Stabilization in the logarithmic Keller-Segel system

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Abstract

The Keller-Segel system

$$\begin{cases} u_t &= D\Delta u - D\chi\nabla\cdot\left(\frac{u}{v}\nabla v\right), & x\in\Omega,\ t>0, \\ v_t &= D\Delta v - v + u, & x\in\Omega,\ t>0, \end{cases}$$

is considered in a bounded domain $\Omega\subset\mathbb{R}^n$, $n\ge 2$, with smooth boundary, where $\chi>0$ and $D>0$.

The main results identify a condition on the parameters $\chi<\sqrt{\frac{2}{n}}$ and $D>0$, essentially reducing to the assumption that $\frac{\chi^2}{D}$ be suitably small, under which for all reasonably regular and positive initial data the corresponding classical solution of an associated Neumann initial-boundary value problem, known to exist globally according to previous findings, approaches the homogeneous steady state $(\overline{u}_0,\overline{u}_0)$ at an exponential rate with respect to the norm in $(L^\infty(\Omega))^2$ as $t\to\infty$, where $\overline{u}_0 := \frac{1}{|\Omega|}\int_\Omega u(\cdot,0)$.

As a particular consequence, this entails a convergence statement of the above flavor in the normalized system with $D=1$ and fixed $\chi<\sqrt{\frac{2}{n}}$, provided that Ω satisfies a certain smallness condition.

Key words: chemotaxis; singular sensitivity; large time behavior
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1 Introduction

Among the rich variety of Keller-Segel-type models discussed in the literature, the problem

$$\begin{cases} U_t = \Delta U - \chi\nabla\cdot\left(\frac{U}{V}\nabla V\right), & y\in\Omega_1,\ t>0,\\ V_t = \Delta V - V + U, & y\in\Omega_1,\ t>0,\\ \frac{\partial U}{\partial\nu} = \frac{\partial V}{\partial\nu} = 0, & y\in\partial\Omega_1,\ t>0,\\ U(y,0)=U_0(y), \quad V(y,0)=V_0(y), & y\in\Omega_1, \end{cases} \tag{1.1}$$

with $\chi > 0$ and given nonnegative functions U_0 and V_0 appears to be of particular interest. This on the one hand reflects that in refinement of the classical minimal Keller-Segel system ([19]) in which the evolution of the cell density U is governed by $U_t = \Delta U - \nabla\cdot(U\nabla S(V))$ with linear S, the choice of the *logarithmic sensitivity* $S(V) = \chi \ln V$ in (1.1) with regard to the chemical signal concentration V is well-adapted to situations in which the chemotactic response of cells respects the long-familiar Weber-Fechner law of stimulus perception ([27], [20]). On the other hand, this modification goes along with significant mathematical challenges which are inter alia due to an apparent loss of a treasured gradient structure that has been underlying essential bodies of the analysis in the case of linear sensitivities (see e.g. [26], [16], [8], [33]).

As a consequence, it seems yet widely unknown to which extent the decay of the derivative $S'(V) = \frac{\chi}{V}$ at large signal densities may suppress phenomena of blow-up which constitute the probably most characteristic qualitative feature of original Keller-Segel models with $S' \equiv const.$ in two- or higher-dimensional spatial domains([15], [33]). After all, partial results indicate a substantial dampening effect at least for small values of the factor χ in (1.1), asserting global existence of bounded classical solutions for widely arbitrary positive initial data, and thus ruling out explosions, when Ω is a bounded domain in $\mathbb{R}^n$, $n \ge 2$, and $\chi < \chi_0(n)$ with some $\chi_0(n) > 0$ which is currently known to satisfy $\chi_0(2) > 1.015$ and $\chi_0(n) \ge \sqrt{\frac{2}{n}}$ for $n \ge 3$ ([21], [3], [32], [10]; cf. also [35], [24]). For larger values of χ, nontrivial global solutions have been constructed only in generalized frameworks so far, thus yet allowing solutions to become unbounded even within finite time, but at least excluding any collapse into persistent Dirac-type singularities. For instance, if $\chi < \sqrt{\frac{n+2}{3n-4}}$ then (1.1) is solvable already within quite a natural weak solution concept ([32]), while under the weaker assumption that $\chi < \sqrt{\frac{n}{n-2}}$, in radially symmetric settings global solutions are known to exist in a slightly more generalized framework ([29]). Only recently, within a yet weaker solution concept this assumption could be further relaxed in the sense that merely requiring

$$\chi < \begin{cases} \infty & \text{if } n=2,\\ \sqrt{8} & \text{if } n=3,\\ \frac{n}{n-2} & \text{if } n\ge 4, \end{cases} \tag{1.2}$$

is sufficient to allow for corresponding global solvability, even without any symmetry assumption ([22]). We remark that for parabolic-elliptic simplifications of (1.1) in which the second equation therein is replaced with $0 = \Delta V - V + U$, somewhat more comprehensive results are available, but beyond this furthermore providing inter alia providing some examples of exploding solutions when $n \ge 3$ and

$\chi > \frac{2n}{n-2}$ ([25]), extending the range of χ in (1.2) for generalized solvability when $n = 3$ ([4]), and especially in the two-dimensional case asserting global classical solvability regardless of the size of χ ([12]); a result of the latter flavor could recently even be carried over to the radial version of the fully parabolic variant of (1.1) with its second equation becoming $\tau V_t = \Delta V - V + U$ for suitably small $\tau > 0$ ([13]).

Beyond these fundamental results on global solvability, however, only little seems known about solutions to (1.1); in particular, their qualitative behavior seems widely unaddressed in the literature, with available exceptions concentrating on the associated steady state problem. Indeed, a large variety of highly nontrivial equilibria have been detected since the seminal work [23] in this direction ([6], [1], [7], [5]), but their role, as actually the role of any stationary solution, in the dynamics of the parabolic problem (1.1) seems unclear up to now.

Main results. The intention of this paper is to accept the challenge of describing the large time behavior in systems of the form (1.1) despite lacking knowledge on any meaningful global gradient flow structure therein. In order to achieve nontrivial progress in this direction going beyond straightforward perturbation arguments leading no further than to local stability and attractivity results, unlike the very few precedent energy-independent cases of large-time analysis in chemotaxis problems which exclusively seem to rely on the availability of comparison arguments (see e.g. [28], [36], [30], [34]), we shall pursue a strategy heuristically motivated by the trivial observation that both the numerator and the denominator in the quotient $\frac{U}{V}$ in (1.1) grow linearly with respect to U and V, respectively. Therefore, namely, naively assuming that up to parabolic smoothing V will become conveniently large wherever U attains large values and vice versa, our goal will be to appropriately bound this crucial quotient, possibly only for suitably large times but widely independent of the particular choice of the initial data. This will enable us to conclude that for sufficiently small values of χ, in suitably small domains essentially covering the entire range $\chi \in (0, \sqrt{\frac{2}{n}})$, for arbitrary initial data the cross-diffusive influence becomes conveniently small at least eventually, thereby enforcing essentially diffusion-driven behavior and thus supporting convergence to constant equilibria.

To make this more precise, instead of addressing (1.1) directly we shall for convenience focus our attention to the family of problems which for a fixed bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, are given by

$$\begin{cases} u_t = D\Delta u - D\chi\nabla \cdot \left(\frac{u}{v}\nabla v\right), & x \in \Omega,\ t > 0, \\ v_t = D\Delta v - v + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

and which are indexed by the parameter $D > 0$. As the special case obtained on letting $D = 1$, this family contains the original problem (1.1), to which (1.3) moreover can be seen to be equivalent by means of the transformation

$$u(x,t) := U(y,t), \quad \text{and} \quad v(x,t) := V(y,t) \qquad \text{with} \qquad x = \frac{y}{R} \quad \text{for } t > 0 \tag{1.4}$$

where $u_0(x) = U_0(Rx)$ and $v_0(x) = V_0(Rx)$ as well as

$$D = \frac{1}{R^2} \qquad \text{and} \qquad \Omega = \frac{1}{R} \cdot \Omega_1. \tag{1.5}$$

As for the initial data in (1.3), we shall assume that

$$\begin{cases} u_0 \in C^0(\overline{\Omega}) \text{ is nonnegative with } u_0 \not\equiv 0, \text{ and that} \\ v_0 \in W^{1,\infty}(\Omega) \text{ is positive in } \overline{\Omega}. \end{cases} \tag{1.6}$$

By concretizing the above strategy in the context of this problem (1.3), our main results on qualitative behavior therein assert global asymptotic stability of spatially homogeneous steady states in the case when in addition to the condition $\chi < \sqrt{\frac{2}{n}}$ known to be sufficient for global existence of a bounded classical solution, a further smallness assumption on the ratio $\frac{\chi^2}{D}$ is satisfied. Here and throughout the sequel, for open bounded $G \subset \mathbb{R}^n$ and $\varphi \in L^1(G)$ we abbreviate the spatial mean of the latter by writing $\overline{\varphi} := \frac{1}{|G|} \int_G \varphi$.

Theorem 1.1 *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ a bounded domain with smooth boundary, and let $\chi_0 \in (0, \sqrt{\frac{2}{n}})$ and $D_0 > 0$. Then there exists $\delta > 0$ with the following property: If $\chi \in (0, \chi_0]$, $D \geq D_0$ and*

$$\frac{\chi^2}{D} \leq \delta, \tag{1.7}$$

and if u_0 and v_0 satisfy (1.6), then the problem (1.3) possesses a uniquely determined global classical solution (u, v) with

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)), \\ v \in \bigcap_{\vartheta > n} C^0([0, \infty); W^{1,\vartheta}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)), \end{cases} \tag{1.8}$$

and for this solution one can find $\kappa > 0$ and $C > 0$ such that

$$\|u(\cdot, t) - \overline{u}_0\|_{L^\infty(\Omega)} \leq Ce^{-\kappa t} \qquad \text{for all } t > 0 \tag{1.9}$$

and

$$\|v(\cdot, t) - \overline{u}_0\|_{L^\infty(\Omega)} \leq Ce^{-\kappa t} \qquad \text{for all } t > 0. \tag{1.10}$$

When re-interpreted in the context of the normalized problem (1.1), this immediately entails the following convergence result under assumptions which in suitably small domains actually reduce to the mere requirement that $\chi < \sqrt{\frac{2}{n}}$:

Proposition 1.2 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with smooth boundary, and let $\chi_0 \in (0, \sqrt{\frac{2}{n}})$. Then there exists $\varepsilon > 0$ with the property that if $\chi > 0$ and $R > 0$ are such that $\chi \leq \chi_0$ and*

$$R\chi \leq \varepsilon,$$

then for $\Omega_1 := R\Omega$ and any choice of $0 \not\equiv U_0 \in C^0(\overline{\Omega}_1)$ and $V_0 \in W^{1,\infty}(\Omega_1)$ with $U_0 \geq 0$ and $V_0 > 0$ in $\overline{\Omega}_1$, for the corresponding global classical solution of (1.1) we have

$$\|U(\cdot, t) - \overline{U_0}\|_{L^\infty(\Omega_1)} \leq Ce^{-\kappa t} \qquad \text{for all } t > 0 \tag{1.11}$$

and

$$\|V(\cdot, t) - \overline{U_0}\|_{L^\infty(\Omega_1)} \leq Ce^{-\kappa t} \qquad \text{for all } t > 0 \tag{1.12}$$

with some $\kappa > 0$ and $C > 0$.

When restated with a focus on possible biological implications, the results from both Theorem 1.1 and Proposition 1.2 in summary identify conditions, merely involving the respective system parameters, under which spatial homogeneity in (1.3) and (1.1) is prevalent in the sense of attracting any trajectory emanating from arbitrarily large initial perturbations. This may be viewed as further underlining the fundamental differences between these logarithmic chemotaxis systems and the minimal Keller-Segel system, with its known potential to destabilize any large-mass equilibrium, and thus particularly each homogeneous state at suitably high level, in the drastic flavor involving blow-up.

Plan of the paper. After giving a basic result on global existence and boundedness of classical solutions to (1.3) in Section 2, we shall derive some asymptotic upper bounds, inter alia with respect to the norms in $L^\infty(\Omega)$, on u and $\frac{1}{v}$ in terms of $\int_\Omega u_0$ in Section 3. This will be achieved by refining knowledge on a basically well-known quasi-energy property which, under the condition $\chi < \sqrt{\frac{2}{n}}$, functionals of the form $\int_\Omega u^p v^{-r}$ have been found to enjoy for some $p > \frac{n}{2}$ when the parameter r is chosen appropriately. In contrast to a previous discovery of this feature in [32], however, our focus will here be on the dependence of correspondingly obtained bounds on the mass functional $\int_\Omega u_0$, and as the main fruit thereof we shall obtain that indeed u can asymptotically be controlled from above by a multiple of $\int_\Omega u_0$ (Lemma 3.6). As a consequence, Section 4 will reveal a key feature of (1.3), expressed in the existence of a universal constant $C > 0$, in particular independent not only of the initial data but also widely independent of χ and D, such that

$$\limsup_{t\to\infty} \left\| \frac{u(\cdot,t)}{v(\cdot,t)} \right\|_{L^\infty(\Omega)} \le C. \tag{1.13}$$

Thereby having at hand a convenient control over the cross-diffusive flux in (1.3), in Section 5 we build our main results on stabilization on the observation that thanks to (1.13), a smallness condition of the form (1.7) warrants the existence of $k > 0$ such that $\int_\Omega (u-\overline{u}_0)^2 + k\int_\Omega (v-\overline{u}_0)^2$ eventually plays the role of a genuine Lyapunov functional along each fixed trajectory, and thereby implies stabilization.

2 Global existence of bounded solutions for $D > 0$ when $\chi < \sqrt{\frac{2}{n}}$

To begin with, let us briefly recall the following from the literature.

Lemma 2.1 *Let $n \ge 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and suppose that $\chi \in (0, \sqrt{\frac{2}{n}})$ and $D > 0$. Then for all u_0 and v_0 fulfilling (1.6), the problem (1.3) admits a global classical solution (u,v), uniquely determined by the inclusions in (1.8), for which $u > 0$ in $\overline{\Omega}\times(0,\infty)$ and $v > 0$ in $\overline{\Omega}\times[0,\infty)$. Moreover, this solution is bounded in the sense that*

$$u \in L^\infty(\Omega\times(0,\infty)) \qquad \textit{and} \qquad v \in L^\infty(\Omega\times(0,\infty)),$$

and moreover

$$\int_\Omega u(\cdot,t) = \int_\Omega u_0 \qquad \textit{for all } t > 0. \tag{2.1}$$

Proof. After a reduction to (1.1) via (1.4) and (1.5), the statements on global existence, boundedness and positivity are precisely covered by corresponding knowledge on (1.1) within the indicated range of χ ([32], [10], [11]). Uniqueness in the class determined by (1.8) can be obtained by following a standard reasoning ([17]), whereas the mass conservation property (2.1) immediately results from (1.1). $\Box$

3 Asymptotic upper bounds on u and $\frac{1}{v}$ in terms of $\int_\Omega u_0$

The principal goal of this section is to prepare the crucial Lemma 4.1 below by providing some information on how the sizes of u and $\frac{1}{v}$ in $L^\infty(\Omega)$, when viewed in the long term, depend on the initial data. Constituting the main outcome in this direction, it will turn out that both these quantities can asymptotically be estimated by means of certain expressions only involving $\int_\Omega u_0$, provided that χ remains suitably below $\sqrt{\frac{2}{n}}$ and D lies significantly above the critical value $D = 0$.

Whereas the corresponding statement on u will be established in Lemma 3.6, the reciprocal of v can be dealt with in quite a straightforward manner by means of an argument based on positivity properties of the Neumann heat semigroup $(e^{\tau\Delta})_{\tau\ge0}$ over Ω.

Lemma 3.1 *Let $D_0 > 0$. Then there exists $K_1 = K_1(D_0) > 0$ such that whenever (u, v) is a global classical solution of (1.3) for some $\chi > 0$ and $D \ge D_0$ and some (u_0, v_0) fulfilling (1.6), the inequality*

$$\liminf_{t\to\infty} \inf_{x\in\Omega} v(x,t) \ge K_1 \int_\Omega u_0 \qquad (3.1)$$

holds.

Proof. Letting $k = k(x, y, \tau)$ denote the heat kernel of the Neumann Laplacian, thanks to the positivity of k on $\overline{\Omega} \times \overline{\Omega} \times [1,\infty)$ (see e.g., [18, Theorem 10.1]) we can pick $c_1 > 0$ such that for all nonnegative $\varphi \in C^0(\overline{\Omega})$,

$$(e^{\tau\Delta}\varphi)(x) = \int_\Omega k(x,y,\tau)\varphi(y)dy \ge c_1 \int_\Omega \varphi(y)dy \qquad \text{for all } x \in \overline{\Omega} \text{ and any } \tau \ge 1$$

(see e.g. [18, Theorem 10.1]; when Ω is convex, and explicit estimate for c_1 can be found in [11, Lemma 2.4]). Since v_0 is nonnegative, according to (2.1) and a variation-of-constants representation associated with the second equation in (1.3) we thus have

$$\begin{aligned} v(\cdot,t) &= e^{-t}e^{tD\Delta}v_0 + \int_0^t e^{-(t-s)}e^{(t-s)D\Delta}u(\cdot,s)ds \\ &\ge \int_0^{t-\frac{1}{D_0}} e^{-(t-s)} \cdot c_1\Big\{\int_\Omega u_0\Big\}ds \\ &= c_1\Big\{\int_\Omega u_0\Big\} \cdot \int_0^{t-\frac{1}{D_0}} e^{-(t-s)}ds \qquad \text{for all } t \ge \frac{1}{D_0}, \end{aligned} \qquad (3.2)$$

because $D \ge D_0$ and therefore $(t-s)D \ge (t-s)D_0 \ge 1$ for $0 \le s \le t - \frac{1}{D_0}$. Here, if even $t \ge \frac{2}{D_0}$, then

$$\int_0^{t-\frac{1}{D_0}} e^{-(t-s)}ds = e^{-\frac{1}{D_0}} - e^{-t} \ge c_2 := e^{-\frac{1}{D_0}} - e^{-\frac{2}{D_0}},$$

so that (3.2) implies the inequality

$$v(\cdot,t) \geq c_1c_2 \int_\Omega u_0 \quad \text{in } \Omega \qquad \text{for all } t \geq \frac{2}{D_0}$$

and thereby establishes (3.1) with $K_1 := c_1c_2$ being independent of $D \geq D_0$ and the particular solution in question. $\Box$

In deriving an asymptotic bound for the quantity $\|u\|_{L^\infty(\Omega)}$, we will inter alia need to appropriately control the cross-diffusive flux in (1.3). A first basic information on this is contained in the following.

Lemma 3.2 *Let $D_0 \geq 0, p \geq 1$ and $q \geq 1$ be such that $q < \frac{np}{(n-p)_+}$. Then one can find $K_2 = K_2(D_0,p,q) > 0$ such that if (u,v) is a global classical solution of (1.3) for some $\chi > 0$, $D \geq D_0$ and (u_0,v_0) satisfying (1.6), then*

$$\limsup_{t\to\infty} \|v(\cdot,t)\|_{W^{1,q}(\Omega)} \leq K_2 \cdot \limsup_{t\to\infty} \|u(\cdot,t)\|_{L^p(\Omega)}. \tag{3.3}$$

Proof. According to a known smoothing property of the Neumann heat semigroup on Ω ([31]), there exists $c_1 > 0$ such that for all $\varphi \in C^0(\overline{\Omega})$,

$$\|e^{\tau\Delta}\varphi\|_{W^{1,q}(\Omega)} \leq c_1(1+\tau^{-\gamma})\|\varphi\|_{L^p(\Omega)} \qquad \text{for all } \tau > 0, \tag{3.4}$$

with $\gamma := \frac{1}{2} + \frac{n}{2}(\frac{1}{p} - \frac{1}{q})$ satisfying $\gamma < 1$ due to our assumption that $q < \frac{np}{(n-p)_+}$. Therefore, $c_2 := \int_0^\infty (1+D_0^{-\gamma}\sigma^{-\gamma})e^{-\sigma}d\sigma$ is finite, and we claim that (3.3) holds if we let $K_2 := 2c_1c_2$.
To see this, given a solution with the indicated properties we can pick $t_0 > 0$ such that

$$\|u(\cdot,t)\|_{L^p(\Omega)} \leq 2L \qquad \text{for all } t > t_0, \tag{3.5}$$

where $L := \limsup_{t\to\infty} \|u(\cdot,t)\|_{L^p(\Omega)}$ is finite and positive due to Lemma 2.1. Now by means of a Duhamel formula for v, twice using (3.4) we can estimate

$$\begin{aligned}
\|v(\cdot,t)\|_{W^{1,q}(\Omega)} &= \left\| e^{-t}e^{tD\Delta}v_0 + \int_0^t e^{-(t-s)}e^{(t-s)D\Delta}u(\cdot,s)ds \right\|_{W^{1,q}(\Omega)} \\
&\leq c_1 \cdot \Big(1+[Dt]^{-\gamma}\Big) \cdot e^{-t}\|v_0\|_{L^p(\Omega)} \\
&\quad + c_1 \int_0^t \Big(1+[D(t-s)]^{-\gamma}\Big) e^{-(t-s)}\|u(\cdot,s)\|_{L^p(\Omega)}ds \\
&= c_1 \cdot (1+D^{-\gamma}t^{-\gamma})e^{-t}\|v_0\|_{L^p(\Omega)} \\
&\quad + c_1 \int_0^t \Big(1+D^{-\gamma}(t-s)^{-\gamma}\Big) e^{-(t-s)}\|u(\cdot,s)\|_{L^p(\Omega)}ds \qquad \text{for all } t > 0,
\end{aligned} \tag{3.6}$$

where clearly

$$c_1(1+D^{-\gamma}t^{-\gamma})e^{-t}\|v_0\|_{L^p(\Omega)} \to 0 \qquad \text{as } t \to \infty. \tag{3.7}$$

Moreover, for $t \geq t_0 + 1$ we have

$$c_1 \int_0^{t_0} \Big(1+D^{-\gamma}(t-s)^{-\gamma}\Big) e^{-(t-s)}\|u(\cdot,s)\|_{L^p(\Omega)}ds \leq c_1(1+D_0^{-\gamma})e^{-t} \int_0^{t_0} e^s\|u(\cdot,s)\|_{L^p(\Omega)}ds$$

and hence also

$$c_1 \int_0^{t_0} \Big(1 + D_0^{-\gamma}(t-s)^{-\gamma}\Big) e^{-(t-s)} \|u(\cdot,s)\|_{L^p(\Omega)} ds \to 0 \qquad \text{as } t \to \infty, \tag{3.8}$$

while finally the corresponding remaining part of the last summand in (3.6) can be controlled by recalling (3.5) and the definition of c_2 according to

$$\begin{aligned} c_1 \int_{t_0}^t \Big(1 + D^{-\gamma}(t-s)^{-\gamma}\Big) e^{-(t-s)} \|u(\cdot,s)\|_{L^p(\Omega)} ds &\le 2c_1 L \int_{t_0}^t \Big(1 + D_0^{-\gamma}(t-s)^{-\gamma}\Big) e^{-(t-s)} ds \\ &= 2c_1 L \int_0^{t-t_0} (1 + D_0^{-\gamma}\sigma^{-\gamma}) e^{-\sigma} d\sigma \\ &\le 2c_1 L \cdot c_2 \qquad \text{for all } t > t_0. \end{aligned}$$

Together with (3.7), (3.8) and (3.6), this shows that

$$\limsup_{t\to\infty} \|v(\cdot,t)\|_{W^{1,q}(\Omega)} \le 2c_1 c_2 L$$

and hence indeed implies (3.3) with the above choice of K_2. $\square$

Now in order to bound the expression appearing on the right-hand side of (3.3), we re-inspect an essentially well-known energy-like property enjoyed by functionals of the form $\int_\Omega u^p v^{-r}$ with some p above the crucial value $\frac{n}{2}$ whenever $\chi < \sqrt{\frac{2}{n}}$ ([32]). Additionally relying on the lower bound from Lemma 3.1, we will thereby achieve an asymptotic upper bound for such functionals which again only involves the mass functional $\int_\Omega u_0$.

Lemma 3.3 *Let $\chi_0 < \sqrt{\frac{2}{n}}$ and $D_0 > 0$. Then there exist $p_0 = p_0(\chi_0) \in (\frac{n}{2}, n)$, $r_0 = r_0(\chi_0) \in (0, p_0)$ and $K_3 = K_3(\chi_0, D_0) > 0$ with the property that if for some $\chi \le \chi_0$, $D \ge D_0$ and (u_0, v_0) fulfilling (1.6) we are given a global classical solution (u, v) of (1.3), then*

$$\limsup_{t\to\infty} \int_\Omega u^{p_0}(\cdot,t) v^{-r_0}(\cdot,t) \le K_3 \cdot \bigg\{ \int_\Omega u_0 \bigg\}^{p_0 - r_0}. \tag{3.9}$$

Proof. Since $\chi_0 < \sqrt{\frac{2}{n}}$, we can fix $p \equiv p_0(\chi_0) \in (\frac{n}{2}, n)$ such that $p\chi_0^2 < 1$, which can readily be seen to ensure that if we let $r \equiv r_0(\chi_0) := \frac{p-1}{2}$, then $r \in (0,p)$ and

$$\frac{[2pr + p(p-1)\chi]^2}{4p(p-1)} < pr\chi + r(r+1) \qquad \text{for all } \chi \le \chi_0.$$

It is therefore possible to find $c_1 > 0$ such that $c_1 < 4p(p-1)$ but such that still

$$c_2 := \inf_{\chi \le \chi_0} \bigg\{ pr\chi + r(r+1) - \frac{[2pr + p(p-1)\chi]^2}{4p(p-1) - c_1} \bigg\} \tag{3.10}$$

is positive. We next write

$$c_3 := \min \bigg\{ \frac{c_1 D_0}{2p^2}, \frac{2c_2 D_0}{r^2} \bigg\}$$

and combine the Gagliardo–Nirenberg inequality with Young's inequality in a standard manner so as to find $c_4 > 0$ fulfilling

$$(r+1)\|\varphi\|^2_{L^2(\Omega)} \le c_3\|\nabla\varphi\|^2_{L^2(\Omega)} + c_4\|\varphi\|^2_{L^{\frac{2}{p}}(\Omega)} \qquad \text{for all } \varphi \in W^{1,2}(\Omega). \tag{3.11}$$

Now assuming (u,v) to be a global classical solution under the indicated circumstances, following [32] we compute

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^p v^{-r} &= -p(p-1)D\int_\Omega u^{p-2}v^{-r}|\nabla u|^2 + [2pr + p(p-1)\chi]D\int_\Omega u^{p-1}v^{-r-1}\nabla u\cdot\nabla v \\ &\quad -[pr\chi + r(r+1)]D\int_\Omega u^p v^{-r-2}|\nabla v|^2 \\ &\quad +r\int_\Omega u^p v^{-r} - r\int_\Omega u^{p+1}v^{-r-1} \qquad \text{for all } t>0, \end{aligned} \tag{3.12}$$

where by Young's inequality and our choice of c_1,

$$\begin{aligned} [2pr + p(p-1)\chi]D\int_\Omega u^{p-1}v^{-r-1}\nabla u\cdot\nabla v &\le \frac{1}{4}[4p(p-1)-c_1]D\int_\Omega u^{p-2}v^{-r}|\nabla u|^2 \\ &\quad + \frac{[2pr+p(p-1)\chi]^2}{4p(p-1)-c_1}D\int_\Omega u^p v^{-r-2}|\nabla v|^2 \qquad \text{for all } t>0. \end{aligned}$$

In view of (3.10) and the nonnegativity of the rightmost summand in (3.12), in view of our definition of c_2 we thus obtain that

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^p v^{-r} + \int_\Omega u^p v^{-r} &\le -\frac{c_1 D_0}{4}\int_\Omega u^{p-2}v^{-r}|\nabla u|^2 - c_2 D_0\int_\Omega u^p v^{-r-2}|\nabla v|^2 \\ &\quad +(r+1)\int_\Omega u^p v^{-r} \qquad \text{for all } t>0, \end{aligned} \tag{3.13}$$

because $\chi \le \chi_0$ and $D \ge D_0$. Here thanks to (3.11),

$$(r+1)\int_\Omega u^p v^{-r} \le c_3\int_\Omega \Big|\nabla(u^{\frac{p}{2}}v^{-\frac{r}{2}})\Big|^2 + c_4\cdot\Big\{\int_\Omega uv^{-\frac{r}{p}}\Big\}^p \qquad \text{for all } t>0, \tag{3.14}$$

where again by Young's inequality, the definition of c_3 warrants that

$$\begin{aligned} c_3\int_\Omega\Big|\nabla(u^{\frac{p}{2}}v^{-\frac{r}{2}})\Big|^2 &= c_3\int_\Omega\Big|\frac{p}{2}u^{\frac{p-2}{2}}v^{-\frac{r}{2}}\nabla u - \frac{r}{2}u^{\frac{p}{2}}v^{-\frac{r+2}{2}}\nabla v\Big|^2 \\ &\le \frac{c_3p^2}{2}\int_\Omega u^{p-2}v^{-r}|\nabla u|^2 + \frac{c_3r^2}{2}\int_\Omega u^p v^{-r-2}|\nabla v|^2 \\ &\le \frac{c_1D_0}{4}\int_\Omega u^{p-2}v^{-r}|\nabla u|^2 + c_2D_0\int_\Omega u^p v^{-r-2}|\nabla v|^2 \qquad \text{for all } t>0, \end{aligned} \tag{3.15}$$

and where due to (2.1),

$$\begin{aligned} c_4\cdot\Big\{\int_\Omega uv^{-\frac{r}{p}}\Big\}^p &\le c_4\Big\{\int_\Omega u\Big\}^p\Big\|\frac{1}{v}\Big\|^r_{L^\infty(\Omega)} \\ &= c_4\Big\{\int_\Omega u_0\Big\}^p\Big\|\frac{1}{v}\Big\|^r_{L^\infty(\Omega)} \qquad \text{for all } t>0. \end{aligned} \tag{3.16}$$

To further estimate this, we recall from Lemma 3.1 that there exists $t_0 > 0$ such that with $K_1 = K_1(D_0) > 0$ taken from (3.1) we have

$$\Big\|\frac{1}{v}\Big\|_{L^\infty(\Omega)} \le \frac{2}{K_1\int_\Omega u_0} \qquad \text{for all } t > t_0,$$

whence combining (3.13)-(3.16) shows that $y(t) := \int_\Omega u^p(\cdot,t)v^{-r}(\cdot,t)$, $t \ge t_0$, satisfies

$$y'(t) + y(t) \le \frac{2^r c_4}{K_1^r} \cdot \Big\{\int_\Omega u_0\Big\}^{p-r} \qquad \text{for all } t > t_0$$

and thus, by integration,

$$y(t) \le y(t_0)e^{-(t-t_0)} + \frac{2^r c_4}{K_1^r} \cdot \Big\{\int_\Omega u_0\Big\}^{p-r} \cdot \int_{t_0}^t e^{-(t-s)}ds \qquad \text{for all } t > t_0.$$

Since $\int_{t_0}^t e^{-(t-s)}ds \le 1$ for all $t > t_0$, this entails that

$$\limsup_{t\to\infty} y(t) \le \frac{2^r c_4}{K_1^r} \cdot \Big\{\int_\Omega u_0\Big\}^{p-r}$$

and thereby completes the proof, because both c_4 and K_1 are independent of the particular choices of $\chi \le \chi_0, D \ge D_0$ and the solution (u, v). $\Box$

By means of a simple argument based on the Hölder inequality and Lemma 3.2, the latter entails the following.

Lemma 3.4 *For all $\chi_0 < \sqrt{\frac{2}{n}}$ and $D_0 > 0$ one can find $p = p(\chi_0) \in (\frac{n}{2}, n)$ and $K_4 = K_4(\chi_0, D_0) > 0$ such that whenever (u, v) is a global classical solution of (1.3) with some $\chi \le \chi_0, D \ge D_0$ and (u_0, v_0) compatible with (1.6), we have*

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^p(\Omega)} \le K_4 \int_\Omega u_0. \tag{3.17}$$

Proof. Using that $\chi_0 < \sqrt{\frac{2}{n}}$, in accordance with Lemma 3.3 we first fix $p_0 = p_0(\chi_0) \in (\frac{n}{2}, n)$, $r_0 = r_0(\chi_0) \in (0, p_0)$ and $K_3 = K_3(\chi_0, D_0)$ with the properties listed there, and next pick any $p \equiv p(\chi_0) \in (\frac{n}{2}, n)$ such that $p < p_0$. We then observe that for $q := \frac{npr_0}{pr_0+n(p_0-p)}$ we have $W^{1,q}(\Omega) \hookrightarrow L^{\frac{pr_0}{p_0-p}}(\Omega)$, so that there exists $c_1 > 0$ such that

$$\|\varphi\|_{L^{\frac{pr_0}{p_0-p}}(\Omega)} \le c_1\|\varphi\|_{W^{1,q}(\Omega)} \qquad \text{for all } \varphi \in W^{1,q}(\Omega). \tag{3.18}$$

Moreover, noting that

$$\frac{(n-p)q}{np} = \frac{(n-p)r_0}{pr_0 + n(p_0-p)} < \frac{(n-\frac{n}{2})r_0}{\frac{n}{2}r_0 + n(p_0-p)} < 1$$

and hence $q < \frac{np}{(n-p)_+}$, we may fix $K_2 = K_2(D_0, p, q) > 0$ as provided by Lemma 3.2.

We now let $\chi \le \chi_0, D \ge D_0$ and (u_0, v_0) be given such that (1.6) holds, and note that then due to Lemma 2.1, again by our assumption that $\chi_0 < \sqrt{\frac{2}{n}}$, the associated solution (u,v) is bounded and hence $L := \limsup_{t\to\infty} \|u(\cdot,t)\|_{L^p(\Omega)}$ finite. In order to quantitatively control the size of L, we use the Hölder inequality in estimating

$$\begin{aligned} \int_\Omega u^p &= \int_\Omega \Big(u^{p_0} v^{-r_0}\Big)^{\frac{p}{p_0}} \cdot v^{\frac{pr_0}{p_0}} \\ &\le \Big\{ \int_\Omega u^{p_0} v^{-r_0} \Big\}^{\frac{p}{p_0}} \cdot \Big\{ \int_\Omega v^{\frac{pr_0}{p_0-p}} \Big\}^{\frac{p_0-p}{p_0}} \qquad \text{for all } t>0, \end{aligned}$$

so that according to Lemma 3.3 and Lemma 3.2,

$$\begin{aligned} L &\le \limsup_{t\to\infty} \Big\{ \int_\Omega u^{p_0} v^{-r_0} \Big\}^{\frac{1}{p_0}} \cdot \limsup_{t\to\infty} \|v\|^{\frac{r_0}{p_0}}_{L^{\frac{pr_0}{p_0-p}}(\Omega)} \\ &\le K_3^{\frac{1}{p_0}} \cdot \Big\{ \int_\Omega u_0 \Big\}^{\frac{p_0-r_0}{p_0}} \cdot c_1^{\frac{r_0}{p_0}} \cdot \limsup_{t\to\infty} \|v\|^{\frac{r_0}{p_0}}_{W^{1,q}(\Omega)} \\ &\le K_3^{\frac{1}{p_0}} \cdot \Big\{ \int_\Omega u_0 \Big\}^{\frac{p_0-r_0}{p_0}} \cdot c_1^{\frac{r_0}{p_0}} \cdot K_2^{\frac{r_0}{p_0}} L^{\frac{r_0}{p_0}}. \end{aligned}$$

Therefore,

$$L^{1-\frac{r_0}{p_0}} \le (c_1^{r_0} K_2^{r_0} K_3)^{\frac{1}{p_0}} \cdot \Big\{ \int_\Omega u_0 \Big\}^{\frac{p_0-r_0}{p_0}},$$

whence relying on the inequality $r_0 < p_0$ we readily end up with (3.17) if we let $K_4 := (c_1^{r_0} K_2^{r_0} K_3)^{\frac{1}{p_0-r_0}}$, for instance. $\square$

In turning this into an estimate for u with respect to a norm even somewhat stronger than that in $L^\infty(\Omega)$, we shall make use of an auxiliary statement on an integral which in a natural manner arises in the analysis of a Duhamel formula associated with (1.3).

Lemma 3.5 *Let $\beta \in (0,1), \gamma \in (0,1)$, $\lambda > 0$ and $a \in [0,1]$. Then there exists $L(\beta,\gamma,\lambda,a) > 0$ such that for all $D > 0$,*

$$D \int_{t_0}^t [D(t-s)]^{-\gamma} e^{-D\lambda(t-s)} \Big(1+[D(s-t_0)]^{-\beta}\Big)^a ds \le L(\beta,\gamma,\lambda,a) \cdot \Big(1+D^{-\beta}(t-t_0)^{-\beta}\Big) \qquad \text{for all } t > t_0. \tag{3.19}$$

PROOF. We first use the evident fact that $(1+\xi)^a \le 1+\xi^a$ for all $\xi \ge 0$ to see that for all $t > t_0$,

$$\begin{aligned} D \int_{t_0}^t [D(t-s)]^{-\gamma} e^{-D\lambda(t-s)} \Big(1+[D(s-t_0)]^{-\beta}\Big)^a ds &\le D^{1-\gamma} \int_{t_0}^t (t-s)^{-\gamma} e^{-D\lambda(t-s)} ds \\ &\quad + D^{1-\gamma-a\beta} \int_{t_0}^t (t-s)^{-\gamma} e^{-D\lambda(t-s)} (s-t_0)^{-a\beta} ds \end{aligned} \tag{3.20}$$

where clearly

$$D^{1-\gamma}\int_{t_0}^t (t-s)^{-\gamma}e^{-D\lambda(t-s)}ds = \int_0^{D(t-t_0)}\sigma^{-\gamma}e^{-\lambda\sigma}d\sigma \le c_1 := \int_0^\infty \sigma^{-\gamma}e^{-\lambda\sigma}d\sigma, \qquad (3.21)$$

and where in the case $t \ge t_0 + \frac{2}{D}$,

$$\begin{aligned}
D^{1-\gamma-a\beta}&\int_{t_0}^t (t-s)^{-\gamma}e^{-D\lambda(t-s)}(s-t_0)^{-a\beta}ds \\
&= D^{1-\gamma-a\beta}\int_{t_0}^{t_0+\frac{1}{D}}(t-s)^{-\gamma}e^{-D\lambda(t-s)}(s-t_0)^{-a\beta}ds \\
&\quad + D^{1-\gamma-a\beta}\int_{t_0+\frac{1}{D}}^{t}(t-s)^{-\gamma}e^{-D\lambda(t-s)}(s-t_0)^{-a\beta}ds \\
&\le D^{1-a\beta}\int_{t_0}^{t_0+\frac{1}{D}}(s-t_0)^{-a\beta}ds + D^{1-\gamma}\int_{t_0+\frac{1}{D}}^{t}(t-s)^{-\gamma}e^{-D\lambda(t-s)}ds \\
&= \frac{1}{1-a\beta} + \int_0^{D(t-t_0)-1}\sigma^{-\gamma}e^{-\lambda\sigma}d\sigma \\
&\le c_2 := \frac{1}{1-a\beta} + c_1. \qquad (3.22)
\end{aligned}$$

Moreover, for $t \in (t_0, t_0+\frac{2}{D})$ we have

$$\begin{aligned}
D^{1-\gamma-a\beta}\int_{t_0}^t (t-s)^{-\gamma}e^{-D\lambda(t-s)}(s-t_0)^{-a\beta}ds &\le D^{1-\gamma-a\beta}\int_{t_0}^t (t-s)^{-\gamma}(s-t_0)^{-a\beta}ds \\
&= c_3 D^{1-\gamma-a\beta}(t-t_0)^{1-\gamma-a\beta} \qquad (3.23)
\end{aligned}$$

with $c_3 := \int_0^1 (1-\sigma)^{-\gamma}\sigma^{-a\beta}d\sigma$, where

$$\begin{aligned}
\frac{c_3 D^{1-\gamma-a\beta}(t-t_0)^{1-\gamma-a\beta}}{D^{-\beta}(t-t_0)^{-\beta}} &= c_3 D^{1-\gamma+(1-a)\beta}(t-t_0)^{1-\gamma+(1-a)\beta} \\
&\le c_4 := c_3 \cdot 2^{1-\gamma+(1-a)\beta} \qquad \text{for all } t \in \Big(t_0, t_0+\frac{2}{D}\Big)
\end{aligned}$$

due to the fact that $1-\gamma+(1-a)\beta$ is nonnegative. In consequence of (3.20)-(3.23), we therefore obtain that

$$D\int_{t_0}^t [D(t-s)]^{-\gamma}e^{-D\lambda(t-s)}\Big(1+[D(s-t_0)]^{-\beta}\Big)^a ds \le c_1 + c_2 + c_4 D^{-\beta}(t-t_0)^{-\beta} \qquad \text{for all } t > t_0,$$

from which (3.19) directly results upon an evident choice of $L(\beta,\gamma,\lambda,a)$. □

We are now in the position to establish an asymptotic estimate for u in $L^\infty(\Omega)$ and thereby complete our provisions for a universal bound on $\frac{u}{v}$ to be formulated in Lemma 4.1 below. In order to simultaneously prepare an interpolation argument pursued in the course of our final convergence proof in Section 5.1, we shall derive a corresponding estimate even a space yet slightly smaller than $L^\infty(\Omega)$.

Lemma 3.6 *Let $\mu \in (0,\lambda_1)$, where $\lambda_1 > 0$ denotes the first nonzero eigenvalue of the Neumann Laplacian in Ω. Then for all $\chi_0 < \sqrt{\frac{2}{n}}$ and $D_0 > 0$ there exist $r = r(\chi_0) > n$, $\alpha = \alpha(\chi_0) \in (\frac{n}{2r}, 1)$ and $K_5 = K_5(\chi_0, D_0) > 0$ such that if $\chi \le \chi_0, D \ge D_0$ and (u_0, v_0) satisfies (1.6), then the global classical solution (u,v) of (1.3) satisfies*

$$\limsup_{t\to\infty} \|A^\alpha u(\cdot,t)\|_{L^r(\Omega)} \le K_5 \int_\Omega u_0, \tag{3.24}$$

where A denotes the sectorial realization of $-\Delta + \mu$ in $L^r(\Omega)$ under homogeneous Neumann boundary conditions.

Proof. Given $\chi_0 < \sqrt{\frac{2}{n}}$ and $D_0 > 0$, we take $K_1 = K_1(D_0)$ from Lemma 3.1 and let $p = p(\chi_0) \in (\frac{n}{2}, n)$ and $K_4 = K_4(\chi_0, D_0) > 0$ be as provided by Lemma 3.4, and using that $p > \frac{n}{2}$ we can fix $q > n$ such that $q < \frac{np}{(n-p)_+}$. In particular, the latter inequality enables us to take $K_2 = K_2(D_0, p, q)$ as given by Lemma 3.2, whereas due to the former we can pick $r \equiv r(\chi_0) > n$ fulfilling $r < q$. As thus $\frac{n}{2r} < \min\{1 - \frac{n}{2}(\frac{1}{p} - \frac{1}{r}), \frac{1}{2}\}$ due to the fact that $p > \frac{n}{2}$, we can find a number $\alpha \equiv \alpha(\chi_0)$ such that $\alpha > \frac{n}{2r}$ and $\alpha < 1 - \frac{n}{2}(\frac{1}{p} - \frac{1}{r})$ as well as $\alpha < \frac{1}{2}$. Therefore, the corresponding fractional power A^α of the operator A introduced above has the property that its domain satisfies $D(A^\alpha) \hookrightarrow L^\infty(\Omega)$ ([14]), whence we can fix $c_1 > 0$ such that

$$\|\varphi\|_{L^\infty(\Omega)} \le c_1 \|A^\alpha \varphi\|_{L^r(\Omega)} \qquad \text{for all } \varphi \in D(A^\alpha). \tag{3.25}$$

From known smoothing properties of the Neumann heat semigroup $(e^{\tau\Delta})_{\tau\ge0} \equiv (e^{-\tau(A-\mu)})_{\tau\ge0}$ and our choice of μ ([31]) we moreover obtain positive constants λ, c_2 and c_3 such that

$$\|A^\alpha e^{\tau\Delta}\varphi\|_{L^r(\Omega)} \le c_2(1+\tau^{-\beta})\|\varphi\|_{L^p(\Omega)} \qquad \text{for all } \tau > 0 \text{ and any } \varphi \in L^p(\Omega) \tag{3.26}$$

as well as

$$\|A^\alpha e^{\tau\Delta}\nabla\cdot\varphi\|_{L^r(\Omega)} \le c_3\tau^{-\gamma}e^{-\lambda\tau}\|\varphi\|_{L^r(\Omega)} \quad \text{for all } \tau > 0 \text{ and each } \varphi \in C^1(\overline{\Omega}) \text{ fulfilling } \varphi\cdot\nu = 0 \text{ on } \partial\Omega, \tag{3.27}$$

with $\beta := \alpha + \frac{n}{2}(\frac{1}{p} - \frac{1}{r})$ and $\gamma := \frac{1}{2} + \alpha$ satisfying $\beta < 1$ and $\gamma < 1$ due to the restrictions that $\alpha < 1 - \frac{n}{2}(\frac{1}{p} - \frac{1}{r})$ and $\alpha < \frac{1}{2}$.

Upon these selections, we now suppose that (u,v) is a global classical solution of (1.3) for some $\chi \le \chi_0, D \ge D_0$ and (u_0, v_0) fulfilling (1.6) and then infer from Lemma 3.1 that

$$\limsup_{t\to\infty} \Big\|\frac{1}{v(\cdot,t)}\Big\|_{L^\infty(\Omega)} \le \frac{1}{K_1\int_\Omega u_0},$$

whereas combining Lemma 3.2 with Lemma 3.4 asserts that

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^p(\Omega)} \le K_4 \int_\Omega u_0$$

and hence

$$\limsup_{t\to\infty} \|v(\cdot,t)\|_{W^{1,q}(\Omega)} \le K_2K_4 \int_\Omega u_0.$$

In particular, these inequalities guarantee the existence of $t_0 > 0$ such that

$$\Big\|\frac{1}{v(\cdot,t)}\Big\|_{L^\infty(\Omega)} \le \frac{2}{K_1\int_\Omega u_0} \qquad \text{for all } t>t_0 \tag{3.28}$$

and

$$\|v(\cdot,t)\|_{W^{1,q}(\Omega)} \le 2K_2K_4\int_\Omega u_0 \qquad \text{for all } t>t_0 \tag{3.29}$$

as well as

$$\|u(\cdot,t_0)\|_{L^p(\Omega)} \le 2K_4\int_\Omega u_0, \tag{3.30}$$

and keeping this value of t_0 fixed henceforth, for arbitrary $T>t_0$ we estimate the finite number

$$M(T) := \sup_{t\in(t_0,T]} \Big(1+D^{-\beta}(t-t_0)^{-\beta}\Big)^{-1}\|A^\alpha u(\cdot,t)\|_{L^r(\Omega)}$$

as follows: By means of a variation-of-constants representation of u, we see that thanks to (3.26), (3.27) and (3.30),

$$\begin{aligned}
\|A^\alpha u(\cdot,t)\|_{L^r(\Omega)} &= \Big\|A^\alpha e^{(t-t_0)D\Delta}u(\cdot,t_0) - D\chi\int_{t_0}^t A^\alpha e^{(t-s)D\Delta}\nabla\cdot\Big(\frac{u(\cdot,s)}{v(\cdot,s)}\nabla v(\cdot,s)\Big)ds\Big\|_{L^r(\Omega)} \\
&\le \|A^\alpha e^{(t-t_0)D\Delta}u(\cdot,t_0)\|_{L^r(\Omega)} + D\chi\int_{t_0}^t \Big\|A^\alpha e^{(t-s)D\Delta}\nabla\cdot\Big(\frac{u(\cdot,s)}{v(\cdot,s)}\nabla v(\cdot,s)\Big)\Big\|_{L^r(\Omega)} ds \\
&\le c_2\cdot\Big(1+[D(t-t_0)]^{-\beta}\Big)\|u(\cdot,t_0)\|_{L^p(\Omega)} \\
&\quad + c_3D\chi\int_{t_0}^t [D(t-s)]^{-\gamma}e^{-\lambda D(t-s)}\Big\|\frac{u(\cdot,s)}{v(\cdot,s)}\nabla v(\cdot,s)\Big\|_{L^r(\Omega)}ds \\
&\le 2c_2K_4\cdot\Big(1+D^{-\beta}(t-t_0)^{-\beta}\Big)\cdot\int_\Omega u_0 \\
&\quad + c_3\chi_0 D\int_{t_0}^t [D(t-s)]^{-\gamma}e^{-\lambda D(t-s)}\Big\|\frac{u(\cdot,s)}{v(\cdot,s)}\nabla v(\cdot,s)\Big\|_{L^r(\Omega)}ds \quad \text{for all } t>t_0,
\end{aligned} \tag{3.31}$$

because $\chi\le\chi_0$. Here in the rightmost integral we employ the Hölder inequality and make use of (3.25), (3.28) and (3.29) as well as the definition of $M(T)$ and (2.1) to see that writing $a := 1-\frac{q-r}{qr}\in(0,1)$ we have

$$\begin{aligned}
\Big\|\frac{u(\cdot,s)}{v(\cdot,s)}\nabla v(\cdot,s)\Big\|_{L^r(\Omega)} &\le \Big\|\frac{1}{v(\cdot,s)}\Big\|_{L^\infty(\Omega)}\|\nabla v(\cdot,s)\|_{L^q(\Omega)}\|u(\cdot,s)\|_{L^{\frac{qr}{q-r}}(\Omega)} \\
&\le \Big\|\frac{1}{v(\cdot,s)}\Big\|_{L^\infty(\Omega)}\|\nabla v(\cdot,s)\|_{L^q(\Omega)}\|u(\cdot,s)\|_{L^1(\Omega)}^{1-a}\|u(\cdot,s)\|_{L^\infty(\Omega)}^a \\
&\le c_1^a\Big\|\frac{1}{v(\cdot,s)}\Big\|_{L^\infty(\Omega)}\|\nabla v(\cdot,s)\|_{L^q(\Omega)}\|u(\cdot,s)\|_{L^1(\Omega)}^{1-a}\|A^\alpha u(\cdot,s)\|_{L^r(\Omega)}^a \\
&\le c_1^a\cdot\frac{2}{K_1\int_\Omega u_0}\cdot 2K_2K_4\int_\Omega u_0\cdot\Big\{\int_\Omega u_0\Big\}^{1-a}\cdot\Big(1+D^{-\beta}(s-t_0)^{-\beta}\Big)^a M^a(T) \\
&= \frac{4c_1^aK_2K_4}{K_1}\cdot\Big\{\int_\Omega u_0\Big\}^{1-a}M^a(T)\cdot\Big(1+D^{-\beta}(s-t_0)^{-\beta}\Big)^a \quad \text{for all } s\in(t_0,T).
\end{aligned}$$

Invoking Lemma 3.5, we hence infer that with $L(\cdot,\cdot,\cdot,\cdot)$ as introduced there and $c_4 := \frac{4c_1^a c_3 K_2 K_4 \chi_0 L(\beta,\gamma,\lambda,a)}{K_1}$,

$$\begin{aligned}
& c_3\chi_0 D \int_{t_0}^t [D(t-s)]^{-\gamma} e^{-\lambda D(t-s)} \Big\| \frac{u(\cdot,s)}{v(\cdot,s)} \nabla v(\cdot,s) \Big\|_{L^r(\Omega)} ds \\
\le\ & \frac{4c_1^a c_3 K_2 K_4 \chi_0}{K_1} \cdot \Big\{ \int_\Omega u_0 \Big\}^{1-a} M^a(T) \cdot D \int_{t_0}^t [D(t-s)]^{-\gamma} e^{-\lambda D(t-s)} \Big(1 + D^{-\beta}(s-t_0)^{-\beta}\Big)^a ds \\
\le\ & c_4 \Big\{ \int_\Omega u_0 \Big\}^{1-a} M^a(T) \cdot \Big(1 + D^{-\beta}(t-t_0)^{-\beta}\Big) \qquad \text{for all } t \in (t_0, T).
\end{aligned}$$

Now since the fact that $a < 1$ warrants that by Young's inequality,

$$\xi\eta \le \frac{1}{2}\xi^{\frac{1}{a}} + c_5 \eta^{\frac{1}{1-a}} \qquad \text{for all } \xi \ge 0 \text{ and } \eta \ge 0$$

with $c_5 := (1-a)\cdot(2a)^{\frac{a}{1-a}}$, together with (3.31) this shows that

$$\begin{aligned}
\Big(1 + D^{-\beta}(t-t_0)^{-\beta}\Big)^{-1} \|A^\alpha u(\cdot,t)\|_{L^r(\Omega)} &\le 2c_2 K_4 \int_\Omega u_0 + c_4 \Big\{ \int_\Omega u_0 \Big\}^{1-a} M^a(T) \\
&\le 2c_2 K_4 \int_\Omega u_0 + \frac{1}{2} M(T) + c_5 \cdot c_4^{\frac{1}{1-a}} \int_\Omega u_0
\end{aligned}$$

for all $t \in (t_0, T)$, and thus

$$M(T) \le c_6 \int_\Omega u_0 \qquad \text{for all } T > t_0,$$

where $c_6 := 2(2c_2K_4 + c_4^{\frac{1}{1-a}} c_5)$ depends on χ_0 and D_0 but neither on $\chi \le \chi_0$ nor on $D \ge D_0$ nor (u,v). As this entails that

$$\begin{aligned}
\|A^\alpha u(\cdot,t)\|_{L^r(\Omega)} &\le \Big(1 + D^{-\beta}(t-t_0)^{-\beta}\Big) M(t) \\
&\le 2c_6 \int_\Omega u_0 \qquad \text{for all } t \ge t_0 + \frac{1}{D},
\end{aligned}$$

the proof becomes complete if we let $K_5 := 2c_6$. □

4 An asymptotic universal estimate for $\frac{u}{v}$

Thanks to the rather precise knowledge on quantitative dependence of the expressions addressed in Lemma 3.1 and Lemma 3.6 on the initial data, the following conclusion of the latter two lemmata is quite straightforward but of fundamental importance for the subsequent arguments.

Lemma 4.1 *Let $\chi_0 < \sqrt{\frac{2}{n}}$ and $D_0 > 0$. Then there exists $K_6 = K_6(\chi_0, D_0) > 0$ such that whenever (u,v) is a global classical solution of (1.3) for some $\chi \le \chi_0, D \ge D_0$ and (u_0, v_0) satisfying (1.6), we have*

$$\limsup_{t\to\infty} \Big\| \frac{u(\cdot,t)}{v(\cdot,t)} \Big\|_{L^\infty(\Omega)} \le K_6. \tag{4.1}$$

PROOF. We let $r > n$, A and $\alpha \in (\frac{n}{2r}, 1)$ be as in Lemma 3.6 and recall that then $D(A^\alpha) \hookrightarrow L^\infty(\Omega)$, and that hence there exists $c_1 > 0$ such that $\|\varphi\|_{L^\infty(\Omega)} \le c_1 \|A^\alpha \varphi\|_{L^r(\Omega)}$ for all $\varphi \in D(A^\alpha)$. Therefore,

$$
\begin{aligned}
\Big\| \frac{u(\cdot,t)}{v(\cdot,t)} \Big\|_{L^\infty(\Omega)} &\le \Big\| \frac{1}{v(\cdot,t)} \Big\|_{L^\infty(\Omega)} \|u(\cdot,t)\|_{L^\infty(\Omega)} \\
&\le c_1 \Big\| \frac{1}{v(\cdot,t)} \Big\|_{L^\infty(\Omega)} \|A^\alpha u(\cdot,t)\|_{L^r(\Omega)} \qquad \text{for all } t > 0,
\end{aligned}
$$

so that the claimed inequality (4.1) immediately results from Lemma 3.1 and Lemma 3.6 if we let $K_6(\chi_0, D_0) := \frac{c_1 K_5(\chi_0, D_0)}{K_1(D_0)}$ with $K_5(\cdot,\cdot)$ and $K_1(\cdot)$ as provided there. $\Box$

5 An asymptotic Lyapunov functional. Proof of the main results

Thanks to the pointwise estimate from Lemma 4.1, a simple linear combination of the expressions $\int_\Omega (u - \overline{u}_0)^2$ and $\int_\Omega (v - \overline{u}_0)^2$ can be seen to play the role of a Lyapunov functional at least after a certain trajectory-dependent waiting time. Indeed, in view of (4.1) the evolution of the first ingredient thereof can be described as follows.

Lemma 5.1 *Let $\chi_0 < \sqrt{\frac{2}{n}}$ and $D_0 > 0$. Then whenever $\chi \le \chi_0, D \ge D_0$ and (u_0, v_0) satisfies (1.6), for the corresponding global classical solution (u, v) one can find $t_0 > 0$ such that*

$$
\frac{d}{dt} \int_\Omega (u - \overline{u}_0)^2 + D \int_\Omega |\nabla u|^2 \le 4K_6^2 D \chi^2 \int_\Omega |\nabla v|^2 \qquad \textit{for all } t > t_0, \tag{5.1}
$$

where $K_6 = K_6(\chi_0, D_0) > 0$ is as in Lemma 4.1.

PROOF. According to Lemma 4.1, we can find $t_0 > 0$ such that

$$
\frac{u(x,t)}{v(x,t)} \le 2K_6 \qquad \text{for all } x \in \Omega \text{ and any } t > t_0.
$$

Therefore, if we test the first equation in (1.3) by $u - \overline{u}_0$, then in the resulting identity

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_\Omega (u - \overline{u}_0)^2 &= \int_\Omega (u - \overline{u}_0) \cdot \Big\{ D\Delta u - D\chi \nabla \cdot \Big(\frac{u}{v} \nabla v \Big) \Big\} \\
&= -D \int_\Omega |\nabla u|^2 + D\chi \int_\Omega \frac{u}{v} \nabla u \cdot \nabla v, \qquad t > 0,
\end{aligned}
$$

by means of Young's inequality we can estimate

$$
\begin{aligned}
D\chi \int_\Omega \frac{u}{v} \nabla u \cdot \nabla v &\le \frac{D}{2} \int_\Omega |\nabla u|^2 + \frac{D\chi^2}{2} \int_\Omega \frac{u^2}{v^2} |\nabla v|^2 \\
&\le \frac{D}{2} \int_\Omega |\nabla u|^2 + 2K_6^2 D \chi^2 \int_\Omega |\nabla v|^2 \qquad \text{for all } t > t_0
\end{aligned}
$$

and conclude. $\Box$

Here the integral on the right-hand side precisely appears as part of the dissipation rate in a corresponding evolutionary inequality for $\int_\Omega (v - \overline{u}_0)$.

Lemma 5.2 *Assume that $\chi < \sqrt{\frac{2}{n}}$, that $D > 0$ and that (1.6) holds. Then the global classical solution (u, v) of (1.3) satisfies*

$$\frac{d}{dt}\int_\Omega (v-\overline{u}_0)^2 + 2D\int_\Omega |\nabla v|^2 + \int_\Omega (v-\overline{u}_0)^2 \le \int_\Omega (u-\overline{u}_0)^2 \qquad \textit{for all } t > 0. \tag{5.2}$$

Proof. We multiply the second equation in (1.3) by $v - \overline{u}_0$ and integrate by parts to see that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega (v-\overline{u}_0)^2 &= \int_\Omega (v-\overline{u}_0)\cdot\Big\{D\Delta v - (v-\overline{u}_0) + (u-\overline{u}_0)\Big\} \\ &= -D\int_\Omega |\nabla v|^2 - \int_\Omega (v-\overline{u}_0)^2 + \int_\Omega (u-\overline{u}_0)(v-\overline{u}_0) \qquad \text{for all } t > 0. \end{aligned}$$

As by Young's inequality,

$$\int_\Omega (u-\overline{u}_0)(v-\overline{u}_0) \le \frac{1}{2}\int_\Omega (v-\overline{u}_0)^2 + \frac{1}{2}\int_\Omega (u-\overline{u}_0)^2 \qquad \text{for all } t > 0,$$

this implies (5.2). □

5.1 Proof of Theorem 1.1 and Proposition 1.2

Based on an appropriate combination of Lemma 5.1 and Lemma 5.2 as well as two interpolation arguments, we can now establish our main results on stabilization in (1.3).

Proof of Theorem 1.1. By means of the Poincaré inequality, we fix $c_1 > 0$ such that

$$\int_\Omega (\varphi - \overline{\varphi})^2 \le c_1 \int_\Omega |\nabla\varphi|^2 \qquad \text{for all } \varphi \in W^{1,2}(\Omega), \tag{5.3}$$

and given $\chi_0 < \sqrt{\frac{2}{n}}$ and $D_0 > 0$ we thereupon let

$$\delta := \frac{1}{4c_1K_6^2} \tag{5.4}$$

with $K_6 := K_6(\chi_0, D_0) > 0$ taken from Lemma 4.1. Then assuming that $\chi \le \chi_0$ and $D \ge D_0$ are such that (1.7) holds, and that (u_0, v_0) complies with (1.6), then from Lemma 2.1 we know that (1.3) possesses a global classical solution (u, v) which is such that (1.8) holds, and for which due to Lemma 5.1 and Lemma 5.2 there exists $t_0 > 0$ satisfying

$$\begin{aligned} &\frac{d}{dt}\Big\{\int_\Omega (u-\overline{u}_0)^2 + \frac{D}{2c_1}\int_\Omega (v-\overline{u}_0)^2\Big\} + D\int_\Omega |\nabla u|^2 + \frac{D^2}{c_1}\int_\Omega |\nabla v|^2 + \frac{D}{2c_1}\int_\Omega (v-\overline{u}_0)^2 \\ &\quad\le\ 4K_6^2 D\chi^2 \int_\Omega |\nabla v|^2 + \frac{D}{2c_1}\int_\Omega (u-\overline{u}_0)^2 \qquad \text{for all } t > t_0. \end{aligned}$$

Since herein

$$D\int_\Omega |\nabla u|^2 \ge \frac{D}{c_1}\int_\Omega (u-\overline{u}_0)^2 \qquad \text{for all } t > 0$$

by (5.3) and (2.1), and since moreover

$$4K_6^2 D\chi^2 \int_\Omega |\nabla v|^2 \le 4\delta K_6^2 D^2 \int_\Omega |\nabla v|^2 = \frac{D^2}{c_1} \int_\Omega |\nabla v|^2 \qquad \text{for all } t > 0$$

according to (1.7) and (5.4), this implies that

$$\frac{d}{dt}\Big\{ \int_\Omega (u-\overline{u}_0)^2 + \frac{D}{2c_1}\int_\Omega (v-\overline{u}_0)^2 \Big\} + \frac{D}{2c_1}\int_\Omega (u-\overline{u}_0)^2 + \frac{D}{2c_1}\int_\Omega (v-\overline{u}_0)^2 \le 0 \qquad \text{for all } t > t_0.$$

Therefore, $y(t) := \int_\Omega (u(\cdot,t)-\overline{u}_0)^2 + \frac{D}{2c_1}\int_\Omega (v(\cdot,t)-\overline{u}_0)^2$, $t \ge t_0$, satisfies

$$y'(t) + c_2 y(t) \le 0 \qquad \text{for all } t > t_0$$

with $c_2 := \min\{\frac{D}{2c_1}\ , 1\}$, on integration showing that

$$y(t) \le c_3 e^{c_1 t} \qquad \text{for all } t \ge t_0$$

and hence

$$\|u(\cdot,t)-\overline{u}_0\|_{L^2(\Omega)} \le c_4 e^{-\frac{c_2}{2}t} \qquad \text{for all } t \ge t_0 \tag{5.5}$$

as well as

$$\|v(\cdot,t)-\overline{u}_0\|_{L^2(\Omega)} \le c_5 e^{-\frac{c_2}{2}t} \qquad \text{for all } t \ge t_0 \tag{5.6}$$

if we let $c_3 := y(t_0)e^{c_2 t_0}$, $c_4 := \sqrt{c_3}$ and $c_5 := \sqrt{\frac{2c_1c_3}{D}}$, for instance. In order to derive (1.9) and (1.10) from this, we proceed by straightforward interpolation relying on Lemma 3.6 and once again on Lemma 3.2 and Lemma 3.4:

Firstly, letting $p = p(\chi_0) \in (\frac{n}{2}, n)$ be as in Lemma 3.4 and choosing any $q > n$ such that $q < \frac{np}{n-p}$, we infer on combining Lemma 3.2 with Lemma 3.4 that with $K_4 = K_4(\chi_0, D_0) > 0$ and $K_2 = K_2(D_0, p, q) > 0$ as introduced there we have

$$\limsup_{t\to\infty} \|v(\cdot,t)\|_{W^{1,q}(\Omega)} \le K_2 \limsup_{t\to\infty} \|u(\cdot,t)\|_{L^p(\Omega)} \le K_2K_4 \int_\Omega u_0$$

and hence

$$\|v(\cdot,t)\|_{W^{1,q}(\Omega)} \le c_6 := 2K_2K_4 \int_\Omega u_0 \qquad \text{for all } t \ge t_1$$

with some appropriately large $t_1 \ge t_0$. Since in view of the restriction $q > n$ the Gagliardo–Nirenberg inequality provides $c_7 > 0$ such that writing $a := \frac{nq}{2q+nq-2n} \in (0,1)$ we have

$$\|v(\cdot,t)-\overline{u}_0\|_{L^\infty(\Omega)} \le c_7 \|v(\cdot,t)-\overline{u}_0\|_{W^{1,q}(\Omega)}^a \|v(\cdot,t)-\overline{u}_0\|_{L^2(\Omega)}^{1-a} \qquad \text{for all } t > 0,$$

estimating

$$\|v(\cdot,t)-\overline{u}_0\|_{W^{1,q}(\Omega)} \le \|v(\cdot,t)\|_{W^{1,q}(\Omega)} + \|\overline{u}_0\|_{W^{1,q}(\Omega)} \le c_8 := c_6 + \|\overline{u}_0\|_{W^{1,q}(\Omega)} \qquad \text{for all } t \ge t_1$$

we obtain that

$$\begin{aligned}\|v(\cdot,t)-\overline{u}_0\|_{l^\infty(\Omega)} &\le c_7c_8^a\|v(\cdot,t)-\overline{u}_0\|_{L^2(\Omega)}^{1-a}\\ &\le c_7c_8^ac_5^{1-a}e^{-\frac{(1-a)c_2}{2}t} \qquad \text{for all } t\ge t_1 \end{aligned} \tag{5.7}$$

according to (5.6).

To prepare a similar argument for the first solution component, we first recall from Lemma 3.6 that there exist $r>n$ and $\alpha\in(\frac{n}{2r},1)$ such that with $K_5(\chi_0,D_0)$ and the operator A defined there we can find $t_2\ge t_0$ such that

$$\|A^\alpha u(\cdot,t)\|_{L^r(\Omega)}\le c_9:=2K_5\int_\Omega u_0 \qquad \text{for all } t\ge t_2. \tag{5.8}$$

Using that $\alpha>\frac{n}{2r}$, we may fix $\alpha_0\in(0,\alpha)$ such that still $\alpha_0>\frac{n}{2r}$, which in view of a known embedding result ([14]) implies that $D(A^{\alpha_0})\hookrightarrow L^\infty(\Omega)$ and hence

$$\|\varphi\|_{L^\infty(\Omega)}\le c_{10}\|A^{\alpha_0}\varphi\|_{L^r(\Omega)} \qquad \text{for all } \varphi\in D(A^{\alpha_0}) \tag{5.9}$$

with some $c_{10}>0$. Now a standard interpolation inequality ([9]) yields $c_{11}>0$ fulfilling

$$\|A^{\alpha_0}\varphi\|_{L^r(\Omega)}\le c_{11}\|A^\alpha\varphi\|_{L^r(\Omega)}^b\|\varphi\|_{L^r(\Omega)}^{1-b} \qquad \text{for all } \varphi\in D(A^\alpha)$$

with $b:=\frac{\alpha_0}{\alpha}\in(0,1)$, whence by the Hölder inequality and (5.9),

$$\|\varphi\|_{L^r(\Omega)}^{1-b}\le\|\varphi\|_{L^\infty(\Omega)}^{\frac{(r-2)(1-b)}{r}}\|\varphi\|_{L^2(\Omega)}^{\frac{2(1-b)}{r}}\le c_{10}^{\frac{(r-2)(1-b)}{r}}\|A^{\alpha_0}\varphi\|_{L^r(\Omega)}^{\frac{(r-2)(1-b)}{r}}\|\varphi\|_{L^2(\Omega)}^{\frac{2(1-b)}{r}} \qquad \text{for all } \varphi\in D(A^{\alpha_0})$$

and thus, altogether,

$$\|A^{\alpha_0}\varphi\|_{L^r(\Omega)}^{1-\frac{(r-2)(1-b)}{r}}\le c_{10}^{\frac{(r-2)(1-b)}{r}}c_{11}\|A^\alpha\varphi\|_{L^r(\Omega)}^b\|\varphi\|_{L^2(\Omega)}^{\frac{2(1-b)}{r}} \qquad \text{for all } \varphi\in D(A^\alpha),$$

that is,

$$\|A^{\alpha_0}\varphi\|_{L^r(\Omega)}\le c_{12}\|A^\alpha\varphi\|_{L^r(\Omega)}^d\|\varphi\|_{L^2(\Omega)}^{1-d} \qquad \text{for all } \varphi\in D(A^\alpha)$$

with $d:=\frac{br}{br+2(1-b)}\in(0,1)$ and $c_{12}:=\Big(c_{10}^{\frac{(r-2)(1-b)}{r}}c_{11}\Big)^{\frac{r}{br+2(1-b)}}>0$. When applied to $\varphi:=u(\cdot,t)-\overline{u}_0$ for $t\ge t_2$, since

$$\begin{aligned}\Big\|A^\alpha(u(\cdot,t)-\overline{u}_0)\Big\|_{L^r(\Omega)} &\le \|A^\alpha u(\cdot,t)\|_{L^r(\Omega)}+\|A^\alpha\overline{u}_0\|_{L^r(\Omega)}\\ &\le c_{13}:=c_9+\|A^\alpha\overline{u}_0\|_{L^r(\Omega)} \qquad \text{for all } t\ge t_2\end{aligned}$$

by (5.8), this entails that

$$\begin{aligned}\Big\|A^{\alpha_0}(u(\cdot,t)-\overline{u}_0)\Big\|_{L^r(\Omega)} &\le c_{12}c_{13}^d\|u(\cdot,t)-\overline{u}_0\|_{L^2(\Omega)}^{1-d}\\ &\le c_4^{1-d}c_{12}c_{13}^de^{-\frac{(1-d)c_2}{2}t} \qquad \text{for all } t\ge t_2\end{aligned}$$

according to (5.5). Once more employing (5.9), from this we obtain (1.9), whereas (1.10) has been established in (5.7). □

The second of our main results can thereafter be obtained by means of a simple reformulation.

PROOF of Proposition 1.2. In view of (1.4) and (1.5), both (1.11) and (1.12) are evident consequences of Theorem 1.1. □

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