37], cf. also Lemma 2.1 below for details of the precise regularity class, and [47] for a related result in the case $\Omega = \mathbb{R}^2$). For $\kappa = 1$, and actually for arbitrary $\kappa \in \mathbb{R}$, any nontrivial of these solutions is moreover known to approach the spatially homogeneous equilibrium $(\overline{n}_0, 0, 0)$ in the large time limit ([38]), at a rate recently found to be exponential ([45]), where $\overline{n}_0 := \frac{1}{|\Omega|}\int_\Omega n_0$.

Now our main result asserts temporally uniform convergence of these solutions in the limit $\kappa \to 0$; more precisely:

**Theorem 1.1** *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with smooth boundary, let $\phi \in W^{2,\infty}(\Omega)$, and suppose that $n_0, c_0$ and $u_0$ satisfy (1.6). Then letting $(n^{(\kappa)}, c^{(\kappa)}, u^{(\kappa)}, P^{(\kappa)})$ denote the solution of (1.3)-(1.5) corresponding to $\kappa \in (-1,1)$, for all $p \in (1,\infty)$ one can find $\mu(p) > 0$ and $C(p) > 0$ with the property that for each $\kappa \in (-1,1)$ we have*

$$\begin{aligned} \Big\| n^{(\kappa)}(\cdot,t) - n^{(0)}(\cdot,t)\Big\|_{L^\infty(\Omega)} + \Big\| c^{(\kappa)}(\cdot,t) - c^{(0)}(\cdot,t)\Big\|_{W^{1,p}(\Omega)} + \Big\| A^\alpha u^{(\kappa)}(\cdot,t) - A^\alpha u^{(0)}(\cdot,t)\Big\|_{L^2(\Omega)} \\ \le C(p)|\kappa| e^{-\mu(p)t} \qquad \textit{for all } t > 0. \end{aligned} \tag{1.7}$$

*In particular, there exist $\mu > 0$ and $C > 0$ such that whenever $\kappa \in (-1,1)$,*

$$\begin{aligned} \Big\| n^{(\kappa)}(\cdot,t) - n^{(0)}(\cdot,t)\Big\|_{L^\infty(\Omega)} + \Big\| c^{(\kappa)}(\cdot,t) - c^{(0)}(\cdot,t)\Big\|_{L^\infty(\Omega)} + \Big\| u^{(\kappa)}(\cdot,t) - u^{(0)}(\cdot,t)\Big\|_{L^\infty(\Omega)} \\ \le C|\kappa| e^{-\mu t} \qquad \textit{for all } t > 0. \end{aligned} \tag{1.8}$$

To our best knowledge, this seems to be the first rigorous mathematical result on a small-convection limit in a chemotaxis-fluid system indeed, thereby supplementing previously gained knowledge mainly based on numerical experiments such as e.g. performed in certain intermediate Reynolds number limits for jet propulsion ([17]).

**Outline of our approach.** The first step in our analysis is based on the observation that for suitably large $a > 0$ not depending on $\kappa \in (-1,1)$,

$$\int_\Omega n \ln n + \frac{1}{2}\int_\Omega \frac{|\nabla c|^2}{c} + a\int_\Omega |u|^2$$

acts as a quasi-energy functional for (1.3). Through a series of subsequent bootstrap arguments, the a priori information thereby obtained finally enables us in Section 2 to derive $\kappa$-independent boundedness properties of solutions to (1.3) with respect to, inter alia, the norm in $C^1(\bar\Omega) \times C^2(\bar\Omega) \times D(A^\alpha)$.

Partially relying on these regularity features, Section 3 will thereupon reveal, as a key feature of (1.3), that the quantity $\|c^{(\kappa)}(\cdot,t)\|_{L^\infty(\Omega)}$ decays in the large time limit, uniformly with respect to the parameter $\kappa \in (-1,1)$. Combined with an analysis of functionals as $\int_\Omega |n^{(\kappa)} - \overline{n}_0|^2$ and $\int_\Omega |u^{(\kappa)}|^2$, thus differing from previously pursued strategy such as that e.g. in [38], in Section 4 this will imply stabilization toward the steady state $(\overline{n}_0, 0, 0)$ at an exponential rate, again uniformly with respect to $\kappa \in (-1,1)$.

Thereafter, the limit behavior as $\kappa \to 0$ in

$$\widehat{n} := n^{(\kappa)} - n^{(0)}, \quad \widehat{c} := c^{(\kappa)} - c^{(0)}, \quad \widehat{u} := u^{(\kappa)} - u^{(0)} \quad \text{and} \quad \widehat{P} := P^{(\kappa)} - P^{(0)}$$

will be examined in Section 5. Here our first step will consist in establishing a corresponding $L^2$ estimate for $(\widehat{n}, \widehat{c}, \widehat{u})$ on the basis of an analysis of the coupled quantity

$$\int_\Omega \widehat{n}^2 + k \int_\Omega \widehat{c}^2 + l \int_\Omega |\widehat{u}|^2$$

with appropriate $k > 0$ and $l > 0$, which will be seen to satisfy an absorptive ODI with certain perturbation terms which thanks to the previously obtained exponential stabilization property decay conveniently fast in the large time limit (Section 5.1). The conclusion thereby gained will thereafter enable us in Section 5.2 to perform two more bootstrap precedures in separately showing that firstly the rightmost summand in (1.7) exhibits the claimed behavior, and that secondly moreover

$$\int_\Omega \widehat{n}^p + \int_\Omega |\nabla \widehat{c}|^p$$

for arbitrary integers $p \geq 4$ satisfies a perturbed absorptive ODI of the above type, implying that also the first two summands in (1.7) can be estimated in the claimed manner.

## 2 Uniform boundedness properties

Let us first make sure that all the problems in question possess globally defined solutions. Indeed, the following result on unique global solvability can be derived by straightforward adaptation of the respective arguments from [37], where only the special case $\kappa = 1$ was detailed but actually the general case of arbitrary $\kappa \in \mathbb{R}$ was covered.

**Lemma 2.1** *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded convex domain with smooth boundary, that $\phi \in W^{2,\infty}(\Omega)$, and that $n_0, c_0$ and $u_0$ satisfy (1.6). Then for any $\kappa \in \mathbb{R}$ there exist uniquely determined functions*

$$\begin{cases} n^{(\kappa)} \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)), \\ c^{(\kappa)} \in \bigcap_{p>2} C^0([0,\infty); W^{1,p}(\Omega)) \cap C^{2,1}(\bar\Omega \times (0,\infty)), \\ u^{(\kappa)} \in C^0([0,\infty); D(A^\alpha)) \cap C^{2,1}(\bar\Omega \times (0,\infty); \mathbb{R}^2), \\ P^{(\kappa)} \in C^{1,0}(\bar\Omega \times (0,\infty)), \end{cases} \tag{2.1}$$

*which are such that $n^{(\kappa)}$ and $c^{(\kappa)}$ are nonnegative in $\Omega \times (0,\infty)$, and such that $(n^{(\kappa)}, c^{(\kappa)}, u^{(\kappa)}, P^{(\kappa)})$ form a classical solution of (1.3)-(1.5).*

Two basic but important properties of these solutions are immediate.

**Lemma 2.2** *For any $\kappa \in \mathbb{R}$, we have*

$$\int_\Omega n^{(\kappa)}(\cdot,t) = \int_\Omega n_0 \qquad \textit{for all } t>0 \tag{2.2}$$

*and*

$$\|c^{(\kappa)}(\cdot,t)\|_{L^\infty(\Omega)} \le \|c_0\|_{L^\infty(\Omega)} \qquad \textit{for all } t>0. \tag{2.3}$$

Proof. The identity (2.2) directly results on integrating the first equation in (1.3), whereas (2.3) follows by an application of the maximum principle to the second equation in (1.3). $\Box$

Based on the analysis of a quasi-energy functional which slightly differs from that used in the global existence analysis from [37], we can assert some first $\kappa$-independent boundedness properties beyond those from Lemma 2.2. We remark that due to a refined construction, including also the fluid velocity field as part of this functional, unlike those from [37] the estimates obtained here will be uniform also with respect to time, and thereby form a cornerstone for the derivation of the global estimates claimed in Theorem 1.1.

**Lemma 2.3** *There exist $C>0$ such that for any choice of $\kappa \in (-1,1)$ we have*

$$\int_\Omega n^{(\kappa)}(\cdot,t)\Big|\ln n^{(\kappa)}(\cdot,t)\Big| \le C \qquad \textit{for all } t>0 \tag{2.4}$$

*and*

$$\int_\Omega |\nabla c^{(\kappa)}(\cdot,t)|^2 \le C \qquad \textit{for all } t>0 \tag{2.5}$$

*and*

$$\int_\Omega |u^{(\kappa)}(\cdot,t)|^2 \le C \qquad \textit{for all } t>0 \tag{2.6}$$

*as well as*

$$\int_t^{t+1}\int_\Omega (n^{(\kappa)})^2 \le C \qquad \textit{for all } t>0 \tag{2.7}$$

*and*

$$\int_t^{t+1}\int_\Omega |\nabla u^{(\kappa)}|^2 \le C \qquad \textit{for all } t>0. \tag{2.8}$$

Proof. Omitting the superscript $\kappa$ for notational convenience, by direct computation using integration by parts and the solenoidality of $u^{(\kappa)}$ we first obtain the identity

$$\begin{aligned}
&\frac{d}{dt}\Big\{\int_\Omega n\ln n + \frac12\int_\Omega \frac{|\nabla c|^2}{c}\Big\} + \int_\Omega \frac{|\nabla n|^2}{n} + \int_\Omega c|D^2\ln c|^2 \\
&= -\frac12\int_\Omega \frac{n}{c}|\nabla c|^2 + \frac12\int_{\partial\Omega}\frac1c\frac{\partial|\nabla c|^2}{\partial\nu} - \int_\Omega \frac1c\nabla c\cdot(\nabla u\cdot\nabla c)
\end{aligned} \tag{2.9}$$

for all $t > 0$ (cf. [37, Lemma 3.2] and [39, Lemma 3.4] for details). Here we make use of the convexity of $\Omega$ to see that since $\frac{\partial c}{\partial \nu} = 0$ on $\partial\Omega$ we have $\frac{\partial |\nabla c|^2}{\partial \nu} \le 0$ on $\partial\Omega$ ([23]), so that the first two summands on the right of (2.9) are nonpositive. In order to estimate the third, we recall that by [37, Lemma 3.3], writing $C_1 := (2+\sqrt{2})^2$ we have

$$\int_\Omega \frac{|\nabla c|^4}{c^3} \le C_1 \int_\Omega c|D^2 \ln c|^2 \qquad \text{for all } t > 0, \tag{2.10}$$

whereupon we apply Young's inequality to see that due to (2.3),

$$\begin{aligned} -\int_\Omega \frac{1}{c}\nabla c \cdot (\nabla u \cdot \nabla c) &\le \frac{1}{2C_1}\int_\Omega \frac{|\nabla c|^4}{c^3} + \frac{C_1}{2}\int_\Omega c|\nabla u|^2 \\ &\le \frac{1}{2C_1}\int_\Omega \frac{|\nabla c|^4}{c^3} + C_2 \int_\Omega |\nabla u|^2 \qquad \text{for all } t > 0 \end{aligned} \tag{2.11}$$

with $C_2 := \frac{C_1}{2}\|c_0\|_{L^\infty(\Omega)}$. We now fix an arbitrary $p > 2$ and test the third equation in (1.3) by $u$, noting that since $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ there exists $C_3 > 0$ such that $\|u\|_{L^p(\Omega)} \le C_3\|\nabla u\|_{L^2(\Omega)}$ for all $t > 0$, and that hence also, by the Hölder inequality,

$$\int_\Omega |u|^2 \le |\Omega|^{\frac{p-2}{p}} \left\{ \int_\Omega |u|^p \right\}^{\frac{2}{p}} \le C_4 \int_\Omega |\nabla u|^2 \qquad \text{for all } t > 0$$

with $C_4 := |\Omega|^{\frac{p-2}{p}} C_3^2$. By using the Hölder inequality and Young's inequality, abbreviating $C_5 := \|\nabla\phi\|_{L^\infty(\Omega)}$ we thereby obtain that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 + \frac{1}{2C_4}\int_\Omega |u|^2 + \frac{1}{2}\int_\Omega |\nabla u|^2 &\le \frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 \\ &= \int_\Omega nu \cdot \nabla\phi \\ &\le C_5 \|n\|_{L^{\frac{p}{p-1}}(\Omega)} \|u\|_{L^p(\Omega)} \\ &\le C_3 C_5 \|n\|_{L^{\frac{p}{p-1}}(\Omega)} \|\nabla u\|_{L^2(\Omega)} \\ &\le \frac{1}{4}\int_\Omega |\nabla u|^2 + C_3^2 C_5^2 \|n\|^2_{L^{\frac{p}{p-1}}(\Omega)} \qquad \text{for all } t > 0, \end{aligned}$$

which combined with (2.9), (2.10) and (2.11) shows that

$$y(t) := \int_\Omega n(\cdot,t)\ln n(\cdot,t) + \frac{1}{2}\int_\Omega \frac{|\nabla c(\cdot,t)|^2}{c(\cdot,t)} + 4C_2 \int_\Omega |u(\cdot,t)|^2, \qquad t \ge 0,$$

satisfies

$$y'(t) + \int_\Omega \frac{|\nabla n|^2}{n} + \frac{1}{2C_1}\int_\Omega \frac{|\nabla c|^4}{c^3} + 4\frac{C_2}{C_4}\int_\Omega |u|^2 + C_2\int_\Omega |\nabla u|^2 \le C_6 \|n\|^2_{L^{\frac{p}{p-1}}(\Omega)} \quad \text{for all } t > 0 \tag{2.12}$$

with $C_6 := 8C_2C_3^2C_5^2$. Here we apply the Gagliardo-Nirenberg inequality along with (2.2) and Young's inequality to see, relying on our restriction $p > 2$, that there exist positive constants $C_7$, $C_8$ and $C_9$ fulfilling

$$\begin{aligned} C_6\|n\|^2_{L^{\frac{p}{p-1}}(\Omega)} = C_6\|\sqrt{n}\|^4_{L^{\frac{2p}{p-1}}(\Omega)} &\le C_7\|\nabla\sqrt{n}\|^{\frac{4}{p}}_{L^2(\Omega)}\|\sqrt{n}\|^{\frac{4(p-1)}{p}}_{L^2(\Omega)} + C_7\|\sqrt{n}\|^4_{L^2(\Omega)} \\ &\le C_8\|\nabla\sqrt{n}\|^{\frac{4}{p}}_{L^2(\Omega)} + C_8 \\ &\le 2\|\nabla\sqrt{n}\|^2_{L^2(\Omega)} + C_9 \\ &= \frac{1}{2}\int_\Omega \frac{|\nabla n|^2}{n} + C_9 \qquad \text{for all } t > 0. \end{aligned} \tag{2.13}$$

A similar argument shows that moreover

$$\begin{aligned} \int_\Omega n^2 = \|\sqrt{n}\|^4_{L^4(\Omega)} &\le C_{10}\|\nabla\sqrt{n}\|^2_{L^2(\Omega)}\|\sqrt{n}\|^2_{L^2(\Omega)} + C_{10}\|\sqrt{n}\|^4_{L^2(\Omega)} \\ &\le C_{11}\int_\Omega \frac{|\nabla n|^2}{n} + C_{11} \qquad \text{for all } t > 0 \end{aligned} \tag{2.14}$$

with some $C_{10} > 0$ and $C_{11} > 0$, so that since $\xi \ln \xi \le \xi^2$ for all $\xi > 0$ we obtain

$$\frac{1}{2}\int_\Omega \frac{|\nabla n|^2}{n} \ge \frac{1}{2C_{11}}\int_\Omega n^2 - \frac{1}{2} \ge \frac{1}{4C_{11}}\int_\Omega n\ln n + \frac{1}{4C_{11}}\int_\Omega n^2 - \frac{1}{2} \qquad \text{for all } t > 0. \tag{2.15}$$

Finally observing that by Young's inequality and (2.3) we have

$$\frac{1}{8C_{11}}\int_\Omega \frac{|\nabla c|^2}{c} \le \frac{1}{2C_1}\int_\Omega \frac{|\nabla c|^4}{c^3} + \frac{C_1}{128C_{11}^2}\int_\Omega c \le \frac{1}{2C_1}\int_\Omega \frac{|\nabla c|^4}{c^3} + \frac{C_1\|c_0\|_{L^\infty(\Omega)}|\Omega|}{128C_{11}^2} \quad \text{for all } t > 0. \tag{2.16}$$

Without loss of generality, we can further choose $C_{11}$ such that $\frac{1}{4C_{11}} \le \frac{1}{C_4}$ and then infer on collecting (2.13)-(2.16) that (2.12) implies the inequality

$$y'(t) + \frac{1}{4C_{11}}y(t) + \frac{1}{4C_{11}}\int_\Omega n^2 + C_2\int_\Omega |\nabla u|^2 \le C_{12} := C_9 + \frac{C_1\|c_0\|_{L^\infty(\Omega)}|\Omega|}{128C_{11}^2} + \frac{1}{2} \quad \text{for all } t > 0. \tag{2.17}$$

By a comparison argument, this firstly entails that

$$y(t) \le C_{13} := \max\left\{y(0)\,,\, 4C_{11}C_{12}\right\} \qquad \text{for all } t > 0, \tag{2.18}$$

and thereafter an integration of (2.17) yields

$$\begin{aligned} \frac{1}{4C_{11}}\int_t^{t+1}\int_\Omega n^2 + C_2\int_t^{t+1}\int_\Omega |\nabla u|^2 &\le C_{12} + y(t) - y(t+1) - \frac{1}{4C_{11}}\int_t^{t+1} y \\ &\le C_{12} + C_{13} + \frac{|\Omega|}{e} + \frac{1}{4C_{11}}\cdot\frac{|\Omega|}{e} \quad \text{for all } t > 0, \end{aligned} \tag{2.19}$$

because $\xi\ln\xi \ge -\frac{1}{e}$ for all $\xi>0$ and hence $y(t) \ge \int_\Omega n\ln n \ge -\frac{|\Omega|}{e}$ for all $t>0$. Since for the same reason we have

$$\int_\Omega n|\ln n| = \int_\Omega n\ln n - 2\int_{\{n<1\}} n\ln n \le \int_\Omega n\ln n + \frac{2|\Omega|}{e} \qquad \text{for all } t>0,$$

from (2.18) together with (2.3) we easily derive (2.4)-(2.6), whereas (2.7) and (2.8) directly result from (2.19). $\square$

By means of a standard testing procedure associates with the Navier-Stokes subsystem of (1.3), from this we can readily derive a higher-order regularity property of the fluid velocity.

**Lemma 2.4** *There exists $C>0$ such that for all $\kappa\in(-1,1)$ we have*

$$\int_\Omega |\nabla u^{(\kappa)}(\cdot,t)|^2 \le C \qquad \textit{for all } t>0. \tag{2.20}$$

PROOF. According to Lemma 2.3, there exist positive constants $C_1, C_2$ and $C_3$ such that again dropping the index $\kappa$ we have

$$\int_\Omega |u|^2 \le C_1, \qquad \int_t^{t+1}\int_\Omega n^2 \le C_2 \qquad \text{and} \qquad \int_t^{t+1}\int_\Omega |\nabla u|^2 \le C_3 \tag{2.21}$$

for all $t>0$. In particular, if we multiply the projected version of the third equation in (1.3), that is, the identity $u_t + Au = \mathcal{P}[n\nabla\phi] - \kappa\mathcal{P}[(u\cdot\nabla)u]$, by $Au$, then making use of Young's inequality, the orthogonal projection property of $\mathcal{P}$, the boundedness of $\nabla\phi$, the Gagliardo-Nirenberg inequality and the Hölder inequality we see that with some $C_4>0$ we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |A^{\frac{1}{2}}u|^2 + \int_\Omega |Au|^2 &= \int_\Omega \mathcal{P}[n\nabla\phi]\cdot Au - \kappa\int_\Omega \mathcal{P}[(u\cdot\nabla)u]\cdot Au \\
&\le \frac{1}{2}\int_\Omega |Au|^2 + \int_\Omega |n\nabla\phi|^2 + \kappa^2\int_\Omega |(u\cdot\nabla)u|^2 \\
&\le \frac{1}{2}\int_\Omega |Au|^2 + \|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega n^2 + \kappa^2\|u\|^2_{L^4(\Omega)}\|\nabla u\|^2_{L^4(\Omega)} \\
&\le \frac{1}{2}\int_\Omega |Au|^2 + \|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega n^2 \\
&\quad + C_4\Big(\|\nabla u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}\Big)\cdot\Big(\|Au\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}\Big) \\
&\le \int_\Omega |Au|^2 + \|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega n^2 + \frac{C_1C_4^2}{2}\Big\{\int_\Omega |\nabla u|^2\Big\}^2 \qquad \text{for all } t>0.
\end{aligned}$$

Since $\int_\Omega |\nabla u|^2 = \int_\Omega |A^{\frac{1}{2}}u|^2$ for all $t\ge 0$, writing $y(t) := \int_\Omega |\nabla u(\cdot,t)|^2$, $g(t) := 2\|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega n^2$ and $h(t) := C_1C_4^2\int_\Omega |\nabla u(\cdot,t)|^2$ for $t\ge 0$, we thus obtain that

$$y'(t) \le g(t) + h(t)y(t) \qquad \text{for all } t>0, \tag{2.22}$$

where by (2.21),

$$\int_t^{t+1} g(s)ds \le C_5 := 2C_2\|\nabla\phi\|^2_{L^\infty(\Omega)} \quad \text{and} \quad \int_t^{t+1} h(s)ds \le C_6 := C_1C_3C_4^2 \qquad \text{for all } t>0.$$

Now for fixed $t > 0$, in view of (2.21) we can find $t_0 = t_0(\kappa, t) \geq 0$ such that $t_0 \in (t-1, t)$ and $y(t_0) \leq C_7 := \max\{C_3, \int_\Omega |\nabla u_0|^2\}$, so that on integrating (2.22) we infer that

$$y(t) \leq y(t_0) e^{\int_{t_0}^t h(s)ds} + \int_{t_0}^t e^{\int_s^t h(\sigma)d\sigma} g(s)ds \leq C_7 e^{C_6} + C_5 e^{C_6},$$

as desired. □

In turning this together with the bounds from Lemma 2.3 into $\kappa$-independent global estimates for $\int_\Omega n^p$ and $\int_\Omega |\nabla c|^{2p}$ with arbitrary $p > 1$, we shall make use of the following special case of a more general Gagliardo-Nirenberg-type interpolation inequality ([31]) which can be proved by straightforward adaptation of the argument introduced in [3].

**Lemma 2.5** *Let $p > \frac{1}{2}$. Then there exists $C(p) > 0$ such that for all $\varepsilon > 0$ one can find $C(p, \varepsilon) > 0$ fulfilling*

$$\|\varphi\|_{L^4(\Omega)}^{\frac{4p}{2p-1}} \leq \varepsilon \|\nabla\varphi\|_{L^2(\Omega)}^2 \Big\|\varphi \cdot |\ln|\varphi||^{\frac{p}{2}}\Big\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{2p-1}} + C(p,\varepsilon) + C(p)\|\varphi\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{4p}{2p-1}}$$

*for all $\varphi \in W^{1,2}(\Omega)$.*

We can thereby make efficient use of the bound on $n$ in $L\log L(\Omega)$ contained in Lemma 2.3 to achieve the following.

**Lemma 2.6** *Let $p > 1$. Then there exists $C(p)$ with the property that whenever $\kappa \in (-1, 1)$,*

$$\int_\Omega |n^{(\kappa)}(\cdot, t)|^p \leq C(p) \qquad \text{for all } t > 0 \tag{2.23}$$

*and*

$$\int_\Omega |\nabla c^{(\kappa)}(\cdot, t)|^{2p} \leq C(p) \qquad \text{for all } t > 0. \tag{2.24}$$

Proof. Once more omitting the index $\kappa$ for convenience, by means of the first two equations in (1.3), Young's inequality and (2.3) we see that for all $t > 0$ we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega n^p + (p-1)\int_\Omega n^{p-2}|\nabla n|^2 = (p-1)\int_\Omega n^{p-1}\nabla n \cdot \nabla c \tag{2.25}$$

$$\leq \frac{p-1}{2}\int_\Omega n^{p-2}|\nabla n|^2 + \frac{p-1}{2}\int_\Omega n^p |\nabla c|^2 \tag{2.26}$$

and

$$\begin{aligned}
\frac{1}{2p}\frac{d}{dt}\int_\Omega |\nabla c|^{2p} &= \int_\Omega |\nabla c|^{2p-2}\nabla c \cdot \nabla(\Delta c - nc - u\cdot\nabla c) \\
&\leq -\int_\Omega |\nabla c|^{2p-2}|D^2 c|^2 + \int_\Omega nc|\nabla c|^{2p-2}\Delta c + 2(p-1)\int_\Omega nc|\nabla c|^{2p-4}\nabla c\cdot(D^2 c\cdot\nabla c) \\
&\quad - \int_\Omega |\nabla c|^{2p-2}\nabla c\cdot(\nabla u\cdot\nabla c) \\
&\leq -\frac{1}{2}\int_\Omega |\nabla c|^{2p-2}|D^2c|^2 + C_1\int_\Omega n^2|\nabla c|^{2p-2} + \int_\Omega |\nabla c|^{2p}|\nabla u| \qquad (2.27)
\end{aligned}$$

with $C_1 := (2+4(p-1)^2)\|c_0\|^2_{L^\infty(\Omega)}$, where we have used that $\nabla\cdot u\equiv 0$, that $\nabla c\cdot\nabla\Delta c\equiv \frac{1}{2}\Delta|\nabla c|^2 - |D^2c|^2$, that $|\Delta c|\le\sqrt{2}|D^2c|$ and that

$$\frac{1}{2}\int_\Omega |\nabla c|^{2p-2}\Delta|\nabla c|^2 = -\frac{p-1}{2}\int_\Omega |\nabla c|^{2p-4}\Big|\nabla|\nabla c|^2\Big|^2 + \frac{1}{2}\int_{\partial\Omega}|\nabla c|^{2p-2}\frac{\partial|\nabla c|^2}{\partial\nu}\le 0 \qquad \text{for all } t>0,$$

again thanks to the fact that $\frac{\partial|\nabla c|^2}{\partial\nu}\le 0$ on $\partial\Omega$. Here due to the Hölder inequality and the Gagliardo-Nirenberg inequality, Lemma 2.3, Young's inequality and the pointwise inequality

$$\Big|\nabla|\nabla c|^p\Big|^2 = \Big|p|\nabla c|^{p-2}D^2c\cdot\nabla c\Big|^2 \le p^2|\nabla c|^{2p-2}|D^2c|^2, \tag{2.28}$$

we see that there exist positive constants $C_2, C_3$ and $C_4$ such that

$$\begin{aligned}
\frac{p-1}{2}\int_\Omega n^p|\nabla c|^2 &\le \frac{p-1}{2}\|n^{\frac{p}{2}}\|^2_{L^4(\Omega)}\Big\||\nabla c|^p\Big\|^{\frac{2}{p}}_{L^{\frac{4}{p}}(\Omega)}\\
&\le C_2\|n^{\frac{p}{2}}\|^2_{L^4(\Omega)}\cdot\Big\{\Big\|\nabla|\nabla c|^p\Big\|^{\frac{1}{p}}_{L^2(\Omega)}\Big\||\nabla c|^p\Big\|^{\frac{1}{p}}_{L^{\frac{2}{p}}(\Omega)} + \Big\||\nabla c|^p\Big\|^{\frac{2}{p}}_{L^{\frac{2}{p}}(\Omega)}\Big\}\\
&\le C_3\|n^{\frac{p}{2}}\|^2_{L^4(\Omega)}\cdot\Big\{\Big\|\nabla|\nabla c|^p\Big\|^{\frac{1}{p}}_{L^2(\Omega)}+1\Big\}\\
&\le \frac{1}{8p^2}\int_\Omega\Big|\nabla|\nabla c|^p\Big|^2 + C_4\|n^{\frac{p}{2}}\|^{\frac{4p}{2p-1}}_{L^4(\Omega)} + C_4\\
&\le \frac{1}{8}\int_\Omega|\nabla c|^{2p-2}|D^2c|^2 + C_4\|n^{\frac{p}{2}}\|^{\frac{4p}{2p-1}}_{L^4(\Omega)} + C_4 \qquad \text{for all } t>0,
\end{aligned}$$

and that, similarly, with some $C_5>0, C_6>0$ and $C_7>0$ we have

$$\begin{aligned}
C_1\int_\Omega n^2|\nabla c|^{2p-2} &\le C_1\|n^{\frac{p}{2}}\|^{\frac{4}{p}}_{L^4(\Omega)}\Big\||\nabla c|^p\Big\|^{\frac{2p-2}{p}}_{L^2(\Omega)}\\
&\le C_5\|n^{\frac{p}{2}}\|^{\frac{4}{p}}_{L^4(\Omega)}\Big\{\Big\|\nabla|\nabla c|^p\Big\|^{\frac{2(p-1)^2}{p^2}}_{L^2(\Omega)}\Big\||\nabla c|^p\Big\|^{\frac{2p-2}{p^2}}_{L^{\frac{2}{p}}(\Omega)} + \Big\||\nabla c|^p\Big\|^{\frac{2p-2}{p}}_{L^{\frac{2}{p}}(\Omega)}\Big\}\\
&\le C_6\|n^{\frac{p}{2}}\|^{\frac{4}{p}}_{L^4(\Omega)}\Big\{\Big\|\nabla|\nabla c|^p\Big\|^{\frac{2(p-1)^2}{p^2}}_{L^2(\Omega)}+1\Big\}\\
&\le \frac{1}{8}\int_\Omega|\nabla c|^{2p-2}|D^2c|^2 + C_7\|n^{\frac{p}{2}}\|^{\frac{4p}{2p-1}}_{L^4(\Omega)} + C_7 \qquad \text{for all } t>0.
\end{aligned}$$

Since using the Cauchy-Schwarz inequality, Lemma 2.3 and the Gagliardo-Nirenberg inequality we moreover find that there exist $C_8>0, C_9>0$ and $C_{10}>0$ such that

$$\begin{aligned}
\int_\Omega|\nabla c|^{2p}|\nabla u| &\le \|\nabla u\|_{L^2(\Omega)}\Big\||\nabla c|^p\Big\|^2_{L^4(\Omega)}\\
&\le C_8\Big\{\Big\|\nabla|\nabla c|^p\Big\|^{\frac{2p-1}{p}}_{L^2(\Omega)}\Big\||\nabla c|^p\Big\|^{\frac{1}{p}}_{L^{\frac{2}{p}}(\Omega)} + \Big\||\nabla c|^p\Big\|^2_{L^{\frac{2}{p}}(\Omega)}\Big\}\\
&\le C_9\Big\{\Big\|\nabla|\nabla c|^p\Big\|^{\frac{2p-1}{p}}_{L^2(\Omega)}+1\Big\}\\
&\le \frac{1}{8}\int_\Omega|\nabla c|^{2p-2}|D^2c|^2 + C_{10} \qquad \text{for all } t>0,
\end{aligned}$$

on adding (2.25) and (2.27) we thus obtain that

$$
\begin{aligned}
&\frac{d}{dt}\Big\{\frac{1}{p}\int_\Omega n^p+\frac{1}{2p}\int_\Omega|\nabla c|^{2p}\Big\}+\frac{2(p-1)}{p^2}\int_\Omega|\nabla n^{\frac{p}{2}}|^2+\frac{1}{8}\int_\Omega|\nabla c|^{2p-2}|D^2c|^2\\
&\qquad\le\ (C_4+C_7)\|n^{\frac{p}{2}}\|_{L^4(\Omega)}^{\frac{4p}{2p-1}}+C_4+C_7+C_{10}\qquad\text{for all } t>0. \qquad (2.29)
\end{aligned}
$$

Here in accordance with Lemma 2.3 we can choose $C_{11}>0$ such that $\int_\Omega n|\ln n^{\frac{p}{2}}|\le C_{11}$ for all $t>0$, and thereupon we combine Lemma 2.5 with Lemma 2.3 to infer the existence of $C_{12}>0$ and $C_{13}>0$ such that

$$
\begin{aligned}
(C_4+C_7)\|n^{\frac{p}{2}}\|_{L^4(\Omega)}^{\frac{4p}{2p-1}} &\le \frac{p-1}{p^2C_{11}^{\frac{p}{2p-1}}}\|\nabla n^{\frac{p}{2}}\|_{L^2(\Omega)}^2\Big\|n^{\frac{p}{2}}|\ln n^{\frac{p}{2}}|^{\frac{p}{2}}\Big\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{2p-1}}+C_{12}+C_{12}\|n^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{4p}{2p-1}}\\
&\le \frac{p-1}{p^2}\int_\Omega|\nabla n^{\frac{p}{2}}|^2+C_{13}\qquad\text{for all } t>0.
\end{aligned}
$$

Since finally, again in view of (2.28), the Poincaré inequality along with Lemma 2.3 provides $C_{14}>0$ and $C_{15}>0$ fulfilling

$$
\frac{p-1}{p^2}\int_\Omega|\nabla n^{\frac{p}{2}}|^2\ge C_{14}\int_\Omega n^p-1\qquad\text{for all } t>0
$$

and

$$
\frac{1}{8}\int_\Omega|\nabla c|^{2p-2}|D^2c|^2\ge C_{15}\int_\Omega|\nabla c|^{2p}-1\qquad\text{for all } t>0,
$$

from (2.29) we conclude that $y(t):=\frac{1}{p}\int_\Omega n^p(\cdot,t)+\frac{1}{2p}\int_\Omega|\nabla c(\cdot,t)|^{2p}$, $t\ge0$, satisfies

$$
y'(t)+C_{16}y(t)\le C_{17}\qquad\text{for all } t>0
$$

with some $C_{16}>0$ and $C_{17}>0$, and that hence $y(t)\le\max\{y(0)\,,\,\frac{C_{17}}{C_{16}}\}$ for all $t>0$. ☐

According to a standard argument, the latter implies a bound for the first component even with respect to the norm in $L^\infty(\Omega)$.

**Lemma 2.7** *There exists $C>0$ such that for any $\kappa\in(-1,1)$,*

$$
\|n^{(\kappa)}(\cdot,t)\|_{L^\infty(\Omega)}\le C\qquad\textit{for all } t>0. \qquad (2.30)
$$

PROOF. We fix an arbitrary $q>2$ and then infer from known smoothing properties of the Neumann heat semigroup $(e^{t\Delta})_{t\ge0}$ in $\Omega$ that there exists $C_1>0$ such that for any $\kappa\in(-1,1)$, again writing $(n,c,u):=(n^{(\kappa)},c^{(\kappa)},u^{(\kappa)})$ we have

$$
\begin{aligned}
\|n(\cdot,t)\|_{L^\infty(\Omega)} &= \Big\|e^{\min\{t,1\}\Delta}n(\cdot,(t-1)_+)-\int_{(t-1)_+}^t e^{(t-s)\Delta}\nabla\cdot\Big(n(\cdot,s)\nabla c(\cdot,s)+n(\cdot,s)u(\cdot,s)\Big)ds\Big\|_{L^\infty(\Omega)}\\
&\le \Big\|e^{\min\{t,1\}\Delta}n(\cdot,(t-1)_+)\Big\|_{L^\infty(\Omega)}\\
&\quad +C_1\int_{(t-1)_+}^t(t-s)^{-\frac{1}{2}-\frac{1}{q}}\Big(\|n(\cdot,s)\nabla c(\cdot,s)\|_{L^q(\Omega)}+\|n(\cdot,s)u(\cdot,s)\|_{L^q(\Omega)}\Big)ds \qquad (2.31)
\end{aligned}
$$

for all $t > 0$, and there herein with some $C_2 > 0$, thanks to the maximum principle and (2.2) we have

$$\left\| e^{\min\{t,1\}\Delta} n(\cdot,(t-1)_+) \right\|_{L^\infty(\Omega)} \le \max\left\{ \|n_0\|_{L^\infty(\Omega)}\,,\, C_2\|n_0\|_{L^1(\Omega)} \right\} =: C_3 \qquad \text{for all } t > 0.$$

Now since $W^{1,2}(\Omega) \hookrightarrow L^{2q}(\Omega)$, combining Lemma 2.6 with Lemma 2.4 we see that there exists $C_4 > 0$ fulfilling

$$\|n\nabla c\|_{L^q(\Omega)} + \|nu\|_{L^q(\Omega)} \le \|n\|_{L^{2q}(\Omega)} \|\nabla c\|_{L^{2q}(\Omega)} + \|n\|_{L^{2q}(\Omega)} \|u\|_{L^{2q}(\Omega)} \le C_4 \qquad \text{for all } t > 0,$$

so that (2.31) implies that

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \le C_3 + C_1C_4 \int_{(t-1)_+}^t (t-s)^{-\frac{1}{2}-\frac{1}{q}} ds \le C_3 + \frac{C_1C_4}{\frac{1}{2}-\frac{1}{q}} \qquad \text{for all } t > 0$$

and thereby establishes (2.30). $\Box$

In particular, this provides some additional boundedness information on the forcing term in the Navier-Stokes system in (1.3) which thereby enjoys a further regularity property:

**Lemma 2.8** *There exists $C > 0$ with the property that whenever $\kappa \in (-1,1)$,*

$$\|A^\alpha u^{(\kappa)}(\cdot,t)\|_{L^2(\Omega)} \le C \qquad \textit{for all } t > 0, \tag{2.32}$$

*and that with some $\theta \in (0,1)$ we have*

$$\|u^{(\kappa)}\|_{C^{\theta,\frac{\theta}{2}}(\bar\Omega\times[t,t+1])} \le C \qquad \textit{for all } t > 0. \tag{2.33}$$

Proof. Using that $\alpha < 1$, we pick $q \in (1,2)$ suitably close to 2 such that $\gamma := \alpha + \frac{1}{q} - \frac{1}{2}$ satisfies $\gamma < 1$, and relying on known regularization properties of the Stokes semigroup in $\Omega$ we obtain $C_1 > 0$ and $C_2 > 0$ such that again dropping the superscript $(\kappa)$ and abbreviating $f := \mathcal{P}[n\nabla\phi] - \kappa\mathcal{P}[(u\cdot\nabla)u]$ we have

$$\begin{aligned} & \|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} \\ = \quad & \left\| A^\alpha e^{-\min\{t,1\}A} u(\cdot,(t-1)_+) + \int_{(t-1)_+}^t A^\alpha e^{-(t-s)A} f(\cdot,s)ds \right\|_{L^2(\Omega)} \\ \le \quad & \left\| A^\alpha e^{-\min\{t,1\}A} u(\cdot,(t-1)_+) \right\|_{L^2(\Omega)} + C_1 \int_{(t-1)_+}^t (t-s)^{-\gamma} \|f(\cdot,s)\|_{L^q(\Omega)} ds \end{aligned} \tag{2.34}$$

and

$$\left\| A^\alpha e^{-\min\{t,1\}A} u(\cdot,(t-1)_+) \right\|_{L^2(\Omega)} \le \max\left\{ \|A^\alpha u_0\|_{L^2(\Omega)}\,,\, C_2 \sup_{\kappa\in(-1,1)} \|u\|_{L^\infty((0,\infty);L^2(\Omega))} \right\} =: C_3$$

for all $t > 0$. Since $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2q}{2-q}}(\Omega)$ and $\mathcal{P}$ is continuous on $L^q(\Omega;\mathbb{R}^2)$ ([13]), Lemma 2.6 and Lemma 2.4 provide $C_4 > 0$ such that

$$\|f\|_{L^q(\Omega)} \le \|\nabla\phi\|_{L^\infty(\Omega)} \|n\|_{L^q(\Omega)} + |\kappa| \|u\|_{L^{\frac{2q}{2-q}}(\Omega)} \|\nabla u\|_{L^2(\Omega)} \le C_4 \qquad \text{for all } t > 0, \tag{2.35}$$

so that (2.34) entails that

$$\|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} \le C_3 + \frac{C_1C_4}{1-\gamma} \qquad \text{for all } t>0$$

and hence proves (2.32), whereupon (2.34) is a consequence of the fact that $\alpha > \frac{1}{2}$ warrants that $D(A^\alpha) \hookrightarrow L^\infty(\Omega;\mathbb{R}^2)$ ([16]).

Similarly, fixing any $\beta \in (\frac{1}{2},\alpha)$, for $t_0 \ge 0$ and $t \in [t_0,t_0+1]$ we can use (2.35) to estimate

$$\begin{aligned}
\|A^\beta u(\cdot,t) - A^\beta u(\cdot,t_0)\|_{L^2(\Omega)} \;\le\; & \Big\|[e^{-tA} - e^{-t_0A}]A^\beta u_0\Big\|_{L^2(\Omega)} \\
& + \int_0^{t_0} \Big\|A^\beta[e^{-(t-s)A} - e^{-(t_0-s)A}]f(\cdot,s)\Big\|_{L^2(\Omega)} ds \\
& + \int_{t_0}^t \Big\|A^\beta e^{-(t-s)A} f(\cdot,s)\Big\|_{L^2(\Omega)} ds \\
=\; & \Big\|\int_{t_0}^t A^{1-\alpha+\beta} e^{-\sigma A} A^\alpha u_0 d\sigma\Big\|_{L^2(\Omega)} \\
& + \int_0^{t_0} \Big\|\int_{t_0}^t A^{1+\beta} e^{-(\sigma-s)A} f(\cdot,s) d\sigma\Big\|_{L^2(\Omega)} ds \\
& + \int_{t_0}^t \Big\|A^\beta e^{-(t-s)A} f(\cdot,s)\Big\|_{L^2(\Omega)} ds \\
\le\; & C_5 \Big\{\int_{t_0}^t \sigma^{-1+\alpha-\beta} d\sigma\Big\} \|A^\alpha u_0\|_{L^2(\Omega)} \\
& + C_5 \int_0^{t_0}\int_{t_0}^t (\sigma-s)^{-1-\beta-(\frac{1}{q}-\frac{1}{2})} \|f(\cdot,s)\|_{L^q(\Omega)} d\sigma ds \\
& + C_5 \int_{t_0}^t (t-s)^{-\beta-(\frac{1}{q}-\frac{1}{2})} \|f(\cdot,s)\|_{L^q(\Omega)} ds \\
\le\; & \frac{C_5}{\alpha-\beta}\Big(t^{\alpha-\beta} - t_0^{\alpha-\beta}\Big)\|A^\alpha u_0\|_{L^2(\Omega)} \\
& + C_4C_5 \int_0^{t_0}\int_{t_0}^t (\sigma-s)^{-1-\beta-(\frac{1}{q}-\frac{1}{2})} d\sigma ds \\
& + C_4C_5 \int_{t_0}^t (t-s)^{-\beta-(\frac{1}{q}-\frac{1}{2})} ds \\
\le\; & \frac{2^{\alpha-\beta}C_5}{\alpha-\beta}(t-t_0)^{\alpha-\beta}\|A^\alpha u_0\|_{L^2(\Omega)} \\
& + \frac{C_4C_5}{(\beta+\frac{1}{q}-\frac{1}{2})(\frac{3}{2}-\beta-\frac{1}{q})}\Big\{t_0^{\frac{3}{2}-\beta-\frac{1}{q}} + (t-t_0)^{\frac{3}{2}-\beta-\frac{1}{q}} - t^{\frac{3}{2}-\beta-\frac{1}{q}}\Big\} \\
& + \frac{C_4C_5}{\frac{3}{2}-\beta-\frac{1}{q}}(t-t_0)^{\frac{3}{2}-\beta-\frac{1}{q}}
\end{aligned} \tag{2.36}$$

with some $C_5 > 0$. Since for any such $\beta$ we have $D(A^\beta) \hookrightarrow C^\theta(\bar{\Omega}; \mathbb{R}^2)$ whenever $\theta \in (0, 2\beta - 1)$, (2.36) together with (2.32) entails (2.33). $\square$

Finally, bounds for solutions in any spaces compatible with the smoothness of $\partial\Omega$ and $\phi$ can be achieved, at least away from the initial time. For our subsequent analysis, the following result in this direction will be sufficient.

**Lemma 2.9** *There exists $C > 0$ such that for all $\kappa \in (-1, 1)$,*

$$\|n^{(\kappa)}(\cdot, t)\|_{C^1(\bar{\Omega})} \le C \qquad \text{for all } t > 1 \tag{2.37}$$

*as well as*

$$\|c^{(\kappa)}(\cdot, t)\|_{C^2(\bar{\Omega})} \le C \qquad \text{for all } t > 1. \tag{2.38}$$

Proof. We first employ a known result on Hölder regularity in scalar parabolic equations ([24]) to obtain $\theta_1 \in (0, 1)$ and $C_1 > 0$ such that for $(n, c, u) := (n^{(\kappa)}, c^{(\kappa)}, u^{(\kappa)})$ with arbitrary $\kappa \in (-1, 1)$ we have

$$\|n\|_{C^{\theta_1, \frac{\theta_1}{2}}(\bar{\Omega} \times [t, t+1])} \le C_1 \qquad \text{for all } t > \frac{1}{2}, \tag{2.39}$$

because combining Lemma 2.7 with Lemma 2.8 and Lemma 2.6 shows that for each $p \in (1, \infty)$ we can find $C_2(p) > 0$ such that $f := n\nabla c + nu$ satisfies $\|f\|_{L^p(\Omega)} \le C_2(p)$ for all $t > 0$. Next, by a standard result on maximal Sobolev regularity for the Neumann problem associated with the inhomogeneous linear heat equation $c_t = \Delta c + g$, $g := -nc - u \cdot \nabla c$ ([15]), we infer that for any $p \in (1, \infty)$ we can pick $C_3(p) > 0$ fulfilling

$$\int_t^{t+1} \Big\{ \|c(\cdot, t)\|^p_{W^{2,p}(\Omega)} + \|c_t(\cdot, t)\|^p_{L^p(\Omega)} \Big\} ds \le C_3(p) \qquad \text{for all } t > \frac{1}{2}, \tag{2.40}$$

because from Lemma 2.7, Lemma 2.8 and Lemma 2.6 we know that for any such $p$ there exists $C_4(p) > 0$ such that $\|g\|_{L^p(\Omega)} \le C_4(p)$ for all $t > 0$. In particular, according to a known embedding property ([1]), (2.40) entails the existence of $\theta_2 \in (0, 1)$ and $C_5 > 0$ fulfilling

$$\|c\|_{C^{1+\theta_2, \theta_2}(\bar{\Omega} \times [t, t+1])} \le C_5 \qquad \text{for all } t > \frac{1}{2}, \tag{2.41}$$

which together with (2.39) and Lemma 2.8 yields $\theta_3 \in (0, 1)$ and $C_6 > 0$ such that $\|f\|_{C^{\theta_3, \frac{\theta_3}{2}}(\bar{\Omega} \times [t, t+1])} \le C_6$ for all $t > \frac{1}{2}$. Therefore, a well-known result on gradient Hölder regularity in general quasilinear parabolic equations ([22]) becomes applicable so as to assert that

$$\|n\|_{C^{1+\theta_4, \frac{1+\theta_4}{2}}(\bar{\Omega} \times [t, t+1])} \le C_7 \qquad \text{for all } t > 1 \tag{2.42}$$

with some $\theta_4 \in (0, 1)$ and $C_7 > 0$. Finally, in view of (2.39), (2.41) and Lemma 2.8 we now know that there exist $\theta_5 \in (0, 1)$ and $C_8 > 0$ such that $\|g\|_{C^{\theta_5, \frac{\theta_5}{2}}(\bar{\Omega} \times [t, t+1])} \le C_8$ for all $t > \frac{1}{2}$, by means of classical parabolic Schauder theory ([20]) implying that actually

$$\|c\|_{C^{2+\theta_6, 1+\frac{\theta_6}{2}}(\bar{\Omega} \times [t, t+1])} \le C_9 \qquad \text{for all } t > 1 \tag{2.43}$$

with some $\theta_6 \in (0, 1)$ and $C_9 > 0$. Whereas (2.42) entails (2.37), from (2.43) we obtain (2.38). $\square$

## 3 Uniform decay of $c^{(\kappa)}$

The purpose of this section consists in deriving a statement on temporal decay of $c^{(\kappa)}$ which will yet be qualitative in that no rate is provided, but which is uniform not only with respect to $x \in \Omega$ but also with regard to $\kappa \in (-1,1)$. In comparison to the original introduction of the strategy pursued here ([38]), our argument yields both a refinement which shows the desired independence of $\kappa \in (-1,1)$, as well as a compactification in presentation.

We first derive, in a way essentially independent from all our above results, the following very weak decay information.

**Lemma 3.1** *For all $\varepsilon > 0$ one can find $T = T(\varepsilon) > 0$ with the property that for all $\kappa \in (-1,1)$ there exists $t_0 = t_0(\varepsilon, \kappa) \in (0,T)$ such that*

$$\int_{t_0}^{t_0+1} \int_\Omega \Big\{ n^{(\kappa)} c^{(\kappa)} + |\nabla c^{(\kappa)}|^2 \Big\} < \varepsilon. \tag{3.1}$$

Proof. Omitting the index $\kappa$ again, we multiply the second equation in (1.3) by 1 and $c$, respectively, to see upon integration that

$$\begin{aligned} \int_0^T \int_\Omega \Big\{ nc + |\nabla c|^2 \Big\} &= \int_\Omega c_0 + \frac{1}{2} \int_\Omega c_0^2 - \int_\Omega c(\cdot,T) - \frac{1}{2} \int_\Omega c^2(\cdot,T) - \int_0^T \int_\Omega nc^2 \\ &\le C_1 := \int_\Omega c_0 + \frac{1}{2} \int_\Omega c_0^2 \qquad \text{for all } T > 0, \end{aligned}$$

and that hence

$$\int_0^\infty \int_\Omega \Big\{ nc + |\nabla c|^2 \Big\} \le C_1. \tag{3.2}$$

Thus, if given $\varepsilon > 0$ we fix an integer $k = k(\varepsilon) \ge 1$ such that $k > \frac{C_1}{\varepsilon}$, then

$$\varepsilon > \frac{C_1}{k} \ge \frac{1}{k} \int_0^k \int_\Omega \Big\{ nc + |\nabla c|^2 \Big\} = \frac{1}{k} \sum_{j=0}^{k-1} \int_j^{j+1} \int_\Omega \Big\{ nc + |\nabla c|^2 \Big\} \ge \min_{j \in \{0,\dots,k-1\}} \int_j^{j+1} \int_\Omega \Big\{ nc + |\nabla c|^2 \Big\},$$

so that we can pick $j_0 = j_0(\varepsilon, \kappa) \in \{0, ..., k-1\}$ such that $\int_{j_0}^{j_0+1} \int_\Omega \{nc + |\nabla c|^2\} < \varepsilon$, thus implying the claimed conclusion if we let $T(\varepsilon) := k(\varepsilon)$ and $t_0(\varepsilon,\kappa) := j_0(\varepsilon,\kappa)$. $\square$

Now by including some of the regularity information gained above, we can indeed assert the following doubly uniform decay property.

**Lemma 3.2** *We have*

$$\sup_{\kappa \in (-1,1)} \|c^{(\kappa)}(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \qquad \textit{as } t \to \infty. \tag{3.3}$$

Proof. The proof will be divided into two steps.

Step 1. We first claim that

$$\sup_{\kappa \in (-1,1)} \|c^{(\kappa)}(\cdot,t)\|_{L^1(\Omega)} \to 0 \qquad \text{as } t \to \infty. \tag{3.4}$$

To see this, we employ a Poincaré inequality and recall Lemma 2.7 to find $C_1 > 0$ and $C_2 > 0$ such that writing $\overline{\varphi} := \frac{1}{|\Omega|}\int_\Omega \varphi$ for $\varphi \in L^1(\Omega)$ we have

$$\int_\Omega |\varphi - \overline{\varphi}|^2 \le C_1 \int_\Omega |\nabla\varphi|^2 \qquad \text{for all } \varphi \in W^{1,2}(\Omega), \tag{3.5}$$

and such that for all $\kappa \in (-1,1)$,

$$\|n^{(\kappa)}(\cdot,t)\|_{L^\infty(\Omega)} \le C_2 \qquad \text{for all } t > 0. \tag{3.6}$$

Then given $\varepsilon > 0$, we fix $\delta > 0$ small enough fulfilling

$$\delta + \sqrt{C_1}C_2\sqrt{|\Omega|}\sqrt{\delta} < \overline{n}_0\varepsilon, \tag{3.7}$$

and thereafter we apply Lemma 3.1 to choose $T = T(\delta) > 0$ such that for all $\kappa \in (-1,1)$ there exists $t_0 = t_0(\delta,\kappa) \in (0,T)$ satisfying

$$\int_{t_0}^{t_0+1}\int_\Omega \Big\{ n^{(\kappa)}c^{(\kappa)} + |\nabla c^{(\kappa)}|^2 \Big\} < \delta. \tag{3.8}$$

Then by (2.2), the Cauchy-Schwarz inequality, (3.5), (3.6) and (3.7), we can estimate

$$\begin{aligned}
\overline{n}_0 \int_{t_0}^{t_0+1}\int_\Omega c^{(\kappa)} &= \int_{t_0}^{t_0+1}\int_\Omega n^{(\kappa)}(x,t)\overline{c^{(\kappa)}(\cdot,t)}dxdt \\
&= \int_{t_0}^{t_0+1}\int_\Omega n^{(\kappa)}(x,t)c^{(\kappa)}(x,t)dxdt - \int_{t_0}^{t_0+1}\int_\Omega n^{(\kappa)}(x,t)\Big(c^{(\kappa)}(x,t) - \overline{c^{(\kappa)}(\cdot,t)}\Big)dxdt \\
&\le \int_{t_0}^{t_0+1}\int_\Omega n^{(\kappa)}(x,t)c^{(\kappa)}(x,t)dxdt \\
&\quad + \Big\{\int_{t_0}^{t_0+1}\int_\Omega |n^{(\kappa)}(x,t)|^2 dxdt\Big\}^{\frac12} \cdot \Big\{\int_{t_0}^{t_0+1}\int_\Omega \Big(c^{(\kappa)}(x,t) - \overline{c^{(\kappa)}(\cdot,t)}\Big)^2 dxdt\Big\}^{\frac12} \\
&\le \int_{t_0}^{t_0+1}\int_\Omega n^{(\kappa)}(x,t)c^{(\kappa)}(x,t)dxdt \\
&\quad + \sqrt{C_1}\Big\{\int_{t_0}^{t_0+1}\int_\Omega |n^{(\kappa)}(x,t)|^2 dxdt\Big\}^{\frac12} \cdot \Big\{\int_{t_0}^{t_0+1}\int_\Omega |\nabla c^{(\kappa)}(x,t)|^2 dxdt\Big\}^{\frac12} \\
&\le \delta + \sqrt{C_1}C_2\sqrt{|\Omega|}\cdot\sqrt{\delta} \\
&< \overline{n}_0\varepsilon \qquad \text{for all } \kappa \in (-1,1).
\end{aligned}$$

As $\frac{d}{dt}\int_\Omega c^{(\kappa)} = -\int_\Omega n^{(\kappa)}c^{(\kappa)} \le 0$ for all $t > 0$ by (1.3), this entails that for all $\kappa \in (-1,1)$ and each $t > T+1$,

$$\int_\Omega c^{(\kappa)}(\cdot,t) \le \int_\Omega c^{(\kappa)}(\cdot,t_0+1) \le \int_{t_0}^{t_0+1}\int_\Omega c^{(\kappa)} < \varepsilon,$$

because any such $t$ satisfies $t > t_0 + 1$.

Step 2. We next make sure that (3.3) holds.

To this end, we only need to observe that by Lemma 2.9 there exists $C_3 > 0$ such that

$$\|c^{(\kappa)}(\cdot,t)\|_{C^1(\bar\Omega)} \le C_3 \qquad \text{for all } t > 1,$$

and that due to the Gagliardo-Nirenberg inequality we can find $C_4 > 0$ fulfilling

$$\|\varphi\|_{L^\infty(\Omega)} \le C_4 \|\varphi\|_{C^1(\bar\Omega)}^{\frac{2}{3}} \|\varphi\|_{L^1(\Omega)}^{\frac{1}{3}} \qquad \text{for all } \varphi \in C^1(\bar\Omega).$$

Therefore, namely,

$$\|c^{(\kappa)}(\cdot,t)\|_{L^\infty(\Omega)} \le C_3^{\frac{2}{3}} C_4 \|c^{(\kappa)}(\cdot,t)\|_{L^1(\Omega)}^{\frac{1}{3}} \qquad \text{for all } t > 1,$$

so that (3.3) results from (3.4). □

# 4 Uniformly exponential stabilization

Our next goal will be to make sure that as a consequence of Lemma 3.2 when combined with the boundedness properties from Section 2, solutions stabilize toward the spatially homogeneous equilibrium $(\overline{n}_0, 0, 0)$ at an exponential rate, again even uniformly with respect to $\kappa \in (-1,1)$. We thereby generalize the results both on mere convergence ([38]) and on exponential stabilization rates ([45]) previously obtained for the particular case $\kappa = 1$. Unlike the strategy in [38] which was essentially based on the observation that whenever $p > 1$, $\int_\Omega \frac{n^p}{(\delta - c)^\gamma}$ acts as a genuine energy functional for (1.3) for $t > t_0(p)$ if $\delta = \delta(p) > 0$ as well as $\gamma = \gamma(p) > 0$ is chosen appropriately, our approach is much more direct in that it mainly relies on an analysis of the functionals $\int_\Omega |n^{(\kappa)} - \overline{n}_0|^2$ and $\int_\Omega |u^{(\kappa)}|^2$.

We first state the following implication, to be used in both Lemma 4.4 and Lemma 4.7 below, of the uniform boundedness property of $n^{(\kappa)}$ asserted in Lemma 2.7.

**Lemma 4.1** *There exists $C > 0$ such that for any $\kappa \in (-1,1)$ we have*

$$\frac{d}{dt}\int_\Omega |n^{(\kappa)}(\cdot,t) - \overline{n}_0|^2 + \frac{1}{C}\int_\Omega |n^{(\kappa)}(\cdot,t) - \overline{n}_0|^2 \le C \int_\Omega |\nabla c^{(\kappa)}(\cdot,t)|^2 \qquad \textit{for all } t > 0. \tag{4.1}$$

Proof. We multiply the first equation in (1.3) by $n^{(\kappa)} - \overline{n}_0$ and integrate by parts to see using Young's inequality that since $\frac{d}{dt}\int_\Omega n = 0$,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |n^{(\kappa)} - \overline{n}_0|^2 + |\nabla n^{(\kappa)}|^2 = \int_\Omega n^{(\kappa)} \nabla n^{(\kappa)} \cdot \nabla c^{(\kappa)} \le \frac{1}{2}\int_\Omega |\nabla n^{(\kappa)}|^2 + \frac{1}{2}\int_\Omega |n^{(\kappa)}|^2 |\nabla c^{(\kappa)}|^2$$

for all $t > 0$, so that

$$\frac{d}{dt}\int_\Omega |n^{(\kappa)} - \overline{n}_0|^2 + \int_\Omega |\nabla n^{(\kappa)}|^2 \le \int_\Omega |n^{(\kappa)}|^2 |\nabla c^{(\kappa)}|^2 \le C_1 \int_\Omega |\nabla c^{(\kappa)}|^2 \qquad \text{for all } t > 0 \tag{4.2}$$

with $C_1 := \sup_{\kappa' \in (-1,1)} \|n^{(\kappa')}\|_{L^\infty(\Omega \times (0,\infty))}^2$ being finite according to Lemma 2.7. Since a Poincaré inequality provides $C_2 > 0$ such that

$$\int_\Omega |n^{(\kappa)} - \overline{n}_0|^2 \le C_2 \int_\Omega |\nabla n^{(\kappa)}|^2 \qquad \text{for all } t > 0,$$

from (4.2) we obtain (4.1) on choosing $C := \max\{C_1, C_2\}$. $\Box$

Now a first $\kappa$-independent estimate for the right-hand side herein can directly be obtained from Lemma 3.2:

**Lemma 4.2** *We have*

$$\sup_{\kappa\in(-1,1)} \int_t^\infty \int_\Omega |\nabla c^{(\kappa)}|^2 \to 0 \qquad \textit{as } t\to\infty. \tag{4.3}$$

PROOF. On testing the second equation in (1.3) by $c^{(\kappa)}$ we obtain

$$\begin{aligned} \int_t^T \int_\Omega |\nabla c^{(\kappa)}|^2 &= \frac{1}{2}\int_\Omega |c^{(\kappa)}(\cdot,t)|^2 - \frac{1}{2}\int_\Omega |c^{(\kappa)}(\cdot,T)|^2 - \int_t^T \int_\Omega n^{(\kappa)} |c^{(\kappa)}|^2 \\ &\le \frac{|\Omega|}{2} \cdot \sup_{\kappa'\in(-1,1)} \|c^{(\kappa')}(\cdot,t)\|^2_{L^\infty(\Omega)} \qquad \text{for all } t>0 \text{ and } T>t, \end{aligned}$$

so that (4.3) results from Lemma 3.2. $\Box$

In Lemma 4.4 we shall need the following statement on uniform decay in families of linearly dampened ODIs, an elementary proof of which can be obtained by straightforward adaptation of the arguments detailed in [9, Lemma 4.6] for the case of a single inequality.

**Lemma 4.3** *Let $I$ be any set and $\lambda > 0$, and for each $\iota \in I$ let $y_\iota \in C^0([0,\infty)) \cap C^1((0,\infty))$ and $f_\iota \in C^0((0,\infty))$ be nonnegative and such that*

$$y_\iota'(t) + \lambda y_\iota(t) \le f_\iota(t) \qquad \textit{for all } t>0 \tag{4.4}$$

*and*

$$\sup_{\iota\in I} y_\iota(0) < \infty, \tag{4.5}$$

*and such that*

$$\sup_{\iota\in I} \|f_\iota\|_{L^\infty((0,\infty))} < \infty \tag{4.6}$$

*as well as*

$$\sup_{\iota\in I} \int_t^{t+1} f_\iota(s)ds \to 0 \qquad \textit{as } t\to\infty. \tag{4.7}$$

*Then*

$$\sup_{\iota\in I} y_\iota(t) \to 0 \qquad \textit{as } t\to\infty. \tag{4.8}$$

We are now in the position to show that also the first solution component stabilizes uniformly with respect to $x \in \Omega$ and $\kappa \in (-1,1)$.

**Lemma 4.4** *The solutions of (1.3) satisfy*

$$\sup_{\kappa\in(-1,1)} \|n^{(\kappa)}(\cdot,t) - \overline{n}_0\|_{L^\infty(\Omega)} \to 0 \qquad \textit{as } t\to\infty. \tag{4.9}$$

PROOF. In view of Lemma 4.3, it follows from Lemma 4.1, Lemma 4.2 and the uniform bound for $\nabla c^{(\kappa)}$ in $L^\infty((0,\infty);L^2(\Omega))$ provided by Lemma 2.6 that

$$\sup_{\kappa\in(-1,1)} \int_\Omega |n^{(\kappa)}(\cdot,t)-\overline{n}_0|^2 \to 0 \qquad \text{as } t\to\infty. \tag{4.10}$$

Since from the Gagliardo-Nirenberg inequality we infer the existence of $C_1>0$ fulfilling

$$\begin{aligned} \|n^{(\kappa)}-\overline{n}_0\|^2_{L^\infty(\Omega)} &\le C_1\|n^{(\kappa)}-\overline{n}_0\|_{C^1(\bar\Omega)}\|n^{(\kappa)}-\overline{n}_0\|_{L^2(\Omega)} \\ &\le C_1\Big(\|n^{(\kappa)}\|_{C^1(\bar\Omega)}+\overline{n}_0\Big)\|n^{(\kappa)}-\overline{n}_0\|_{L^2(\Omega)} \qquad \text{for all } t>1, \end{aligned}$$

on combining (4.10) with Lemma 2.9 we directly obtain (4.9). $\square$

Making essential use of the uniformity in the above statement with respect to $x\in\Omega$, by means of a straightforward comparison argument we can derive the following improvement of Lemma 3.2 which now contains an exponential rate of convergence.

**Lemma 4.5** *There exist $\mu>0$ and $C>0$ such that for any choice of $\kappa\in(-1,1)$ we have*

$$\|c^{(\kappa)}(\cdot,t)\|_{L^\infty(\Omega)} \le Ce^{-\mu t} \qquad \textit{for all } t>0. \tag{4.11}$$

PROOF. We first apply Lemma 4.4 to obtain $t_0>0$ such that for each $\kappa\in(-1,1)$,

$$n^{(\kappa)}(x,t)\ge C_1 := \frac{\overline{n}_0}{2} \qquad \text{for all } x\in\Omega \text{ and } t>t_0.$$

Therefore,

$$c_t^{(\kappa)} \le \Delta c^{(\kappa)} - C_1 c^{(\kappa)} \qquad \text{in } \Omega\times(t_0,\infty),$$

so that by means of the comparison principle and (2.3) we easily infer that

$$c^{(\kappa)}(\cdot,t) \le \|c^{(\kappa)}(\cdot,t_0)\|_{L^\infty(\Omega)}e^{-C_1(t-t_0)} \le \|c_0\|_{L^\infty(\Omega)}e^{-C_1(t-t_0)} \qquad \text{for all } t>t_0,$$

and hence again (2.3) asserts that (4.11) is valid actually for all $t>0$ if we let $C:=\|c_0\|_{L^\infty(\Omega)}e^{C_1t_0}$ and $\mu:=\frac{\overline{n}_0}{2}$, noting that $\mu$ is positive according to (1.6). $\square$

By interpolation with Lemma 2.9, this entails exponential decay also of $\nabla c^{(\kappa)}$ in the following sense.

**Lemma 4.6** *For all $p>1$, there exist $\mu>0$ and $C>0$ such that*

$$\|c^{(\kappa)}(\cdot,t)\|_{W^{1,p}(\Omega)} \le Ce^{-\mu t} \qquad \textit{for all } t>0 \tag{4.12}$$

*whenever $\kappa\in(-1,1)$.*

PROOF. Assuming without loss of generality that $p>2$, by the Gagliardo-Nirenberg inequality we obtain $C_1>0$ such that

$$\|c^{(\kappa)}(\cdot,t)\|_{W^{1,p}(\Omega)} \le C_1\|c^{(\kappa)}(\cdot,t)\|^{\frac{p-2}{2p}}_{C^2(\bar\Omega)}\|c^{(\kappa)}(\cdot,t)\|^{\frac{p+2}{2p}}_{L^\infty(\Omega)} \qquad \text{for all } t>0.$$

Therefore, Lemma 4.5 in conjunction with Lemma 2.9 yields (4.12). $\square$

This in turn improves our knowledge on temporal decay of the integral on the right of (4.1), as compared to the outcome from Lemma 4.2. Using this, we obtain that also the stabilization asserted by Lemma 4.4 occurs at an exponential rate.

**Lemma 4.7** *There exist $\mu > 0$ and $C > 0$ with the property that for each $\kappa \in (-1, 1)$,*

$$\|n^{(\kappa)}(\cdot, t) - \overline{n}_0\|_{L^\infty(\Omega)} \le Ce^{-\mu t} \qquad \textit{for all } t > 0. \tag{4.13}$$

PROOF. In view of Lemma 4.6, Lemma 4.1 says that with some $C_1 > 0$, $C_2 > 0$ and $\mu_1 \in (0, C_1)$, the function $y \in C^0([0,\infty)) \cap C^1((0,\infty))$ defined by $y(t) := \int_\Omega |n^{(\kappa)}(\cdot, t) - \overline{n}_0|^2$, $t \ge 0$, satisfies

$$y'(t) + C_1 y(t) \le C_2 e^{-\mu_1 t} \qquad \text{for all } t > 0.$$

On integration, this shows that writing $C_3 := \int_\Omega |n_0 - \overline{n}_0|^2$ we have

$$\begin{aligned} y(t) &\le C_3 e^{-C_1 t} + C_2 \int_0^t e^{-C_1(t-s)} e^{-\mu_1 s} ds \\ &= C_3 e^{-C_1 t} + \frac{C_2}{C_1 - \mu_1}(e^{-\mu_1 t} - e^{-C_1 t}) \\ &\le \Big(C_3 + \frac{C_2}{C_1 - \mu_1}\Big) e^{-\mu_1 t} \qquad \text{for all } t > 0, \end{aligned}$$

because $\mu_1 < C_1$. Again by interpolation using Lemma 2.9, this implies (4.13). □

The latter, finally, implies exponential decay also in the fluid component.

**Lemma 4.8** *There exist $\mu > 0$ and $C > 0$ such that for any $\kappa \in (-1, 1)$,*

$$\|u^{(\kappa)}(\cdot, t)\|_{L^\infty(\Omega)} \le Ce^{-\mu t} \qquad \textit{for all } t > 0. \tag{4.14}$$

PROOF. Using that by the Poincaré inequality there exists $C_1 > 0$ such that

$$\int_\Omega |\nabla u^{(\kappa)}|^2 \ge C_1 \int_\Omega |u^{(\kappa)}|^2 \qquad \text{for all } t > 0,$$

on testing the third equation in (1.3) by $u^{(\kappa)}$ we see that writing $C_2 := \|\nabla\phi\|_{L^\infty(\Omega)}$, thanks to Young's inequality we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |u^{(\kappa)}|^2 + \frac{C_1}{2}\int_\Omega |u^{(\kappa)}|^2 &\le \frac{1}{2}\frac{d}{dt}\int_\Omega |u^{(\kappa)}|^2 + \frac{1}{2}\int_\Omega |\nabla u^{(\kappa)}|^2 \\ &= -\frac{1}{2}\int_\Omega |\nabla u^{(\kappa)}|^2 + \int_\Omega (n^{(\kappa)} - \overline{n}_0) u^{(\kappa)} \cdot \nabla\phi \\ &\le -\frac{1}{2}\int_\Omega |\nabla u^{(\kappa)}|^2 + C_2\sqrt{|\Omega|}\|n^{(\kappa)} - \overline{n}_0\|_{L^\infty(\Omega)}\|u^{(\kappa)}\|_{L^2(\Omega)} \\ &\le -\frac{1}{2}\int_\Omega |\nabla u^{(\kappa)}|^2 + \frac{C_2\sqrt{|\Omega|}}{\sqrt{C_1}}\|n^{(\kappa)} - \overline{n}_0\|_{L^\infty(\Omega)}\|\nabla u^{(\kappa)}\|_{L^2(\Omega)} \\ &\le \frac{C_2^2|\Omega|}{2C_1} \cdot \|n^{(\kappa)} - \overline{n}_0\|_{L^\infty(\Omega)}^2 \qquad \text{for all } t > 0. \end{aligned}$$

Due to Lemma 4.7, we thus obtain $C_3 > 0$ and $\mu_1 \in (0, C_1)$ such that $y(t) := \int_\Omega |u^{(\kappa)}(\cdot, t)|^2$, $t \ge 0$, satisfies

$$y'(t) + C_1 y(t) \le C_3 e^{-\mu_1 t} \qquad \text{for all } t > 0,$$

from which by integration we infer that

$$y(t) \le C_4 e^{-C_1 t} + C_3 \int_0^t e^{-C_1(t-s)} e^{-\mu_1 s} ds \le C_5 e^{-\mu_1 t} \qquad \text{for all } t > 0$$

with $C_4 := \int_\Omega |u_0|^2$ and $C_5 := C_4 + \frac{C_3}{C_1 - \mu_1}$. We now recall Lemma 2.8 to find $C_6 > 0$ such that

$$\|A^\alpha u^{(\kappa)}(\cdot,t)\|_{L^2(\Omega)} \le C_6 \qquad \text{for all } t > 0,$$

and fixing an arbitrary $\beta \in (\frac{1}{2}, \alpha)$ we apply a known interpolation result ([10]) to obtain $C_7 > 0$ fulfilling

$$\|A^\beta u^{(\kappa)}(\cdot,t)\|_{L^2(\Omega)} \le C_7 \|A^\alpha u^{(\kappa)}(\cdot,t)\|_{L^2(\Omega)}^{\frac{\beta}{\alpha}} \|u^{(\kappa)}(\cdot,t)\|_{L^2(\Omega)}^{1-\frac{\beta}{\alpha}} \le C_7 C_6^{\frac{\beta}{\alpha}} C_5^{1-\frac{\beta}{\alpha}} e^{-(1-\frac{\beta}{\alpha})\mu_1 t} \qquad \text{for all } t > 0.$$

As $D(A^\beta) \hookrightarrow L^\infty(\Omega; \mathbb{R}^2)$ ([16]), this establishes (4.14). $\square$

# 5 Convergence as $\kappa \to 0$. Proof of Theorem 1.1

We next return to our original purpose by deriving estimates for the differences addressed in Theorem 1.1. In order to keep readability, throughout the sequel we abbreviate

$$\widehat{n} := n^{(\kappa)} - n^{(0)}, \qquad \widehat{c} := c^{(\kappa)} - c^{(0)}, \qquad \widehat{u} := u^{(\kappa)} - u^{(0)} \qquad \text{and} \qquad \widehat{P} := P^{(\kappa)} - P^{(0)}, \tag{5.1}$$

for $\kappa \in (-1, 1)$, and observe that according to (1.3), (1.4) and (1.5) we have

$$\begin{cases} \widehat{n}_t = \Delta \widehat{n} - \nabla \cdot (\widehat{n} \nabla c^{(\kappa)}) - \nabla \cdot (n^{(0)} \nabla \widehat{c}) - u^{(\kappa)} \cdot \nabla \widehat{n} - \widehat{u} \cdot \nabla (n^{(0)} - \overline{n}_0), & x \in \Omega, \ t > 0, \\ \widehat{c}_t = \Delta \widehat{c} - n^{(\kappa)} \widehat{c} - \widehat{n} c^{(0)} - u^{(\kappa)} \cdot \nabla \widehat{c} - \widehat{u} \cdot \nabla c^{(0)}, & x \in \Omega, \ t > 0, \\ \widehat{u}_t = \Delta \widehat{u} - \nabla \widehat{P} + \widehat{n} \nabla \phi - \kappa (u^{(\kappa)} \cdot \nabla) u^{(\kappa)}, & x \in \Omega, \ t > 0, \\ \nabla \cdot \widehat{u} = 0, & x \in \Omega, \ t > 0, \end{cases} \tag{5.2}$$

and

$$\frac{\partial \widehat{n}}{\partial \nu} = 0, \quad \frac{\partial \widehat{c}}{\partial \nu} = 0 \quad \text{and} \quad \widehat{u} = 0, \qquad x \in \partial\Omega, \ t > 0,$$

as well as

$$\widehat{n}(x,0) = 0, \quad \widehat{c}(x,0) = 0 \quad \text{and} \quad \widehat{u}(x,0) = 0, \qquad x \in \Omega.$$

As a preparation for our analysis of $(\widehat{n}, \widehat{c}, \widehat{u})$, let us separately state the following auxiliary lemma on exponential decay in a linear absorptive ODI with certain exponentially decreasing perturbations, to be used in both Lemma 5.6 and Lemma 5.9 below.

**Lemma 5.1** *Let $a > 0, b > 0, \mu_1 > 0, \mu_2 > 0$ and $\mu_3 \in (0, \mu_1)$, and suppose that $y \in C^0([0,\infty)) \cap C^1((0,\infty))$ is a nonnegative function satisfying $y(0) = 0$ and*

$$y'(t) + \mu_1 y(t) \le a e^{-\mu_2 t} y(t) + b e^{-\mu_3 t} \qquad \textit{for all } t > 0. \tag{5.3}$$

*Then*

$$y(t) \le \frac{b}{\mu_1 - \mu_3} e^{\frac{a}{\mu_2}} e^{-\mu_3 t} \qquad \textit{for all } t > 0. \tag{5.4}$$

PROOF. As $y(0)=0$, an integration of (5.3) shows that

$$y(t) \le b\int_0^t \exp\left\{\int_s^t \Big(ae^{-\mu_2\sigma}-\mu_1\Big)d\sigma\right\}\cdot e^{-\mu_3 s}ds \qquad \text{for all } t>0,$$

where

$$\begin{aligned}\int_s^t \Big(ae^{-\mu_2\sigma}-\mu_1\Big)d\sigma &= \frac{a}{\mu_2}\Big(e^{-\mu_2 s}-e^{-\mu_2 t}\Big)-\mu_1(t-s)\\ &\le \frac{a}{\mu_2}-\mu_1(t-s) \qquad \text{for all } t>0 \text{ and each } s\in(0,t).\end{aligned}$$

Therefore, thanks to our hypothesis $\mu_1>\mu_3$ we obtain

$$\begin{aligned}y(t) &\le be^{\frac{a}{\mu_2}}\cdot\int_0^t e^{-\mu_1(t-s)}e^{-\mu_3 s}ds\\ &= \frac{b}{\mu_1-\mu_3}e^{\frac{a}{\mu_2}}\cdot\Big(e^{-\mu_3 t}-e^{-\mu_1 t}\Big) \qquad \text{for all } t>0,\end{aligned}$$

which implies (5.4). □

## 5.1 Convergence with respect to spatial $L^2$ norms

Our first crucial step toward Theorem 1.1 will consist in the derivation of a corresponding estimate for $(\widehat{n},\widehat{c},\widehat{u})$ with respect to the norm in $(L^2(\Omega))^4$. This will be accomplished in Lemma 5.5 and Lemma 5.6 on the basis of an ODI of the above structure for

$$\int_\Omega \widehat{n}^2+k\int_\Omega \widehat{c}^2+l\int_\Omega|\widehat{u}|^2, \qquad t\ge 0,$$

with suitably chosen $k>0$ and $l>0$. In order to motivate our selections of these parameters, we separately state the respective results of the three associated testing procedures in the following three lemmata, the first of which is concerned with the first solution component.

**Lemma 5.2** *There exist $\mu>0$ and $C>0$ such that for each $\kappa\in(-1,1)$,*

$$\frac{d}{dt}\int_\Omega \widehat{n}^2+\int_\Omega|\nabla\widehat{n}|^2 \le C\int_\Omega|\nabla\widehat{c}|^2+Ce^{-\mu t}\cdot\left\{\int_\Omega\widehat{n}^2+\int_\Omega|\widehat{u}|^2\right\} \qquad \textit{for all } t>0. \tag{5.5}$$

PROOF. We multiply the first equation in (5.2) by $\widehat{n}$ and integrate by parts over $\Omega$ to obtain

$$\frac12\frac{d}{dt}\int_\Omega\widehat{n}^2+\int_\Omega|\nabla\widehat{n}|^2=\int_\Omega\widehat{n}\nabla\widehat{n}\cdot\nabla c^{(\kappa)}+\int_\Omega n^{(0)}\nabla\widehat{n}\cdot\nabla\widehat{c}+\int_\Omega(n^{(0)}-\overline{n}_0)\widehat{u}\cdot\nabla\widehat{n} \qquad \text{for all } t>0, \tag{5.6}$$

where we have made use of the fact that $\nabla\cdot u^{(\kappa')}\equiv 0$ for all $\kappa'\in(-1,1)$. On the right-hand side herein, employing Young's inequality we see that

$$\int_\Omega\widehat{n}\nabla\widehat{n}\cdot\nabla c^{(\kappa)}\le\frac18\int_\Omega|\nabla\widehat{n}|^2+2\int_\Omega\widehat{n}^2|\nabla c^{(\kappa)}|^2 \qquad \text{for all } t>0, \tag{5.7}$$

where by the Cauchy-Schwarz inequality and the Gagliardo-Nirenberg inequality, we find that

$$\begin{aligned} 2\int_\Omega \widehat{n}^2|\nabla c^{(\kappa)}|^2 &\le 2\|\widehat{n}\|_{L^4(\Omega)}^2\|\nabla c^{(\kappa)}\|_{L^4(\Omega)}^2 \\ &\le C_1\|\nabla\widehat{n}\|_{L^2(\Omega)}\|\widehat{n}\|_{L^2(\Omega)}\|\nabla c^{(\kappa)}\|_{L^4(\Omega)}^2 \qquad \text{for all } t>0 \end{aligned} \tag{5.8}$$

with some $C_1 > 0$, bearing in mind that according to (2.2) we know that

$$\int_\Omega \widehat{n} = 0 \qquad \text{for all } t > 0.$$

Now since Lemma 4.6 provides $\mu_1 > 0$ and $C_2 > 0$ fulfilling

$$\int_\Omega |\nabla c^{(\kappa)}|^4 \le C_2 e^{-\mu_1 t} \qquad \text{for all } t > 0,$$

again using Young's inequality in (5.8) we can proceed to estimate

$$\begin{aligned} C_1\|\nabla\widehat{n}\|_{L^2(\Omega)}\|\widehat{n}\|_{L^2(\Omega)}\|\nabla c^{(\kappa)}\|_{L^4(\Omega)}^2 &\le \frac{1}{8}\int_\Omega|\nabla\widehat{n}|^2 + 2C_1^2\|\widehat{n}\|_{L^2(\Omega)}^2\|\nabla c^{(\kappa)}\|_{L^4(\Omega)}^4 \\ &\le \frac{1}{8}\int_\Omega|\nabla\widehat{n}|^2 + 2C_1^2C_2e^{-\mu_1 t}\int_\Omega\widehat{n}^2 \qquad \text{for all } t>0, \end{aligned}$$

whence by (5.7),

$$\int_\Omega \widehat{n}\nabla\widehat{n}\cdot\nabla c^{(\kappa)} \le \frac{1}{4}\int_\Omega|\nabla\widehat{n}|^2 + 2C_1^2C_2e^{-\mu_1 t}\int_\Omega\widehat{n}^2 \qquad \text{for all } t>0. \tag{5.9}$$

Next, recalling that due to Lemma 2.7 there exists $C_3 > 0$ such that

$$\|n^{(0)}\|_{L^\infty(\Omega)} \le C_3 \qquad \text{for all } t > 0,$$

once more thanks to Young's inequality we see that the second summand on the right of (5.6) can be controlled according to

$$\begin{aligned} \int_\Omega n^{(0)}\nabla\widehat{n}\cdot\nabla\widehat{c} &\le \frac{1}{8}\int_\Omega|\nabla\widehat{n}|^2 + 2\int_\Omega|n^{(0)}|^2|\nabla\widehat{c}|^2 \\ &\le \frac{1}{8}\int_\Omega|\nabla\widehat{n}|^2 + 2C_3^2\int_\Omega|\nabla\widehat{c}|^2 \qquad \text{for all } t>0. \end{aligned} \tag{5.10}$$

Finally, since Lemma 4.7 yields $\mu_2 > 0$ and $C_4 > 0$ fulfilling

$$\|n^{(0)} - \overline{n}_0\|_{L^\infty(\Omega)} \le C_4e^{-\mu_2 t} \qquad \text{for all } t > 0,$$

by Young's inequality we obtain

$$\begin{aligned} \int_\Omega (n^{(0)}-\overline{n}_0)\widehat{u}\cdot\nabla\widehat{n} &\le \frac{1}{8}\int_\Omega|\nabla\widehat{n}|^2 + 2\int_\Omega|n^{(0)}-\overline{n}_0|^2|\widehat{u}|^2 \\ &\le \frac{1}{8}\int_\Omega|\nabla\widehat{n}|^2 + 2\|n^{(0)}-\overline{n}_0\|_{L^\infty(\Omega)}^2\int_\Omega|\widehat{u}|^2 \\ &\le \frac{1}{8}\int_\Omega|\nabla\widehat{n}|^2 + 2C_4^2e^{-2\mu_2 t}\int_\Omega|\widehat{u}|^2 \qquad \text{for all } t>0. \end{aligned}$$

Together with (5.9) and (5.10) inserted into (5.6), this shows that

$$\frac{d}{dt}\int_\Omega \widehat{n}^2 + \int_\Omega |\nabla\widehat{n}|^2 \le 4C_1^2C_2e^{-\mu_1 t}\int_\Omega \widehat{n}^2 + 4C_3^2\int_\Omega |\nabla\widehat{c}|^2 + 4C_4^2e^{-2\mu_2 t}\int_\Omega |\widehat{u}|^2 \qquad \text{for all } t>0$$

and thereby establishes (5.5). □

Fortunately, the first integral on the right of (5.5) appears as part of the dissipation rate in a corresponding inequality derived on testing the second equation in (5.2) against $\widehat{c}$.

**Lemma 5.3** *There exists $\mu > 0$ with the property that for any $\varepsilon > 0$ one can find $C(\varepsilon) > 0$ such that for all $\kappa \in (-1,1)$ we have*

$$\frac{d}{dt}\int_\Omega \widehat{c}^2 + \int_\Omega |\nabla\widehat{c}|^2 + \overline{n}_0\int_\Omega \widehat{c}^2 \le \varepsilon\int_\Omega |\nabla\widehat{u}|^2 + C(\varepsilon)e^{-\mu t}\cdot\left\{\int_\Omega \widehat{n}^2 + \int_\Omega \widehat{c}^2 + \int_\Omega |\widehat{u}|^2\right\} \qquad \text{for all } t>0. \tag{5.11}$$

Proof. Testing the second equation in (5.2) by $\widehat{c}$ we see that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \widehat{c}^2 + \int_\Omega |\nabla\widehat{c}|^2 = -\int_\Omega n^{(\kappa)}\widehat{c}^2 - \int_\Omega \widehat{n}c^{(0)}\widehat{c} - \int_\Omega \widehat{c}\widehat{u}\cdot\nabla c^{(0)} \qquad \text{for all } t>0, \tag{5.12}$$

again because $\nabla\cdot u^{(\kappa)} \equiv 0$. Here an appropriate absorptive term containing a genuine spatial $L^2$ norm of $\widehat{c}$ can be created by splitting

$$-\int_\Omega n^{(\kappa)}\widehat{c}^2 = -\overline{n}_0\int_\Omega \widehat{c}^2 - \int_\Omega (n^{(\kappa)} - \overline{n}_0)\widehat{c}^2 \qquad \text{for all } t>0,$$

where thanks to Lemma 4.7,

$$\|n^{(\kappa)} - \overline{n}_0\|_{L^\infty(\Omega)} \le C_1e^{-\mu_1 t} \qquad \text{for all } t>0$$

with some $\mu_1 > 0$ and $C_1 > 0$, so that

$$-\int_\Omega n^{(\kappa)}\widehat{c}^2 \le -\overline{n}_0\int_\Omega \widehat{c}^2 + C_1e^{-\mu_1 t}\int_\Omega \widehat{c}^2 \qquad \text{for all } t>0. \tag{5.13}$$

Next, in Lemma 4.5 we have seen that there exist $\mu_2 > 0$ and $C_2 > 0$ fulfilling

$$\|c^{(0)}\|_{L^\infty(\Omega)} \le C_2e^{-\mu_2 t} \qquad \text{for all } t>0,$$

by using Young's inequality we see that

$$\begin{aligned} -\int_\Omega \widehat{n}c^{(0)}\widehat{c} &\le \|c^{(0)}\|_{L^\infty(\Omega)}\cdot\left\{\frac{1}{2}\int_\Omega \widehat{n}^2 + \frac{1}{2}\int_\Omega \widehat{c}\right\} \\ &\le \frac{C_2}{2}e^{-\mu_2 t}\cdot\left\{\int_\Omega \widehat{n}^2 + \int_\Omega \widehat{c}^2\right\} \qquad \text{for all } t>0. \end{aligned} \tag{5.14}$$

For adequately treating the rightmost summand in (5.12), we first note that due to Lemma 4.6 we can find $\mu_3 > 0$ and $C_3 > 0$ such that

$$\|\nabla c^{(0)}\|_{L^4(\Omega)} \le C_3e^{-\mu_3 t} \qquad \text{for all } t>0,$$

whence by Young's inequality and the Cauchy-Schwarz inequality we infer that

$$
\begin{aligned}
-\int_\Omega \widehat{c}\widehat{u}\cdot\nabla c^{(0)} &\le \frac{\overline{n}_0}{2}\int_\Omega \widehat{c}^2 + \frac{1}{2\overline{n}_0}\int_\Omega |\widehat{u}|^2|\nabla c^{(0)}|^2 \\
&\le \frac{\overline{n}_0}{2}\int_\Omega \widehat{c}^2 + \frac{1}{2\overline{n}_0}\|\widehat{u}\|_{L^4(\Omega)}^2\|\nabla c^{(0)}\|_{L^4(\Omega)}^2 \\
&\le \frac{\overline{n}_0}{2}\int_\Omega \widehat{c}^2 + \frac{C_3^2}{2\overline{n}_0}e^{-2\mu_3 t}\|\widehat{u}\|_{L^4(\Omega)}^2 \qquad \text{for all } t>0.
\end{aligned}
$$

As in view of the inclusion $\widehat{u}(\cdot,t)\in W_0^{1,2}(\Omega;\mathbb{R}^2)$ for all $t>0$ the Gagliardo-Nirenberg inequality provides $C_4>0$ such that

$$
\|\widehat{u}\|_{L^4(\Omega)}^2 \le C_4\|\nabla\widehat{u}\|_{L^2(\Omega)}\|\widehat{u}\|_{L^2(\Omega)} \qquad \text{for all } t>0,
$$

by means of Young's inequality we thus infer that given $\varepsilon>0$ we have

$$
\begin{aligned}
-\int_\Omega \widehat{c}\widehat{u}\cdot\nabla c^{(0)} &\le \frac{\overline{n}_0}{2}\int_\Omega \widehat{c}^2 + \frac{C_3^2C_4}{2\overline{n}_0}e^{-2\mu_3 t}\|\nabla\widehat{u}\|_{L^2(\Omega)}\|\widehat{u}\|_{L^2(\Omega)} \\
&\le \frac{\overline{n}_0}{2}\int_\Omega \widehat{c}^2 + \frac{\varepsilon}{2}\int_\Omega |\nabla\widehat{u}|^2 + \frac{C_3^4C_4^2}{8\overline{n}_0^2\varepsilon}e^{-4\mu_3 t}\int_\Omega |\widehat{u}|^2 \qquad \text{for all } t>0.
\end{aligned}
$$

Along with (5.13) and (5.14), this shows that (5.12) entails (5.11). $\square$

Finally, the first summand on the right of (5.11) will be compensated by using the standard energy inequality associated with the Stokes subsystem of (5.2), in our framework leading to the following.

**Lemma 5.4** *There exist $\mu>0$ and $C>0$ such that for any $\kappa\in(-1,1)$,*

$$
\frac{d}{dt}\int_\Omega |\widehat{u}|^2 + \int_\Omega |\nabla\widehat{u}|^2 \le C\int_\Omega \widehat{n}^2 + C\kappa^2e^{-\mu t} \qquad \textit{for all } t>0. \tag{5.15}
$$

Proof. We use $\widehat{u}$ as a test function in the third equation in (5.2) to find on applying the Cauchy-Schwarz inequality that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\widehat{u}|^2 + \int_\Omega |\nabla\widehat{u}|^2 \\
=\ &\int_\Omega \widehat{n}\widehat{u}\cdot\nabla\phi - \kappa\int_\Omega \widehat{u}\cdot(u^{(\kappa)}\cdot\nabla)u^{(\kappa)} \\
\le\ &\|\widehat{u}\|_{L^2(\Omega)}\cdot\Big\{C_1\|\widehat{n}\|_{L^2(\Omega)} + |\kappa|\cdot\|(u^{(\kappa)}\cdot\nabla)u^{(\kappa)}\|_{L^2(\Omega)}\Big\} \quad \text{for all } t>0
\end{aligned} \tag{5.16}
$$

with $C_1:=\|\nabla\phi\|_{L^\infty(\Omega)}$. Here we observe that by the Poincaré inequality we can find $C_2>0$ such that

$$
\|\widehat{u}\|_{L^2(\Omega)} \le C_2\|\nabla\widehat{u}\|_{L^2(\Omega)} \qquad \text{for all } t>0,
$$

and that Lemma 4.8 and Lemma 2.4 warrant that

$$
\|u^{(\kappa)}\|_{L^\infty(\Omega)} \le C_3e^{-\mu_1 t} \qquad \text{for all } t>0
$$

and

$$\|\nabla u^{(\kappa)}\|_{L^2(\Omega)} \le C_4 \qquad \text{for all } t>0$$

with some positive constants $\mu_1, C_3$ and $C_4$. Therefore, on the right-hand side of (5.16) we can estimate by means of Young's inequality according to

$$\begin{aligned}
&\|\widehat{u}\|_{L^2(\Omega)} \cdot \Big\{ C_1 \|\widehat{n}\|_{L^2(\Omega)} + |\kappa| \cdot \|(u^{(\kappa)} \cdot \nabla) u^{(\kappa)}\|_{L^2(\Omega)} \Big\} \\
&\qquad \le C_2 \|\nabla \widehat{u}\|_{L^2(\Omega)} \cdot \Big\{ C_1 \|\widehat{n}\|_{L^2(\Omega)} + |\kappa| \cdot \|u^{(\kappa)}\|_{L^\infty(\Omega)} \|\nabla u^{(\kappa)}\|_{L^2(\Omega)} \Big\} \\
&\qquad \le \frac{1}{2} \int_\Omega |\nabla \widehat{u}|^2 + \frac{C_2^2}{2} \cdot \Big\{ C_1 \|\widehat{n}\|_{L^2(\Omega)} + |\kappa| \cdot \|u^{(\kappa)}\|_{L^\infty(\Omega)} \|\nabla u^{(\kappa)}\|_{L^2(\Omega)} \Big\}^2 \\
&\qquad \le \frac{1}{2} \int_\Omega |\nabla \widehat{u}|^2 + C_1^2 C_2^2 \int_\Omega \widehat{n}^2 + C_2^2 \kappa^2 \|u^{(\kappa)}\|_{L^\infty(\Omega)}^2 \|\nabla u^{(\kappa)}\|_{L^2(\Omega)}^2 \\
&\qquad \le \frac{1}{2} \int_\Omega |\nabla \widehat{u}|^2 + C_1^2 C_2^2 \int_\Omega \widehat{n}^2 + C_2^2 C_3^2 C_4^2 \kappa^2 e^{-2\mu_1 t} \qquad \text{for all } t>0,
\end{aligned}$$

whence (5.15) results from (5.16). $\square$

Now taking a suitable linear combination of the inequalities from Lemma 5.2, Lemma 5.3 and Lemma 5.4, we obtain the following quasi-energy inequality of the structure as addressed in Lemma 5.1.

**Lemma 5.5** *There exist positive constants $k, l, \mu$ and $C$ such that for any choice of $\kappa \in (-1,1)$ we have*

$$\begin{aligned}
\frac{d}{dt} \Big\{ \int_\Omega \widehat{n}^2 + k \int_\Omega \widehat{c}^2 + l \int_\Omega |\widehat{u}|^2 \Big\} \;&+\; \frac{1}{C} \cdot \Big\{ \int_\Omega \widehat{n}^2 + k \int_\Omega \widehat{c}^2 + l \int_\Omega |\widehat{u}|^2 \Big\} \\
&\le\; C e^{-\mu t} \cdot \Big\{ \int_\Omega \widehat{n}^2 + k \int_\Omega \widehat{c}^2 + l \int_\Omega |\widehat{u}|^2 + \kappa^2 \Big\} \qquad (5.17)
\end{aligned}$$

*for all $t>0$.*

Proof. In order to prepare our definition of $k$ and $l$, according to two versions of the Poincaré inequality let us fix $C_1>0$ and $C_2>0$ such that

$$\int_\Omega \varphi^2 \le C_1 \int_\Omega |\nabla \varphi|^2 \qquad \text{for all } \varphi \in W^{1,2}(\Omega) \text{ such that } \int_\Omega \varphi = 0, \qquad (5.18)$$

and that

$$\int_\Omega |\varphi|^2 \le C_2 \int_\Omega |\nabla \varphi|^2 \qquad \text{for all } \varphi \in W_0^{1,2}(\Omega;\mathbb{R}^2). \qquad (5.19)$$

Moreover, employing Lemma 5.2 and Lemma 5.4 we can find positive constants $\mu_1, \mu_2, C_3$ and $C_4$ such that

$$\frac{d}{dt} \int_\Omega \widehat{n}^2 + \int_\Omega |\nabla \widehat{n}|^2 \le C_3 \int_\Omega |\nabla \widehat{c}|^2 + C_3 e^{-\mu_1 t} \cdot \Big\{ \int_\Omega \widehat{n}^2 + \int_\Omega |\widehat{u}|^2 \Big\} \qquad \text{for all } t>0 \qquad (5.20)$$

and

$$\frac{d}{dt} \int_\Omega |\widehat{u}|^2 + \int_\Omega |\nabla \widehat{u}|^2 \le C_4 \int_\Omega \widehat{n}^2 + C_4 \kappa^2 e^{-\mu_2 t} \qquad \text{for all } t>0. \qquad (5.21)$$

We now fix $k > 0$ large enough and $l > 0$ suitably small fulfilling

$$k \geq C_3 \tag{5.22}$$

and

$$l \leq \frac{1}{2C_1C_4}, \tag{5.23}$$

and then obtain from Lemma 5.3 when applied to $\varepsilon := \frac{l}{2k}$ that there exist $\mu_3 > 0$ and $C_5 > 0$ satisfying

$$\frac{d}{dt}\int_\Omega \widehat{c}^2 + \int_\Omega |\nabla\widehat{c}|^2 + \overline{n}_0 \int_\Omega \widehat{c}^2 \leq \frac{l}{2k}\int_\Omega |\nabla\widehat{u}|^2 + C_5 e^{-\mu_3 t}\cdot\Big\{\int_\Omega \widehat{n}^2 + \int_\Omega \widehat{c}^2 + \int_\Omega |\widehat{u}|^2\Big\} \qquad \text{for all } t > 0.$$

On adding this to (5.20) and (5.21), thanks to (5.18) and (5.19) as well as our restrictions (5.22) and (5.23) we thus infer that

$$\begin{aligned}
\frac{d}{dt}\Big\{\int_\Omega \widehat{n}^2 + k\int_\Omega \widehat{c}^2 + l\int_\Omega |\widehat{u}|^2\Big\} \quad &+ \quad \frac{1}{C_1}\int_\Omega \widehat{n}^2 + k\int_\Omega |\nabla\widehat{c}|^2 + k\overline{n}_0\int_\Omega \widehat{c}^2 + l\cdot\Big\{\frac{1}{2C_2}\int_\Omega |\widehat{u}|^2 + \frac{1}{2}\int_\Omega |\nabla\widehat{u}|^2\Big\} \\
&\leq \quad C_3\int_\Omega |\nabla\widehat{c}|^2 + C_3 e^{-\mu_1 t}\cdot\Big\{\int_\Omega \widehat{n}^2 + \int_\Omega |\widehat{u}|^2\Big\} \\
&\qquad + C_4 l\int_\Omega \widehat{n}^2 + C_4 l\kappa^2 e^{-\mu_2 t} \\
&\qquad + \frac{l}{2}\int_\Omega |\nabla\widehat{u}|^2 + C_5 k e^{-\mu_3 t}\cdot\Big\{\int_\Omega \widehat{n}^2 + \int_\Omega \widehat{c}^2 + \int_\Omega |\widehat{u}|^2\Big\} \\
&\leq \quad C_3\int_\Omega |\nabla\widehat{c}|^2 + \frac{1}{2C_1}\int_\Omega \widehat{n}^2 + \frac{l}{2}\int_\Omega |\nabla\widehat{u}|^2 \\
&\qquad + C_6 e^{-\mu_4 t}\cdot\Big\{\int_\Omega \widehat{n}^2 + \int_\Omega \widehat{c}^2 + \int_\Omega |\widehat{u}|^2\Big\} \\
&\qquad + C_4 l\kappa^2 e^{-\mu_2 t} \qquad \text{for all } t > 0
\end{aligned}$$

with $C_6 := C_3 + C_3 k$ and $\mu_4 := \min\{\mu_1, \mu_3\}$. On straightforward rearrangement, this simplifies to

$$\begin{aligned}
\frac{d}{dt}\Big\{\int_\Omega \widehat{n}^2 + k\int_\Omega \widehat{c}^2 + l\int_\Omega |\widehat{u}|^2\Big\} \quad &+ \quad \frac{1}{2C_1}\int_\Omega \widehat{n}^2 + k\overline{n}_0\int_\Omega \widehat{c}^2 + \frac{l}{2C_2}\int_\Omega |\widehat{u}|^2 \\
&\leq \quad C_6 e^{-\mu_4 t}\cdot\Big\{\int_\Omega \widehat{n}^2 + \int_\Omega \widehat{c}^2 + \int_\Omega |\widehat{u}|^2\Big\} + C_4 l\kappa^2 e^{-\mu_2 t}
\end{aligned}$$

for all $t > 0$, and thereby leads to (5.17) if we let $\mu := \min\{\mu_2, \mu_4\}$ and $C := \max\{2C_1, \frac{1}{k\overline{n}_0}, \frac{2C_2}{l}, C_6, C_4 l\}$, for instance. $\square$

By means of Lemma 5.1, the latter thus entails the following.

**Lemma 5.6** *There exist $\mu > 0$ and $C > 0$ such that for any $\kappa \in (-1, 1)$,*

$$\|\widehat{n}(\cdot,t)\|_{L^2(\Omega)} + \|\widehat{c}(\cdot,t)\|_{L^2(\Omega)} + \|\widehat{u}(\cdot,t)\|_{L^2(\Omega)} \leq C|\kappa|e^{-\mu t} \qquad \textit{for all } t > 0. \tag{5.24}$$

Proof. From Lemma 5.5 we know that with $k > 0$ and $l > 0$ as introduced there, we can find $C_1 > 0$, $C_2 > 0$ and $\mu_1 \in (0, C_1)$ such that for any choice of $\kappa \in (-1, 1)$, the fucntion defined by

$$y(t) := \int_\Omega \widehat{n}^2(\cdot, t) + k \int_\Omega \widehat{c}^2(\cdot, t) + l \int_\Omega |\widehat{u}(\cdot, t)|^2, \qquad t \geq 0,$$

satisfies

$$y'(t) + C_1 y(t) \leq C_2 e^{-\mu_1 t} y(t) + C_2 \kappa^2 e^{-\mu_1 t} \qquad \text{for all } t > 0.$$

As $y(0) = 0$ and $\mu_1 < C_1$, Lemma 5.1 applies so as to show that therefore

$$y(t) \leq \frac{C_2 \kappa^2}{C_1 - \mu_1} e^{\frac{C_2}{\mu_1}} e^{-\mu_1 t} \qquad \text{for all } t > 0,$$

and that thus (5.24) holds with $\mu := \frac{\mu_1}{2}$ and some appropriately large $C > 0$. $\square$

### 5.2 Higher norms. Proof of Theorem 1.1

We next turn our attention to the convergence statements involving the norms appearing in Theorem 1.1. Unlike in the previous section, in our analysis we will now be able to address the solution components of (5.2) more separately. Indeed, as a first part of our final result we will obtain the respective estimate for $\widehat{u}$ claimed in (1.7) on the basis of Lemma 5.6 and the following elementary inequality.

**Lemma 5.7** *Let $\beta \in [0, 1)$, $\mu_1 > 0$ and $\mu_2 \in (0, \mu_1)$. Then for all $\mu \in (0, \mu_2)$ one can find $C(\mu) > 0$ such that*

$$\int_0^t (t-s)^{-\beta} e^{-\mu_1 (t-s)} e^{-\mu_2 s} ds \leq C(\mu) e^{-\mu t} \qquad \textit{for all } t > 0. \tag{5.25}$$

Proof. We fix $\theta \in (0, 1)$ such that $\theta > \frac{\mu}{\mu_2}$, and given $t \geq 1$ we then split

$$I(t) := \int_0^t (t-s)^{-\beta} e^{-\mu_1 (t-s)} e^{-\mu_2 s} ds = \int_0^{\theta t} (t-s)^{-\beta} e^{-\mu_1 (t-s)} e^{-\mu_2 s} ds + \int_{\theta t}^t (t-s)^{-\beta} e^{-\mu_1 (t-s)} e^{-\mu_2 s} ds,$$

where

$$\begin{aligned}
\int_0^{\theta t} (t-s)^{-\beta} e^{-\mu_1 (t-s)} e^{-\mu_2 s} ds &\leq (t - \theta t)^{-\beta} \int_0^{\theta t} e^{-\mu_1 (t-s)} e^{-\mu_2 s} ds \\
&= (1-\theta)^{-\beta} t^{-\beta} \cdot \frac{1}{\mu_1 - \mu_2} e^{-\mu_1 t} \Big( e^{(\mu_1 - \mu_2)\theta t} - 1 \Big) \\
&\leq \frac{(1-\theta)^{-\beta}}{\mu_1 - \mu_2} t^{-\beta} e^{-(1-\theta)\mu_1 t} e^{-\mu_2 \theta t} \\
&\leq \frac{(1-\theta)^{-\beta}}{\mu_1 - \mu_2} e^{-\mu_2 \theta t} \qquad \text{for all } t \geq 1
\end{aligned} \tag{5.26}$$

and

$$\begin{aligned}\int_{\theta t}^t (t-s)^{-\beta} e^{-\mu_1(t-s)} e^{-\mu_2 s} ds &\le e^{-\mu_2\theta t}\int_{\theta t}^t (t-s)^{-\beta} ds \\ &= \frac{(1-\theta)^{1-\beta}}{1-\beta} t^{1-\beta} e^{-\mu_2\theta t} \qquad \text{for all } t\ge 1. \end{aligned} \tag{5.27}$$

Since our restriction $\theta > \frac{\mu}{\mu_2}$ warrants that with some $C_1>0$ we have

$$e^{-\mu_2\theta t} \le t^{1-\beta} e^{-\mu_2\theta t} \le C_1 e^{-\mu t} \qquad \text{for all } t\ge 1,$$

and since clearly

$$I(t) \le \int_0^t (t-s)^{-\beta} ds = \frac{t^{1-\beta}}{1-\beta} \le \frac{e^{\mu}}{1-\beta} e^{-\mu t} \qquad \text{for all } t\in(0,1),$$

from (5.26) and (5.27) we obtain (5.25). □

In fact, combining Lemma 5.6 with Lemma 4.8 and Lemma 2.4 we thereby obtain the following.

**Lemma 5.8** *There exist $\mu>0$ and $C>0$ such that whenever $\kappa\in(-1,1)$,*

$$\|A^\alpha \widehat{u}(\cdot,t)\|_{L^2(\Omega)} \le C|\kappa| e^{-\mu t} \qquad \textit{for all } t>0 \tag{5.28}$$

*and*

$$\|\widehat{u}(\cdot,t)\|_{L^\infty(\Omega)} \le C|\kappa| e^{-\mu t} \qquad \textit{for all } t>0. \tag{5.29}$$

Proof. We represent $A^\alpha\widehat{u}$ according to

$$A^\alpha \widehat{u}(\cdot,t) = \int_0^t A^\alpha e^{-(t-s)A} f(\cdot,s) ds, \qquad t>0,$$

where

$$f := \mathcal{P}[\widehat{n}\nabla\phi] - \kappa\mathcal{P}[(u^{(\kappa)}\cdot\nabla)u^{(\kappa)}],$$

and use known smoothing properties of the Stokes semigroup to find $\mu_1>0$ and $C_1>0$ such that

$$\|A^\alpha \widehat{u}(\cdot,t)\|_{L^2(\Omega)} \le C_1 \int_0^t (t-s)^{-\alpha} e^{-\mu_1(t-s)} \|f(\cdot,s)\|_{L^2(\Omega)} ds \qquad \text{for all } t>0. \tag{5.30}$$

Here we observe that

$$\begin{aligned}\int_\Omega |f(\cdot,t)|^2 &= \int_\Omega \Big|\widehat{n}(\cdot,t)\nabla\phi - \kappa(u^{(\kappa)}(\cdot,t)\cdot\nabla)u^{(\kappa)}(\cdot,t)\Big|^2 \\ &\le 2\|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega \widehat{n}^2(\cdot,t) + 2\kappa^2\|u^{(\kappa)}(\cdot,t)\|^2_{L^\infty(\Omega)}\int_\Omega |\nabla u^{(\kappa)}(\cdot,t)|^2 \qquad \text{for all } t>0,\end{aligned}$$

where using Lemma 4.8 and Lemma 2.4 we can find positive constants $\mu_2, C_2$ and $C_3$ such that

$$\|u^{(\kappa)}(\cdot,t)\|_{L^\infty(\Omega)} \le C_2 e^{-\mu_2 t} \qquad \text{for all } t>0$$

and

$$\int_\Omega |\nabla u^{(\kappa)}(\cdot,t)|^2 \le C_3 \qquad \text{for all } t>0,$$

and that we now thanks to Lemma 5.6 moreover know that

$$\int_\Omega \widehat{n}^2(\cdot,t) \le C_4\kappa^2 e^{-\mu_3 t} \qquad \text{for all } t>0$$

with some $C_4>0$ and $\mu_3 \in (0,\mu_1)$. We thus conclude that if we let $\mu_4 := \min\{\mu_3, 2\mu_2\}$ and $C_5 := 2\|\nabla\phi\|^2_{L^\infty(\Omega)} C_4 + 2C_2^2 C_3$, then

$$\int_\Omega |f(\cdot,t)|^2 \le C_5\kappa^2 e^{-\mu_4 t} \qquad \text{for all } t>0,$$

and that hence, by (5.30),

$$\|A^\alpha \widehat{u}(\cdot,t)\|_{L^2(\Omega)} \le C_1 C_5 \kappa^2 \int_0^t (t-s)^{-\alpha} e^{-\mu_1(t-s)} e^{-\mu_4 s} ds \qquad \text{for all } t>0. \tag{5.31}$$

Here as $\mu_4 \le \mu_3 < \mu_1$, Lemma 5.7 applies so as to yield $C_6>0$ such that

$$\int_0^t (t-s)^{-\alpha} e^{-\mu_1(t-s)} e^{-\mu_4 s} ds \le C_6 e^{-\frac{\mu_4}{2}t} \qquad \text{for all } t>0,$$

so that (5.28) follows from (5.31). Once more using that $D(A^\alpha) \hookrightarrow L^\infty(\Omega;\mathbb{R}^2)$, from this we immediately obtain (5.29). □

Now the core of this section can be found in the next lemma within which a functional of the form

$$\int_\Omega \widehat{n}^p + \int_\Omega |\nabla\widehat{c}|^p, \qquad t\ge 0,$$

is shown to satisfy an ODI of the form in Lemma 5.1 for arbitrary even integers $p\ge 4$. We remark that coupling densities and chemoattractive gradients at such equal integrability powers seems rather unusual in the context of Keller-Segel systems in which, as e.g. done in Lemma 2.6, terms of the form $\int_\Omega n^p$ are commonly combined with integrals of the type $\int_\Omega |\nabla c|^{2p}$ for $p>1$.

**Lemma 5.9** *For all $p\ge 2$ there exist $\mu(p)>0$ and $C(p)>0$ with the property that*

$$\|\widehat{n}(\cdot,t)\|_{L^p(\Omega)} + \|\widehat{c}(\cdot,t)\|_{W^{1,p}(\Omega)} \le C(p)|\kappa| e^{-\mu(p)t} \qquad \textit{for all } t>0 \tag{5.32}$$

*and any $\kappa\in(-1,1)$.*

PROOF. It is evident that we may restrict ourselves to the convenient case when $p\ge 4$ is an even integer, in which we first note that due to Lemma 4.7, Lemma 4.5, Lemma 4.6, Lemma 4.8, Lemma 2.7, Lemma 5.6 and Lemma 5.8 we may fix positive constants $\mu_1,...,\mu_4$ and $C_1,...,C_7$ such that for any $\kappa\in(-1,1)$,

$$\|n^{(\kappa)} - \overline{n}_0\|_{L^\infty(\Omega)} \le C_1 e^{-\mu_1 t} \qquad \text{for all } t>0 \tag{5.33}$$

and

$$\|c^{(\kappa)}(\cdot,t)\|_{L^\infty(\Omega)} + \|\nabla c^{(\kappa)}(\cdot,t)\|_{L^{2p}(\Omega)} \le C_2 e^{-\mu_2 t} \qquad \text{for all } t>0 \tag{5.34}$$

and

$$\|u^{(\kappa)}(\cdot,t)\|_{L^\infty(\Omega)} \le C_3 e^{-\mu_3 t} \qquad \text{for all } t>0, \tag{5.35}$$

that

$$\|n^{(\kappa)}(\cdot,t)\|_{L^\infty(\Omega)} \le C_4 \qquad \text{for all } t>0, \tag{5.36}$$

and that

$$\|\widehat{n}(\cdot,t)\|_{L^2(\Omega)} \le C_5|\kappa| e^{-\mu_4 t} \qquad \text{for all } t>0 \tag{5.37}$$

and

$$\|\widehat{c}(\cdot,t)\|_{L^2(\Omega)} \le C_6|\kappa| \qquad \text{for all } t>0 \tag{5.38}$$

as well as

$$\|\widehat{u}(\cdot,t)\|_{L^{2p}(\Omega)} \le C_7|\kappa| \qquad \text{for all } t>0. \tag{5.39}$$

Now for the functions $\widehat{n}, \widehat{c}$ and $\widehat{u}$ introduced in (5.1), we use (5.2) to compute

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega \widehat{n}^p + \int_\Omega \widehat{n}^p + (p-1)\int_\Omega \widehat{n}^{p-2}|\nabla\widehat{n}|^2 \quad &= \quad (p-1)\int_\Omega \widehat{n}^{p-1}\nabla\widehat{n}\cdot\nabla c^{(\kappa)} \\
&\quad +(p-1)\int_\Omega n^{(0)}\widehat{n}^{p-2}\nabla\widehat{n}\cdot\nabla\widehat{c} + \int_\Omega \widehat{n}^p \\
&\quad +(p-1)\int_\Omega (n^{(0)}-\overline{n}_0)\widehat{n}^{p-2}\widehat{u}\cdot\nabla\widehat{n}
\end{aligned} \tag{5.40}$$

for all $t>0$, because $\nabla\cdot\widehat{u}\equiv 0$. Here using Young's inequality and Hölder's inequality as well as (5.34), we can estimate

$$\begin{aligned}
(p-1)\int_\Omega \widehat{n}^{p-1}\nabla\widehat{n}\cdot\nabla c^{(\kappa)} \quad &\le \quad \frac{p-1}{2}\int_\Omega \widehat{n}^{p-2}|\nabla\widehat{n}|^2 + \frac{p-1}{2}\int_\Omega \widehat{n}^p|\nabla c^{(\kappa)}|^2 \\
&\le \quad \frac{p-1}{2}\int_\Omega \widehat{n}^{p-2}|\nabla\widehat{n}|^2 + \frac{p-1}{2}|\Omega|^{\frac{p-2}{2p}}\Big\{\int_\Omega \widehat{n}^{2p}\Big\}^{\frac12}\cdot\Big\{\int_\Omega |\nabla c^{(\kappa)}|^{2p}\Big\}^{\frac1p} \\
&\le \quad \frac{p-1}{2}\int_\Omega \widehat{n}^{p-2}|\nabla\widehat{n}|^2 + \frac{p-1}{2}|\Omega|^{\frac{p-2}{2p}} C_2^2 e^{-2\mu_2 t}\Big\{\int_\Omega \widehat{n}^{2p}\Big\}^{\frac12}
\end{aligned} \tag{5.41}$$

for all $t>0$, where by the Gagliardo-Nirenberg inequality, (5.37) and Young's inequality with some $C_8>0$ and $C_9>0$ we have

$$\begin{aligned}
\frac{p-1}{2}|\Omega|^{\frac{p-2}{2p}} C_2^2 e^{-2\mu_2 t}\Big\{\int_\Omega \widehat{n}^{2p}\Big\}^{\frac12} \quad &= \quad \frac{p-1}{2}|\Omega|^{\frac{p-2}{2p}} C_2^2 e^{-2\mu_2 t}\|\widehat{n}^{\frac p2}\|^2_{L^4(\Omega)} \\
&\le \quad C_8 e^{-\mu_2 t}\cdot\Big\{\|\nabla\widehat{n}^{\frac p2}\|^{\frac{2(p-1)}{p}}_{L^2(\Omega)}\|\widehat{n}^{\frac p2}\|^{\frac2p}_{L^{\frac4p}(\Omega)} + \|\widehat{n}^{\frac p2}\|^2_{L^{\frac4p}(\Omega)}\Big\} \\
&\le \quad C_8 e^{-\mu_2 t}\cdot\Big\{C_5|\kappa|\|\nabla\widehat{n}^{\frac p2}\|^{\frac{2(p-1)}{p}}_{L^2(\Omega)} + C_5^p\kappa^p\Big\} \\
&\le \quad \frac{p-1}{p^2}\|\nabla\widehat{n}^{\frac p2}\|^2_{L^2(\Omega)} + C_9\kappa^p e^{-p\mu_2 t} + C_5^p C_8\kappa^p e^{-\mu_2 t} \\
&\le \quad \frac{p-1}{4}\int_\Omega \widehat{n}^{p-2}|\nabla\widehat{n}|^2 + (C_9 + C_5^p C_8)\kappa^p e^{-\mu_2 t}
\end{aligned} \tag{5.42}$$

for all $t > 0$. On the right-hand side of (5.40) we next use Young's inequality along with (5.36) and the Hölder inequality to estimate

$$
\begin{aligned}
(p-1)\int_\Omega n^{(0)}\widehat{n}^{p-2}\nabla\widehat{n}\cdot\nabla\widehat{c}+\int_\Omega \widehat{n}^p \ &\le\ \frac{p-1}{8}\int_\Omega \widehat{n}^{p-2}|\nabla\widehat{n}|^2+2(p-1)\int_\Omega |n^{(0)}|^2\widehat{n}^{p-2}|\nabla\widehat{c}|^2+\int_\Omega \widehat{n}^p \\
&\le\ \frac{p-1}{8}\int_\Omega \widehat{n}^{p-2}|\nabla\widehat{n}|^2+2(p-1)C_4^2\int_\Omega \widehat{n}^{p-2}|\nabla\widehat{c}|^2+\int_\Omega \widehat{n}^p \\
&\le\ \frac{p-1}{8}\int_\Omega \widehat{n}^{p-2}|\nabla\widehat{n}|^2 \\
&\quad +2(p-1)C_4^2\Big\{\int_\Omega \widehat{n}^p\Big\}^{\frac{p-2}{p}}\cdot\Big\{\int_\Omega |\nabla\widehat{c}|^p\Big\}^{\frac{2}{p}}+\int_\Omega \widehat{n}^p \\
&\le\ \frac{p-1}{8}\int_\Omega \widehat{n}^{p-2}|\nabla\widehat{n}|^2+\frac{\overline{n}_0}{2}\int_\Omega |\nabla\widehat{c}|^p+C_{10}\int_\Omega \widehat{n}^p \qquad (5.43)
\end{aligned}
$$

for all $t > 0$ with some $C_{10} > 0$, where again due to the Gagliardo-Nirenberg inequality, (5.37) and Young's inequality we see that there exist $C_{11} > 0$ and $C_{12} > 0$ fulfilling

$$
\begin{aligned}
C_{10}\int_\Omega \widehat{n}^p \ &=\ C_{10}\|\widehat{n}^{\frac{p}{2}}\|_{L^2(\Omega)}^2 \\
&\le\ C_{11}\|\nabla\widehat{n}^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{2(p-2)}{p}}\|\widehat{n}^{\frac{p}{2}}\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{4}{p}}+C_{11}\|\widehat{n}^{\frac{p}{2}}\|_{L^{\frac{4}{p}}(\Omega)}^2 \\
&\le\ C_{11}C_5^2\kappa^2e^{-2\mu_4t}\|\nabla\widehat{n}^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{2(p-2)}{p}}+C_{11}C_5^p\kappa^pe^{-p\mu_4t} \\
&\le\ \frac{p-1}{4p^2}\|\nabla\widehat{n}^{\frac{p}{2}}\|_{L^2(\Omega)}^2+C_{12}\kappa^pe^{-p\mu_4t} \\
&=\ \frac{p-1}{16}\int_\Omega \widehat{n}^{p-2}|\nabla\widehat{n}|^2+C_{12}\kappa^pe^{-p\mu_4t} \qquad \text{for all } t>0. \qquad (5.44)
\end{aligned}
$$

Finally, in the rightmost summand in (5.40) we once more rely on Young's inequality and the Cauchy-Schwarz inequality and use (5.33) as well as (5.39) to see that for all $t > 0$,

$$
\begin{aligned}
(p-1)\int_\Omega (n^{(0)}-\overline{n}_0)\widehat{n}^{p-2}\widehat{u}\cdot\nabla\widehat{n} \ &\le\ \frac{p-1}{16}\int_\Omega \widehat{n}^{p-2}|\nabla\widehat{n}|^2+4(p-1)\int_\Omega |n^{(0)}-\overline{n}_0|^2\widehat{n}^{p-2}|\widehat{u}|^2 \\
&\le\ \frac{p-1}{16}\int_\Omega \widehat{n}^{p-2}|\nabla\widehat{n}|^2+4(p-1)C_1^2e^{-2\mu_1t}\int_\Omega \widehat{n}^{p-2}|\widehat{u}|^2 \\
&\le\ \frac{p-1}{16}\int_\Omega \widehat{n}^{p-2}|\nabla\widehat{n}|^2+4(p-1)C_1^2e^{-2\mu_1t}\cdot\Big\{\int_\Omega \widehat{n}^p+\int_\Omega |\widehat{u}|^p\Big\} \\
&\le\ \frac{p-1}{16}\int_\Omega \widehat{n}^{p-2}|\nabla\widehat{n}|^2+4(p-1)C_1^2e^{-2\mu_1t}\cdot\Big\{\int_\Omega \widehat{n}^p+\sqrt{|\Omega|}\cdot C_7^p\kappa^p\Big\},
\end{aligned}
$$

which combined with (5.40)-(5.44) shows that there exist $\mu_5 > 0$ and $C_{13} > 0$ such that

$$
\frac{1}{p}\frac{d}{dt}\int_\Omega \widehat{n}^p+\int_\Omega \widehat{n}^p\le\frac{\overline{n}_0}{2}\int_\Omega |\nabla\widehat{c}|^p+C_{13}e^{-\mu_5t}\cdot\Big\{\int_\Omega \widehat{n}^p+\kappa^p\Big\} \qquad \text{for all } t>0. \qquad (5.45)
$$

In order to adequately compensate the first summand on the right-hand side herein, we use the second equation in (5.2) to calculate

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega |\nabla\widehat{c}|^p &= \int_\Omega |\nabla\widehat{c}|^{p-2}\nabla\widehat{c}\cdot\nabla\widehat{c}_t \\
&= \int_\Omega |\nabla\widehat{c}|^{p-2}\nabla\widehat{c}\cdot\nabla\Delta\widehat{c} - \overline{n}_0\int_\Omega |\nabla\widehat{c}|^p + \int_\Omega (n^{(\kappa)}-\overline{n}_0)\widehat{c}\nabla\cdot(|\nabla\widehat{c}|^{p-2}\nabla\widehat{c}) \\
&\quad + \int_\Omega \widehat{n}c^{(0)}\nabla\cdot(|\nabla\widehat{c}|^{p-2}\nabla\widehat{c}) \\
&\quad + \int_\Omega (u^{(\kappa)}\cdot\nabla\widehat{c})\nabla\cdot(|\nabla\widehat{c}|^{p-2}\nabla\widehat{c}) + \int_\Omega (\widehat{u}\cdot\nabla c^{(0)})\nabla\cdot(|\nabla\widehat{c}|^{p-2}\nabla\widehat{c}) \qquad (5.46)
\end{aligned}$$

for all $t>0$, where again by convexity of $\Omega$,

$$\begin{aligned}
\int_\Omega |\nabla\widehat{c}|^{p-2}\nabla\widehat{c}\cdot\nabla\Delta\widehat{c} &= \frac{1}{2}\int_\Omega |\nabla\widehat{c}|^{p-2}\Delta|\nabla\widehat{c}|^2 - \int_\Omega |\nabla\widehat{c}|^{p-2}|D^2\widehat{c}|^2 \\
&\le -\frac{p-2}{4}\int_\Omega |\nabla\widehat{c}|^{p-4}\Big|\nabla|\nabla\widehat{c}|^2\Big|^2 - \int_\Omega |\nabla\widehat{c}|^{p-2}|D^2\widehat{c}|^2 \\
&\le -\int_\Omega |\nabla\widehat{c}|^{p-2}|D^2\widehat{c}|^2 \qquad \text{for all } t>0.
\end{aligned}$$

Now since $|\Delta\widehat{c}|\le\sqrt{2}|D^2\widehat{c}|$, we may use the pointwise estimate

$$\begin{aligned}
\Big|\nabla\cdot(|\nabla\widehat{c}|^{p-2}\nabla\widehat{c})\Big| &= \Big||\nabla\widehat{c}|^{p-2}\Delta\widehat{c} + \frac{p-2}{2}|\nabla\widehat{c}|^{p-4}\nabla\widehat{c}\cdot\nabla|\nabla\widehat{c}|^2\Big| \\
&\le C_{14}|\nabla\widehat{c}|^{p-2}|D^2\widehat{c}| \qquad \text{in } \Omega\times(0,\infty), \qquad (5.47)
\end{aligned}$$

valid with $C_{14}:=\sqrt{2}+p-2$, to see by Young's inequality and (5.33) that there exists $C_{15}>0$ fulfilling

$$\begin{aligned}
\int_\Omega (n^{(\kappa)}-\overline{n}_0)\widehat{c}\nabla\cdot(|\nabla\widehat{c}|^{p-2}\nabla\widehat{c}) &\le C_{14}\int_\Omega |n^{(\kappa)}-\overline{n}_0|\cdot|\widehat{c}|\cdot|\nabla\widehat{c}|^{p-2}|D^2\widehat{c}| \\
&\le \frac{1}{2}\int_\Omega |\nabla\widehat{c}|^{p-2}|D^2\widehat{c}|^2 + \frac{1}{2}C_{14}^2\int_\Omega |n^{(\kappa)}-\overline{n}_0|^2\widehat{c}^2|\nabla\widehat{c}|^{p-2} \\
&\le \frac{1}{2}\int_\Omega |\nabla\widehat{c}|^{p-2}|D^2\widehat{c}|^2 + \frac{1}{2}C_1^2C_{14}^2e^{-2\mu_1 t}\int_\Omega \widehat{c}^2|\nabla\widehat{c}|^{p-2} \\
&\le \frac{1}{2}\int_\Omega |\nabla\widehat{c}|^{p-2}|D^2\widehat{c}|^2 + \frac{1}{2}C_1^2C_{14}^2e^{-2\mu_1 t}\cdot\Big\{\int_\Omega \widehat{c}^p + \int_\Omega |\nabla\widehat{c}|^p\Big\} \\
&\le \frac{1}{2}\int_\Omega |\nabla\widehat{c}|^{p-2}|D^2\widehat{c}|^2 + C_{15}e^{-2\mu_1 t}\cdot\Big\{\int_\Omega |\nabla\widehat{c}|^p + \kappa^p\Big\} \qquad (5.48)
\end{aligned}$$

for all $t>0$, because according to the Gagliardo-Nirenberg inequality, Young's inequality and (5.38) we see that with some $C_{16}>0$ we have

$$\begin{aligned}
\int_\Omega \widehat{c}^p &\le C_{16}\|\nabla\widehat{c}\|_{L^p(\Omega)}^{\frac{p(p-2)}{2p-2}}\|\widehat{c}\|_{L^2(\Omega)}^{\frac{p^2}{2p-2}} + C_{16}\|\widehat{c}\|_{L^2(\Omega)}^p \\
&\le C_{16}\|\nabla\widehat{c}\|_{L^p(\Omega)}^p + 2C_{16}\|\widehat{c}\|_{L^2(\Omega)}^p \\
&\le C_{16}\int_\Omega |\nabla\widehat{c}|^p + 2C_6^pC_{16}\kappa^p \qquad \text{for all } t>0.
\end{aligned}$$

Similarly, using (5.47) together with Young's inequality, (5.34) and (5.35) we obtain

$$\begin{aligned}
\int_\Omega \widehat{n} c^{(0)} \nabla \cdot (|\nabla \widehat{c}|^{p-2} \nabla \widehat{c}) &\le C_{14} \int_\Omega |\widehat{n}| c^{(0)} |\nabla \widehat{c}|^{p-2} |D^2 \widehat{c}| \\
&\le \frac{1}{4} \int_\Omega |\nabla \widehat{c}|^{p-2} |D^2 \widehat{c}|^2 + C_{14}^2 \int_\Omega \widehat{n}^2 |c^{(0)}|^2 |\nabla \widehat{c}|^{p-2} \\
&\le \frac{1}{4} \int_\Omega |\nabla \widehat{c}|^{p-2} |D^2 \widehat{c}|^2 + C_2^2 C_{14}^2 e^{-2\mu_2 t} \int_\Omega \widehat{n}^2 |\nabla \widehat{c}|^{p-2} \\
&\le \frac{1}{4} \int_\Omega |\nabla \widehat{c}|^{p-2} |D^2 \widehat{c}|^2 + C_2^2 C_{14}^2 e^{-2\mu_2 t} \cdot \left\{ \int_\Omega \widehat{n}^p + \int_\Omega |\nabla \widehat{c}|^p \right\} \qquad (5.49)
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega (u^{(\kappa)} \cdot \nabla \widehat{c}) \nabla \cdot (|\nabla \widehat{c}|^{p-2} \nabla \widehat{c}) &\le C_{14} \int_\Omega |u^{(\kappa)}| \cdot |\nabla \widehat{c}|^{p-1} |D^2 \widehat{c}| \\
&\le \frac{1}{8} \int_\Omega |\nabla \widehat{c}|^{p-2} |D^2 \widehat{c}|^2 + 2C_{14}^2 \int_\Omega |u^{(\kappa)}|^2 |\nabla \widehat{c}|^p \\
&\le \frac{1}{8} \int_\Omega |\nabla \widehat{c}|^{p-2} |D^2 \widehat{c}|^2 + 2C_3^2 C_{14}^2 e^{-2\mu_3 t} \int_\Omega |\nabla \widehat{c}|^p, \qquad (5.50)
\end{aligned}$$

and that thanks to Young's inequality, the Hölder inequality, (5.34) and (5.39),

$$\begin{aligned}
\int_\Omega (\widehat{u} \cdot \nabla c^{(0)}) \nabla \cdot (|\nabla \widehat{c}|^{p-2} \nabla \widehat{c}) &\le C_{14} \int_\Omega |\widehat{u}| \cdot |\nabla c^{(0)}| \cdot |\nabla \widehat{c}|^{p-2} |D^2 \widehat{c}| \\
&\le \frac{1}{8} \int_\Omega |\nabla \widehat{c}|^{p-2} |D^2 \widehat{c}|^2 + 2C_{14}^2 \int_\Omega |\widehat{u}|^2 |\nabla c^{(0)}|^2 |\nabla \widehat{c}|^{p-2} \\
&\le \frac{1}{8} \int_\Omega |\nabla \widehat{c}|^{p-2} |D^2 \widehat{c}|^2 + 2C_{14}^2 \|\widehat{u}\|_{L^{2p}(\Omega)}^2 \|\nabla c^{(0)}\|_{L^{2p}(\Omega)}^2 \|\nabla \widehat{c}\|_{L^p(\Omega)}^{p-2} \\
&\le \frac{1}{8} \int_\Omega |\nabla \widehat{c}|^{p-2} |D^2 \widehat{c}|^2 + 2C_2^2 C_7^2 C_{14}^2 \kappa^2 e^{-2\mu_2 t} \|\nabla \widehat{c}\|_{L^p(\Omega)}^{p-2} \\
&\le \frac{1}{8} \int_\Omega |\nabla \widehat{c}|^{p-2} |D^2 \widehat{c}|^2 + 2C_2^2 C_7^2 C_{14}^2 e^{-2\mu_2 t} \cdot \left\{ \int_\Omega |\nabla \widehat{c}|^p + \kappa^p \right\} \qquad (5.51)
\end{aligned}$$

for all $t > 0$. Collecting (5.48)-(5.51), from (5.46) we thus infer the existence of $\mu_6 > 0$ and $C_{17} > 0$ such that

$$\frac{1}{p} \frac{d}{dt} \int_\Omega |\nabla \widehat{c}|^p + \overline{n}_0 \int_\Omega |\nabla \widehat{c}|^p \le C_{17} e^{-\mu_6 t} \cdot \left\{ \int_\Omega \widehat{n}^p + \int_\Omega |\nabla \widehat{c}|^p + \kappa^p \right\} \qquad \text{for all } t > 0,$$

whence in view of (5.45) we obtain that if we let $C_{18} := p \min\{1, \frac{\overline{n}_0}{2}\}$, $C_{19} := 2p(C_{13} + C_{17})$ and choose any $\mu_7 \in (0, C_{18})$ such that $\mu_7 \le \min\{\mu_5, \mu_6\}$, then

$$y(t) := \int_\Omega \widehat{n}^p(\cdot, t) + \int_\Omega |\nabla \widehat{c}(\cdot, t)|^p, \qquad t \ge 0,$$

satisfies

$$y'(t) + C_{18} y(t) \le C_{19} e^{-\mu_7 t} y(t) + C_{19} \kappa^p e^{-\mu_7 t} \qquad \text{for all } t > 0.$$

Since $y(0) = 0$, using that $\mu_7 < C_{18}$ we may invoke Lemma 5.1 to conclude that this entails the inequality

$$y(t) \le \frac{C_{19}\kappa^p}{C_{18}-\mu_7} e^{\frac{C_{19}}{\mu_7}} e^{-\mu_7 t} \qquad \text{for all } t > 0$$

and thereby proves the lemma. $\square$

Again, passing to a corresponding estimate for $\widehat{n}$ with respect to spatial $L^\infty$ norms is quite straightforward.

**Lemma 5.10** *There exist $\mu > 0$ and $C > 0$ such that*

$$\|\widehat{n}(\cdot,t)\|_{L^\infty(\Omega)} \le C|\kappa|e^{-\mu t} \qquad \textit{for all } t > 0 \tag{5.52}$$

*whenever $\kappa \in (-1,1)$.*

PROOF. Using that $\nabla \cdot u^{(\kappa)} \equiv 0$ for all $\kappa \in (-1,1)$, we may rewrite the first equation in (5.2) in the form

$$\widehat{n}_t = \Delta\widehat{n} - \nabla \cdot f(x,t), \qquad x \in \Omega,\ t > 0,$$

where

$$f := \widehat{n}\nabla c^{(\kappa)} + n^{(0)}\nabla\widehat{c} + \widehat{n}u^{(\kappa)} + n^{(0)}\widehat{u}$$

satisfying $f \cdot \nu = 0$ on $\partial\Omega$, so that if we fix an arbitrary $p > 2$, then a known regularization feature of the Neumann heat semigroup over $\Omega$ ([11, Lemma 3.3] and [36, Lemma 1.3]) applies so as to yield $\mu_1 > 0$ and $C_1 > 0$ such that

$$\|\widehat{n}(\cdot,t)\|_{L^\infty(\Omega)} \le C_1 \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{p}} e^{-\mu_1(t-s)} \|f(\cdot,s)\|_{L^p(\Omega)} ds \qquad \text{for all } t > 0. \tag{5.53}$$

We now estimate

$$\|f\|_{L^p(\Omega)} \le \|\widehat{n}\|_{L^{2p}(\Omega)}\|\nabla c^{(\kappa)}\|_{L^{2p}(\Omega)} + \|n^{(0)}\|_{L^\infty(\Omega)}\|\nabla\widehat{c}\|_{L^p(\Omega)} + \|\widehat{n}\|_{L^p(\Omega)}\|u^{(\kappa)}\|_{L^\infty(\Omega)} + \|n^{(0)}\|_{L^p(\Omega)}\|\widehat{u}\|_{L^\infty(\Omega)}$$

for $t > 0$, so that combining the decay estimates provided by Lemma 5.8 and Lemma 5.9 with the boundedness properties from Lemma 2.6, Lemma 2.7 and Lemma 2.8 we obtain $\mu_2 \in (0,\mu_1)$ and $C_2 > 0$ such that

$$\|f(\cdot,t)\|_{L^p(\Omega)} \le C_2|\kappa|e^{-\mu_2 t} \qquad \text{for all } t > 0.$$

In view of Lemma 5.7, from (5.53) we therefore obtain that with some $C_3 > 0$ we have

$$\begin{aligned}
\|\widehat{n}(\cdot,t)\|_{L^\infty(\Omega)} &\le C_1C_2|\kappa| \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{p}} e^{-\mu_1(t-s)} e^{-\mu_2 s} ds \\
&\le C_3|\kappa|e^{-\frac{\mu_2}{2}t} \qquad \text{for all } t > 0,
\end{aligned}$$

as desired. $\square$

It remains to summarize:

Proof of Theorem 1.1. We only need to collect the outcomes of Lemma 5.10, Lemma 5.9 and Lemma 5.8, and once more make use of the continuity of the embeddings $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $p > 2$ and $D(A^\alpha) \hookrightarrow L^\infty(\Omega; \mathbb{R}^2)$. $\square$

**Acknowledgement.** This work was initiated while the second author visited Xihua University and the University of Electronic Science and Technology of China in Chengdu in January 2016. He is very grateful for the warm hospitality. Y. Wang was supported by Xihua University Young Scholars Training Program and the NNSF of China (no. 11501457). Z. Xiang was supported by the Fundamental Research Funds for the Central Universities (no. ZYGX2015J101) and the NNSF of China (no. 11571063, 11501086).

# References

[1] Amann, H.: *Compact embeddings of vector-valued Sobolev and Besov spaces.* Glasnik Mat. **35** (55), 161-177 (2000)

[2] Bellomo, N., Bellouquid, A., Tao, Y., Winkler, M.: *Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues.* Math. Mod. Meth. Appl. Sci. **25**, 1663-1763 (2015)

[3] Biler, P., Hebisch, W., Nadzieja, T.: *The Debye system: Existence and large time behavior of solutions.* Nonlinear Analysis, TMA **23** (9), 1189-1209 (1994)

[4] Cao, X., Lankeit, J.: *Global classical small-data solutions for a three-dimensional chemotaxis Navier-Stokes system involving matrix-valued sensitivities.* Preprint

[5] Chae, M., Kang, K., Lee, J.: *Global Existence and temporal decay in Keller-Segel models coupled to fluid equations.* Comm. Part. Diff. Eqs., **39**, 1205-1235 (2014)

[6] Difrancesco, M., Lorz, A., Markowich, P. A.: *Chemotaxis-fluid coupled model for swimming bacteria with nonlinear diffusion: global existence and asymptotic behavior.* Discr. Cont. Dyn. Syst. A, **28**, 1437-1453 (2010)

[7] Duan, R., Lorz, A., Markowich, P. A.: *Global solutions to the coupled chemotaxis-fluid equations.* Comm. Part. Diff. Eqs., **35**, 1635-1673 (2010)

[8] Duan, R., Xiang, Z.: *A note on global existence for the chemotaxis-Stokes model with nonlinear diffusion.* Int. Math. Res. Not. IMRN, **2014**, 1833-1852 (2014)

[9] Espejo, E.E., Winkler, M.: *Global classical solvability and stabilization in a two-dimensional chemotaxis-Navier-Stokes system modeling coral ferilization.* Preprint

[10] Friedman, A.: *Partial Differential Equations.* Holt, Rinehart & Winston, New York, 1969

[11] Fujie, K., Ito, A., Winkler, M., Yokota, T.: *Stabilization in a chemotaxis model for tumor invasion.* Discrete Contin. Dyn. Syst. **36**, 151-169 (2016)

[12] Fujita, H., Kato, T.: *On the Navier-Stokes initial value problem. I.* Arch. Rat. Mech. Anal. **16**, 269-315 (1964)

[13] Fujiwara, D., Morimoto, H.: *An $L^r$-theorem of the Helmholtz decomposition of vector fields.* J. Fac. Sci. Univ. Tokyo **24**, 685-700 (1977)

[14] Giga, Y.: *Solutions for semilinear parabolic equations in $L_p$ and regularity of weak solutions of the Navier-Stokes system.* J. Differential Equations **61**, 186-212 (1986)

[15] Giga, Y., Sohr, H.: *Abstract $L^p$ estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains.* J. Functional Analysis **102**, 72-94 (1991)

[16] Henry, D.: *Geometric Theory of Semilinear Parabolic Equations.* Springer, Berlin/Heidelberg, 1981.

[17] Herschlag, G., Miller, M.: *Reynolds number limits for jet propulsion: A numerical study of simplified jellyfish.* J. Theoretical Biology **285**, 84-95 (2011)

[18] Kozono, H., Miura, M., Sugiyama, Y.: *Existence and uniqueness theorem on mild solutions to the Keller-Segel system coupled with the Navier-Stokes fluid.* J. Functional Analysis 2**70**, 1663-1683 (2016)

[19] Leray, J.: *Sur le mouvement d'un liquide visqueus amplissant l'espace.* Acta Math. **63**, 193-248 (1934)

[20] Ladyzenskaja, O. A., Solonnikov, V. A., Ural'ceva, N. N.: *Linear and Quasi-Linear Equations of Parabolic Type.* Amer. Math. Soc. Transl., Vol. **23**, Providence, RI, 1968

[21] Liu, J., Lorz, A.: *A coupled chemotaxis-fluid model: Global existence.* Ann. Inst. H. Poincaré Anal. Non Linéaire **28**, 643-652 (2011)

[22] Lieberman, G.: *Hölder continuity of the gradient of solutions of uniformly parabolic equations with conormal boundary conditions.* Ann. Mat. Pura Appl. **148**, 77-99 (1987)

[23] Lions, P.L.: *Résolution de problèmes elliptiques quasilinéaires.* Arch. Rat. Mech. Anal. **74**, 335-353 (1980)

[24] Porzio, M.M., Vespri, V.: *Hölder estimates for local solutions of some doubly nonlinear degenerate parabolic equations.* J. Differential Equations **103** (1), 146-178 (1993)

[25] Sohr, H.: *The Navier-Stokes Equations. An Elementary Functional Analytic Approach.* Birkhäuser, Basel, 2001

[26] Tan, Z., Zhang, X.: *Decay estimates of the coupled chemotaxis-fluid equations in $\mathbb{R}^3$.* J. Math. Anal. Appl., **410**(1), 27-38 (2014)

[27] Tao, Y.: *Boundedness in a chemotaxis model with oxygen consumption by bacteria.* J. Math. Anal. Appl. **381**, 521-529 (2011)

[28] Tao, Y., Winkler, M.: *Eventual smoothness and stabilization of large-data solutions in a three-dimensional chemotaxis system with consumption of chemoattractant.* J. Differential Equations **252**, 2520-2543 (2012)

[29] Tao, Y., Winkler, M.: *Global existence and boundedness in a Keller-Segel-Stokes model with arbitrary porous medium diffusion.* Discr. Cont. Dyn. Syst. A **32** (5), 1901-1914 (2012)

[30] Tao, Y., Winkler, M: *Locally bounded global solutions in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion.* Ann. Inst. H. Poincaré, Anal. Non Linéaire **30**, 157-178 (2013)

[31] Tao, Y., Winkler, M.: *Energy-type estimates and global solvability in a two-dimensional chemotaxis-haptotaxis model with remodeling of non-diffusible attractant.* J. Differential Equations **257**, 784-815 (2014)

[32] Tuval, I., Cisneros, L., Dombrowski, C., Kessler, J. O., Goldstein R. E.: *Bacterial swimming and oxygen transport near contact lines.* Proc. Nat. Acad. Sci. USA., **102**, 2277-2282 (2005)

[33] Vorotnikov, D.: *Weak solutions for a bioconvection model related to Bacillus subtilis.* Commun. Math. Sci. **12**, 545-563 (2014)

[34] Wang, Y., Cao, X.: *Global classical solutions of a 3D chemotaxis-Stokes system with rotation.* Discr. Cont. Dyn. Syst. B **20**, 3235-3254 (2015)

[35] Wang, Y., Xiang, Z.: *Global existence and boundedness in a Keller-Segel-Stokes system involving a tensor-valued sensitivity with saturation.* J. Differential Equations **259**, 7578-7609 (2015)

[36] Winkler, M.: *Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model.* J. Differential Equations **248**, 2889-2905 (2010)

[37] Winkler, M.: *Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops.* Comm. Partial Differ. Equations **37**, 319-351 (2012)

[38] Winkler, M.: *Stabilization in a two-dimensional chemotaxis-Navier-Stokes system.* Arch. Ration. Mech. Anal. **211** (2), 455-487 (2014)

[39] Winkler, M.: *Global weak solutions in a three-dimensional chemotaxis-Navier-Stokes system.* Ann. Inst. Henri Poincaré, Anal. Non Linéaire, to appear

[40] Winkler, M.: *How far do chemotaxis-driven forces influence regularity in the Navier-Stokes system?* Trans. Amer. Math. Soc., to appear

[41] Winkler, M.: *Boundedness and large time behavior in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion and general sensitivity.* Calculus of Variations **54**, 3789-3828 (2015)

[42] Winkler, M.: *A two-dimensional chemotaxis-Stokes system with rotational flux: Global solvability, eventual smoothness and stabilization.* Preprint

[43] Winkler, M.: *A three-dimensional Keller-Segel-Navier-Stokes system with logistic source: Global weak solutions and asymptotic stabilization.* Preprint.

[44] Ye X.: *Existence and decay of global smooth solutions to the coupled chemotaxis-fluid model.* J. Math. Anal. Appl. **427**, 60-73 (2015)

[45] Zhang, Q., Li, Y.: *Convergence rates of solutions for a two-dimensional chemotaxis-Navier-Stokes system.* Discrete Contin. Dyn. Syst. Ser. B **20**, 2751-2759 (2015).

[46] Zhang, Q., Li, Y.: *Global weak solutions for the three-dimensional chemotaxis-Navier-Stokes system with nonlinear diffusion.* J. Differential Equations **259**, 3730-3754 (2015)

[47] Zhang Q., Zheng X.: *Global well-posedness for the two-dimensional incompressible chemotaxis-Navier-Stokes equations.* SIAM J. Math. Anal. **46**, 3078-3105 (2014)