A critical blow-up exponent in a chemotaxis system with nonlinear signal production

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Abstract

This paper is concerned with radially symmetric solutions of the Keller-Segel system with nonlinear signal production, as given by

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v), \\ 0 = \Delta v - \mu(t) + f(u), \qquad \mu(t) := \frac{1}{|\Omega|}\int_\Omega f(u(\cdot,t)), \end{cases}$$

in the ball $\Omega = B_R(0) \subset \mathbb{R}^n$ for $n \ge 1$ and $R > 0$, where f is a suitably regular function generalizing the prototype determined by the choice $f(u) = u^\kappa$, $u \ge 0$, with $\kappa > 0$.

The main results assert that if in this setting the number κ satisfies

$$\kappa > \frac{2}{n}, \qquad (\star)$$

then for any prescribed mass level $m > 0$, there exist initial data u_0 with $\int_\Omega u_0 = m$, for which the solution of the corresponding Neumann initial-boundary value problem blows up in finite time.

That the condition in $(\star)$ is essentially optimal is indicated by a complementary result according to which in the case $\kappa < \frac{2}{n}$, for widely arbitrary initial data a global bounded classical solution can always be found.

Key words: chemotaxis; nonlinear signal production; finite-time blow-up
MSC 2010: 35B44 (primary); 35B33, 35K65, 92C17 (secondary)

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1 Introduction

In the literature on Keller-Segel-type chemotaxis systems, understanding the destabilizing potential of the respective cross-diffusion terms therein plays a dominant role. Indeed, since their introduction in the early 1970s such systems have been stimulating mathematical analysis at various levels, but a main focus has been on the question how far chemotactic cross-diffusion may enforce the spontaneous formation of singular structures. Here, after concentrating on the apparently simplest versions of the fully parabolic Keller-Segel model ([27])

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v), \\ v_t = \Delta v - v + u, \end{cases} \tag{1.1}$$

and parabolic-elliptic simplifications thereof, and clarifying that such explosion phenomena may occur therein if and only if the spatial setup is two- or higher-dimensional ([26], [20], [36], [22], [23], [50]), in the past few years the analysis has proceeded to studying corresponding questions for more elaborate chemotaxis systems accounting for certain types of mechanisms which are relevant in various application contexts but which, as suggested by refined developments in modeling, may not appropriately be captured by minimal Keller-Segel systems ([21]).

In fact, a steadily increasing literature has been providing considerable knowledge in this direction, but it can broadly be observed that up to very few exemptions, existing results on such more complex chemotaxis systems focus on identifying conditions as sufficient to rule out blow-up phenomena (corresponding summaries far from complete can be found in [21] and [2], for instance). Thus typically leading to statements on global existence and boundedness of solutions under appropriate assumptions on the particular system ingredients, findings of this form are usually not accompanied by associated complementary results indicating optimality of the respectively obtained conditions. This may be regarded as reflecting the considerable mathematical difficulties inherent to blow-up detection in any partially dissipative evolution system in general, and especially in chemotaxis systems. Indeed, already in simple minimal Keller-Segel systems the discovery of unboundedness phenomena has given rise to substantial challenges which could so far be overcome only by making subtle use of particular structural properties which are rather fragile and quite unstable with regard to modifications in the model (cf. the discussions in the surveys [24] and [2], for instance).

Only in few exceptional cases, critical relationships between cross-diffusion and dissipative mechanisms, thus providing an essentially clear picture about the strength of the respective taxis process, could be identified up to now. An important class of such examples is given by the quasilinear Keller-Segel system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (uS(u)\nabla v), \\ v_t = \Delta v - v + u, \end{cases} \tag{1.2}$$

which allows for a rather comprehensive knowledge with regard to the occurrence of blow-up: As already anticipated by some early partial results (see [41], [28] and [25], and also [10]) namely, if the positive parameter functions D and S in (1.2) satisfy, besides some further technical conditions such as at most algebraic decay of $D(u)$ as $u \to \infty$, the crucial inequality $\frac{uS(u)}{D(u)} \le Cu^{\frac{2}{n}-\varepsilon}$ for some $C > 0$ and $\varepsilon > 0$ and all $u > 1$, then for all reasonably regular initial data the corresponding Neumann problem in a smoothly bounded domain $\Omega \subset \mathbb{R}^n$ possesses a global bounded classical solution ([42],

[46]; cf. also [15] for some recent findings concerned with exponentially decaying diffusion rates); on the other hand, if $\frac{uS(u)}{D(u)} \ge Cu^{\frac{2}{n}+\varepsilon}$ for all $u > 1$ with some $C > 0$ and $\varepsilon > 0$, then under mild additional structural assumptions some unbounded solutions can be constructed ([47]), which in some cases are even known to blow up in finite time ([11], [13], [12]). In corresponding parabolic-elliptic variants of (1.2) in which the second equation is replaced by either $0 = \Delta v - v + u$ or $0 = \Delta v - \mu + u$ with $\mu = \frac{1}{|\Omega|}\int_\Omega u_0$, partially even more detailed results are available ([9], [8], [16]).

In comparison to this, the knowledge on other biologically relevant variants of (1.1) is much less complete: As for the important class of Keller-Segel systems with logarithmic sensitivity, for instance, as obtained from (1.1) upon replacing the first equation therein with $u_t = \Delta u - \chi\nabla \cdot (\frac{u}{v}\nabla v)$, the size of the parameter χ is known to play a crucial role with regard to the occurrence of blow-up: On the one hand, exclusively global smooth solutions exist if $n \ge 2$ and $\chi < \chi_\star(n)$ with some $\chi_\star(n) \in (0, \infty]$ depending on whether the setting is fully parabolic or parabolic-elliptic, and on a possible restriction to radially symmetric solutions ([5], [19], [18], [17], [30], [48]); e.g. in the simple parabolic-elliptic and radial case, it is known that $\chi_\star(n) \ge \frac{2}{n-2}$ ([37]). On the other hand, in the latter simplified framework it has been found that if $n \ge 3$ and $\chi > \frac{2n}{n-2}$, then some initial data lead to finite-time blow-up of solutions ([37]); however, even in this reduced setup the optimal value of $\chi_\star(n)$ ensuring the above conclusion seems yet unknown whenever $n \ge 3$.

Similar observations concern modifications of (1.1) accounting for logistic-type cell proliferation and death, as modeled e.g. by additional summands of the form $\lambda u - \mu u^\kappa$ with $\lambda \in \mathbb{R}, \mu > 0$ and $\kappa > 1$ in respective the first equation: For such systems, in the quadratic degradation case $\kappa = 2$ solutions exist globally and remain bounded when either $n \le 2$ and $\mu > 0$ is arbitrary ([39]), or $n \ge 3$ and $\mu > \mu_\star(n, \lambda)$ with some $\mu_\star(n, \lambda) \ge 0$ ([44]). Complementing results on the occurrence of blow-up solutions in three- and higher-dimensional domains, however, are available only in certain cases of subquadratic degradation, namely for $\kappa < \frac{3}{2} + \frac{1}{2n-2}$ when $n \ge 5$ ([49]), and alternatively for $\kappa < \frac{7}{6}$ when $n \in \{3, 4\}$ ([52]), thus yet leaving a considerable gap even with respect to the exponent κ.

In fact, it seems that beyond (1.2) only very few chemotaxis systems have been understood to a comparably comprehensive extent with regard to correspondingly critical relationships between cross-diffusion and further ingredients. Exceptions seem restricted to systems involving external chemoattractant sources ([6], [45]), to Keller-Segel models including some gradient-dependent flux limitations ([1], [3], [4]), and to extended three-component systems reflecting certain indirect signal production mechanisms ([43]).

Main results. In the present work, we shall consider a class of Keller-Segel-type systems in which, as compared to (1.1), the process of signal production through cells need no longer depend on the population density in a linear manner, but may e.g. account for saturation effects at large densities, as discussed in [21, Section 2.6], for instance (cf. also [33] and [34]). Resorting to a simplified parabolic-elliptic and spatially radial framework, we shall more precisely be concerned with the initial-boundary value problem

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v), & & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \mu(t) + f(u), & \mu(t) := \frac{1}{|\Omega|}\int_\Omega f(u(\cdot, t)), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & & x \in \partial\Omega,\ t > 0, \\ u(x, 0) = u_0(x), & & x \in \Omega, \end{cases} \tag{1.3}$$

in the ball $\Omega := B_R(0) \subset \mathbb{R}^n$ with $n \ge 1$ and $R > 0$, where

$$f \in \bigcup_{\vartheta\in(0,1)} C^{\vartheta}_{loc}([0,\infty)) \cap C^1((0,\infty)) \quad \text{is nonnegative and nondecreasing,} \tag{1.4}$$

and where

$$u_0 \in \bigcup_{\vartheta\in(0,1)} C^{\vartheta}(\overline{\Omega}) \quad \text{is nonnegative, radially symmetric and nonincreasing with respect to } |x|. \tag{1.5}$$

Having in mind the prototypical case determined by the choice

$$f(u) = u^{\kappa}, \qquad u \ge 0, \tag{1.6}$$

with some $\kappa > 0$, our main interest will be in the question for which values of κ herein the self-enhancement of chemotactic attraction is strong enough so as to enforce finite-time blow-up of some solutions. From the analysis of (1.1) and parabolic-elliptic analogues it is clear that when $\kappa = 1$, exploding solutions with arbitrarily small total mass can be expected only when $n \ge 3$ ([40], [38]), while if $n = 2$ then critical mass phenomena accordingly known for such minimal Keller-Segel systems, becoming manifest in findings on small-mass global bounded solutions ([38]) and on blow-up for certain large-mass initial data ([36], [5], [20], [26]) indicate that $\kappa = 1$ might be critical in that planar case. This is further underlined by a recent result on global existence and boundedness in a fully parabolic counterpart of (1.3) under the assumption that f satisfies $f(u) \le Ku^{\kappa}$ for all $u \ge 1$ with some $\kappa < \frac{2}{n}$ ([32]).

The goal of this work consists in confirming that the exponent $\kappa_c = \frac{2}{n}$ appearing herein indeed is critical in this recpect. In order to precisely formulate our corresponding main results, let us first note that suitable adaptation of well-known arguments, based e.g. on fixed point properties in appropriate functional frameworks (see [35], [16] and [14], for instance), readily yields the following basic result on local existence and extensibility of radial classical solutions to (1.3):

Proposition 1.1 *Let $\Omega = B_R(0) \subset \mathbb{R}^n$ with some $n \ge 1$ and $R > 0$, and assume that f and u_0 satisfy (1.4) and (1.5). Then there exist $T_{max} \in (0,\infty]$ and a classical solution (u,v) of (1.3) in $\Omega\times(0,T_{max})$, for each $T \in (0,T_{max})$ uniquely determined by the inclusions*

$$\begin{cases} u \in C^0(\overline{\Omega}\times[0,T]) \cap C^{2,1}(\overline{\Omega}\times(0,T)), \\ v \in \bigcap_{q>n} L^{\infty}((0,T);W^{1,q}(\Omega)) \cap C^{2,0}(\overline{\Omega}\times(0,T)), \end{cases}$$

and the identity

$$\int_\Omega v(\cdot,t) = 0 \qquad \text{for all } t \in (0,T),$$

such that $u \ge 0$ in $\Omega\times(0,T_{max})$ that $u(\cdot,t)$ and $v(\cdot,t)$ are radially symmetric with respect to $x = 0$, and that

$$\text{if } T_{max} < \infty, \quad \text{then} \quad \limsup_{t\nearrow T_{max}} \|u(\cdot,t)\|_{L^{\infty}(\Omega)} = \infty. \tag{1.7}$$

Moreover,

$$\int_\Omega u(x,t)dx = \int_\Omega u_0 dx \qquad \text{for all } t \in (0,T_{max}). \tag{1.8}$$

Now our main result asserts that if f satisfies a condition conveniently generalizing (1.6) with some $\kappa > \frac{2}{n}$, then at each arbitrarily small mass level some initial data can be found which lead to finite-time blow-up of the corresponding solution:

Theorem 1.2 *Let $n \geq 1, R > 0$ and $\Omega = B_R(0) \subset \mathbb{R}^n$, and suppose that f satisfies (1.4) as well as*

$$f(u) \geq ku^\kappa \qquad \text{for all } u \geq 1 \tag{1.9}$$

with some $k > 0$ and

$$\kappa > \frac{2}{n}. \tag{1.10}$$

Then for all $m > 0$ there exist $\varepsilon = \varepsilon(k, \kappa, m, R) \in (0, m)$ and $r_\star = r_\star(k, \kappa, m, R) \in (0, R)$ with the property that whenever u_0 satisfies (1.5) and is such that

$$\int_\Omega u_0 dx = m \qquad \text{but} \qquad \int_{B_{r_\star}(0)} u_0 dx \geq m - \varepsilon, \tag{1.11}$$

the corresponding solution (u, v) of (1.3) from Proposition 1.1 blows up in finite time in the sense that in Proposition 1.1 we have $T_{max} < \infty$.

We will finally include a short proof of the following statement indicating that the above assumptions on f are indeed essentially optimal, and that hence the exponent $\kappa_c = \frac{2}{n}$ may be viewed critical with respect to the possibility of blow-up phenomena in (1.3):

Proposition 1.3 *Let $n \geq 1, R > 0$ and $\Omega = B_R(0) \subset \mathbb{R}^n$, and suppose that f satisfies (1.4) and is such that*

$$f(u) \leq Ku^\kappa \qquad \text{for all } u \geq 1 \tag{1.12}$$

with some $K > 0$ and

$$\kappa < \frac{2}{n}. \tag{1.13}$$

Then for each u_0 fulfilling (1.5), the solution (u, v) of (1.3) from Proposition 1.1 is global and bounded in the sense that in Proposition 1.1 we have $T_{max} = \infty$ and

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \qquad \text{for all } t > 0 \tag{1.14}$$

with some $C > 0$.

As the proof of Proposition 1.3 in Section 4 will show, its result can easily be carried over to the case when instead of satisfying (1.5), u_0 merely is assumed to be nonnegative and suitably regular, but not necessarily radially symmetric nor monotonic in any direction.

Main ideas. Following precedents concerned with the corresponding parabolic-ellitpic variant of (1.1) ([26], [5]), the main body of our blow-up analysis will be launched by the substitution

$$w(s, t) := \int_0^{s^{\frac{1}{n}}} r^{n-1} u(r, t) dr, \qquad s \in [0, R^n], \ t \in [0, T_{max}), \tag{1.15}$$

which can readily be seen to transform (1.3) into the Dirichlet problem

$$\begin{cases} w_t(s,t) = n^2 s^{2-\frac{2}{n}} w_{ss}(s,t) + w_s(s,t) \int_0^s f(nw_s(\sigma,t))d\sigma - \mu(t) s w_s(s,t), & s \in (0,R^n),\ t \in (0,T_{max}), \\ w(0,t) = 0, \quad w(R^n,t) = \frac{m}{\omega_n}, & t \in (0,T_{max}), \end{cases} \tag{1.16}$$

along with an evident initial condition, and with $m = \int_\Omega u_0 dx$ as well as

$$\mu(t) = \frac{1}{|\Omega|} \int_\Omega f(u(x,t))dx \equiv \frac{1}{R^n} \int_0^{R^n} f(nw_s(s,t))ds, \qquad t \in (0,T_{max}) \tag{1.17}$$

(cf. also (2.1)). Here and below, we abbreviate $\omega_n := n|B_1(0)|$, and whenever this appears convenient we switch between the notation $\varphi(x)$ and $\varphi(r)$ for functions φ defined on Ω but actually depending on the radial variable $r = |x|$ only.

Unlike in the situation when f is linear, in presence of general f the driving nonlinearity in (1.16) is genuinely nonlocal in space. Moreover, in the case of sublinearly growing f which apparently is of particular interest, trivially estimating the crucial integral $\int_0^s f(nw_s(\sigma,t))d\sigma$ therein e.g. via the Hölder inequality, if possible at all, only yields upper bounds therefor, which seem essentially useless. A major challenge will thus be linked to the question how the positive part of the cross-diffusive contribution to (1.16) can be controlled *from below* as efficiently as possible, and as a first step toward achieving this we shall make use of the monotonicity assumption in (1.5) in observing by means of a maximum principle argument that w remains spatially concave throughout evolution (Lemma 2.2). In deriving this solution property crucial for our subsequent analysis, through a favorable structure of the parabolic equation (2.8) satisfied by the derivative u_r we will make essential use of the fact that in the second equation in (1.3), the quantity v does not additionally appear in further zero-order expressions such as in other parabolic-elliptic simplifications of Keller-Segel systems in which the respective second equations are of the form $0 = \Delta v - v + f(u)$.

Besides providing a favorable upper bound on the numbers μ in (1.17) for widely arbitrary f (Lemma 3.2), this will entail that in the main part of our analysis, concerned with the time evolution of the localized moment-like functional given by

$$\phi(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s) w(s,t) ds, \qquad t \in [0,T_{max}),$$

with suitably chosen $\gamma \in (-\infty,1)$ and $s_0 \in (0,R^n)$, the respective nonlocal contributions can conveniently be estimated from below against

$$\psi(t) := \frac{1}{2} \int_0^{s_0} s^{1-\gamma}(s_0 - s) w_s(s,t) f(nw_s(s,t)) ds, \qquad t \in (0,T_{max})$$

(Lemma 3.4). A simple but crucial observation will thereafter reveal a pointwise upper inequality for $w(\cdot,t)$ in terms of this expression $\psi(t)$ for each time t at which $\psi(t)$ lies above some suitable lower bound (Corollary 3.6). At any such time, this will enable us to appropriately control the respective explosion-counteracting integrals co-determining the evolution of ϕ (Lemmata 3.8 and 3.9), as well as to relate ϕ itself to a superlinear power of ψ (Lemma 3.7). As seen in Lemma 3.10, collecting these properties will entail a superlinearly forced ODI for ϕ within a suitably restricted set of times

$t \in (0, T_{max})$, which in the course of the proof of Theorem 1.2 will actually be seen to necessarily coincide with all of $(0, \infty)$ whenever (u, v) is global, and to thus in fact contradict the latter in the case when the initial data satisfy (1.11) with sufficiently small ε.

2 Concavity of w

Throughout the sequel, assuming (1.4) and (1.5) to hold we let (u, v), T_{max} and w be as in Proposition 1.1 and (1.15). Computing

$$w_s(s,t) = \frac{1}{n} u(s^{\frac{1}{n}}, t) \quad \text{and} \quad w_{ss}(s,t) = \frac{1}{n^2} s^{\frac{1}{n}-1} u_r(s^{\frac{1}{n}}, t), \qquad s \in (0, R^n),\ t \in (0, T_{max}), \tag{2.1}$$

we therefore immediately obtain that w satisfies (1.16), and that

$$w_s(s,t) \ge 0 \qquad \text{for all } s \in (0, R^n) \text{ and } t \in (0, T_{max}).$$

The purpose of this section is to make sure that moreover, as a consequence of (1.5), $w(\cdot, t)$ actually remains concave throughout evolution. As a preparation for this, let us note the following simple but useful observation.

Lemma 2.1 *Suppose that (1.4) and (1.5) hold. Then*

$$v_r(r,t) = \frac{1}{n} \mu(t) r - r^{1-n} \int_0^r \rho^{n-1} f(u(\rho, t)) d\rho \qquad \text{for all } r \in (0, R) \text{ and } t \in (0, T_{max}). \tag{2.2}$$

In particular,

$$v_r(r,t) \le \frac{1}{n} \mu(t) r \qquad \text{for all } r \in (0, R) \text{ and } t \in (0, T_{max}). \tag{2.3}$$

Proof. Rewritten in radial variables, the second equation in (1.3) becomes

$$(r^{n-1} v_r)_r = r^{n-1} \mu(t) - r^{n-1} f(u) \qquad \text{for all } r \in (0, R) \text{ and } t \in (0, T_{max}).$$

On integration, this yields

$$r^{n-1} v_r(r,t) = \mu(t) \cdot \int_0^r \rho^{n-1} d\rho - \int_0^r \rho^{n-1} f(u(\rho, t)) d\rho \qquad \text{for all } r \in (0, R) \text{ and } t \in (0, T_{max})$$

and thereby implies both (2.2) and, due to the nonnegativity of f, also (2.3). $\square$

Making use of the one-sided information in (2.3) now enables us to, we can now combine a reasoning based on maximum-principle-type arguments with an additional approximation procedure to assert that under the general assumptions in (1.4) and (1.5), u indeed inherits downward radial monotonicity from its initial data.

Lemma 2.2 *Assume (1.4) and (1.5). Then*

$$u_r(r,t) \le 0 \qquad \text{for all } r \in (0, R) \text{ and each } t \in (0, T_{max}) \tag{2.4}$$

and, hence for w as in (1.15) we have

$$w_{ss}(s,t) \le 0 \qquad \text{for all } s \in (0, R^n) \text{ and any } t \in (0, T_{max}). \tag{2.5}$$

Proof. In view of (2.1) we only need to establish (2.4), which will be achieved through two steps.

Step 1. We first verify (2.4) under the assumption that beyond (1.4) and (1.5), f and u_0 in addition have the properties that $f \in C^2([0,\infty))$ and that

$$u_0 \in C^2(\overline{\Omega}) \qquad \text{with} \qquad \frac{\partial u_0}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{2.6}$$

Then, namely, well-known theory on higher regularity in scalar parabolic equations ([29], [31]) warrants that not only u but also u_r belongs to $C^0([0,R] \times [0,T_{max})) \cap C^{2,1}((0,R) \times (0,T_{max}))$, and using that

$$v_{rr} + \frac{n-1}{r} v_r = \mu(t) - f(u) \qquad \text{for all } r \in (0,R) \text{ and } t \in (0,T_{max}) \tag{2.7}$$

according to (1.3), expanding the first equation in (1.3) we obtain that

$$\begin{aligned} u_t &= u_{rr} + \frac{n-1}{r} u_r - u_r v_r - u\Big(v_{rr} + \frac{n-1}{r} v_r\Big) \\ &= u_{rr} + \frac{n-1}{r} u_r - u_r v_r - \mu(t) u + u f(u) \qquad \text{for all } r \in (0,R) \text{ and } t \in (0,T_{max}) \end{aligned}$$

and hence, on differentiation,

$$u_{rt} = u_{rrr} + a(r,t) u_{rr} + b(r,t) u_r \qquad \text{for all } r \in (0,R) \text{ and } t \in (0,T_{max}) \tag{2.8}$$

with

$$a(r,t) := \frac{n-1}{r} - v_r(r,t)$$

and

$$b(r,t) := -\frac{n-1}{r^2} - v_{rr}(r,t) - \mu(t) + f(u(r,t)) + u(r,t) f'(u(r,t)) \tag{2.9}$$

for $r \in (0,R)$ and $t \in (0,T_{max})$.

Now for fixed $T \in (0,T_{max})$, using the continuity of u in $[0,R] \times [0,T]$ we can pick $\lambda > 0$ large such that

$$\lambda \geq 4\|f(u)\|_{L^\infty((0,R)\times(0,T))} + 2\|u f'(u)\|_{L^\infty((0,R)\times(0,T))}, \tag{2.10}$$

and for $\varepsilon > 0$ we thereupon let

$$z(r,t) := u_r(r,t) - \varepsilon e^{\lambda t}, \qquad r \in [0,R],\ t \in [0,T].$$

Then according to the above, z lies in $C^0([0,R] \times [0,T])$ and satisfies

$$z(r,0) = u_{0r}(r) - \varepsilon < 0 \qquad \text{for all } r \in [0,R]$$

due to (1.5), and also

$$z(0,t) = z(R,t) = -\varepsilon e^{\lambda t} < 0 \qquad \text{for all } t \in [0,T]$$

by (1.3), because clearly $u_r(0,t) = 0$ for all $t \in (0,T_{max})$. To show that actually

$$z(r,t) < 0 \qquad \text{for all } r \in [0,R] \text{ and any } t \in [0,T], \tag{2.11}$$

assuming this to be false, by following the standard initial step of maximum principle-based resonings we could therefore find some $r_0 \in (0,R)$ and $t_0 \in (0,T]$ such that

$$z(r_0,t_0)=0, \qquad z_r(r_0,t_0)=0, \qquad z_{rr}(r_0,t_0)\le 0 \qquad \text{and} \qquad z_t(r_0,t_0)\ge 0. \tag{2.12}$$

As from (2.8) we know that

$$z_t = z_{rr} + a(r,t)z_r + b(r,t)\cdot(z+\varepsilon e^{\lambda t}) - \lambda\varepsilon e^{\lambda t} \qquad \text{for all } r\in(0,R) \text{ and } t\in(0,T),$$

it follows from (2.12) that at (r_0,t_0) we have

$$0 \le z_t(r_0,t_0) \le \Big\{ b(r_0,t_0) - \lambda \Big\} \cdot \varepsilon e^{\lambda t_0}. \tag{2.13}$$

Here we go back to the definition (2.9) of b, which together with (2.7) shows that for all $r \in (0,R)$ and $t \in (0,T_{max})$,

$$\begin{aligned} b(r,t) &= -\frac{n-1}{r^2} - \Big\{ -\frac{n-1}{r}v_r(r,t) + \mu(t) - f(u(r,t)) \Big\} - \mu(t) + f(u(r,t)) + u(r,t)f'(u(r,t)) \\ &= -\frac{n-1}{r^2} + \frac{n-1}{r}v_r(r,t) - 2\mu(t) + 2f(u(r,t)) + u(r,t)f'(u(r,t)), \end{aligned}$$

so that since

$$\frac{n-1}{r}v_r(r,t) \le \frac{n-1}{n}\mu(t) \qquad \text{for all } r\in(0,R) \text{ and } t\in(0,T_{max})$$

by Lemma 2.1, b satisfies the one-sided estimate

$$\begin{aligned} b(r,t) &\le -\frac{n-1}{r^2} - \frac{n+1}{n}\mu(t) + 2f(u(r,t)) + u(r,t)f'(u(r,t)) \\ &\le 2f(u(r,t)) + u(r,t)f'(u(r,t)) \\ &\le \frac{\lambda}{2} \qquad \text{for all } r\in(0,R) \text{ and } t\in(0,T] \end{aligned}$$

according to (2.10). Therefore, (2.13) leads to the absurd conclusion that

$$0 \le -\frac{\lambda}{2}\cdot\varepsilon e^{\lambda t_0}$$

and hence implies that indeed (2.11) holds, from which in turn (2.4) results on taking $\varepsilon \searrow 0$ and then $T \nearrow T_{max}$.

Step 2. We proceed to derive (2.4) for arbitrary f and u_0 merely fulfilling (1.4) and (1.5).

To this end, given $T \in (0,T_{max})$ we let $c_1 > 0$ be large enough such that $u \le c_1$ in $\Omega \times (0,T)$. This boundedness feature enables us to fix $(f_j)_{j\in\mathbb{N}} \subset C^2([0,\infty))$ and $(u_{0j})_{j\in\mathbb{N}} \subset C^2(\overline{\Omega})$ such that for each $j \in \mathbb{N}$, f_j is nondecreasing with

$$0 \le f_j \le f(c_1+1) \qquad \text{on } [0,\infty), \tag{2.14}$$

and that u_{0j} is nonnegative, radially symmetric, nonincreasing with respect to $|x|$ and such that $\frac{\partial u_{0j}}{\partial \nu} = 0$ on $\partial\Omega$, and thanks to the regularity properties of f and u_0 we can moreover achieve that with some $\vartheta \in (0,1)$ we have

$$f_j \to f \quad \text{in } C^{\vartheta}_{loc}([0,\infty)) \qquad \text{and} \qquad u_{0j} \to u_0 \quad \text{in } C^{\vartheta}(\overline{\Omega}) \qquad \text{as } j \to \infty. \tag{2.15}$$

Then due to (2.14), it can readily be verified by means of straightforward regularity arguments well-established in the analysis of chemotaxis systems (see e.g. [42], [51, Section 7] and also Proposition 1.3) that each of the corresponding solutions (u_j, v_j) of (1.3) is actually global in time, and that furthermore with some $\theta \in (0,1)$, $(u_j)_{j\in\mathbb{N}}$ and $(v_j)_{j\in\mathbb{N}}$ are bounded in $L^\infty(\Omega \times (0,\infty)) \cap C^{\theta,\frac{\theta}{2}}_{loc}(\overline{\Omega} \times [0,\infty)) \cap C^{2+\theta,1+\frac{\theta}{2}}_{loc}(\overline{\Omega} \times (0,\infty))$ and in $C^{2+\theta,\frac{\theta}{2}}_{loc}(\overline{\Omega} \times (0,\infty)) \cap L^\infty((0,\infty); W^{1,\infty}(\Omega))$, respectively. Therefore, using the Arzelà-Ascoli theorem we easily infer that on passing to a subsequence if necessary, as $j \to \infty$ we have

$$u_j \to \widetilde{u} \qquad \text{in } C^0_{loc}(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}_{loc}(\overline{\Omega} \times (0,\infty)) \tag{2.16}$$

as well as $v_j \to \widetilde{v}$ in $C^{2,0}_{loc}(\overline{\Omega} \times (0,\infty))$ and $v_j \overset{\star}{\rightharpoonup} \widetilde{v}$ in $L^\infty((0,\infty); W^{1,\infty}(\Omega))$ with some limit pair $(\widetilde{u}, \widetilde{v})$ which thanks to (2.15) clearly forms a classical solution of (1.3) in $\{(x,t) \in \Omega \times (0,\infty) \mid \widetilde{u}(x,t) \le c_1 + 1\}$ and thus can easily be seen to coincide with (u,v) in $\Omega \times (0,T)$ according to the uniqueness statement in Proposition 1.1. Since Step 1 warrants that $u_{jr}(r,t) \le 0$ for all $r \in (0,R)$ and $t \in (0,T)$, in view of (2.16) we immediately obtain (2.4) on taking $j \to \infty$ and then $T \nearrow T_{max}$ herein. $\square$

3 A conditional superlinear ODI for ϕ. Proof of Theorem 1.2

Our approach toward the blow-up result in Theorem 1.2 will be based on a contradictory argument which, given a solution (u,v) of (1.3), at its core analyzes the time evolution of

$$\phi(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s) w(s,t) ds, \qquad t \in [0, T_{max}), \tag{3.1}$$

for appropriately chosen values of $\gamma \in (-\infty, 1)$ and $s_0 \in (0, R^n)$, and with w as defined through (1.15). As u and u_t are continuous in $\overline{\Omega} \times [0, T_{max})$ and in $\overline{\Omega} \times (0, T_{max})$, respectively, it can readily be verified e.g. by means of the dominated convergence theorem that since for any such γ and s_0 the mapping $(0, s_0) \ni s \mapsto s^{-\gamma}(s_0 - s)$ is integrable, the function ϕ is well-defined and belongs to $C^0([0, T_{max})) \cap C^1((0, T_{max}))$.

In order to prepare our subsequent analysis of ϕ, given $\gamma \in (-\infty, 1)$ and $s_0 \in (0, R^n)$ let us furthermore introduce

$$\psi(t) := \frac{1}{2} \int_0^{s_0} s^{1-\gamma}(s_0 - s) w_s(s,t) f(n w_s(s,t)) ds, \qquad t \in (0, T_{max}), \tag{3.2}$$

as well as the sets

$$S_\phi := \left\{ t \in (0, T_{max}) \;\middle|\; \phi(t) \ge \frac{1}{(1-\gamma)(2-\gamma)\omega_n} \cdot \left\{ \int_\Omega u_0 dx - s_0 \right\} \cdot s_0^{2-\gamma} \right\} \tag{3.3}$$

and

$$S_\psi := \left\{ t \in (0, T_{max}) \;\middle|\; \psi(t) \ge s_0^{3-\gamma} \right\}. \tag{3.4}$$

Our main goal, to be achieved in Lemma 3.10 and the proof of Theorem 1.2, will consist in deriving a superlinearly forced ODI for ϕ which is *conditional* in the sense of being restricted to the set of times at which both conditions appearing in the definions of (3.3) and (3.4) simultaneously hold.

3.1 An upper inequality for $\mu(t)$

As a preparation, let us first establish an upper bound for the expression $\mu(t)$ in (1.17). Unlike in the case when f grows in a sublinear manner, for widely arbitrary f merely fulfilling (1.4) this essentially amounts to appropriately estimating the part of $\int_0^{R^n} f(nw_s(s,t))ds$ in which the integration variable is close to the origin $s = 0$. To this end, we begin by making use of the defining property of S_ϕ in the following.

Lemma 3.1 *Assume that f and u_0 satisfy (1.4) and (1.5), and let $\gamma \in (-\infty, 1)$ and $s_0 \in (0, R^n)$. Then writing $m := \int_\Omega u_0 dx$, we have*

$$w\Big(\frac{s_0}{2}, t\Big) \ge \frac{1}{\omega_n} \cdot \Big(m - \frac{4s_0}{2^\gamma(3-\gamma)}\Big) \qquad \textit{for all } t \in S_\phi. \tag{3.5}$$

Proof. If (3.5) was false for some $t \in S_\phi$, then necessarily $\delta := \frac{4s_0}{2^\gamma(3-\gamma)}$ would satisfy $\delta < m$, and by monotonicity of $w(\cdot, t)$ we would obtain that $w(s,t) < \frac{m-\delta}{\omega_n}$ for all $s \in (0, \frac{s_0}{2})$ and thus, since $w(s,t) \le \frac{m}{\omega_n}$ for all $s \in (0, R^n)$,

$$\begin{aligned}
\phi(t) &< \frac{m-\delta}{\omega_n} \cdot \int_0^{\frac{s_0}{2}} s^{-\gamma}(s_0 - s)ds + \frac{m}{\omega_n} \cdot \int_{\frac{s_0}{2}}^{s_0} s^{-\gamma}(s_0 - s)ds \\
&= \frac{m}{\omega_n} \cdot \int_0^{s_0} s^{-\gamma}(s_0 - s)ds - \frac{\delta}{\omega_n} \cdot \int_0^{\frac{s_0}{2}} s^{-\gamma}(s_0 - s)ds \\
&= \frac{m}{\omega_n} \cdot \Big\{ s_0 \cdot \frac{s_0^{1-\gamma}}{1-\gamma} - \frac{s_0^{2-\gamma}}{2-\gamma} \Big\} - \frac{\delta}{\omega_n} \cdot \Big\{ s_0 \cdot \frac{(\frac{s_0}{2})^{1-\gamma}}{1-\gamma} - \frac{(\frac{s_0}{2})^{2-\gamma}}{2-\gamma} \Big\} \\
&= \frac{m}{(1-\gamma)(2-\gamma)\omega_n} \cdot s_0^{2-\gamma} - \frac{2^\gamma(3-\gamma)\delta}{4(1-\gamma)(2-\gamma)\omega_n} \cdot s_0^{2-\gamma}.
\end{aligned}$$

In view of the inclusion $t \in S_\phi$ and (3.3), however, this would entail that

$$m - s_0 < m - \frac{2^\gamma(3-\gamma)\delta}{4},$$

contradicting our definition of δ. $\square$

We can thereby estimate μ against a conveniently small part of the positive nonlocal contribution to (1.16).

Lemma 3.2 *Assume that (1.4) and (1.5) hold, and let $\gamma \in (-\infty, 1)$ and $s_0 > 0$ satisfy*

$$s_0 \le \frac{R^n}{4}. \tag{3.6}$$

Then the function μ in (1.17) has the property that

$$\mu(t) \le f_\gamma + \frac{1}{2s}\int_0^s f(nw_s(\sigma,t))d\sigma \qquad \textit{for all } s\in(0,s_0) \textit{ and any } t\in S_\phi, \tag{3.7}$$

where

$$f_\gamma := f\Big(\frac{8n}{2^\gamma(3-\gamma)\omega_n}\Big). \tag{3.8}$$

Proof. For any fixed $t\in S_\phi$, we may invoke Lemma 3.1 to see that again abbreviating $m:=\int_\Omega u_0dx$ and $\delta := \frac{4s_0}{2^\gamma(3-\gamma)}$, we have

$$w\Big(\frac{s_0}{2},t\Big) \ge \frac{m-\delta}{\omega_n}$$

and thus, as $w\le\frac{m}{\omega_n}$,

$$\frac{w(s_0,t)-w(\frac{s_0}{2},t)}{\frac{s_0}{2}} \le \frac{\frac{m}{\omega_n}-\frac{m-\delta}{\omega_n}}{\frac{s_0}{2}} = \frac{2\delta}{\omega_n s_0}.$$

On the other hand, by concavity of $w(\cdot,t)$, as asserted by Lemma 2.2,

$$\frac{w(s_0,t)-w(\frac{s_0}{2},t)}{\frac{s_0}{2}} \ge w_s(s_0,t) \ge w_s(s,t) \qquad \text{for all } s\in(s_0,R^n),$$

so that since $f'\ge 0$, in the identity

$$\mu(t) = \frac{1}{R^n}\int_0^{s_0} f(nw_s(\sigma,t))d\sigma + \frac{1}{R^n}\int_{s_0}^{R^n} f(nw_s(\sigma,t))d\sigma \tag{3.9}$$

we can estimate the last summand according to

$$\frac{1}{R^n}\int_{s_0}^{R^n} f(nw_s(\sigma,t))d\sigma \le \frac{R^n-s_0}{R^n}\cdot f\Big(n\cdot\frac{2\delta}{\omega_n s_0}\Big) \le f\Big(n\cdot\frac{2\delta}{\omega_n s_0}\Big) = f_\gamma \tag{3.10}$$

by (3.8), because $n\cdot\frac{2\delta}{\omega_n s_0} = \frac{8n}{2^\gamma(3-\gamma)\omega_n}$ due to our choice of δ.
To estimate the first integral in (3.9), we once more split

$$\int_0^{s_0} f(nw_s(\sigma,t))d\sigma = \int_0^s f(nw_s(\sigma,t))d\sigma + \int_s^{s_0} f(nw_s(\sigma,t))d\sigma, \qquad s\in(0,s_0), \tag{3.11}$$

and again use the downward monotonicity of $w_s(\cdot,t)$ along with the inequality $f'\ge 0$ to see that

$$\int_s^{s_0} f(nw_s(\sigma,t))d\sigma \le (s_0-s)f(nw_s(s,t)) \le s_0 f(nw_s(s,t)) \qquad \text{for all } s\in(0,s_0),$$

and that

$$\int_0^s f(nw_s(\sigma,t))d\sigma \ge s\cdot f(nw_s(s,t)) \qquad \text{for all } s\in(0,s_0).$$

Therefore, from (3.11) we infer that

$$\begin{aligned}\frac{1}{R^n}\int_0^{s_0} f(nw_s(\sigma,t))d\sigma &\le \frac{1}{R^n}\int_0^s f(nw_s(\sigma,t))d\sigma + \frac{s_0}{R^n} f(nw_s(s,t)) \\ &\le \frac{1}{R^n}\int_0^s f(nw_s(\sigma,t))d\sigma + \frac{s_0}{R^n s}\int_0^s f(nw_s(\sigma,t))d\sigma \qquad \text{for all } s\in(0,s_0).\end{aligned}$$

Since (3.6) guarantees that

$$\frac{1}{R^n} \le \frac{1}{4s_0} \le \frac{1}{4s} \qquad \text{for all } s\in(0,s_0)$$

and that also

$$\frac{s_0}{R^n s} \le \frac{1}{4s} \qquad \text{for all } s\in(0,s_0),$$

together with (3.11) and (3.9) this readily implies (3.7). □

Accordingly, w actually satisfies a parabolic inequality which, apart from degenerate diffusion and a yet nonlocal driving nonlinearity, contains a multiple of sw_s as a subsequently well-controllable additional summand:

Corollary 3.3 *Assume (1.4) and (1.5), and let $\gamma\in(-\infty,1)$ and $s_0>0$ be such that $s_0\le\frac{R^n}{4}$. Then with f_γ and S_ϕ as in (3.8) and (3.3),*

$$w_t(s,t) \ \ge\ n^2 s^{2-\frac{2}{n}} w_{ss}(s,t) + \frac{1}{2} w_s(s,t)\cdot\int_0^s f(nw_s(\sigma,t))d\sigma - f_\gamma s w_s(s,t) \qquad \textit{for all } s\in(0,R^n) \textit{ and any } t\in S_\phi. \tag{3.12}$$

Proof. In view of (1.16) and the nonnegativity of w_s, this is an immediate consequence of Lemma 3.2. □

3.2 A basic differential inequality for ϕ inside S_ϕ

On the basis of (3.12) and several integrations by parts, by making essential use of the inequality $w_{ss}\le 0$ we can now establish a basic differential inequality for our target functional, at this point only subject to the single condition that the considered times be contained in S_ϕ.

Lemma 3.4 *Let f and u_0 satisfy (1.4) and (1.5), and let $\gamma\in(-\infty,1)$ and $s_0>0$ be such that $\gamma<2-\frac{2}{n}$ and $s_0\le\frac{R^n}{4}$. Then the function ϕ introduced in (3.1) satisfies*

$$\begin{aligned}\phi'(t) \ \ge\ & \frac{1}{2}\int_0^{s_0} s^{1-\gamma}(s_0-s)w_s(s,t)f(nw_s(s,t))ds \\ & -2n^2\Big(2-\frac{2}{n}-\gamma\Big)s_0\int_0^{s_0} s^{-\gamma-\frac{2}{n}}w(s,t)ds - f_\gamma\int_0^{s_0} s^{1-\gamma}w(s,t)ds \qquad \textit{for all } t\in S_\phi,\end{aligned} \tag{3.13}$$

where f_γ and S_ϕ are as given by (3.8) and (3.3).

PROOF. According to (3.12), at each $t \in S_\phi$ we have

$$\begin{aligned}\phi'(t) &\geq n^2 \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0-s)w_{ss}ds \\ &\quad +\frac{1}{2}\int_0^{s_0} s^{-\gamma}(s_0-s)w_s \cdot \Big\{ \int_0^s f(nw_s(\sigma,t))d\sigma \Big\} ds - f_\gamma \int_0^{s_0} s^{1-\gamma}(s_0-s)w_s ds. \qquad (3.14)\end{aligned}$$

Since $\gamma < 2 - \frac{2}{n}$, and since thus particularly $s^{1-\frac{2}{n}-\gamma}w(s,t) \to 0$ as $s \searrow 0$ for each $t \in (0,T_{max})$, on integrating by parts we obtain that herein for all $t \in S_\phi$,

$$\begin{aligned}n^2 \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0-s)w_{ss}ds &= -n^2\Big(2-\frac{2}{n}-\gamma\Big)\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s)w_s ds + n^2 \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}w_s ds \\ &\quad + n^2 s^{2-\frac{2}{n}-\gamma}(s_0-s)w_s\Big|_0^{s_0} \\ &= n^2\Big(2-\frac{2}{n}-\gamma\Big)\Big(1-\frac{2}{n}-\gamma\Big)\int_0^{s_0} s^{-\gamma-\frac{2}{n}}(s_0-s)wds \\ &\quad -2n^2\Big(2-\frac{2}{n}-\gamma\Big)\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}wds \\ &\quad -n^2\Big(2-\frac{2}{n}-\gamma\Big)s^{1-\frac{2}{n}-\gamma}(s_0-s)w\Big|_0^{s_0} + n^2 s^{2-\frac{2}{n}-\gamma}w\Big|_0^{s_0} \\ &\geq n^2\Big(2-\frac{2}{n}-\gamma\Big)\Big(1-\frac{2}{n}-\gamma\Big)\int_0^{s_0} s^{-\gamma-\frac{2}{n}}(s_0-s)wds \\ &\quad -2n^2\Big(2-\frac{2}{n}-\gamma\Big)\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}wds \\ &= n^2\Big(2-\frac{2}{n}-\gamma\Big)\Big(1-\frac{2}{n}-\gamma\Big)s_0\int_0^{s_0} s^{-\gamma-\frac{2}{n}}wds \\ &\quad -n^2\Big(2-\frac{2}{n}-\gamma\Big)\Big(3-\frac{2}{n}-\gamma\Big)\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}wds,\end{aligned}$$

where using that $\gamma < 3 - \frac{2}{n}$ we may estimate $\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}wds \leq s_0 \int_0^{s_0} s^{-\frac{2}{n}-\gamma}wds$ to see that in fact

$$\begin{aligned}n^2 \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0-s)w_{ss}ds &\geq n^2\Big(2-\frac{2}{n}-\gamma\Big)\cdot\Big\{\Big(1-\frac{2}{n}-\gamma\Big)-\Big(3-\frac{2}{n}-\gamma\Big)\Big\}\cdot s_0\int_0^{s_0} s^{-\gamma-\frac{2}{n}}wds \\ &= -2n^2\Big(2-\frac{2}{n}-\gamma\Big)s_0\int_0^{s_0} s^{-\gamma-\frac{2}{n}}wds \qquad \text{for all } t \in S_\phi. \qquad (3.15)\end{aligned}$$

Likewise, for the rightmost summand in (3.14) we find that

$$\begin{aligned}-f_\gamma \int_0^{s_0} s^{1-\gamma}(s_0-s)w_s ds &= (1-\gamma)f_\gamma \int_0^{s_0} s^{-\gamma}(s_0-s)wds - f_\gamma \int_0^{s_0} s^{1-\gamma}wds \\ &\quad - f_\gamma s^{1-\gamma}(s_0-s)w\Big|_0^{s_0} \\ &\geq -f_\gamma \int_0^{s_0} s^{1-\gamma}wds \qquad \text{for all } t \in S_\phi, \qquad (3.16)\end{aligned}$$

again since $\gamma < 1$ and since f_γ is nonnegative.
Finally, using that $w_{ss} \le 0$ in $(0,R^n) \times (0,T_{max})$ according to Lemma 2.2, we can use the upward monotonicity of f in estimating

$$\int_0^s f(nw_s(\sigma,t))d\sigma \ge \int_0^s f(nw_s(s,t))d\sigma = sf(nw_s(s,t)) \qquad \text{for all } s \in (0,R^n) \text{ and } t \in (0,T_{max}),$$

so that by nonnegativity of w_s,

$$\frac{1}{2}\int_0^{s_0} s^{-\gamma}(s_0-s)w_s \cdot \left\{ \int_0^s f(nw_s(\sigma,t))d\sigma \right\} ds \ge \frac{1}{2}\int_0^{s_0} s^{1-\gamma}(s_0-s)w_s \cdot f(nw_s)ds \qquad \text{for each } t \in S_\phi.$$

Together with (3.15) and (3.16) inserted into (3.14), this yields (3.13). $\Box$

3.3 A pointwise inequality for w in terms of ψ

In order to adequately relate both ϕ itself as well as the negative summands on the right of (3.13) to the positive first term on the right-hand-side therein, let us make sure that the latter expression, actually coinciding with $\psi(t)$ from (3.2), suitably controls $w(s,t)$ for all $t \in (0,T_{max})$ and any $s \in (0,s_0)$, up to an additive correction $\frac{1}{n}s$. Here, the condition (3.17) enters as a first restriction on γ which, along with a second one to be encountered in Lemma 3.8, will finally determine the range of κ admissible in Theorem 1.2.

Lemma 3.5 *Suppose that f satisfies (1.4) and (1.9) with some $k > 0$ and $\kappa > 0$, and let $\gamma \in (-\infty,1)$ be such that $\gamma < 2 - \frac{2}{n}$ and*

$$\gamma > 1 - \kappa. \tag{3.17}$$

Then for each u_0 fulfilling (1.5), and for any choice of $s_0 \in (0,R^n)$, the function ψ defined in (3.2) has the property that

$$\begin{aligned} w(s,t) \quad &\le \quad \frac{1}{n}\cdot s + \frac{1}{n^{\frac{\kappa}{\kappa+1}}} \cdot \left(\frac{2}{k}\right)^{\frac{1}{\kappa+1}} \cdot \left(\frac{\kappa}{\kappa+\gamma-1}\right)^{\frac{\kappa}{\kappa+1}} \cdot s^{\frac{\kappa+\gamma-1}{\kappa+1}}(s_0-s)^{-\frac{1}{\kappa+1}}\psi^{\frac{1}{\kappa+1}}(t) \\ &\qquad \textit{for all } s \in (0,s_0) \textit{ and } t \in (0,T_{max}). \end{aligned} \tag{3.18}$$

Proof. We first observe that by nonnegativity of both w_s and f we may use (1.9) to see that

$$\begin{aligned} \psi(t) \quad &\ge \quad \frac{1}{2}\int_0^{s_0} \chi_{\{nw_s(\cdot,t)\ge 1\}}(s) \cdot s^{1-\gamma}(s_0-s)w_s(s,t)f(nw_s(s,t))ds \\ &\ge \quad \frac{kn^\kappa}{2}\int_0^{s_0} \chi_{\{nw_s(\cdot,t)\ge 1\}}(s) \cdot s^{1-\gamma}(s_0-s)w_s^{\kappa+1}(s,t)ds \qquad \text{for all } t \in (0,T_{max}), \end{aligned} \tag{3.19}$$

where as usual, χ_M denotes the characteristic function of the set $M \subset \mathbb{R}$. Accordingly, for $t \in (0,T_{max})$ and $s \in (0,s_0)$ using that $w(0,t) = 0$ we split

$$\begin{aligned} w(s,t) \quad &= \quad \int_0^s w_s(\sigma,t)d\sigma \\ &= \quad \int_0^s \chi_{\{nw_s(\cdot,t)<1\}}(\sigma) \cdot w_s(\sigma,t)d\sigma + \int_0^s \chi_{\{nw_s(\cdot,t)\ge 1\}}(\sigma) \cdot w_s(\sigma,t)d\sigma, \end{aligned} \tag{3.20}$$

where

$$\int_0^s \chi_{\{nw_s(\cdot,t)<1\}}(\sigma)\cdot w_s(\sigma,t)d\sigma \le \frac{1}{n}\int_0^s \chi_{\{nw_s(\cdot,t)<1\}}(\sigma)d\sigma \le \frac{1}{n}\cdot s. \tag{3.21}$$

Moreover, by means of the Hölder inequality we can estimate

$$\begin{aligned}
&\int_0^s \chi_{\{nw_s(\cdot,t)\ge 1\}}(\sigma)\cdot w_s(\sigma,t)d\sigma \\
&= \int_0^s \Big\{\chi_{\{nw_s(\cdot,t)\ge 1\}}(\sigma)\cdot\sigma^{1-\gamma}(s_0-\sigma)w_s^{\kappa+1}(\sigma,t)\Big\}^{\frac{1}{\kappa+1}}\cdot\sigma^{-\frac{1-\gamma}{\kappa+1}}(s_0-\sigma)^{-\frac{1}{\kappa+1}}d\sigma \\
&\le (s_0-s)^{-\frac{1}{\kappa+1}}\cdot\int_0^s \Big\{\chi_{\{nw_s(\cdot,t)\ge 1\}}(\sigma)\cdot\sigma^{1-\gamma}(s_0-\sigma)w_s^{\kappa+1}(\sigma,t)\Big\}^{\frac{1}{\kappa+1}}\cdot\sigma^{-\frac{1-\gamma}{\kappa+1}}d\sigma \\
&\le (s_0-s)^{-\frac{1}{\kappa+1}}\cdot\Big\{\int_0^s \chi_{\{nw_s(\cdot,t)\ge 1\}}(\sigma)\cdot\sigma^{1-\gamma}(s_0-\sigma)w_s^{\kappa+1}(\sigma,t)d\sigma\Big\}^{\frac{1}{\kappa+1}}\cdot\Big\{\int_0^s\sigma^{-\frac{1-\gamma}{\kappa}}d\sigma\Big\}^{\frac{\kappa}{\kappa+1}}
\end{aligned}$$

for all $s\in(0,s_0)$ and $t\in(0,T_{max})$. Since our assumption (3.17) warrants that $\frac{1-\gamma}{\kappa}<1$, herein we have

$$\int_0^s \sigma^{-\frac{1-\gamma}{\kappa}}d\sigma = \frac{\kappa}{\kappa+\gamma-1}s^{\frac{\kappa+\gamma-1}{\kappa}} \qquad \text{for all } s\in(0,s_0),$$

so that using (3.19) we obtain that for all $s\in(0,s_0)$ and $t\in(0,T_{max})$,

$$\begin{aligned}
\int_0^s \chi_{\{nw_s(\cdot,t)\ge 1\}}(\sigma)\cdot w_s(\sigma,t)d\sigma &\le (s_0-s)^{-\frac{1}{\kappa+1}}\cdot\Big\{\frac{2\psi(t)}{kn^\kappa}\Big\}^{\frac{1}{\kappa+1}}\cdot\Big\{\frac{\kappa}{\kappa+\gamma-1}\cdot s^{\frac{\kappa+\gamma-1}{\kappa}}\Big\}^{\frac{\kappa}{\kappa+1}} \\
&= \frac{1}{n^{\frac{\kappa}{\kappa+1}}}\cdot\Big(\frac{2}{k}\Big)^{\frac{1}{\kappa+1}}\cdot\Big(\frac{\kappa}{\kappa+\gamma-1}\Big)^{\frac{\kappa}{\kappa+1}}\cdot s^{\frac{\kappa+\gamma-1}{\kappa+1}}(s_0-s)^{-\frac{1}{\kappa+1}}\psi^{\frac{1}{\kappa+1}}(t).
\end{aligned}$$

In combination with (3.21) and (3.20), this precisely yields (3.18). □

Now when resorting to the set of times satisfying the restriction defining S_ψ, this correcting summand $\frac{1}{n}s$ can conveniently be estimated in terms of the essential rightmost part in (3.18).

Corollary 3.6 *Assume that (1.4), (1.9) and (1.5) hold with some $k>0$ and $\kappa>0$, let $\gamma\in(-\infty,1)$ satisfy $1-\kappa<\gamma<2-\frac{2}{n}$, and let $s_0\in(0,R^n)$. Then with ψ and S_ψ taken from (3.2) and (3.4), respectively, we have*

$$w(s,t)\le Ls^{\frac{\kappa+\gamma-1}{\kappa+1}}(s_0-s)^{-\frac{1}{\kappa+1}}\psi^{\frac{1}{\kappa+1}}(t) \qquad \textit{for all } s\in(0,s_0) \textit{ and } t\in S_\psi, \tag{3.22}$$

where we have set

$$L\equiv L(k,\kappa,\gamma):=\frac{1}{n}+\frac{1}{n^{\frac{\kappa}{\kappa+1}}}\cdot\Big(\frac{2}{k}\Big)^{\frac{1}{\kappa+1}}\cdot\Big(\frac{\kappa}{\kappa+\gamma-1}\Big)^{\frac{\kappa}{\kappa+1}}. \tag{3.23}$$

Proof. According to Lemma 3.5, we only need to observe that since $\gamma<2$ and by the defining property of S_ψ, the first summand on the right of (3.18) can be controlled in terms of the second one according to

$$\begin{aligned}
\frac{s}{s^{\frac{\kappa+\gamma-1}{\kappa+1}}(s_0-s)^{-\frac{1}{\kappa+1}}\psi^{\frac{1}{\kappa+1}}(t)} &= s^{\frac{2-\gamma}{\kappa+1}}(s_0-s)^{\frac{1}{\kappa+1}}\psi^{-\frac{1}{\kappa+1}}(t) \\
&\le s_0^{\frac{2-\gamma}{\kappa+1}}\cdot s_0^{\frac{1}{\kappa+1}}\cdot (s_0^{3-\gamma})^{-\frac{1}{\kappa+1}} \\
&= 1 \qquad \text{for all } s\in(0,s_0)
\end{aligned}$$

whenever $t\in S_\psi$. □

3.4 Estimating ϕ and the negative summands on the right of (3.13) in terms of ψ

By means of three applications of Corollary 3.6, inside S_ψ we can now use ψ to conveniently control the integrals under consideration, up to certain correcting factors containing s_0. We first relate ψ to a superlinear power of ϕ.

Lemma 3.7 *Let f and u_0 be such that (1.4), (1.9) and (1.5) are valid with some $k>0$ and $\kappa>0$, let $\gamma\in(-\infty,1)$ be such that $1-\kappa<\gamma<2-\frac{2}{n}$, and let $s_0\in(0,R^n)$. Then taking ϕ,ψ, S_ψ and L from (3.1), (3.2), (3.4) and (3.23), we have*

$$\psi(t)\ge\Big(\frac{(2-\gamma)\kappa}{(\kappa+1)L}\Big)^{\kappa+1}\cdot s_0^{-(3-\gamma)\kappa}\cdot\phi^{\kappa+1}(t) \qquad \textit{for all } t\in S_\psi. \tag{3.24}$$

PROOF. For any $t\in S_\psi$, the statement from Corollary 3.6 applies so as to warrant that

$$\phi(t)=\int_0^{s_0}s^{-\gamma}(s_0-s)w(s,t)ds\le L\cdot\Big\{\int_0^{s_0}s^{-\gamma+\frac{\kappa+1-\gamma}{\kappa+1}}(s_0-s)^{\frac{\kappa}{\kappa+1}}ds\Big\}\cdot\psi^{\frac{1}{\kappa+1}}(t). \tag{3.25}$$

Here the inequality $\gamma<2$ ensures that

$$\begin{aligned}
\int_0^{s_0}s^{-\gamma+\frac{\kappa+1-\gamma}{\kappa+1}}(s_0-s)^{\frac{\kappa}{\kappa+1}}ds &\le s_0^{\frac{\kappa}{\kappa+1}}\cdot\int_0^{s_0}s^{-\gamma+\frac{\kappa+\gamma-1}{\kappa+1}}ds \\
&= \frac{\kappa+1}{(2-\gamma)\kappa}\cdot s_0^{\frac{(3-\gamma)\kappa}{\kappa+1}},
\end{aligned}$$

so that after a straightforward rearrangement we infer (3.24) from (3.25). □

We next turn to the diffusive contribution to (3.13). In its appropriate handling, we will need a second condition on γ, unlike (3.17) this time requiring an additional restriction from above.

Lemma 3.8 *Suppose that (1.4), (1.9) and (1.5) hold with some $k>0$ and $\kappa>0$, and assume that $\gamma\in(-\infty,1)$ is such that $1-\kappa<\gamma<2-\frac{2}{n}$ as well as*

$$\gamma<2-\frac{2}{n}-\frac{2}{n\kappa}. \tag{3.26}$$

Then for any choice of $s_0\in(0,R^n)$,

$$s_0\int_0^{s_0}s^{-\gamma-\frac{2}{n}}w(s,t)ds\le L\cdot B\Big(\frac{(2-\frac{2}{n}-\gamma)\kappa-\frac{2}{n}}{\kappa+1},\frac{\kappa}{\kappa+1}\Big)\cdot s_0^{\frac{(3-\frac{2}{n}-\gamma)\kappa-\frac{2}{n}}{\kappa+1}}\cdot\psi^{\frac{1}{\kappa+1}}(t) \qquad \textit{for all } t\in S_\psi, \tag{3.27}$$

where B denotes Euler's Beta function, and where L, ψ and S_ψ are as in (3.23), (3.2) and (3.4), respectively.

Proof. Again by means of Corollary 3.6, for arbitrary $t \in S_\psi$ we see that

$$s_0 \int_0^{s_0} s^{-\gamma-\frac{2}{n}} w(s,t)ds \le L s_0 \cdot \left\{ \int_0^{s_0} s^{-\gamma-\frac{2}{n}+\frac{\kappa+\gamma-1}{\kappa+1}} (s_0-s)^{-\frac{1}{\kappa+1}} ds \right\} \cdot \psi^{\frac{1}{\kappa+1}}(t). \tag{3.28}$$

As (3.26) warrants that

$$1-\gamma-\frac{2}{n}+\frac{\kappa+\gamma-1}{\kappa+1} = \frac{(2-\frac{2}{n}-\gamma)\kappa-\frac{2}{n}}{\kappa+1} > 0,$$

herein we have

$$s_0 \cdot \left\{ \int_0^{s_0} s^{-\gamma-\frac{2}{n}+\frac{\kappa+\gamma-1}{\kappa+1}} (s_0-s)^{-\frac{1}{\kappa+1}} ds \right\} = B\Big(\frac{(2-\frac{2}{n}-\gamma)\kappa-\frac{2}{n}}{\kappa+1}, \frac{\kappa}{\kappa+1}\Big) \cdot s_0^{\frac{(3-\frac{2}{n}-\gamma)\kappa-\frac{2}{n}}{\kappa+1}},$$

whence (3.28) implies (3.27). $\Box$

Finally, also the rightmost term in (3.13) is dominated essentially by a sublinear power of ψ.

Lemma 3.9 *Let (1.4), (1.9) and (1.5) be valid with some $k>0$ and $\kappa>0$, and let $\gamma \in (-\infty,1)$ satisfy $1-\kappa<\gamma<2-\frac{2}{n}$. Then for any $s_0 \in (0,R^n)$,*

$$\int_0^{s_0} s^{1-\gamma} w(s,t)ds \le \frac{\kappa+1}{\kappa} L s_0^{\frac{(3-\gamma)\kappa}{\kappa+1}} \psi^{\frac{1}{\kappa+1}}(t) \qquad \textit{for all } t \in S_\psi, \tag{3.29}$$

where again L, ψ and S_ψ have been taken from (3.23), (3.2) and (3.4), respectively.

Proof. Once more by Corollary 3.6, we can estimate

$$\int_0^{s_0} s^{1-\gamma} w(s,t)ds \le L \cdot \left\{ \int_0^{s_0} s^{1-\gamma+\frac{\kappa+\gamma-1}{\kappa+1}} (s_0-s)^{-\frac{1}{\kappa+1}} ds \right\} \cdot \psi^{\frac{1}{\kappa+1}}(t) \qquad \text{for all } t \in S_\psi,$$

which immediately entails (3.29), for

$$\begin{aligned} \int_0^{s_0} s^{1-\gamma+\frac{\kappa+\gamma-1}{\kappa+1}} (s_0-s)^{-\frac{1}{\kappa+1}} ds &= \int_0^{s_0} s^{\frac{(2-\gamma)\kappa}{\kappa+1}} (s_0-s)^{-\frac{1}{\kappa+1}} ds \\ &\le s_0^{\frac{(2-\gamma)\kappa}{\kappa+1}} \int_0^{s_0} (s_0-s)^{-\frac{1}{\kappa+1}} ds \\ &= \frac{\kappa+1}{\kappa} s_0^{\frac{(3-\gamma)\kappa}{\kappa+1}} \end{aligned}$$

due to the fact that $\gamma<2$. $\Box$

3.5 Conclusion. Proof of Theorem 1.2

Now for any f complying with the assumptions from Theorem 1.2, one can find γ admissible in both Lemma 3.4 as well as in all inequalities from the previous section. Upon suitable combination of the latter and under the restriction that t belongs to both S_ϕ and S_ψ, we can therefore establish the following ODI for ϕ, besides a superlinear source yet containing an absorptive correction.

Lemma 3.10 *Assume that (1.4) and (1.9) hold with some $k > 0$ and some*

$$\kappa > \frac{2}{n}. \tag{3.30}$$

Then there exist $\gamma = \gamma(\kappa) \in (-\infty, 1)$ and $C = C(k, \kappa, R) > 0$ such that for each u_0 fulfilling (1.5) and for any choice of $s_0 > 0$ such that $s_0 \le \frac{R^n}{4}$, the function ϕ defined in (3.1) satisfies

$$\phi'(t) \ge \frac{1}{C} \cdot s_0^{-(3-\gamma)\kappa} \phi^{\kappa+1}(t) - C \cdot s_0^{\frac{(3-\frac{2}{n}-\gamma)\kappa - \frac{2}{n}}{\kappa}} \qquad \textit{for all } t \in S_\phi \cap S_\psi, \tag{3.31}$$

where S_ϕ and S_ψ are as determined by (3.3), and (3.4), respectively.

Proof. Given $\kappa > \frac{2}{n}$, we note that evidently $1 - \kappa < 1$ and $1 - \kappa < 2 - \frac{2}{n}$, and that moreover

$$\begin{aligned} \frac{(2-\frac{2}{n})\kappa - \frac{2}{n}}{\kappa} - (1-\kappa) &= \frac{1}{\kappa} \cdot \Big\{ \kappa^2 + \Big(1 - \frac{2}{n}\Big)\kappa - \frac{2}{n} \Big\} \\ &= \frac{(\kappa+1)(\kappa - \frac{2}{n})}{\kappa} \\ &> 0 \end{aligned}$$

and hence also $1 - \kappa < \frac{(2-\frac{2}{n})\kappa - \frac{2}{n}}{\kappa}$, so that it is possible to find $\gamma = \gamma(\kappa) \in (-\infty, 1)$ such that $\gamma < 2 - \frac{2}{n}$ as well as

$$1 - \kappa < \gamma < \frac{(2-\frac{2}{n})\kappa - \frac{2}{n}}{\kappa}. \tag{3.32}$$

Futhermore fixing $c_1 \equiv c_1(\kappa) > 0$ large enough such that according to Young's inequality we have

$$\xi\eta \le \frac{1}{4}\xi^{\kappa+1} + c_1 \eta^{\frac{\kappa+1}{\kappa}} \qquad \text{for all } \xi \ge 0 \text{ and } \eta \ge 0, \tag{3.33}$$

we let u_0 be given such that (1.5) holds, and for $s_0 > 0$ fulfilling $s_0 \le \frac{R^n}{4}$ we take ϕ, ψ, S_ϕ and S_ψ as thereupon defined in (3.1), (3.2), (3.3) and (3.4).
Then since $\gamma < 1$ and $\gamma < 2 - \frac{2}{n}$, Lemma 3.4 applies to say that with $c_2 \equiv c_2(\kappa) := 2n^2(2 - \frac{2}{n} - \gamma)$ and $f_\gamma > 0$ as in (3.8),

$$\phi'(t) \ge \psi(t) - c_2 s_0 \int_0^{s_0} s^{-\gamma - \frac{2}{n}} w ds - f_\gamma \int_0^{s_0} s^{1-\gamma} w ds \qquad \text{for all } t \in S_\phi. \tag{3.34}$$

Here relying on the second inequality in (3.32) we may employ Lemma 3.8, which in conjunction with (3.33) namely ensures that with $c_3 \equiv c_3(k, \kappa) := L(k, \kappa, \gamma) \cdot B(\frac{(2-\frac{2}{n}-\gamma)\kappa - \frac{2}{n}}{\kappa+1}, \frac{\kappa}{\kappa+1})$ and $c_4 \equiv c_4(k, \kappa) := c_1 (c_2 c_3)^{\frac{\kappa+1}{\kappa}}$ we have

$$\begin{aligned} c_2 s_0 \int_0^{s_0} s^{-\gamma - \frac{2}{n}} w ds &\le c_2 c_3 s_0^{\frac{(3-\frac{2}{n}-\gamma)\kappa - \frac{2}{n}}{\kappa+1}} \psi^{\frac{1}{\kappa+1}}(t) \\ &\le \frac{1}{4}\psi(t) + c_4 s_0^{\frac{(3-\frac{2}{n}-\gamma)\kappa - \frac{2}{n}}{\kappa}} \qquad \text{for all } t \in S_\psi. \end{aligned} \tag{3.35}$$

Likewise, the left inequality in (3.32) enables us to invoke Lemma 3.9 to see that again due to (3.33),

$$\begin{aligned} f_\gamma \int_0^{s_0} s^{1-\gamma} w ds &\le f_\gamma \cdot \frac{\kappa+1}{\kappa} L s_0^{\frac{(3-\gamma)\kappa}{\kappa+1}} \psi^{\frac{1}{\kappa+1}}(t) \\ &\le \frac{1}{4}\psi(t) + c_5 s_0^{3-\gamma} \qquad \text{for all } t \in S_\psi \end{aligned} \tag{3.36}$$

with $c_5 \equiv c_5(k,\kappa) := c_1 \cdot (\frac{\kappa}{\kappa+1} f_\gamma L)^{\frac{\kappa+1}{\kappa}}$.
Finally, once more due to the first restriction on γ in (3.32) we may conclude from Lemma 3.7 that if we abbreviate $c_6 \equiv c_6(k,\kappa) := \frac{1}{2}(\frac{(2-\gamma)\kappa}{(\kappa+1)L})^{\kappa+1}$, then

$$\frac{1}{2}\psi(t) \ge c_6 s_0^{-(3-\gamma)\kappa} \phi^{\kappa+1}(t) \qquad \text{for all } t \in S_\psi. \tag{3.37}$$

When combined with (3.35) and (3.36), this shows that (3.34) implies that

$$\phi'(t) \ge c_6 s_0^{-(3-\gamma)\kappa} \phi^{\kappa+1}(t) - c_4 s_0^{\frac{(3-\frac{2}{n}-\gamma)\kappa - \frac{2}{n}}{\kappa}} - c_5 s_0^{3-\gamma} \qquad \text{for all } t \in S_\phi \cap S_\psi,$$

whence observing that

$$\begin{aligned} s_0^{3-\gamma} &= s_0^{\frac{2(\kappa+1)}{n\kappa}} \cdot s_0^{\frac{(3-\frac{2}{n}-\gamma)\kappa - \frac{2}{n}}{\kappa}} \\ &\le R^{\frac{2(\kappa+1)}{\kappa}} \cdot s_0^{\frac{(3-\frac{2}{n}-\gamma)\kappa - \frac{2}{n}}{\kappa}}, \end{aligned}$$

from this we infer that indeed (3.31) holds if we let $C \equiv C(k,\kappa,R) := \max\{\frac{1}{c_6}, c_4 + c_5 R^{\frac{2(\kappa+1)}{\kappa}}\}$. ☐

For initial data fulfilling (1.11) with suitably small ε, the initial value of ϕ turns out to be conveniently large so as to allow for the conclusion that if (u,v) was global, then (3.31) in fact would imply a genuinely superlinear ODI without any absorptive contribution, actually valid for all positive times. The corresponding contradictory argument will thereby establish our main result on the occurrence of finite-time blow-up in (1.3):

Proof of Theorem 1.2. We start by only fixing Ω and f such that (1.4) and (1.9) hold with some $k > 0$ and $\kappa > \frac{2}{n}$, and then apply Lemma 3.10 to find $\gamma \equiv \gamma(\kappa) \in (-\infty, 1)$, $c_1 = c_1(k,\kappa,R) > 0$ and $c_2 = c_2(k,\kappa,R) > 0$ with the property that for arbitrary u_0 fulfilling (1.5) and any $s_0 > 0$ such that $s_0 \le \frac{R^n}{4}$, the function ϕ from (3.1) satisfies

$$\phi'(t) \ge c_1 s_0^{-(3-\gamma)\kappa} \phi^{\kappa+1}(t) - c_2 s_0^{\frac{(3-\frac{2}{n}-\gamma)\kappa - \frac{2}{n}}{\kappa}} \qquad \text{for all } t \in S_\phi \cap S_\psi \tag{3.38}$$

with S_ϕ, ψ and S_ψ given by (3.3), (3.2) and (3.4). Moreover, Lemma 3.7 provides $c_3 = c_3(k,\kappa) > 0$ such that for any such u_0 and s_0,

$$\psi(t) \ge c_3 s_0^{-(3-\gamma)\kappa} \phi^{\kappa+1}(t) \qquad \text{for all } t \in S_\psi. \tag{3.39}$$

Next, to specify our choice of s_0 herein we let $m > 0$ be given and use (1.10) in picking $s_0 = s_0(k,\kappa,m,R) > 0$ suitably small such that $s_0 \le \frac{R^n}{4}$, that

$$s_0 \le \frac{m}{2} \tag{3.40}$$

and that

$$s_0^{\frac{(\kappa+1)(\kappa-\frac{2}{n})}{\kappa}} \le \frac{c_1}{2c_2} \cdot \Big(\frac{m}{2(1-\gamma)(2-\gamma)\omega_n}\Big)^{\kappa+1} \tag{3.41}$$

as well as

$$s_0 \le c_3^{\frac{1}{\kappa+1}} \cdot \frac{m}{2(1-\gamma)(2-\gamma)\omega_n}. \tag{3.42}$$

We then fix $\varepsilon = \varepsilon(k, \kappa, m, R) > 0$ small such that $\varepsilon < \frac{s_0}{2}$, and observe that since obviously

$$\int_\delta^{s_0} s^{-\gamma}(s_0 - s)ds \nearrow \frac{s_0^{2-\gamma}}{(1-\gamma)(2-\gamma)} \qquad \text{as } \delta \searrow 0,$$

it thereupon becomes possible to find $s_\star = s_\star(k, \kappa, m, R) \in (0, s_0)$ such that

$$\frac{m-\varepsilon}{\omega_n} \cdot \int_{s_\star}^{s_0} s^{-\gamma}(s_0 - s)ds > \frac{m - s_0}{(1-\gamma)(2-\gamma)\omega_n} \cdot s_0^{2-\gamma}. \tag{3.43}$$

Setting $r_\star \equiv r_\star(k, \kappa, m, R) := s_\star^{\frac{1}{n}} \in (0, R)$ as a final preparation, we now suppose that u_0 satisfies (1.5) and is such that (1.11) holds, and we assume for contradiction that then in Proposition 1.1 we have $T_{max} = \infty$. Then the corresponding expression in (3.1) is defined throughout $[0, \infty)$, and writing

$$S := \Big\{ T > 0 \ \Big| \ \phi(t) > \frac{m - s_0}{(1-\gamma)(2-\gamma)\omega_n} \cdot s_0^{2-\gamma} \quad \text{for all } t \in [0, T] \Big\},$$

we note that by continuity of ϕ, S is not empty due to the fact that by (1.11) we have

$$w(s, 0) \ge w(s_\star, 0) = \frac{1}{\omega_n} \cdot \int_{B_{r_\star}(0)} u_0 dx \ge \frac{m-\varepsilon}{\omega_n} \qquad \text{for all } s \in (s_\star, R^n)$$

and hence

$$\begin{aligned} \phi(0) &\ge \int_{s_\star}^{s_0} s^{-\gamma}(s_0 - s)w(s, 0)ds \\ &\ge \frac{m-\varepsilon}{\omega_n} \cdot \int_{s_\star}^{s_0} s^{-\gamma}(s_0 - s)ds \\ &> \frac{m - s_0}{(1-\gamma)(2-\gamma)\omega_n} \cdot s_0^{2-\gamma} \end{aligned} \tag{3.44}$$

thanks to (3.43). Therefore, $T := \sup S$ is a well-defined element of $(0, \infty]$ which is such that $\phi(t) > \frac{m-s_0}{(1-\gamma)(2-\gamma)\omega_n} \cdot s_0^{2-\gamma}$ for all $t \in (0, T)$ and that thus, in particular,

$$\phi(t) \ge \frac{m}{2(1-\gamma)(2-\gamma)\omega_n} \cdot s_0^{2-\gamma} \qquad \text{for all } t \in (0, T) \tag{3.45}$$

because of (3.40). As two consequences thereof, we observe that (3.41) ensures that

$$\frac{c_1}{2} s_0^{-(3-\gamma)\kappa} \phi^{\kappa+1}(t) - c_2 s_0^{\frac{(3-\frac{2}{n}-\gamma)\kappa-\frac{2}{n}}{\kappa}}$$

$$\geq \frac{c_1}{2} \cdot \Big(\frac{m}{2(1-\gamma)(2-\gamma)\omega_n}\Big)^{\kappa+1} \cdot s_0^{-(3-\gamma)\kappa+(2-\gamma)(\kappa+1)} - c_2 s_0^{\frac{(3-\frac{2}{n}-\gamma)\kappa-\frac{2}{n}}{\kappa}}$$
$$= \left\{\frac{c_1}{2c_2} \cdot \Big(\frac{m}{2(1-\gamma)(2-\gamma)\omega_n}\Big)^{\kappa+1} - s_0^{\frac{(\kappa+1)(\kappa-\frac{2}{n})}{\kappa}}\right\} \cdot c_2 s_0^{-\kappa-\gamma+2}$$
$$\geq 0 \qquad \text{for all } t \in (0,T), \tag{3.46}$$

and that by (3.39) and (3.42), the accordingly defined function ψ from (3.2) satisfies

$$\psi(t) \geq c_3 \cdot \Big(\frac{m}{2(1-\gamma)(2-\gamma)\omega_n}\Big)^{\kappa+1} \cdot s_0^{-(3-\gamma)\kappa+(2-\gamma)(\kappa+1)}$$
$$= \left\{c_3^{\frac{1}{\kappa+1}} \cdot \frac{m}{2(1-\gamma)(2-\gamma)\omega_n} \cdot \frac{1}{s_0}\right\}^{\kappa+1} \cdot s_0^{3-\gamma}$$
$$\geq s_0^{3-\gamma} \qquad \text{for all } t \in (0,T). \tag{3.47}$$

In particular, (3.45) and (3.47) guarantee that for the sets S_ϕ and S_ψ in (3.3) and (3.4) we have

$$(0,T) \subset S_\psi \cap S_\psi,$$

whence combining (3.38) with (3.46) shows that actually any t within the entire interval $(0,T)$ has the property that

$$\phi'(t) \geq \frac{c_1}{2} s_0^{-(3-\gamma)\kappa} \phi^{\kappa+1}(t). \tag{3.48}$$

On the one hand, this, clearly entails that indeed we must have $T=\infty$, for otherwise, by definition of S and again by continuity of ϕ, we could conclude that $\phi(T) = \frac{m-s_0}{(1-\gamma)(2-\gamma)\omega_n} \cdot s_0^{2-\gamma}$ which is incompatible with the fact that $\phi(t) \geq \phi(0) > \frac{m-s_0}{(1-\gamma)(2-\gamma)\omega_n} \cdot s_0^{2-\gamma}$ according to (3.48) and (3.44).
On the other hand, however, due to the positivity of both κ and $\phi(0)$, upon integration the inequality (3.48) enforces that in contradiction to the latter,

$$T \leq \frac{2}{c_1 \kappa \phi^\kappa(0)}.$$

In consequence, for any such u_0 we infer that actually T_{max} must be finite, as intended. □

4 Global boundedness. Proof of Proposition 1.3

Let us finally include a short argument confirming the statement on global existence and boundedness contained in Proposition 1.3, hence asserting essential optimality of Theorem 1.2 with respect to the hypotheses on f made therein.

PROOF of Proposition 1.3. In view of (1.7) and well-established regularity arguments based e.g. on a Moser-type iteration ([42, Lemma A.1]) or on standard $L^p - L^q$ estimates for the Neumann heat semigroup ([2, Lemma 3.2]), it is sufficient to make sure that for each u_0 fulfilling (1.5) and any $p>1$ satisfying $p > \frac{(n-2)\kappa}{2}$ one can find $c_1 = c_1(p) > 0$ such that

$$\int_\Omega u^p(x,t)dx \leq c_1 \qquad \text{for all } t \in (0,T_{max}). \tag{4.1}$$

To verify this, given any such p we use (1.3) and integrate by parts to see that writing $c_2 \equiv c_2(p) := \frac{p-1}{p}K|\Omega|$ we have

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx + (p-1)\int_\Omega u^{p-2}|\nabla u|^2 dx &= (p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla v dx \\
&= -\frac{p-1}{p}\int_\Omega u^p \Delta v dx \\
&= \frac{p-1}{p}\int_\Omega u^p f(u) dx - \frac{p-1}{p}\mu(t)\int_\Omega u^p dx \\
&\le \frac{p-1}{p}\int_{\{u\ge 1\}} u^p f(u) dx + \frac{p-1}{p}\int_{\{u<1\}} u^p f(u) dx \\
&\le \frac{p-1}{p}K\int_\Omega u^{p+\kappa} dx + c_2 \qquad \text{for all } t\in(0,T_{max}), \qquad (4.2)
\end{aligned}$$

because $\mu \ge 0$ by nonnegativity of f, and because f is nondecreasing and satisfies (1.12). Now since (1.13) entails that $a := \frac{np}{2}\cdot\frac{p+\kappa-1}{p+\kappa}\cdot(1-\frac{n}{2}+\frac{np}{2})^{-1}$ has the property that $\frac{2(p+\kappa)}{p}a < 2$, we may combine that Gagliardo-Nirenberg with (1.8) and Young's inequality to fix positive constants $c_3 = c_3(p), c_4 = c_4(p)$ and $c_5 = c_5(p)$ such that

$$\begin{aligned}
\frac{p-1}{p}K\int_\Omega u^{p+\kappa} dx &= \frac{p-1}{p}K\|u^{\frac{p}{2}}\|^{\frac{2(p+\kappa)}{p}}_{L^{\frac{2(p+\kappa)}{p}}(\Omega)} \\
&\le c_3\|\nabla u^{\frac{p}{2}}\|^{\frac{2(p+\kappa)}{p}a}_{L^2(\Omega)}\|u^{\frac{p}{2}}\|^{\frac{2(p+\kappa)}{p}(1-a)}_{L^{\frac{2}{p}}(\Omega)} + c_3\|u^{\frac{p}{2}}\|^{\frac{2(p+\kappa)}{p}}_{L^{\frac{2}{p}}(\Omega)} \\
&\le c_4\|\nabla u^{\frac{p}{2}}\|^{\frac{2(p+\kappa)}{p}a}_{L^2(\Omega)} + c_4 \\
&\le \frac{p-1}{2}\int_\Omega u^{p-2}|\nabla u|^2 dx + c_5 \qquad \text{for all } t\in(0,T_{max}).
\end{aligned}$$

In quite a similar manner, we obtain $c_6 = c_6(p) > 0$ fulfilling

$$\int_\Omega u^p dx \le \frac{p-1}{2}\int_\Omega u^{p-2}|\nabla u|^2 dx + c_6 \qquad \text{for all } t\in(0,T_{max}),$$

so that from (4.2) we conclude that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx + \int_\Omega u^p dx \le c_2 + c_5 + c_6 \qquad \text{for all } t\in(0,T_{max}),$$

which immediately yields (4.1) by means of an ODE comparison argument and thereby completes the proof. □

Acknowledgement. The author would like to thank the anonymous reviewers as well as Youshan Tao for several helpful comments which have led to a considerable improvement of this paper. The author furthermore acknowledges support of the *Deutsche Forschungsgemeinschaft* in the context of the project *Analysis of chemotactic cross-diffusion in complex frameworks*.

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