Renormalized radial large-data solutions to the higher-dimensional Keller-Segel system with singular sensitivity and signal absorption

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Abstract

The chemotaxis system

$$\begin{cases} u_t = \Delta u - \nabla \cdot \left(\frac{u}{v}\nabla v\right), \\ v_t = \Delta v - uv, \end{cases} \qquad (\star)$$

is considered under homogeneous Neumann boundary conditions in the ball $\Omega = B_R(0) \subset \mathbb{R}^n$, where $R > 0$ and $n \geq 2$.

Despite its great relevance as a model for the spontaneous emergence of spatial structures in populations of primitive bacteria, since its introduction by Keller and Segel in 1971 this system has been lacking a satisfactory theory even at the level of the basic questions from the context of well-posedness; global existence results in the literature are restricted to spatially one- or two-dimensional cases so far, or alternatively require certain smallness hypotheses on the initial data.

For all suitably regular and radially symmetric initial data (u_0, v_0) satisfying $u_0 \geq 0$ and $v_0 > 0$, the present paper establishes the existence of a globally defined pair (u, v) of radially symmetric functions which are continuous in $(\bar{\Omega} \setminus \{0\}) \times [0, \infty)$ and smooth in $(\bar{\Omega} \setminus \{0\}) \times (0, \infty)$, and which solve the corresponding initial-boundary value problem for $(\star)$ with $(u(\cdot, 0), v(\cdot, 0)) = (u_0, v_0)$ in an appropriate generalized sense. To the best of our knowledge, this in particular provides the first result on global existence for the three-dimensional version of $(\star)$ involving arbitrarily large initial data.

Key words: chemotaxis, global existence, renormalized solution, generalized solution
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1 Introduction

This work is concerned with the initial-boundary value problem

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\frac{u}{v}\nabla v), & x \in \Omega,\ t>0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t>0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t>0, \\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

in a domain $\Omega \subset \mathbb{R}^n$, where our main focus will be on the case when $\Omega = B_R(0) \subset \mathbb{R}^n$ with $R>0$ and $n \ge 2$, and on solutions which are radially symmetric with respect to $|x|$ and nonnegative in their first and positive in their second component.

The PDE system in (1.1) has been proposed by Keller and Segel in the second of their seminal works ([15]) in order to describe the spatio-temporal behavior in populations of cells which, besides diffusing randomly, partially orient their movement toward increasing concentrations of a diffusible signal substance which they consume upon contact; in this framework, $u=u(x,t)$ denotes the density of cells and $v=v(x,t)$ represents the nutrient concentration (cf. [24], [18], [14], [27] and [19] for further modeling aspects, also in different biological contexts). As indicated by formal and numerical as well as rigorous analytical results on existence and stability properties of wave-like solutions ([13], [22], [21], [10]), the interplay of chemotaxis and absorption mechanisms as modeled by (1.1) is indeed able to support the emergence of spatially heterogeneous structures, as known from experimental observations to be a striking feature of such simple biological settings ([1]).

Despite its evident relevance in biological applications, a comprehensive solution theory for (1.1) seems yet lacking. In fact, the singular behavior near $v=0$ of the chemotactic sensitivity function, in (1.1) chosen as $S(u,v)=\frac{u}{v}$, may considerably enhance the relative strength of cross-diffusion at each point where the signal concentration becomes small, and the second equation in (1.1) suggests to conjecture that the set of such points should become substantially large during evolution. Accordingly, it is still an open problem to decide whether in spatially higher-dimensional cases, (1.1) may enforce the spontaneous formation of singularities, as known to occur e.g. in the classical Keller-Segel system obtained on choosing $S(u)=u$ as chemotactic sensitivity, and replacing the second equation in (1.1) by the equation $v_t = \Delta v - v + u$ modeling signal production by cells (see [11], [35] and also [3] for a survey on this and related problems). In presence of non-singular chemotactic cross-diffusion determined by the choice $S(u)=u$, the dissipative action due to a signal absorption mechanism as in (1.1) is known to entirely suppress such blow-up phenomena at least in two-dimensional situations, and in the case $n=3$ at least certain global weak solutions can always be constructed ([30]), being classical whenever v is suitably small ([29], [39]). However, the destabilizing potential of singular sensitivities of the form in (1.1) is far from understood even in systems which account for signal production and hence counteract the tendency of the quantity v to attain small values; for such systems in the case $n \ge 2$, namely, global existence and boundedness of classical solutions is guaranteed only when in the more general sensitivity function given by $S(u,v)=\chi\frac{u}{v}$ with $\chi>0$ we have $\chi < \sqrt{\frac{2}{n}}$ ([4], [34], [7], [40]) or $n=2$ and $\chi<\chi_0$ with some not explicitly known $\chi_0 > 1.015$ ([17]), whereas only certain generalized solutions are known to exist globally under less restrictive restrictions on the size of χ, after all allowing for choosing any $\chi>0$ in planar radial cases ([34], [28]).

Results on global existence of classical solutions to the PDE system in (1.1) currently are available for arbitrarily large initial data only in spatially one-dimensional frameworks ([31], [23]), whereas in higher-dimensional cases such solutions have been constructed only under appropriate smallness conditions on the initial data so far, even in the simplified version of (1.1) obtained on neglecting diffusion in the second equation therein ([38], [20]).

In particular, the Cauchy problem for (1.1) in $\Omega = \mathbb{R}^n$ is known to have global smooth solutions when $n \in \{2,3\}$ and both $\|u_0 - a\|_{W^{1,2}(\mathbb{R}^n)}$ and $\|\ln v_0\|_{W^{1,2}(\mathbb{R}^n)}$ are suitably small with some $a > 0$ ([32]); for the Neumann problem (1.1) in bounded convex planar domains, even an essentially explicit smallness condition on the initial data, involving only the quantities $\int_\Omega u_0 \ln u_0$ and $\int_\Omega |\nabla \ln v_0|^2$, can be identified to ensure global classical solvability ([37]). To the best of our knowledge, however, global existence of solutions to (1.1) for large initial data has been achieved only in the spatially two-dimensional case so far: As recently found, for smoothly bounded convex domains $\Omega \subset \mathbb{R}^2$ and all suitably regular initial data, (1.1) possesses at least one globally defined generalized solution in an appropriate framework ([36]).

Main results. It is the purpose of the present paper to demonstrate that in radially symmetric settings, (1.1) always admits global generalized solutions without any restriction on the spatial dimension nor the size of the initial data. To make this more precise, throughout the sequel we shall assume that $\Omega = B_R(0) \subset \mathbb{R}^n$ with some $n \ge 2$ and $R > 0$, and that the initial data in (1.1) are such that

$$\begin{cases} u_0 \in C^0(\bar\Omega) \quad \text{is radially symmetric with } u_0 \ge 0 \text{ in } \Omega, \qquad \text{and} \\ v_0 \in W^{1,\infty}(\Omega) \quad \text{is radially symmetric with } v_0 > 0 \text{ in } \bar\Omega. \end{cases} \tag{1.2}$$

Then within the framework of *renormalized solutions* to be specified in Definition 4.2 below, (1.1) in fact is globally solvable:

Theorem 1.1 *Let $n \ge 2$, $R > 0$ and $\Omega := B_R(0) \subset \mathbb{R}^n$, and suppose that u_0 and v_0 satisfy (1.2). Then there exists at least one pair (u,v) of radially symmetric functions*

$$\begin{cases} u \in L^\infty((0,\infty); L^1(\Omega)) \cap C^0((\bar\Omega \setminus \{0\}) \times [0,\infty)) \cap C^{2,1}((\bar\Omega \setminus \{0\}) \times (0,\infty)), \\ v \in L^\infty(\Omega \times (0,\infty)) \cap C^0((\bar\Omega \setminus \{0\}) \times [0,\infty)) \cap C^{2,1}((\bar\Omega \setminus \{0\}) \times (0,\infty)), \end{cases} \tag{1.3}$$

such that $u \ge 0$ and $v > 0$ in $(\bar\Omega \setminus \{0\}) \times [0,\infty)$, that

$$\frac{\nabla u}{u+1} \in L^2_{loc}(\bar\Omega \times [0,\infty)) \qquad \text{and} \qquad \frac{\nabla v}{v} \in L^2_{loc}(\bar\Omega \times [0,\infty)), \tag{1.4}$$

and that (u,v) is a global renormalized solution of (1.1) in the sense of Definition 4.2 below. Moreover, (u,v) solves (1.1) classically in $(\bar\Omega \setminus \{0\}) \times [0,\infty)$.

Main ideas. Any analytical approach to (1.1) needs to cope with the circumstance that in higher-dimensional cases, (1.1) apparently lacks any energy-like structure providing regularity information sufficient for deriving a priori estimates in suitably strong topologies. Even the functional given by $\mathcal{F}_\mu(u,v) := \int_\Omega u \ln \frac{u}{\mu} + \frac{1}{2}\int_\Omega \frac{|\nabla v|^2}{v^2}$ for $\mu > 0$, known to constitute a genuine Lyapunov functional for the variant of (1.1) with second equation simplifed to $v_t = -uv$, and serving as an energy functional for the full system (1.1) in the two-dimensional case at least along small-data trajectories ([37]), seems to lose any such property when $n \ge 3$. Accordingly, as a starting point of our analysis we will resort to the quite weak but dimension-independent global dissipative structure formally expressed in the identity

$$\frac{d}{dt}\int_\Omega \Big(-\ln v\Big) + \int_\Omega \frac{|\nabla v|^2}{v^2} = \int_\Omega u \equiv \int_\Omega u_0, \tag{1.5}$$

as satisfied by suitably regular solutions of (1.1), as well as the inequality

$$\int_0^t \int_\Omega \frac{|\nabla u|^2}{(u+1)^2} \le - \int_\Omega \ln\Big(\frac{v_0}{\|v_0\|_{L^\infty(\Omega)}}\Big) + 2\int_\Omega u_0 + \Big(\int_\Omega u_0\Big)\cdot t, \tag{1.6}$$

which can formally be derived from this (see Lemma 2.1 and Lemma 2.2). These properties have been used in the two-dimensional setting in [36] already, but unlike in the latter situation it seems that in the case $n \ge 3$ e.g. the inequality (1.6) is insufficient to imply any further useful global regularity information on the quantity u itself, rather than for $\ln(u+1)$, in adequate Lebesgue spaces.

The challenge in our analysis will thus consist in making appropriate use of the one-dimensional structure of the radial version of (1.1) in order to successively derive higher regularity properties of radial solutions $(u_\varepsilon, v_\varepsilon)$ to a suitably regularized variant of (1.1) (see (2.1)) from the preliminary estimates implied by the correspondingly obtained rigorous counterparts of (1.5) and (1.6), at least locally outside the spatial origin. This will be achieved in Section 3 by means of a bootstrapping procedure involving one-dimensional interpolation arguments as well as several types of results from parabolic regularity theory for the Neumann problem associated with the inhomogeneous linear heat equation in the interval $(0,R)$, where a key role will be played by an ε-independent pointwise lower estimate, locally with respect to $x \in \bar\Omega \setminus \{0\}$ and $t \in [0,\infty)$ (Lemma 3.2). Based on estimates for both u_ε and v_ε in corresponding local $C^{2+\theta,1+\frac{\theta}{2}}$ spaces (Lemma 3.15 and Lemma 3.14), through a straightforward extraction procedure we will thereafter obtain limit functions u and v which form a smooth and classical solution away from the origin (Lemma 4.1), and which due to the global regularity properties connected to (1.5) and (1.6) can be seen to actually solve (1.1) in a generalized sense that can be viewed as an adaptation of the well-known concept of renormalized solutions ([6]) to the present setting (see Definition 4.2 and Lemma 4.3).

2 Approximation of solutions

Following the approach in [36], for $\varepsilon \in (0,1)$ let us consider the regularized problems

$$\begin{cases} u_{\varepsilon t} = \Delta u_\varepsilon - \nabla \cdot \big(\frac{u_\varepsilon f_\varepsilon'(u_\varepsilon)}{v_\varepsilon}\nabla v_\varepsilon\big), & x \in \Omega,\ t > 0, \\ v_{\varepsilon t} = \Delta v_\varepsilon - f_\varepsilon(u_\varepsilon) v_\varepsilon, & x \in \Omega,\ t > 0, \\ \frac{\partial u_\varepsilon}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x,0) = u_0(x), \quad v_\varepsilon(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{2.1}$$

where with a fixed nonincreasing function $\rho \in C^\infty([0,\infty))$ fulfilling $\rho \equiv 1$ in $[0,1]$ and $\rho \equiv 0$ in $[2,\infty)$, we have set

$$f_\varepsilon(s) := \int_0^s \rho(\varepsilon\sigma)d\sigma, \qquad s \ge 0. \tag{2.2}$$

Then the evident properties that for arbitrary $\varepsilon \in (0,1)$ we have

$$f_\varepsilon(s) = s \quad \text{for all } s \in \Big[0, \frac{1}{\varepsilon}\Big] \qquad \text{and} \qquad f'_\varepsilon(s) = 0 \quad \text{for all } s \geq \frac{2}{\varepsilon} \tag{2.3}$$

and that for any $s \geq 0$,

$$f_\varepsilon(s) \nearrow s \quad \text{and} \quad f'_\varepsilon(s) \nearrow 1 \qquad \text{as } \varepsilon \searrow 0,$$

warrant that (2.1) indeed formally approaches the original system (1.1) in the limit $\varepsilon \searrow 0$, and that moreover each individual among the problems (2.1) possesses a globally defined classical solution, as can readily be verified by means of standard extension arguments (cf. [36, Lemma 2.2] for details in the particular two-dimensional case). Clearly, this solution $(u_\varepsilon, v_\varepsilon)$ inherits radial symmetry from the initial data, and from the maximum principle and (1.2) it follows that $u_\varepsilon \geq 0$ in $\bar\Omega \times [0,\infty)$, and that

$$0 < v_\varepsilon(x,t) \leq \|v_0\|_{L^\infty(\Omega)} \qquad \text{for all } x \in \bar\Omega \text{ and } t \geq 0. \tag{2.4}$$

In some places below, it will be convenient to consider instead of the second solution component the nonnegative normalized logarithmic variant thereof given by

$$w_\varepsilon := -\ln\Big(\frac{v_\varepsilon}{\|v_0\|_{L^\infty(\Omega)}}\Big), \tag{2.5}$$

whereupon (2.1) transforms to

$$\begin{cases} u_{\varepsilon t} = \Delta u_\varepsilon + \nabla\cdot(u_\varepsilon f'_\varepsilon(u_\varepsilon)\nabla w_\varepsilon), & x\in\Omega,\ t>0,\\ w_{\varepsilon t} = \Delta w_\varepsilon - |\nabla w_\varepsilon|^2 + f_\varepsilon(u_\varepsilon), & x\in\Omega,\ t>0,\\ \frac{\partial u_\varepsilon}{\partial\nu} = \frac{\partial w_\varepsilon}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u_\varepsilon(x,0) = u_0(x), \quad w_\varepsilon(x,0) = w_0(x) := -\ln\frac{v_0(x)}{\|v_0\|_{L^\infty(\Omega)}}, & x\in\Omega. \end{cases} \tag{2.6}$$

Some elementary but crucial regularity properties of the solutions to (2.1) can be summarized as follows.

Lemma 2.1 *For each $\varepsilon \in (0,1)$ we have*

$$\int_\Omega u_\varepsilon(x,t)dx = \int_\Omega u_0 \qquad \textit{for all } t>0 \tag{2.7}$$

as well as

$$\int_\Omega w_\varepsilon(x,t)dx \leq \int_\Omega w_0 + \Big(\int_\Omega u_0\Big)\cdot t \qquad \textit{for all } t>0 \tag{2.8}$$

and

$$\int_0^T\int_\Omega |\nabla w_\varepsilon(x,t)|^2dxdt \leq \int_\Omega w_0 + \Big(\int_\Omega u_0\Big)\cdot T \qquad \textit{for all } T>0. \tag{2.9}$$

Proof. The identity (2.7) immediately results on integrating the first equation in (2.1) over $\Omega \times (0,t)$. Thereupon, integrating the second equation in (2.6) shows that

$$\begin{aligned}
\int_\Omega w_\varepsilon(\cdot,t) + \int_0^t \int_\Omega |\nabla w_\varepsilon|^2 &= \int_\Omega w_0 + \int_0^t \int_\Omega f_\varepsilon(u_\varepsilon) \\
&\le \int_\Omega w_0 + \int_0^t \int_\Omega u_\varepsilon \\
&= \int_\Omega w_0 + \Big(\int_\Omega u_0 \Big) \cdot t \qquad \text{for all } t > 0,
\end{aligned}$$

from which both (2.8) and (2.9) result thanks to the nonnegativity of w_ε. □

As already observed in [36], (2.7) and (2.9) entail an important consequence on the regularity of the spatial gradient of u_ε.

Lemma 2.2 *For any $\varepsilon \in (0,1)$,*

$$\int_0^T \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} \le \int_\Omega w_0 + 2\int_\Omega u_0 + \Big(\int_\Omega u_0\Big) \cdot T \qquad \textit{for all } T > 0. \tag{2.10}$$

Proof. On testing the first equation in (2.6) against $\frac{1}{u_\varepsilon+1}$, this can be derived by using the inequality (2.9) to appropriately absorb the resulting expression stemming from the cross-diffusive interaction; for details we may refer to Lemma 2.4 in [36] which is formulated there for the case $n = 2$, but can easily be verified to hold actually for arbitrary $n \ge 1$. □

By means of the Cauchy-Schwarz inequality, the latter in conjunction with (2.7) entails an integral estimate for the gradient of the power-type function $\sqrt{u_\varepsilon + 1}$ of u_ε, rather than merely for a logarithmically transformed version of u_ε.

Corollary 2.3 *We have*

$$\int_0^T \bigg\{ \int_\Omega \Big|\nabla\sqrt{u_\varepsilon(\cdot,t)+1}\Big| \bigg\}^2 dt \le \frac{\int_\Omega u_0 + |\Omega|}{2} \cdot \bigg\{ \int_\Omega w_0 + 2\int_\Omega u_0 + \Big(\int_\Omega u_0\Big) \cdot T \bigg\} \qquad \textit{for all } T > 0 \tag{2.11}$$

whenever $\varepsilon \in (0,1)$.

Proof. We apply the Cauchy-Schwarz inequality and recall (2.7) to estimate

$$\begin{aligned}
\int_0^T \bigg\{ \int_\Omega \Big|\nabla\sqrt{u_\varepsilon(\cdot,t)+1}\Big| \bigg\}^2 dt &= \frac{1}{2} \int_0^T \bigg\{ \int_\Omega \frac{\nabla u_\varepsilon(\cdot,t)}{u_\varepsilon(\cdot,t)+1} \cdot \Big(u_\varepsilon(\cdot,t)+1\Big) \bigg\}^2 dt \\
&\le \frac{1}{2} \int_0^T \bigg\{ \int_\Omega \frac{|\nabla u_\varepsilon(\cdot,t)|^2}{(u_\varepsilon(\cdot,t)+1)^2} \bigg\} \cdot \bigg\{ \int_\Omega \Big(u_\varepsilon(\cdot,t)+1\Big) \bigg\} dt \\
&= \frac{\int_\Omega u_0 + |\Omega|}{2} \int_0^T \int_\Omega \frac{|\nabla u_\varepsilon(\cdot,t)|^2}{(u_\varepsilon(\cdot,t)+1)^2} dt
\end{aligned}$$

for all $T > 0$. Therefore, (2.11) is implied by (2.10). □

Apart from the above, let us finally also state one further property which does not rely on the radial symmetry of solutions.

Lemma 2.4 *For all $q \in [1, \frac{n}{n-1})$ and any $T > 0$ there exists $C(q,T) > 0$ such that for each $\varepsilon \in (0,1)$,*

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \le C(q,T) \qquad \text{for all } t \in (0,T). \tag{2.12}$$

Proof. Following a standard argument (see e.g. [12, Lemma 4.1]), by means of the Neumann heat semigroup $(e^{\tau\Delta})_{\tau\ge 0}$ on Ω we represent ∇v_ε according to

$$\nabla v_\varepsilon(\cdot,t) = \nabla e^{t\Delta} v_0 - \int_0^t \nabla e^{(t-s)\Delta}\Big(f_\varepsilon(u_\varepsilon(\cdot,s))v_\varepsilon(\cdot,s)\Big)ds, \qquad t > 0,$$

and recall known regularization properties of $(e^{\tau\Delta})_{\tau\ge 0}$ ([33, Lemma 1.3]) to find $c_1 = c_1(q) > 0$ and $c_2 = c_2(q) > 0$ such that

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \le c_1\|v_0\|_{W^{1,q}(\Omega)} + c_2\int_0^t \Big(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\Big)\Big\|f_\varepsilon(u_\varepsilon(\cdot,s))v_\varepsilon(\cdot,s)\Big\|_{L^1(\Omega)} ds \tag{2.13}$$

for all $t > 0$. Since $0 \le f_\varepsilon(u_\varepsilon) \le u_\varepsilon$ by (2.2) and hence

$$\Big\|f_\varepsilon(u_\varepsilon(\cdot,s))v_\varepsilon(\cdot,s)\Big\|_{L^1(\Omega)} \le \|f_\varepsilon(u_\varepsilon(\cdot,s))\|_{L^1(\Omega)}\|v_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^1(\Omega)}\|v_0\|_{L^\infty(\Omega)} \qquad \text{for all } s > 0$$

according to (2.7) and (2.4), from (2.13) we therefore readily obtain (2.12) on observing that $\frac{1}{2} + \frac{n}{2}(1 - \frac{1}{q}) < 1$ due to our hypothesis $q < \frac{n}{n-1}$. □

3 Local estimates outside the origin for radial solutions

From now on we explicitly focus on the framework of radial symmetry, and our goal in this section will consist in providing ε-independent estimates for the solutions of the approximate problems in the annular regions $\bar\Omega \setminus B_\delta(0)$ with arbitrary $\delta \in (0,R)$. For frequent reference at several stages of this procedure, let us fix a cut-off function $\zeta \in C^\infty([0,\infty))$ satisfying

$$\zeta \equiv 0 \quad \text{in } [0, \tfrac{1}{2}], \qquad \zeta \equiv 1 \quad \text{in } [1,\infty) \tag{3.1}$$

as well as

$$0 \le \zeta' \le 4 \quad \text{and} \quad |\zeta''| \le 32 \qquad \text{on } [0,\infty), \tag{3.2}$$

and introduce

$$\zeta_\delta(r) := \zeta\Big(\frac{r}{\delta}\Big) \qquad \text{for } r \ge 0 \text{ and } \delta > 0. \tag{3.3}$$

In order to avoid abundant efforts in presentation, throughout the sequel we shall use the standard radial notation by e.g. writing $u_\varepsilon(r,t)$ in referring to $u_\varepsilon(x,t)$ for arbitrary $x \in \partial B_r(0)$ whenever $r \in [0,R]$ and $t \ge 0$.

Let us first draw some immediate consequences of Lemma 2.2, Corollary 2.3 and Lemma 2.4 for the resulting functions u_ε, v_ε and w_ε when accordingly viewed as functions of the one-dimensional spatial variable $r \in [0,R]$ only.

Lemma 3.1 *For all $\delta \in (0,R)$ there exists $C(\delta) > 0$ such that for each $\varepsilon \in (0,1)$ we have*

$$\|u_\varepsilon(\cdot,t)\|_{L^1((\delta,R))} \le C(\delta) \qquad \text{for all } t > 0, \tag{3.4}$$

and for each $\delta \in (0,R)$ and arbitrary $T > 0$ one can find $C(\delta,T) > 0$ fulfilling

$$\|w_\varepsilon(\cdot,t)\|_{L^1((\delta,R))} \le C(\delta,T) \qquad \text{for all } t \in (0,T) \tag{3.5}$$

as well as

$$\int_0^T \left\{ \int_\delta^R \left| \partial_r \sqrt{u_\varepsilon(r,t)+1} \right| dr \right\}^2 dt \le C(\delta,T) \tag{3.6}$$

and

$$\int_0^T \int_\delta^R w_{\varepsilon r}^2(r,t) dr dt \le C(\delta,T) \tag{3.7}$$

for all $\varepsilon \in (0,1)$. Moreover, for any choice of $q \in [1, \frac{n}{n-1})$, $\delta \in (0,R)$ and $T > 0$ one can fix $C(q,\delta,T) > 0$ such that

$$\|v_{\varepsilon r}(\cdot,t)\|_{L^q((\delta,R))} \le C(q,\delta,T) \qquad \text{for all } t \in (0,T) \tag{3.8}$$

whenever $\varepsilon \in (0,1)$.

Proof. As

$$\int_\delta^R u_\varepsilon(r,t) dr \le \delta^{1-n} \int_\delta^R r^{n-1} u_\varepsilon(r,t) dr \qquad \text{for all } t > 0,$$

the inequality (3.4) is obviously impled by (2.7). Likewise, (3.5), (3.6), (3.7) and (3.8) directly result from (2.8), (2.11), (2.9) and (2.12), respectively. □

In order to successively improve our knowledge on the regularity of $u_\varepsilon, v_\varepsilon$ and w_ε away from $r = 0$, we observe that given any $\xi \in C^\infty([0,R] \times [0,\infty))$ fulfilling $\operatorname{supp} \xi_r \subset (0,R) \times [0,\infty)$, these functions satisfy the inhomogeneous linear initial-boundary value problems

$$\begin{cases} (\xi u_\varepsilon)_t = (\xi u_\varepsilon)_{rr} + a_{1r}(r,t) + a_2(r,t) + a_3(r,t), & r \in (0,R),\ t > 0, \\ (\xi u_\varepsilon)_r(0,t) = (\xi u_\varepsilon)_r(R,t) = 0, & t > 0, \\ (\xi u_\varepsilon)(r,0) = \xi(r,0) u_0(r), & r \in (0,R), \end{cases} \tag{3.9}$$

with

$$a_1(r,t) \equiv a_1(r,t;\xi,\varepsilon) := \xi(r,t) u_\varepsilon(r,t) f'_\varepsilon(u_\varepsilon(r,t)) w_{\varepsilon r}(r,t), \qquad r \in (0,R),\ t > 0, \tag{3.10}$$

and

$$a_2(r,t) \equiv a_2(r,t;\xi,\varepsilon) := \frac{n-1}{r} \xi(r,t) u_{\varepsilon r}(r,t) - 2\xi_r(r,t) u_{\varepsilon r}(r,t), \qquad r \in (0,R),\ t > 0, \tag{3.11}$$

as well as

$$\begin{aligned} a_3(r,t) \equiv a_3(r,t;\xi,\varepsilon) \ &:= \ \xi_t(r,t) u_\varepsilon(r,t) - \xi_{rr}(r,t) u_\varepsilon(r,t) \\ &\quad - \xi_r(r,t) u_\varepsilon(r,t) f'_\varepsilon(u_\varepsilon(r,t)) w_{\varepsilon r}(r,t) \\ &\quad + \frac{n-1}{r} \xi(r,t) u_\varepsilon(r,t) f'_\varepsilon(u_\varepsilon(r,t)) w_{\varepsilon r}(r,t), \qquad r \in (0,R),\ t > 0, \end{aligned} \tag{3.12}$$

and

$$\begin{cases} (\xi v_\varepsilon)_t = (\xi v_\varepsilon)_{rr} + b_1(r,t) + b_2(r,t), & r \in (0,R),\ t>0, \\ (\xi v_\varepsilon)_r(0,t) = (\xi v_\varepsilon)_r(R,t) = 0, & t>0, \\ (\xi v_\varepsilon)(r,0) = \xi(r,0) v_0(r), & r \in (0,R), \end{cases} \tag{3.13}$$

where

$$b_1(r,t) \equiv b_1(r,t;\xi,\varepsilon) := -2\xi_r(r,t) v_{\varepsilon r}(r,t) + \frac{n-1}{r}\xi(r,t) v_{\varepsilon r}(r,t), \qquad r \in (0,R),\ t>0, \tag{3.14}$$

and

$$\begin{aligned} b_2(r,t) \equiv b_2(r,t;\xi,\varepsilon) \ &:= \ \xi_t(r,t) v_\varepsilon(r,t) - \xi_{rr}(r,t) v_\varepsilon(r,t) \\ &\quad -\xi(r,t) f_\varepsilon(u_\varepsilon(r,t)) v_\varepsilon(r,t), \qquad r \in (0,R),\ t>0, \end{aligned} \tag{3.15}$$

and finally

$$\begin{cases} (\xi w_\varepsilon)_t = (\xi w_\varepsilon)_{rr} + d_1(r,t) + d_2(r,t) - d_3(r,t), & r \in (0,R),\ t>0, \\ (\xi w_\varepsilon)_r(0,t) = (\xi w_\varepsilon)_r(R,t) = 0, & t>0, \\ (\xi w_\varepsilon)(r,0) = \xi(r,0) w_0(r), & r \in (0,R), \end{cases} \tag{3.16}$$

with

$$d_1(r,t) \equiv d_1(r,t;\xi,\varepsilon) := -2\xi_r(r,t) w_{\varepsilon r}(r,t) + \frac{n-1}{r}\xi(r,t) w_{\varepsilon r}(r,t), \qquad r \in (0,R),\ t>0, \tag{3.17}$$

and

$$d_2(r,t) \equiv d_2(r,t;\xi,\varepsilon) := \xi_t(r,t) w_\varepsilon(r,t) - \xi_{rr}(r,t) w_\varepsilon(r,t) + \xi(r,t) f_\varepsilon(u_\varepsilon(r,t)) \qquad r \in (0,R),\ t>0, \tag{3.18}$$

as well as

$$d_3(r,t) \equiv d_3(r,t;\xi,\varepsilon) := \xi(r,t) w_{\varepsilon r}^2(r,t), \qquad r \in (0,R),\ t>0, \tag{3.19}$$

for $\varepsilon \in (0,1)$.

In what follows, by A we abbreviate the formal differential operator $-(\cdot)_{rr}$ under homogeneous Neumann boundary conditions in the one-dimensional interval $(0,R)$, along with its realizations in $L^p((0,R))$ for $p \in (1,\infty)$, and the corresponding analytic semigroups $(e^{-\tau A})_{\tau \geq 0}$ which are clearly independent of p when acting on suitably smooth functions e.g. lying in $C^0([0,R])$.

3.1 A pointwise lower bound for v_ε

As a first crucial step in our analysis, let us make sure that in regions away from the origin, the component v_ε remains bounded from below by a positive constant, at least locally in time, but uniformly with respect to $\varepsilon \in (0,1)$. In view of (2.5), this is equivalent to deriving a corresponding pointwise upper bound for w_ε, which is at the core of the argument in the following lemma.

Lemma 3.2 *Let $\delta \in (0,R)$ and $T>0$. Then there exists $C(\delta,T)>0$ such that for all $\varepsilon \in (0,1)$ we have*

$$v_\varepsilon(r,t) \geq C(\delta,T) \qquad \textit{for all } r \in (\delta,R) \textit{ and } t \in (0,T). \tag{3.20}$$

Proof. From standard estimates on the smoothing action of $(e^{-\tau A})_{\tau\ge 0}$ ([26], [33, Lemma 1.3]) we know that there exist $c_1>0$ and $c_2>0$ such that for all $\tau>0$ we have

$$\|e^{-\tau A}\varphi\|_{L^\infty((0,R))} \le c_1(1+\tau^{-\frac{1}{4}})\|\varphi\|_{L^2(\Omega)} \qquad \text{for all } \varphi\in L^2((0,R)) \tag{3.21}$$

and

$$\|e^{-\tau A}\varphi\|_{L^\infty((0,R))} \le c_2(1+\tau^{-\frac{1}{2}})\|\varphi\|_{L^1(\Omega)} \qquad \text{for all } \varphi\in L^1((0,R)). \tag{3.22}$$

Now given $\delta\in(0,R)$, in (3.16)-(3.18) we choose $\xi(r,t):=\zeta_\delta(r)$, and in order to estimate the product $\zeta_\delta w_\varepsilon$ thus addressed in (3.16), for fixed $T>0$ we first apply Lemma 3.1 to gain $c_3=c_3(\delta)>0$, $c_4=c_4(\delta,T)>0$ and $c_5=c_5(\delta,T)>0$ such that

$$\int_{\frac{\delta}{2}}^R u_\varepsilon(r,t)dr \le c_3 \qquad \text{for all } t\in(0,T) \tag{3.23}$$

and

$$\int_{\frac{\delta}{2}}^R w_\varepsilon(r,t)dr \le c_4 \qquad \text{for all } t\in(0,T) \tag{3.24}$$

as well as

$$\int_0^T\int_{\frac{\delta}{2}}^R w_{\varepsilon r}^2(r,t)drdt \le c_5. \tag{3.25}$$

Since $0\le\zeta_\delta\le 1$ as well as $|\zeta_{\delta r}|\le\frac{4}{\delta}$ and $|\zeta_{\delta rr}|\le\frac{32}{\delta^2}$ on $(0,R)$ by (3.1)-(3.3), and since $0\le f_\varepsilon(u_\varepsilon)\le u_\varepsilon$ according to (2.2), from (3.23) and (3.24) we obtain that in (3.18) we have

$$\begin{aligned}\|d_2(\cdot,t)\|_{L^1((0,R))} &\le \int_0^R|\zeta_{\delta rr}(r)|w_\varepsilon(r,t)dr+\int_0^R\zeta_\delta(r)f_\varepsilon(u_\varepsilon(r,t))dr\\ &\le c_6:=\frac{32}{\delta^2}\cdot c_4+c_3 \qquad \text{for all } t\in(0,T),\end{aligned} \tag{3.26}$$

whereas (3.25) warrants that in (3.17) we can estimate

$$\begin{aligned}\int_0^T\|d_1(\cdot,t)\|_{L^2((0,R))}^2dt &\le 2\int_0^T\int_0^R 4\zeta_{\delta r}^2(r)w_{\varepsilon r}^2(r,t)drdt+2\int_0^T\int_0^R\frac{(n-1)^2}{r^2}\zeta_\delta^2(r)w_{\varepsilon r}^2(r,t)drdt\\ &\le \frac{128}{\delta^2}\int_0^T\int_{\frac{\delta}{2}}^R w_{\varepsilon r}^2(r,t)drdt+\frac{8(n-1)^2}{\delta^2}\int_0^T\int_{\frac{\delta}{2}}^R w_{\varepsilon r}^2(r,t)drdt\\ &\le c_7:=\frac{128}{\delta^2}\cdot c_5+\frac{8(n-1)^2}{\delta^2}\cdot c_5.\end{aligned} \tag{3.27}$$

On the basis of the variation-of-constants representation of w_ε associated with (3.16), that is, of the identity

$$\begin{aligned}\zeta_\delta w_\varepsilon(\cdot,t) &= e^{-tA}(\zeta_\delta w_0)+\int_0^t e^{-(t-s)A}d_1(\cdot,s)ds+\int_0^t e^{-(t-s)A}d_2(\cdot,s)ds,\\ &\quad -\int_0^t e^{-(t-s)A}d_3(\cdot,s)ds, \qquad t>0,\end{aligned}$$

by means of (3.21), (3.22) and the maximum principle we now obtain, using that $w_\varepsilon \ge 0$ and that by (3.19) also d_3 is nonnegative, that

$$\begin{aligned} \|\zeta_\delta w_\varepsilon(\cdot,t)\|_{L^\infty((0,R))} &\le \|\zeta_\delta w_0\|_{L^\infty((0,R))} + c_1 \int_0^t \Big(1+(t-s)^{-\frac14}\Big)\|d_1(\cdot,s)\|_{L^2((0,R))}ds \\ &\quad + c_2 \int_0^t \Big(1+(t-s)^{-\frac12}\Big)\|d_2(\cdot,s)\|_{L^1((0,R))}ds \qquad \text{for all } t>0. \end{aligned} \tag{3.28}$$

Again since $0 \le \zeta_\delta \le 1$, we herein have

$$\|\zeta_\delta w_0\|_{L^\infty((0,R))} \le \|w_0\|_{L^\infty((0,R))},$$

and using (3.26) we see that

$$\begin{aligned} c_2 \int_0^t \Big(1+(t-s)^{-\frac12}\Big)\|d_2(\cdot,s)\|_{L^1((0,R))}ds &\le c_2 c_6 \int_0^t \Big(1+(t-s)^{-\frac12}\Big)ds \\ &\le c_8 := c_2 c_6 (T + 2T^{\frac12}) \qquad \text{for all } t \in (0,T). \end{aligned}$$

Moreover, the Cauchy-Schwarz inequality along with (3.27) guarantees that

$$\begin{aligned} & c_1 \int_0^t \Big(1+(t-s)^{-\frac14}\Big)\|d_1(\cdot,s)\|_{L^2((0,R))}ds \\ &\quad \le c_1 \cdot \bigg\{\int_0^t \Big(1+(t-s)^{-\frac14}\Big)^2 ds\bigg\}^{\frac12} \cdot \bigg\{\int_0^t \|d_1(\cdot,s)\|^2_{L^2((0,R))}ds\bigg\}^{\frac12} \\ &\quad \le c_1 c_7^{\frac12} \cdot \bigg\{2\int_0^t \Big(1+(t-s)^{-\frac12}\Big)ds\bigg\}^{\frac12} \\ &\quad \le c_9 := c_1 c_7^{\frac12} (2T + 4T^{\frac12})^{\frac12} \qquad \text{for all } t \in (0,T). \end{aligned}$$

According to (3.28), we thus infer that

$$\|\zeta_\delta w_\varepsilon(\cdot,t)\|_{L^\infty((0,R))} \le \|w_0\|_{L^\infty((0,R))} + c_9 + c_8 \qquad \text{for all } t \in (0,T),$$

which implies (3.20) due to (2.5) and the fact that $\zeta_\delta \equiv 1$ on (δ, R). □

3.2 Regularity properties of $v_{\varepsilon r}$ and $w_{\varepsilon r}$ in dependece on L^p bounds for u_ε

Our next purpose will be to improve the regularity information on the chemotactic grandient $v_{\varepsilon r}$ from Lemma 2.4. As a preliminary step toward this, the following lemma provides a criterion for L^q boundedness of this quantity in dependence of a supposedly known spatial L^p bound for u_ε. This will firstly be used in Corollary 3.4 for arbitrary $q < \infty$, and a second application in Corollary 3.8 will later yield a corresponding L^∞ bound.

Lemma 3.3 *Let $p \ge 1$ and $q \ge 1$ be such that*

$$\begin{cases} q < \infty & \text{if } p = 1, \\ q \le \infty & \text{if } p > 1. \end{cases} \tag{3.29}$$

Then for all $\delta \in (0,R)$ and any $T>0$ there exists $C(p,q,\delta,T)>0$ with the property that for all $\varepsilon \in (0,1)$ we have

$$\|v_{\varepsilon r}(\cdot,t)\|_{L^q((\delta,R))} \le C(p,q,\delta,T)\cdot\Big\{1+\sup_{s\in(0,t)}\|u_\varepsilon(\cdot,t)\|_{L^p((\frac{\delta}{2},R))}\Big\} \qquad \textit{for all } t\in(0,T). \tag{3.30}$$

PROOF. Without loss of generality we may assume that $p<\frac{n}{n-1}$ and $q\ge p$. Then since for each $\varphi \in W^{1,q}((0,R))$ and any $\tau>0$ we clearly have $\partial_r e^{-\tau A}\varphi = e^{-\tau A_D}\varphi_r$ with A_D denoting the operator $-(\cdot)_{rr}$ under homogeneous Dirichlet boundary conditions in $(0,R)$, so that

$$\|\partial_r e^{-\tau A}\varphi\|_{L^q((0,R))} \le \|\varphi_r\|_{L^q((0,R))} \qquad \text{for all } \tau>0 \text{ and each } \varphi\in W^{1,q}((0,R)). \tag{3.31}$$

Moreover, again relying on known regularizing features of $(e^{-\tau A})_{\tau\ge0}$ we can find $c_1>0$ such that

$$\|\partial_r e^{-\tau A}\varphi\|_{L^q((0,R))} \le c_1(1+\tau^{-\gamma})\|\varphi\|_{L^p((0,R))} \qquad \text{for all } \tau>0 \text{ and any } \varphi\in L^p((0,R)), \tag{3.32}$$

where in both cases $q<\infty$ and $q=\infty$, our assumption (3.29) warrants that

$$\gamma := \begin{cases} \frac12+\frac12(\frac1p-\frac1q) & \text{if } q<\infty, \\ \frac12+\frac{1}{2p} & \text{if } q=\infty, \end{cases}$$

satisfies $\gamma<1$. Therefore, in the identity

$$\partial_r(\zeta_\delta v_\varepsilon(\cdot,t)) = \partial_r e^{-tA}(\zeta_\delta v_0) + \int_0^t \partial_r e^{-(t-s)A} b_1(\cdot,s)ds + \int_0^t \partial_r e^{-(t-s)A} b_2(\cdot,s)ds, \qquad t>0,$$

as obtained from (3.13)-(3.15) on choosing $\xi(r,t):=\zeta_\delta(r)$ for $(r,t)\in[0,R]\times[0,\infty)$ and fixed $\delta\in(0,R)$, we can estimate

$$\begin{aligned} \|v_{\varepsilon r}(\cdot,t)\|_{L^q((\delta,R))} &\le \|\partial_r(\zeta_\delta v_\varepsilon(\cdot,t))\|_{L^q((0,R))} \\ &\le \|\partial_r(\zeta_\delta v_0)\|_{L^q((0,R))} + c_1\int_0^t \Big(1+(t-s)^{-\gamma}\Big)\|b_1(\cdot,s)\|_{L^p((0,R))}ds \\ &\quad + c_1\int_0^t \Big(1+(t-s)^{-\gamma}\Big)\|b_2(\cdot,s)\|_{L^p((0,R))}ds \end{aligned} \tag{3.33}$$

for $t>0$. Here since $p<\frac{n}{n-1}$, Lemma 3.1 applies so as to yield $c_2=c_2(p,\delta,T)>0$ fulfilling

$$\|v_{\varepsilon r}(\cdot,s)\|_{L^p((\frac{\delta}{2},R))} \le c_2 \qquad \text{for all } s\in(0,T),$$

whence again using that $\zeta_\delta\equiv0$ in $(0,\frac{\delta}{2})$ and $0\le\zeta_\delta\le1$ as well as $|\zeta_{\delta r}|\le\frac{4}{\delta}$, we see that

$$\begin{aligned} \|b_1(\cdot,s)\|_{L^p((0,R))} &\le \Big\|2\zeta_{\delta r}v_{\varepsilon r}(\cdot,s)\Big\|_{L^p((0,R))} + \Big\|\frac{n-1}{r}\zeta_\delta v_{\varepsilon r}(\cdot,s)\Big\|_{L^p((0,R))} \\ &\le \frac{8}{\delta}\|v_{\varepsilon r}(\cdot,s)\|_{L^p((\frac{\delta}{2},R))} + \frac{2(n-1)}{\delta}\|v_{\varepsilon r}(\cdot,s)\|_{L^p((\frac{\delta}{2},R))} \\ &\le c_3 := \frac{8}{\delta}c_2 + \frac{2(n-1)}{\delta}c_2 \qquad \text{for all } s\in(0,T), \end{aligned}$$

and that hence

$$\begin{aligned} c_1 \int_0^t \Big(1 + (t-s)^{-\gamma}\Big) \|b_1(\cdot,s)\|_{L^p((0,R))} ds &\le c_1 c_3 \int_0^t \Big(1+(t-s)^{-\gamma}\Big) ds \\ &= c_1 c_3 \Big(t + \frac{t^{1-\gamma}}{1-\gamma}\Big) \\ &\le c_1 c_3 \Big(T + \frac{T^{1-\gamma}}{1-\gamma}\Big) \qquad \text{for all } t \in (0,T), \end{aligned} \tag{3.34}$$

because $\gamma < 1$. Similarly, as $|\zeta_{\delta rr}| \le \frac{32}{\delta^2}$ and $0 \le f_\varepsilon(u_\varepsilon) \le u_\varepsilon$, thanks to (2.4) we find that

$$\begin{aligned} \|b_2(\cdot,s)\|_{L^p((0,R))} &\le \|\zeta_{\delta rr} v_\varepsilon(\cdot,s)\|_{L^p((0,R))} + \|\zeta_\delta f_\varepsilon(u_\varepsilon(\cdot,s)) v_\varepsilon(\cdot,s)\|_{L^p((0,R))} \\ &\le \frac{32}{\delta^2} R^{\frac{1}{p}} \|v_\varepsilon(\cdot,s)\|_{L^\infty((0,R))} + \|u_\varepsilon(\cdot,s)\|_{L^p((\frac{\delta}{2},R))} \|v_\varepsilon(\cdot,s)\|_{L^\infty((0,R))} \\ &\le \frac{32}{\delta^2} R^{\frac{1}{p}} \|v_0\|_{L^\infty((0,R))} + \|u_\varepsilon(\cdot,s)\|_{L^p((\frac{\delta}{2},R))} \|v_0\|_{L^\infty((0,R))} \\ &\le c_4 \cdot \Big\{ 1 + \sup_{\sigma\in(0,T)} \|u_\varepsilon(\cdot,\sigma)\|_{L^p((\frac{\delta}{2},R))} \Big\} \qquad \text{for all } s \in (0,T) \end{aligned}$$

with an evident choice of $c_4 = c_4(p,\delta)$, whence also

$$c_1 \int_0^t \Big(1+(t-s)^{-\gamma}\Big) \|b_2(\cdot,s)\|_{L^p((0,R))} ds \le c_1 c_4 \cdot \Big(T + \frac{T^{1-\gamma}}{1-\gamma}\Big) \cdot \Big\{ 1 + \sup_{s\in(0,T)} \|u_\varepsilon(\cdot,s)\|_{L^p((\frac{\delta}{2},R))} \Big\} \tag{3.35}$$

for all $t \in (0,T)$. Since finally the number $c_5 \equiv c_5(q,\delta) := \|\partial_r(\zeta_\delta v_0)\|_{L^q((0,R))}$ is finite thanks to the assumed inclusion $v_0 \in W^{1,\infty}((0,R))$ asserted by (1.2), it follows from (3.33), (3.34) and (3.35) that indeed (3.30) holds if we choose $C(p,q,\delta,T) := c_5 + c_1(c_3+c_4)(T + \frac{T^{1-\gamma}}{1-\gamma})$, for instance. $\Box$

By means of Lemma 2.1, a first consequence of the latter is immediate.

Corollary 3.4 *Let $q \in [1,\infty)$. Then for all $\delta \in (0,R)$ and $T > 0$ there exists $C(q,\delta,T) > 0$ such that*

$$\|w_{\varepsilon r}(\cdot,t)\|_{L^q((\delta,R))} \le C(q,\delta,T) \qquad \textit{for all } t \in (0,T) \tag{3.36}$$

whenever $\varepsilon \in (0,1)$.

Proof. In view of (3.4), we may apply Lemma 3.3 to $p := 1$ to find $c_1 = c_1(q,\delta,T) > 0$ such that for all $\varepsilon \in (0,1)$,

$$\|v_{\varepsilon r}(\cdot,t)\|_{L^q((\delta,R))} \le c_1 \qquad \text{for all } t \in (0,T).$$

Since Lemma 3.2 implies the existence of $c_2 = c_2(\delta,T) > 0$ such that

$$v_\varepsilon(r,t) \ge c_2 \qquad \text{for all } r \in (\delta,R),\ t \in (0,T) \text{ and } \varepsilon \in (0,1),$$

in view of the identity $w_{\varepsilon r} = -\frac{v_{\varepsilon r}}{v_\varepsilon}$ this directly yields (3.36). $\Box$

3.3 Time-independent bounds for u_ε in L^2. Space-time L^2 estimates for $u_{\varepsilon r}$

We shall next be concerned with the regularity properties of u_ε. Here the standard approach of pursuing the time evolution of spatial L^p norms thereof can easily be seen to require quite thorough knowledge on the regularity of the gradient of w_ε, going far beyond (2.9) even for small positive p when the corresponding procedure is carried out at a spatially global level. In light of e.g the outcome of Corollary 3.4, however, an adequately localized testing procedure may be expected to provide some information on regularity at least away from the origin. Our analysis of certain spatially weighted functionals of this type is prepared by the following observation which will below be applied twice, namely first to some $p \in (1,2)$ in Lemma 3.7, and thereafter to $p := 2$ in Lemma 3.9.

Lemma 3.5 *Let $p > 1$. Then for all $\delta \in (0,R)$ one can pick $C(p,\delta) > 0$ such that for all $\varepsilon \in (0,1)$ we have*

$$\begin{aligned} \frac{d}{dt}\int_0^R r^{n-1}\zeta_\delta(r)u_\varepsilon^p(r,t)dr &+ \frac{p(p-1)\delta^{n-1}}{2}\int_\delta^R u_\varepsilon^{p-2}(r,t)u_{\varepsilon r}^2(r,t)dr \\ &\le C(p,\delta)\int_{\frac{\delta}{2}}^R u_\varepsilon^p(r,t)\Big(w_{\varepsilon r}^2(r,t)+1\Big)dr \qquad \textit{for all } t>0. \end{aligned} \tag{3.37}$$

Proof. We multiply the radial version of the first equation in (2.6) by $\zeta_\delta(r)u_\varepsilon^{p-1}(r,t)$ and integrate by parts over $r \in (0,R)$ to obtain

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int_0^R r^{n-1}\zeta_\delta(r)u_\varepsilon^p(r,t)dr &+ (p-1)\int_0^R r^{n-1}\zeta_\delta(r)u_\varepsilon^{p-2}(r,t)u_{\varepsilon r}^2(r,t)dr \\ = & -\int_0^R r^{n-1}\zeta_{\delta r}(r)u_\varepsilon^{p-1}(r,t)u_{\varepsilon r}(r,t)dr \\ & -(p-1)\int_0^R r^{n-1}\zeta_\delta(r)u_\varepsilon^{p-1}(r,t)f_\varepsilon'(u_\varepsilon(r,t))u_{\varepsilon r}(r,t)w_{\varepsilon r}(r,t)dr \\ & -\int_0^R r^{n-1}\zeta_{\delta r}(r)u_\varepsilon^p(r,t)f_\varepsilon'(u_\varepsilon(r,t))w_{\varepsilon r}(r,t)dr \qquad \text{for all } t>0. \end{aligned} \tag{3.38}$$

Here, another integration by parts shows that in view of (3.1)-(3.3) we can find $c_1 = c_1(\delta) > 0$ such that

$$\begin{aligned} -\int_0^R r^{n-1}\zeta_{\delta r}(r)u_\varepsilon^{p-1}(r,t)u_{\varepsilon r}(r,t)dr &= \frac{1}{p}\int_0^R (r^{n-1}\zeta_{\delta r})_r u_\varepsilon^p(r,t)dr \\ &\le c_1\int_{\frac{\delta}{2}}^R u_\varepsilon^p(r,t)dr \qquad \text{for all } t>0, \end{aligned} \tag{3.39}$$

whereas two applications of Young's inequality reveal that

$$\begin{aligned} -(p-1)\int_0^R r^{n-1}\zeta_\delta(r)u_\varepsilon^{p-1}(r,t)f_\varepsilon'(u_\varepsilon(r,t))u_{\varepsilon r}(r,t)w_{\varepsilon r}(r,t)dr &\le \frac{p-1}{2}\int_0^R r^{n-1}\zeta_\delta(r)u_\varepsilon^{p-2}(r,t)u_{\varepsilon r}^2(r,t)dr \\ &+\frac{p-1}{2}\int_0^R r^{n-1}\zeta_\delta(r)u_\varepsilon^p(r,t)w_{\varepsilon r}^2(r,t)dr \end{aligned}$$

$$\le \frac{p-1}{2}\int_0^R r^{n-1}\zeta_\delta(r)u_\varepsilon^{p-2}(r,t)u_{\varepsilon r}^2(r,t)dr + \frac{p-1}{2}R^{n-1}\int_{\frac{\delta}{2}}^R u_\varepsilon^p(r,t)w_{\varepsilon r}^2(r,t)dr \qquad (3.40)$$

and

$$\begin{aligned} -\int_0^R r^{n-1}\zeta_{\delta r}(r)u_\varepsilon^p(r,t)f_\varepsilon'(u_\varepsilon(r,t))w_{\varepsilon r}(r,t)dr &\le \frac{1}{2}\int_0^R r^{n-1}\zeta_{\delta r}(r)u_\varepsilon^p(r,t)\Big(w_{\varepsilon r}^2(r,t)+1\Big)dr \\ &\le \frac{2R^{n-1}}{\delta}\int_{\frac{\delta}{2}}^R u_\varepsilon^p(r,t)\Big(w_{\varepsilon r}^2(r,t)+1\Big)dr \qquad (3.41)\end{aligned}$$

for all $t > 0$, again because of (3.1)-(3.3) and the fact that $0 \le f_\varepsilon' \le 1$. Combined with (3.39)-(3.41), once more in view of (3.1) the identity (3.38) thus entails (3.37). □

In estimating the integral on the right of (3.37) appropriately for $p < 2$ in Lemma 3.7, we will make use of the following consequence of Lemma 3.1 when combined with a one-dimensional interpolation argument.

Lemma 3.6 *Let $\delta \in (0, R)$ and $T > 0$. Then there exists $C(\delta, T) > 0$ such that*

$$\int_0^T\int_\delta^R u_\varepsilon^2(r,t)drdt \le C(\delta,T) \qquad \textit{for all } \varepsilon \in (0,1). \qquad (3.42)$$

PROOF. According to the one-dimensional version of the Gagliardo-Nirenberg inequality, there exists $c_1 = c_1(\delta) > 0$ such that

$$\|\varphi\|_{L^4((\delta,R))}^4 \le c_1\|\varphi_r\|_{L^1((\delta,R))}^2\|\varphi\|_{L^2((\delta,R))}^2 + c_1\|\varphi\|_{L^2((\delta,R))}^4 \qquad \text{for all } \varphi \in W^{1,1}((\delta,R)).$$

When applied to $\varphi := \sqrt{u_\varepsilon(\cdot,t)+1}$ and integrated with respect to $t \in (0,T)$, this shows that

$$\begin{aligned}\int_0^T\int_\delta^R \Big(u_\varepsilon(r,t)+1\Big)^2 drdt &\le c_1\int_0^T\Big\{\int_\delta^R\Big|\partial_r\sqrt{u_\varepsilon(r,t)+1}\Big|dr\Big\}^2\cdot\Big\{\int_\delta^R\Big(u_\varepsilon(r,t)+1\Big)dr\Big\}dt \\ &\quad + c_1\int_0^T\Big\{\int_\delta^R\Big(u_\varepsilon(r,t)+1\Big)dr\Big\}^2 dt\end{aligned}$$

for all $T > 0$. Since Lemma 3.1 provides $c_2 = c_2(\delta) > 0$ fulfilling

$$\int_\delta^R\Big(u_\varepsilon(r,t)+1\Big)dr \le c_2 \qquad \text{for all } t > 0,$$

from this we particularly infer that

$$\int_0^T\int_\delta^R u_\varepsilon^2(r,t)dr \le c_1c_2\int_0^T\Big\{\int_\delta^R\Big|\partial_r\sqrt{u_\varepsilon(r,t)+1}\Big|dr\Big\}^2dt + c_1c_2^2T \qquad \text{for all } T > 0,$$

and that thus (3.42) is a consequence of (3.6). □

With this estimate at hand, from Lemma 3.6 we can now derive a spatially local L^p bound for u_ε whenever $p < 2$.

Lemma 3.7 *Let $p \in (1,2)$. Then for all $\delta \in (0,R)$ and each $T>0$ there exists $C(p,\delta,T)>0$ fulfilling*

$$\int_\delta^R u_\varepsilon^p(r,t)dr \le C(p,\delta,T) \qquad \textit{for all } t \in (0,T) \textit{ and any } \varepsilon \in (0,1). \tag{3.43}$$

Proof. Given $p \in (1,2)$, $\delta \in (0,R)$ and $T>0$, from Lemma 3.6 we obtain $c_1 = c_1(\delta,T)>0$ such that for all $\varepsilon \in (0,1)$,

$$\int_0^T \int_{\frac{\delta}{2}}^R u_\varepsilon^2(r,t)drdt \le c_1, \tag{3.44}$$

whereas since $p<2$ we may invoke Corollary 3.4 to find $c_2 = c_2(p,\delta,T)>0$ satisfying

$$\int_{\frac{\delta}{2}}^R |w_{\varepsilon r}(r,t)|^{\frac{4}{2-p}} dr \le c_2 \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1). \tag{3.45}$$

Therefore, using Young's inequality we can estimate the integral appearing on the right-hand side of (3.37) according to

$$\begin{aligned} \int_{\frac{\delta}{2}}^R u_\varepsilon^p(r,t)\Big(w_{\varepsilon r}^2(r,t)+1\Big)dr &\le 2\int_{\frac{\delta}{2}}^R u_\varepsilon^2(r,t) + \int_{\frac{\delta}{2}}^R |w_{\varepsilon r}(r,t)|^{\frac{4}{2-p}}dr + R \\ &\le 2\int_{\frac{\delta}{2}}^R u_\varepsilon^2(r,t) + c_2 + R \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1), \end{aligned}$$

whence from Lemma 3.5 we infer the existence of $c_3 = c_3(p,\delta,T)>0$ such that for all $\varepsilon \in (0,1)$ we have

$$\int_0^R r^{n-1}\zeta_\delta(r)u_\varepsilon^p(r,t)dr \le \int_0^R r^{n-1}\zeta_\delta(r)u_0^p(r)dr + c_3 \cdot \left\{1 + \int_0^T \int_{\frac{\delta}{2}}^R u_\varepsilon^2(r,t)drdt\right\} \qquad \text{for all } t \in (0,T).$$

In view of (3.44) and (3.1), this immediately leads to (3.43). □

In view of our preparation made in Lemma 3.3, from the above we can conclude without any further efforts an L^∞ bound for $v_{\varepsilon r}$, and hence also for $w_{\varepsilon r}$, outside the origin.

Corollary 3.8 *For all $\delta \in (0,R)$ and $T>0$ one can find $C(\delta,T)>0$ such that for each $\varepsilon \in (0,1)$ we have*

$$|v_{\varepsilon r}(r,t)| \le C(\delta,T) \qquad \textit{for all } r \in (\delta,R) \textit{ and } t \in (0,T) \tag{3.46}$$

and

$$|w_{\varepsilon r}(r,t)| \le C(\delta,T) \qquad \textit{for all } r \in (\delta,R) \textit{ and } t \in (0,T). \tag{3.47}$$

Proof. Choosing any $p \in (1,2)$ in Lemma 3.5, we see that then (3.46) becomes a consequence of the latter when combined with Lemma 3.3. Thereafter, (3.47) results from (3.46) on recalling Lemma 3.2. □

As a particular outcome of Corollary 3.8, we obtain that even for the value $p=2$, yet excluded in Lemma 3.7, the integral on the right of (3.37) can be adequately controlled by means of Lemma 3.6, in consequence leading to the following main result of this section.

Lemma 3.9 *Let $\delta \in (0,R)$ and $T>0$. Then one can choose $C(\delta,T)>0$ with the property that whenever $\varepsilon \in (0,1)$,*

$$\int_\delta^R u_\varepsilon^2(r,t)dr \le C(\delta,T) \qquad \textit{for all } t \in (0,T) \tag{3.48}$$

and

$$\int_0^T \int_\delta^R u_{\varepsilon r}^2(r,t)drdt. \tag{3.49}$$

Proof. By means of Corollary 3.8 taking $c_1 = c_1(\delta,T)>0$ such that $|w_{\varepsilon r}| \le c_1$ in $(\frac{\delta}{2},R)\times(0,T)$ for all $\varepsilon \in (0,1)$, from Lemma 3.5 we see that with some $c_2 = c_2(\delta)>0$ we have

$$\begin{aligned}
\frac{d}{dt}\int_0^R r^{n-1}\zeta_\delta(r)u_\varepsilon^2(r,t)dr + \delta^{n-1}\int_\delta^R u_{\varepsilon r}^2(r,t)dr \\
&\le c_2 \int_{\frac{\delta}{2}}^R u_\varepsilon^2(r,t)\Big(w_{\varepsilon r}^2(r,t)+1\Big)dr \\
&\le c_2\cdot(c_1^2+1)\int_{\frac{\delta}{2}}^R u_\varepsilon^2(r,t)dr \qquad \text{for all } t\in(0,T)
\end{aligned} \tag{3.50}$$

and each $\varepsilon \in (0,1)$. As again Lemma 3.6 yields $c_3 = c_3(\delta,T)>0$ such that

$$\int_0^T \int_{\frac{\delta}{2}}^R u_\varepsilon^2(r,t)drdt \le c_3 \qquad \text{for all } \varepsilon\in(0,1),$$

on integrating (3.50) in time we infer that for all $\varepsilon \in (0,1)$,

$$\begin{aligned}
\int_0^R r^{n-1}\zeta_\delta(r)u_\varepsilon^2(r,t)dr + \delta^{n-1}\int_0^t\int_\delta^R u_{\varepsilon r}^2(r,s)drds \\
&\le \int_0^R r^{n-1}\zeta_\delta(r)u_0^2(r)dr + c_2\cdot(c_1^2+1)c_3 \qquad \text{for all } t\in(0,T),
\end{aligned}$$

from which both (3.48) and (3.49) follow thanks to (3.1). □

We remark that in light of this new information, on suitably interpolating between (3.48) and (3.49) it would be possible to improve Lemma 3.6 so as to provide a bound for u_ε in corresponding spatio-temporal L^6 spaces, thereafter go back to Lemma 3.5 to derive a spatially local bound for u_ε in L^6, and repeat this bootstrapping procedure so as to finally obtain boundedness of $(u_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^\infty_{loc}([0,\infty);L^p_{loc}((0,R]))$ for any $p\in[1,\infty)$. In view of a stronger result to be obtained more directly in Lemma 3.11 and Corollary 3.12 below, however, we do not pursue this any further here.

3.4 A Hölder estimate for v_ε

In order to verify the claimed continuity property of the second solution component stated in Theorem 1.1, let us note a uniform Hölder regularity feature enjoyed by v_ε according to standard parabolic regularity results and the estimates provided by Lemma 3.9 and Corollary 3.8.

Lemma 3.10 *For all $\delta \in (0,R)$ and $T>0$ there exist $\theta=\theta(\delta,T)\in(0,1)$ and $C(\delta,T)>0$ such that*

$$\|v_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}([\delta,R]\times[0,T])} \le C(\delta,T) \qquad \textit{for all } \varepsilon\in(0,1). \tag{3.51}$$

PROOF. For fixed $\delta\in(0,R)$ we choose $\xi(r,t):=\zeta_\delta(r)$, $(r,t)\in[0,R]\times[0,\infty)$, in (3.13). Then applying Lemma 3.9 and Corollary 3.8 to fix $c_1=c_1(\delta,T)>0$ and $c_2=c_2(\delta,T)>0$ such that for all $\varepsilon\in(0,1)$ we have

$$\int_{\frac{\delta}{2}}^R u_\varepsilon^2(r,t)dr \le c_1(\delta,T)$$

and

$$|v_{\varepsilon r}(r,t)|\le c_2 \qquad \text{for all } r\in(\tfrac{\delta}{2},R) \text{ and } t\in(0,T),$$

we see using (3.1)-(3.3) and (2.4) that in (3.14) and (3.15) we can estimate

$$\begin{aligned} |b_1(r,t)| &\le 2|\zeta_{\delta r}(r)|\cdot|v_{\varepsilon r}(r,t)| + \frac{2(n-1)}{\delta}\zeta_\delta(r)|v_{\varepsilon r}(r,t)| \\ &\le \frac{8}{\delta}c_2 + \frac{2(n-1)}{\delta}c_2 \qquad \text{for all } r\in(0,R), t\in(0,T) \text{ and } \varepsilon\in(0,1) \end{aligned}$$

and

$$\begin{aligned} \int_0^R b_2^2(r,t)dr &\le 2\int_{\frac{\delta}{2}}^R \frac{1024}{\delta^4}v_\varepsilon^2(r,t)dr + 2\int_{\frac{\delta}{2}}^R u_\varepsilon^2(r,t)v_\varepsilon^2(r,t)dr \\ &\le \frac{2048}{\delta^4}\|v_0\|^2_{L^\infty((0,R))}R + 2\|v_0\|^2_{L^\infty((0,R))}c_1 \qquad \text{for all } t\in(0,T) \text{ and each } \varepsilon\in(0,1). \end{aligned}$$

As this implies an estimate for b_1+b_2 in $L^{\hat r}((0,T);L^{\hat q}((0,R)))$ with $\hat r:=\infty$ and $\hat q:=2$ fulfilling $\frac{1}{\hat r}+\frac{1}{2\hat q}=\frac14<1$, a known result on Hölder regularity properties of the Neumann problem for the inhomogeneous linear heat equation ([25, Theorem 1.3, Remark 1.4]) yields $\theta_1=\theta_1(\delta,T)\in(0,1)$ and $c_3=c_3(\delta,T)>0$ such that

$$\|\zeta_\delta v_\varepsilon\|_{C^{\theta_1,\frac{\theta_1}{2}}([0,R)\times[0,T])} \le c_3 \qquad \text{for all } \varepsilon\in(0,1).$$

As $\zeta_\delta\equiv 1$ in $[\delta,R]$, this clearly implies (3.51). $\square$

3.5 L^∞ and Hölder bounds for u_ε

Now an important consequence of the two estimates from Lemma 3.9, along with the chemotactic gradient bounds from Corollary 3.8, can be obtained by means of a proper exploitation of standard smoothing estimates for the heat semigroup associated with the linear inhomogeneous Neumann problem (3.9). We formulate our result in this direction in such a manner that it will firstly imply an L^∞ bound for u_ε in $(\delta,R)\times(0,T)$ for arbitrary $\delta\in(0,R)$ and $T>0$ in Corollary 3.12, and that later on it can moreover be used to guarantee continuity of the correspondingly obtained limit function down to $t=0$ in Lemma 4.1.

Lemma 3.11 *For all $\delta \in (0,R)$ and $T>0$ there exists $C(\delta,T)>0$ such that whenever $\varepsilon \in (0,1)$, we have*

$$\Big\|\zeta_\delta u_\varepsilon(\cdot,t) - e^{-tA}(\zeta_\delta u_0)\Big\|_{L^\infty((0,R))} \le C(\delta,T)\cdot t^{\frac{1}{4}} \qquad \text{for all } t\in(0,T). \tag{3.52}$$

PROOF. We once more recall known smoothing properties of the Neumann heat semigroup in $(0,R)$ ([33], [8]) to find $c_1>0$ and $c_2>0$ such that for any $\tau>0$ we have

$$\|e^{-\tau A}\varphi\|_{L^\infty((0,R))} \le c_1(1+\tau^{-\frac{1}{4}})\|\varphi\|_{L^2((0,R))} \qquad \text{for all } \varphi\in L^2((0,R)) \tag{3.53}$$

and

$$\|e^{-\tau A}\varphi_r\|_{L^\infty((0,R))} \le c_2(1+\tau^{-\frac{3}{4}})\|\varphi\|_{L^2((0,R))} \qquad \text{for all } \varphi\in W^{1,2}((0,R)) \text{ satisfying } \varphi(0)=\varphi(R)=0. \tag{3.54}$$

Now for fixed $\delta\in(0,R)$, in (3.9) we choose $\xi(r,t):=\zeta_\delta(r)$, $(r,t)\in[0,R]\times[0,\infty)$, to obtain from an associated variation-of-constants representation that with a_1,a_2 and a_3 as correspondingly defined in (3.10)-(3.12) we have

$$\zeta_\delta u_\varepsilon(\cdot,t) = e^{-tA}(\zeta_\delta u_0) + \int_0^t e^{-(t-s)A}\Big\{a_{1r}(\cdot,s)+a_2(\cdot,s)+a_3(\cdot,s)\Big\}ds$$

and hence

$$\begin{aligned}
\Big\|\zeta_\delta u_\varepsilon(\cdot,t) - e^{-tA}(\zeta_\delta u_0)\Big\|_{L^\infty((0,R))} &\le \int_0^t \|e^{-(t-s)A}a_{1r}(\cdot,s)\|_{L^\infty((0,R))}ds \\
&\quad + \int_0^t \|e^{-(t-s)A}a_2(\cdot,s)\|_{L^\infty((0,R))}ds \\
&\quad + \int_0^t \|e^{-(t-s)A}a_3(\cdot,s)\|_{L^\infty((0,R))}ds \\
&=: I_1(t)+I_2(t)+I_3(t)
\end{aligned} \tag{3.55}$$

for $t>0$. To estimate I_1, I_2 and I_3 over $(0,T)$ for given $T>0$, we invoke Lemma 3.9 and Corollary 3.8 to fix positive constants c_3, c_4 and c_5, all possibly depending on δ and T, such that for all $\varepsilon\in(0,1)$,

$$\int_{\frac{\delta}{2}}^R u_\varepsilon^2(r,t)dr \le c_3 \qquad \text{for all } t\in(0,T) \tag{3.56}$$

and

$$\int_0^T \int_{\frac{\delta}{2}}^R u_{\varepsilon r}^2(r,t)drdt \le c_4 \tag{3.57}$$

as well as

$$|w_{\varepsilon r}(r,t)| \le c_5 \qquad \text{for all } r\in(\tfrac{\delta}{2},R) \text{ and } t\in(0,T). \tag{3.58}$$

In view of (3.1)-(3.3) and (2.2), we can therefore estimate

$$\begin{aligned}
\int_0^R a_1^2(r,t)dr &= \int_0^R \zeta_\delta^2(r)u_\varepsilon^2(r,t)f_\varepsilon'^2(u_\varepsilon(r,t))w_{\varepsilon r}^2(r,t)dr \\
&\le \int_{\frac{\delta}{2}}^R u_\varepsilon^2(r,t)w_{\varepsilon r}^2(r,t)dr \\
&\le c_6 := c_3c_5^2 \qquad \text{for all } t\in(0,T)
\end{aligned} \tag{3.59}$$

and

$$\begin{aligned}
\int_0^T\int_0^R a_2^2(r,t)drdt &\le 2\int_0^T\int_0^R \frac{(n-1)^2}{r^2}\zeta_\delta^2(r)u_{\varepsilon r}^2(r,t)drdt + 2\int_0^T\int_0^R 4\zeta_{\delta r}^2(r)u_{\varepsilon r}^2(r,t)drdt \\
&\le \frac{8(n-1)^2}{\delta^2}\int_0^T\int_{\frac{\delta}{2}}^R u_{\varepsilon r}^2(r,t)drdt + \frac{128}{\delta^2}\int_0^T\int_{\frac{\delta}{2}}^R u_{\varepsilon r}^2(r,t)drdt \\
&\le c_7 := \frac{8(n-1)^2}{\delta^2}c_4 + \frac{128}{\delta^2}c_4 \qquad (3.60)
\end{aligned}$$

as well as

$$\begin{aligned}
\int_0^R a_3^2(r,t)dr &\le 3\int_0^R \zeta_{\delta rr}^2(r)u_\varepsilon^2(r,t)dr \\
&\quad +3\int_0^R \zeta_{\delta r}^2(r)u_\varepsilon^2(r,t)f_\varepsilon'^2(u_\varepsilon(r,t))w_{\varepsilon r}^2(r,t)dr \\
&\quad +3\int_0^R \frac{(n-1)^2}{r^2}\zeta_\delta^2(r)u_\varepsilon^2(r,t)f_\varepsilon'^2(u_\varepsilon(r,t))w_{\varepsilon r}^2(r,t)dr \\
&\le \frac{3072}{\delta^4}\int_{\frac{\delta}{2}}^R u_\varepsilon^2(r,t)dr \\
&\quad +\frac{48}{\delta^2}\int_{\frac{\delta}{2}}^R u_\varepsilon^2(r,t)w_{\varepsilon r}^2(r,t)dr \\
&\quad +\frac{12(n-1)^2}{\delta^2}\int_{\frac{\delta}{2}}^R u_\varepsilon^2(r,t)w_{\varepsilon r}^2(r,t)dr \\
&\le c_8 := \frac{3072}{\delta^4}c_3 + \frac{48}{\delta^2}c_3c_5^2 + \frac{12(n-1)^2}{\delta^2}c_3c_5^2 \qquad \text{for all } t\in(0,T) \qquad (3.61)
\end{aligned}$$

whenever $\varepsilon\in(0,1)$. Consequently, using (3.54) and that $a_1(0,t)=a_1(R,t)=0$ due to the fact that $w_{\varepsilon r}(0,t)=w_{\varepsilon r}(R,t)=0$ for all $t>0$, we find that

$$\begin{aligned}
I_1(t) &\le c_2\int_0^t \Big(1+(t-s)^{-\frac{3}{4}}\Big)\|a_1(\cdot,s)\|_{L^2((0,R))}ds \\
&\le c_2c_6^{\frac{1}{2}}\int_0^t \Big(1+(t-s)^{-\frac{3}{4}}\Big)ds \\
&= c_2c_6^{\frac{1}{2}}(t+4t^{\frac{1}{4}}) \\
&\le c_2c_6^{\frac{1}{2}}(T^{\frac{3}{4}}+4)\cdot t^{\frac{1}{4}} \qquad \text{for all } t\in(0,T),
\end{aligned}$$

whence combining (3.53) with the Cauchy-Schwarz inequality implies that

$$\begin{aligned}
I_2(t) &\le c_1\int_0^t \Big(1+(t-s)^{-\frac{1}{4}}\Big)\|a_2(\cdot,s)\|_{L^2((0,R))}ds \\
&\le c_1\cdot\bigg\{\int_0^t \Big(1+(t-s)^{-\frac{1}{4}}\Big)^2ds\bigg\}^{\frac{1}{2}}\cdot\bigg\{\int_0^t \|a_2(\cdot,s)\|_{L^2((0,R))}^2ds\bigg\}^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\le c_1 \cdot \left\{ 2\int_0^t \Big(1+(t-s)^{-\frac12}\Big)ds \right\}^{\frac12} \cdot \left\{ \int_0^t \|a_2(\cdot,s)\|_{L^2((0,R))}^2 ds \right\}^{\frac12} \\
&\le c_1 \cdot (2c_7)^{\frac12} \cdot \left\{ \int_0^t \Big(1+(t-s)^{-\frac12}\Big)ds \right\}^{\frac12} \\
&= c_1 \cdot (2c_7)^{\frac12} \cdot (t+2t^{\frac12})^{\frac12} \\
&\le c_1 \cdot (2c_7)^{\frac12} \cdot (T^{\frac12}+2)^{\frac12} \cdot t^{\frac14} \qquad \text{for all } t\in(0,T).
\end{aligned}
$$

As (3.53) moreover warrants that

$$
\begin{aligned}
I_3(t) &\le c_1 \int_0^t \Big(1+(t-s)^{-\frac14}\Big)\|a_3(\cdot,s)\|_{L^2((0,R))}ds \\
&\le c_1 c_8^{\frac12} \int_0^t \Big(1+(t-s)^{-\frac14}\Big)ds \\
&= c_1 c_8^{\frac12} \Big(t+\frac43 t^{\frac34}\Big) \\
&\le c_1 c_8^{\frac12} \Big(T^{\frac34}+\frac43 T^{\frac12}\Big)\cdot t^{\frac14} \qquad \text{for all } t\in(0,T),
\end{aligned}
$$

from (3.55) we thus conclude that (3.52) holds. $\Box$

The following conclusion from the latter is evident.

Corollary 3.12 *For all $\delta\in(0,R)$ and any $T>0$ there exists $C(\delta,T)>0$ such that*

$$
u_\varepsilon(r,t) \le C(\delta,T) \qquad \textit{for all } r\in(\delta,R), t\in(0,T) \textit{ and } \varepsilon\in(0,1). \tag{3.62}
$$

Proof. Recalling that $\zeta_\delta\equiv 1$ in (δ,R) and observing that $e^{-tA}(\zeta_\delta u_0)\le \|u_0\|_{L^\infty((0,R))}$ in $(0,R)$ for all $t>0$ according to the maximum principle, one readily derives (3.62) as a consequence of Lemma 3.11. $\Box$

With this boundedness property at hand, we can next invoke standard regularity theory for inhomogeneous linear parabolic equations, as formulated e.g. in [25] even in a more general quasilinear framework, so as to obtain Hölder estimates for u_ε. Since we do not assume higher regularity of u_0 beyond continuity, these estimates need to remain local also with respect to $t\in(0,T]$ for given $T>0$, and accordingly their derivation involves a further cut-off procedure involving the time variable.

Lemma 3.13 *Let $\delta\in(0,R), T>0$ and $\tau\in(0,T)$. Then there exist $\theta=\theta(\delta,T,\tau)\in(0,1)$ and $C(\delta,T,\tau)>0$ such that*

$$
\|u_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}([\delta,R]\times[\tau,T])} \le C(\delta,T,\tau) \qquad \textit{for all } \varepsilon\in(0,1). \tag{3.63}
$$

Proof. We let

$$
\xi(r,t)\equiv \xi_{\delta\tau}(r,t) := \zeta_\delta(r)\zeta_\tau(t), \qquad r\in[0,R],\ t\ge 0,
$$

and apply Corollary 3.12, Corollary 3.8 and Lemma 3.9 to fix $c_1=c_1(\delta,T)>0$, $c_2=c_2(\delta,T)>0$ and $c_3=c_3(\delta,T)>0$ such that

$$
u_\varepsilon(r,t)\le c_1 \qquad \text{for all } r\in(\tfrac{\delta}{2},R) \text{ and } t\in(0,T) \tag{3.64}
$$

and

$$|w_{\varepsilon r}(r,t)| \le c_2 \qquad \text{for all } r \in (\tfrac{\delta}{2}, R) \text{ and } t \in (0,T), \tag{3.65}$$

and that

$$\int_0^T \int_{\frac{\delta}{2}}^R u_{\varepsilon r}^2(r,t)drdt \le c_3 \tag{3.66}$$

for all $\varepsilon \in (0,1)$. Then due to (3.1)-(3.3) and (2.2), the functions a_1, a_2 and a_3 in (3.9) defined by (3.10)-(3.12) satisfy

$$|a_1(r,t)| \le u_\varepsilon(r,t) f'_\varepsilon(u_\varepsilon(r,t)) |w_{\varepsilon r}(r,t)| \le c_1 c_2 \qquad \text{for all } r \in (0,R) \text{ and } t \in (0,T)$$

and

$$\begin{aligned} \int_0^T \int_0^R a_2^2(r,t)drdt &\le 2\int_0^T \int_{\frac{\delta}{2}}^R \frac{(n-1)^2}{r^2} u_{\varepsilon r}^2(r,t)drdt + 2\int_0^T \int_{\frac{\delta}{2}}^R \frac{16}{\delta^2} u_{\varepsilon r}^2(r,t)drdt \\ &\le \frac{8(n-1)^2}{\delta^2} c_3 + \frac{32}{\delta^2} c_3 \end{aligned} \tag{3.67}$$

as well as

$$\begin{aligned} |a_3(r,t)| &\le \frac{4}{\tau} u_\varepsilon(r,t) + \frac{32}{\delta^2} u_\varepsilon(r,t) \\ &\quad + \frac{4}{\delta} u_\varepsilon(r,t) f'_\varepsilon(u_\varepsilon(r,t)) |w_{\varepsilon r}(r,t)| + \frac{2(n-1)}{\delta} u_\varepsilon(r,t) f'_\varepsilon(u_\varepsilon(r,t)) |w_{\varepsilon r}(r,t)| \\ &\le \frac{4}{\tau} c_1 + \frac{32}{\delta^2} c_1 + \frac{4}{\delta} c_1 c_2 + \frac{2(n-1)}{\delta} c_1 c_2 \qquad \text{for all } r \in (0,R) \text{ and } t \in (0,T) \end{aligned} \tag{3.68}$$

whenever $\varepsilon \in (0,1)$. We now interpret (3.9) as a PDE for $z := \xi u_\varepsilon$ of the form

$$z_t = \partial_r a(r,t,z_r) + b(r,t)$$

with $a(r,t,p) := p + a_1(r,t)$ and $b(r,t) := a_2(r,t) + a_3(r,t)$ for $(r,t) \in (0,R) \times (0,T)$ and $p \in \mathbb{R}$, and estimate

$$a(r,t,p) \cdot p = p^2 + a_1(r,t)p \ge \frac{1}{2}p^2 - \frac{1}{2}a_1^2(r,t) \ge \frac{1}{2}p^2 - \frac{1}{2}c_1^2 c_2^2 \qquad \text{for all } (r,t,p) \in (0,R) \times (0,T) \times \mathbb{R}$$

and

$$|a(r,t,p)| \le p + c_1 c_2 \qquad \text{for all } (r,t,p) \in (0,R) \times (0,T) \times \mathbb{R}.$$

Since (3.67) and (3.68) imply a bound for b in $L^{\hat{r}}((0,T); L^{\hat{q}}((0,R)))$ with $\hat{q} := \hat{r} := 2$ satisfying $\frac{1}{\hat{r}} + \frac{1}{2\hat{q}} = \frac{3}{4} < 1$, we may conclude (3.63) directly from a standard result on parabolic Hölder regularity ([25, Theorem 1.3, Remark 1.3]). □

3.6 Estimates in $C^{2+\theta,1+\frac{\theta}{2}}$

The goal of this section is to complete the series of our regularity arguments so as to finally provide estimates in $C^{2+\theta,1+\frac{\theta}{2}}$ spaces involving finite time intervals, locally away from both the spatial and the temporal origin. In first addressing this topic for the second solution component, we shall employ a two-step bootstrap procedure, at the first stage relying on maximal Sobolev regularity estimates for the linear inhomogeneous heat equation ([9]) along with an embedding property of suitable Sobolev spaces into $C^{1+\theta,\theta}$ spaces ([2]), and at the second involving classical Schauder theory ([16]).

Lemma 3.14 *For any $\delta \in (0,R), T > 0$ and $\tau \in (0,T)$ one can pick $\theta = \theta(\delta,T,\tau) \in (0,1)$ and $C(\delta,T,\tau) > 0$ such that*

$$\|v_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}([\delta,R]\times[\tau,T])} \le C(\delta,T,\tau) \tag{3.69}$$

and

$$\|w_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}([\delta,R]\times[\tau,T])} \le C(\delta,T,\tau) \tag{3.70}$$

for all $\varepsilon \in (0,1)$.

Proof. The proof proceeds in two steps.

<u>Step 1.</u> We first claim that for any choice of $\delta \in (0,R), T > 0$ and $\tau \in (0,T)$ there exist $\theta_1 \in (0,1)$ and $c_1 = c_1(\delta,T,\tau) > 0$ such that

$$\|v_\varepsilon\|_{C^{1+\theta_1,\theta_1}([\delta,R]\times[\tau,T])} \le c_1 \tag{3.71}$$

for all $\varepsilon \in (0,1)$.

To verify this, for such δ, T and τ we again choose $\xi(r,t) \equiv \xi_{\delta\tau}(r,t) := \zeta_\delta(r)\zeta_\tau(t)$, $(r,t) \in [0,R]\times[0,\infty)$, and then infer from Corollary 3.8, Corollary 3.12 and (2.4) that for the correspondingly defined functions b_1 and b_2 in (3.14) and (3.15) we can find $c_2 = c_2(\delta,T,\tau) > 0$ such that for all $\varepsilon \in (0,1)$ we have

$$|b_1(r,t)| + |b_2(r,t)| \le c_2 \qquad \text{for all } r \in (0,R) \text{ and } t \in (0,T),$$

again because of the fact that $\zeta \equiv 0$ on $[0,\frac{1}{2}]$. As $(\xi_{\delta\tau}v_\varepsilon)_r = 0$ for $(r,t) \in \{0,R\} \times (0,\infty)$ and $(\xi_{\delta\tau}v_\varepsilon)(r,0) = 0$ for all $r \in (0,R)$, an application of known estimates on maximal Sobolev regularity in the Neumann problem for the inhomogeneous linear heat equation in $(0,R)$ ([9]) shows that given any $p \in (1,\infty)$ we can find $c_3 = c_3(p,\delta,T,\tau) > 0$ with the property that

$$\int_0^T \Big\|(\xi_{\delta\tau}v_\varepsilon)(\cdot,t)\Big\|^p_{W^{2,p}((0,R))} dt + \int_0^T \Big\|(\xi_{\delta\tau}v_\varepsilon)_t(\cdot,t)\Big\|^p_{L^p((0,R))} dt \le c_3$$

for all $\varepsilon \in (0,1)$. According to a known embedding result ([2]), on choosing p suitably large here we can achieve that for some $\theta_1 \in (0,1)$ and $c_4 = c_4(\delta,T,\tau) > 0$ we have

$$\|\xi_{\delta\tau}v_\varepsilon\|_{C^{1+\theta_1,\theta_1}([0,R]\times[0,T])} \le c_4 \qquad \text{for all } \varepsilon \in (0,1),$$

from which (3.71) immediately follows with $c_1 := c_4$.

Step 2. We proceed to show that the desired conlcusion holds.

For this purpose, given $\delta \in (0,R), T > 0$ and $\tau \in (0,T)$ we let $\xi_{\delta\tau}$ be as above and then obtain from Step 1 in conjunction with Lemma 3.13 that in fact we can pick $\theta_2 = \theta_2(\delta,T,\tau) \in (0,1)$ and $c_5 = c_5(\delta,T,\tau) > 0$ such that the functions in (3.14) and (3.15) satisfy

$$\|b_1 + b_2\|_{C^{\theta_2,\frac{\theta_2}{2}}([0,R]\times[0,T])} \le c_5 \qquad \text{for all } \varepsilon \in (0,1).$$

Therefore, classical parabolic Schauder estimates ([16]) provide $c_6 = c_6(\delta,T,\tau) > 0$ fulfilling

$$\|\xi_{\delta\tau} v_\varepsilon\|_{C^{2+\theta_2,1+\frac{\theta_2}{2}}([0,R]\times[0,T])} \le c_6 \qquad \text{for all } \varepsilon \in (0,1),$$

which again due to the properties of $\xi_{\delta\tau}$ implied by (3.1)-(3.3), and due to (2.4) and (2.5), readily entails both (3.69) and (3.70). $\square$

Utilizing similar tools from standard parabolic regularity theory, we can finally achieve a similar conclusion also for u_ε. In comparison to the above, the more delicate coupling through the chemotactic term, according to (3.9) interpreted as an (r,t)-dependent given source here, apparently requires the use of an iterative procedure rather than a one-step reasoning at the level of the argument involving maximal Sobolev regularity.

Lemma 3.15 *Let $\delta \in (0,R), T > 0$ and $\tau \in (0,T)$. Then there exist $\theta = \theta(\delta,T,\tau) \in (0,1)$ and $C(\delta,T,\tau) > 0$ such that*

$$\|u_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}([\delta,R]\times[\tau,T])} \le C(\delta,T,\tau) \qquad \textit{for all } \varepsilon \in (0,1). \tag{3.72}$$

Proof. The argument will be divided into four steps, where throughout the proof, given $\delta \in (0,R), T > 0$ and $\tau \in (0,T)$ we once again abbreviate $\xi(r,t) \equiv \xi_{\delta\tau}(r,t) := \zeta_\delta(r)\zeta_\tau(t)$, $(r,t) \in [0,R] \times [0,\infty)$, and in accordance with Corollary 3.12, Corollary 3.8, Lemma 3.14 and Lemma 3.9 we fix positive constants $c_1 = c_1(\delta,T), c_2 = c_2(\delta,T)$, $c_3 = c_3(\delta,T,\tau)$ and $c_4 = c_4(\delta,T)$ fulfilling

$$u_\varepsilon(r,t) \le c_1 \qquad \text{for all } r \in (\tfrac{\delta}{2},R) \text{ and } t \in (0,T) \tag{3.73}$$

and

$$|w_{\varepsilon r}(r,t)| \le c_2 \qquad \text{for all } r \in (\tfrac{\delta}{2},R) \text{ and } t \in (0,T) \tag{3.74}$$

as well as

$$|w_{\varepsilon rr}(r,t)| \le c_3 \qquad \text{for all } r \in (\tfrac{\delta}{2},R) \text{ and } t \in (\tfrac{\tau}{2},T) \tag{3.75}$$

and

$$\int_0^T \int_{\frac{\delta}{2}}^R u_{\varepsilon r}^2(r,t)\,drdt \le c_4 \tag{3.76}$$

for all $\varepsilon \in (0,1)$.

Step 1. We first claim that for all $\delta \in (0,R), T > 0$ and $\tau \in (0,T)$ there exists $c_5 = c_5(\delta,T,\tau) > 0$ with the property that for each $\varepsilon \in (0,1)$ we have

$$\int_\tau^T \|u_\varepsilon(\cdot,t)\|^2_{W^{2,2}((\delta,R))}dt + \int_\tau^T \|u_{\varepsilon t}\|^2_{L^2((\delta,R))}dt \le c_5. \tag{3.77}$$

To see this, we observe that as a consequence of (3.73)-(3.76) and (3.1)-(3.3) as well as (2.2), the functions a_1, a_2 and a_3 in (3.9) satisfy

$$\begin{aligned}
\int_0^T \int_0^R a_{1r}^2(r,t)drdt &= \int_0^T \int_0^R \Big\{ (\xi_{\delta\tau})_r(r,t) u_\varepsilon(r,t) f'_\varepsilon(u_\varepsilon(r,t)) w_{\varepsilon r}(r,t) \\
&\qquad + \xi_{\delta\tau}(r,t) u_{\varepsilon r}(r,t) f'_\varepsilon(u_\varepsilon(r,t)) w_{\varepsilon r}(r,t) \\
&\qquad + \xi_{\delta\tau}(r,t) u_\varepsilon(r,t) f''_\varepsilon(u_\varepsilon(r,t)) u_{\varepsilon r}(r,t) w_{\varepsilon r}(r,t) \\
&\qquad + \xi_{\delta\tau}(r,t) u_\varepsilon(r,t) f'_\varepsilon(u_\varepsilon(r,t)) w_{\varepsilon rr}(r,t) \Big\}^2 drdt \\
&\le 4 \int_0^T \int_{\frac{\delta}{2}}^R \frac{16}{\delta^2} u_\varepsilon^2(r,t) w_{\varepsilon r}^2(r,t) drdt + 4 \int_0^T \int_{\frac{\delta}{2}}^R u_{\varepsilon r}^2(r,t) w_{\varepsilon r}^2(r,t) drdt \\
&\quad + 4 \int_0^T \int_{\frac{\delta}{2}}^R u_\varepsilon^2(r,t) \cdot \varepsilon^2 \|\rho'\|_{L^\infty((0,\infty))}^2 u_{\varepsilon r}^2(r,t) w_{\varepsilon r}^2(r,t) drdt \\
&\quad + 4 \int_0^T \int_{\frac{\delta}{2}}^R u_\varepsilon^2(r,t) w_{\varepsilon rr}^2(r,t) drdt \\
&\le \frac{64}{\delta^2} c_1^2 c_2^2 RT + 4 c_2^2 c_4 + 4 \|\rho'\|_{L^\infty((0,\infty))}^2 c_1^2 c_2^2 c_4 + 4 c_1^2 c_3^2 RT
\end{aligned} \tag{3.78}$$

and, similarly,

$$\begin{aligned}
\int_0^T \int_0^R a_2^2(r,t)drdt &\le 2 \int_0^T \int_{\frac{\delta}{2}}^R \frac{(n-1)^2}{r^2} u_{\varepsilon r}^2(r,t) drdt + 2 \int_0^T \int_{\frac{\delta}{2}}^R \frac{64}{\delta^2} u_{\varepsilon r}^2(r,t) drdt \\
&\le \frac{8(n-1)^2}{\delta^2} c_4 + \frac{128}{\delta^2} c_4
\end{aligned} \tag{3.79}$$

as well as

$$\begin{aligned}
\int_0^T \int_0^R a_3^2(r,t)drdt &\le 4 \int_0^T \int_{\frac{\delta}{2}}^R \frac{16}{\tau^2} u_\varepsilon^2(r,t) drdt + 4 \int_0^T \int_{\frac{\delta}{2}}^R \frac{1024}{\delta^4} u_\varepsilon^2(r,t) drdt \\
&\quad + 4 \int_0^T \int_{\frac{\delta}{2}}^R \frac{16}{\delta^2} u_\varepsilon^2(r,t) f_\varepsilon'^2(u_\varepsilon(r,t)) w_{\varepsilon r}^2(r,t) drdt \\
&\quad + 4 \int_0^T \int_{\frac{\delta}{2}}^R \frac{(n-1)^2}{r^2} u_\varepsilon^2(r,t) f_\varepsilon'^2(u_\varepsilon(r,t)) w_{\varepsilon r}^2(r,t) drdt \\
&\le \frac{64}{\tau^2} c_1^2 RT + \frac{4096}{\delta^4} c_1^2 RT + \frac{64}{\delta^2} c_1^2 RT + \frac{16(n-1)^2}{\delta^2} c_1^2 c_2^2 RT
\end{aligned} \tag{3.80}$$

whenever $\varepsilon \in (0,1)$. Now since $(\xi_{\delta\tau} u_\varepsilon)_r(0,t) = (\xi_{\delta\tau} u_\varepsilon)_r(R,t) = 0$ for all $t > 0$ and $(\xi_{\delta\tau} u_\varepsilon)(r,0) = 0$ for all $r \in (0,R)$, known maximal Sobolev regularity properties of the Neumann heat semigroup on $(0,R)$ imply the existence of $c_6 > 0$, possibly depending on $T > 0$, such that

$$\begin{aligned}
&\int_0^T \Big\| (\xi_{\delta\tau} u_\varepsilon)(\cdot,t) \Big\|_{W^{2,2}((0,R))}^2 dt + \int_0^T \Big\| (\xi_{\delta\tau} u_\varepsilon)_t(\cdot,t) \Big\|_{L^2((0,R))}^2 dt \\
&\qquad \le c_6 \int_0^T \Big\| a_{1r}(\cdot,t) + a_2(\cdot,t) + a_3(\cdot,t) \Big\|_{L^2((0,R))}^2 dt
\end{aligned}$$

for all $\varepsilon \in (0,1)$. Therefore, the claimed conclusion is a consequence of (3.78), (3.79) and (3.80).

Step 2. Our next goal is to make sure that for each positive integer k and any $\delta \in (0,R), T>0$ and $\tau \in (0,T)$ there exists $c_7 = c_7(k,\delta,T,\tau) > 0$ such that

$$\int_\tau^T \|u_\varepsilon(\cdot,t)\|^{p_k}_{W^{2,p_k}((\delta,R))} dt + \int_\tau^T \|u_{\varepsilon t}\|^{p_k}_{L^{p_k}((\delta,R))} dt \le c_7 \qquad \text{for all } \varepsilon \in (0,1), \tag{3.81}$$

where $p_k := 2^k$.

For an inductive verification thereof on the basis of Step 1 as a starting point, assuming this property to be valid for some $k \ge 1$, given $\delta \in (0,R), T>0$ and $\tau \in (0,T)$ we can particularly fix $c_8 = c_8(k,\delta,T,\tau) > 0$ such that

$$\int_0^T \|u_\varepsilon(\cdot,t)\|^{p_k}_{W^{2,p_k}((\frac{\delta}{2},R))} dt \le c_8 \qquad \text{for all } \varepsilon \in (0,1).$$

Since by means of the Gagliardo-Nirenberg inequality we can find $c_9 = c_9(k,\delta) > 0$ fulfilling

$$\int_0^T \|u_{\varepsilon r}(\cdot,t)\|^{2p_k}_{L^{2p_k}((\frac{\delta}{2},R))} dt \le c_9 \int_0^T \|u_\varepsilon(\cdot,t)\|^{p_k}_{W^{2,p_k}((\frac{\delta}{2},R))} \|u_\varepsilon(\cdot,t)\|^{p_k}_{L^\infty((\frac{\delta}{2},R))} dt \qquad \text{for all } \varepsilon \in (0,1),$$

along with (3.73) this yields $c_{10} = c_{10}(k,\delta,T,\tau) > 0$ such that

$$\int_0^T \int_{\frac{\delta}{2}}^R |u_{\varepsilon r}(r,t)|^{p_{k+1}} dr dt \le c_{10} \qquad \text{for all } \varepsilon \in (0,1).$$

Together with (3.73)-(3.75) and the properties of $\xi_{\delta\tau}$ entailed by (3.1)-(3.3), this now provides an estimate in $L^{p_{k+1}}((0,R)\times(0,T))$ for the functions a_{1r} and a_2 determined by (3.10) and (3.11), whereas a corresponding bound for a_3 can independently be obtained solely from (3.73)-(3.75). Another application of maximal Sobolev regularity estimates on the Neumann problem for (3.9) with vanishing initial data thus readily yields $c_{11} = c_{11}(k,\delta,T,\tau) > 0$ such that

$$\int_0^T \Big\|(\xi_{\delta\tau}u_\varepsilon)(\cdot,t)\Big\|^{p_{k+1}}_{W^{2,p_{k+1}}((0,R))} dt + \int_0^T \Big\|(\xi_{\delta\tau}u_\varepsilon)_t(\cdot,t)\Big\|^{p_{k+1}}_{L^{p_{k+1}}((0,R))} dt \le c_{11} \qquad \text{for all } \varepsilon \in (0,1)$$

and thereby completes Step 2.

Step 3. Let us next verify for any $\delta \in (0,R), T>0$ and $\tau \in (0,T)$ there exist $\theta_1 \in (0,1)$ and $c_{12} = c_{12}(\delta,T,\tau) > 0$ fulfilling

$$\|u_\varepsilon\|_{C^{1+\theta_1,\theta_1}([\delta,R]\times[\tau,T])} \le c_{12} \qquad \text{for all } \varepsilon \in (0,1). \tag{3.82}$$

Indeed, this again results from a known embedding property stated in [2] by fixing some appropriately large $k \in \mathbb{N}$ in Step 2.

Step 4. We finally derive the conclusion of the lemma.

To achieve this, we fix $\delta \in (0,R), T>0$ and $\tau \in (0,T)$ and then obtain from Step 3 combined with the outcome of Lemma 3.13 and Lemma 3.14 that there exists $\theta_2 = \theta_2(\delta,T,\tau) \in (0,1)$ such that in

(3.9) all the functions a_{1r}, a_2 and a_3 satisfy estimates in $C^{\theta_2 \frac{\theta_2}{2}}([0,R]\times[0,T])$ which are independent of $\varepsilon \in (0,1)$. Classical parabolic Schauder estimates ([16]) therefore yield $c_{13} = c_{13}(\delta, T, \tau) > 0$ such that

$$\|\xi_{\delta\tau} u_\varepsilon\|_{C^{2+\theta_2, 1+\frac{\theta_2}{2}}([0,R]\times[0,T])} \le c_{13} \qquad \text{for all } \varepsilon \in (0,1),$$

which obviously entails (3.72). □

4 Global renormalized solutions. Proof of Theorem 1.1

In this section we shall construct a globally defined object (u,v) as a limit of solutions to (2.1), and investigate its properties with regard to the original problem (1.1), thereby proving Theorem 1.1.

4.1 Identification of a locally smooth limit

Let us first exploit the estimates gathered in the previous section to identify, via straightforward compactness and extraction arguments, a pair of limit functions u and v which are smooth outside the origin and solve (1.1) classically in this region.

Lemma 4.1 *There exist $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,1)$ fulfilling $\varepsilon_k \searrow 0$ as $k \to \infty$, as well as a couple (u,v) of nonnegative functions*

$$\begin{cases} u \in C^0((0,R]\times[0,\infty)) \cap C^{2,1}((0,R]\times(0,\infty)), \\ v \in C^0((0,R]\times[0,\infty)) \cap C^{2,1}((0,R]\times(0,\infty)), \end{cases} \tag{4.1}$$

such that

$$u_\varepsilon \to u \quad \text{and} \quad v_\varepsilon \to v \qquad \text{in } C^0_{loc}((0,R]\times[0,\infty)) \cap C^{2,1}_{loc}((0,R]\times(0,\infty)) \qquad \text{as } \varepsilon = \varepsilon_k \searrow 0, \tag{4.2}$$

that $v > 0$ in $(0,R]\times[0,\infty)$, and such that (u,v) solves the radial version of (1.1) in the classical sense in $(0,R]\times[0,\infty)$; that is, we have

$$\begin{cases} u_t = \frac{1}{r^{n-1}}(r^{n-1}u_r)_r - \frac{1}{r^{n-1}}(r^{n-1}\frac{u}{v}v_r)_r, & r \in (0,R], t>0, \\ v_t = \frac{1}{r^{n-1}}(r^{n-1}v_r)_r - uv, & r\in(0,R], t>0, \\ u_r(R,t) = v_r(R,t) = 0, \qquad t>0, \\ u(r,0) = u_0(r), \quad v(r,0) = v_0(r), \qquad r \in (0,R]. \end{cases} \tag{4.3}$$

PROOF. According to Lemma 3.15 and Lemma 3.14, an application of the Arzelà-Ascoli theorem provides $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,1)$ with $\varepsilon_k \searrow 0$ as $k\to\infty$, and nonnegative functions u and v on $(0,r]\times[0,\infty)$ which have the properties in (4.1) and are such that

$$u_\varepsilon \to u \quad \text{and} \quad v_\varepsilon \to v \qquad \text{in } C^{2,1}_{loc}((0,R]\times(0,\infty)) \tag{4.4}$$

as $\varepsilon = \varepsilon_k \searrow 0$. Thanks to Lemma 3.10, we know that actually also

$$v_\varepsilon \to v \qquad \text{in } C^0_{loc}((0,R]\times[0,\infty)) \tag{4.5}$$

as $\varepsilon = \varepsilon_k \searrow 0$, and the uniform lower bound on v_ε implied by Lemma 3.2 guarantees that v is positive in $(0, R] \times [0, \infty)$. Consequently, we may let $\varepsilon = \varepsilon_k \searrow 0$ in both boundary value problems in the corresponding radial version of (2.1) separately to conclude that in fact (u, v) is a classical solution of the boundary value problem in (4.3) in $(0, R] \times (0, \infty)$.
According to (4.5), it is clear that furthermore $v \in C^0((0, R] \times [0, \infty))$ with $v(r, 0) = v_0(r)$ for all $r \in (0, R]$, so that it remains to draw the corresponding conclusion also for u. To this end, we fix $\delta \in (0, R)$ and go back to Lemma 3.11 to obtain $c_1 = c_1(\delta) > 0$ such that

$$\left\|\zeta_\delta u_\varepsilon(\cdot, t) - e^{-tA}(\zeta_\delta u_0)\right\|_{L^\infty((0,R))} \le c_1 t^{\frac{1}{4}} \qquad \text{for all } t \in (0, 1) \tag{4.6}$$

whenever $\varepsilon \in (0, 1)$. By well-known continuity properties of solutions to the Neumann problem for the heat equation in $(0, R)$, however, we see that

$$e^{-tA}(\zeta_\delta u_0) \to \zeta_\delta u_0 \qquad \text{in } L^\infty((0, R)) \qquad \text{as } t \searrow 0, \tag{4.7}$$

because $\zeta_\delta u_0$ is continuous on $[0, R]$ due to (1.2). As (4.6) implies that

$$\begin{aligned}
\sup_{t\in(0,\tau)} \left\|\zeta_\delta u(\cdot, t) - \zeta_\delta u_0\right\|_{L^\infty((0,R))} &\le \sup_{t\in(0,\tau)} \lim_{\varepsilon=\varepsilon_k\searrow 0} \left\|\zeta_\delta u_\varepsilon(\cdot, t) - \zeta_\delta u_0\right\|_{L^\infty((0,R))} \\
&\le \sup_{t\in(0,\tau)} \lim_{\varepsilon=\varepsilon_k\searrow 0} \left\|\zeta_\delta u_\varepsilon(\cdot, t) - e^{-tA}(\zeta_\delta u_0)\right\|_{L^\infty((0,R))} \\
&\quad + \sup_{t\in(0,\tau)} \left\|e^{-tA}(\zeta_\delta u_0) - \zeta_\delta u_0\right\|_{L^\infty((0,R))} \\
&\le c_1 \tau^{\frac{1}{4}} + \sup_{t\in(0,\tau)} \left\|e^{-tA}(\zeta_\delta u_0) - \zeta_\delta u_0\right\|_{L^\infty((0,R))} \qquad \text{for all } \tau \in (0, 1),
\end{aligned}$$

from (4.7) we infer that

$$\zeta_\delta u(\cdot, t) \to \zeta_\delta u_0 \quad \text{in } L^\infty((0, R)) \qquad \text{as } t \searrow 0$$

and thus, since $\delta \in (0, R)$ was arbitrary, conclude that indeed also u belongs to $C^0((0, R] \times [0, \infty))$ and satisfies $u(r, 0) = u_0(r)$ for all $r \in (0, R]$. □

4.2 A spatially global solution concept. Renormalized solutions

In order to assign the above functions u and v the role of a solution to (1.1) in an appropriate spatially global sense, let us introduce the following generalized solution concept for (1.1) which follows the tradition of renormalized solutions of scalar evolution problems ([6]), and which actually does not require any restriction on radial symmetry or the space dimension.

Definition 4.2 *Suppose that $n \ge 1$, that $\Omega \subset \mathbb{R}^n$ is a bounded domain and that $u_0 \in L^1(\Omega)$ and $v_0 \in L^1(\Omega)$ are nonnegative. Then a pair (u, v) of functions*

$$\begin{cases} u \in L^1_{loc}(\bar\Omega \times [0, \infty)), \\ v \in L^\infty_{loc}(\bar\Omega \times [0, \infty)), \end{cases} \tag{4.8}$$

satisfying $u \geq 0$ *and* $v > 0$ *a.e. in* $\Omega \times (0,\infty)$*, will be called a* global renormalized solution *of (1.1) if*

$$\begin{cases} \chi_{\{u<M\}}\nabla u \in L^2_{loc}(\bar\Omega \times [0,\infty)) \quad \textit{for all } M > 0 \qquad \textit{and} \\ \frac{\nabla v}{v} \in L^2_{loc}(\bar\Omega \times [0,\infty)), \end{cases} \tag{4.9}$$

if for all $\phi \in C^\infty([0,\infty))$ *with* $\phi' \in C_0^\infty([0,\infty))$ *we have*

$$\begin{aligned} -\int_0^\infty\int_\Omega \phi(u)\varphi_t - \int_\Omega \phi(u_0)\varphi(\cdot,0) \quad &= \quad -\int_0^\infty\int_\Omega \phi''(u)|\nabla u|^2\varphi - \int_0^\infty\int_\Omega \phi'(u)\nabla u\cdot\nabla\varphi \\ &\quad + \int_0^\infty\int_\Omega u\phi''(u)\Big(\nabla u\cdot\frac{\nabla v}{v}\Big)\varphi + \int_0^\infty\int_\Omega u\phi'(u)\frac{\nabla v}{v}\cdot\nabla\varphi \end{aligned} \tag{4.10}$$

for all $\varphi \in C_0^\infty(\bar\Omega\times[0,\infty))$*, and if moreover the identity*

$$\int_0^\infty\int_\Omega v\varphi_t + \int_\Omega v_0\varphi(\cdot,0) = \int_0^\infty\int_\Omega \nabla v\cdot\nabla\varphi + \int_0^\infty\int_\Omega uv\varphi \tag{4.11}$$

is valid for any $\varphi \in C_0^\infty(\bar\Omega\times[0,\infty))$*.*

Remark. It can readily be verified that under the integrability assumptions in (4.8) and (4.9), each of the summands appearing in (4.10) and (4.11) indeed is well-defined. Moreover, it follows from standard arguments that any pair (u,v) which is a global renormalized solution of (1.1) in the sense of Definition 4.2, and which in addition is suitably smooth in the sense that $(u,v) \in (C^0(\bar\Omega\times[0,\infty))\cap C^{2,1}(\bar\Omega\times(0,\infty)))^2$, in fact already is a solution of (1.1) in $\Omega\times(0,\infty)$ in the classical sense.

Now combining the local solution properties of (u,v) outside the origin, as stated in Lemma 4.1, with the global regularity properties implied by Lemma 2.1 and Lemma 2.2, it is possible to show that (u,v) actually is a solution of (1.1) in the above sense.

Lemma 4.3 *Let* $n \geq 2, R > 0$ *and* $\Omega := B_R(0) \subset \mathbb{R}^n$*, and suppose that* u_0 *and* v_0 *satisfy (1.2). Then the functions* u *and* v *obtained in Lemma 4.1 form a global renormalized solution of (1.1). Moreover,* u *and* v *have the additional regularity properties in (1.3) and (1.4).*

PROOF. With $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,1)$ taken from Lemma 4.1, combining (2.7) and (2.4) with (4.2) we see that $u \in L^\infty((0,\infty);L^1(\Omega))$ and $v \in L^\infty(\Omega\times(0,\infty))$, which clearly entails that both (4.8) and (1.3) are satisfied. From (4.2) it moreover follows that $w_\varepsilon \to w := -\ln\frac{v}{\|v_0\|_{L^\infty(\Omega)}}$ a.e. in $\Omega\times(0,\infty)$ as $\varepsilon = \varepsilon_k \searrow 0$, so that using (2.9) and Fatou's lemma we infer that

$$\int_0^T\int_\Omega \frac{|\nabla v|^2}{v^2} \leq \liminf_{\varepsilon=\varepsilon_k\searrow 0}\int_0^T\int_\Omega |\nabla w_\varepsilon|^2 \leq \int_\Omega w_0 + \Big(\int_\Omega u_0\Big)\cdot T \qquad \text{for all } T > 0, \tag{4.12}$$

and that hence $\frac{\nabla v}{v}$ belongs to $L^2_{loc}(\bar\Omega\times[0,\infty))$.
Finally, given $T > 0$ we may invoke (2.10) which, again by Fatou's lemma, guarantees that there exists $c_1 = c_1(T) > 0$ such that

$$\int_0^T\int_\Omega \frac{|\nabla u|^2}{(u+1)^2} \leq c_1(T).$$

In conjunction with (4.12) this establishes (1.4), and for arbitrary $M > 0$ we can moreover estimate

$$\begin{aligned}
\int_0^T \int_\Omega \Big|\chi_{\{u<M\}} \nabla u\Big|^2 &= \int_0^T \int_\Omega \chi_{\{u<M\}} (u+1)^2 \cdot \frac{|\nabla u|^2}{(u+1)^2} \\
&\le (M+1)^2 \int_0^T \int_\Omega \chi_{\{u<M\}} \frac{|\nabla u|^2}{(u+1)^2} \\
&\le (M+1)^2 c_1(T),
\end{aligned}$$

and hence conclude that also $\chi_{\{u<M\}}\nabla u \in L^2_{loc}(\bar\Omega \times [0,\infty))$ for any such M, thereby completing the verification of (4.9).

Next, in order to derive (4.10) and (4.11) we let $\varphi \in C_0^\infty(\bar\Omega \times [0,\infty))$ and $\phi \in C^\infty([0,\infty))$ with $\phi' \in C_0^\infty([0,\infty))$ be arbitrary, and for $\delta \in (0,R)$ we define $\xi_\delta(x) := \zeta_\delta(|x|)$, $x \in \bar\Omega$, and

$$\varphi_\delta(x) := \xi_\delta(x)\varphi(x,t), \qquad x \in \bar\Omega,\ t \ge 0, \tag{4.13}$$

with ζ_δ as introduced in (3.3). Then since $\varphi_\delta \equiv 0$ in $B_{\frac{\delta}{2}}(0) \times [0,\infty)$, and since (u,v) is a classical solution of (1.1) in $(\bar\Omega \setminus \{0\}) \times [0,\infty)$ according to Lemma 4.1, it follows from testing the respective sub-problems of (1.1) against φ_δ that

$$\int_0^\infty \int_\Omega v\varphi_{\delta t} + \int_\Omega v_0 \varphi_\delta(\cdot,0) = \int_0^\infty \int_\Omega \nabla v \cdot \nabla\varphi_\delta + \int_0^\infty \int_\Omega uv\varphi_\delta \tag{4.14}$$

and

$$\begin{aligned}
-\int_0^\infty \int_\Omega \phi(u)\varphi_{\delta t} - \int_\Omega \phi(u_0)\varphi_\delta(\cdot,0) &= -\int_0^\infty \int_\Omega \phi''(u)|\nabla u|^2 \varphi_\delta - \int_0^\infty \int_\Omega \phi'(u)\nabla u \cdot \nabla\varphi_\delta \\
&\quad + \int_0^\infty \int_\Omega u\phi''(u)\Big(\nabla u \cdot \frac{\nabla v}{v}\Big)\varphi_\delta \\
&\quad + \int_0^\infty \int_\Omega u\phi'(u)\frac{\nabla v}{v} \cdot \nabla\varphi_\delta
\end{aligned} \tag{4.15}$$

for all $\delta \in (0,R)$. Here, fixing $T > 0$ large such that $\varphi \equiv 0$ in $\Omega \times (T,\infty)$ we note that thanks to (4.8) and (4.9) we in particular know that

$$\nabla v \in L^2(\Omega \times (0,T)), \tag{4.16}$$

so that in

$$\int_0^\infty \int_\Omega \nabla v \cdot \nabla\varphi_\delta = \int_0^\infty \int_\Omega \xi_\delta \nabla v \cdot \nabla\varphi + \int_0^\infty \int_\Omega \varphi\nabla v \cdot \nabla\xi_\delta$$

we have

$$\int_0^\infty \int_\Omega \xi_\delta \nabla v \cdot \nabla\varphi \to \int_0^\infty \int_\Omega \nabla v \cdot \nabla\varphi \qquad \text{as } \delta \searrow 0 \tag{4.17}$$

according to the dominated convergence theorem. As $|\nabla\xi_\delta| \le \frac{4}{\delta}$ in Ω and $\operatorname{supp}\nabla\xi_\delta \subset \bar{B}_\delta(0)$ by (3.1) and (3.2), we can moreover use the Cauchy-Schwarz inequality to estimate

$$\begin{aligned} \left| \int_0^\infty \int_\Omega \varphi\nabla v\cdot\nabla\xi_\delta \right| &\le \|\varphi\|_{L^\infty(\Omega\times(0,\infty))} \left\{ \int_0^T \int_{B_\delta(0)} |\nabla v|^2 \right\}^{\frac12} \cdot \left\{ \int_0^T \int_{B_\delta(0)} |\nabla\xi_\delta|^2 \right\}^{\frac12} \\ &\le \|\varphi\|_{L^\infty(\Omega\times(0,\infty))} \left\{ \int_0^T \int_{B_\delta(0)} |\nabla v|^2 \right\}^{\frac12} \cdot \frac{4}{\delta} T^{\frac12} |B_\delta(0)|^{\frac12} \end{aligned}$$

for all $\delta \in (0,R)$, so that since $\sup_{\delta\in(0,R)} \frac{|B_\delta(0)|^{\frac12}}{\delta}$ is finite thanks to our overall assumption that $n \ge 2$, we obtain

$$\int_0^\infty \int_\Omega \varphi\nabla v\cdot\nabla\xi_\delta \to 0 \qquad \text{as } \delta \searrow 0, \tag{4.18}$$

because $\int_0^T \int_{B_\delta(0)} |\nabla v|^2 \to 0$ as $\delta \searrow 0$ by (4.16) and e.g. again the dominated convergence theorem. Since (4.8) entails the inclusions $v \in L^1(\Omega\times(0,T))$ and $uv \in L^1(\Omega\times(0,T))$, and since $v_0 \in L^1(\Omega)$, on three further applications of the dominated convergence theorem we moreover infer that

$$\int_0^\infty \int_\Omega v\varphi_{\delta t} \to \int_0^\infty \int_\Omega v\varphi_t$$

and

$$\int_0^T \int_\Omega uv\varphi_\delta \to \int_0^\infty \int_\Omega uv\varphi$$

as well as

$$\int_\Omega v_0\varphi_\delta(\cdot,0) \to \int_\Omega v_0\varphi(\cdot,0)$$

as $\delta \searrow 0$. Combining this with (4.17) and (4.18) we thus conclude from (4.14) that indeed (4.11) is valid.

Similarly, for the derivation of (4.10) we first observe that by (4.9) and the boundedness of $\operatorname{supp}\phi'$ we know that

$$\phi'(u)\nabla u \in L^2(\Omega\times(0,T)) \qquad \text{and} \qquad u\phi'(u)\frac{\nabla v}{v} \in L^2(\Omega\times(0,T)), \tag{4.19}$$

and that

$$\phi(u) \in L^1(\Omega\times(0,T)), \qquad \phi''(u)|\nabla u|^2 \in L^1(\Omega\times(0,T)) \qquad \text{and} \qquad u\phi''(u)\nabla u\cdot\frac{\nabla v}{v} \in L^1(\Omega\times(0,T)). \tag{4.20}$$

Hence, arguing as in (4.17) and (4.18) we obtain from (4.19) that

$$\begin{aligned} -\int_0^\infty \int_\Omega \phi'(u)\nabla u\cdot\nabla\varphi_\delta &= -\int_0^\infty \int_\Omega \xi_\delta\phi'(u)\nabla u\cdot\nabla\varphi - \int_0^\infty \int_\Omega \varphi\phi'(u)\nabla u\cdot\nabla\xi_\delta \\ &\to -\int_0^\infty \int_\Omega \phi'(u)\nabla u\cdot\nabla\varphi \end{aligned}$$

and

$$\begin{aligned}\int_0^\infty\int_\Omega u\phi'(u)\frac{\nabla v}{v}\cdot\nabla\varphi_\delta &= \int_0^\infty\int_\Omega \xi_\delta u\phi'(u)\frac{\nabla v}{v}\cdot\nabla\varphi + \int_0^\infty\int_\Omega \varphi u\phi'(u)\frac{\nabla v}{v}\cdot\nabla\xi_\delta \\ &\to \int_0^\infty\int_\Omega u\phi(u)\frac{\nabla v}{v}\cdot\nabla\varphi\end{aligned}$$

as $\delta\searrow 0$, whereas (4.20) in conjunction with the dominated convergence theorem guarantees that

$$-\int_0^\infty\int_\Omega \phi(u)\varphi_{\delta t}\to -\int_0^\infty\int_\Omega \phi(u)\varphi_t$$

and

$$-\int_\Omega \phi(u_0)\varphi_\delta(\cdot,0)\to -\int_\Omega \phi(u_0)\varphi(\cdot,0)$$

as well as

$$-\int_0^\infty\int_\Omega \phi''(u)|\nabla u|^2\varphi_\delta\to -\int_0^\infty\int_\Omega \phi''(u)|\nabla u|^2\varphi$$

and

$$\int_0^\infty\int_\Omega u\phi''(u)\Big(\nabla u\cdot\frac{\nabla v}{v}\Big)\varphi_\delta\to \int_0^\infty\int_\Omega u\phi''(u)\Big(\nabla u\cdot\frac{\nabla v}{v}\Big)\varphi$$

as $\delta\searrow 0$. In consequence, (4.15) thus implies (4.10). □

Finally, our main result thereby becomes an immediate consequnce.

PROOF of Theorem 1.1. All claimed statements are covered by the results from Lemma 4.1 and Lemma 4.3. □

4.3 Concluding remarks

This study provides a first step into the analysis of (1.1) in presence of large initial data in higher-dimensional settings. It has been seen that despite the enhanced destabilizing potential therein, originating from nutrient-oriented chemotactic attraction with sensitivity singular at vanishing signal densities, a basic solution theory can be established at least in spatially radial frameworks.

The presented approach, yielding solutions smooth outside the origin but resorting to a strongly generalized solution concept at a spatially global level, seems to rely on both diffusion mechanisms in (1.1) through appropriate regularizing actions of the respective heat semigroups; in particular, an interesting question left open for future research is how far similar results can be derived in the case when the second equation in (1.1) is replaced with the diffusion-free ODE $v_t=-uv$.

Apart from that, the challenging topic of describing qualitative aspects, and especially the large time behavior such as partially addressed in [36] and [37] for two-dimensional versions of (1.1), of the obtained solutions, has not yet been addressed in the course of the present analysis which seems to be restricted to providing estimates essentially local in time only.

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