Global mass-preserving solutions in a two-dimensional chemotaxis-Stokes system with rotational flux components

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Abstract

In a bounded domain $\Omega \subset \mathbb{R}^2$, we consider the the chemotaxis-Stokes system

$$\begin{cases} n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot \Big(nS(x,n,c) \cdot \nabla c \Big), & x \in \Omega,\ t > 0, \\ c_t + u \cdot \nabla c &= \Delta c - nf(c), & x \in \Omega,\ t > 0, \\ u_t &= \Delta u + \nabla P + n \nabla \phi, \qquad \nabla \cdot u = 0, & x \in \Omega,\ t > 0, \end{cases} \qquad (\star)$$

which arises as a model for populations of aerobic bacteria swimming in a sessile water drop. In accordance with refined modeling approaches, we do not necessarily assume the chemotactic sensitivity S herein to be a scalar function, but rather allow S to attain values in $\mathbb{R}^{2\times 2}$.

As compared to the well-studied case of scalar-valued sensitivities in which an analysis can be based on favorable energy-type inequalities, this modification brings about significant new challenges which require to adequately cope with only little a priori information on regularity of solutions of $(\star)$. The present work creates a functional setup which despite this allows for the construction of certain global mass-preserving generalized solutions to an associated initial-boundary value problem in planar convex domains with smooth boundary, provided that the initial data and the parameter functions S, f and ϕ are sufficiently smooth, and that S is bounded and f is nonnegative with $f(0) = 0$.

Key words: chemotaxis; Stokes; global existence; generalized solution
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1 Introduction

We consider a model for the spatio-temporal evolution in populations of microscopic organisms, surrounded by a liquid medium, which partially orient their movement according to concentration gradients of a chemical that they consume upon contact. As indicated by striking experimental findings, even in quite primitive setups such types of interplay may be sufficient to generate considerably complex forms of collective behavior, inter alia enforcing phenomena of self-concentration and large-scale dynamic coherence ([9]). In particular, it can be observed that in populations of *Bacillus subtilis*, bacterial aggregates spontaneously emerge, and that these may move at speeds considerably higher

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than the speeds of the individual organisms, leading to Péclet numbers significantly greater than 1.

Accordingly, a fundamental modeling approach presented in [31] postulates that for understanding such phenomena it is essential to consider the mutual interaction between cells and the surrounding water, rather than merely the motion of bacteria in an otherwise passive fluid such as in standard descriptions of cell movement, even in complex processes e.g. involving bioconvection (cf. [16] and also [19], for instance). Thus, besides the hypothesis that the motion of individual cells is chemotactically biased by concentration gradients of dissolved oxygen which they consume, constitutive modeling assumptions in [31] are that moreover cells and oxygen are transported through water via convection, and that the swimming bacteria, having a slightly higher density than water, affect the fluid motion through bouyant forces.

In cases of low Reynolds numbers when nonlinear convection can be disregarded, these suppositions lead to systems of evolution equations which upon convenient parameter normalization take the form

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot\Big(nS(x,n,c)\cdot\nabla c\Big), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - nf(c), & x\in\Omega,\ t>0,\\ u_t = \Delta u + \nabla P + n\nabla\phi, \qquad \nabla\cdot u = 0, & x\in\Omega,\ t>0, \end{cases} \tag{1.1}$$

in the physical domain $\Omega\subset\mathbb{R}^N$ ([31], [24]). Here, $n=n(x,t), c=c(x,t), u=u(x,t)$ and $P=P(x,t)$ represent the population density, the oxygen concentration, the fluid velocity field and the associated pressure, respectively, and the chemotactic sensitivity S, the oxygen consumption rate f and the gravitational potential ϕ are supposed to be given functions.

In particular, the original version of the model (1.1) proposed in [31] adapts to the classical Keller-Segel system of chemotaxis in supposing S to be a scalar function, thus assuming that chemotactic cross-diffusion is exclusively directed toward increasing signal concentrations. In this setting, analytical approaches address issues of global well-posedness in appropriate initial-boundary value problems for (1.1) in various functional frameworks, either in bounded two- or three-dimensional domains, or in the respective entire spaces, and under diverse, more or less restrictive assumptions on the parameter functions S, f and ϕ ([10], [5], [20], [39]; cf. also [7], [32], [11], [33] and [29] for results on related models with nonlinear diffusion). The knowledge in this respect has become quite comprehensive in the spatially two-dimensional setting in which even Navier-Stokes fluid evolution may lead to global existence of uniquely determined classical solutions ([23], [39]); in the case $N=3$, after all, certain global weak and at least eventually smooth solutions can be constructed under mild hypotheses on the respective model ingredients ([39], [44], [45]; see also [21] for a related study including effects of logistic sources).

Chemotaxis with rotational flux: Loss of energy structure – and of mass conservation? With regard to structure generation, the above results in particular indicate that in sharp contrast to the classical Keller-Segel model in which cells produce the signal instead of consuming it ([15], [40]), phenomena of finite-time blow-up, typifying the apparently most extreme facet of bacterial aggregation, do not occur in (1.1). Going beyond this, known results suggest that in both its two- and three-dimensional version, (1.1) may not even be able to describe structure formation on large time scales. Indeed, under certain conditions on the the parameter functions, weak enough so as to allow for the prototypical choices $S\equiv const., f(c)=c$ for $c\ge 0$ and $\nabla\phi\equiv const.$, it has been shown in [41] and [48] that in bounded convex planar domains, all solutions emanating from reasonably smooth nontrivial initial data remain globally bounded and approach the spatially homogeneous equilibrium $(\fint_\Omega n_0, 0, 0)$ at an exponential rate in the large time limit, even in the more complex variant of (1.1) obtained on replacing the Stokes by the Navier-Stokes system; in the three-dimensional version of the latter problem, a corresponding result on asymptotic homogeneization has recently been established in [45].

Taking into account the experimental observation that spatial inhomogeneities typically originate from regions near the boundary of the fluid ([9]), in this work we shall follow a more recent modeling approach which assumes that bacterial chemotaxis in such boundary regions is not precisely oriented along signal gradients, but rather involves rotational flux components. According to the detailed model derivation in [47] and [46], this requires the sensitivity function S in (1.1) to be a general 2×2 matrix which at positions x close to $\partial\Omega$ possibly contains nontrivial off-diagonal entries such as in the prototype

$$S = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad a > 0,\ b \in \mathbb{R}.$$

As compared to the above case of scalar sensitivities, this generalization brings about considerable mathematical challenges resulting from the fact that thereby (1.1) apparently loses a favorable quasi-energy structure. In fact, large bodies of the analysis for scalar-valued S are based on corresponding inequalities which e.g. in the case when $S \equiv 1$ and $f(c) = c$ for $c \geq 0$ take the form

$$\frac{d}{dt}\left\{ \int_\Omega n \ln n + \frac{1}{2}\int_\Omega \frac{|\nabla c|^2}{c} \right\} + \int_\Omega \frac{|\nabla n|^2}{n} + \int_\Omega c|D^2 \ln c|^2 \leq C \int_\Omega |u|^4, \qquad t > 0, \tag{1.2}$$

with some $C > 0$ ([39]). Combined with appropriate regularization properties of the fluid flow, these may be used to derive a priori estimates which may form the starting point for global existence theories (see [10], [49], [39]) and also [29], for instance) as well as for the description of the large time asymptotics of solutions ([41], [45]).

The lack of appropriate analogues of (1.2) in presence of general tensor-valued sensitivities significantly complicates the analysis even in pure chemotaxis systems without any fluid interaction. In the case of scalar-valued S, energy-based arguments can be applied to the variant of (1.1) obtained on letting $u \equiv 0$ ([28]); likewise, the current knowledge also on the classical parabolic Keller-Segel system, in contrast to (1.1) accounting for signal production rather than consumption through cells, to a considerable extent relies on the use of a corresponding energy functional ([25], [17], [40]), and accordingly in this latter context only few results, exclusively relying on quite strong assumptions on the system ingredients, address situations the sensitivity is allowed to be tensor-valued ([1], [4], [36], [37], [26]). A similar shortfall in knowledge can be observed in chemotaxis systems with signal consumption as soon as tensor-valued sensitivities are involved: Even in the case $N = 2$, global bounded smooth solutions to the fluid-free version of (1.1) so far have only been constructed under an additional smallness assumption on c in $L^\infty(\Omega)$ ([22]), or if diffusion is nonlinear and enhanced at large densities ([3]; cf. also [4], [2], [34], [35] and [43] for extensions involving fluid interaction and addressing three-dimensional settings). For arbitrarily large data in the original chemotaxis model with linear diffusion, at least certain global generalized solutions are known to exist for any choice of $N \geq 1$, but their regularity properties may be rather poor ([42]); in particular, it seems not even known whether these solutions describe mass conservation in the flavor of the identity

$$\int_\Omega n(\cdot,t) = \int_\Omega n_0 \qquad \text{for } t>0, \tag{1.3}$$

constituting a property which, although very basic, apparently is of fundamental biological importance.

Main results. It is the goal of the present work to develop a method which in the spatially two-dimensional case and under essentially minimal assumptions, in particular involving basically no structural hypothesis on the matrix-valued sensitivity S, allows for establishing a basic theory on global existence of solutions to (1.1) which do conserve mass, even when fluid interaction is accounted for. In order to formulate our main results in this respect, let us specify the precise problem setting by considering (1.1) in a bounded convex domain $\Omega \subset \mathbb{R}^2$ along with the boundary conditions

$$\nabla n \cdot \nu = n\Big(S(x,n,c)\nabla c\Big)\cdot \nu, \quad \nabla c \cdot \nu = 0, \quad u = 0, x \in \partial\Omega,\ t>0, \tag{1.4}$$

and the initial conditions

$$n(x,0) = n_0(x), \quad c(x,0) = c_0(x), \quad u(x,0) = u_0(x), x \in \Omega. \tag{1.5}$$

Throughout this paper, we shall assume that

$$f \in C^1([0,\infty)) \quad \text{is nonnegative with} \quad f(0) = 0 \tag{1.6}$$

and that

$$\phi \in W^{2,\infty}(\Omega), \tag{1.7}$$

and we shall suppose that $S = (S_{ij})_{i,j\in\{1,2\}}$ is such that

$$S_{ij} \in C^2(\bar\Omega \times [0,\infty) \times [0,\infty)) \qquad \text{for } i,j \in \{1,2\} \tag{1.8}$$

and that

$$|S(x,n,c)| \le S_0(c) \quad \text{for all } (x,n,c) \in \bar\Omega \times [0,\infty)^2 \qquad \text{with some nondecreasing } S_0 : [0,\infty) \to \mathbb{R}. \tag{1.9}$$

Concerning the initial data, our standing assumptions will be that

$$\begin{cases} n_0 \in C^\iota(\bar\Omega) \quad \text{for some } \iota > 0 \text{ with } n_0 \ge 0 \text{ in } \Omega, \quad \text{that} \\ c_0 \in W^{1,\infty}(\Omega) \quad \text{satisfies } c_0 \ge 0 \text{ in } \Omega, \quad \text{and that} \\ u_0 \in D(A_2^\vartheta) \quad \text{for some } \vartheta \in (\frac12, 1), \end{cases} \tag{1.10}$$

where A_2 denotes the Stokes operator in the Hilbert space $L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega) \mid \nabla\cdot\varphi = 0\}$ of all solenoidal functions in $L^2(\Omega)$, with the natural domain of A_2 given by $D(A_2) := W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \cap L^2_\sigma(\Omega)$ (cf. also Section 3.2 below).

We shall see that under these assumptions, (1.1) indeed is globally solvable by mass-conserving functions:

Theorem 1.1. *Suppose that f, ϕ and S satisfy (1.6), (1.7), (1.8) and (1.9), and that n_0, c_0 and u_0 comply with (1.10). Then there exists at least one triple of functions*

$$\begin{cases} n \in L^\infty([0,\infty); L^1(\Omega)), \\ c \in L^\infty(\Omega \times (0,\infty)) \cap L^2_{loc}([0,\infty); W^{1,2}(\Omega)) \qquad \textit{and} \\ u \in L^2_{loc}(\bar\Omega \times [0,\infty)) \cap \bigcap_{p\in[1,2)} L^p_{loc}([0,\infty); W_0^{1,p}(\Omega)), \end{cases}$$

such that (n,c,u) is a global mass-preserving generalized solution of (1.1), (1.4), (1.5) in the sense of Definition 2.1 below; in particular,

$$\int_\Omega n(\cdot,t) = \int_\Omega n_0 \qquad \textit{for a.e. } t>0. \tag{1.11}$$

Main ideas. One particular technical challenge will stem from the observation that with regard to the crucial solution component n, standard testing procedures to track the time evolution of e.g. $\int_\Omega \Phi(n(\cdot,t))$, constitutive of basically any known method in the analysis of chemotaxis systems, apparently fail for all reasonable choices of superlinearly growing Φ in the present general situation. On the other hand, an estimate for n in $L^\infty((0,\infty); L^1(\Omega))$, as resulting from the formally evident mass conservation property (1.3), is apparently insufficient to provide adequate compactness properties in any reasonable approximation process, allowing e.g. for the conclusion that the limit would in fact satisfy (1.3).

Of fundamental importance for our approach will thus be an inequality of the form

$$\int_0^\infty \int_\Omega \frac{|\nabla n|^2}{(n+1)^2} \le C \tag{1.12}$$

(Lemma 3.1), which upon an application of the Moser-Trudinger inequality can be turned into an estimate for $\int_t^{t+1} \ln \|n(\cdot,s)+1\|^p_{L^p(\Omega)} ds$, $t>0$, with some conveniently chosen $p>2$ in the present two-dimensional context (Lemma 4.1). Along with appropriate compactness properties for c and u, essentially resulting from (1.3) due to the smoothing effects of the heat and Stokes semigroups (cf. Sections 3.1 and 3.2), this additional integrability information will be used in the important Lemma 4.2 to establish strong compactness in $L^1_{loc}(\bar\Omega\times[0,\infty))$ of the first component in a sequence of solutions to suitably regularized problems (cf. (2.8)). This in turn will entail a corresponding strong compactness property also for ∇c in $L^2_{loc}(\bar\Omega\times[0,\infty))$ (Section 4.2) and thereby allow for constructing a solution fulfilling (1.11) through a standard extraction procedure (Section 5).

2 A generalized solution concept, regularization and basic estimates

In the sequel we shall pursue the following solution concept which is very weak, especially with regard to the solution properties of the first component, and which partially resembles the notion of renormalized solutions ([8]) but differs from the latter in decisive details (cf. e.g. the *in*equality appearing in (2.4)). This non-standard choice is due to our sparse knowledge on regularity features of solutions; in particular, we will not be able to decide whether for the solution we shall construct below, the corresponding cross-diffusive flux $nS(x,n,c)\cdot\nabla c$ in the first equation in (1.1) is integrable. After all, our concept complies with the basic requirement that smooth functions which solve (1.1) in our generalized sense are also classical solutions, cf. the remark subsequent to the following definition. All this partly parallels the situation in the pure chemotaxis system obtained when $u \equiv 0$, as studied in [42] in the general N-dimensional framework with a similar notion of solution. As compared to the latter, an important improvement that can and will be gained in the present two-dimensional case will be that our solutions will satisfy the natural mass conservation identity (2.3), rather than merely an inequality as in [42].

Definition 2.1. *Suppose that*

$$
\begin{aligned}
&n \in L^\infty((0,\infty); L^1(\Omega)),\\
&c \in L^\infty_{loc}(\bar\Omega \times [0,\infty)) \cap L^2_{loc}([0,\infty); W^{1,2}(\Omega)) \qquad \text{and}\\
&u \in L^1_{loc}([0,\infty); (W^{1,1}_0(\Omega))^2)
\end{aligned}
\tag{2.1}
$$

are such that $n \geq 0$ and $c \geq 0$ a.e. in $\Omega \times (0,\infty)$ and

$$
\ln(n+1) \in L^2_{loc}([0,\infty); W^{1,2}(\Omega)), \tag{2.2}
$$

that

$$
\int_\Omega n(x,t)dx = \int_\Omega n_0(x) \qquad \text{for a.e. } t>0, \tag{2.3}
$$

and that $\nabla \cdot u = 0$ a.e. in $\Omega \times (0,\infty)$. Then the triple (n,c,u) will be called a global mass-preserving generalized solution *of (1.1), (1.4), (1.5) if the inequality*

$$
\begin{aligned}
-\int_0^\infty \int_\Omega \ln(n+1)\varphi_t - \int_\Omega \ln(n_0+1)\varphi(\cdot,0) \quad \geq \quad & \int_0^\infty \int_\Omega \ln(n+1)\Delta\varphi + \int_0^\infty \int_\Omega |\nabla \ln(n+1)|^2\varphi \\
& - \int_0^\infty \int_\Omega \frac{n}{n+1} \nabla \ln(n+1) \cdot \Big(S(x,n,c)\cdot\nabla c\Big)\varphi \\
& + \int_0^\infty \int_\Omega \frac{n}{n+1} \Big(S(x,n,c)\cdot\nabla c\Big)\cdot\nabla\varphi \\
& + \int_0^\infty \int_\Omega \ln(n+1)(u\cdot\nabla\varphi)
\end{aligned}
\tag{2.4}
$$

holds for each nonnegative $\varphi \in C_0^\infty(\bar\Omega \times [0,\infty))$ with $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega \times (0,\infty)$, if moreover

$$
\int_0^\infty \int_\Omega c\varphi_t + \int_\Omega c_0\varphi(\cdot,0) = \int_0^\infty \int_\Omega \nabla c\cdot\nabla\varphi + \int_0^\infty \int_\Omega nf(c)\varphi - \int_0^\infty \int_\Omega c(u\cdot\nabla\varphi) \tag{2.5}
$$

for any $\varphi \in L^\infty(\Omega \times (0,\infty)) \cap L^2((0,\infty); W^{1,2}(\Omega))$ having compact support in $\bar\Omega \times [0,\infty)$ with $\varphi_t \in L^2(\Omega \times (0,\infty))$, and if finally

$$
-\int_0^\infty \int_\Omega u\cdot\varphi_t - \int_\Omega u_0\cdot\varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla u\cdot\nabla\varphi + \int_0^\infty \int_\Omega n\nabla\phi\cdot\varphi \tag{2.6}
$$

for all $\varphi \in C_0^\infty(\Omega \times [0,\infty); \mathbb{R}^2)$ with $\nabla\cdot\varphi \equiv 0$.

Remark. i) As can readily be seen, the regularity requirements in (2.1) and (2.2) along with (1.6) and (1.9) indeed ensure that all integrals appearing in (2.4), (2.5) and (2.6) are well-defined.

ii) It is well-known ([30]) that under the assumptions of Definition 2.1 there exists a distribution P on $\Omega \times (0,\infty)$ such that $u_t = \Delta u + \nabla P + n\nabla\phi$ holds in $\mathcal{D}'(\Omega \times (0,\infty))$.

iii) Evidently, the inequality (2.4) expresses a supersolution property of n with regard to the first equation in (1.1) only, rather than a solution property. In conjunction with the mass conservation identity (2.3), however, this in fact constitutes a proper, albeit quite weak, solution concept for the respective sub-problem of (1.1), (1.4), (1.5): Namely, whenever n and c are nonnegative functions from $C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty))$ and $u \in C^0(\bar\Omega \times [0,\infty);\mathbb{R}^2) \cap C^{2,1}(\Omega \times (0,\infty))$ such that $\nabla \cdot u \equiv 0$ and such that (n,c,u) is a global mass-preserving generalized solution of (1.1), (1.4), (1.5) in the sense of Definition 2.1, there exists $P \in C^{1,0}(\Omega \times (0,\infty))$ such that (n,c,u,P) also is a classical solution of (1.1), (1.4), (1.5) in $\Omega \times (0,\infty)$. In fact, this can be shown using the fact that the function $[0,\infty) \ni \xi \mapsto \ln(\xi+1)$ has its derivative strictly positive throughout $[0,\infty)$, following the proof of a corresponding statement in the special case $u \equiv 0$ which is detailed in [42, Lemma 2.1].

In order to construct a generalized solution of (1.1), (1.4), (1.5) in the above sense, following the approaches in [22] and [42] our initial step consists in approximating (1.1), (1.4), (1.5) by problems which firstly are globally solvable, and in which secondly the no-flux boundary condition for the first solution component, in (1.4) being coupled to c in quite a complicated manner, reduces to a homogeneous Neumann condition. To achieve this, we fix families $(\rho_\varepsilon)_{\varepsilon\in(0,1)}$ and $(\chi_\varepsilon)_{\varepsilon\in(0,1)}$ of cut-off functions in Ω and $[0,\infty)$, respectively, having the properties that

$$\rho_\varepsilon \in C_0^\infty(\Omega) \quad \text{is such that} \quad 0 \le \rho_\varepsilon \le 1 \text{ in } \Omega \quad \text{and} \quad \rho_\varepsilon \nearrow 1 \text{ in } \Omega \text{ as } \varepsilon \searrow 0$$

and

$$\chi_\varepsilon \in C_0^\infty([0,\infty)) \quad \text{is such that} \quad 0 \le \chi_\varepsilon \le 1 \text{ in } [0,\infty) \quad \text{and} \quad \chi_\varepsilon \nearrow 1 \text{ in } [0,\infty) \text{ as } \varepsilon \searrow 0.$$

For $\varepsilon \in (0,1)$, we then define

$$S_\varepsilon(x,n,c) := \rho_\varepsilon(x)\cdot\chi_\varepsilon(n)\cdot S(x,n,c), \qquad (x,n,c) \in \bar\Omega \times [0,\infty)^2, \tag{2.7}$$

and thereupon consider the regularized problems

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon = \Delta n_\varepsilon - \nabla \cdot \Big(n_\varepsilon S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon\Big), & x \in \Omega,\ t>0, \\ c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - n_\varepsilon f(c_\varepsilon), & x \in \Omega,\ t>0, \\ u_{\varepsilon t} = \Delta u_\varepsilon + \nabla P_\varepsilon + n_\varepsilon \nabla\phi, & x \in \Omega,\ t>0, \\ \nabla \cdot u_\varepsilon = 0, & x \in \Omega,\ t>0, \\ \frac{\partial n_\varepsilon}{\partial\nu} = \frac{\partial c_\varepsilon}{\partial\nu} = 0, \quad u_\varepsilon = 0, & x \in \partial\Omega,\ t>0, \\ n_\varepsilon(x,0) = n_0(x), \quad c_\varepsilon(x,0) = c_0(x), \quad u_\varepsilon(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.8}$$

each of which is indeed globally solvable in the classical sense:

Lemma 2.2. *Let $\varepsilon \in (0,1)$. Then there exist functions*

$$
\begin{aligned}
&n_\varepsilon \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)),\\
&c_\varepsilon \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)),\\
&u_\varepsilon \in C^0(\bar\Omega \times [0,\infty);\mathbb{R}^2) \cap C^{2,1}(\bar\Omega \times (0,\infty);\mathbb{R}^2),\\
&P_\varepsilon \in C^{1,0}(\Omega \times (0,\infty)),
\end{aligned}
$$

such that n_ε and c_ε are nonnegative, and such that $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ is a classical solution of (2.8).

Proof. By a straightforward application of the contraction mapping principle in an appropriate framework, (2.8) can be seen to possess a classical solution in a cylinder $\Omega \times (0, T_{max,\varepsilon})$ with some maximal $T_{max,\varepsilon} \in (0,\infty]$ which satisfies either $T_{max,\varepsilon} = \infty$, or

$$
\limsup_{t \nearrow T_{max,\varepsilon}} \Big(\|n_\varepsilon(\cdot,t)\|_{C^2(\bar\Omega)} + \|c_\varepsilon(\cdot,t)\|_{C^2(\bar\Omega)} + \|u_\varepsilon(\cdot,t)\|_{C^2(\bar\Omega)} \Big) = \infty
$$

(cf. e.g. [39, Lemma 2.1] for details). But here we actually must have $T_{max,\varepsilon} = \infty$, because the fact that $S_\varepsilon(x,n,c) = 0$ for all suitably large $n > 0$ enables us to perform standard a priori estimation procedures ([39, Sect. 5], [18]) to show that for each $\varepsilon \in (0,1)$ and any $T > 2$ one can find $C(\varepsilon,T) > 0$ fulfilling

$$
\begin{aligned}
&\|n_\varepsilon(\cdot,t)\|_{C^2(\bar\Omega)} + \|c_\varepsilon(\cdot,t)\|_{C^2(\bar\Omega)} + \|u_\varepsilon(\cdot,t)\|_{C^2(\bar\Omega)} \le C(\varepsilon,T)\\
&\qquad\qquad\qquad\text{for all } t \in \Big(\min\Big\{1, \frac{1}{2}T_{max,\varepsilon}\Big\}, \min\Big\{T, T_{max,\varepsilon}\Big\}\Big).
\end{aligned}
$$

Therefore, $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ in fact is a global classical solution of (2.8). □

Some basic properties of these approximate solutions can be summarized as follows.

Lemma 2.3. *Let $\varepsilon \in (0,1)$. Then*

$$
\int_\Omega n_\varepsilon(x,t)dx = \int_\Omega n_0 \qquad \textit{for all } t > 0, \tag{2.9}
$$

and for each $p \in [1,\infty]$ we have

$$
\|c_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le \|c_\varepsilon(\cdot,s)\|_{L^p(\Omega)} \qquad \textit{for all } s \ge 0 \textit{ and each } t \ge s. \tag{2.10}
$$

Moreover,

$$
\int_0^\infty \int_\Omega |\nabla c_\varepsilon|^2 \le \frac{1}{2} \int_\Omega c_0^2 \tag{2.11}
$$

and

$$
\int_0^\infty \int_\Omega n_\varepsilon f(c_\varepsilon) \le \int_\Omega c_0 \tag{2.12}
$$

as well as

$$
|S_\varepsilon(x, n_\varepsilon, c_\varepsilon)| \le S_1 := S_0(\|c_0\|_{L^\infty(\Omega)}) \qquad \textit{for all } x \in \Omega \textit{ and } t \ge 0. \tag{2.13}
$$

Proof. The identity (2.9) immediately results upon integration in the first equation in (2.8). We next fix $p \in [1, \infty)$ and then obtain from the second equation in (2.8) that

$$\frac{1}{p}\int_\Omega c_\varepsilon^p + (p-1)\int_0^t\int_\Omega c_\varepsilon^{p-2}|\nabla c_\varepsilon|^2 + \int_0^t\int_\Omega n_\varepsilon c_\varepsilon^{p-1} f(c_\varepsilon) = \frac{1}{p}\int_\Omega c_0^p \qquad \text{for all } t > 0, \tag{2.14}$$

which for any such p establishes downward monotonicity of $\|c_\varepsilon(\cdot,t)\|_{L^p(\Omega)}$. The proof of (2.10) thus becomes complete on taking $p \to \infty$, and thereupon (2.13) is an immediate consequence of this, (2.7) and (1.9). Finally, the specific choices $p = 2$ and $p = 1$ in (2.14) yield the inequalities (2.11) and (2.12), respectively. $\square$

3 Further ε-independent estimates. Construction of a limit (n, c, u)

3.1 Estimates for n_ε

Let us first establish an estimate on the spatial gradient of $\ln(n_\varepsilon + 1)$ which will be fundamental for our existence theory. Its derivation is almost identical to that of the corresponding statement in the fluid-free case, as addressed in [42, Lemma 4.1], and therefore we may confine ourselves to sketching a proof here only.

Lemma 3.1. *Writing $C := 2\int_\Omega n_0 + \frac{S_1^2}{2}\cdot\int_\Omega c_0^2$ with S_1 taken from (2.13), we have*

$$\int_0^\infty\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} \le C \qquad \textit{for all } \varepsilon \in (0,1). \tag{3.1}$$

Proof. We test the first equation in (2.8) against $\frac{1}{n_\varepsilon+1}$. Since $\nabla\cdot u_\varepsilon \equiv 0$ entails that $\int_\Omega u_\varepsilon \cdot \frac{\nabla n_\varepsilon}{n_\varepsilon+1} = 0$, on applying Young's inequality and (2.13) we thereby obtain that

$$\begin{aligned} \frac{d}{dt}\int_\Omega \ln(n_\varepsilon+1) &= \int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} - \int_\Omega \frac{n_\varepsilon}{(n_\varepsilon+1)^2}\nabla n_\varepsilon\cdot\Big(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big) \\ &\ge \frac{1}{2}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} - \frac{S_1^2}{2}\int_\Omega |\nabla c_\varepsilon|^2 \quad \text{for all } t > 0. \end{aligned}$$

Integrating with respect to time and using that

$$\int_\Omega \ln\Big(n_\varepsilon(x,t)+1\Big)dx - \int_\Omega \ln\Big(n_0(x)+1\Big)dx \le \int_\Omega n_\varepsilon(x,t)dx = \int_\Omega n_0 \qquad \text{for all } t > 0$$

due to (2.9) and the validity of $0 \le \ln(\xi+1) \le \xi$ for all $\xi \ge 0$, recalling that $\int_0^\infty\int_\Omega |\nabla c_\varepsilon|^2 \le \frac{1}{2}\int_\Omega c_0^2$ by (2.11), we readily end up with (3.1). $\square$

3.2 Estimates for u_ε

We next plan to derive appropriate estimates for u_ε. To prepare our arguments in this direction, let us recall some well-known facts from the context of the Stokes operator when considered in the subspaces $L^p_\sigma(\Omega) = \{\varphi \in L^p(\Omega) \mid \nabla\cdot\varphi = 0\}$ of all solenoidal vector fields in $L^p(\Omega)$ for arbitrary $p \in (1,\infty)$ ([13],

[12], [27]). Indeed, for any such p the Helmholtz projection acts as a bounded linear operator $\mathcal{P}_p$ from $L^p(\Omega)$ onto $L^p_\sigma(\Omega)$, and the corresponding realization $A_p := -\mathcal{P}_p\Delta$ of the Stokes operator with domain $D(A_p) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \cap L^p_\sigma(\Omega)$ is sectorial in $L^p_\sigma(\Omega)$. In particular, for any $\beta \in \mathbb{R}$ this operator possesses closed fractional powers A_p^β with dense domains $D(A_p^\beta)$, and A_p generates an analytic semigroup $(e^{-tA_p})_{t\ge 0}$ in $L^p_\sigma(\Omega)$. Since $\mathcal{P}_p$ and A_p^β as well as e^{-tA_p} are actually independent of $p \in (1,\infty)$ whenever applied to smooth functions, we may and will omit an explicit inclusion of the index p whenever there is no danger of confusion.

The inequality in the following auxiliary lemma may be interpreted as an estimate which, loosely speaking, indicates that up to projection to divergence-free vector fields, functions from $L^1(\Omega)$ can be viewed as elements of $D(A_p^{-\alpha})$ for $p > 1$ and suitably large $\alpha > 0$. A similar statement in the three-dimensional context can be found in [43, Lemma 3.3].

Lemma 3.2. *Let $p > 1$ and $\alpha > \frac{p-1}{p}$. Then there exists $C > 0$ such that*

$$\|A^{-\alpha}\mathcal{P}\varphi\|_{L^p(\Omega)} \le C\|\varphi\|_{L^1(\Omega)} \qquad \textit{for all } \varphi \in L^1(\Omega). \tag{3.2}$$

PROOF. Let $\chi \in C_0^\infty(\Omega)$. Then since $A^{-\alpha}\mathcal{P}\varphi \in L^p_\sigma(\Omega)$ and $A^{-\alpha}$ is symmetric, we have

$$\int_\Omega A^{-\alpha}\mathcal{P}\varphi \cdot \chi = \int_\Omega A^{-\alpha}\mathcal{P}\varphi \cdot \mathcal{P}\chi = \int_\Omega \mathcal{P}\varphi \cdot A^{-\alpha}\mathcal{P}\chi = \int_\Omega \varphi \cdot A^{-\alpha}\mathcal{P}\chi.$$

Since writing $p' := \frac{p}{p-1}$ we know that $2\alpha - \frac{2}{p'}$ is positive by assumption on α, we have $D(A^\alpha_{p'}) \hookrightarrow L^\infty(\Omega)$ ([12], [14]), and hence

$$\begin{aligned}\left|\int_\Omega A^{-\alpha}\mathcal{P}\varphi \cdot \chi\right| &\le \|\varphi\|_{L^1(\Omega)}\|A^{-\alpha}\mathcal{P}\chi\|_{L^\infty(\Omega)} \\ &\le C_1\|\varphi\|_{L^1(\Omega)}\|\mathcal{P}\chi\|_{L^{p'}(\Omega)} \\ &\le C_2\|\varphi\|_{L^1(\Omega)}\|\chi\|_{L^{p'}(\Omega)} \qquad \text{for all } \chi \in C_0^\infty(\Omega)\end{aligned}$$

with some $C_1 > 0$ and $C_2 > 0$, because $\mathcal{P}$ is continuous in $L^{p'}(\Omega)$. By a standard duality and completion argument, this implies (3.2). $\square$

Based on the latter and the projected version $u_{\varepsilon t} + Au_\varepsilon = \mathcal{P}[n_\varepsilon\nabla\phi]$ of the Stokes system in (2.8), we can establish some consequences of the mass conservation property (2.9) on the regularity of u_ε.

Lemma 3.3. *i) Given $p \in (1,\infty)$, one can find $C(p) > 0$ such that whenever $\varepsilon \in (0,1)$,*

$$\|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le C(p) \qquad \textit{for all } t > 0. \tag{3.3}$$

ii) Let $p \in (1,2)$. Then for all $\delta > 0$ there exists $C(p,\delta) > 0$ such that for each $\varepsilon \in (0,1)$ we have

$$\|u_\varepsilon(\cdot,t)\|_{W^{1,p}(\Omega)} \le C(p,\delta)\cdot\left(1 + t^{-\frac{1}{2}-\delta}\right) \qquad \textit{for all } t > 0. \tag{3.4}$$

PROOF. We only detail the derivation of (3.4), because (3.3) can thereafter easily be proved by obvious modifications of the reasoning which thereby actually even becomes slightly simpler.
In order to estimate u_ε on the basis of its variation-of-constants representation,

$$u_\varepsilon(\cdot,t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}\mathcal{P}[n_\varepsilon(\cdot,s)\nabla\phi]ds, \qquad t \ge 0, \tag{3.5}$$

let us first fix some $\alpha > 0$ such that

$$\frac{p-1}{p} < \alpha < \frac{1}{2}, \tag{3.6}$$

which is possible since $p < 2$, and then choose $\eta \in (0, \delta]$ small fulfilling

$$\frac{1}{2} + \eta + \alpha < 1. \tag{3.7}$$

Since this ensures that $D(A_p^{\frac{1}{2}+\eta}) \hookrightarrow W^{1,p}(\Omega)$ ([12], [14]), we can find $C_1 > 0$ such that

$$\|\varphi\|_{W^{1,p}(\Omega)} \le C_1 \|A^{\frac{1}{2}+\eta}\varphi\|_{L^p(\Omega)} \qquad \text{for all } \varphi \in D(A_p), \tag{3.8}$$

and recalling Lemma 3.2 we can fix $C_2 > 0$ satisfying

$$\|A^{-\alpha}\mathcal{P}\varphi\|_{L^p(\Omega)} \le C_2\|\varphi\|_{L^1(\Omega)} \qquad \text{for all } \varphi \in L^1(\Omega), \tag{3.9}$$

because $\alpha > \frac{p-1}{p}$. Moreover, known smoothing and decay properties of that Stokes semigroup ([14]) assert the existence of $\lambda > 0$ such that for all $\beta \ge 0$ one can find $C_3(\beta) > 0$ fulfilling

$$\|A^\beta e^{-tA}\varphi\|_{L^p(\Omega)} \le C_3(\beta) t^{-\beta} e^{-\lambda t} \|\varphi\|_{L^p(\Omega)} \qquad \text{for all } \varphi \in L^p_\sigma(\Omega). \tag{3.10}$$

Therefore,

$$\begin{aligned} \|e^{-tA}u_0\|_{W^{1,p}(\Omega)} &\le C_1 \cdot C_3\Big(\frac{1}{2}+\eta\Big) \cdot t^{-\frac{1}{2}-\eta}\, e^{-\lambda t} \|u_0\|_{L^p(\Omega)} \\ &\le C_4 t^{-\frac{1}{2}-\delta} \qquad \text{for all } t > 0 \end{aligned} \tag{3.11}$$

with some $C_4 > 0$, because $\eta \le \delta$ and $u_0 \in L^p_\sigma(\Omega)$ as a consequence of (1.10). Furthermore, (3.8), (3.10) and (3.9) yield

$$\begin{aligned} &\Big\| \int_0^t e^{-(t-s)A}\mathcal{P}[n_\varepsilon(\cdot,s)\nabla\phi]ds \Big\|_{W^{1,p}(\Omega)} \\ &= \Big\| \int_0^t A^\alpha e^{-(t-s)A} A^{-\alpha}\mathcal{P}[n_\varepsilon(\cdot,s)\nabla\phi]ds \Big\|_{W^{1,p}(\Omega)} \\ &\le C_1 \Big\| \int_0^t A^{\frac{1}{2}+\eta+\alpha} e^{-(t-s)A} A^{-\alpha}\mathcal{P}[n_\varepsilon(\cdot,s)\nabla\phi]ds \Big\|_{L^p(\Omega)} \\ &\le C_1C_3\Big(\frac{1}{2}+\eta+\alpha\Big) \cdot \int_0^t (t-s)^{-\frac{1}{2}-\eta-\alpha}\, e^{-\lambda(t-s)} \|A^{-\alpha}\mathcal{P}[n_\varepsilon(\cdot,s)\nabla\phi]\|_{L^p(\Omega)}ds \\ &\le C_1C_2C_3\Big(\frac{1}{2}-\eta+\alpha\Big) \cdot \int_0^t (t-s)^{-\frac{1}{2}-\eta-\alpha}\, e^{-\lambda(t-s)} \|n_\varepsilon(\cdot,s)\nabla\phi\|_{L^1(\Omega)}ds \\ &\le C_1C_2C_3\Big(\frac{1}{2}-\eta+\alpha\Big) \cdot \|n_0\|_{L^1(\Omega)} \cdot \|\nabla\phi\|_{L^\infty(\Omega)} \cdot \int_0^\infty \sigma^{-\frac{1}{2}-\eta-\alpha}\, e^{-\lambda\sigma} d\sigma \qquad \text{for all } t > 0. \end{aligned}$$

Combining this with (3.11) and (3.5) proves (3.4). □

3.3 Estimates of time derivatives. Construction of limit functions

In our limit procedure below it will be important to achieve certain pointwise convergence properties at least for the first two solution components. We therefor derive bounds for the respective time derivatives in $L^1_{loc}([0,\infty);H^\star)$ with some Hilbert space H, thus preparing a standard argument based on a corresponding version of the Aubin-Lions lemma ([30, Theorem 2.3]).

Lemma 3.4. *For all $T>0$ there exists $C(T)>0$ such that for all $\varepsilon\in(0,1)$ we have*

$$\int_0^T \|\partial_t \ln(n_\varepsilon(\cdot,t)+1)\|_{(W_0^{2,2}(\Omega))^\star} dt \le C(T) \tag{3.12}$$

and

$$\int_0^T \|c_{\varepsilon t}(\cdot,t)\|_{(W_0^{2,2}(\Omega))^\star} dt \le C(T). \tag{3.13}$$

Proof. In order to establish (3.12), we fix $t>0$ and multiply the first equation in (2.8) by $\frac{\psi(x)}{n_\varepsilon(x,t)+1}$ for $\psi\in C_0^\infty(\Omega)$. On integrating by parts this yields

$$\begin{aligned}
\int_\Omega \partial_t \ln(n_\varepsilon(x,t)+1)\cdot\psi(x)dx \;=\; & -\int_\Omega \frac{1}{n_\varepsilon+1}\nabla n_\varepsilon\cdot\nabla\psi + \int_\Omega \frac{1}{(n_\varepsilon+1)^2}|\nabla n_\varepsilon|^2\psi \\
& + \int_\Omega \frac{n_\varepsilon}{n_\varepsilon+1}\Big(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big)\cdot\nabla\psi \\
& - \int_\Omega \frac{n_\varepsilon}{(n_\varepsilon+1)^2}\nabla n_\varepsilon\cdot\Big(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big)\psi \\
& - \int_\Omega u_\varepsilon\cdot\frac{\nabla n_\varepsilon}{n_\varepsilon+1}\psi.
\end{aligned}$$

Here making repeated use of the Cauchy-Schwarz inequality and (2.13), we can estimate each of the integrals on the right in a straightforward manner so as to end up with

$$\begin{aligned}
&\left|\int_\Omega \partial_t \ln(n_\varepsilon(x,t)+1)\cdot\psi(x)dx\right| \\
&\quad\le \left\{\Big(\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2}\Big)^{\frac12} + \int_\Omega \frac{|\nabla n_\varepsilon|^2}{(u_\varepsilon+1)^2} + S_1\cdot\Big(\int_\Omega |\nabla c_\varepsilon|^2\Big)^{\frac12}\right. \\
&\qquad\left. + S_1\cdot\Big(\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2}\Big)^{\frac12}\cdot\Big(\int_\Omega|\nabla c_\varepsilon|^2\Big)^{\frac12} + \Big(\int_\Omega |u_\varepsilon|^2\Big)^{\frac12}\cdot\Big(\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2}\Big)^{\frac12}\right\}\times \\
&\qquad\times\Big(\|\nabla\psi\|_{L^2(\Omega)} + \|\psi\|_{L^\infty(\Omega)}\Big) \qquad \text{for all } \psi\in C_0^\infty(\Omega).
\end{aligned}$$

Since in the present two-dimensional context we have $W_0^{2,2}(\Omega)\hookrightarrow L^\infty(\Omega)$, after some applications of Young's inequality and taking the supremum over all $\psi\in C_0^\infty(\Omega)$ with $\|\psi\|_{W^{2,2}(\Omega)}\le 1$ this shows that with some $C_1>0$ we have

$$\Big\|\partial_t \ln(n_\varepsilon(\cdot,t)+1)\Big\|_{(W_0^{2,2}(\Omega))^\star} \le C_1\cdot\left\{1+\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} + \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |u_\varepsilon|^2\right\} \qquad \text{for all } t>0.$$

On integration in time, in view of the estimates provided by Lemma 3.1, Lemma 2.3 and Lemma 3.3 this readily implies (3.12).

Next, (3.13) can be derived quite similarly: For $\psi \in C_0^\infty(\Omega)$ and $t \in (0,T)$, the second equation in (2.8) along with Lemma 2.3 and (1.6) yields

$$\begin{aligned}\left|\int_\Omega c_{\varepsilon t}(x,t)\psi(x)dx\right| &= \left|-\int_\Omega \nabla c_\varepsilon\cdot\nabla\psi - \int_\Omega n_\varepsilon f(c_\varepsilon)\psi - \int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)\psi\right| \\ &\le \left\{\left(\int_\Omega |\nabla c_\varepsilon|^2\right)^{\frac{1}{2}} + C_2 + \left(\int_\Omega |u_\varepsilon|^2\right)^{\frac{1}{2}}\cdot\left(\int_\Omega |\nabla c_\varepsilon|^2\right)^{\frac{1}{2}}\right\}\times \\ &\quad \times\Big(\|\nabla\psi\|_{L^2(\Omega)} + \|\psi\|_{L^\infty(\Omega)}\Big)\end{aligned}$$

with $C_2 := \|n_0\|_{L^1(\Omega)}\cdot\|f\|_{L^\infty((0,\|c_0\|_{L^\infty(\Omega)}))}$. Again since $W_0^{2,2}(\Omega)\hookrightarrow L^\infty(\Omega)$, using Young's inequality we thus find $C_3>0$ such that

$$\int_0^T \|c_{\varepsilon t}(\cdot,t)\|_{(W_0^{2,2}(\Omega))^\star}dt \le C_3\int_0^T\left\{1+\int_\Omega|\nabla c_\varepsilon|^2+\int_\Omega|u_\varepsilon|^2\right\}dt,$$

which together with Lemma 2.3 and Lemma 3.3 implies (3.13). $\Box$

Collecting compactness properties of $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ implied by the estimates gathered so far, we are now ready to construct a limit object (n,c,u) through a limit process along an appropriate sequence $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$, and to show that the second equation in (1.1) is satisfied in the spirit of Definition 2.1. However, the respective convergence properties of n_ε and ∇c_ε asserted in (3.14) seem yet too weak to allow for a verification of the corresponding weak relations in Definition 2.1 concerning n and u; in particular, (3.14) is apparently even insufficient to warrant conservation of mass for the limit (n,c,u), as required in (2.3).

Lemma 3.5. *There exists $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ such that $\varepsilon_j\searrow 0$ as $j\to\infty$ and*

$$\begin{cases} n_\varepsilon \to n & a.e.\ in\ \Omega\times(0,\infty), \\ \ln(n_\varepsilon+1)\rightharpoonup \ln(n+1) & in\ L^2_{loc}([0,\infty);W^{1,2}(\Omega)), \\ c_\varepsilon\to c & in\ L^2_{loc}(\bar\Omega\times[0,\infty))\ \ and\ a.e.\ in\ \Omega\times(0,\infty), \\ c_\varepsilon(\cdot,t)\to c(\cdot,t) & in\ L^2(\Omega)\quad for\ a.e.\ t>0, \\ c_\varepsilon\rightharpoonup c & in\ L^2_{loc}([0,\infty);W^{1,2}(\Omega)) \qquad and \\ u_\varepsilon\rightharpoonup u & in\ L^2_{loc}(\bar\Omega\times[0,\infty)) \\ & and\ in\ L^p_{loc}([0,\infty);W_0^{1,p}(\Omega))\quad for\ all\ p\in(1,2)\end{cases} \tag{3.14}$$

as $\varepsilon=\varepsilon_j\searrow 0$ with some limit functions n, c and u defined on $\Omega\times(0,\infty)$ and satisfying $n\ge 0$ and $c\ge 0$ as well as $\nabla\cdot u=0$ a.e. in $\Omega\times(0,\infty)$.
Moreover, these limit functions satisfy the identity (2.5) in Definition 2.1 for all test functions from the class indicated there.

Proof. According to Lemma 3.1, Lemma 3.4 and the Aubin-Lions lemma ([30]), the family $(\ln(n_\varepsilon+1))_{\varepsilon\in(0,1)}$ is relatively compact in $L^2_{loc}([0,\infty);W^{1,2}(\Omega))$ with respect to weak convergence,

and relatively compact in $L^2_{loc}(\bar\Omega \times [0,\infty))$ with respect to the strong topology therein. Likewise, Lemma 2.3 combined with Lemma 3.4 warrants relative compactness of $(c_\varepsilon)_{\varepsilon\in(0,1)}$ with respect to the weak topology in $L^2_{loc}([0,\infty); W^{1,2}(\Omega))$, and with respect to the strong topology in $L^2_{loc}(\bar\Omega \times [0,\infty))$. Finally, in light of Lemma 3.3 and Lemma 3.4 we know that $(u_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded, and hence relatively compact with respect to weak convergence, in $L^2_{loc}(\bar\Omega \times [0,\infty))$ and also in $L^p_{loc}([0,\infty); W^{1,p}_0(\Omega))$ for each $p \in (1,2)$, because given any such p, on choosing $\delta > 0$ small enough such that $\delta < \frac{2-p}{2p}$, we can achieve that the expression on the right of (3.4) belongs to $L^p_{loc}([0,\infty))$.
By means of standard extraction procedures, these compactness properties allow for choosing a sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ such that (3.14) holds for some limit triple (n,c,u), where clearly n and c inherit nonnegativity from n_ε and c_ε, and where $\nabla\cdot u = 0$ a.e. in $\Omega\times(0,\infty)$ due to the fact that $\nabla \cdot u_\varepsilon \equiv 0$. Based on (3.14), the verification of the integral identity in (2.5) is now straightforward, so that we may omit giving details here. □

4 Strong precompactness of $(n_{\varepsilon_j})_{j\in\mathbb{N}}$ in L^1 and of $(\nabla c_{\varepsilon_j})_{j\in\mathbb{N}}$ in L^2

4.1 Strong convergence of $(n_{\varepsilon_j})_{j\in\mathbb{N}}$

In order to prepare Lemma 4.2 on strong L^1 convergence of the sequence $(n_{\varepsilon_j})_{j\in\mathbb{N}}$, we draw the following consequence of the spatio-temporal L^2 estimate on the gradient of $\ln(n_\varepsilon+1)$ gained in Lemma 3.1. Being based on the Moser-Trudinger inequality, our argument essentially relies on the fact that the considered space dimension is two.

Lemma 4.1. *Let $p>0$. Then there exists $C = C(p) > 0$ such that for all $\varepsilon\in(0,1)$ we have*

$$\int_0^T \ln\left\{\frac{1}{|\Omega|}\int_\Omega \Big(n_\varepsilon(x,s)+1\Big)^p dx\right\} ds \le C\cdot(T+1) \qquad \textit{for all } T>0. \tag{4.1}$$

Proof. According to the Moser-Trudinger inequality ([6]), we can find $C_1>0$ and $C_2>0$ such that

$$\int_\Omega e^{|\varphi(x)|}dx \le C_1 e^{C_2\|\varphi\|^2_{W^{1,2}(\Omega)}} \qquad \text{for all } \varphi\in W^{1,2}(\Omega), \tag{4.2}$$

and since Ω is bounded, the Poincaré inequality yields $C_3>0$ fulfilling

$$\|\varphi\|^2_{W^{1,2}(\Omega)} \le C_3\cdot\left\{\int_\Omega|\nabla\varphi|^2 + \Big(\int_\Omega|\varphi|\Big)^2\right\} \qquad \text{for all } \varphi\in W^{1,2}(\Omega). \tag{4.3}$$

Therefore, given any $\varepsilon\in(0,1)$ and $t>0$ we can estimate

$$\begin{aligned}
\int_\Omega \Big(n_\varepsilon(\cdot,t)+1\Big)^p dx &= \int_\Omega e^{|p\ln(n_\varepsilon(x,t)+1)|}dx \\
&\le C_1\exp\left\{p^2C_2\|\ln(n_\varepsilon(\cdot,t)+1)\|^2_{W^{1,2}(\Omega)}\right\} \\
&\le C_1\exp\left\{p^2C_2C_3\Big\{\int_\Omega\frac{|\nabla n_\varepsilon(x,t)|^2}{(n_\varepsilon(x,t)+1)^2}dx + \Big(\int_\Omega\ln(n_\varepsilon(x,t)+1)dx\Big)^2\Big\}\right\},
\end{aligned} \tag{4.4}$$

where from (2.9) we know that with $m := \int_\Omega n_0$ we have

$$\int_\Omega \ln(n_\varepsilon(x,t)+1)dx \le \int_\Omega (n_\varepsilon(x,t)+1)dx = m + |\Omega|,$$

because $\ln(\xi+1) \le \xi$ for all $\xi \ge 0$. Hence, (4.4) implies that

$$\begin{aligned}\int_0^T \ln\Big\{\frac{1}{|\Omega|}\int_\Omega \Big(n_\varepsilon(x,s)+1\Big)^p dx\Big\}ds \quad &\le \quad \Big(\ln\frac{C_1}{|\Omega|}\Big)\cdot T + p^2 C_2 C_3 \int_0^T\int_\Omega \frac{|\nabla n_\varepsilon(x,t)|^2}{(n_\varepsilon(x,s)+1)^2}dxds \\ &\quad + p^2 C_2 C_3 \cdot (m+|\Omega|)^2 \cdot T \qquad \text{for all } T>0,\end{aligned}$$

which upon invoking Lemma 3.1 entails (4.1). $\square$

Now by means of the Vitali convergence theorem, the pointwise convergence property of $(n_{\varepsilon_j})_{j\in\mathbb{N}}$ in (3.14) can be combined with Lemma 4.1 so as to yield the desired statement on strong convergence.

Lemma 4.2. *Let $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ be as provided by Lemma 3.5. Then*

$$n_\varepsilon \to n \quad \text{in } L^1_{loc}(\bar\Omega\times[0,\infty)) \qquad \text{as } \varepsilon=\varepsilon_j \searrow 0. \tag{4.5}$$

Proof. Since we already know from Lemma 3.5 that $n_\varepsilon \to n$ a.e. in $\Omega\times(0,\infty)$ as $\varepsilon=\varepsilon_j\searrow 0$, in view of the Vitali convergence theorem it is sufficient to show that for all $T>0$, given any $\eta>0$ we can find $\delta=\delta(\eta,T)>0$ with the property that for measurable $E\subset\Omega\times(0,T)$ we can conclude that

$$\text{whenever } |E|<\delta, \text{ we have } \iint_E n_\varepsilon < \eta. \tag{4.6}$$

To verify this, we let $T>0$ and $\eta>0$ be given and then obtain from Lemma 4.1 that there exists $C_1 = C_1(T)>0$ fulfilling

$$\int_0^T \ln\Big\{\frac{1}{|\Omega|}\int_\Omega \Big(n_\varepsilon(x,s)+1\Big)^3 dx\Big\}ds \le C_1, \tag{4.7}$$

for all $\varepsilon\in(0,1)$, whence it is possible to fix $N=N(T,\eta)>1$ large enough such that with $m:=\int_\Omega n_0$ we have

$$\frac{C_1}{\ln\frac{N}{|\Omega|}}\cdot m < \frac{\eta}{2}. \tag{4.8}$$

For each $\varepsilon\in(0,1)$, we thereupon decompose $(0,T)$ by introducing

$$S_{N,\varepsilon} := \Big\{t\in(0,T)\ \Big|\ \int_\Omega n_\varepsilon^2(x,t)dx \le N\Big\} \qquad \text{and}$$
$$R_{N,\varepsilon} := \Big\{t\in(0,T)\ \Big|\ \int_\Omega n_\varepsilon^2(x,t)dx > N\Big\}.$$

Then indeed $(0,T) = S_{N,\varepsilon} \cup R_{N,\varepsilon}$, and Lemma 4.1 allows us to estimate the size of $R_{N,\varepsilon}$: Namely, by (4.7) we have

$$
\begin{aligned}
C_1 &\ge \int_{R_{N,\varepsilon}} \ln \left\{ \frac{1}{|\Omega|} \int_\Omega \Big(n_\varepsilon(x,t) + 1 \Big)^3 dx \right\} dt \\
&\ge \int_{R_{N,\varepsilon}} \ln \left\{ \frac{1}{|\Omega|} \int_\Omega n_\varepsilon^2(x,t) dx \right\} dt \\
&\ge |R_{N,\varepsilon}| \cdot \ln \frac{N}{|\Omega|}
\end{aligned}
$$

and hence

$$
|R_{N,\varepsilon}| \cdot m \le \frac{C_1}{\ln \frac{N}{|\Omega|}} \cdot m < \frac{\eta}{2} \qquad \text{for all } \varepsilon \in (0,1) \tag{4.9}
$$

by (4.8). We now choose $\mu = \mu(T,\eta) > 0$ small enough satisfying

$$
\sqrt{N} \mu T < \frac{\eta}{4}, \tag{4.10}
$$

and finally fix $\delta = \delta(T,\eta) > 0$ suitably small such that

$$
\frac{\sqrt{N}}{4\mu} \cdot \delta < \frac{\eta}{4}, \tag{4.11}
$$

and to see that then the implication in (4.6) is valid, we suppose that $E \subset \Omega \times (0,T)$ is measurable with $|E| < \delta$, and let $E(t) := \{x \in \Omega \mid (x,t) \in E\}$ for $t \in (0,T)$. Then $E(t)$ is measurable for a.e. $t \in (0,T)$, and for all $\varepsilon \in (0,1)$ we have

$$
\begin{aligned}
\iint_E n_\varepsilon &= \int_0^T \int_{E(t)} n_\varepsilon(x,t) dx dt \\
&= \int_{S_{N,\varepsilon}} \int_{E(t)} n_\varepsilon(x,t) dx dt + \int_{R_{N,\varepsilon}} \int_{E(t)} n_\varepsilon(x,t) dx dt \\
&=: I_1(\varepsilon) + I_2(\varepsilon).
\end{aligned} \tag{4.12}
$$

Here, (4.9) along with (2.9) asserts that

$$
|I_2(\varepsilon)| \le \int_{R_{N,\varepsilon}} \int_\Omega n_\varepsilon(x,t) dx dt = \int_{R_{N,\varepsilon}} m dt = |R_{N,\varepsilon}| \cdot m < \frac{\eta}{2} \qquad \text{for all } \varepsilon \in (0,1). \tag{4.13}
$$

In estimating $I_1(\varepsilon)$, we first use the Cauchy-Schwarz inequality, the definition of $S_{N,\varepsilon}$ and Young's inequality to obtain

$$
\begin{aligned}
|I_1(\varepsilon)| &\le \int_{S_{N,\varepsilon}} \Big(\int_{E(t)} n_\varepsilon^2(x,t)dx\Big)^{\frac{1}{2}} \cdot |E(t)|^{\frac{1}{2}} dt \\
&\le \sqrt{N} \int_{R_{N,\varepsilon}} |E(t)|^{\frac{1}{2}} dt \\
&\le \sqrt{N} \int_{S_{N,\varepsilon}} \Big(\mu + \frac{1}{4\mu}|E(t)|\Big) dt \\
&= \sqrt{N}\mu |S_{N,\varepsilon}| + \frac{\sqrt{N}}{4\mu} \int_{S_{N,\varepsilon}} |E(t)| dt \qquad \text{for all } \varepsilon \in (0,1).
\end{aligned}
$$

Since $|S_{N,\varepsilon}| \le T$ and $\int_{S_{N,\varepsilon}} |E(t)|dt = |E| < \delta$, we may thus apply (4.10) and (4.11) to infer that

$$
|I_1(\varepsilon)| \le \sqrt{N}\mu T + \frac{\sqrt{N}}{4\mu} \cdot \delta < \frac{\eta}{4} + \frac{\eta}{4} = \frac{\eta}{2} \qquad \text{for all } \varepsilon \in (0,1).
$$

Combined with (4.13) and (4.12), this shows (4.6). $\square$

4.2 Strong convergence of $(\nabla c_{\varepsilon_j})_{j\in\mathbb{N}}$

In order to pass to the limit in the cross-diffusive term in the first equation of (2.8), in view of Lemma 3.5 it will be convenient to know that the family $(\nabla c_\varepsilon)_{\varepsilon\in(0,1)}$, by Lemma 2.3 known to be bounded and hence relatively compact in $L^2_{loc}(\bar\Omega \times [0,\infty))$ with respect to weak convergence, is actually strongly precompact in this space. That this indeed is true can be verified by an argument identical to that presented in [42, Section 8] for the fluid-free case $u \equiv 0$. We therefore may confine ourselves to sketching the main ideas here only, the first of which is contained in the following lemma that generalizes Lemma 8.1 in [42] to the present framework.

Lemma 4.3. *There exists a null set $N \subset (0,\infty)$ such that for n, c and u as given by Lemma 3.5 we have the inequality*

$$
\frac{1}{2}\int_\Omega c^2(\cdot,T) - \frac{1}{2}\int_\Omega c_0^2 + \int_0^T \int_\Omega |\nabla c|^2 \ge -\int_0^T \int_\Omega ncf(c) \qquad \textit{for all } T \in (0,\infty) \setminus N. \tag{4.14}
$$

Proof. We define $N := (0,\infty) \setminus L$, where L denotes the set of all Lebesgue points of $(0,\infty) \ni t \mapsto \int_\Omega c^2(x,t)dx,\ t > 0$. Then given $T > 0$, $\delta \in (0,1)$, $h \in (0,1)$ and $k \in \mathbb{N}$, we let

$$
\varphi(x,t) := \zeta_\delta(t) \cdot (A_h \tilde c_k)(x,t), \qquad (x,t) \in \Omega \times (0,\infty),
$$

where ζ_δ denotes the continuous piecewise linear function on $[0,\infty)$ satisfying $\zeta \equiv 1$ on $[0,T]$ and $\zeta \equiv 0$ on $[T+\delta,\infty)$, where

$$
\tilde c_k(x,t) := \begin{cases} c(x,t), & (x,t) \in \Omega \times (0,\infty), \\ c_{0k}(x), & (x,t) \in \Omega \times (-1,0], \end{cases}
$$

for $k \in \mathbb{N}$, with $(c_{0k})_{k\in\mathbb{N}} \subset C^1(\bar\Omega)$ satisfying $c_{0k} \to c_0$ in $L^2(\Omega)$, and where

$$(A_h\tilde c_k)(x,t) := \frac{1}{h}\int_{t-h}^t \tilde c_k(x,s)ds, \qquad (x,t) \in \Omega \times (0,\infty).$$

Then since Lemma 3.5 along with Lemma 2.3 ensures that $c \in L^\infty(\Omega\times(0,\infty)) \cap L^2((0,\infty);W^{1,2}(\Omega))$, it follows that φ has the regularity properties required for (2.5) in Definition 2.1. In view of the solution property of (n,c,u) with regard to the second equation in (1.1), as asserted by Lemma 3.5, we may thus evaluate (2.5) for this particular choice of φ. In one of the resulting integrals containing φ_t, we use Young's inequality to estimate

$$\begin{aligned}\frac{1}{h}\int_0^\infty\int_\Omega \zeta_\delta(t)\tilde c_k(v,t)\tilde c_k(x,t-h)dxdt \ &\le\ \frac{1}{2h}\int_0^\infty\int_\Omega \zeta_\delta(t)\tilde c_k^2(x,t)dxdt \\ &\quad +\frac{1}{2h}\int_0^\infty\int_\Omega \zeta_\delta(t)\tilde c_k^2(x,t-h)dxdt,\end{aligned}$$

and thereupon consecutively take $h \searrow 0$, then $k \to \infty$ and finally $\delta \searrow 0$. Using known properties of the averaging operators A_h with respect to the convergence process $h \searrow 0$, (2.5) thereby implies (4.14); for more details, we may refer to [42, Lemma 8.1]. □

The strong compactness feature in question thereby becomes a consequence of the strong convergence property of $(n_{\varepsilon_j})_{j\in\mathbb{N}}$ asserted by Lemma 4.2.

Lemma 4.4. *With c and $(\varepsilon_j)_{j\in\mathbb{N}}$ as given by Lemma 3.5, we have*

$$\nabla c_\varepsilon \to \nabla c \quad \text{in } L^2(\Omega\times(0,T)) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0. \tag{4.15}$$

PROOF. We take $N \subset (0,\infty)$ from Lemma 4.3 and apply Lemma 3.5 to find that there exists a null set $\tilde N \subset (0,\infty)$ such that $\tilde N \supset N$ and $\int_\Omega c_\varepsilon^2(\cdot,t) \to \int_\Omega c^2(\cdot,T)$ for all $T \in (0,\infty)\setminus\tilde N$. Fixing any such T, we note that as $\varepsilon = \varepsilon_j \searrow 0$, we have $n_\varepsilon \to n$ in $L^1(\Omega\times(0,T))$ by Lemma 4.2, and that clearly $c_\varepsilon f(c_\varepsilon) \overset{\star}{\rightharpoonup} cf(c)$ in $L^\infty(\Omega\times(0,T))$ due to (2.10) and Lemma 3.5. Since consequently $\int_0^T\int_\Omega n_\varepsilon c_\varepsilon f(c_\varepsilon) \to \int_0^T\int_\Omega ncf(c)$ as $\varepsilon=\varepsilon_j\searrow 0$, using Lemma 4.3 and testing the second equation in (2.8) by c_ε we obtain that

$$\begin{aligned}\int_0^T\int_\Omega|\nabla c|^2 \ &\ge\ \lim_{\varepsilon=\varepsilon_j\searrow 0}\Big\{-\frac12\int_\Omega c_\varepsilon^2(\cdot,T)+\frac12\int_\Omega c_0^2-\int_0^T\int_\Omega n_\varepsilon c_\varepsilon f(c_\varepsilon)\Big\} \\ &=\ \lim_{\varepsilon=\varepsilon_j\searrow 0}\int_0^T\int_\Omega|\nabla c_\varepsilon|^2.\end{aligned}$$

Since on the other hand $\int_0^T\int_\Omega|\nabla c|^2 \le \liminf_{\varepsilon=\varepsilon_j\searrow 0}\int_0^T\int_\Omega|\nabla c_\varepsilon|^2$ by Lemma 3.5 and lower semicontinuity of the norm in $L^2(\Omega\times(0,T))$, this implies (4.15) and thereby completes the proof. □

5 Solution properties of (n,c,u). Proof of Theorem 1.1

With the above preparations at hand, we are now in the position to verify (2.3), (2.4) and (2.6), and thereby to complete the proof of the fact that (n,c,u) is a global mass-preserving generalized solution in the sense of Definition 2.1.

Proof of Theorem 1.1. The regularity properties in (2.1) and (2.2) are immediate consequences of Lemma 3.5. Moreover, using that

$$n_\varepsilon \to n \quad \text{in } L^1_{loc}(\bar\Omega \times [0,\infty)) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0 \tag{5.1}$$

and that

$$u_\varepsilon \rightharpoonup u \quad \text{in } L^1_{loc}([0,\infty); W^{1,1}_0(\Omega)) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0 \tag{5.2}$$

by Lemma 4.2 and Lemma 3.5, respectively, we immediately obtain from (2.9) and the fact that $\nabla \cdot u_\varepsilon \equiv 0$ that (2.3) holds, and that $\nabla \cdot u = 0$ a.e. in $\Omega \times (0,\infty)$.
Since the validity of (2.5) has been asserted by Lemma 3.5 already, we are left with the verification of (2.4) and (2.6), where we note that the latter directly results on taking $\varepsilon = \varepsilon_j \searrow 0$ in the third equation in (2.8) and again making use of (5.1) and (5.2).
To derive the inequality in (2.4) for all nonnegative $\varphi \in C_0^\infty(\bar\Omega \times [0,\infty))$ with $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega \times (0,\infty)$, given any such φ we multiply the first equation in (2.8) by $\frac{\varphi}{n_\varepsilon+1}$ to obtain on integrating by parts that

$$\begin{aligned}
\int_0^\infty \int_\Omega |\nabla \ln(n_\varepsilon+1)|^2 \varphi \;=\; & -\int_0^\infty \int_\Omega \ln(n_\varepsilon+1)\varphi_t - \int_\Omega \ln(n_0+1)\varphi(\cdot,0) - \int_0^\infty \int_\Omega \ln(n_\varepsilon+1)\Delta\varphi \\
& + \int_0^\infty \int_\Omega \frac{n_\varepsilon}{n_\varepsilon+1} \nabla \ln(n_\varepsilon+1) \cdot \Big(S(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big)\varphi \\
& - \int_0^\infty \int_\Omega \frac{n_\varepsilon}{n_\varepsilon+1} \Big(S(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big)\cdot\nabla\varphi \\
& - \int_0^\infty \int_\Omega \ln(n_\varepsilon+1)(u_\varepsilon\cdot\nabla\varphi)
\end{aligned} \tag{5.3}$$

for each $\varepsilon \in (0,1)$. Here we note that (5.1) combined with the Lipschitz continuity of $[0,\infty) \ni \xi \mapsto \ln^2(\xi+1)$ entails that $\int_0^T \int_\Omega \ln^2(n_\varepsilon+1) \to \int_0^T \int_\Omega \ln^2(n+1)$, and that according to Lemma 3.5 hence

$$\ln(n_\varepsilon+1) \to \ln(n+1) \quad \text{in } L^2_{loc}(\bar\Omega \times [0,\infty))$$

as $\varepsilon = \varepsilon_j \searrow 0$. Since the pointwise convergence properties in (3.14) combined with the strong convergence in Lemma 4.4 and (1.9) ensure that

$$\frac{n_\varepsilon}{n_\varepsilon+1}\Big(S(\cdot,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big) \to \frac{n}{n+1}\Big(S(\cdot,n,c)\cdot\nabla c\Big) \qquad \text{in } L^2_{loc}(\bar\Omega \times [0,\infty))$$

(cf. [42, Lemma 10.4]), and since Lemma 3.5 moreover warrants that $\nabla(n_\varepsilon+1) \rightharpoonup \nabla(n+1)$ in $L^2_{loc}(\bar\Omega \times [0,\infty))$ and that $u_\varepsilon \rightharpoonup u$ in $L^2_{loc}(\bar\Omega \times [0,\infty))$ as $\varepsilon = \varepsilon_j \searrow 0$, it follows that each of the integrals on the right of (5.3) has the respective limit with $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ replaced by (n,c,u) as $\varepsilon = \varepsilon_j \searrow 0$. Since $\int_0^\infty \int_\Omega |\nabla \ln(n+1)|^2 \le \liminf_{\varepsilon=\varepsilon_j\searrow 0} \int_0^\infty \int_\Omega |\nabla \ln(n_\varepsilon+1)|^2$ by nonnegativity of φ and a standard argument involving lower semicontinuity of norms with respect to weak convergence in Hilbert spaces, this establishes (2.4) and thereby completes the proof. $\square$

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