Can rotational fluxes impede the tendency toward spatial homogeneity in nutrient taxis(-Stokes) systems?

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Abstract

We consider the spatially two-dimensional version of the model

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot\big(nS(x,n,c)\cdot\nabla c\big), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - nf(c), & x\in\Omega,\ t>0,\\ u_t = \Delta u + \nabla P + n\nabla\phi, \qquad \nabla\cdot u = 0, & x\in\Omega,\ t>0, \end{cases} \qquad (\star)$$

for nutrient taxis processes, possibly interacting with liquid environments. Here the particular focus is on the situation when the chemotactic sensitivity S is not a scalar function but rather attains general values in $\mathbb{R}^{2\times 2}$, thus accounting for rotational flux components in accordance with experimental findings and recent modeling approaches.

Reflecting significant new challenges which mainly stem from an apparent loss of energy-like structures, especially for initial data with large size the knowledge on $(\star)$ so far seems essentially restricted to results on global existence of certain generalized solutions with possibly quite poor boundedness and regularity properties; widely unaddressed seem aspects related to possible effects of such non-diagnonal taxis mechanisms on the qualitative solution behavior, especially with regard to the fundamental question whether spatial structures may thereby be supported.

The present work answers the latter in the negative in the following sense: Under the assumptions that the initial data (n_0, c_0, u_0) and the parameter functions S, f and ϕ are sufficiently smooth, and that S is bounded and f is positive on $(0,\infty)$ with $f(0)=0$, it is shown that any nontrivial of these solutions eventually becomes smooth and satisfies

$$n(\cdot,t)\to \fint_\Omega n_0, \quad c(\cdot,t)\to 0 \quad \text{and} \quad u(\cdot,t)\to 0 \qquad \text{as } t\to\infty,$$

uniformly with respect to $x\in\Omega$. By not requiring any smallness condition on the initial data, the latter seems new even in the corresponding fluid-free version obtained on letting $u\equiv 0$ in $(\star)$.

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1 Introduction

Understanding causal nexus leading to the emergence of structures in living systems has provided enduring motivation for theoretical biologists and applied mathematicians for decades ([33]). Identifying the respectively responsible mechanisms seems particularly challenging in situations when the considered system consists of exceptionally primitive individuals. In experimental contexts, a paradigmatic role in this regard is played by the bacterial species *Bacillus subtilis*, about which it is known that cells are well able to actively move toward a present nutrient which they consume upon contact, but that beyond this their movement occurs essentially at random. Nevertheless, populations of this species have been observed to form quite complex spatial patterns either on thin agar plates in absence of further components ([32], [11]), or also when suspended to sessile drops of water ([6], [40]). Phenomena of structure generation in connection with such nutrient taxis mechanisms have also been found in various other primitive bacterial species, including *Proteus mirabilis* and *Bacillus cereus* ([11], [19]).

In their apparently simplest form, macroscopic models for such processes concentrate on describing the cell population density $n = n(x,t)$ and the nutrient concentration $c = c(x,t)$, e.g. by means of chemotaxis systems of the form

$$\begin{cases} n_t = \Delta n - \nabla \cdot (n\nabla c), \\ c_t = \Delta c - nc, \end{cases} \tag{1.1}$$

thus consituting a derivate of the classical Keller-Segel system, differing from (1.1) in its second equation $c_t = \Delta c - c + n$ and thus accounting for signal production through cells, with its well-known ability to enforce the emergence of structures even in the extreme sense of finite-time singularity formation ([15], [16], [48]). In comparison to the latter, however, (1.1) has been found to exhibit a substantially stronger tendency toward supporting spatial homogeneity: Namely, when considered along with no-flux boundary conditions in smoothly bounded planar domains, then for any reasonably regular nontrivial initial data (1.1) possesses a bounded global classical solution which asymptotically becomes spatially constant in the sense that

$$n(\cdot,t) \to n_\infty \quad \text{and} \quad c(\cdot,t) \to 0 \quad \text{in } L^\infty(\Omega) \qquad \text{as } t \to \infty \tag{1.2}$$

with $n_\infty \equiv ⨍_\Omega n_0$ ([37]). For the three-dimensional analogue, it is known that after all certain global weak solutions exist, and that these at least after some initial waiting time become smooth and classical and satisfy (1.2) ([37]).

Similar trends toward homogeneity have been found to dominate the qualitative behavior also in several generalizations of (1.1), even in some cases in which an additional buoyancy-induced coupling to the surrounding fluid, described through its velocity field $u = u(x,t)$ and the associated pressure $P = P(x,t)$, is accounted for. Indeed, numerical ([30]) and also recent analytical studies ([20], [21], [22]) have revealed remarkable effects of fluid interaction on chemotaxis systems, inter alia concerning prevention of blow-up and efficiency of mixing, at least in cases involving signal production. However, corresponding Neumann-Neumann-Dirichlet initial-boundary value problems in N-dimensional domains for the general nutrient taxis(-fluid) system

$$\begin{cases} n_t + u\cdot\nabla n &= \Delta n - \nabla \cdot \Big(nS(c)\nabla c\Big) + g(n,c), \\ c_t + u\cdot\nabla c &= \Delta c - nf(c), \\ u_t + \kappa(u\cdot\nabla)u &= \Delta u + \nabla P + n\nabla\phi, \qquad \nabla\cdot u = 0, \end{cases} \tag{1.3}$$

have been found to possess global solutions satisfying (1.2) with some constant n_∞ for various choices of the nonnegative chemotactic sensitivity function S, under mild assumptions on the gravitational potential ϕ, and in either of the cases $\kappa = 0$ and $\kappa = 1$ in which the fluid flow is governed by the Stokes or the full Navier-Stokes system. When $g \equiv 0$, for instance, global classical solutions fulfilling (1.2) with $n_\infty = \fint_\Omega n_0$ are known to exist when $N = 2$, even in the case $\kappa = 1$, under appropriate structural assumptions on S, including the prototypical choice $S(c) = c$ ([7], [47], [49], [57], [18]; cf. also [9]). For the three-dimensional counterpart, global weak solutions have been constructed in [52], and in [53] it has recently been shown that any such solution becomes eventually smooth and again satisfies (1.2) with $n_\infty = \fint_\Omega n_0$.

Even the inclusion of logistic-type cell proliferation terms of the form $g(n, c) = \rho n - \mu n^2$ in the latter three-dimensional chemotaxis-Navier-Stokes system does not substantially affect this property of allowing for eventually smooth global weak solutions which stabilize in the flavor of (1.2) ([26], see also [41]). Some additional results on further particular versions of (1.3), albeit yet far from creating a complete picture, indicate that neither purely productive nutrient-supported cell kinetics, as modeled in the simplest case by the source term $g(n, c) = nc$, nor singular behavior of S, such as present in the classical logarithmic sensitivity given by $S(c) = \frac{\chi_0}{c}$ with some $\chi_0 > 0$, substantially reduce this large-time prevalence of spatial flatness ([1]). For related results on global existence and asymptotic homogenization in variants of (1.3), inter alia accounting for nonlinear cell diffusion mechanisms, the reader may consult [4], [23], [3], [29], [5], [8], [58] and [38], for instance.

Chemotaxis(-fluid) systems with rotational fluxes. The above results may be viewed as reflecting a certain dominance of the dissipative signal absorption mechanism in (1.3), together with diffusion, over any destabilizing action of the cross-diffusion process described therein. Mathematically, this becomes manifest in certain entropy-like structures which in an appropriate manner supplement the basic decay properties

$$\int_0^\infty \int_\Omega n f(c) \le \int_\Omega c_0 \qquad \text{and} \qquad \int_0^\infty \int_\Omega |\nabla c|^2 \le \frac{1}{2} \int_\Omega c_0^2 \tag{1.4}$$

formally associated with (1.3) (see also Lemma 2.1 below). When $S \equiv 1, f(c) = c$ and $g \equiv 0$, for instance, inequalities of the form

$$\frac{d}{dt}\Big\{ \int_\Omega n \ln n + \frac{1}{2} \int_\Omega \frac{|\nabla c|^2}{c} + a \int_\Omega |u|^2 \Big\} + \frac{1}{C} \cdot \Big\{ \int_\Omega \frac{|\nabla n|^2}{n} + \int_\Omega \frac{|\nabla c|^4}{c^3} + \int_\Omega |\nabla u|^2 \Big\} \le C \qquad t > 0, \tag{1.5}$$

have been found to hold with some $a > 0$ and $C > 0$, and to thereby provide additional regularity information sufficient to turn (1.4) into (1.2) even in the full three-dimensional Navier-Stokes version of (1.3) ([53], see also [49]).

Favorable complementary structures of this form are apparently lost when the chemotactic sensitivity S in (1.3) is no longer assumed to be a scalar function but rather allowed to be matrix-valued. Indeed, more recent modeling approaches ([56], [55]) suggest to describe chemotactic bacterial motion near surfaces by means of certain tensor-valued sensitivities $S = S(x, n, c) \in \mathbb{R}^{N\times N}$ which in particular involve rotational flux components, in the most prototypical case near boundary points of two-dimensional domains taking the form

$$S = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad a > 0, \ b \in \mathbb{R}. \tag{1.6}$$

By impeding the derivation of appropriate *a priori* estimates e.g. from inequalities of type (1.5), even at the level of mere existence theory such generalizations bring about significant analytical challenges, and accordingly very little is known beyond quite basic aspects, unless additional regularizing mechanisms such as nonlinear diffusion enhancement or saturation of cross-diffusive fluxes at large cell densities are introduced (see e.g. [2], [43], [42] and [51], but also [44] and [45] for some examples). Without such changes in the system, only under appropriate smallness assumptions on the initial data global smooth solutions have been found to exist both in the corresponding fluid-free variant of (1.3) when $N = 2$, $u \equiv 0$, $f(c) = c, g \equiv 0$ and S is suitably smooth and bounded ([27]), as well as in the Stokes counterpart for $N \in \{2, 3\}$ with nontrivial u and $\kappa = 0$ ([3]). For initial data of arbitrary size, only certain generalized solutions have been found to exist globally when either $N \geq 1$ and $u \equiv 0$ ([50]), or when $N = 2$ and the nontrivial fluid is governed by the Stokes equations ([54]).

Main results. The purpose of the present work consists in examining qualitative properties of the latter solutions in spatially planar frameworks, with a particular focus on the question how far off-diagonal flux components such as in (1.6) may affect the tendency toward asymptotic honogenization. For this purpose, including both the corresponding fluid-free chemotaxis-only system addressed in [50] as well as the associated chemotaxis-Stokes model studied in [54], we shall subsequently be concerned with the initial-boundary value problem

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot \Big(nS(x,n,c) \cdot \nabla c \Big), & x \in \Omega,\ t > 0, \\ c_t + u \cdot \nabla c = \Delta c - nf(c), & x \in \Omega,\ t > 0, \\ u_t = \Delta u + \nabla P + n\nabla\phi, \qquad \nabla \cdot u = 0, & x \in \Omega,\ t > 0, \\ \nabla n \cdot \nu = n\Big(S(x,n,c) \cdot \nabla c \Big) \cdot \nu, \quad \nabla c \cdot \nu = 0, \quad u = 0, & x \in \partial\Omega,\ t > 0, \\ n(x,0) = n_0(x), \quad c(x,0) = c_0(x), \quad u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.7}$$

in a bounded convex domain $\Omega \subset \mathbb{R}^2$. Here in order to make the existence theory from [54] applicable we will assume that

$$f \in C^1([0,\infty)) \quad \text{is nonnegative with} \quad f(0) = 0, \tag{1.8}$$

that

$$\phi \in W^{2,\infty}(\Omega), \tag{1.9}$$

and that $S = (S_{ij})_{i,j \in \{1,2\}}$ satisfies

$$S_{ij} \in C^2(\bar{\Omega} \times [0,\infty) \times [0,\infty)) \qquad \text{for } i,j \in \{1,2\} \tag{1.10}$$

as well as

$$|S(x,n,c)| \leq S_0(c) \quad \text{for all } (x,n,c) \in \bar{\Omega} \times [0,\infty)^2 \qquad \text{with some nondecreasing } S_0 : [0,\infty) \to \mathbb{R}. \tag{1.11}$$

As for the initial data in (1.7), we shall suppose that

$$\begin{cases} n_0 \in C^\iota(\bar{\Omega}) \quad \text{for some } \iota > 0 \text{ with } n_0 \geq 0 \text{ in } \Omega, \quad \text{that} \\ c_0 \in W^{1,\infty}(\Omega) \quad \text{satisfies } c_0 \geq 0 \text{ in } \Omega, \quad \text{and that} \\ u_0 \in D(A^\vartheta) \quad \text{for some } \vartheta \in (\frac{1}{2}, 1), \end{cases} \tag{1.12}$$

where with $\mathcal{P} : L^2(\Omega;\mathbb{R}^2) \to L^2_\sigma(\Omega)$ denoting the Helmholtz projection of $L^2(\Omega;\mathbb{R}^2)$ onto its solenoidal subspace $L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega;\mathbb{R}^2) \mid \nabla\cdot\varphi = 0\}$, A represents the realization of the Stokes operator $-\mathcal{P}\Delta$ in $L^2_\sigma(\Omega)$ with its natural domain given by $D(A) := W^{2,2}(\Omega;\mathbb{R}^2) \cap W^{1,2}_0(\Omega;\mathbb{R}^2) \cap L^2_\sigma(\Omega)$ ([36]).

Indeed, in this setting me may recall the following result on global solvability from [54, Theorem 1.1], referring to a concept of solvability that will explicitly be recalled in Definition 6.1 below for completeness.

Theorem A. *Suppose that f, ϕ and S satisfy (1.8), (1.9), (1.10) and (1.11), and that n_0, c_0 and u_0 comply with (1.12). Then there exists at least one triple of functions*

$$\begin{cases} n \in L^\infty([0,\infty);L^1(\Omega)), \\ c \in L^\infty(\Omega\times(0,\infty)) \cap L^2_{loc}([0,\infty);W^{1,2}(\Omega)) \qquad \textit{and} \\ u \in L^2_{loc}(\bar\Omega\times[0,\infty);\mathbb{R}^2) \cap \bigcap_{p\in[1,2)} L^p_{loc}([0,\infty);W^{1,p}_0(\Omega;\mathbb{R}^2)), \end{cases} \tag{1.13}$$

such that (n,c,u) is a global mass-preserving generalized solution of (1.7) in the sense of [54, Definition 2.1].

The information (1.13) on the regularity properties of the above solution is rather sparse, falling far short of corresponding knowledge in typical cases of scalar-valued S ([37]). In particular, the inclusions in (1.13) seem insufficient to warrant uniqueness of the obtained generalized solutions through any of the approaches used in corresponding previous analysis of chemotaxis-fluid systems (see e.g. [23] or also [47]). Yet more drastically, even finite-time blow-up e.g. of the first solution component n with respect to the norm in $L^\infty(\Omega)$ is by no means ruled out through Theorem A; after all, (1.13) precludes any collapse of n into an afterwards persisting Dirac-type measure, as known to occur in two-dimensional parabolic-elliptic Keller-Segel systems ([31], [39]). It remains an interesting open issue to clarify how far the occurrence of weaker types of blow-up phenomena may be enforced by non-scalar chemotactic sensitivities.

Anyhow, the first of our main results states that each individual nontrivial solution from Theorem A will become smooth eventually, provided that f satisfies an additional mild positivity assumption:

Theorem 1.1. *Assume (1.8), (1.9), (1.10) and (1.11), and suppose that moreover*

$$f > 0 \quad \textit{on } (0,\infty). \tag{1.14}$$

Then whenever n_0, c_0 and u_0 are such that (1.12) holds and $n_0 \not\equiv 0$, one can find $T > 0$ such that the global generalized solution (n,c,u) constructed in Theorem A satisfies

$$(n,c,u) \in C^{2,1}(\bar\Omega\times[T,\infty)) \times C^{2,1}(\bar\Omega\times[T,\infty)) \times C^{2,1}(\bar\Omega\times[T,\infty);\mathbb{R}^2), \tag{1.15}$$

and such that there exists $P \in C^{1,0}(\bar\Omega\times[T,\infty))$ with the property that (n,c,u,P) is a classical solution of the boundary value problem in (1.7) in $\bar\Omega\times[T,\infty)$.

The second among our main results asserts that under the same hypotheses, each of the above solutions stabilizes toward the unique spatially constant steady state at the respective mass level, similar to the case of scalar S ([49]).

Theorem 1.2. *Let (1.8), (1.9), (1.10), (1.11) and (1.14) hold, and assume that (n_0, c_0, u_0) satisfies (1.12) with $n_0 \not\equiv 0$. Then the solution (n, c, u) of (1.7) from Theorem A satisfies*

$$n(\cdot, t) \to \overline{n_0} \qquad in\ L^\infty(\Omega) \tag{1.16}$$

and

$$c(\cdot, t) \to 0 \qquad in\ L^\infty(\Omega) \tag{1.17}$$

as well as

$$u(\cdot, t) \to 0 \qquad in\ L^\infty(\Omega) \tag{1.18}$$

as $t \to \infty$, where $\overline{n_0} := \fint_\Omega n_0$.

In conclusion, even matrix-valued bounded S will not enforce the large-time formation of structures in the two-dimensional version (1.7) and thus, in particular, neither in its two-dimensional chemotaxis-only counterpart; however, our results do not exclude the possibility of relevant nontrivial dynamical effects on intermediate time scales such as, for instance, transient growth or oscillation phenomena as detected for some particular chemotaxis systems without evident energy structure ([34], [25]).

Main ideas. The challenge of coping with poor regularity information. Due to the superlinear structure of the nonlinearities in (1.7), the proofs of Theorem 1.1 and Theorem 1.2 will be closely linked to each other. Especially in view of precedent derivations of similar results for simpler problems, it may already be expected that a key role will be played by the decay property (1.17) of c. Indeed, in cases exclusively involving diagonal fluxes such as in (1.3), this could be achieved on the basis of some (temporally integral) regularity information on $\sqrt{n}$ in $W^{1,2}(\Omega)$ and on $\nabla \sqrt[4]{c}$ in $W^{1,4}(\Omega)$, as gained from the dissipation process expressed in inequalities of the form (1.5); inter alia by essentially relying on the compact embedding of $W^{1,4}(\Omega)$ into $L^\infty(\Omega)$, through suitable interpolation this can indeed be used to turn the basic decay information contained in (1.4) into the uniform decay statement in (1.17) in such cases ([37], [49]).

In stark contrast to this apparently lacking any global property as convenient as (1.5), the present situation will require to adequately make use of regularity information which, at a first stage, will be substantially weaker in actually reducing to the poor features described in (1.4). After all, as already observed in [54], this entails a further dissipative property which is formally expressed in the inequality

$$\int_0^\infty \int_\Omega \frac{|\nabla n|^2}{(n+1)^2} \le 2\int_\Omega n_0 + \frac{S_1^2}{2} \cdot \int_\Omega c_0^2 \tag{1.19}$$

(Lemma 2.2), and which will serve as a crucial fundament for our analysis. In the present two-dimensional setting, namely, this can firstly be turned into a basic statement on decay of $\int_\Omega |n(\cdot,t) - \kappa(t)|$ with some appropriate $\kappa(t) > 0$ (Lemma 3.1). Since, secondly, the Moser-Trudinger inequality allows us to infer from (1.19) a certain integrability property of n involving arbitrary spatial L^p norms (Lemma 2.3), together with (1.4) and suitable interpolation this can be seen to warrant smallness of the functional $y(t) := \int_\Omega |\nabla c(\cdot,t)|^2 + \int_\Omega |n(\cdot,t) - \kappa(t)|^2$ at some particular but arbitrarily large times (Lemma 4.2). By means of appropriate testing techniques, in Section 4.2 we shall derive an ODI for y which ensures that such a smallness property is essentially maintained throughout a time interval of fixed length (Lemma 4.5). Since this ODI (4.31) additionally contains $\int_\Omega |\Delta c|^2$ and $\int_\Omega |\nabla n|^2$ as absorptive terms, correspondingly obtained bounds for the latter provide sufficient regularity, inter alia for c in a space now again compactly sitting in $L^\infty(\Omega)$, so as to allow for the claimed conclusion on decay of c in Section 4.3. This knowledge on eventual uniform smallness of c will enable us to assert suitable prevalence of diffusion in the first two equations in (1.7), which will finally yield both Theorem 1.1 (Section 4.4) and Theorem 1.2 (Section 5).

2 Preliminary results from existence theory

To begin with, let us recall the particular construction through which the solution (n,c,u) from Theorem A has been obtained in [54]; indeed, all our results for (n,c,u) will be derived on the basis of arguments referring to approximate solutions of appropriately regularized problems, upon taking limits. Following [54, Definition 2.1], for $\varepsilon \in (0,1)$ let us accordingly consider

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon = \Delta n_\varepsilon - \nabla \cdot \Big(n_\varepsilon S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon \Big), & x\in\Omega,\ t>0, \\ c_{\varepsilon t} + u_\varepsilon\cdot\nabla c_\varepsilon = \Delta c_\varepsilon - n_\varepsilon f(c_\varepsilon), & x\in\Omega,\ t>0, \\ u_{\varepsilon t} = \Delta u_\varepsilon + \nabla P_\varepsilon + n_\varepsilon \nabla\phi, \qquad \nabla\cdot u_\varepsilon = 0, & x\in\Omega,\ t>0, \\ \frac{\partial n_\varepsilon}{\partial\nu} = \frac{\partial c_\varepsilon}{\partial\nu} = 0, \quad u_\varepsilon = 0, & x\in\partial\Omega,\ t>0, \\ n_\varepsilon(x,0)=n_0(x), \quad c_\varepsilon(x,0)=c_0(x), \quad u_\varepsilon(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{2.1}$$

with

$$S_\varepsilon(x,n,c) := \rho_\varepsilon(x)\cdot\chi_\varepsilon(n)\cdot S(x,n,c), \qquad (x,n,c)\in\bar\Omega\times[0,\infty)^2, \tag{2.2}$$

where

$$\rho_\varepsilon \in C_0^\infty(\Omega) \quad \text{is such that} \quad 0\le\rho_\varepsilon\le 1 \text{ in } \Omega \quad \text{and} \quad \rho_\varepsilon \nearrow 1 \text{ in } \Omega \text{ as } \varepsilon\searrow 0$$

and where

$$\chi_\varepsilon \in C_0^\infty([0,\infty)) \quad \text{is such that} \quad 0\le\chi_\varepsilon\le 1 \text{ in } [0,\infty) \quad \text{and} \quad \chi_\varepsilon \nearrow 1 \text{ in } [0,\infty) \text{ as } \varepsilon\searrow 0.$$

According to [54, Lemma 2.2], each of these problems possesses a global classical solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ such that

$$\begin{cases} n_\varepsilon \in C^0(\bar\Omega\times[0,\infty)) \cap C^{2,1}(\bar\Omega\times(0,\infty)), \\ c_\varepsilon \in C^0(\bar\Omega\times[0,\infty)) \cap C^{2,1}(\bar\Omega\times(0,\infty)), \\ u_\varepsilon \in C^0(\bar\Omega\times[0,\infty);\mathbb{R}^2) \cap C^{2,1}(\bar\Omega\times(0,\infty);\mathbb{R}^2) \qquad \text{and} \\ P_\varepsilon \in C^{1,0}(\Omega\times(0,\infty)), \end{cases}$$

and such that n_ε and c_ε are nonnegative. Moreover, Lemma 3.5, Lemma 4.2 and Lemma 4.4 in [54] ensure the existence of $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_j \searrow 0$ as $j\to\infty$, and such that with (n,c,u) as in Theorem A we have

$$\begin{cases} n_\varepsilon \to n & \text{in } L^1_{loc}(\bar\Omega\times[0,\infty)) \text{ and a.e. in } \Omega\times(0,\infty), \\ c_\varepsilon \to c & \text{in } L^2_{loc}([0,\infty);W^{1,2}(\Omega)) \text{ and a.e. in } \Omega\times(0,\infty) \qquad \text{and} \\ u_\varepsilon \rightharpoonup u & \text{in } L^2_{loc}(\bar\Omega\times[0,\infty)) \end{cases} \tag{2.3}$$

as $\varepsilon=\varepsilon_j\searrow 0$. Some basic properties of these approximate solutions, inter alia containing a rigorous counterpart of (1.4), are immediate from (2.1) (cf. [54, Lemma 2.3]).

Lemma 2.1. *Let $\varepsilon \in (0,1)$. Then*

$$\int_\Omega n_\varepsilon(x,t)dx = \int_\Omega n_0 \qquad \textit{for all } t > 0, \tag{2.4}$$

and for each $p \in [1,\infty]$ we have

$$\|c_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le \|c_\varepsilon(\cdot,s)\|_{L^p(\Omega)} \qquad \textit{for all } s \ge 0 \textit{ and each } t \ge s. \tag{2.5}$$

Moreover,

$$\int_0^\infty \int_\Omega |\nabla c_\varepsilon|^2 \le \frac{1}{2}\int_\Omega c_0^2 \tag{2.6}$$

and

$$\int_0^\infty \int_\Omega n_\varepsilon f(c_\varepsilon) \le \int_\Omega c_0 \tag{2.7}$$

as well as

$$|S_\varepsilon(x,n_\varepsilon,c_\varepsilon)| \le S_1 := S_0(\|c_0\|_{L^\infty(\Omega)}) \qquad \textit{for all } x \in \Omega \textit{ and } t \ge 0. \tag{2.8}$$

The following consequence of (2.6) on the spatial gradient of $\ln(n_\varepsilon+1)$ has already played an important role in the development of the existence theory in [54]. Since it involves an integral over the whole infinite time interval $(0,\infty)$, together with (2.6) and (2.7) it will moreover form the basis for our derviation of Theorem 1.1 and Theorem 1.2.

Lemma 2.2. *With S_1 taken from (2.8), we have*

$$\int_0^\infty \int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} \le 2\int_\Omega n_0 + \frac{S_1^2}{2}\cdot\int_\Omega c_0^2 \qquad \textit{for all } \varepsilon \in (0,1). \tag{2.9}$$

By means of the Moser-Trudinger inequality, and hence essentially relying on the planarity of the current setting, the latter can readily be seen to imply the following ([54, Lemma 4.1]).

Lemma 2.3. *Let $p > 0$. Then there exists $C = C(p) > 0$ such that for all $\varepsilon \in (0,1)$ we have*

$$\int_0^T \ln\left\{\frac{1}{|\Omega|}\int_\Omega \Big(n_\varepsilon(x,s)+1\Big)^p dx\right\}ds \le C\cdot(T+1) \qquad \textit{for all } T > 0. \tag{2.10}$$

As a last preliminary, let us furthermore recall from [54, Lemma 3.3] a consequence of the mass conservation property (2.4) on the regularity of u_ε, emphasizing that this conclusion strongly relies on the Stokes simplification underlying (1.7); in fact, the main reason for not addressing here the associated full chemotaxis-Navier-Stokes system, as e.g. corresponding to the choice $\kappa = 1$ in (1.3), consists in an apparent lack of any appropriate similar regularity feature in the latter.

Lemma 2.4. *Let $p \in (1,\infty)$. Then there exists $C = C(p) > 0$ such that whenever $\varepsilon \in (0,1)$,*

$$\|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le C \qquad \textit{for all } t > 0. \tag{2.11}$$

3 Further basic estimates

3.1 Weak decay properties

An important consequence of Lemma 2.2 is a first quantitative information, albeit yet quite weak, on asymptotic homogenization of n_ε.

Lemma 3.1. *Given $\varepsilon \in (0,1)$, we let*

$$w_\varepsilon(t) := \fint_\Omega \sqrt{n_\varepsilon(x,t)+1}\,dx \quad \textit{and} \quad \kappa_\varepsilon(t) := w_\varepsilon^2(t) - 1 \qquad \textit{for } t \ge 0. \tag{3.1}$$

Then

$$\kappa_\varepsilon(t) \le \overline{n_0} \qquad \textit{for all } t \ge 0 \textit{ and each } \varepsilon \in (0,1), \tag{3.2}$$

and there exists $C > 0$ such that

$$\int_0^\infty \bigg(\int_\Omega |n_\varepsilon(x,t) - \kappa_\varepsilon(t)| dx \bigg)^2 dt \le C \qquad \textit{for all } \varepsilon \in (0,1). \tag{3.3}$$

PROOF. In order to verify (3.2), we only need to apply the Cauchy-Schwarz inequality in estimating

$$\begin{aligned} \kappa_\varepsilon(t) &= \Big(\frac{1}{|\Omega|} \int_\Omega \sqrt{n_\varepsilon(x,t)+1} dx \Big)^2 - 1 \\ &\le \frac{1}{|\Omega|} \int_\Omega \Big(n_\varepsilon(x,t) + 1 \Big) dx - 1 \qquad \text{for all } t \ge 0 \text{ and each } \varepsilon \in (0,1), \end{aligned}$$

and observe that here the right-hand side indeed coincides with $\overline{n_0}$ thanks to (2.4).

To prove (3.3), according to Lemma 2.2 we first pick $C_1 > 0$ fulfilling

$$\int_0^\infty \int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} dxdt \le C_1 \qquad \text{for all } \varepsilon \in (0,1). \tag{3.4}$$

In order to make appropriate use of this, we again invoke the Cauchy-Schwarz inequality and (2.4) to estimate

$$\begin{aligned} \int_\Omega |\nabla \sqrt{n_\varepsilon + 1}| &= \frac{1}{2} \int_\Omega \frac{|\nabla n_\varepsilon|}{\sqrt{n_\varepsilon+1}} \\ &\le \frac{1}{2} \bigg(\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} \bigg)^{\frac{1}{2}} \cdot \bigg(\int_\Omega (n_\varepsilon + 1) \bigg)^{\frac{1}{2}} \\ &= C_2 \bigg(\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} \bigg)^{\frac{1}{2}} \qquad \text{for all } t > 0 \end{aligned} \tag{3.5}$$

with $C_2 := \frac{1}{2}\sqrt{\int_\Omega n_0 + |\Omega|}$. Now thanks to the continuity of the embedding $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$ in the present two-dimensional setting, by the corresponding Poincaré-Sobolev inequality we can fix $C_3 > 0$ such that

$$\int_\Omega \Big| \varphi(x) - \fint_\Omega \varphi \Big|^2 dx \le C_3 \bigg(\int_\Omega |\nabla \varphi(x)| dx \bigg)^2 \qquad \text{for all } \varphi \in W^{1,1}(\Omega).$$

Therefore, (3.5) and (3.4) show that

$$\begin{aligned}\int_0^\infty \int_\Omega \Big|\sqrt{n_\varepsilon(x,t)+1} - w_\varepsilon(t)\Big|^2 dxdt &\le C_3C_2^2 \int_0^\infty \int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} \\ &\le C_1C_2^2C_3 \qquad \text{for all } \varepsilon\in(0,1). \end{aligned} \tag{3.6}$$

Again due to the Cauchy-Schwarz inequality,

$$\begin{aligned}\int_\Omega \Big|(n_\varepsilon(x,t)+1) - w_\varepsilon^2(t)\Big| dx &= \int_\Omega \Big|\sqrt{n_\varepsilon(x,t)+1} - w_\varepsilon(t)\Big| \cdot \Big|\sqrt{n_\varepsilon(x,t)+1} + w_\varepsilon(t)\Big| dx \\ &\le \bigg(\int_\Omega \Big|\sqrt{n_\varepsilon(x,t)+1} - w_\varepsilon(t)\Big|^2 dx\bigg)^{\frac12} \times \\ &\quad \times \bigg(\int_\Omega \Big|\sqrt{n_\varepsilon(x,t)+1} + w_\varepsilon(t)\Big|^2 dx\bigg)^{\frac12} \qquad \text{for all } t>0,\end{aligned}$$

where by the same token and (2.4) we have

$$\begin{aligned}\int_\Omega \Big|\sqrt{n_\varepsilon(x,t)+1} + w_\varepsilon(t)\Big|^2 dx &\le 2\int_\Omega \Big(n_\varepsilon(x,t)+1+w_\varepsilon^2(t)\Big) dx \\ &= 2\int_\Omega n_0 + 2|\Omega| + 2w_\varepsilon^2(t)|\Omega| \\ &= 2\int_\Omega n_0 + 2|\Omega| + \frac{2}{|\Omega|}\bigg(\int_\Omega \sqrt{n_\varepsilon(x,t)+1} dx\bigg)^2 \\ &\le 2\int_\Omega n_0 + 2|\Omega| + 2\bigg(\int_\Omega \Big(n_\varepsilon(x,t)+1\Big) dx\bigg) \\ &= C_4 := 4\int_\Omega n_0 + 4|\Omega| \qquad \text{for all } t>0.\end{aligned}$$

From (3.6) we thus infer that

$$\int_0^\infty \bigg(\int_\Omega \Big|n_\varepsilon(x,t)+1-w_\varepsilon^2(t)\Big| dx\bigg)^2 dt \le C_1C_2^2C_3C_4 \qquad \text{for all } \varepsilon\in(0,1)$$

and conclude. □

We next derive from Lemma 3.1 a decay property of u_ε in some integral sense, involving certain fractional powers of the Stokes operator A as introduced in the context of (1.12) above.

Lemma 3.2. *Let $\alpha>0$. Then there exists $C>0$ such that for any $\varepsilon\in(0,1)$ we have*

$$\int_0^\infty \int_\Omega \big|A^{\frac12-\alpha} u_\varepsilon(x,t)\big|^2 dxdt \le C. \tag{3.7}$$

PROOF. Upon applying the Helmholtz projection, from the third equation in (2.1) we obtain that $u_{\varepsilon t} + Au_\varepsilon = \mathcal{P}[n_\varepsilon\nabla\phi]$ in $\Omega\times(0,\infty)$, so that according to the projection property of $\mathcal{P}$, taking κ_ε from (3.1) we can herein decompose

$$\mathcal{P}[n_\varepsilon(\cdot,t)\nabla\phi] = \mathcal{P}\Big[(n_\varepsilon(\cdot,t)-\kappa_\varepsilon(t))\nabla\phi\Big] + \kappa_\varepsilon(t)\mathcal{P}[\nabla\phi] = \mathcal{P}\Big[(n_\varepsilon(\cdot,t)-\kappa_\varepsilon(t))\nabla\phi\Big] \quad \text{in } \Omega$$

for all $t > 0$, whence actually

$$u_{\varepsilon t} + Au_\varepsilon = \mathcal{P}\Big[(n_\varepsilon(\cdot,t) - \kappa_\varepsilon(t))\nabla\phi\Big] \qquad \text{in } \Omega \times (0,\infty).$$

Given $\alpha > 0$, we multiply this by $A^{-2\alpha}u_\varepsilon$ and integrate over Ω to see using the symmetry of the operators $A^{-\beta}$, $\beta > 0$, that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |A^{-\alpha}u_\varepsilon|^2 + \int_\Omega |A^{\frac{1-2\alpha}{2}}u_\varepsilon|^2 &= \int_\Omega A^{-2\alpha}u_\varepsilon \cdot u_{\varepsilon t} + \int_\Omega A^{-2\alpha}u_\varepsilon \cdot Au_\varepsilon \\ &= \int_\Omega A^{-2\alpha}u_\varepsilon \cdot \mathcal{P}\Big[(n_\varepsilon(\cdot,t) - \kappa_\varepsilon(t))\nabla\phi\Big] \\ &= \int_\Omega A^{-2\alpha}u_\varepsilon \cdot (n_\varepsilon(\cdot,t) - \kappa_\varepsilon(t))\nabla\phi \qquad \text{for all } t > 0, \end{aligned} \tag{3.8}$$

again because of the fact that $\mathcal{P}$ is a projector.
Now if we let B denote the realization of $-\Delta$ in $L^2(\Omega)$ under homogeneous Dirichlet boundary conditions on Ω, then it is well-known that for the respective domains we have $D(A^\beta) = D(B^\beta) \cap L^2_\sigma(\Omega)$ for all $\beta > 0$ ([12]). Correspondingly, in the presently considered two-dimensional case we have $D(A^\beta) \hookrightarrow L^\infty(\Omega;\mathbb{R}^2)$ whenever $\beta > \frac{1}{2}$ ([14]), so that in particular we can fix $C_1 > 0$ satisfying

$$\|\varphi\|_{L^\infty(\Omega)} \le C_1\|A^{\frac{1+2\alpha}{2}}\varphi\|_{L^2(\Omega)} \qquad \text{for all } \varphi \in D(A^{\frac{1+2\alpha}{2}}).$$

Applying this to $\varphi := A^{-2\alpha}u_\varepsilon(\cdot,t)$, we can estimate the integral on the right of (3.8) according to

$$\begin{aligned} \int_\Omega A^{-2\alpha}u_\varepsilon \cdot (n_\varepsilon(\cdot,t) - \kappa_\varepsilon(t))\nabla\phi &\le \|A^{-2\alpha}u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\|n_\varepsilon(\cdot,t) - \kappa_\varepsilon(t)\|_{L^1(\Omega)}\|\nabla\phi\|_{L^\infty(\Omega)} \\ &\le C_2\|A^{\frac{1-2\alpha}{2}}u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\|n_\varepsilon(\cdot,t) - \kappa_\varepsilon(t)\|_{L^1(\Omega)} \qquad \text{for all } t > 0 \end{aligned}$$

with $C_2 := C_1\|\nabla\phi\|_{L^\infty(\Omega)}$. By means of Young's inequality, (3.8) thus implies that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |A^{-\alpha}u_\varepsilon|^2 + \frac{1}{2}\int_\Omega |A^{\frac{1-2\alpha}{2}}u_\varepsilon|^2 \le \frac{C_2^2}{2}\bigg(\int_\Omega |n_\varepsilon(\cdot,t) - \kappa_\varepsilon(t)|\bigg)^2 \qquad \text{for all } t > 0,$$

which upon integration yields

$$\begin{aligned} &\int_\Omega |A^{-\alpha}u_\varepsilon(x,t)|^2dx + \int_0^t\int_\Omega |A^{\frac{1-2\alpha}{2}}u_\varepsilon(x,s)|^2dxds \\ &\qquad \le \int_\Omega |A^{-\alpha}u_0(x)|^2dx + C_2^2\int_0^t\bigg(\int_\Omega |n_\varepsilon(x,s) - \kappa_\varepsilon(s)|dx\bigg)^2ds \qquad \text{for all } t > 0 \end{aligned}$$

and thereby implies (3.7) in view of Lemma 3.1. □

3.2 A further boundedness property of ∇c_ε

Let us next interpret the second equation in (2.1) as an inhomogeneous heat equation $c_{\varepsilon t} = \Delta c_\varepsilon + h_\varepsilon(x,t)$, and turn the regularity information on n_ε and u_ε collected so far into an estimate for ∇c_ε. With (2.4) apparently being the only relevant information for n_ε that is currently available, the range $[1,2)$ admissible for q in the following lemma seems natural in the two-dimensional context. The fact that here $h_\varepsilon = -n_\varepsilon f(c_\varepsilon) - u_\varepsilon \cdot \nabla c_\varepsilon$ itself contains ∇c_ε gives rise to an argument more involved than in related cases without such a dependence on the estimated quantity (see e.g. [17, Lemma 4.1]).

Lemma 3.3. *Let $q \in [1,2)$. Then there exists $C > 0$ such that for all $\varepsilon \in (0,1)$ we have*

$$\|\nabla c_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \le C \qquad \textit{for all } t > 0. \tag{3.9}$$

Proof. We may assume that $q > 1$, and we represent c_ε according to

$$c_\varepsilon(\cdot,t) = c_{\varepsilon 1}(\cdot,t) + c_{\varepsilon 2}(\cdot,t) + c_{\varepsilon 3}(\cdot,t), \qquad t > 0,$$

where

$$\begin{aligned} c_{\varepsilon 1}(\cdot,t) &:= e^{t\Delta} c_0, \\ c_{\varepsilon 2}(\cdot,t) &:= -\int_0^t e^{(t-s)\Delta} n_\varepsilon(\cdot,s) f(c_\varepsilon(\cdot,s)) ds \qquad \text{and} \\ c_{\varepsilon 3}(\cdot,t) &:= -\int_0^t e^{(t-s)\Delta} (u_\varepsilon \cdot \nabla c_\varepsilon)(\cdot,s) ds \end{aligned}$$

for $t > 0$, with $(e^{\tau\Delta})_{\tau \ge 0}$ denoting the Neumann heat semigroup over Ω. We moreover fix any $\mu \in (0,\lambda_1)$, where λ_1 denotes the first nonzero eigenvalue of the Neumann Laplacian in Ω. Then it is known ([14]) that the realization B_q of the operator $-\Delta + \mu$ under homogeneous Neumann boundary conditions in Ω is sectorial in $L^q(\Omega)$, and that the domains of the associated fractional powers satisfy

$$D(B_q^\gamma) \hookrightarrow W^{1,r}(\Omega) \qquad \text{whenever } r \ge 1 \text{ and } \gamma > \frac{1}{2} + \frac{1}{q} - \frac{1}{r}. \tag{3.10}$$

In particular, applying this to $r := q$ shows that in order to prove (3.9) it is sufficient to pick $\alpha \in (\frac{1}{2}, \frac{1}{q})$ and then verify the existence of $C_1 > 0$ such that

$$\|\nabla c_{\varepsilon 1}(\cdot,t)\|_{L^q(\Omega)} \le C_1, \quad \|\nabla c_{\varepsilon 2}(\cdot,t)\|_{L^q(\Omega)} \le C_1 \quad \text{and} \quad \|B_q^\alpha c_{\varepsilon 3}(\cdot,t)\|_{L^q(\Omega)} \le C_1 \qquad \text{for all } t > 0. \tag{3.11}$$

To achieve this, let us choose $p \in (q,2)$ close enough to q such that

$$\frac{1}{2} + \frac{1}{q} - \frac{1}{p} < \alpha, \tag{3.12}$$

and thereupon fix $\beta > 0$ such that

$$\frac{1}{2} + \frac{1}{q} - \frac{1}{p} < \beta < \alpha. \tag{3.13}$$

Then from (3.10) and (3.13) we infer that there exists $C_2 > 0$ satisfying

$$\|\nabla\varphi\|_{L^p(\Omega)} \le C_2 \|B_q^\beta \varphi\|_{L^q(\Omega)} \qquad \text{for all } \varphi \in C^2(\bar\Omega) \text{ such that } \tfrac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega, \tag{3.14}$$

and since $\alpha > \beta$, by interpolation ([10]) we can find $C_3 > 0$ such that

$$\|B_q^\beta \varphi\|_{L^q(\Omega)} \le C_3 \|B_q^\alpha \varphi\|_{L^q(\Omega)}^{\frac{\beta}{\alpha}} \cdot \|\varphi\|_{L^q(\Omega)}^{\frac{\alpha-\beta}{\alpha}} \qquad \text{for all } \varphi \in C^2(\bar\Omega) \text{ such that } \tfrac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega. \tag{3.15}$$

Moreover, in view of the fact that $\mu < \lambda_1$, known smoothing estimates for the Neumann heat semigroup ([10], [35], [46]) provide positive constants δ, C_4 and C_5 such that for all $t > 0$ we have

$$\|B_q^\alpha e^{t\Delta} \varphi\|_{L^q(\Omega)} \le C_4 t^{-\alpha} e^{-\delta t} \|\varphi\|_{L^q(\Omega)} \qquad \text{for all } \varphi \in L^q(\Omega) \text{ fulfilling } \int_\Omega \varphi = 0 \tag{3.16}$$

and

$$\|\nabla e^{t\Delta}\varphi\|_{L^q(\Omega)} \le C_5 t^{-\frac{3}{2}+\frac{1}{q}} e^{-\delta t}\|\varphi\|_{L^1(\Omega)} \qquad \text{for all } \varphi \in L^q(\Omega). \tag{3.17}$$

Since $p < 2$, we can now use the Hölder inequality and our assumption that $c_0 \in W^{1,2}(\Omega)$ implied by (1.12) to find $C_6 > 0$ and $C_7 > 0$ such that

$$\|\nabla c_{\varepsilon 1}(\cdot,t)\|_{L^p(\Omega)} \le C_6\|\nabla e^{t\Delta}c_0\|_{L^2(\Omega)} \le C_6\|\nabla c_0\|_{L^2(\Omega)} \le C_7 \qquad \text{for all } t > 0, \tag{3.18}$$

because it is clear that $\frac{d}{dt}\int_\Omega |\nabla e^{t\Delta}\varphi|^2 \le 0$ for all $t > 0$ and each $\varphi \in W^{1,2}(\Omega)$. As $p > q$, (3.18) in particular entails the first inequality in (3.11).
Similarly, a sharper variant of the second estimate in (3.11) can be obtained using (3.17), (2.4) and (2.5), which show that

$$\begin{aligned}
\|\nabla c_{\varepsilon 2}(\cdot,t)\|_{L^p(\Omega)} &\le \int_0^t \|\nabla e^{(t-s)\Delta} n_\varepsilon(\cdot,s) f(c_\varepsilon(\cdot,s))\|_{L^p(\Omega)} ds \\
&\le C_5 \int_0^t (t-s)^{-\frac{3}{2}+\frac{1}{q}} e^{-\delta(t-s)} \|n_\varepsilon(\cdot,s) f(c_\varepsilon(\cdot,s))\|_{L^1(\Omega)} ds \\
&\le C_5 \|n_0\|_{L^1(\Omega)} \cdot \|f\|_{L^\infty((0,\|c_0\|_{L^\infty(\Omega)}))} \cdot \int_0^t \sigma^{-\frac{3}{2}+\frac{1}{q}} e^{-\delta\sigma} d\sigma \qquad \text{for all } t > 0,
\end{aligned}$$

and that hence for some $C_8 > 0$ we have

$$\|\nabla c_{\varepsilon 2}(\cdot,t)\|_{L^p(\Omega)} \le C_8 \qquad \text{for all } t > 0, \tag{3.19}$$

because $-\frac{3}{2} + \frac{1}{q} > -1$ thanks to our hypothesis $q < 2$.
Having thereby also proved the second inequality in (3.11), we again make use of (3.18) and (3.19) in verifying the third: Namely, for each $\varepsilon \in (0,1)$ let us set

$$M_\varepsilon(T) := \sup_{t\in(0,T)} \|B_q^\alpha c_{\varepsilon 3}(\cdot,t)\|_{L^q(\Omega)}, \qquad T > 0.$$

Then given any $T > 0$ and $t \in (0,T)$ we can substitute $c_\varepsilon = c_{\varepsilon 1} + c_{\varepsilon 2} + c_{\varepsilon 3}$ in the definition of $c_{\varepsilon 3}$ to estimate

$$\begin{aligned}
\|B_q^\alpha c_{\varepsilon 3}(\cdot,t)\|_{L^q(\Omega)} &\le \sum_{i=1}^3 \left\| B_q^\alpha \int_0^t e^{(t-s)\Delta}\Big(u_\varepsilon(\cdot,s)\cdot\nabla c_{\varepsilon i}(\cdot,s)\Big) ds \right\|_{L^q(\Omega)} \\
&\le \sum_{i=1}^3 \int_0^t (t-s)^{-\alpha} e^{-\delta(t-s)} \Big\| u_\varepsilon(\cdot,s)\cdot\nabla c_{\varepsilon i}(\cdot,s)\Big\|_{L^q(\Omega)} ds
\end{aligned} \tag{3.20}$$

for all $t > 0$ by means of (3.16), noting that $\int_\Omega u_\varepsilon \cdot \nabla c_{\varepsilon i} = 0$ thanks to the solenoidality of u_ε. Here in view of the fact that $C_9 := \sup_{\varepsilon\in(0,1)} \|u_\varepsilon\|_{L^\infty((0,\infty);L^{\frac{pq}{p-q}}(\Omega))}$ is finite due to Lemma 2.4, upon applying the Hölder inequality we find that

$$\begin{aligned}
\Big\| u_\varepsilon(\cdot,s)\cdot\nabla c_{\varepsilon i}(\cdot,s)\Big\|_{L^q(\Omega)} &\le \|u_\varepsilon(\cdot,s)\|_{L^{\frac{pq}{p-q}}(\Omega)} \|\nabla c_{\varepsilon i}(\cdot,s)\|_{L^p(\Omega)} \\
&\le C_9 \|\nabla c_{\varepsilon i}(\cdot,s)\|_{L^p(\Omega)} \\
&\qquad \text{for all } s > 0 \text{ and } i \in \{1,2,3\},
\end{aligned} \tag{3.21}$$

where in the case $i=3$ we recall (3.14) and interpolate on the basis of (3.15) to see that according to (2.5),

$$\begin{aligned}\|\nabla c_{\varepsilon 3}(\cdot,s)\|_{L^p(\Omega)} &\le C_2\|B_q^\beta c_{\varepsilon 3}(\cdot,s)\|_{L^q(\Omega)} \\ &\le C_2C_3\|B_q^\alpha c_{\varepsilon 3}(\cdot,s)\|_{L^q(\Omega)}^{\frac{\beta}{\alpha}}\|c_{\varepsilon 3}(\cdot,s)\|_{L^q(\Omega)}^{\frac{\alpha-\beta}{\alpha}} \\ &\le C_{10}\|B_q^\alpha c_{\varepsilon 3}(\cdot,s)\|_{L^q(\Omega)}^{\frac{\beta}{\alpha}} \\ &\le C_{10}M_\varepsilon^{\frac{\beta}{\alpha}}(T) \qquad \text{for all } s\in(0,T)\end{aligned}$$

with $C_{10}:=C_2C_3\|c_0\|_{L^q(\Omega)}^{\frac{\alpha-\beta}{\alpha}}$.
Therefore, (3.20) and (3.21) combined with (3.18) and (3.19) yield

$$\|B_q^\alpha c_{\varepsilon 3}(\cdot,t)\|_{L^q(\Omega)} \le C_9\cdot\Big(C_7+C_8+C_{10}M_\varepsilon^{\frac{\beta}{\alpha}}(T)\Big)\cdot\int_0^t \sigma^{-\alpha}e^{-\delta\sigma}d\sigma \qquad \text{for all } t\in(0,T).$$

Since $\alpha<1$, this shows that with some $C_{11}>0$ we have

$$M_\varepsilon(T)\le C_{11}\cdot\Big(1+M_\varepsilon^{\frac{\beta}{\alpha}}(T)\Big) \qquad \text{for all } T>0 \text{ and } \varepsilon\in(0,1).$$

As $\beta<\alpha$ ensures that $\frac{\beta}{\alpha}<1$, from this we obtain $C_{12}>0$ such that

$$M_\varepsilon(T)\le C_{12} \qquad \text{for all } T>0 \text{ and } \varepsilon\in(0,1),$$

which implies the third inequality in (3.11) and hence completes the proof. □

4 Eventual smoothness. Proof of Theorem 1.1

We shall now turn our attention to the regularity and convergence properties of (n,c,u) claimed in Theorem 1.1 and Theorem 1.2. Accordingly, we shall tacitly assume that the additional hypotheses made therein are satisfied; that is, throughout the sequel we shall require that

$$f>0 \quad \text{on } (0,\infty) \qquad \text{and} \qquad n_0\not\equiv 0,$$

noting that this will actually be necessary only from Section 4.3 on.

4.1 Basic decay properties of $n-\overline{n_0}$, c and u

Let us list some essentially direct consequences of the spatio-temporal integrability properties in Lemma 2.1, Lemma 3.1 and Lemma 3.2, which may be viewed as providing some first asymptotic properties of the limit (n,c,u). The estimates (4.2) and (4.3) will be used in Lemma 4.7 to prove decay of c, whereas (4.4) and (4.5) will imply the claimed convergence properties of n and u in Theorem 1.2.

Lemma 4.1. *Let*

$$w(t) := \fint_\Omega \sqrt{n(x,t)+1}\,dx \quad \textit{and} \quad \kappa(t) := w^2(t) - 1 \qquad \textit{for } t>0. \tag{4.1}$$

Then

$$\int_0^\infty \int_\Omega |\nabla c|^2 < \infty \tag{4.2}$$

and

$$\int_0^\infty \int_\Omega n f(c) < \infty, \tag{4.3}$$

and moreover we have

$$\int_0^\infty \left(\int_\Omega \Big| n(x,t) - \kappa(t) \Big| dx \right)^2 dt < \infty \tag{4.4}$$

as well as

$$\int_0^\infty \int_\Omega |u|^2 < \infty. \tag{4.5}$$

Proof. Since thanks to (2.3) we know that as $\varepsilon = \varepsilon_j \searrow 0$ we have $n_\varepsilon \to n, f(c_\varepsilon) \to f(c)$ and $\nabla c_\varepsilon \to \nabla c$ a.e. in $\Omega \times (0,\infty)$, (4.2) and (4.3) are immediate from Lemma 2.1 in view of Fatou's lemma. Moreover, the strong convergence $n_\varepsilon \to n$ in $L^1_{loc}(\bar\Omega \times [0,\infty))$ entails that for a.e. $t>0$ we have $n_\varepsilon(\cdot,t) \to n(\cdot,t)$ in $L^1(\Omega)$ and hence clearly also $\int_\Omega \sqrt{n_\varepsilon(x,t)+1}dx \to \int_\Omega \sqrt{n(x,t)+1}dx$ as $\varepsilon = \varepsilon_j \searrow 0$. Accordingly, with w_ε and κ_ε as defined in Lemma 3.1 we see that $w_\varepsilon(t) \to w(t)$ and $\kappa_\varepsilon(t) \to \kappa(t)$ for a.e. $t>0$ as $\varepsilon = \varepsilon_j \searrow 0$, so that (4.4) again becomes a consequence of Fatou's lemma when applied to (3.3).
Finally, invoking Lemma 3.2 with $\alpha := \frac{1}{2}$ we obtain $C_1 > 0$ fulfilling

$$\int_0^\infty \int_\Omega |u_\varepsilon(x,t)|^2 dxdt \le C_1 \qquad \text{for all } \varepsilon \in (0,1),$$

which allows us to pick another subsequence of $(\varepsilon_j)_{j\in\mathbb{N}}$ along which $u_\varepsilon \rightharpoonup z$ in $L^2(\Omega \times (0,\infty))$ with some $z \in L^2(\Omega \times (0,\infty);\mathbb{R}^2)$ which due to (2.3) clearly must coincide with u. By lower semicontinuity of the norm in $L^2(\Omega \times (0,\infty);\mathbb{R}^2)$ with respect to weak convergence, this entails (4.5). □

4.2 Eventual smallness of ∇n_ε and Δc_ε

The purpose of this section will be to prove that both $g_\varepsilon(t) := \int_\Omega |\nabla n_\varepsilon(\cdot,t)|^2$ and $h_\varepsilon(t) := \int_\Omega |\Delta c_\varepsilon(\cdot,t)|^2$ eventually become small in the sense that for some suitably large $T>0$, given any $t_0 \ge 1$ one can pick $t_\varepsilon \in (t_0, t_0+T)$ such that $\int_{t_\varepsilon}^{t_\varepsilon+T}(g_\varepsilon(t)+h_\varepsilon(t))dt$ lies below the arbitrarily prescribed number 1.
This result, to be established in Lemma 4.5, will be prepared by three lemmata, the first of which encounters the challenge of turning the inequalities (4.8) and (4.9) gained from Lemma 2.1 and Lemma 3.1 into some quantitative decay information which is essentially independent of $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$. Here in order to achieve an estimate involving n_ε in the space $L^2(\Omega)$, rather than merely in $L^1(\Omega)$ as suggested by (3.3), we make use of the inequality (4.10) implied by Lemma 2.3.

Lemma 4.2. *Let $\delta > 0$. Then there exists $T > 1$ with the property that for all $t_0 > 0$ and each $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ one can find $t_\varepsilon \in (t_0, t_0+T)$ fulfilling*

$$\int_\Omega |\nabla c_\varepsilon(x,t_\varepsilon)|^2 dx < \frac{\delta}{2} \tag{4.6}$$

and

$$\int_\Omega \Big| n_\varepsilon(x,t_\varepsilon) - \kappa_\varepsilon(t_\varepsilon)\Big|^2 dx < \frac{\delta}{2}, \tag{4.7}$$

where κ_ε is as defined in (3.1).

Proof. According to Lemma 2.1, writing $C_1 := \frac{1}{2}\int_\Omega c_0^2$ we know that

$$\int_0^\infty \int_\Omega |\nabla c_\varepsilon(x,t)|^2 dxdt \le C_1 \qquad \text{for all } \varepsilon \in (0,1), \tag{4.8}$$

and Lemma 3.1 provides $C_2 > 0$ such that

$$\int_0^\infty \bigg(\int_\Omega \Big| n_\varepsilon(x,t) - \kappa_\varepsilon(t)\Big| dx\bigg)^2 dt \le C_2 \qquad \text{for all } \varepsilon \in (0,1), \tag{4.9}$$

whereas from Lemma 2.3 we obtain $C_3 > 0$ satisfying

$$\int_t^{t+1} \ln\bigg\{ \frac{1}{|\Omega|} \int_\Omega \Big(n_\varepsilon(x,t)+1\Big)^3 dx \bigg\} dt \le C_3 \qquad \text{for all } \varepsilon \in (0,1) \text{ and each } t \ge 0. \tag{4.10}$$

We thereupon abbreviate

$$C_4 := \Big(e^{\frac{4C_3}{3}} \cdot |\Omega|^{\frac{1}{3}} + \overline{n_0}\cdot|\Omega|^{\frac{1}{3}}\Big)^{\frac{3}{2}} \tag{4.11}$$

with $\overline{n_0} := \fint_\Omega n_0$, choose $l \in \mathbb{N}$ such that

$$l \ge \frac{8C_1}{\delta} \qquad \text{and} \qquad l \ge \frac{64 C_2 C_4^4}{\delta^4}, \tag{4.12}$$

and we shall see that the desired conclusion holds if we let

$$T := 2l+2. \tag{4.13}$$

To this end, given $t_0 > 0$ we fix $k_0 \in \mathbb{N}$ such that

$$t_0 < k_0 \le t_0 + 1 \tag{4.14}$$

and introduce the numbers

$$a_{k,\varepsilon} := \int_k^{k+1} \int_\Omega |\nabla c_\varepsilon(x,t)|^2 dxdt$$

and

$$b_{k,\varepsilon} := \int_k^{k+1} \bigg(\int_\Omega \Big| n_\varepsilon(x,t) - \kappa_\varepsilon(t)\Big| dx\bigg)^2 dt$$

for $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ and nonnegative integers k, where again κ_ε has been taken from (3.1).
We then claim that

$$\text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}} \text{ there exists } k_\varepsilon \in \{k_0, ..., k_0+2l\} \text{ such that}$$
$$a_{k_\varepsilon,\varepsilon} < \frac{\delta}{8} \qquad \text{and} \qquad b_{k_\varepsilon,\varepsilon} < \frac{\delta^4}{64C_4^4}. \tag{4.15}$$

Indeed, if this was false then there would exist some $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ such that for each $k \in \{k_0, ..., k_0+2l\}$ we would either have $a_{k,\varepsilon} \ge \frac{\delta}{8}$ or $b_{k,\varepsilon} \ge \frac{\delta^4}{64C_4^4}$. Since the index set $I := \{k_0, ..., k_0+2l\}$ has more than $2l$ elements, this would mean that either

$$I_a := \left\{ k \in I \;\middle|\; a_{k,\varepsilon} \ge \frac{\delta}{8} \right\}$$

or

$$I_b := \left\{ k \in I \;\middle|\; b_{k,\varepsilon} \ge \frac{\delta^4}{64C_4^4} \right\}$$

would contain more than l elements. In the former case, this would contradict (4.8), because by definition of I_a and the left inequality in (4.12) we then would have

$$C_1 \ge \int_0^\infty \int_\Omega |\nabla c_\varepsilon|^2 = \sum_{k=0}^\infty a_{k,\varepsilon} \ge \sum_{k\in I_a} a_{k,\varepsilon} \ge \frac{\delta}{8} \cdot |I_a| > \frac{\delta}{8} \cdot l \ge C_1.$$

Likewise, if $|I_b| > l$ then by (4.9) and the right inequality in (4.12),

$$C_2 \ge \int_0^\infty \left(\int_\Omega \Big| n_\varepsilon(x,t) - \kappa_\varepsilon(t) \Big| dx \right)^2 dt = \sum_{k=0}^\infty b_{k,\varepsilon} \ge \sum_{k\in I_b} b_{k,\varepsilon} \ge \frac{\delta^4}{64C_4^4} \cdot |I_b| > \frac{\delta^4}{64C_4^4} \cdot l \ge C_2,$$

which again is absurd.

Having thereby proved (4.15), we introduce the sets

$$Q_\varepsilon := \left\{ t \in (k_\varepsilon, k_\varepsilon+1) \;\middle|\; \int_\Omega |\nabla c_\varepsilon(x,t)|^2 dx < \frac{\delta}{2} \right\}$$

and

$$R_\varepsilon := \left\{ t \in (k_\varepsilon, k_\varepsilon+1) \;\middle|\; \left(\int_\Omega \Big| n_\varepsilon(x,t) - \kappa_\varepsilon(t) \Big| dx \right)^2 < \frac{\delta^4}{16C_4^4} \right\}$$

as well as

$$S_\varepsilon := \left\{ t \in (k_\varepsilon, k_\varepsilon+1) \;\middle|\; \ln \left\{ \frac{1}{|\Omega|} \int_\Omega \Big(n_\varepsilon(x,t) + 1 \Big)^3 dx \right\} < 4C_3 \right\}$$

for $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$. Then the first inequality in (4.15) implies that

$$\begin{aligned}
\frac{\delta}{8} &> a_{k_\varepsilon,\varepsilon} \\
&= \int_{k_\varepsilon}^{k_\varepsilon+1} \int_\Omega |\nabla c_\varepsilon(x,t)|^2 dxdt \\
&\ge \int_{(k_\varepsilon,k_\varepsilon+1)\setminus Q_\varepsilon} \int_\Omega |\nabla c_\varepsilon(x,t)|^2 dxdt \\
&\ge \frac{\delta}{2} \cdot \Big|(k_\varepsilon, k_\varepsilon+1) \setminus Q_\varepsilon\Big| \\
&= \frac{\delta}{2} \cdot \Big(1 - |Q_\varepsilon|\Big),
\end{aligned}$$

from which we conclude that

$$|Q_\varepsilon| > \frac{3}{4}. \tag{4.16}$$

Likewise, from the second inequality in (4.15) we obtain that

$$\begin{aligned}
\frac{\delta^4}{64C_4^4} &> b_{k_\varepsilon,\varepsilon} \\
&= \int_{k_\varepsilon}^{k_\varepsilon+1} \bigg(\int_\Omega \Big| n_\varepsilon(x,t) - \kappa_\varepsilon(t) \Big| dx \bigg)^2 dt \\
&\ge \int_{(k_\varepsilon,k_\varepsilon+1)\setminus R_\varepsilon} \bigg(\int_\Omega \Big| n_\varepsilon(x,t) - \kappa_\varepsilon(t) \Big| dx \bigg)^2 dt \\
&\ge \frac{\delta^4}{16C_4^4} \cdot \Big|(k_\varepsilon, k_\varepsilon+1) \setminus R_\varepsilon\Big| \\
&= \frac{\delta^4}{16C_4^4} \cdot \Big(1 - |R_\varepsilon|\Big),
\end{aligned}$$

that is,

$$|R_\varepsilon| > \frac{3}{4}. \tag{4.17}$$

Finally, (4.10) ensures that

$$\begin{aligned}
C_3 &\ge \int_{k_\varepsilon}^{k_\varepsilon+1} \ln \bigg\{ \frac{1}{|\Omega|} \int_\Omega \Big(n_\varepsilon(x,t)+1 \Big)^3 dx \bigg\} dt \\
&\ge \int_{(k_\varepsilon,k_\varepsilon+1)\setminus S_\varepsilon} \ln \bigg\{ \frac{1}{|\Omega|} \int_\Omega \Big(n_\varepsilon(x,t)+1 \Big)^3 dx \bigg\} dt \\
&\ge 4C_3 \cdot \Big|(k_\varepsilon, k_\varepsilon+1) \setminus S_\varepsilon\Big| \\
&\ge 4C_3 \cdot \Big(1 - |S_\varepsilon|\Big)
\end{aligned}$$

and hence also

$$|S_\varepsilon| > \frac{3}{4}. \tag{4.18}$$

Combining (4.16)-(4.18), we see that $|Q_\varepsilon \cap R_\varepsilon \cap S_\varepsilon| > \frac{1}{4}$, so that for each $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ we can find at least one t_ε belonging to $Q_\varepsilon \cap R_\varepsilon \cap S_\varepsilon$.

Thereupon, (4.6) is an evident consequence of the inclusion $t_\varepsilon \in Q_\varepsilon$ and the definition of Q_ε. In order to show that also (4.7) holds, we first invoke the Hölder inequality to estimate

$$\begin{aligned}\int_\Omega \Big| n_\varepsilon(x,t_\varepsilon) - \kappa_\varepsilon(t_\varepsilon)\Big|^2 dx &\le \|n_\varepsilon(\cdot,t_\varepsilon) - \kappa_\varepsilon(t_\varepsilon)\|_{L^3(\Omega)}^{\frac{3}{2}} \cdot \|n_\varepsilon(\cdot,t_\varepsilon) - \kappa_\varepsilon(t_\varepsilon)\|_{L^1(\Omega)}^{\frac{1}{2}} \\ &\le \Big(\|n_\varepsilon(\cdot,t_\varepsilon)\|_{L^3(\Omega)} + \kappa_\varepsilon(t_\varepsilon)\cdot|\Omega|^{\frac{1}{3}}\Big)^{\frac{3}{2}} \cdot \bigg(\int_\Omega \Big|n_\varepsilon(x,t_\varepsilon) - \kappa_\varepsilon(t_\varepsilon)\Big| dx\bigg)^{\frac{1}{2}}.\end{aligned}$$

Thus, using that $\kappa_\varepsilon(t) \le \overline{n_0}$ for all $t \ge 0$ and $\varepsilon \in (0,1)$ by (3.2), from the fact that $t_\varepsilon \in R_\varepsilon \cap S_\varepsilon$ and the definition of C_4 we infer that indeed

$$\begin{aligned}\int_\Omega \Big| n_\varepsilon(x,t_\varepsilon) - \kappa_\varepsilon(t_\varepsilon)\Big|^2 dx &< \Big(e^{\frac{4C_3}{3}} \cdot |\Omega|^{\frac{1}{3}} + \overline{n_0}\cdot|\Omega|^{\frac{1}{3}}\Big)^{\frac{3}{2}} \cdot \Big(\frac{\delta^4}{16C_4^4}\Big)^{\frac{1}{4}} \\ &= C_4 \cdot \frac{\delta}{2C_4} = \frac{\delta}{2}.\end{aligned}$$

Finally, from the construction of k_ε, (4.14) and the fact that $t_\varepsilon \in (k_\varepsilon, k_\varepsilon+1)$ it follows that

$$t_\varepsilon > k_\varepsilon \ge k_0 > t_0$$

and, by (4.13), that moreover

$$t_\varepsilon < k_\varepsilon + 1 \le k_0 + 2l + 1 \le t_0 + 2l + 2 = t_0 + T,$$

so that in fact $t_\varepsilon \in (t_0, t_0+T)$, as claimed. □

Our goal will be to use the above inequalities (4.6) and (4.7) to control, for a certain $\mu \ge 0$, the initial size of $y_\varepsilon(t) := \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |n_\varepsilon - \mu|^2$ when considered as a function for $t \ge t_\varepsilon$. Indeed, in Lemma 4.5 we shall see that an appropriately small initial bound on y_ε will remain essentially unchanged throughout some time interval with small but ε-independent size. This will be accomplished in Lemma 4.5 on the basis of a suitable ODI for y_ε which will be prepared by separately tracking the time evolution of the two summands making up y_ε. Let us begin by deriving an inequality for the time derivative of $\int_\Omega |\nabla c_\varepsilon|^2$ by means of a standard testing procedure.

Lemma 4.3. *There exists $C > 0$ such that for all $\varepsilon \in (0,1)$ we have*

$$\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |\Delta c_\varepsilon|^2 \le \frac{1}{2}\int_\Omega |\nabla n_\varepsilon|^2 + C\int_\Omega |\nabla c_\varepsilon|^2 \qquad \textit{for all } t > 1. \tag{4.19}$$

Proof. We multiply the second equation in (2.1) by $-\Delta c_\varepsilon$ to see upon integrating by parts that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |\Delta c_\varepsilon|^2 &= \int_\Omega n_\varepsilon c_\varepsilon \Delta c_\varepsilon + \int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)\Delta c_\varepsilon \\ &= -\int_\Omega n_\varepsilon |\nabla c_\varepsilon|^2 - \int_\Omega c_\varepsilon \nabla n_\varepsilon \cdot \nabla c_\varepsilon + \int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon)\Delta c_\varepsilon \end{aligned} \tag{4.20}$$

for all $t > 0$. Here by Young's inequality and (2.5) we obtain

$$
\begin{aligned}
-\int_\Omega c_\varepsilon \nabla n_\varepsilon \cdot \nabla c_\varepsilon &\le \frac{1}{4}\int_\Omega |\nabla n_\varepsilon|^2 + \int_\Omega c_\varepsilon^2 |\nabla c_\varepsilon|^2 \\
&\le \frac{1}{4}\int_\Omega |\nabla n_\varepsilon|^2 + \|c_0\|_{L^\infty(\Omega)}^2 \int_\Omega |\nabla c_\varepsilon|^2 \qquad \text{for all } t > 0. \qquad (4.21)
\end{aligned}
$$

Moreover, using the Hölder inequality we find that

$$
\int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon)\Delta c_\varepsilon \le \Big(\int_\Omega |u_\varepsilon|^4\Big)^{\frac{1}{4}} \cdot \Big(\int_\Omega |\nabla c_\varepsilon|^4\Big)^{\frac{1}{4}} \cdot \Big(\int_\Omega |\Delta c_\varepsilon|^2\Big)^{\frac{1}{2}} \qquad \text{for all } t > 0, \qquad (4.22)
$$

where an application of the Gagliardo-Nirenberg inequality along with standard elliptic regularity theory provides $C_1 > 0$ fulfilling

$$
\Big(\int_\Omega |\nabla c_\varepsilon|^4\Big)^{\frac{1}{4}} \le C_1 \Big(\int_\Omega |\Delta c_\varepsilon|^2\Big)^{\frac{1}{4}} \cdot \Big(\int_\Omega |\nabla c_\varepsilon|^2\Big)^{\frac{1}{4}} \qquad \text{for all } t > 0. \qquad (4.23)
$$

The integral involving u_ε can be controlled using Lemma 2.4, which yields $C_2 > 0$ such that

$$
\|u_\varepsilon(\cdot,t)\|_{L^4(\Omega)} \le C_2 \qquad \text{for all } t > 0, \qquad (4.24)
$$

so that combining this with (4.22) and (4.23), once more by Young's inequality we obtain $C_3 > 0$ satisfying

$$
\begin{aligned}
\int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon)\Delta c_\varepsilon &\le C_1 C_2 \Big(\int_\Omega |\Delta c_\varepsilon|^2\Big)^{\frac{3}{4}} \cdot \Big(\int_\Omega |\nabla c_\varepsilon|^2\Big)^{\frac{1}{4}} \\
&\le \frac{1}{2}\int_\Omega |\Delta c_\varepsilon|^2 + C_3 \int_\Omega |\nabla c_\varepsilon|^2 \qquad \text{for all } t > 1.
\end{aligned}
$$

Inserting this together with (4.21) into (4.20), upon dropping a nonpositive term we readily arrive at (4.19). □

The first integral on the right of (4.19), precluding the latter to become an autonomous ODI itself, can fortunately be compensated by adding the result of a suitable testing procedure in the first equation in (2.1). The appearance of the norm of $n_\varepsilon - \mu$ in $L^2(\Omega)$ in the following lemma a posteriori explains our choice of the norm in (4.7).

Lemma 4.4. *Let $\mu_0 > 0$. Then there exists $C > 0$ such that whenever $\mu \in (0, \mu_0]$ and $\varepsilon \in (0,1)$, we have*

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega \Big|n_\varepsilon(\cdot,t) - \mu\Big|^2 + \int_\Omega |\nabla n_\varepsilon|^2 &\le C\int_\Omega |\nabla c_\varepsilon|^2 + C \\
&+ C\Big(\int_\Omega \Big|n_\varepsilon(\cdot,t) - \mu\Big|^2\Big) \cdot \Big(\int_\Omega |\Delta c_\varepsilon|^2\Big) \cdot \Big(\int_\Omega |\nabla c_\varepsilon|^2\Big) \qquad \textit{for all } t > 0. \qquad (4.25)
\end{aligned}
$$

Proof. Testing the first equation in (2.1) against $n_\varepsilon - \mu$ and using Young's inequality and (2.8) yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |n_\varepsilon-\mu|^2 + \int_\Omega |\nabla n_\varepsilon|^2 &= \int_\Omega n_\varepsilon \nabla n_\varepsilon \cdot \Big(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big) \\
&= \int_\Omega (n_\varepsilon-\mu)\nabla n_\varepsilon \cdot \Big(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big) \\
&\quad +\mu\int_\Omega \nabla n_\varepsilon \cdot \Big(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big) \\
&\le \frac{1}{8}\int_\Omega |\nabla n_\varepsilon|^2 + C_1\int_\Omega |n_\varepsilon-\mu|^2\cdot|\nabla c_\varepsilon|^2 \\
&\quad +\frac{1}{8}\int_\Omega |\nabla n_\varepsilon|^2 + C_2\int_\Omega |\nabla c_\varepsilon|^2 \qquad \text{for all } t>0 \qquad (4.26)
\end{aligned}$$

with positive constants C_1 and C_2 possibly depending on μ_0 and the number S_1 in (2.8). By means of the Hölder inequality and a Gagliardo-Nirenberg interpolation, again involving standard elliptic regularity theory, we can further estimate

$$\begin{aligned}
C_1\int_\Omega |n_\varepsilon-\mu|^2\cdot|\nabla c_\varepsilon|^2 &\le C_1\Big(\int_\Omega |n_\varepsilon-\mu|^4\Big)^{\frac{1}{2}}\cdot\Big(\int_\Omega|\nabla c_\varepsilon|^4\Big)^{\frac{1}{2}} \\
&\le C_3\Big(\int_\Omega |n_\varepsilon-\mu|^4\Big)^{\frac{1}{2}}\cdot\Big(\int_\Omega|\Delta c_\varepsilon|^2\Big)^{\frac{1}{2}}\cdot\Big(\int_\Omega|\nabla c_\varepsilon|^2\Big)^{\frac{1}{2}} \qquad (4.27)
\end{aligned}$$

for all $t>0$ with some $C_3>0$, and once more invoking the Gagliardo-Nirenberg inequality and (2.4) we find $C_4>0$ and $C_5>0$ fulfilling

$$\begin{aligned}
\Big(\int_\Omega |n_\varepsilon-\mu|^4\Big)^{\frac{1}{2}} &= \|n_\varepsilon-\mu\|^2_{L^4(\Omega)} \\
&\le C_4\|\nabla n_\varepsilon\|_{L^2(\Omega)}\cdot\|n_\varepsilon-\mu\|_{L^2(\Omega)} + C_4\|n_\varepsilon-\mu\|^2_{L^1(\Omega)} \\
&\le C_4\|\nabla n_\varepsilon\|_{L^2(\Omega)}\cdot\|n_\varepsilon-\mu\|_{L^2(\Omega)} + C_5 \qquad \text{for all } t>0.
\end{aligned}$$

Inserted into (4.27), in view of Young's inequality this shows that there exists $C_6>0$ such that

$$C_1\int_\Omega |n_\varepsilon-\mu|^2\cdot|\nabla c_\varepsilon|^2 \le \frac{1}{4}\int_\Omega|\nabla n_\varepsilon|^2 + C_6 + C_6\Big(\int_\Omega|n_\varepsilon-\mu|^2\Big)\cdot\Big(\int_\Omega|\Delta c_\varepsilon|^2\Big)\cdot\Big(\int_\Omega|\nabla c_\varepsilon|^2\Big)$$

for all $t>0$. Along with (4.26), this proves (4.25). □

Now combining the previous two lemmata, we can indeed derive a favorable differential inequality for the coupled functional y_ε discussed above. According to (4.19) and (4.25), this inequality will in addition contain some dissipated quantity, and the bound correspondingly obtained for the latter will precisely yield (4.28) as the main outcome of this section.

Lemma 4.5. *There exist $T>0$ and $\tau\in(0,1)$ with the following property: For each $t_0\ge 1$ and all $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}$ one can find $t_\varepsilon\in(t_0,t_0+T)$ fulfilling*

$$\int_{t_\varepsilon}^{t_\varepsilon+\tau}\int_\Omega |\nabla n_\varepsilon(x,t)|^2dxdt + \int_{t_\varepsilon}^{t_\varepsilon+\tau}\int_\Omega|\Delta c_\varepsilon(x,t)|^2dxdt \le 1. \qquad (4.28)$$

PROOF. We invoke Lemma 4.3 and apply Lemma 4.4 to $\mu_0 := \overline{n_0} := \fint_\Omega n_0$ to find $C_1 > 0$ and $C_2 > 0$ such that for each $\varepsilon \in (0,1)$,

$$\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |\Delta c_\varepsilon|^2 \le \frac{1}{2}\int_\Omega |\nabla n_\varepsilon|^2 + C_1 \int_\Omega |\nabla c_\varepsilon|^2 \qquad \text{for all } t > 1, \tag{4.29}$$

and such that if moreover $\mu \in [0, \overline{n_0}]$ then

$$\begin{aligned} \frac{d}{dt}\int_\Omega \Big|n_\varepsilon(\cdot,t) - \mu\Big|^2 + \int_\Omega |\nabla n_\varepsilon|^2 &\le C_2 \int_\Omega |\nabla c_\varepsilon|^2 + C_2 \\ &+ C_2\Big(\int_\Omega \Big|n_\varepsilon(\cdot,t)-\mu\Big|^2\Big)\cdot\Big(\int_\Omega |\Delta c_\varepsilon|^2\Big)\cdot\Big(\int_\Omega |\nabla c_\varepsilon|^2\Big) \qquad \text{for all } t > 0. \end{aligned} \tag{4.30}$$

Adding both these inequalities we see that for any such μ,

$$\begin{aligned} &\frac{d}{dt}\Big\{\int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega \Big|n_\varepsilon(\cdot,t)-\mu\Big|^2\Big\} + \int_\Omega |\Delta c_\varepsilon|^2 + \frac{1}{2}\int_\Omega |\nabla n_\varepsilon|^2 \\ &\le (C_1 + C_2)\int_\Omega |\nabla c_\varepsilon|^2 + C_2 \\ &\quad + C_2 \cdot \Big(\int_\Omega \Big|n_\varepsilon(\cdot,t)-\mu\Big|^2\Big)\cdot\Big(\int_\Omega |\Delta c_\varepsilon|^2\Big)\cdot\Big(\int_\Omega |\nabla c_\varepsilon|^2\Big) \qquad \text{for all } t > 1. \end{aligned} \tag{4.31}$$

We now apply Lemma 4.2 to $\delta := \min\big\{\frac{1}{\sqrt{2C_2}}, 1\big\}$ and thereby obtain $T > 0$ such that for any choice of $t_0 > 1$ and $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ we can find $t_\varepsilon \in (t_0, t_0 + T)$ such that the function y_ε defined by

$$y_\varepsilon(t) := \int_\Omega |\nabla c_\varepsilon(x,t)|^2 dx + \int_\Omega \Big|n_\varepsilon(x,t) - \kappa_\varepsilon(t_\varepsilon)\Big|^2 dx, \qquad t > 0,$$

with κ_ε as in (3.1), satisfies

$$y_\varepsilon(t_\varepsilon) \le \frac{\delta}{4}. \tag{4.32}$$

We shall see that then the conclusion of the lemma is valid if we fix any $\tau \in (0,1)$ satisfying

$$\tau \le \frac{\delta}{4\big\{(C_1+C_2)\delta + C_2\big\}}. \tag{4.33}$$

To verify this, we note that due to (4.32),

$$\tau_\varepsilon := \sup\Big\{\tilde\tau_\varepsilon \in [0,\tau] \ \Big|\ y_\varepsilon(t) < \delta \text{ for all } t \in [t_\varepsilon, t_\varepsilon + \tilde\tau_\varepsilon)\Big\}$$

is a well-defined element of $[0,\tau]$, and we first claim that that actually $\tau_\varepsilon = \tau$ for any $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$. Indeed, by definition of τ_ε and continuity of y_ε we have

$$y_\varepsilon \le \delta \qquad \text{on } [t_\varepsilon, t_\varepsilon + \tau_\varepsilon]. \tag{4.34}$$

Along with our choice of δ, this implies that writing $\mu := \kappa_\varepsilon(t_\varepsilon)$ we have

$$\begin{aligned} C_2 \cdot \Big(\int_\Omega \Big| n_\varepsilon(\cdot,t) - \mu \Big|^2 \Big) \cdot \Big(\int_\Omega |\Delta c_\varepsilon|^2 \Big) \cdot \Big(\int_\Omega |\nabla c_\varepsilon|^2 \Big) &\le C_2 y_\varepsilon^2(t) \cdot \int_\Omega |\Delta c_\varepsilon|^2 \\ &\le C_2 \delta^2 \cdot \int_\Omega |\Delta c_\varepsilon|^2 \\ &\le \frac{1}{2} \int_\Omega |\Delta c_\varepsilon|^2 \qquad \text{for all } t \in [t_\varepsilon, t_\varepsilon + \tau_\varepsilon]. \end{aligned}$$

Since $\mu \le \overline{n_0}$ according to (3.2), we can thus invoke (4.31) to derive, again using (4.34), that

$$\begin{aligned} y_\varepsilon'(t) + \frac{1}{2} \int_\Omega |\Delta c_\varepsilon|^2 + \frac{1}{2} \int_\Omega |\nabla n_\varepsilon|^2 &\le (C_1 + C_2) \cdot \int_\Omega |\nabla c_\varepsilon|^2 + C_2 \\ &\le (C_1 + C_2) \cdot y_\varepsilon(t) + C_2 \\ &\le (C_1 + C_2) \cdot \delta + C_2 \qquad \text{for all } t \in [t_\varepsilon, t_\varepsilon + \tau_\varepsilon]. \end{aligned}$$

An integration using (4.32) thus implies that

$$\begin{aligned} y_\varepsilon(t) + \frac{1}{2} \int_{t_\varepsilon}^t \int_\Omega |\Delta c_\varepsilon|^2 + \int_{t_\varepsilon}^t \int_\Omega |\nabla n_\varepsilon|^2 &\le y_\varepsilon(t_\varepsilon) + \Big\{ (C_1 + C_2)\delta + C_2 \Big\} \cdot (t - t_\varepsilon) \\ &\le \frac{\delta}{4} + \Big\{ (C_1 + C_2)\delta + C_2 \Big\} \cdot \tau_\varepsilon \\ &< \frac{\delta}{4} + \Big\{ (C_1 + C_2)\delta + C_2 \Big\} \cdot \tau \\ &= \frac{\delta}{2} \qquad \text{for all } t \in [t_\varepsilon, t_\varepsilon + \tau_\varepsilon]. \end{aligned} \tag{4.35}$$

In particular, assuming for contradiction that $\tau_\varepsilon < \tau$ for some $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, from (4.35) we could derive the inequality $y_\varepsilon(t_\varepsilon + \tau_\varepsilon) \le \frac{\delta}{2}$ which is incompatible with the continuity of y_ε, the latter in this case implying that necessarily $y_\varepsilon(t_\varepsilon + \tau_\varepsilon) = \delta$.
Hence knowing that in fact $\tau_\varepsilon = \tau$, from (4.35) and the fact that $\delta \le 1$ we easily obtain (4.28), because y_ε is nonnegative. □

4.3 Doubly uniform decay of c_ε

We shall next make sure that under the extra assumptions $f > 0$ on $(0, \infty)$ and $n_0 \not\equiv 0$, as made in both Theorem 1.1 and Theorem 1.2, we have $c_\varepsilon(\cdot, t) \to 0$ as $t \to \infty$, and it will be of crucial importance for our proof of eventual smoothness that this convergence is not only uniform with respect to $x \in \Omega$, but in addition also uniform with respect to sufficiently small $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$.
To this end, in Lemma 4.7 we shall first prove a decay statement in a weaker topology for the single limit function c by combining the finiteness of the integrals in (4.2) and (4.3) with the following additional regularity property implied by Lemma 4.5.

Corollary 4.6. *Let $\tau \in (0, 1)$ be as given by Lemma 4.5. Then there exists $(t_k)_{k\in\mathbb{N}} \subset (1, \infty)$ such that*

$$\int_{t_k}^{t_k + \tau} \int_\Omega |\nabla n(x,t)|^2 dx dt \le 1 \qquad \textit{for all } k \in \mathbb{N}. \tag{4.36}$$

PROOF. This is an immediate consequence of Lemma 4.5. $\square$

Indeed, we can thereby show the following statement on temporal decay of c.

Lemma 4.7. *Let $\tau \in (0,1)$ and $(t_k)_{k\in\mathbb{N}}$ be as given by Lemma 4.5 and Corollary 4.6, respectively. Then*

$$\int_{t_k}^{t_k+\tau} \|c(\cdot,t)\|_{L^1(\Omega)} dt \to 0 \qquad \text{as } k \to \infty. \tag{4.37}$$

PROOF. Let us define

$$n_k(x,s) := n(x,t_k+s) \quad \text{and} \quad c_k(x,s) := c(x,t_k+s) \qquad \text{for } (x,s) \in \Omega \times (0,\tau) \text{ and } k \in \mathbb{N}$$

as well as

$$\lambda(t) := \begin{cases} \int_\Omega c(x,t)dx & \text{if } c(\cdot,t) \in L^1(\Omega), \\ 0 & \text{else,} \end{cases} \tag{4.38}$$

and

$$\lambda_k(s) := \lambda(t_k+s) \qquad \text{for } s \in (0,\tau) \text{ and } k \in \mathbb{N}.$$

Then since $[0,\infty) \ni t \mapsto \|c_\varepsilon(\cdot,t)\|_{L^1(\Omega)}$ in nonincreasing by Lemma 2.1, we know that $\int_{t_k}^{t_k+\tau} \lambda(t)dt$ is nonincreasing with respect to k and thus

$$\int_{t_k}^{t_k+\tau} \lambda(t)dt \searrow \lambda_\infty \qquad \text{as } k \to \infty \tag{4.39}$$

with some some constant $\lambda_\infty \geq 0$. In order to show that actually $\lambda_\infty = 0$, we first use the Poincaré inequality to find $C_1 > 0$ such that

$$\begin{aligned} \int_0^\tau \|c_k(\cdot,s) - \lambda_k(s)\|^2_{L^2(\Omega)} ds &= \int_{t_k}^{t_k+\tau} \|c(\cdot,t) - \lambda(t)\|^2_{L^2(\Omega)} dt \\ &\leq C_1 \int_{t_k}^{t_k+\tau} \|\nabla c(\cdot,t)\|^2_{L^2(\Omega)} dt \qquad \text{for all } k \in \mathbb{N}, \end{aligned}$$

whence recalling that $t_k \to \infty$ as $k \to \infty$ and that $\int_0^\infty \int_\Omega |\nabla c|^2 < \infty$ by Lemma 4.1 we obtain

$$\int_0^\tau \|c_k(\cdot,s) - \lambda_k(s)\|^2_{L^2(\Omega)} ds \to 0 \qquad \text{as } k \to \infty. \tag{4.40}$$

Since moreover $\lambda(t) \leq C_2 := \int_\Omega c_0(x)dx$ for all $t \geq 0$, (4.39) entails that

$$\begin{aligned} \int_{t_k}^{t_k+\tau} \int_\Omega |\lambda(t) - \lambda_\infty|^2 dxdt &= |\Omega| \cdot \int_{t_k}^{t_k+\tau} \Big(\lambda(t) + \lambda_\infty\Big) \cdot \Big(\lambda(t) - \lambda_\infty\Big) dt \\ &\leq 2C_2|\Omega| \cdot \left\{ \int_{t_k}^{t_k+\tau} \lambda(t)dt - \lambda_\infty \cdot \tau \right\} \\ &\to 0 \qquad \text{as } k \to \infty. \end{aligned}$$

Combined with (4.40), this shows that

$$\int_0^\tau \int_\Omega \Big| c_k(x,s) - \lambda_\infty \Big|^2 dxds \quad \le \quad 2\int_0^\tau \int_\Omega \Big| c_k(x,s) - \lambda_k(s) \Big|^2 dxds + 2\int_0^\tau \int_\Omega \Big| \lambda_k(s) - \lambda_\infty \Big|^2 dxds$$
$$\to \quad 0 \qquad \text{as } k \to \infty;$$

that is, we have

$$c_k \to \lambda_\infty \quad \text{in } L^2(\Omega \times (0,\tau)) \qquad \text{as } k \to \infty, \tag{4.41}$$

which clearly implies that also

$$f(c_k) \to f(\lambda_\infty) \quad \text{in } L^2(\Omega \times (0,\tau)) \qquad \text{as } k \to \infty, \tag{4.42}$$

because $\|c_k\|_{L^\infty(\Omega\times(0,\tau))} \le \|c_0\|_{L^\infty(\Omega)}$ thanks to (2.5) and (2.3). Now according to our choice of $(t_k)_{k\in\mathbb{N}}$, Corollary 4.6 says that

$$\int_0^\tau \int_\Omega |\nabla n_k(x,s)|^2 dxds = \int_{t_k}^{t_k+\tau} \int_\Omega |\nabla n(x,t)|^2 dxdt \le 1 \qquad \text{for all } k \in \mathbb{N},$$

which combined with (6.3) entails that $(n_k)_{k\in\mathbb{N}}$ is bounded in $L^2(\Omega\times(0,\tau))$. Passing to a subsequence if necessary we may thus assume that for some $n_\infty \in L^2(\Omega \times (0,\tau))$ we have

$$n_k \rightharpoonup n_\infty \quad \text{in } L^2(\Omega \times (0,\tau)) \qquad \text{as } k \to \infty. \tag{4.43}$$

In particular, again using (6.3) we obtain that with $m := \int_\Omega n_0$ we have

$$m\tau = \int_0^\tau \int_\Omega n_k(x,s) dxds \to \int_0^\tau \int_\Omega n_\infty(x,s) dxds \qquad \text{as } k \to \infty,$$

and that hence

$$\int_0^\tau \int_\Omega n_\infty(x,s) dxds = m\tau.$$

Therefore, (4.43) along with (4.42) implies that

$$\int_0^\tau \int_\Omega n_k(x,s) f(c_k)(x,s) dxds \quad \to \quad \int_0^\tau n_\infty(x,s) \cdot f(\lambda_\infty) dxds$$
$$= \quad f(\lambda_\infty) \cdot m\tau \qquad \text{as } k \to \infty. \tag{4.44}$$

On the other hand, from Lemma 4.1 we also know that $\int_0^\infty \int_\Omega nf(c) < \infty$, which entails that

$$\int_0^\tau \int_\Omega n_k(x,s) f(c_k(x,s)) dxds = \int_{t_k}^{t_k+\tau} \int_\Omega n(x,t) f(c(x,t)) dxdt \to 0 \qquad \text{as } k \to \infty.$$

Combined with (4.44), this shows that $f(\lambda_\infty) \cdot m\tau = 0$, which in light of our assumptions that $n_0 \not\equiv 0$ and that $f > 0$ on $(0,\infty)$ means that in fact we must have $\lambda_\infty = 0$. By definition (4.38) of λ, using that $c(\cdot,t) \in L^1(\Omega)$ for a.e. $t > 0$ we thus infer that (4.37) holds. □

Now in view of the convergence properties of $(c_\varepsilon)_{\varepsilon\in(0,1)}$ as $\varepsilon = \varepsilon_j \searrow 0$, we can combine the above decay statement with the ε-independent bound on Δc_ε guaranteed by Lemma 4.5 to obtain by means of an interpolation argument the main result of this section.

Lemma 4.8. *For each $\delta > 0$ one can find $T_\delta > 0$ and $\varepsilon_\star \in (0,1)$ such that whenever $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ satisfies $\varepsilon < \varepsilon_\star$, then*

$$\|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \delta \qquad \textit{for all } t \ge T_\delta. \tag{4.45}$$

Proof. In order to prepare our choice of T_δ, let us first apply the Gagliardo-Nirenberg inequality and elliptic estimates to find $C_1 > 0$ satisfying

$$\|\varphi\|_{L^\infty(\Omega)} \le C_1 \|\Delta\varphi\|_{L^2(\Omega)}^{\frac{2}{3}} \|\varphi\|_{L^1(\Omega)}^{\frac{1}{3}} + C_1\|\varphi\|_{L^1(\Omega)} \qquad \text{for all } \varphi \in C^2(\bar\Omega) \text{ such that } \tfrac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega, \tag{4.46}$$

and take $T > 0$ and $\tau \in (0,1)$ as provided by Lemma 4.5. Then given $\delta > 0$, we can fix $\eta > 0$ small enough such that

$$(2\eta\tau)^{\frac{1}{3}} \cdot C_1 + 2\eta C_1 \le \delta\tau, \tag{4.47}$$

and thereupon invoke Lemma 4.7 to obtain some $t_0 \in \mathbb{N}$ fulfilling

$$\int_{t_0}^{t_0+\tau} \|c(\cdot,t)\|_{L^1(\Omega)} dt \le \eta. \tag{4.48}$$

We claim that then (4.45) holds for all sufficiently small $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ if we let

$$T_\delta := t_0 + T + \tau. \tag{4.49}$$

To verify this, we first note that as a consequence of the strong convergence statement concerning c in (2.3) we know that $c_\varepsilon \to c$ in $L^1(\Omega \times (t_0, t_0+\tau))$, whence (4.48) entails that if we fix $\varepsilon_\star \in (0,1)$ suitably small, then

$$\int_{t_0}^{t_0+\tau} \|c_\varepsilon(\cdot,t)\|_{L^1(\Omega)} dt \le 2\eta \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}} \text{ with } \varepsilon < \varepsilon_\star. \tag{4.50}$$

Next, an application of Lemma 4.5 yields some $t_\varepsilon \in (t_0, t_0+T)$ such that

$$\int_{t_\varepsilon}^{t_\varepsilon+\tau} \|\Delta c_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^2 dt \le 1 \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}. \tag{4.51}$$

Now since (4.46) combined with the Hölder inequality shows that

$$\begin{aligned}
\int_{t_\varepsilon}^{t_\varepsilon+\tau} \|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} dt &\le C_1 \int_{t_\varepsilon}^{t_\varepsilon+\tau} \|\Delta c_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^{\frac{2}{3}} \|c_\varepsilon(\cdot,t)\|_{L^1(\Omega)}^{\frac{1}{3}} dt \\
&\quad + C_1 \int_{t_\varepsilon}^{t_\varepsilon+\tau} \|c_\varepsilon(\cdot,t)\|_{L^1(\Omega)} dt \\
&\le C_1 \cdot \left(\int_{t_\varepsilon}^{t_\varepsilon+\tau} \|\Delta c_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^2 dt \right)^{\frac{1}{3}} \cdot \left(\int_{t_\varepsilon}^{t_\varepsilon+\tau} \|c_\varepsilon(\cdot,t)\|_{L^1(\Omega)} dt \right)^{\frac{1}{3}} \cdot t^{\frac{1}{3}} \\
&\quad + C_1 \int_{t_\varepsilon}^{t_\varepsilon+\tau} \|c_\varepsilon(\cdot,t)\|_{L^1(\Omega)} dt \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}},
\end{aligned}$$

from (4.50), (4.51) and (4.47) we obtain that

$$\int_{t_\varepsilon}^{t_\varepsilon+\tau} \|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} dt \le C_1 \cdot 1 \cdot (2\eta)^{\frac{1}{3}} \cdot \tau^{\frac{1}{3}} + C_1 \cdot 2\eta \le \delta\tau$$

for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ such that $\varepsilon < \varepsilon_\star$. By the mean value theorem, for any such ε this entails the existence of some $\widehat{t_\varepsilon} \in (t_\varepsilon, t_\varepsilon+\tau)$ satisfying

$$\|c_\varepsilon(\cdot,\widehat{t_\varepsilon})\|_{L^\infty(\Omega)} \le \delta$$

and hence, by the downward monotonicity of $(0,\infty) \ni t \mapsto \|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$ asserted by Lemma 2.1, shows that

$$\|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \delta \qquad \text{for all } t \ge \hat{t_\varepsilon}.$$

As the inclusions $\widehat{t_\varepsilon} \in (t_\varepsilon, t_\varepsilon+\tau)$ and $t_\varepsilon \in (t_0, t_0+T)$ imply that

$$\widehat{t_\varepsilon} < t_\varepsilon + \tau < t_0 + T + \tau,$$

recalling (4.49) we see that indeed (4.45) holds for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ fulfilling $\varepsilon < \varepsilon_\star$. □

4.4 Proof of Theorem 1.1

Now knowing that c_ε is conveniently small beyond some waiting time which is independent of suitably small $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, we see that proving Theorem 1.1 essentially reduces to deriving suitable higher-order estimates for solutions of (2.1) emanating from initial data which are small in their second component. To achieve this, we follow a strategy similar to that pursued in [27], where corresponding estimates for such small-data solutions have been established for the associated chemotaxis-only model with $u \not\equiv 0$. In particular, at the core of our analysis in this direction we will study the evolution of the functional $\int_\Omega n^p + \int_\Omega |\nabla c|^{2p}$ for arbitrarily large $p > 2$, making essential use of the following interpolation inequality which has been proved in [27].

Lemma 4.9. *Let $r > 0$, $p \ge 1$ and $s \ge 2(p+1)$. Then there exists $C > 0$ such that the inequality*

$$\|\nabla\varphi\|_{L^s(\Omega)} \le C\Big\{ \Big\| |\nabla\varphi|^{p-1} D^2\varphi \Big\|_{L^2(\Omega)}^{\frac{s-2}{ps}} + \|\nabla\varphi\|_{L^r(\Omega)}^{\frac{s-2}{s}} \Big\} \cdot \|\varphi\|_{L^\infty(\Omega)}^{\frac{2}{s}} \tag{4.52}$$

holds for all $\varphi \in C^2(\bar\Omega)$ satisfying $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$.

This enables us to prove the following regularity statement.

Lemma 4.10. *Let $p > 2$. Then there exist $\delta = \delta(p) > 0$ and $C = C(p) > 0$ with the property that whenever $t_0 \ge 1$ and $\varepsilon \in (0,1)$ are such that*

$$\|c_\varepsilon(\cdot,t_0)\|_{L^\infty(\Omega)} \le \delta, \tag{4.53}$$

we have

$$\int_\Omega n_\varepsilon^p(x,t)dx + \int_\Omega |\nabla c_\varepsilon(x,t)|^{2p} dx \le C \qquad \text{for all } t \ge t_0 + 1. \tag{4.54}$$

PROOF. With $\delta > 0$ to be specified below, let us assume that (4.53) holds for some $\varepsilon \in (0,1)$ and $t_0 \geq 1$. We then observe that according to Lemma 2.1,

$$\|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq \delta \qquad \text{for all } t \geq t_0, \tag{4.55}$$

and that, moreover, testing the fist equation in (2.1) against n_ε^{p-1} and using (2.8) yields

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int_\Omega n_\varepsilon^p + (p-1)\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 &= (p-1)\int_\Omega n_\varepsilon^{p-1}\nabla n_\varepsilon \cdot \Big(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big) \\ &\leq \frac{p-1}{2}\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + \frac{p-1}{2}S_1\cdot\int_\Omega n_\varepsilon^p|\nabla c_\varepsilon|^2 \qquad \text{for all } t>0. \end{aligned} \tag{4.56}$$

Here by Hölder's and Young's inequalities, the Gagliardo-Nirenberg inequality, Lemma 4.9 and (4.55), we can find $C_1 > 0$ and $C_2 > 0$ such that

$$\begin{aligned} \frac{p-1}{2}S_1\cdot\int_\Omega n_\varepsilon^p|\nabla c_\varepsilon|^2 &\leq \frac{p-1}{2}S_1\Big(\int_\Omega n_\varepsilon^{p+1}\Big)^{\frac{p}{p+1}}\cdot\Big(\int_\Omega|\nabla c_\varepsilon|^{2(p+1)}\Big)^{\frac{1}{p+1}} \\ &= \frac{p-1}{2}S_1\|n_\varepsilon^{\frac{p}{2}}\|^2_{L^{\frac{2(p+1)}{p}}(\Omega)}\cdot\|\nabla c_\varepsilon\|^2_{L^{2(p+1)}(\Omega)} \\ &\leq C_1\cdot\bigg(\|\nabla n_\varepsilon^{\frac{p}{2}}\|^{\frac{2p}{p+1}}_{L^2(\Omega)}\cdot\|n_\varepsilon^{\frac{p}{2}}\|^{\frac{2}{p+1}}_{L^{\frac{2}{p}}(\Omega)} + \|n_\varepsilon^{\frac{p}{2}}\|^2_{L^{\frac{2}{p}}(\Omega)}\bigg)\times \\ &\quad\times\bigg(\Big\||\nabla c_\varepsilon|^{p-1}D^2c_\varepsilon\Big\|^{\frac{2}{p+1}}_{L^2(\Omega)} + \|\nabla c_\varepsilon\|^{\frac{2p}{p+1}}_{L^1(\Omega)}\bigg)\cdot\|c_\varepsilon\|^{\frac{2}{p+1}}_{L^\infty(\Omega)} \\ &\leq C_2\delta^{\frac{2}{p+1}}\cdot\bigg(\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + \int_\Omega|\nabla c_\varepsilon|^{2p-2}|D^2c_\varepsilon|^2 + 1\bigg) \end{aligned} \tag{4.57}$$

for all $t > t_0$, because $\|n_\varepsilon^{\frac{p}{2}}(\cdot,t)\|^{\frac{2}{p}}_{L^{\frac{2}{p}}(\Omega)} = \int_\Omega n_0$ for all $t > 0$ by (2.4) and $\sup_{\varepsilon\in(0,1)}\|\nabla c_\varepsilon\|_{L^\infty((0,\infty);L^1(\Omega))} < \infty$ by Lemma 3.3.

Likewise, from the Gagliardo-Nirenberg inequality and (2.4) we obtain $C_3 > 0$ and $C_4 > 0$ fulfilling

$$\begin{aligned} \int_\Omega n_\varepsilon^p = \|n_\varepsilon^{\frac{p}{2}}\|^2_{L^2(\Omega)} &\leq C_3\|\nabla n_\varepsilon^{\frac{p}{2}}\|^{\frac{2(p-1)}{p}}_{L^2(\Omega)}\cdot\|n_\varepsilon^{\frac{p}{2}}\|^{\frac{2}{p}}_{L^{\frac{2}{p}}(\Omega)} + C_3\|n_\varepsilon^{\frac{p}{2}}\|^2_{L^{\frac{2}{p}}(\Omega)} \\ &\leq C_4\cdot\Big(\|\nabla n_\varepsilon^{\frac{p}{2}}\|^2_{L^2(\Omega)} + 1\Big)^{\frac{p-1}{p}} \qquad \text{for all } t>0 \end{aligned}$$

and hence

$$\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 \geq C_5\Big(\int_\Omega n_\varepsilon^p\Big)^{\frac{p}{p-1}} - 1 \qquad \text{for all } t>0 \tag{4.58}$$

with some $C_5 > 0$. Combining (4.56)-(4.58) and assuming that

$$C_3\delta^{\frac{2}{p+1}} \leq \frac{p-1}{4}, \tag{4.59}$$

we thus find $C_6 > 0$ and $C_7 > 0$ such that

$$\frac{d}{dt}\int_\Omega n_\varepsilon^p + C_6\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + C_6\bigg(\int_\Omega n_\varepsilon^p\bigg)^{\frac{p}{p-1}} \le C_7\delta^{\frac{2}{p+1}}\cdot\int_\Omega |\nabla c_\varepsilon|^{2p-2}|D^2c_\varepsilon|^2 + C_7 \qquad \text{for all } t > t_0. \tag{4.60}$$

We next multiply the differentiated version of the second equation in (2.1), that is, the identity

$$\partial_t|\nabla c_\varepsilon|^2 = \Delta|\nabla c_\varepsilon|^2 - 2|D^2c_\varepsilon|^2 - 2\nabla c_\varepsilon\cdot\nabla(n_\varepsilon f(c_\varepsilon)) - 2\nabla c_\varepsilon\cdot(u_\varepsilon\cdot\nabla c_\varepsilon), \qquad x\in\Omega,\ t>0,$$

by $|\nabla c_\varepsilon|^{2p-2}$ to see that

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega|\nabla c_\varepsilon|^{2p} + (p-1)\int_\Omega|\nabla c_\varepsilon|^{2p-4}\Big|\nabla|\nabla c_\varepsilon|^2\Big|^2 + 2\int_\Omega|\nabla c_\varepsilon|^{2p-2}|D^2c_\varepsilon|^2 &\\ \le\ -2\int_\Omega|\nabla c_\varepsilon|^{2p-2}\nabla c_\varepsilon\cdot\nabla(n_\varepsilon f(c_\varepsilon)) &\\ -2\int_\Omega|\nabla c_\varepsilon|^{2p-2}\nabla c_\varepsilon\cdot\nabla(u_\varepsilon\cdot\nabla c_\varepsilon) \qquad \text{for all } t>0. &\end{aligned} \tag{4.61}$$

Here we note that the corresponding boundary integral is nonpositive, because due to the convexity of Ω, the boundary condition $\frac{\partial c_\varepsilon}{\partial\nu} = 0$ on $\partial\Omega$ implies that $\frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu} \le 0$ on $\partial\Omega$ ([28]). Since $\sup_{\varepsilon\in(0,1)}\|f(c_\varepsilon)\|_{L^\infty(\Omega\times(0,\infty))}$ is finite thanks to (1.8) and (2.5), integrating by parts and using the pointwise inequality $|\Delta c_\varepsilon| \le \sqrt{2}|D^2c_\varepsilon|$ we obtain $C_8 > 0$ such that

$$\begin{aligned}-2\int_\Omega|\nabla c_\varepsilon|^{2p-2}\nabla c_\varepsilon\cdot\nabla(n_\varepsilon f(c_\varepsilon)) &= 2\int_\Omega n_\varepsilon f(c_\varepsilon)|\nabla c_\varepsilon|^{2p-2}\Delta c_\varepsilon\\ &\quad +4(p-1)\int_\Omega n_\varepsilon f(c_\varepsilon)|\nabla c_\varepsilon|^{2p-4}\nabla c_\varepsilon\cdot(D^2c_\varepsilon\cdot\nabla c_\varepsilon)\\ &\le C_8\int_\Omega n_\varepsilon|\nabla c_\varepsilon|^{2p-2}|D^2c_\varepsilon|^2\\ &\le \frac14\int_\Omega|\nabla c_\varepsilon|^{2p-p}|D^2c_\varepsilon|^2 + C_8^2\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^{2p-2}\end{aligned} \tag{4.62}$$

for all $t > t_0$. Here we again interpolate using Hölder's and Young's inequalities, the Gagliardo-Nirenberg inequality and Lemma 4.9 as well as (2.4), (4.55) and Lemma 3.3 to find positive constants C_9 and C_{10} satisfying

$$\begin{aligned}C_8^2\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^2 &\le C_8^2\|n_\varepsilon^{\frac p2}\|^{\frac4p}_{L^{\frac{2(p+1)}{p}}(\Omega)}\cdot\|\nabla c_\varepsilon\|^{2(p-1)}_{L^{2(p+1)}(\Omega)}\\ &\le C_9\cdot\bigg(\|\nabla n_\varepsilon^{\frac p2}\|^{\frac{4}{p+1}}_{L^2(\Omega)}\cdot\|n_\varepsilon^{\frac p2}\|^{\frac{4}{p(p+1)}}_{L^{\frac2p}(\Omega)} + \|n_\varepsilon^{\frac p2}\|^{\frac4p}_{L^{\frac2p}(\Omega)}\bigg)\times\\ &\quad\times\bigg(\Big\||\nabla c_\varepsilon|^{p-1}D^2c_\varepsilon\Big\|^{\frac{2(p-1)}{p+1}}_{L^2(\Omega)} + \|\nabla c_\varepsilon\|^{\frac{2p(p-1)}{p+1}}_{L^1(\Omega)}\bigg)\cdot\|c_\varepsilon\|^{\frac{2(p-1)}{p+1}}_{L^\infty(\Omega)}\\ &\le C_{10}\delta^{\frac{2(p-1)}{p+1}}\cdot\bigg(\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + \int_\Omega|\nabla c_\varepsilon|^{2p-2}|D^2c_\varepsilon|^2 + 1\bigg)\end{aligned} \tag{4.63}$$

for all $t > t_0$. In the last integral in (4.61), we once more integrate by parts and proceed in a way similar to that in (4.62) to obtain $C_{11} > 0$ such that

$$\begin{aligned} -2\int_\Omega |\nabla c_\varepsilon|^{2p-2}\nabla c_\varepsilon\cdot\nabla(u_\varepsilon\cdot\nabla c_\varepsilon) &= 2\int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)|\nabla c_\varepsilon|^{2p-2}\Delta c_\varepsilon \\ &\quad +4(p-1)\int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)|\nabla c_\varepsilon|^{2p-4}\nabla c_\varepsilon\cdot(D^2c_\varepsilon\cdot\nabla c_\varepsilon) \\ &\le \frac{1}{4}\int_\Omega |\nabla c_\varepsilon|^{2p-2}|D^2c_\varepsilon|^2 + C_{11}\int_\Omega |u_\varepsilon|^2|\nabla c_\varepsilon|^{2p} \qquad \text{for all } t>0, \end{aligned}$$

where since $t_0 \ge 1$, and since Lemma 2.4 entails that $\sup_{\varepsilon\in(0,1)}\|u_\varepsilon\|_{L^\infty((1,\infty);L^{2(p+1)}(\Omega)}$ is finite, again by means of the Hölder inequality, Lemma 4.9, Lemma 3.3 and (4.55) we can find $C_{12} > 0$ and $C_{13} > 0$ such that

$$\begin{aligned} C_{11}\int_\Omega |u_\varepsilon|^2|\nabla c_\varepsilon|^{2p} &\le C_{11}\Big(\int_\Omega |u_\varepsilon|^{2(p+1)}\Big)^{\frac{1}{p+1}}\cdot\Big(\int_\Omega |\nabla c_\varepsilon|^{2(p+1)}\Big)^{\frac{p}{p+1}} \\ &\le C_{12}\cdot\bigg(\Big\||\nabla c_\varepsilon|^{p-1}D^2c_\varepsilon\Big\|_{L^2(\Omega)}^{\frac{2p}{p+1}} + \|\nabla c_\varepsilon\|_{L^1(\Omega)}^{\frac{2p^2}{p+1}}\bigg)\cdot\|c_\varepsilon\|_{L^\infty(\Omega)}^{\frac{2p}{p+1}} \\ &\le C_{13}\delta^{\frac{2p}{p+1}}\cdot\Big(\int_\Omega |\nabla c_\varepsilon|^{2p-2}|D^2c_\varepsilon|^2+1\Big) \qquad \text{for all } t>t_0. \end{aligned} \tag{4.64}$$

As moreover the Gagliardo-Nirenberg inequality and Lemma 3.3 provide positive constants C_{14}, C_{15} and C_{16} such that

$$\begin{aligned} \int_\Omega |\nabla c_\varepsilon|^{2p} = \Big\||\nabla c_\varepsilon|^p\Big\|_{L^2(\Omega)}^2 &\le C_{14}\Big\|\nabla|\nabla c_\varepsilon|^p\Big\|_{L^2(\Omega)}^{\frac{2p-1}{p}}\cdot\Big\||\nabla c_\varepsilon|^p\Big\|_{L^{\frac{1}{p}}(\Omega)}^{\frac{1}{p}} + C_{14}\Big\||\nabla c_\varepsilon|^p\Big\|_{L^{\frac{1}{p}}(\Omega)}^2 \\ &\le C_{15}\cdot\Big(\int_\Omega \Big|\nabla|\nabla c_\varepsilon|^p\Big|^2+1\Big)^{\frac{2p-1}{2p}} \\ &\le C_{16}\cdot\Big(\int_\Omega |\nabla c_\varepsilon|^{2p-4}\Big|\nabla|\nabla c_\varepsilon|^2\Big|^2+1\Big)^{\frac{2p-1}{2p}} \qquad \text{for all } t>0, \end{aligned}$$

and since thus

$$\int_\Omega |\nabla c_\varepsilon|^{2p-4}\Big|\nabla|\nabla c_\varepsilon|^2\Big|^2 \ge C_{17}\Big(\int_\Omega |\nabla c_\varepsilon|^{2p}\Big)^{\frac{2p}{2p-1}} - 1 \qquad \text{for all } t>0$$

with some $C_{17} > 0$, the inequalities (4.61), (4.62), (4.63) and (4.64) ensure that under the assumptions

$$C_{10}\delta^{\frac{2(p-1)}{p+1}} \le \frac{1}{2} \qquad \text{and} \qquad C_{13}\delta^{\frac{2p}{p+1}} \le \frac{1}{2}, \tag{4.65}$$

we can find $C_{18} > 0$ and $C_{19} > 0$ fulfilling

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^{2p} + C_{18}\int_\Omega |\nabla c_\varepsilon|^{2p-2}|D^2c_\varepsilon|^2 + C_{18}\cdot\Big(\int_\Omega |\nabla c_\varepsilon|^{2p}\Big)^{\frac{2p}{2p-1}} \\ &\quad\le C_{19}\delta^{2(p-1)}p+1\cdot\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + C_{19} \qquad \text{for all } t>t_0. \end{aligned}$$

Adding this to (4.60), we infer that if furthermore

$$C_7\delta^{\frac{2}{p+1}} \le C_{18} \qquad \text{and} \qquad C_{19}\delta^{\frac{2(p-1)}{p+1}} \le C_6,$$

then

$$\frac{d}{dt}\Big\{ \int_\Omega n_\varepsilon^p + \int_\Omega |\nabla c_\varepsilon|^{2p} \Big\} + C_6 \cdot \Big(\int_\Omega n_\varepsilon^p \Big)^{\frac{p}{p-1}} + C_{18} \cdot \Big(\int_\Omega |\nabla c_\varepsilon|^{2p} \Big)^{\frac{2p}{2p-1}} \le C_7 + C_{19}$$

for all $t > t_0$, which implies that $y(t) := \int_\Omega n_\varepsilon(\cdot,t) + \int_\Omega |\nabla c_\varepsilon(\cdot,t)|^{2p}$, $t \ge t_0$, satisfies

$$y'(t) + C_{20} \cdot y^\lambda(t) \le C_{21} \qquad \text{for all } t > t_0$$

with certain positive constants C_{20} and C_{21} and $\lambda := \min\Big\{ \frac{p}{p-1}\, , \, \frac{2p}{2p-1} \Big\} \equiv \frac{2p}{2p-1} > 1$. A straightforward ODE comparison shows that therefore

$$y(t) \le \max \left\{ \Big(\frac{2C_{21}}{C_{20}}\Big)\frac{1}{\lambda}\, , \, \Big(\frac{(\lambda-1)C_{20}}{2}\Big)^{-\frac{1}{\lambda-1}} \cdot (t-t_0)^{-\frac{1}{\lambda-1}} \right\} \qquad \text{for all } t > t_0,$$

which immediately yields (4.54). □

Now higher regularity properties beyond an adequately large time can be obtained by standard arguments.

Lemma 4.11. *There exist $T > 0$, $\theta \in (0,1)$ and $C > 0$ such that*

$$\|n\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar\Omega\times[t,t+1])} + \|c\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar\Omega\times[t,t+1])} + \|u\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar\Omega\times[t,t+1])} \le C \qquad \textit{for all } t \ge T. \tag{4.66}$$

Proof. We fix any $p > 2$ and apply Lemma 4.10 to find $\delta > 0$ with the property stated therein. Then Lemma 4.8 provides $T_0 > 0$ and $\varepsilon_\star \in (0,1)$ such that $\|c_\varepsilon(\cdot,T_0)\|_{L^\infty(\Omega)} \le \delta$ for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ such that $\varepsilon < \varepsilon_\star$. Then writing $T_1 := T_0 + 1$, from Lemma 4.10 we obtain $C_1 > 0$ such that whenever $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ is such that $\varepsilon < \varepsilon_\star$, we have

$$\|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le C_1 \qquad \text{for all } t \ge T_1 \tag{4.67}$$

and

$$\|\nabla c_\varepsilon(\cdot,t)\|_{L^{2p}(\Omega)} \le C_1 \qquad \text{for all } t \ge T_1. \tag{4.68}$$

In particular, using that $p > 1$, from the variation-of-constants representation

$$u_\varepsilon(\cdot,t) = e^{-(t-T_1)A}u_\varepsilon(\cdot,T_1) + \int_{T_1}^t e^{-(t-s)A}\mathcal{P}[n_\varepsilon(\cdot,s)\nabla\phi]ds, \qquad t > T_1,$$

and standard regularity arguments involving well-known smoothing properties of the Stokes semigroup ([13, p.201]) one can readily derive the existence of $\theta_1 \in (0,1)$ and $C_2 > 0$ such that for any such ε,

$$\|u_\varepsilon\|_{C^{\theta_1,\frac{\theta_1}{2}}(\bar\Omega\times[t,t+1])} \le C_2 \qquad \text{for all } t \ge T_2 := T_1 + 1. \tag{4.69}$$

Along with (4.67) and (4.68), this provides a bound in $L^\infty((T_2,\infty);L^p(\Omega))$ for the inhomogeneity $h_\varepsilon := n_\varepsilon f(c_\varepsilon) - u_\varepsilon\cdot\nabla c_\varepsilon$ in the equation $c_{\varepsilon t} = \Delta c_\varepsilon + h_\varepsilon$, whence standard $L^p - L^q$ estimates for the Neumann heat semigroup $(e^{\tau\Delta})_{\tau\ge 0}$ (see e.g. [46] for versions covering the present situation) allow for using the identity

$$\nabla c_\varepsilon(\cdot,t) = \nabla e^{(t-T_2)\Delta}c_\varepsilon(\cdot,T_2) + \int_{T_2}^t \nabla e^{(t-s)\Delta}h_\varepsilon(\cdot,s)ds, \qquad t>T_2,$$

to gain $\theta_2\in(0,1)$ and $C_3>0$ such that

$$\|\nabla c_\varepsilon(\cdot,t)\|_{C^{\theta_2,\frac{\theta_2}{2}}(\bar\Omega\times[t,t+1])} \le C_3 \qquad \text{for all } t\ge T_3 := T_2+1 \tag{4.70}$$

for all $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}$ such that $\varepsilon<\varepsilon_\star$.
We may now use that $\nabla\cdot(n_\varepsilon u_\varepsilon) = u_\varepsilon\cdot\nabla n_\varepsilon$ thanks to the fact that $\nabla\cdot u_\varepsilon\equiv 0$, and that hence

$$n_\varepsilon(\cdot,t) = e^{(t-T_3)\Delta}n_\varepsilon(\cdot,T_3) - \int_{T_3}^t e^{(t-s)\Delta}\nabla\cdot\Big\{n_\varepsilon\Big(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big) + n_\varepsilon u_\varepsilon\Big\}(\cdot,s)ds, \qquad t>T_3,$$

to obtain from (4.67), (4.70) and (4.69) that for some $\theta_3\in(0,1)$ and $C_4>0$ we have

$$\|n_\varepsilon\|_{C^{\theta_3,\frac{\theta_3}{2}}(\bar\Omega\times[t,t+1])} \le C_4 \qquad \text{for all } t\ge T_4 := T_3+1. \tag{4.71}$$

Together with (4.69), (4.70) and standard parabolic Schauder theory ([24]), this now guarantees a bound for c_ε in $C^{2+\theta_4,1+\frac{\theta_4}{2}}(\bar\Omega\times[t,t+1])$ with some $\theta_4\in(0,1)$ and all $t\ge T_5 := T_4+1$, whereupon by the same token we obtain an estimate for n_ε in $C^{2+\theta_5,1+\frac{\theta_5}{2}}(\bar\Omega\times[t,t+1])$ for $t\ge T_6 := T_5+1$ and some $\theta_5\in(0,1)$. Finally, the regularizing effect of the Stokes semigroup thereafter implies a bound for u_ε in $C^{2+\theta_6,1+\frac{\theta_6}{2}}(\bar\Omega\times[t,t+1];\mathbb{R}^2)$ for $t\ge T_7 := T_6+1$ and some $\theta_6\in(0,1)$. Taking $\varepsilon=\varepsilon_j\searrow 0$, we easily arrive at the desired conclusion. □

Our main result on eventual smoothness thereby becomes immediate.

Proof of Theorem 1.1. By means of (4.66) and an application of the Arzelà-Ascoli theorem, we infer that the pointwise convergence processes in (2.3) actually take place in the respective spaces indicated in (1.15). Along with a standard construction of an associated pressure function, this also implies the claimed solution properties. □

5 Stabilization. Proof of Theorem 1.2

In light of the regularity estimates asserted by Lemma 4.11, the weak decay properties indicated by the inequalities (4.4) and (4.5) in Lemma 4.1 can now easily be turned into uniform stabilization of n and u toward the claimed constant limit functions in the large time limit:

Proof of Theorem 1.2. In view of (2.3), the statement (1.17) is an immediate consequence of Lemma 4.8.
To prove (1.18), let us assume on the contrary that there exist $C_1>0$ and $(t_k)_{k\in\mathbb{N}}\subset(0,\infty)$ such that $t_k\to\infty$ as $k\to\infty$ and

$$\|u(\cdot,t_k)\|_{L^\infty(\Omega)} \ge C_1 \qquad \text{for all } k\in\mathbb{N},$$

where we clearly may assume that $t_{k+1} > t_k + 1$ for all $k \in \mathbb{N}$. Then Lemma 4.11 combined with the Arzelà-Ascoli theorem allows us to pick a nontrivial $u_\infty \in C^0(\bar{\Omega})$ and a subsequence of $(t_k)_{k\in\mathbb{N}}$, again denoted by $(t_k)_{k\in\mathbb{N}}$ for notational convenience, along which we have $u(\cdot, t_k) \to u_\infty$ in $L^\infty(\Omega)$ and hence also in $L^2(\Omega)$. Writing $C_2 := \|u_\infty\|_{L^2(\Omega)}$, we thus obtain that $\|u(\cdot, t_k)\|_{L^2(\Omega)} \geq \frac{C_2}{2}$ for all sufficiently large $k \in \mathbb{N}$. Now using the temporal Hölder estimate contained in (4.66) we easily inter the existence of $\tau \in (0, 1)$ and $k_0 \in \mathbb{N}$ such that

$$\|u(\cdot, t)\|_{L^2(\Omega)} \geq \frac{C_2}{4} \qquad \text{for all } t \in [t_k, t_k + \tau] \text{ and each } k \geq k_0,$$

which in particular implies that for any $k_1 > k_0$ we have

$$\begin{aligned} \int_0^\infty \|u(\cdot, t)\|^2_{L^2(\Omega)} dt &\geq \int_0^{t_{k_1}+1} \|u(\cdot, t)\|^2_{L^2(\Omega)} dt \\ &\geq \sum_{k=k_0}^{k_1} \int_{t_k}^{t_k+\tau} \|u(\cdot, t)\|^2_{L^2(\Omega)} dt \\ &\geq \frac{C_2^2}{16} \cdot \tau \cdot (k_1 - k_0). \end{aligned} \tag{5.1}$$

Taking $k_1 \to \infty$ yields a contradiction to the inclusion $u \in L^2(\Omega \times (0, \infty); \mathbb{R}^2)$ asserted by Lemma 4.1 and thereby shows that actually (1.18) must be valid.

Similarly, (1.16) can be proved on combining Lemma 4.11 with the integrability property (4.4) in Lemma 4.1: Indeed, if (1.16) was false then there would exist $C_3 > 0$ and a sequence $(\tilde{t}_k)_{k\in\mathbb{N}}$ such that $\tilde{t}_{k+1} > \tilde{t}_k + 1$ for all $k \in \mathbb{N}$ and

$$\|n(\cdot, \tilde{t}_k) - \overline{n_0}\|_{L^\infty(\Omega)} \geq C_3 \qquad \text{for all } k \in \mathbb{N}. \tag{5.2}$$

By means of Lemma 4.11 we may extract a subsequence of $(\tilde{t}_k)_{k\in\mathbb{N}}$, again denoted by $(\tilde{t}_k)_{k\in\mathbb{N}}$, such that

$$n(\cdot, \tilde{t}_k) \to n_\infty \quad \text{in } L^\infty(\Omega) \qquad \text{as } k \to \infty \tag{5.3}$$

with some nonnegative $n_\infty \in C^0(\bar{\Omega})$. Here the uniformity of the convergence in (5.3) warrants that firstly, with $\kappa(t) = (\fint_\Omega \sqrt{n(\cdot, t) + 1})^2 - 1$ as in (4.1) we have

$$\kappa(\tilde{t}_k) \to \kappa_\infty := \Big(\fint_\Omega \sqrt{n_\infty + 1} \Big)^2 - 1 \qquad \text{as } k \to \infty, \tag{5.4}$$

and that secondly

$$\fint_\Omega n_\infty = \lim_{k\to\infty} \fint_\Omega n(\cdot, t_k) = \overline{n_0} \tag{5.5}$$

because of (6.3). Let us next make sure that

$$\|n(\cdot, \tilde{t}_k) - \kappa(\tilde{t}_k)\|_{L^1(\Omega)} \to 0 \qquad \text{as } k \to \infty. \tag{5.6}$$

In fact, if this was false then for some $C_4 > 0$ we could assume on passing to a further subsequence if necessary that

$$\|n(\cdot, \tilde{t}_k) - \kappa(\tilde{t}_k)\|_{L^1(\Omega)} \geq C_4 \qquad \text{for all } k \in \mathbb{N}.$$

Relying on the temporal Hölder estimate for n implied by (4.66), we could then find $\tilde{\tau} \in (0,1)$ and $\tilde{k}_0 \in \mathbb{N}$ such that

$$\|n(\cdot,t) - \kappa(t)\|_{L^1(\Omega)} \ge \frac{C_4}{2} \qquad \text{for all } t \in [\tilde{t}_k, \tilde{t}_k + \tilde{\tau}] \text{ and any } k \ge k_0,$$

which, by an argument similar to that in (5.1), would entail that

$$\int_0^\infty \|n(\cdot,t) - \kappa(t)\|_{L^1(\Omega)}^2 dt = \infty$$

and thereby contradict the outcome of Lemma 4.1.

Having thereby verified (5.6), we may combine this with (5.3) and (5.4) to conclude that

$$n_\infty \equiv \kappa_\infty \qquad \text{in } \Omega.$$

In light of (5.5), this identifies the constant κ_∞ according to

$$\kappa_\infty = \fint_\Omega n_0 = \overline{n_0},$$

and thereby shows that $n_\infty \equiv \overline{n_0}$, which is evidently incompatible with (5.2) and (5.3). The proof of (1.16) is thus complete. □

6 Appendix: The underlying solution concept

Although nowhere explicitly referred to in our analysis, for completeness let us recall from [54, Definition 2.1] the precise generalized solution concept underlying Theorem A, and hence also Theorem 1.1 and Theorem 1.2).

Definition 6.1. *Suppose that*

$$\begin{aligned} & n \in L^\infty((0,\infty); L^1(\Omega)), \\ & c \in L^\infty_{loc}(\bar{\Omega} \times [0,\infty)) \cap L^2_{loc}([0,\infty); W^{1,2}(\Omega)) \qquad \text{and} \\ & u \in L^1_{loc}([0,\infty); (W_0^{1,1}(\Omega))^2) \end{aligned} \tag{6.1}$$

are such that $n \ge 0$ *and* $c \ge 0$ *a.e. in* $\Omega \times (0,\infty)$ *and*

$$\ln(n+1) \in L^2_{loc}([0,\infty); W^{1,2}(\Omega)), \tag{6.2}$$

that

$$\int_\Omega n(x,t)dx = \int_\Omega n_0(x) \qquad \text{for a.e. } t > 0, \tag{6.3}$$

and that $\nabla \cdot u = 0$ *a.e. in* $\Omega \times (0,\infty)$. *Then the triple* (n,c,u) *will be called a* global mass-preserving generalized solution *of (1.7) if the inequality*

$$\begin{aligned} -\int_0^\infty \int_\Omega \ln(n+1)\varphi_t - \int_\Omega \ln(n_0+1)\varphi(\cdot,0) \quad \ge \quad & \int_0^\infty \int_\Omega \ln(n+1)\Delta\varphi + \int_0^\infty \int_\Omega |\nabla \ln(n+1)|^2 \varphi \\ & - \int_0^\infty \int_\Omega \frac{n}{n+1} \nabla \ln(n+1) \cdot \Big(S(x,n,c) \cdot \nabla c\Big)\varphi \\ & + \int_0^\infty \int_\Omega \frac{n}{n+1} \Big(S(x,n,c) \cdot \nabla c\Big) \cdot \nabla\varphi \\ & + \int_0^\infty \int_\Omega \ln(n+1)(u \cdot \nabla\varphi) \end{aligned} \tag{6.4}$$

holds for each nonnegative $\varphi \in C_0^\infty(\bar\Omega \times [0,\infty))$ *with* $\frac{\partial\varphi}{\partial\nu} = 0$ *on* $\partial\Omega \times (0,\infty)$*, if moreover*

$$\int_0^\infty \int_\Omega c\varphi_t + \int_\Omega c_0\varphi(\cdot,0) = \int_0^\infty \int_\Omega \nabla c \cdot \nabla\varphi + \int_0^\infty \int_\Omega nf(c)\varphi - \int_0^\infty \int_\Omega c(u \cdot \nabla\varphi) \tag{6.5}$$

for any $\varphi \in L^\infty(\Omega\times(0,\infty)) \cap L^2((0,\infty); W^{1,2}(\Omega))$ *having compact support in* $\bar\Omega\times[0,\infty)$ *with* $\varphi_t \in L^2(\Omega\times(0,\infty))$*, and if finally*

$$-\int_0^\infty \int_\Omega u \cdot \varphi_t - \int_\Omega u_0 \cdot \varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla u \cdot \nabla\varphi + \int_0^\infty \int_\Omega n\nabla\phi \cdot \varphi \tag{6.6}$$

for all $\varphi \in C_0^\infty(\Omega \times [0,\infty); \mathbb{R}^2)$ *with* $\nabla \cdot \varphi \equiv 0$*.*

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