Finite-time blow-up in low-dimensional Keller-Segel systems with logistic-type superlinear degradation

Michael Winkler*
Institut für Mathematik, Universität Paderborn,
33098 Paderborn, Germany

Abstract

We consider radially symmetric solutions of the Keller-Segel system with generalized logistic source given by

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + \lambda u - \mu u^\kappa, \\ 0 = \Delta v - v + u, \end{cases} \qquad (\star)$$

under homogeneous Neumann boundary conditions in the ball $\Omega = B_R(0) \subset \mathbb{R}^n$ for $n \geq 3$ and $R > 0$, where $\lambda \in \mathbb{R}, \mu > 0$ and $\kappa > 1$.

Under the assumption that

$$\kappa < \begin{cases} \frac{7}{6} & \text{if } n \in \{3,4\}, \\ 1 + \frac{1}{2(n-1)} & \text{if } n \geq 5, \end{cases}$$

a condition on the initial data is derived which is seen to be sufficient to ensure the occurrence of finite-time blow-up for the corresponding solution of $(\star)$. Moreover, this criterion is shown to be mild enough so as to allow for the conclusion that in fact any positive continuous radial function on $\overline{\Omega}$ is the limit in $L^1(\Omega)$ of a sequence $(u_{0k})_{k \in \mathbb{N}}$ of continuous radial initial data which are such that for each $k \in \mathbb{N}$ the associated initial-boundary value problem for $(\star)$ exhibits a finite-time explosion phenomenon in the above sense.

In particular, this apparently provides the first rigorous detection of blow-up in a superlinearly dampened but otherwise essentially original Keller-Segel system in the physically relevant three-dimensional case.

Key words: chemotaxis; logistic source; finite-time blow-up
MSC 2010: 35B44 (primary); 35K65, 92C17 (secondary)

*michael.winkler@math.uni-paderborn.de

1 Introduction

The interaction between cross-diffusion and logistic kinetics plays an important role at various levels in population dynamics ([12], [30]), with applications including pattern formation in bacterial colonies ([50]) and also in populations of macroscopic individuals ([30]), tumor invasion processes ([7]), as well as self-organization during embryonic development ([28]). In the particular context of Keller-Segel-type chemotactic cross-diffusion, a prototypical version of a corresponding rudimentary model for such processes, when reduced to a parabolic-elliptic framework reflecting comparatively fast diffusion of the respective signal substance, is given by the initial-boundary value problem

$$\begin{cases} u_t &= \Delta u - \nabla\cdot(u\nabla v) + \lambda u - \mu u^\kappa, & x\in\Omega,\ t>0,\\ 0 &= \Delta v - v + u, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, & & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x), & & x\in\Omega, \end{cases} \tag{1.1}$$

with parameters $\lambda\in\mathbb{R}$, $\mu>0$ and $\kappa>1$, and with $\Omega\subset\mathbb{R}^n$, $n\ge 1$, denoting the underlying physical habitat. Systems of this type can in fact be found as subsystems in an abundant number of more elaborate models for taxis mechanisms under the influence of spontaneous proliferation and death in more complex frameworks (see [21], [2] or also [31] for some recent examples).

The mathematical understanding already of the comparatively simple problem (1.1), however, seems far from complete especially with regard to the fundamental question how far the superlinear absorption mechanism expressed in the first equation therein is capable of suppressing taxis-driven explosions. Indeed, while phenomena of blow-up in finite time constitute one of the certainly most characteristic effects of chemotactic cross-diffusion in frameworks of pure Keller-Segel systems without cell kinetics ([13], [25], [11], [45]), for the more complex system (1.1) the literature essentially concentrates on situations in which any such singularity formation is entirely ruled out by the presence of superlinear death terms.

In the particular setting of (1.1), for instance, it is known that in the case of the most standard choice $\kappa=2$ and for any λ, if either $n\le 2$ and $\mu>0$ is arbitrary, or if $n\ge 3$ and μ lies above a critical number μ_c satisfying $\mu_c\le\frac{n-2}{n}$, then no blow-up occurs in the sense that for all reasonably regular initial data the problem (1.1) possesses a globally defined classical solution which is moreover even bounded in both components in $\Omega\times(0,\infty)$; when $\kappa>2$, the same conclusion holds without any rectriction on the size of $\mu>0$ ([35]). Extending results of this flavor, ensuring global existence of smooth bounded solutions under quite similar assumptions on κ,λ and μ, is possible also to the case when $\Omega=\mathbb{R}^n$ ([29]), and even to the fully parabolic counterpart of (1.1) obtained upon replacing the elliptic subproblem for v therein by an initial-boundary value problem for $v_t=\Delta v-v+u$ ([26], [43]). In the latter context, for $\kappa=2$ and arbitrary $n\ge 3$ it could more recently be shown in [17] that any $\mu>0$ is sufficient to warrant global solvability at least in the natural framework of weak solutions which in the case $n=3$ moreover, though possibly undergoing some explosions within finite time, at least eventually become smooth and classical. Resorting to some yet weaker solutions concepts, without imposing restrictions on the size of the initial data some global solutions could be constructed, and some asymptotic regularization effects be detected, also for certain subquadratic degradation terms, namely satisfying $\kappa>2-\frac{1}{n}$ for $n\ge 2$ only, both for (1.1) and its parabolic-parabolic analogue ([42],

[36], [37], [38]). Apart from that, the literature has identified numerous situations in which the regularizing effect of logistic growth restrictions is sufficient to warrant global solvability of solutions to several modifications of (1.1), inter alia accounting for nonlinear cell diffusion ([51], [6], [41], [55], [58]), for variants in the cross-diffusive interaction ([53], [20], [4], [39], [57]) or also in the signal evolution, or even containing couplings to further mechanisms and components such as haptotactic cues ([5], [40], [32]), liquid environments ([15], [34], or competing species ([52], [1]).

Beyond these results essentially concerned with issues related to global well-posedness, the literature has so far provided only few results rigorously confirming the possibility of aggregation phenomena in models of type (1.1), as revealed in numerical experiments and moreover indicated by formal analysis ([27], [22]). Indeed, the analysis on qualitative aspects of solution behavior in chemotaxis systems with logistic kinetics seems widely dominated by studies concerned with effects already known from the corresponding taxis-free and hence diagnonal reaction-diffusion systems; accordingly available findings on asymptotic negligibility of cross-diffusion under appropriate assumptions typically assert stabilization toward spatially constant equilibria in bounded domains ([10], [19], [3], [47], [33], [56]), or also large-time invasion of the whole space by means of wave-like propagation ([23], [29]).

The few available exceptions address the taxis-driven spontaneous emergence of arbitrarily large population densities, possibly in the form of transient phenomena, in presence of suitably small diffusion and death parameters even in spatially one-dimensional, but also in higher-dimensional Keller-Segel systems with quadratic degradation either of parabolic-elliptic ([16], [14], [46]) or of fully parabolic type ([48]). The detection of genuine explosion phenomena, however, could so far be accomplished only in a high-dimensional variant of (1.1), which in comparison to the latter moreover is further simplified in that the signal evolution is considered as governed by the equation $0 = \Delta v - \mu(t) + u$ with the spatially constant average $\mu(t) := \frac{1}{|\Omega|}\int_\Omega u(\cdot,t)$; for the radially symmetric version of this problem posed in n-dimensional balls, namely, it is known that when

$$n \geq 5 \qquad \text{and} \qquad \kappa < \frac{3}{2} + \frac{1}{2n-2}, \tag{1.2}$$

some initial data can be found which finite-time blow-up occurs ([44], cf. also [54] for an analogue addressing a quasilinear generalization of (1.1)).

Main results. The principal purpose of the present work is to provide some rigorous evidence indicating that also in low-dimensional spatial settings, in particular including three-dimensional cases, chemotactic cross-diffusion of the form in (1.1) is strong enough so as to potentially overbalance even superlinear logistic-type dampening in a substantial manner. More precisely, we shall be concerned with the problem (1.1) in the ball $\Omega = B_R(0) \subset \mathbb{R}^n$ with $n \geq 3$ and $R > 0$, and resorting to a spatially radial setting we shall assume that

$$u_0 \in C^0(\overline{\Omega}) \quad \text{is radially symmetric and nonnegative.} \tag{1.3}$$

Our main result will then, in fact for any $n \geq 3$, identify a certain dimension-dependent range of numbers $\kappa > 1$ in which blow-up can be observed:

Theorem 1.1 *Let $\Omega = B_R(0) \subset \mathbb{R}^n$ with $n \geq 3$ and $R > 0$, and let $\lambda \in \mathbb{R}, \mu > 0$ and $\kappa > 1$ be such that*

$$\kappa < \begin{cases} \frac{7}{6} & \text{if } n \in \{3,4\}, \\ 1 + \frac{1}{2(n-1)} & \text{if } n \geq 5. \end{cases} \tag{1.4}$$

Then for all $L > 0$, $m > 0$ and $m_0 \in (0, m)$ one can find $r_0 = r_0(R, \lambda, \mu, \kappa, L, m, m_0) \in (0, R)$ with the property that whenever u_0 satisfies (1.3) and is such that

$$u_0(x) \le L|x|^{-n(n-1)} \qquad \text{for all } x \in \Omega \tag{1.5}$$

as well as

$$\int_\Omega u_0 \le m \qquad \text{but} \qquad \int_{B_{r_0}(0)} u_0 \ge m_0, \tag{1.6}$$

there exist $T_{max} \in (0, \infty)$ and a classical solution (u, v) of (1.1), uniquely determined by the inclusions

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})) \qquad \text{and} \\ v \in \bigcap_{q>n} L^\infty_{loc}([0, T_{max}); W^{1,q}(\Omega)) \cap C^{2,0}(\overline{\Omega} \times (0, T_{max})), \end{cases}$$

which blows up at $t = T_{max}$ in the sense that

$$\limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{1.7}$$

The following consequence underlines that the latter in fact entails blow-up throughout a considerably large class of initial data in the indicated range of κ.

Corollary 1.2 *Let $\Omega = B_R(0) \subset \mathbb{R}^n$ with $n \ge 3$ and $R > 0$, and assume that $\lambda \in \mathbb{R}, \mu > 0$ and $\kappa > 1$ are such that (1.4) holds. Then for any positive function u_0 fulfilling (1.3), there exist initial data u_{0k}, $k \in \mathbb{N}$, which satisfy (1.3) as well as*

$$u_{0k} \to u_0 \quad \text{in } L^1(\Omega) \qquad \text{as } k \to \infty, \tag{1.8}$$

and which are such that for each $k \in \mathbb{N}$, the problem (1.1) possesses a classical solution (u_k, v_k) with $u_k|_{t=0} = u_{0k}$ and blowing up in finite time in the sense specified in Theorem 1.1.

Let us remark that even in the case $n \ge 5$ in which the assumption (1.4) evidently is more restrictive than that in (1.2), this provides progress in comparison to the results in ([44]) which, indeed, do not explicitly contain a density statement of the flavor in Corollary 1.2, and which actually refer to a further simplified problem apparently somewhat simpler than (1.1).

Plan of the paper. Following a well-established approach originating from [13], we will base our analysis on a transformation of (1.1) into a Dirichlet problem for the scalar parabolic equation

$$w_t = n^2 s^{2-\frac{2}{n}} w_{ss} + n w w_s - n z w_s + \lambda w - n^{\kappa-1} \mu \int_0^s w_s^\kappa(\sigma, t) d\sigma, \tag{1.9}$$

satisfied by the mass accumulation functions defined by letting $w(s,t) := \int_{B_{s^{1/n}}(0)} u dx$ and $z(s,t) := \int_{B_{s^{1/n}}(0)} v dx$, $s \in [0, R^n], t \in [0, T_{max})$, with $T_{max} \in (0, \infty]$ denoting the maximal existence of a given radial solution (u, v) to (1.1).

Here a substantial challenge will consist in appropriately making use of the directional character of the summand $+nww_s$, as originating from the cross-diffusive interaction in (1.1), in comparison to

both dampening effects stemming from the local second-order expression reflecting diffusion, as well as from the nonlocal nonlinear absorption term in (1.9). That this can in fact be achieved in the course of an analsyis of the generalized moment functional ϕ defined by

$$\phi(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s)w(s,t)ds, \qquad t \in [0, T_{max}),$$

will be essentially due to the fact that the respective superlinear dissipative term contributing to its time evolution (see Lemma 4.1) can be controlled through an inequality of the form

$$u(x,t) \le C(\varepsilon)|x|^{-n(n-1)-\varepsilon}, \qquad x \in \Omega,\ t \in (0, T_{max}).$$

Adequately exploiting the latter estimate, valid as a consequence of a recent result on pointwise singularity control in radial Keller-Segel systems ([49], cf. also Lemma 3.3 below), will form one focus of Section 4.1 in which inter alia, as the main reason for our restriction (1.4) on κ, the crucial assumption (4.12) on the parameter γ will be required. Thereafter, the Sections 4.2 and 4.3 will be devoted to studying how far the cross-diffusive contribution to the evolution of ϕ may overbalance the respective expression originating from the summand in (1.9) which contains z and hence depends, implicitly, on w in a nonlocal manner as well.

In Lemma 4.9, these preparations will be seen to allow for deriving an autonomous Riccati-type ODI for ϕ, and in Section 5 a suitable adjustment of the yet free parameter s_0 will thereupon yield the blow-up result from Theorem 1.1, followed by a verification of Corollary 1.2 via an appropriate approximation of arbitrary initial data by those falling among the class identified in Theorem 1.1.

2 Local existence and transformation to a scalar problem

The following basic statement on local existence, uniqueness and extensibilty of classical solutions can be obtained by straightforward adaptation of standard procedures based e.g. on reasonings in appropriate fixed point frameworks, as detailed e.g. in [24], [9] and [8] for closely related problems; we may therefore omit repeating arguments of this type here.

Lemma 2.1 *Let $n \ge 1$, $R > 0$, $\lambda \in \mathbb{R}, \mu > 0$ and $\kappa > 1$, and assume that u_0 satisfies (1.3). Then there exist $T_{max} \in (0, \infty]$ and uniquely determined radially symmetric nonnegative functions*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})), \\ v \in \bigcap_{q>n} L^\infty_{loc}([0, T_{max}); W^{1,q}(\Omega)) \cap C^{2,0}(\overline{\Omega} \times (0, T_{max})), \end{cases}$$

such that (u, v) forms a classical solution of (1.1) in $\Omega \times (0, T_{max})$, and such that

$$\text{if } T_{max} < \infty, \quad \text{then} \quad \limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{2.1}$$

Throughout the sequel, given some u_0 fulfilling (1.3) we let $(u, v) = (u(r,t), v(r,t))$ denote the the corresponding local solution of (1.1), as obtained in Lemma 2.1 and extended up to its maximal existence time $T_{max} \le \infty$, and following [13] we set

$$w(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho, t)d\rho, \qquad s \in [0, R^n],\ t \in [0, T_{max}), \tag{2.2}$$

as well as

$$z(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} v(\rho,t)d\rho, \qquad s \in [0,R^n],\ t \in [0,T_{max}), \tag{2.3}$$

Then

$$w_s(s,t) = \frac{1}{n} u(s^{\frac{1}{n}},t) \quad \text{and} \quad w_{ss}(s,t) = \frac{1}{n^2} s^{\frac{1}{n}-1} u_r(s^{\frac{1}{n}},t), \qquad s \in (0,R^n),\ t \in (0,T_{max}), \tag{2.4}$$

and similarly

$$z_s(s,t) = \frac{1}{n} v(s^{\frac{1}{n}},t) \quad \text{and} \quad z_{ss}(s,t) = \frac{1}{n^2} s^{\frac{1}{n}-1} v_r(s^{\frac{1}{n}},t), \qquad s \in (0,R^n),\ t \in (0,T_{max}), \tag{2.5}$$

whence the second equation in (1.1) particularly implies that

$$r^{n-1} v_r(r,t) = z(r^n,t) - w(r^n,t) \qquad \text{for all } r \in (0,R) \text{ and } t \in (0,T_{max}). \tag{2.6}$$

Therefore, an integration in the first equation in (1.1) shows that

$$w_t = n^2 s^{2-\frac{2}{n}} w_{ss} + nww_s - nzw_s + \lambda w - n^{\kappa-1}\mu \int_0^s w_s^\kappa(\sigma,t)d\sigma, \qquad \text{for all } s \in (0,R^n) \text{ and } t \in (0,T_{max}), \tag{2.7}$$

while clearly

$$0 = w(0,t) \le w(s,t) \le w(R^n,t) = \frac{1}{\omega_n} \int_\Omega u(\cdot,t) \qquad \text{for all } s \in (0,R^n) \text{ and } t \in (0,T_{max}),$$

and

$$w_s(s,t) \ge 0 \qquad \text{for all } s \in (0,R^n) \text{ and } t \in (0,T_{max}),$$

where here and throughout the sequel we make use of the abbreviation $\omega_n := n|B_1(0)|$.

3 Basic estimates on mass evolution and singular behavior

To begin with, we derive an upper bound for the total mass functional $\int_\Omega u$ within conveniently short time intervals.

Lemma 3.1 *Let $n \ge 3$, $R > 0$, $\lambda \in \mathbb{R}, \mu > 0$ and $\kappa > 1$, and assume (1.3). Then for the solution (u,v) of (1.1) from Lemma 2.1 we have*

$$\int_\Omega u(x,t)dx \le e^{\lambda_+} \int_\Omega u_0 \qquad \textit{for all } t \in (0,\widehat{T}_{max}), \tag{3.1}$$

where $\widehat{T}_{max} := \min\{1, T_{max}\}$ and $\lambda_+ := \max\{0,\lambda\}$.

Proof. As by (1.1) we have

$$\frac{d}{dt}\int_\Omega u = \lambda\int_\Omega u - \mu\int_\Omega u^\kappa \le \lambda\int_\Omega u \qquad \text{for all } t\in(0,T_{max})$$

and hence

$$\int_\Omega u(\cdot,t) \le e^{\lambda t}\cdot\int_\Omega u_0 \qquad \text{for all } t\in(0,T_{max}),$$

in both cases $\lambda \ge 0$ and $\lambda < 0$ this is evident. □

In the particular parabolic-elliptic setting of (1.1), for radial solutions this L^1 control of the inhomogeneity u therein implies pointwise bounds for v and its gradient which are slightly sharper than those known for fully parabolic analogues ([45], [49]).

Lemma 3.2 *Let $n \ge 3$, $R > 0$, $\lambda \in \mathbb{R}, \mu > 0$ and $\kappa > 1$. Then for all $m > 0$ there exists $C = C(R,\lambda,m) > 0$ such that if (1.3) holds with $\int_\Omega u_0 \le m$, then*

$$|v_r(r,t)| \le Cr^{1-n} \qquad \textit{for all } r\in(0,R) \textit{ and } t\in(0,\widehat{T}_{max}) \tag{3.2}$$

as well as

$$v(r,t) \le Cr^{2-n} \qquad \textit{for all } r\in(0,R) \textit{ and } t\in(0,\widehat{T}_{max}), \tag{3.3}$$

where $\widehat{T}_{max} = \min\{1, T_{max}\}$.

Proof. We first note that since clearly $\int_\Omega v(\cdot,t) = \int_\Omega u(\cdot,t)$ for all $t\in(0,T_{max})$ by (1.1), the inequality (3.1) entails that

$$\int_\Omega u(\cdot,t) \le c_1 := e^{\lambda_+}m \quad \text{and} \quad \int_\Omega v(\cdot,t) \le c_1 \qquad \text{for all } t\in(0,\widehat{T}_{max}). \tag{3.4}$$

Due to the nonnegativity of both u and v, this implies that the functions w and z defined in (2.2) and (2.3) satisfy

$$0 \le w(s,t) \le w(R^n,t) = \frac{1}{\omega_n}\int_\Omega u(\cdot,t) \le \frac{c_1}{\omega_n} \qquad \text{for all } s\in(0,R^n) \text{ and } t\in(0,\widehat{T}_{max})$$

and

$$0 \le z(s,t) \le z(R^n,t) = \frac{1}{\omega_n}\int_\Omega v(\cdot,t) \le \frac{c_1}{\omega_n} \qquad \text{for all } s\in(0,R^n) \text{ and } t\in(0,\widehat{T}_{max}),$$

whence according to (2.6),

$$r^{n-1}v_r(r,t) \le z(r^n,t) \le \frac{c_1}{\omega_n} \qquad \text{for all } r\in(0,R) \text{ and } t\in(0,\widehat{T}_{max}) \tag{3.5}$$

as well as

$$r^{n-1}v_r(r,t) \ge -w(r^n,t) \ge -\frac{c_1}{\omega_n} \qquad \text{for all } r\in(0,R) \text{ and } t\in(0,\widehat{T}_{max}). \tag{3.6}$$

This directly yields (3.2), while (3.3) readily results due to the fact that thus for all $r_0 \in (0,R), r \in (0,R)$ and $t \in (0,\widehat{T}_{max})$,

$$|v(r,t)-v(r_0,t)| \;=\; \left|\int_{r_0}^r v_r(\rho,t)d\rho\right| \le \frac{c_1}{\omega_n}\left|\int_{r_0}^r \rho^{1-n}d\rho\right| = \frac{c_1}{(n-2)\omega_n}|r^{2-n}-r_0^{2-n}|,$$

and that again by (3.4),

$$\min_{r_0\in[\frac{R}{2},R]} v(r_0,t) \le \frac{c_1}{|\Omega\setminus B_{\frac{R}{2}}(0)|}$$

for all $t \in (0,T_{max})$. □

We next utilize a recent general result on pointwise bounds for radial solutions to heat equations, perturbed by linear drift terms with possibly singular behavior controlled in the flavor of (3.2), to turn the latter into a pointwise upper bound for u.

Lemma 3.3 *Let $n \ge 3, R > 0, \lambda \in \mathbb{R}, \mu > 0$ and $\kappa > 1$. Then for all $m > 0, L > 0$ and $\varepsilon > 0$ there exists $C = C(R,\lambda,L,m,\varepsilon) > 0$ such that if u_0 satisfies (1.3) and*

$$\int_\Omega u_0 \le m \tag{3.7}$$

as well as

$$u_0(r) \le Lr^{-n(n-1)} \qquad \text{for all } r \in (0,R), \tag{3.8}$$

then

$$u(r,t) \le Cr^{-n(n-1)-\varepsilon} \qquad \text{for all } r \in (0,R) \text{ and each } t \in (0,\widehat{T}_{max}), \tag{3.9}$$

where again $\widehat{T}_{max} = \min\{1, T_{max}\}$.

Proof. Given $m > 0$, we first invoke Lemma 3.2 to find $c_1 = c_1(R,\lambda,m) > 0$ such that if (1.3) and (3.7) hold, then

$$|v_r(r,t)| \le c_1 r^{1-n} \qquad \text{for all } r \in (0,R) \text{ and } t \in (0,\widehat{T}_{max}),$$

which entails that if for fixed $\varepsilon > 0$ we pick $q > n$ large enough such that

$$\alpha := n(n-1)+\varepsilon > \frac{n(n-1)q}{q-n}, \tag{3.10}$$

then

$$\begin{aligned}\int_\Omega |x|^{(n-1)q}|\nabla v(x,t)|^q dx &= \omega_n\int_0^R r^{(n-1)(q+1)}|v_r(r,t)|^q dr\\ &\le c_1^q\omega_n\int_0^R r^{n-1}dr\\ &= \frac{c_1^q\omega_n R^n}{n} \qquad \text{for all } t \in (0,\widehat{T}_{max}).\end{aligned} \tag{3.11}$$

Now letting $U(x,t) := e^{-\lambda t}u(x,t)$ for $(x,t) \in \overline{\Omega} \times [0,T_{max})$, from (1.1) we obtain that

$$\begin{aligned} U_t &= e^{-\lambda t}u_t - \lambda e^{-\lambda t}u \\ &= \Big\{\Delta U - \nabla\cdot(U\nabla v) + \lambda U - \mu e^{-\lambda t}u^\kappa\Big\} - \lambda U \\ &\le \Delta U - \nabla\cdot(U\nabla v) \qquad \text{in } \Omega\times(0,T_{max}) \end{aligned}$$

with $\frac{\partial U}{\partial \nu} = 0$ on $\partial\Omega \times (0,T_{max})$ and $\int_\Omega U(\cdot,0) = \int_\Omega u_0 \le m$. Therefore, (3.10) and (3.11) warrant that [49, Theorem 1.1] becomes applicable so as to show that due to (3.7) and (3.8) we can find $c_2 = c_2(R,\lambda,L,m,\varepsilon) > 0$ fulfilling

$$U(x,t) \le c_2|x|^{-\alpha} \qquad \text{for all } x\in\Omega \text{ and } t\in(0,\widehat{T}_{max}),$$

which immediately implies (3.9) on choosing $C(R,\lambda,L,m,\varepsilon) := c_2 e^{\lambda_+}$. □

4 A differential inequality for a moment-type functional

We proceed to introduce the main object that will serve as a means for the derivation of our blow-up result, namely the functional ϕ given by (4.1) which may be viewed as a generalized moment of the function w from (2.2), conveniently localized to a region near the origin. A basic differential inequality describing the evolution of ϕ can be obtained through straightforward integration by parts and an application of Fubini's theorem in the respective nonlocal expression from (2.7).

Lemma 4.1 *Let $n \ge 3$, $R > 0$, $\lambda \in \mathbb{R}, \mu > 0$ and $\kappa > 1$, and assume (1.3). Then for any choice of $\gamma \in (1-\frac{2}{n}, 1)$ and $s_0 \in (0,R^n)$, the function $\phi:\ [0,T_{max}) \to \mathbb{R}$ defined by*

$$\phi(t) := \int_0^{s_0} s^{-\gamma}(s_0-s)w(s,t)ds, \qquad t\in[0,T_{max}), \tag{4.1}$$

belongs to $C^0([0,T_{max})) \cap C^1((0,T_{max}))$ and satisfies

$$\begin{aligned} \phi'(t) \ \ge\ & n\int_0^{s_0} s^{-\gamma}(s_0-s)w(s,t)w_s(s,t)ds \\ & -n(\gamma+1)s_0\int_0^{s_0} s^{-\gamma-1}z(s,t)w(s,t)ds \\ & -n^2\Big(2-\frac{2}{n}-\gamma\Big)\Big(\gamma+\frac{2}{n}\Big)s_0\int_0^{s_0} s^{-\gamma-\frac{2}{n}}w(s,t)ds \\ & -\lambda_-\int_0^{s_0} s^{-\gamma}(s_0-s)w(s,t)ds \\ & -\frac{n^{\kappa-1}\mu}{1-\gamma}s_0^{1-\gamma}\int_0^{s_0}(s_0-s)w_s^\kappa(s,t)ds \qquad \textit{for all } t\in(0,T_{max}), \end{aligned} \tag{4.2}$$

where $\lambda_- := \max\{0,-\lambda\}$.

Proof. For fixed $\tau \in (0,T_{max})$ and $T \in (\tau,T_{max})$, it follows from Lemma 2.1 that u_t is continuous in $\overline{\Omega}\times[\tau,T]$ and that hence, by (2.7), w_t is continuous in $[0,R^n]\times[\tau,T]$. As for any choice of $\gamma\in(0,1)$

and $s_0 \in (0, R^n)$ the function $(0, s_0) \ni s \mapsto s^{-\gamma}(s_0 - s)$ is integrable, a standard reasoning based on the dominated convergence theorem shows that indeed for all such γ and s_0, the function ϕ in (4.1) is not only continuous on $[0, T_{max})$ but moreover also continuously differentiable in $(0, T_{max})$ with

$$\begin{aligned}
\phi'(t) &= \int_0^{s_0} s^{-\gamma}(s_0 - s) w_t ds \\
&= n^2 \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0 - s) w_{ss} ds \\
&\quad + n \int_0^{s_0} s^{-\gamma}(s_0 - s) w w_s ds - n \int_0^{s_0} s^{-\gamma}(s_0 - s) z w_s ds \\
&\quad + \lambda \int_0^{s_0} s^{-\gamma}(s_0 - s) w ds \\
&\quad - n^{\kappa-1}\mu \int_0^{s_0} s^{-\gamma}(s_0 - s) \cdot \Big\{ \int_0^s w_s^\kappa(\sigma, t) d\sigma \Big\} ds \qquad \text{for all } t \in (0, T_{max}) \qquad (4.3)
\end{aligned}$$

by (2.2). Here two integrations by parts yield

$$\begin{aligned}
n^2 \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0 - s) w_{ss} ds &= -n^2\Big(2 - \frac{2}{n} - \gamma\Big) \int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0 - s) w_s ds + n^2 \int_0^{s_0} s^{2-\frac{2}{n}-\gamma} w_s ds \\
&\quad + n^2 s^{2-\frac{2}{n}-\gamma}(s_0 - s) w_s \Big|_0^{s_0} \\
&\geq -n^2\Big(2 - \frac{2}{n} - \gamma\Big) \int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0 - s) w_s ds \\
&= -n^2\Big(2 - \frac{2}{n} - \gamma\Big)\Big(\gamma - 1 + \frac{2}{n}\Big) \int_0^{s_0} s^{-\gamma-\frac{2}{n}}(s_0 - s) w ds \\
&\quad - n^2\Big(2 - \frac{2}{n} - \gamma\Big) \int_0^{s_0} s^{1-\frac{2}{n}-\gamma} w ds \\
&\quad + \liminf_{\delta \searrow 0} \Big\{ n^2\Big(2 - \frac{2}{n} - \gamma\Big) \delta^{1-\frac{2}{n}-\gamma}(s_0 - \delta) w(\delta, t) \Big\} \\
&\geq -n^2\Big(2 - \frac{2}{n} - \gamma\Big)\Big(\gamma - 1 + \frac{2}{n}\Big) \int_0^{s_0} s^{-\gamma-\frac{2}{n}} s_0 w ds \\
&\quad - n^2\Big(2 - \frac{2}{n} - \gamma\Big) \int_0^{s_0} s_0 \cdot s^{-\gamma-\frac{2}{n}} w ds \\
&= -n^2\Big(2 - \frac{2}{n} - \gamma\Big)\Big(\gamma + \frac{2}{n}\Big) s_0 \int_0^{s_0} s^{-\gamma-\frac{2}{n}} w ds \qquad (4.4)
\end{aligned}$$

for all $t \in (0, T_{max})$, because w_s is nonnegative and locally bounded in $[0, R^n] \times [0, T_{max})$ by Lemma 2.1, and because $2 - \frac{2}{n} - \gamma$ and $\gamma - 1 + \frac{2}{n}$ are positive thanks to our restrictions that $\gamma < 1$ and $n \geq 3$, and that $\gamma > 1 - \frac{2}{n}$.

Next, once more integrating by parts we obtain that

$$\begin{aligned}
-n \int_0^{s_0} s^{-\gamma}(s_0 - s) z w_s ds &= n \int_0^{s_0} s^{-\gamma}(s_0 - s) z_s w ds - n\gamma \int_0^{s_0} s^{-\gamma-1}(s_0 - s) z w ds \\
&\quad - n \int_0^{s_0} s^{-\gamma} z w ds + \liminf_{\delta \searrow 0} \Big\{ n\delta^{-\gamma}(s_0 - \delta) z(\delta, t) w(\delta, t) \Big\}
\end{aligned}$$

$$
\begin{aligned}
&\geq -n\gamma \int_0^{s_0} s^{-\gamma-1}(s_0-s)zwds - n\int_0^{s_0} s^{-\gamma}zwds \\
&\geq -n\gamma \int_0^{s_0} s^{-\gamma-1}s_0zwds - n\int_0^{s_0} s_0\cdot s^{-\gamma-1}zwds \\
&= -n(\gamma+1)s_0\int_0^{s_0} s^{-\gamma-1}zwds \qquad \text{for all } t\in(0,T_{max}) \qquad (4.5)
\end{aligned}
$$

due to the fact that z_s, z and w are all nonnegative.
In the rightmost summand in (4.3) we apply the Fubini theorem to see that for all $t\in(0,T_{max})$,

$$
\int_0^{s_0} s^{-\gamma}(s_0-s)\cdot\left\{\int_0^s w_s^\kappa(\sigma,t)d\sigma\right\}ds = \int_0^{s_0}\left\{\int_\sigma^{s_0} s^{-\gamma}(s_0-s)ds\right\}\cdot w_s^\kappa(\sigma,t)d\sigma, \qquad (4.6)
$$

where again using that $\gamma<1$ we can estimate

$$
\int_\sigma^{s_0} s^{-\gamma}(s_0-s)ds \leq (s_0-\sigma)\int_0^{s_0} s^{-\gamma} = \frac{s_0^{1-\gamma}}{1-\gamma}\cdot(s_0-\sigma) \qquad \text{for all } \sigma\in 0,s_0). \qquad (4.7)
$$

As clearly

$$
\lambda\int_0^{s_0} s^{-\gamma}(s_0-s)wds \geq -\lambda_-\int_0^{s_0} s^{-\gamma}(s_0-s)wds \qquad \text{for all } t\in(0,T_{max}),
$$

combining (4.3)-(4.7) we immediately arrive at (4.2). □

4.1 Estimating the last three integrals in (4.2)

Our key toward our derivation of approppriate estimates for the three rightmost negative summands in (4.2) consists in the following basic but helpful observation which provides a pointwise upper bound for widely arbitrary functions in terms of an expression essentially resembling the positive and cross-diffusion-induced positive contribution on the right of (4.2).

Lemma 4.2 *Let $\gamma\in(0,2)$ and $s_0>0$, and let $\varphi\in C^1([0,s_0])$ be nonnegative with $\varphi(0)=0$ and $\varphi'(s)\geq 0$ for all $s\in(0,s_0)$. Then*

$$
\varphi(s) \leq \sqrt{2}\cdot s^{\frac{\gamma}{2}}(s_0-s)^{-\frac{1}{2}}\cdot\left\{\int_0^{s_0}\sigma^{-\gamma}(s_0-\sigma)\varphi(\sigma)\varphi'(\sigma)d\sigma\right\}^{\frac{1}{2}} \qquad \textit{for all } s\in(0,s_0). \qquad (4.8)
$$

Proof. As $\gamma<2$, $\varphi(0)=0$ and $\varphi'\in C^0([0,s_0])$, it follows that $\psi(s):=\frac{1}{2}s^{-\gamma}(s_0-s)\varphi^2(s)$, $s\in(0,s_0]$, actually defines a function $\psi\in C^0([0,s_0])\cap C^1((0,s_0))$ with $\psi(0)=0$, whence the fundamental theorem of elementary calculus applies so as to warrant that

$$
\begin{aligned}
\psi(s) &= \int_0^s \psi'(\sigma)d\sigma \\
&= \int_0^s\left\{\sigma^{-\gamma}(s_0-\sigma)\varphi(\sigma)\varphi'(\sigma) - \frac{\gamma}{2}\sigma^{\frac{\gamma}{2}-1}(s_0-\sigma)\varphi^2(\sigma) - \frac{1}{2}\sigma^{-\gamma}\varphi^2(\sigma)\right\}d\sigma \\
&\leq \int_0^{s_0}\sigma^{-\gamma}(s_0-\sigma)\varphi(\sigma)\varphi'(\sigma)d\sigma \qquad \text{for all } s\in(0,s_0),
\end{aligned}
$$

because $\gamma > 0$ and φ' is nonnegative. By definition of ψ, this directly implies (4.8). $\square$

A first application yields an estimate for the integral in (4.2) originating from the cell diffusion process in (1.1), provided that the exponent γ is conveniently small.

Lemma 4.3 *Let $n \geq 3$ and $\gamma \in (0, 2 - \frac{4}{n})$. Then there exists $C = C(\gamma) > 0$ such that if $R > 0, \lambda \in \mathbb{R}, \mu > 0$ and $\kappa > 1$, and if u_0 is such that (1.3) holds, then for arbitrary $s_0 \in (0, R^n)$ we have*

$$s_0 \int_0^{s_0} s^{-\gamma-\frac{2}{n}} w(s,t)ds \leq C s_0^{\frac{3-\gamma}{2}-\frac{2}{n}} \cdot \left\{ \int_0^{s_0} s^{-\gamma}(s_0 - s)w(s,t)w_s(s,t)ds \right\}^{\frac{1}{2}} \qquad \text{for all } t \in (0, T_{max}). \tag{4.9}$$

Proof. Using Lemma 4.2, we estimate

$$s_0 \int_0^{s_0} s^{-\gamma-\frac{2}{n}} w ds \leq \sqrt{2} \cdot \left\{ \int_0^{s_0} s^{-\gamma}(s_0 - s)ww_s ds \right\}^{\frac{1}{2}} \cdot s_0 \int_0^{s_0} s^{-\frac{\gamma}{2}-\frac{2}{n}}(s_0 - s)^{-\frac{1}{2}} ds \qquad \text{for all } t \in (0, T_{max}), \tag{4.10}$$

where by a simple variable transformation,

$$\begin{aligned} s_0 \int_0^{s_0} s^{-\frac{\gamma}{2}-\frac{2}{n}}(s_0 - s)^{-\frac{1}{2}} ds &= s_0 \int_0^1 (s_0\sigma)^{-\frac{\gamma}{2}-\frac{2}{n}} [s_0(1-\sigma)]^{-\frac{1}{2}} \cdot s_0 d\sigma \\ &= s_0^{\frac{3-\gamma}{2}-\frac{2}{n}} B\Big(1 - \frac{\gamma}{2} - \frac{2}{n}, \frac{1}{2}\Big). \end{aligned}$$

Since herein the latter expression involving Euler's Beta function B is well-defined due to the fact that our assumption $\gamma < 2 - \frac{4}{n}$ ensures that $\frac{\gamma}{2} + \frac{2}{n} < 1$, (4.9) directly results upon an evident definition of C. $\square$

Secondly, Lemma 4.2 can be used to control the second last summand in (4.2) as follows.

Lemma 4.4 *Let $n \geq 3$ and $\gamma \in (0, 2)$. Then there exists $C = C(\gamma) > 0$ such that if $R > 0, \lambda \in \mathbb{R}, \mu > 0$ and $\kappa > 1$, for any choice of u_0 fulfilling (1.3) and each $s_0 \in (0, R^n)$ we have*

$$\int_0^{s_0} s^{-\gamma}(s_0 - s)w(s,t)ds \leq C \cdot s_0^{\frac{3-\gamma}{2}} \cdot \left\{ \int_0^{s_0} s^{-\gamma}(s_0 - s)w(s,t)w_s(s,t)ds \right\}^{\frac{1}{2}} \qquad \text{for all } t \in (0, T_{max}). \tag{4.11}$$

Proof. In a way quite similar to that in Lemma 4.3, from Lemma 4.2 we derive the inequality

$$\begin{aligned} \int_0^{s_0} s^{-\gamma}(s_0 - s)wds &\leq s_0 \int_0^{s_0} s^{-\gamma} w ds \\ &\leq \sqrt{2} \cdot \left\{ \int_0^{s_0} s^{-\gamma}(s_0 - s)ww_s ds \right\}^{\frac{1}{2}} \cdot s_0 \int_0^{s_0} s^{-\frac{\gamma}{2}}(s_0 - s)^{-\frac{1}{2}} ds \\ &= \sqrt{2} \cdot \left\{ \int_0^{s_0} s^{-\gamma}(s_0 - s)ww_s ds \right\}^{\frac{1}{2}} \cdot s_0^{\frac{3-\gamma}{2}} B\Big(1 - \frac{\gamma}{2}, \frac{1}{2}\Big) \qquad \text{for all } t \in (0, T_{max}), \end{aligned}$$

because $\frac{\gamma}{2} < 1$. $\square$

When combined with a pointwise inequality for w_s implied by Lemma 3.3, a third application of Lemma 4.2 shows that if unlike in Lemma 4.3 γ now is suitably large, then also the dampening action of the logistic death term can be estimated in terms of the positive summand on the right of (4.2).

Lemma 4.5 *Let $n \geq 3$, $R > 0$, $\lambda \in \mathbb{R}, \mu > 0, \kappa > 1$ and $\gamma \in (0,2)$ be such that*

$$(n-1)(\kappa-1) < \frac{\gamma}{2}. \tag{4.12}$$

Then for all $m > 0$ and $L > 0$ and any choice of $\varepsilon > 0$ one can find $C = C(R, \lambda, \kappa, \gamma, L, m, \varepsilon) > 0$ such that if u_0 satisfies (1.3) and $\int_\Omega u_0 \leq m$ as well as (3.8), then for any $s_0 \in (0, R^n)$ and all $t \in (0, \widehat{T}_{max})$,

$$s_0^{1-\gamma} \int_0^{s_0} (s_0 - s) w_s^\kappa(s,t) ds \leq C s_0^{-(n-1)(\kappa-1)+\frac{3-\gamma}{2}-\varepsilon} \cdot \left\{ \int_0^{s_0} s^{-\gamma}(s_0 - s) w(s,t) w_s(s,t) ds \right\}^{\frac{1}{2}}, \tag{4.13}$$

where $\widehat{T}_{max} = \min\{1, T_{max}\}$.

Proof. Given $\varepsilon > 0$, according to (4.12) we can fix $\eta > 0$ small enough such that both

$$\frac{\eta}{n} \cdot (\kappa - 1) \leq \min\{\varepsilon, 1\} \tag{4.14}$$

and

$$(n-1)(\kappa-1) + \frac{\eta}{n} \cdot (\kappa - 1) < \frac{\gamma}{2}, \tag{4.15}$$

and thereupon invoke Lemma 3.3 to find $c_1 = c_1(R, \lambda, L, m, \varepsilon) > 0$ such that whenever u_0 satisfies (1.3) and $\int_\Omega u_0 \leq m$ as well as (3.8), we have

$$u(r,t) \leq c_1 r^{-n(n-1)-\eta} \qquad \text{for all } r \in (0,R) \text{ and } t \in (0, T_{max}).$$

For any such u_0, rewritten through (2.2) this means that

$$\begin{aligned} w_s^{\kappa-1}(s,t) &= \left(\frac{u(s^{\frac{1}{n}}, t)}{n} \right)^{\kappa-1} \\ &\leq c_2 s^{-(n-1)(\kappa-1)-\frac{\eta}{n}(\kappa-1)} \qquad \text{for all } s \in (0, R^n) \text{ and } t \in (0, \widehat{T}_{max}) \end{aligned}$$

with $c_2 := (\frac{c_1}{n})^{\kappa-1}$. In the integral under consideration this enables us to integrate by parts to see that thanks to (4.14), for all $t \in (0, \widehat{T}_{max})$ we have

$$\begin{aligned} s_0^{1-\gamma} \int_0^{s_0} (s_0 - s) w_s^\kappa ds &\leq c_2 s_0^{1-\gamma} \int_0^{s_0} s^{-(n-1)(\kappa-1)-\frac{\eta}{n}(\kappa-1)} (s_0 - s) w_s ds \\ &= \left[(n-1)(\kappa-1) + \frac{\eta}{n}(\kappa-1) \right] c_2 s_0^{1-\gamma} \int_0^{s_0} s^{-(n-1)(\kappa-1)-1-\frac{\eta}{n}(\kappa-1)} (s_0 - s) w ds \\ &\quad + c_2 s_0^{1-\gamma} \int_0^{s_0} s^{-(n-1)(\kappa-1)-\frac{\eta}{n}(\kappa-1)} w ds \\ &\quad - \liminf_{\delta \searrow 0} \left\{ c_2 s_0^{1-\gamma} \delta^{(n-1)(\kappa-1)-\frac{\eta}{n}(\kappa-1)} (s_0 - \delta) w(\delta, t) \right\} \\ &\leq [(n-1)(\kappa-1) + 2] c_2 s_0^{2-\gamma} \int_0^{s_0} s^{-(n-1)(\kappa-1)-1-\frac{\eta}{n}(\kappa-1)} w ds, \end{aligned} \tag{4.16}$$

where again by means of Lemma 4.2 we can estimate

$$\begin{aligned}
&\int_0^{s_0} s^{-(n-1)(\kappa-1)-1-\frac{\eta}{n}(\kappa-1)} w ds \\
&\quad \le \sqrt{2}\cdot\left\{\int_0^{s_0} s^{-\gamma}(s_0-s)ww_s ds\right\}^{\frac12}\cdot\int_0^{s_0} s^{-(n-1)(\kappa-1)-1+\frac{\gamma}{2}-\frac{\eta}{n}(\kappa-1)}(s_0-s)^{-\frac12}ds \\
&\quad \le \sqrt{2}\cdot\left\{\int_0^{s_0} s^{-\gamma}(s_0-s)ww_s ds\right\}^{\frac12}\cdot s_0^{-(n-1)(\kappa-1)+\frac{\gamma-1}{2}-\frac{\eta}{n}(\kappa-1)}\cdot c_3 \quad \text{for all } t\in(0,\widehat{T}_{max}) \qquad (4.17)
\end{aligned}$$

with $c_3 := B\Big(\frac{\gamma}{2}-(n-1)(\kappa-1)-\frac{\eta}{n}(\kappa-1)\,,\,\frac12\Big)$ being well-defined and finite because of the fact that $\frac{\gamma}{2}-(n-1)(\kappa-1)-\frac{\eta}{n}(\kappa-1)$ is positive by (4.15).
Combining (4.16) with (4.17) thus shows that for all $t\in(0,\widehat{T}_{max})$,

$$\begin{aligned}
&s_0^{1-\gamma}\int_0^{s_0}(s_0-s)w_s^\kappa ds \\
&\quad \le \sqrt{2}[(n-1)(\kappa-1)+2]c_2c_3 s_0^{-(n-1)(\kappa-1)+\frac{3-\gamma}{2}-\frac{\eta}{n}(\kappa-1)}\cdot\left\{\int_0^{s_0} s^{-\gamma}(s_0-s)ww_s ds\right\}^{\frac12} \\
&\quad \le \sqrt{2}[(n-1)(\kappa-1)+2]c_2c_3 R^{n\varepsilon-\eta(\kappa-1)} s_0^{-(n-1)(\kappa-1)+\frac{3-\gamma}{2}-\varepsilon}\cdot\left\{\int_0^{s_0} s^{-\gamma}(s_0-s)ww_s ds\right\}^{\frac12},
\end{aligned}$$

for $\frac{\eta}{n}(\kappa-1)\le\varepsilon$ due to (4.14). □

4.2 A pointwise inequality for z

In order to prepare an adequate estimation of the second integral on the right of (4.2), in this section we derive an upper bound for the function z appearing therein. As through (1.1) this quantity depends on w in a nonlocal manner obtained by two successive integrations, let us first focus on the essential part appearing therein:

Lemma 4.6 *Let $\alpha\in(1,2)$ and $\beta\in(0,1)$. Then there exists $C=C(\alpha,\beta)>0$ such that if $s_0>0$, then*

$$\int_0^s\int_\sigma^{s_0}\xi^{-\alpha}(s_0-\xi)^{-\beta}d\xi d\sigma \le Cs_0^{-\beta}s^{2-\alpha} \qquad \textit{for all } s\in(0,s_0). \tag{4.18}$$

Proof. By means of the Fubini theorem, we rewrite

$$J(s) := \int_0^s\int_\sigma^{s_0}\xi^{-\alpha}(s_0-\xi)^{-\beta}d\xi d\sigma, \qquad s\in[0,s_0],$$

according to

$$\begin{aligned}
J(s) &= \int_0^s\left\{\int_0^\xi d\sigma\right\}\cdot\xi^{-\alpha}(s_0-\xi)^{-\beta}d\xi + \int_s^{s_0}\left\{\int_0^s d\sigma\right\}\cdot\xi^{-\alpha}(s_0-\xi)^{-\beta}d\xi \\
&= \int_0^s \xi^{1-\alpha}(s_0-\xi)^{-\beta}d\xi + s\cdot\int_s^{s_0}\xi^{-\alpha}(s_0-\xi)^{-\beta}d\xi \\
&=: J_1(s)+J_2(s) \qquad \text{for all } s\in[0,s_0]. \qquad (4.19)
\end{aligned}$$

Here, first concentrating on the case when $s \le \frac{s_0}{2}$ we may use that $\alpha < 2$ to see that

$$J_1(s) \le \Big(\frac{s_0}{2}\Big)^{-\beta} \int_0^s \xi^{1-\alpha} d\xi = \frac{2^\beta}{2-\alpha} s_0^{-\beta} s^{2-\alpha} \qquad \text{for all } s \in \Big[0, \frac{s_0}{2}\Big], \tag{4.20}$$

and rely on the assumptions that $\alpha > 1$ and $\beta < 1$ in estimating

$$\begin{aligned} J_2(s) &= s \cdot \int_s^{\frac{s_0}{2}} \xi^{-\alpha}(s_0-\xi)^{-\beta} d\xi + s \cdot \int_{\frac{s_0}{2}}^{s_0} \xi^{-\alpha}(s_0-\xi)^{-\beta} d\xi \\ &\le \Big(\frac{s_0}{2}\Big)^{-\beta} s \cdot \int_s^\infty \xi^{-\alpha} d\xi + s \cdot \Big(\frac{s_0}{2}\Big)^{-\alpha} \cdot \int_{\frac{s_0}{2}}^{s_0} (s_0-\xi)^{-\beta} d\xi \\ &= \frac{2^\beta}{\alpha-1} s_0^{-\beta} s^{2-\alpha} + \frac{1}{1-\beta}\Big(\frac{s_0}{2}\Big)^{1-\alpha-\beta} \cdot s \qquad \text{for all } s \in \Big[0, \frac{s_0}{2}\Big]. \end{aligned}$$

As within this range we have $s = s^{\alpha-1} \cdot s^{2-\alpha} \le (\frac{s_0}{2})^{\alpha-1} s^{2-\alpha}$, this shows that in fact

$$J_2(s) \le \frac{2^\beta}{\alpha-1} s_0^{-\beta} s^{2-\alpha} + \frac{1}{1-\beta}\Big(\frac{s_0}{2}\Big)^{-\beta} s^{2-\alpha} \qquad \text{for all } s \in \Big[0, \frac{s_0}{2}\Big],$$

which combined with (4.20) and (4.19) entails that

$$J(s) \le \Big(\frac{2^\beta}{2-\alpha} + \frac{2^\beta}{\alpha-1} + \frac{2^\beta}{1-\beta}\Big) s_0^{-\beta} s^{2-\alpha} \qquad \text{for all } s \in \Big[0, \frac{s_0}{2}\Big]. \tag{4.21}$$

For larger values of s, however, we only need to observe that $[0, s_0] \ni s \mapsto J(s)$ is nondecreasing and hence

$$J(s) \le J(s_0) = J_1(s_0) = \int_0^{s_0} \xi^{1-\alpha}(s_0-\xi)^{-\beta} d\xi = c_1 s_0^{2-\alpha-\beta} \qquad \text{for all } s \in \Big[\frac{s_0}{2}, s_0\Big],$$

with finiteness of $c_1 := B(2-\alpha, 1-\beta)$ guaranteed by the inequalities $\alpha < 2$ and $\beta < 1$. Therefore, namely, we obtain that

$$J(s) \le c_1 s_0^{-\beta} s_0^{2-\alpha} \le c_1 s_0^{-\beta} \cdot (2s)^{2-\alpha} \qquad \text{for all } s \in \Big[\frac{s_0}{2}, s_0\Big],$$

which together with (4.21) establishes (4.18) upon an obvious definition of C. □

Through an appropriate representation formula based on (1.1), we can thereby achieve the following pointwise inequality for z, again containing the first integral from the right-hand side in (4.2), and again under the restriction on γ from Lemma 4.3.

Lemma 4.7 *Let $n \ge 3$, $R > 0, \lambda \in \mathbb{R}, \mu > 0$ and $\kappa > 1$, and suppose that $\gamma > 0$ is such that $\gamma < 2 - \frac{4}{n}$. Then for all $m > 0$ there exists $C = C(R, \lambda, m, \gamma) > 0$ such that whenever u_0 satisfies (1.3) and $\int_\Omega u_0 \le m$, then for any $s_0 \in (0, R^n)$, the function z defined in (2.3) has the property that*

$$z(s,t) \le C s_0^{\frac{2}{n}-1} s + C s_0^{-\frac{1}{2}} s^{\frac{2}{n}+\frac{\gamma}{2}} \cdot \Big\{ \int_0^{s_0} \sigma^{-\gamma}(s_0-\sigma) w(\sigma,t) w_s(\sigma,t) d\sigma \Big\}^{\frac{1}{2}}$$

$$\text{for all } s \in (0, s_0) \text{ and any } t \in (0, \widehat{T}_{max}), \tag{4.22}$$

where as before $\widehat{T}_{max} := \min\{1, T_{max}\}$.

Proof. We first employ Lemma 3.2 to find $c_1 = c_1(R, \lambda, m) > 0$ such that if u_0 is such that (1.3) holds with $\int_\Omega u_0 \le m$, then

$$v(r,t) \le c_1 r^{2-n} \qquad \text{for all } r \in (0,R) \text{ and } t \in (0, \widehat{T}_{max}). \tag{4.23}$$

Moreover, going back to (2.6) we obtain the one-sided inequality

$$r^{n-1} v_r(r,t) \ge -w(r^n, t) \qquad \text{for all } r \in (0,R) \text{ and any } t \in (0, T_{max}),$$

which when rephrased in terms of the variables z and s says that

$$z_{ss}(s,t) \ge -\frac{1}{n^2} s^{\frac{2}{n}-2} w(s,t) \qquad \text{for all } s \in (0, R^n) \text{ and } t \in (0, T_{max}).$$

Upon two integrations, for arbitrary $s_0 \in (0, R^n)$ this entails that due to (4.23),

$$\begin{aligned} z_s(s,t) &= z_s(s_0,t) - \int_s^{s_0} z_{ss}(\sigma,t) d\sigma \\ &= \frac{1}{n} v(s_0^{\frac{1}{n}}, t) - \int_s^{s_0} z_{ss}(\sigma,t) d\sigma \\ &\le \frac{c_1}{n} s_0^{\frac{2}{n}-1} + \frac{1}{n^2} \int_s^{s_0} \sigma^{\frac{2}{n}-2} w(\sigma,t) d\sigma \qquad \text{for all } s \in (0,s_0) \text{ and } t \in (0, \widehat{T}_{max}) \end{aligned}$$

and that hence

$$\begin{aligned} z(s,t) &= \int_0^s z_s(\sigma,t) d\sigma \\ &\le \frac{c_1}{n} s_0^{\frac{2}{n}-1} s + \frac{1}{n^2} \int_0^s \int_\sigma^{s_0} \xi^{\frac{2}{n}-2} w(\xi,t) d\xi d\sigma \qquad \text{for all } s \in (0,s_0) \text{ and } t \in (0, \widehat{T}_{max}). \end{aligned}$$

Once more by means of Lemma 4.2, we thus infer that for all $s \in (0,s_0)$ and $t \in (0, \widehat{T}_{max})$,

$$\begin{aligned} z(s,t) \le\ & \frac{c_1}{n} s_0^{\frac{2}{n}-1} s \\ & + \frac{\sqrt{2}}{n^2} \cdot \left\{ \int_0^{s_0} \sigma^{-\gamma} (s_0 - \sigma) w(\sigma,t) w_s(\sigma,t) d\sigma \right\}^{\frac{1}{2}} \cdot \int_0^s \int_\sigma^{s_0} \xi^{\frac{2}{n}-2+\frac{\gamma}{2}} (s_0 - \xi)^{-\frac{1}{2}} d\xi d\sigma, \end{aligned} \tag{4.24}$$

where since our assumption $\gamma < 2 - \frac{4}{n}$ warrants that $2 > -\frac{2}{n} + 2 - \frac{\gamma}{2} > -\frac{2}{n} + 2 - \frac{2-\frac{4}{n}}{2} = 1$, we may invoke Lemma 4.6 to pick $c_2 = c_2(\gamma) > 0$ fulfilling

$$\int_0^s \int_\sigma^{s_0} \xi^{\frac{2}{n}-2+\frac{\gamma}{2}} (s_0 - \xi)^{-\frac{1}{2}} d\xi d\sigma \le c_2 s_0^{-\frac{1}{2}} s^{\frac{2}{n}+\frac{\gamma}{2}} \qquad \text{for all } s \in (0,s_0).$$

Therefore, (4.22) is a consequence of (4.24). □

4.3 Estimating the fourth last integral in (4.2)

Now once more invoking Lemma 4.2, in quite a straightforward manner we obtain the following from Lemma 4.7.

Lemma 4.8 *Let $n \ge 3, R > 0, \lambda \in \mathbb{R}, \mu > 0$ and $\kappa > 1$, and let $\gamma \in (0,1)$ be such that $\gamma < 2 - \frac{4}{n}$. Then for all $m > 0$ one can find $K = K(R,\lambda,\gamma,m) > 0$ such that if (1.3) holds with $\int_\Omega u_0 \le m$, then for all $s_0 \in (0,R^n)$, with $\widehat{T}_{max} = \min\{1, T_{max}\}$ we have*

$$n(\gamma+1)s_0 \int_0^{s_0} s^{-\gamma-1} z(s,t)w(s,t)ds \le K s_0^{\frac{2}{n}+1-\gamma} + K s_0^{\frac{2}{n}} \int_0^{s_0} s^{-\gamma}(s_0-s)w(s,t)w_s(s,t)ds \tag{4.25}$$

for all $t \in (0,\widehat{T}_{max})$.

Proof. According to Lemma 4.7, there exists $c_1 = c_1(R,\lambda,m,\gamma) > 0$ such that if (1.3) holds and $\int_\Omega u_0 \le m$, then for any choice of $s_0 \in (0,R^n)$,

$$z(s,t) \le c_1 s_0^{\frac{2}{n}-1} s + c_1 s_0^{-\frac{1}{2}} s^{\frac{2}{n}+\frac{\gamma}{2}} \cdot \left\{ \int_0^{s_0} \sigma^{-\gamma}(s_0-\sigma)ww_s d\sigma \right\}^{\frac{1}{2}} \qquad \text{for all } s \in (0,s_0) \text{ and } t \in (0,\widehat{T}_{max}), \tag{4.26}$$

whereas Lemma 3.1 provides $c_2 = c_2(\lambda,m) > 0$ such that

$$w(s,t) \le c_2 \qquad \text{for all } s \in (0,R^n) \text{ and each } t \in (0,\widehat{T}_{max}). \tag{4.27}$$

Thus, by (4.26) we see that for all $t \in (0,\widehat{T}_{max})$,

$$s_0 \int_0^{s_0} s^{-\gamma-1} zwds \le c_1 s_0^{\frac{2}{n}} \int_0^{s_0} s^{-\gamma} wds + c_1 \left\{ \int_0^{s_0} s^{-\gamma}(s_0-s)ww_s ds \right\}^{\frac{1}{2}} \cdot s_0^{\frac{1}{2}} \int_0^{s_0} s^{\frac{2}{n}-1-\frac{\gamma}{2}} wds, \tag{4.28}$$

where due to (4.27) and the restriction that $\gamma < 1$,

$$\begin{aligned} c_1 s_0^{\frac{2}{n}} \int_0^{s_0} s^{-\gamma} wds &\le c_1 c_2 s_0^{\frac{2}{n}} \int_0^{s_0} s^{-\gamma} ds \\ &= \frac{c_1 c_2}{1-\gamma} s_0^{\frac{2}{n}+1-\gamma} \qquad \text{for all } t \in (0,\widehat{T}_{max}). \end{aligned} \tag{4.29}$$

Furthermore, again relying on Lemma 4.2 we can estimate

$$\begin{aligned} s_0^{\frac{1}{2}} \int_0^{s_0} s^{\frac{2}{n}-1-\frac{\gamma}{2}} wds &\le \sqrt{2} \cdot \left\{ \int_0^{s_0} s^{-\gamma}(s_0-s)ww_s ds \right\}^{\frac{1}{2}} \cdot s_0^{\frac{1}{2}} \int_0^{s_0} s^{\frac{2}{n}-1}(s_0-s)^{-\frac{1}{2}} ds \\ &= \sqrt{2} \cdot \left\{ \int_0^{s_0} s^{-\gamma}(s_0-s)ww_s ds \right\}^{\frac{1}{2}} \cdot s_0^{\frac{2}{n}} B\Big(\frac{2}{n},\frac{1}{2}\Big) \qquad \text{for all } t \in (0,\widehat{T}_{max}), \end{aligned}$$

which when together with (4.29) inserted into (4.28) yields (4.25). □

4.4 An autonomous superlinear differential inequality for ϕ

Now it turns out that if κ satisfies the requirements from Theorem 1.1, then one can find some $\gamma > 0$ which is simultaneously admissible in all of the previously gained estimates, provided that s_0 is chosen suitably small. Upon collecting, in summary we can thereby develop the inequality from Lemma 4.1 into an autonomous ODI for ϕ which contains a quadratic source term.

Lemma 4.9 *Let $n \ge 3, R > 0, \lambda \in \mathbb{R}$ and $\mu > 0$, and let $\kappa > 1$ be such that*

$$\kappa < \begin{cases} \frac{7}{6} & \text{if } n \in \{3,4\}, \\ 1 + \frac{1}{2(n-1)} & \text{if } n \ge 5. \end{cases} \tag{4.30}$$

Then there exists $\gamma = \gamma(\kappa) \in (1 - \frac{2}{n}, 1)$ with the property that for all $m > 0$ and $L > 0$ one can find $s_\star = s_\star(R, \lambda, m) \in (0, R^n)$ and $C = C(R, \lambda, \mu, \kappa, L, m) > 0$ such that whenever u_0 satisfies (1.3) as well as $\int_\Omega u_0 \le m$ and (3.8), for any choice of $s_0 \in (0, s_\star)$ the function ϕ defined in (4.1) satisfies

$$\phi'(t) \ge \frac{1}{C} s_0^{\gamma-3} \phi^2(t) - C s_0^{\frac{2}{n}+1-\gamma} \qquad \text{for all } t \in (0, \widehat{T}_{max}) \tag{4.31}$$

with $\widehat{T}_{max} = \min\{1, T_{max}\}$.

Proof. In the case $n = 3$, (4.30) entails that $(n-1)(\kappa-1) = 2(\kappa-1) < \frac{1}{3} = \frac{2-\frac{4}{n}}{2}$, while if $n \ge 4$ then from (4.30) we know that $(n-1)(\kappa-1) < (n-1) \cdot \frac{1}{2(n-1)} = \frac{1}{2}$. In both cases it is therefore possible to pick $\gamma = \gamma(\kappa) \in (1 - \frac{2}{n}, 1)$ such that $\gamma < 2 - \frac{4}{n}$ and that still

$$(n-1)(\kappa-1) < \frac{\gamma}{2}. \tag{4.32}$$

Keeping this value of γ fixed henceforth and choosing any $\varepsilon > 0$ fulfilling

$$2\varepsilon \le 1 - \frac{2}{n}, \tag{4.33}$$

given $m > 0$ and $L > 0$ we take $K = K(R, \lambda, \gamma, m) > 0$ as provided by Lemma 4.8, and we claim that the desired conclusion is valid if we let $s_\star = s_\star(R, \lambda, m) \in (0, R^n)$ be small enough such that

$$K s_\star^{\frac{2}{n}} \le \frac{n}{8}. \tag{4.34}$$

To this end, we invoke Lemma 4.1 to find $c_1 = c_1(\lambda, \mu, \kappa) > 0$ such that for any u_0 fulfilling (1.3) as well as $\int_\Omega u_0 \le m$ and (3.8), for ϕ as in (4.1) we have

$$\begin{aligned} \phi'(t) \quad \ge \quad & n\psi(t) - n(\gamma+1)s_0 \int_0^{s_0} s^{-\gamma-1} zw ds - c_1 s_0 \int_0^{s_0} s^{-\gamma-\frac{2}{n}} w ds \\ & - c_1 \int_0^{s_0} s^{-\gamma}(s_0 - s) w ds - c_1 s_0^{1-\gamma} \int_0^{s_0} (s_0 - s) w_s^\kappa ds \qquad \text{for all } t \in (0, \widehat{T}_{max}), \end{aligned} \tag{4.35}$$

where

$$\psi(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s) w w_s ds, \qquad t \in (0, T_{max}).$$

Here since $\gamma < 2 - \frac{4}{n}$, Lemma 4.3 provides $c_2 = c_2(\lambda, \mu, \kappa) > 0$ such that for any such u_0 and all $s_0 \in (0, R^n)$, due to Young's inequality we can estimate

$$\begin{aligned} c_1 s_0 \int_0^{s_0} s^{-\gamma-\frac{2}{n}} w ds &\leq c_2 s_0^{\frac{3-\gamma}{2}-\frac{2}{n}} \sqrt{\psi(t)} \\ &\leq \frac{n}{8}\psi(t) + \frac{2c_2^2}{n} s_0^{3-\gamma-\frac{4}{n}} \qquad \text{for all } t \in (0, \widehat{T}_{max}), \end{aligned} \tag{4.36}$$

whereas thanks to (4.32) we may employ Lemma 4.5 to similarly conclude that for some $c_3 = c_3(R, \lambda, \mu, \kappa, L, m) > 0$ and all $s_0 \in (0, R^n)$,

$$\begin{aligned} c_1 s_0^{1-\gamma} \int_0^{s_0} (s_0 - s) w_s^\kappa ds &\leq c_3 s_0^{-(n-1)(\kappa-1)+\frac{3-\gamma}{2}-\varepsilon} \sqrt{\psi(t)} \\ &\leq \frac{n}{8}\psi(t) + \frac{2c_3^2}{n} s_0^{-2(n-1)(\kappa-1)+3-\gamma-2\varepsilon} \qquad \text{for all } t \in (0, \widehat{T}_{max}). \end{aligned} \tag{4.37}$$

We next use Lemma 4.8 to see that whenever $s_0 \in (0, s_\star)$, according to (4.34) we have

$$\begin{aligned} n(\gamma+1) s_0 \int_0^{s_0} s^{-\gamma-1} z w ds &\leq K s_0^{\frac{2}{n}} \psi(t) + K s_0^{\frac{2}{n}+1-\gamma} \\ &\leq \frac{n}{8}\psi(t) + K s_0^{\frac{2}{n}+1-\gamma} \qquad \text{for all } t \in (0, \widehat{T}_{max}). \end{aligned} \tag{4.38}$$

Finally, Lemma 4.4 yields $c_4 = c_4(\gamma) > 0$ such that if $s_0 \in (0, R^n)$, then

$$\int_0^{s_0} s^{-\gamma}(s_0 - s) w ds \leq c_4 s_0^{\frac{3-\gamma}{2}} \sqrt{\psi(t)} \qquad \text{for all } t \in (0, T_{max}),$$

which by Young's inequality firstly entails that

$$c_1 \int_0^{s_0} s^{-\gamma}(s_0 - s) w ds \leq \frac{n}{8}\psi(t) + \frac{2c_4^2}{n} s_0^{3-\gamma} \qquad \text{for all } t \in (0, T_{max}), \tag{4.39}$$

and which secondly ensures that

$$\frac{n}{2}\psi(t) \geq \frac{n}{2c_4^2} s_0^{\gamma-3} \phi^2(t) \qquad \text{for all } t \in (0, T_{max}). \tag{4.40}$$

Summarizing, from (4.35)-(4.40) we infer that whenever $s_0 \in (0, s_\star)$ and $t \in (0, \widehat{T}_{max})$,

$$\begin{aligned} \phi'(t) &\geq \frac{n}{2c_4^2} s_0^{\gamma-3} \phi^2(t) \\ &\quad - \frac{2c_2^2}{n} s_0^{3-\gamma-\frac{4}{n}} - \frac{2c_3^2}{n} s_0^{-2(n-1)(\kappa-1)+3-\gamma-2\varepsilon} - K s_0^{\frac{2}{n}+1-\gamma} - \frac{2c_4^2}{n} s_0^{3-\gamma}, \end{aligned} \tag{4.41}$$

where we note that since $n \geq 3$,

$$c_5 := (3-\gamma) - \left(\frac{2}{n} + 1 - \gamma\right) > \left(3 - \gamma - \frac{4}{n}\right) - \left(\frac{2}{n} + 1 - \gamma\right) = 2 - \frac{6}{n} \geq 0,$$

and that due to (4.33), (4.32) and the inequality $\gamma < 1$ also

$$c_6 := \Big(-2(n-1)(\kappa-1) + 3 - \gamma - 2\varepsilon \Big) - \Big(\frac{2}{n} + 1 - \gamma \Big) \begin{array}{l} = -2(n-1)(\kappa-1) + 2 - \frac{2}{n} - 2\varepsilon \\ > 1 - \frac{2}{n} - 2\varepsilon \geq 0. \end{array}$$

We may therefore estimate

$$\begin{array}{l} \frac{2c_2^2}{n} s_0^{3-\gamma-\frac{4}{n}} + \frac{2c_3^2}{n} s_0^{-2(n-1)(\kappa-1)+3-\gamma-2\varepsilon} + K s_0^{\frac{2}{n}+1-\gamma} + \frac{2c_4^2}{n} s_0^{3-\gamma} \\ \quad \leq \Big(\frac{2c_2^2}{n} R^{(2-\frac{6}{n})n} + \frac{2c_3^2}{n} R^{c_6 n} + K + \frac{2c_4^2}{n} R^{c_5 n} \Big) \cdot s_0^{\frac{2}{n}+1-\gamma} \qquad \text{for all } s_0 \in (0, s_\star) \end{array}$$

and hence derive (4.31) from (4.41). □

5 Conclusion. Proof of Theorem 1.1 and Corollary 1.2

By suitably making use of the properties of $\phi(0)$ enforced by (1.4), we can thus establish our main result on blow-up in (1.1) by means of a contradictory argument.

Proof of Theorem 1.1. For fixed $L > 0$ and $m > 0$, according to (1.4) we may apply Lemma 4.9 to find $\gamma = \gamma(\kappa) \in (1 - \frac{2}{n}, 1)$, $s_\star = s_\star(R, \lambda, m) \in (0, R^n)$ and $c_i = c_i(R, \lambda, \mu, \kappa, L, m) > 0$, $i \in \{1, 2\}$, such that for any $s_0 \in (0, s_\star)$ and arbitrary u_0 fulfilling $\int_\Omega u_0 \leq m$ and (1.5), the function ϕ introduced in (4.1) satisfies

$$\phi'(t) \geq c_1 s_0^{\gamma-3} \phi^2(t) - c_2 s_0^{\frac{2}{n}+1-\gamma} \qquad \text{for all } t \in (0, \widehat{T}_{max}), \tag{5.1}$$

where once more $\widehat{T}_{max} = \min\{1, T_{max}\}$.
Next, given $m_0 \in (0, m)$ we specify our selection of an appropriate value of s_0 herein by choosing $s_0 = s_0(R, \lambda, \mu, \kappa, L, m, m_0) \in (0, s_\star)$ small enough fulfilling

$$s_0^{\frac{2}{n}} \leq \frac{2^{2\gamma-7} c_1 m_0^2}{c_2 \omega_n^2} \tag{5.2}$$

and

$$s_0 \leq \frac{c_1 m_0}{2^{6-\gamma} \omega_n}, \tag{5.3}$$

and thereupon we let

$$r_0 := \Big(\frac{s_0}{4} \Big)^{\frac{1}{n}} \in (0, R). \tag{5.4}$$

We now suppose that u_0 is a fixed function complying with (1.3) as well as with (1.5) and (1.6), and let (u, v) denote the associated solution of (1.1) from Lemma 2.1, defined up to its maximal existence time $T_{max} \in (0, \infty]$. To see that actually $T_{max} \leq \frac{1}{2}$, assuming on the contrary that $T_{max} > \frac{1}{2}$ we note that restated in the correspondingly transformed variables in (2.2), due to (5.4) the second restriction in (1.6) means that

$$w(s, 0) \geq w\Big(\frac{s_0}{4}, 0 \Big) \geq \frac{m_0}{\omega_n} \qquad \text{for all } s \in \Big(\frac{s_0}{4}, R^n \Big),$$

which in particular implies that with ϕ as in (4.1) we have

$$\begin{aligned}\phi(0) &= \int_0^{s_0} s^{-\gamma}(s_0-s)w(s,0)ds \\ &\geq \int_{\frac{s_0}{4}}^{\frac{s_0}{2}} \left(\frac{s_0}{2}\right)^{-\gamma} \cdot \frac{s_0}{2} \cdot \frac{m_0}{\omega_n} ds \\ &= \frac{2^{\gamma-3}m_0}{\omega_n} \cdot s_0^{2-\gamma}. \end{aligned} \tag{5.5}$$

In view of (5.2), this entails that

$$\begin{aligned}\frac{\frac{c_1}{2}s_0^{\gamma-3}\phi^2(0)}{c_2 s_0^{\frac{2}{n}+1-\gamma}} &= \frac{c_1}{2c_2} s_0^{2\gamma-4-\frac{2}{n}}\phi^2(0) \\ &\geq \frac{2^{2\gamma-7}c_1 m_0^2}{c_2\omega_n^2} \cdot s_0^{-\frac{2}{n}} \\ &\geq 1\end{aligned}$$

and hence

$$c_1 s_0^{\gamma-3}\phi^2(0) - c_2 s_0^{\frac{2}{n}+1-\gamma} \geq \frac{c_1}{2} s_0^{\gamma-3}\phi^2(0).$$

Therefore, applying a straightforward ODE comparison argument to (5.1) shows that

$$c_1 s_0^{\gamma-3}\phi^2(t) - c_2 s_0^{\frac{2}{n}+1-\gamma} \geq \frac{c_1}{2} s_0^{\gamma-3}\phi^2(t) \qquad \text{for all } t \in \left(0, \frac{1}{2}\right)$$

and that thus

$$\phi'(t) \geq \frac{c_1}{2} s_0^{\gamma-3}\phi^2(t) \qquad \text{for all } t \in \left(0, \frac{1}{2}\right),$$

afer integration implying that

$$\frac{c_1}{2} s_0^{\gamma-3} t \leq -\frac{1}{\phi(t)} + \frac{1}{\phi(0)} \leq \frac{1}{\phi(0)} \qquad \text{for all } t \in \left(0, \frac{1}{2}\right).$$

Again by (5.5), however, according to (5.3) this leads to the absurd conclusion that

$$t < \frac{2}{c_1} s_0^{3-\gamma} \cdot \frac{\omega_n}{2^{\gamma-3}m_0} s_0^{\gamma-2} = \frac{2^{4-\gamma}\omega_n}{c_1 m_0} \cdot s_0 \leq \frac{1}{4} \qquad \text{for all } t \in \left(0, \frac{1}{2}\right)$$

and thereby shows that indeed we actually must have $T_{max} \leq \frac{1}{2}$. Therefore, both the statement on local existence of a classical solution and the claimed blow-up property (1.7) immediately result from Lemma 2.1. □

Suitably approximating widely arbitrary initial data finally shows that indeed blow-up in (1.1) occurs for a considerably large set of solutions:

Proof of Corollary 1.2. We fix any $\beta \in [n, n(n-1)]$, and given a positive u_0 fulfilling (1.3) we write $m := \int_\Omega u_0$ and pick $(m_k)_{k\in\mathbb{N}} \subset (0,m)$ such that

$$2^{\frac{\beta+4}{2}} m_k \le 1 \qquad \text{for all } k \in \mathbb{N} \tag{5.6}$$

and

$$m_k \searrow 0 \qquad \text{as } k \to \infty. \tag{5.7}$$

With

$$L := \|u_0\|_{L^\infty(\Omega)} R^{n(n-1)}, \tag{5.8}$$

for $k \in \mathbb{N}$ we then let $r_{0k} := r_0(R, \lambda, \mu, \kappa, L, m+1, m_k) \in (0,R)$ be as thereupon provided by Theorem 1.1, and using the positivity of u_0 we choose $r_k \in (0, r_{0k}]$ small enough such that

$$r_k^n \le \frac{c_1 m_k}{I} \tag{5.9}$$

and

$$\frac{1}{2} u_0(0) \le u_0(r_k) \le 2u_0(0), \tag{5.10}$$

where

$$I := \int_0^1 \xi^{n-1} (\xi^2+1)^{-\frac{\beta}{2}} d\xi \qquad \text{and} \qquad c_1 := \frac{2}{\omega_n u_0(0)}. \tag{5.11}$$

Observing that then for any $k \in \mathbb{N}$,

$$I_k(\eta) := \int_0^1 \xi^{n-1} \Big(\xi^2 + \frac{\eta^2}{r_k^2}\Big)^{-\frac{\beta}{2}} d\xi, \qquad \eta \in (0, r_k],$$

defines a continuous function satisfying

$$I_k(r_k) = I \le c_1 m_k r_k^{-n} \qquad \text{and} \qquad I_k(\eta) \nearrow +\infty \quad \text{as } \eta \searrow 0$$

due to (5.9) and our restriction that $\beta \ge n$, for each $k \in \mathbb{N}$ we can finally fix $\eta_k \in (0, r_k]$ such that

$$I_k(\eta_k) = c_1 m_k r_k^{-n}. \tag{5.12}$$

Then for $k \in \mathbb{N}$ letting

$$u_{0k}(r) := \begin{cases} u_0(r_k) \cdot \Big(\frac{r_k^2+\eta_k^2}{r^2+\eta_k^2}\Big)^{\frac{\beta}{2}} & \text{if } r \in [0, r_k], \\ u_0(r) & \text{if } r \in (r_k, R], \end{cases} \tag{5.13}$$

we clearly obtain a sequence of positive functions u_{0k} which all satisfy (1.3), and about which thanks to (5.12) and (5.11) we moreover know that

$$\begin{aligned} \int_{B_{r_k}(0)} u_{0k} &= \omega_n u_0(r_k) \cdot (r_k^2+\eta_k^2)^{\frac{\beta}{2}} \int_0^{r_k} r^{n-1} (r^2+\eta_k^2)^{-\frac{\beta}{2}} dr \\ &= \omega_n u_0(r_k) \cdot (r_k^2+\eta_k^2)^{\frac{\beta}{2}} \cdot r_k^{n-\beta} I_k(\eta_k) \\ &= \omega_n u_0(r_k) \cdot \Big(1 + \frac{\eta_k^2}{r_k^2}\Big)^{\frac{\beta}{2}} \cdot c_1 m_k \\ &= 2 \cdot \frac{u_0(r_k)}{u_0(0)} \cdot \Big(1 + \frac{\eta_k^2}{r_k^2}\Big)^{\frac{\beta}{2}} \cdot m_k. \end{aligned} \tag{5.14}$$

Using (5.10) and that $\eta_k \le r_k$ for all $k \in \mathbb{N}$, from this and (5.13) we particularly infer that

$$\int_{B_{r_k}(0)} u_{0k} \le 2^{\frac{\beta+4}{2}} m_k \qquad \text{for all } k \in \mathbb{N} \tag{5.15}$$

and that hence, by (5.6),

$$\int_\Omega u_{0k} \le \int_\Omega u_0 + \int_{B_{r_k}(0)} u_{0k} \le m+1 \qquad \text{for all } k \in \mathbb{N}, \tag{5.16}$$

whereas our definition (5.8) of L together with the restriction $\beta \le n(n-1)$ ensures that

$$\begin{aligned} r^{n(n-1)} u_{0k}(r) &\le u_0(r_k) \cdot r^{n(n-1)} \Big(\frac{r_k^2 + \eta_k^2}{r^2 + \eta_k^2} \Big)^{\frac{\beta}{2}} \\ &\le u_0(r_k) \cdot r^{n(n-1)} \Big(\frac{r_k^2}{r^2} \Big)^{\frac{\beta}{2}} \\ &= u_0(r_k) \cdot r_k^\beta r^{n(n-1)-\beta} \\ &\le \|u_0\|_{L^\infty(\Omega)} R^\beta \cdot R^{n(n-1)-\beta} \\ &\le L \qquad \text{for all } r \in (0, r_k] \end{aligned} \tag{5.17}$$

and

$$r^{n(n-1)} u_{0k}(r) = r^{n(n-1)} u_0(r) \le R^{n(n-1)} \|u_0\|_{L^\infty(\Omega)} \le L \qquad \text{for all } r \in (r_k, R]. \tag{5.18}$$

Since moreover the inequality $r_k \le r_{0k}$ guarantees that as a further consequence of (5.14) when combined with (5.10) we obtain that

$$\int_{B_{r_{0k}}(0)} u_{0k} \ge \int_{B_{r_k}(0)} u_{0k} \ge 2 \cdot \frac{1}{2} m_k = m_k \qquad \text{for all } k \in \mathbb{N},$$

we may use this along with (5.16), (5.17) and (5.18) to conclude on applying Theorem 1.1 that in fact for all $k \in \mathbb{N}$ the problem (1.1) admits a classical solution (u_k, v_k) emanating from $u_k|_{t=0} = u_{0k}$ and blowing up in finite time.
To finally verify the approximation property (1.8), we only need to go back to (5.15) once again, which in conjunction with (5.13) and (5.7) namely shows that indeed

$$\begin{aligned} \|u_{0k} - u_0\|_{L^1(\Omega)} = \|u_{0k} - u_0\|_{L^1(B_{r_k}(0)} &\le \int_{B_{r_k}(0)} u_{0k} + \|u_0\|_{L^\infty(\Omega)} \cdot |B_{r_k}(0)| \\ &\le 2^{\frac{\beta+4}{2}} m_k + \|u_0\|_{L^\infty(\Omega)} \cdot |B_{r_k}(0)| \\ &\to 0 \qquad \text{as } k \to \infty \end{aligned}$$

due to the fact that clearly $r_k \to 0$ as $k \to \infty$ by (5.9) and (5.10). □

Acknowledgement. The author acknowledges support of the *Deutsche Forschungsgemeinschaft* in the context of the project *Analysis of chemotactic cross-diffusion in complex frameworks.*

References

[1] Bai, X., Winkler, M.: *Equilibration in a fully parabolic two-species chemotaxis system with competitive kinetics.* Indiana Univ. Math. J. **65**, 553-583 (2016)

[2] Bianchi, A., Painter., K.J., Sherratt, J.A.: *Spatio-temporal models of lymphangiogenesis in wound healing.* Bull. Math. Biol. **78**, 1904-1941 (2016)

[3] Black, T., Lankeit, J., Mizukami, M.: *On the weakly competitive case in a two-species chemotaxis model.* IMA J. Appl. Math. **81**, 860-876 (2016)

[4] Cao, X.: *Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with logistic source.* J. Math. Anal. Appl. **412**, 181-188 (2014)

[5] Cao, X.: *Boundedness in a three-dimensional chemotaxis-haptotaxis model.* Z. Angew. Math. Phys. **67**, Art. 11, 13 pp. (2016)

[6] Cao, X., Zheng, S.: *Boundedness of solutions to a quasilinear parabolic-elliptic Keller-Segel system with logistic source.* Math. Meth. Appl. Sci. **37**, 2326-2330 (2014)

[7] Chaplain, M.A.J., Lolas, G.: *Mathematical modelling of cancer invasion of tissue: The role of the urokinase plasminogen activation system.* Math. Mod. Meth. Appl. Sci. **15**, 1685-1734 (2005)

[8] Cieślak, T., Winkler, M.: *Finite-time blow-up in a quasilinear system of chemotaxis.* Nonlinearity **21**, 1057-1076 (2008)

[9] Djie, K., Winkler, M.: *Boundedness and finite-time collapse in a chemotaxis system with volume-filling effect.* Nonlin. Anal. TMA **72** (2), 1044-1064 (2010)

[10] He, X., Zheng, S.: *Convergence rate estimates of solutions in a higher dimensional chemotaxis system with logistic source.* J. Math. Anal. Appl. **436**, 970-982 (2016)

[11] Herrero, M.A., Velázquez, J.J.L.: *A blow-up mechanism for a chemotaxis model.* Ann. Scu. Norm. Super. Pisa Cl. Sci. **24**, 663-683 (1997)

[12] Hillen, T., Painter, K.J.: *A User's Guide to PDE Models for Chemotaxis.* J. Math. Biol. **58** (1), 183-217 (2009)

[13] Jäger, W., Luckhaus, S.: *On explosions of solutions to a system of partial differential equations modelling chemotaxis.* Trans. Am. Math. Soc. **329**, 819-824 (1992)

[14] Kang, K., Stevens, A.: *Blowup and global solutions in a chemotaxis-growth system.* Nonlin. Anal. TMA **135**, 57-72 (2016)

[15] Kiselev, A., Ryzhik, L.: *Biomixing by chemotaxis and enhancement of biological reactions.* Comm. Partial Differ. Eq. **37** (1-3), 298-318 (2012)

[16] Lankeit, J.: *Chemotaxis can prevent thresholds on population density.* Discr. Cont. Dyn. Syst. B **20**, 1499-1527 (2015)

[17] Lankeit, J.: *Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source.* J. Differential Eq. **258**, 1158-1191 (2015)

[18] Lankeit, J.: *Long-term behaviour in a chemotaxis-fluid system with logistic source.* Math. Mod. Meth. Appl. Sci. **26**, 2071-2109 (2016)

[19] Lankeit, J., Mizukami, M.: *How far does small chemotactic interaction perturb the Fisher-KPP dynamics?* J. Math. Anal. Appl. **452**, 429-442 (2017)

[20] Li, X., Xiang, Z.: *Boundedness in quasilinear Keller-Segel equations with nonlinear sensitivity and logistic source.* Discr. Cont. Dyn. Syst. **35**, 3503-3531 (2015)

[21] Meral, G., Stinner, C., Surulescu, C.: *On a multiscale model involving cell contractivity and its effects on tumor invasion.* Discrete Cont. Dyn. Syst. Ser. B **20**, 189-213 (2015)

[22] Mimura, M., Tsujikawa, T.: *Aggregating pattern dynamics in a chemotaxis model including growth.* Physica A **230**, 499-543 (1996)

[23] Nadin, G., Perthame, B., Ryzhik, L.: *Lenya Traveling waves for the Keller-Segel system with Fisher birth terms.* Interfaces Free Bound. **10**, 517-538 (2008)

[24] Nagai, T.: *Blow-up of radially symmetric solutions to a chemotaxis system.* Adv. Math. Sci. Appl. **5**, 581-601 (1995)

[25] Nagai, T.: *Blowup of Nonradial Solutions to Parabolic-Elliptic Systems Modeling Chemotaxis in Two-Dimensional Domains.* J. Inequal. Appl. **6**, 37-55 (2001)

[26] Osaki, K., Tsujikawa, T., Yagi, A., Mimura, M.: *Exponential attractor for a chemotaxis-growth system of equations.* Nonlinear Analysis **51**, 119-144 (2002)

[27] Painter, K.J., Hillen, T.: *Spatio-Temporal Chaos in a Chemotaxis Model.* Physica D **240**, 363-375 (2011)

[28] Painter, K.J., Maini, P.K., Othmer, H.G.: *Complex spatial patterns in a hybrid chemotaxis reaction-diffusion model.* J. Math. Biol. **41** (4), 285-314 (2000)

[29] Salako, R., Shen, W.: *Global existence and asymptotic behavior of classical solutions to a parabolic-elliptic chemotaxis system with logistic source on $\mathbb{R}^N$.* J. Differential Eq. **262**, 5635-5690 (2017)

[30] Shigesada, N., Kawasaki, K., Teramoto, E.: *Spatial Segregation of Interacting Species.* J. Theor. Biol. **79**, 83-99 (1979)

[31] Strohm, S., Tyson, R.C., Powell, J.A.: *Pattern formation in a model for mountain pine beetle dispersal: linking model predictions to data.* Bull. Math. Biol. **75**, 1778-1797 (2013)

[32] Tao, Y., Winkler, M.: *A chemotaxis-haptotaxis model: the roles of porous medium diffusion and logistic source.* SIAM J. Math. Anal. **43**, 685-704 (2011)

[33] Tao, Y., Winkler, M.: *Boundedness and stabilization in a multi-dimensional chemotaxis-haptotaxis model.* Proc. Roy. Soc. Edinburgh Sect. A **144**, 1067-1084 (2014)

[34] Tao, Y., Winkler, M.: *Boundedness and decay enforced by quadratic degradation in a three-dimensional chemotaxis-fluid system.* Z. Angew. Math. Phys. **66**, 2555-3573 (2015)

[35] Tello, J.I., Winkler, M.: *A chemotaxis system with logistic source.* Comm. Part. Differential Eq. **32** (6), 849-877 (2007)

[36] Viglialoro, G.: *Very weak global solutions to a parabolic-parabolic chemotaxis-system with logistic source.* J. Math. Anal. Appl. **439**, 197-212 (2016)

[37] Viglialoro, G.: *Boundedness properties of very weak solutions to a fully parabolic chemotaxis system with logistic source.* Nonlin. Anal. Real World Appl. **34**, 520-535 (2017)

[38] Viglialoro, G., Woolley, T.E.: *Eventual smoothness and asymptotic behaviour of solutions to a chemotaxis system perturbed by a logistic growth.* Discr. Cont. Dyn. Syst. B, to appear. doi:10.3934/dcdsb.201-7199

[39] Wang, L., Li, Y., Mu, C.: *Boundedness in a parabolic-parabolic quasilinear chemotaxis system with logistic source.* Discr. Cont. Dyn. Syst. A **34**, 789-802 (2014)

[40] Wang, L., Mu, C., Hu, X., Tian, Y.: *Boundedness in a quasilinear chemotaxis-haptotaxis system with logistic source.* Math. Meth. Appl. Sci. **40**, 3000-3016 (2017)

[41] Wang, L., Mu, C., Zheng, P.: *On a quasilinear parabolic-elliptic chemotaxis system with logistic source.* J. Differential Eq. **256**, 1847-1872 (2014)

[42] Winkler, M.: *Chemotaxis with logistic source: Very weak global solutions and boundedness properties.* J. Math. Anal. Appl. **348**, 708-729 (2008)

[43] Winkler, M.: *Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source.* Comm. Part. Differential Eq. **35**, 1516-1537 (2010)

[44] Winkler, M.: *Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction.* J. Math. Anal. Appl. **384**, 261-272 (2011)

[45] Winkler, M.: *Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system.* J. Math. Pures Appl. **100**, 748-767 (2013), `arXiv:1112.4156v1`

[46] Winkler, M.: *How far can chemotactic cross-diffusion enforce exceeding carrying capacities?* J. Nonlinear Sci. **24**, 809-855 (2014)

[47] Winkler, M.: *Global asymptotic stability of constant equilibria in a fully parabolic chemotaxis system with strong logistic dampening.* J. Differential Eq. **257**, 1056-1077 (2014)

[48] Winkler, M.: *Emergence of large population densities despite logistic growth restrictions in fully parabolic chemotaxis systems.* Discr. Cont. Dyn. Syst. Ser. B **22**, 2777-2793 (2017)

[49] Winkler, M.: *Blow-up profiles and life beyond blow-up in the fully parabolic Keller-Segel system.* Preprint

[50] Woodward, D.E., Tyson, R., Myerscough, M.R., Murray, J.D., Budrene, E.O., Berg, H.C.: *Spatiotemporal patterns generated by Salmonella typhimurium.* Biophys. J. **68** (5), 2181-2189 (1995)

[51] Yang, C., Cao, X., Jiang, Z., Zheng, S.: *Boundedness in a quasilinear fully parabolic Keller-Segel system of higher dimension with logistic source.* J. Math. Anal. Appl. **430**, 585-591 (2015)

[52] Zhang, Q., Li, Y.: *Global boundedness of solutions to a two-species chemotaxis system.* Z. Angew. Math. Phys. **66**, 83-93 (2015)

[53] Zhao, X., Zheng, S.: *Global boundedness to a chemotaxis system with singular sensitivity and logistic source.* Z. Angew. Math. Phys. **68**, Art. 2, 13 pp. (2017)

[54] Zheng, P., Mu, C., Hu., X.: *Boundedness and blow-up for a chemotaxis system with generalized volume-filling effect and logistic source.* Discr. Cont. Dyn. Syst. A **35**, 2299-2323 (2015)

[55] Zheng, J.: *Boundedness of solutions to a quasilinear parabolic-elliptic Keller-Segel system with logistic source.* J. Differential Eq. **259**, 120-140 (2015)

[56] Zheng, J.: *Boundedness and global asymptotic stability of constant equilibria in a fully parabolic chemotaxis system with nonlinear logistic source.* J. Math. Anal. Appl. **450**,, 1047-1061 (2017)

[57] Zheng, J.: *A note on boundedness of solutions to a higher-dimensional quasi-linear chemotaxis system with logistic source,* Z. Angew. Math. Mech. **97**, 414-421 (2017)

[58] Zheng, J., Wang, Y.: *Boundedness and decay behavior in a higher-dimensional quasilinear chemotaxis system with nonlinear logistic source.* Comput. Math. Appl. **72**, 2604-2619 (2016)