A Gagliardo-Nirenberg-type inequality and its applications to decay estimates for solutions of a degenerate parabolic equation

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Abstract

We establish a Gagliardo-Nirenberg-type inequality in $\mathbb{R}^n$ for functions which decay fast as $|x| \to \infty$. We use this inequality to derive upper bounds for the decay rates of solutions of a degenerate parabolic equation. Moreover, we show that these upper bounds, hence also the Gagliardo-Nirenberg-type inequality, are sharp in an appropriate sense.

Key words: Gagliardo-Nirenberg inequality; degenerate parabolic equation; decay rates of solutions
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1 Introduction

A Gagliardo-Nirenberg-type inequality. The Gagliardo-Nirenberg inequalities ([30], [31], [41]) play an important role in studying partial differential equations (cf. [13], for instance). Consider the special case when $1 \le r < q < \infty$, and $0 \le \theta \le 1$ are such that

$$\frac{1}{q} = \frac{\theta}{r} + (1-\theta)\left(\frac{1}{2} - \frac{1}{n}\right).$$

Then there is a constant $c > 0$ which depends only on n, q and r, such that any $\varphi \in L^r(\mathbb{R}^n)$ with $\nabla\varphi \in L^2(\mathbb{R}^n)$ satisfies

$$\|\varphi\|_{L^q(\mathbb{R}^n)} \le c\|\varphi\|^{\theta}_{L^r(\mathbb{R}^n)} \|\nabla\varphi\|^{1-\theta}_{L^2(\mathbb{R}^n)}. \tag{1.1}$$

Our aim is to establish a new optimal inequality of a similar type by replacing the term $\|\nabla\varphi\|^{1-\theta}_{L^2(\mathbb{R}^n)}$ with $\|\nabla\varphi\|_{L^2(\mathbb{R}^n)} F\left(\|\nabla\varphi\|_{L^2(\mathbb{R}^n)}\right)$, where F is some positive function with the properties that

$$F(s) \to \infty \text{ as } s \to 0 \qquad \text{and} \qquad s^{\theta}F(s) \to 0 \text{ as } s \to 0 \text{ for any } \theta > 0.$$

The term $c\|\varphi\|^{\theta}_{L^r(\mathbb{R}^n)}$ will then be replaced by a constant which depends only on n, q and $\int_{\mathbb{R}^n} \mathcal{L}(\varphi)$, where $\mathcal{L}$ is a suitable function related to F. The integrability of $\mathcal{L}(\varphi)$ will require fast (exponential-like) decay of φ.

The Gagliardo-Nirenberg inequalities (GNI) from [30], [31], [41] have been improved and extended in many different directions. We shall mention some examples below without trying to give an exhaustive list. Sharp constants in GNI in $\mathbb{R}^n$ were studied in [2, 3, 4, 17, 21, 22, 23, 24, 35, 36, 37, 49] and in GNI on Riemannian manifolds in [1, 14, 15]. The sharp constant in an anisotropic GNI with fractional derivatives was found in [28]. A pointwise GNI can be found in [40], a weighted GNI in [27] and a GNI on manifolds in [6]. GNI in Orlicz spaces were established in [32, 33, 34], in Besov spaces of negative order in [38], in weak Lebesgue spaces in [39] and in spaces of functions with bounded mean oscillation in [39, 44]. An affine GNI was derived in [37, 49] and a nonlinear GNI in [44]. Connections between logarithmic Sobolev inequalities and generalizations of GNI were investigated in [11].

We are not aware, however, of any example of a GNI in the literature which to an essentially optimal extent makes use of a presupposed superalgebraically fast decay of the involved function. Addressing this problem, as the main result of this paper we shall obtain the following.

Theorem 1.1 *Assume that $s_0 > 0$ and $\mathcal{L} \in C^0([0,\infty)) \cap C^1((0,s_0))$ is positive on $(0,\infty)$, bounded, nondecreasing and such that the following holds:*

(H) *There exist $a, \lambda_0 > 0$ such that*

$$\mathcal{L}(s) \le (1+a\lambda)\mathcal{L}(s^{1+\lambda}) \qquad \textit{for all } s \in (0,s_0) \textit{ and } \lambda \in (0,\lambda_0). \tag{1.2}$$

Then for any $n \ge 1$, $K > 0$ and $q > 0$ such that $q < \frac{2n}{(n-2)_+}$ there exists $C = C(n,q,K) > 0$ such that if $0 \not\equiv \varphi \in W^{1,2}(\mathbb{R}^n)$ is a nonnegative function satisfying

$$\int_{\mathbb{R}^n} \mathcal{L}(\varphi) \le K, \tag{1.3}$$

then

$$\|\varphi\|_{L^q(\mathbb{R}^n)} \le C\|\nabla\varphi\|_{L^2(\mathbb{R}^n)}\mathcal{L}^{-(\frac{1}{q}-\frac{n-2}{2n})}\Big(\|\nabla\varphi\|^2_{L^2(\mathbb{R}^n)}\Big). \tag{1.4}$$

Typical examples of functions fulfilling (H) are given by

$$\mathcal{L}(s) = \ln^{-\kappa}\frac{M}{s} \qquad \text{or} \qquad \mathcal{L}(s) = \ln^{-\kappa}\ln\frac{M}{s}, \qquad s>0, \kappa>0, M>e, \tag{1.5}$$

see Lemmata 3.9 and 3.10. With such a choice of $\mathcal{L}$, (1.3) will be satisfied if there are positive constants c_0, α, β and γ such that

$$\varphi(x) \le c_0\, e^{-\alpha|x|^\beta} \qquad \text{or} \qquad \varphi(x) \le c_0\, \exp\Big\{-\alpha\exp\big(\beta|x|^\gamma\big)\Big\} \qquad \text{for all } x\in\mathbb{R}^n,$$

respectively. Notice that if $n>2$, $r := \frac{n}{n-2}$ and $\mathcal{L}(s) = s^r$, then (1.4) with

$$C = c\|\varphi\|^\theta_{L^r(\mathbb{R}^n)}, \qquad \theta = \frac{2n}{q(n-2)} - 1,$$

corresponds to (1.1).

We shall next show that the exponent $\frac{1}{q} - \frac{n-2}{2n}$ in (1.4) is sharp. We accomplish that in the context of an analysis of temporal decay rates in a degenerate parabolic equation, for which Theorem 1.1 will yield certain upper bounds that thereafter, essentially by means of arguments based on parabolic comparison principles, will be seen to be optimal in an appropriate sense.

Applications to decay estimates for a degenerate parabolic equation. For $n\ge 1$ and $p\ge 1$, consider the Cauchy problem

$$\begin{cases} u_t = u^p\Delta u, & x\in\mathbb{R}^n,\ t>0, \\ u(x,0) = u_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.6}$$

where $u_0 \in C^0(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$. Our purpose is to study the large time behavior of global classical solutions under the assumption that

$$u_0(x)\to 0 \qquad \text{as } |x|\to\infty, \tag{1.7}$$

and our particular focus is on describing in a quantitative manner how various types of decay of u_0 affect the asymptotic behavior of $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}$ as $t\to\infty$.

Before addressing this, as a caveat we need to note that even in the framework of smooth positive solutions, uniqueness does not hold for (1.6). After all, however, (1.6) always possesses a *minimal* global classical solution u for any positive continuous and bounded initial data ([29]). This solution is minimal in the sense that whenever $T\in(0,\infty]$ and $\widetilde{u}\in C^0(\mathbb{R}^n\times[0,T))\cap C^{2,1}(\mathbb{R}^n\times(0,T))$ are such that $\widetilde{u}$ is positive and solves (1.6) classically in $\mathbb{R}^n\times(0,T)$ then we have $u\le\widetilde{u}$ in $\mathbb{R}^n\times(0,T)$.

Now for any initial data decaying sufficiently fast in space, this minimal solution is known to approach zero at a temporal rate which at its leading order is determined by the algebraic function $t^{-\frac{1}{p}}$, but which in fact must involve a subalgebraic correction. More precisely, the following was shown in [29].

Theorem 1.2 *If $p \geq 1$ and $u_0 \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ satisfies $u_0 \in \bigcap_{q_0>0} L^{q_0}(\mathbb{R}^n)$, then for any $\delta > 0$ one can find $C(\delta) > 0$ such that for the minimal solution u of* (1.6) *we have*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq C(\delta) t^{-\frac{1}{p}+\delta} \qquad \textit{for all } t > 0. \tag{1.8}$$

Any global positive classical u of (1.6) *has the property that for every $R > 0$,*

$$\inf_{|x|<R} \left\{ t^{\frac{1}{p}} u(x,t) \right\} \to +\infty \qquad \textit{as } t \to \infty.$$

This theorem suggests that logarithmic terms may occur in the sharp decay rate of $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}$ if the decay of u_0 is fast enough. We show that Theorem 1.1 implies an upper bound which supports this conjecture.

Theorem 1.3 *Suppose that $p \geq 1$, that $s_0 > 0$, and that $\mathcal{L} \in C^0([0,\infty)) \cap C^2((0,s_0))$ is positive and nondecreasing on $(0,\infty)$ with $\mathcal{L}(0) = 0$ and such that* (H) *is valid, and such that furthermore*

$$s\mathcal{L}''(s) \geq -\frac{3p+q_0-2}{p+q_0} \mathcal{L}'(s) \qquad \textit{for all } s \in (0,s_0) \tag{1.9}$$

with a certain $q_0 > 0$. Moreover, assume that $u_0 \in C^0(\mathbb{R}^n)$ is positive, radially symmetric and nonincreasing with respect to $|x|$ and such that

$$u_0 < \min \left\{ s_0^{\frac{2}{p}}, s_0^{\frac{2}{p+q_0}} \right\} \qquad \textit{in } \mathbb{R}^n \tag{1.10}$$

as well as

$$\int_{\mathbb{R}^n} \mathcal{L}(u_0) < \infty. \tag{1.11}$$

Then there exist $t_0 > 0$ and $C > 0$ such that the minimal solution u of (1.6) *satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-\frac{1}{p}} \mathcal{L}^{-\frac{2}{np}} \left(\frac{1}{t}\right) \qquad \textit{for all } t \geq t_0. \tag{1.12}$$

As observed in Lemmata 3.9 and 3.10 below, the condition (1.9) is indeed satisfied by the functions from (1.5). Firstly, concentrating on the first choice therein, as a consequence of Theorem 1.3 we obtain the following.

Corollary 1.4 *Let $p \geq 1$, and assume that $u_0 \in C^0(\mathbb{R}^n)$ is positive, radially symmetric and nondecreasing with respect to $|x|$ with $u_0(x) \to 0$ as $|x| \to \infty$ and*

$$\int_{\{u_0<\frac{1}{2}\}} \ln^{-\kappa} \frac{1}{u_0(x)} dx < \infty \tag{1.13}$$

for some $\kappa > 0$. Then there exist $t_0 > 1$ and $C > 0$ such that the minimal solution of (1.6) *satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-\frac{1}{p}} \ln^{\frac{2\kappa}{np}} t \qquad \textit{for all } t \geq t_0. \tag{1.14}$$

Corollary 1.5 *Let $p \geq 1$, and suppose that $u_0 \in C^0(\mathbb{R}^n)$ be positive and such that*

$$u_0(x) \leq c_0\, e^{-\alpha|x|^\beta} \qquad \text{for all } x \in \mathbb{R}^n \tag{1.15}$$

with positive constants c_0, α and β. Then for any $\delta > 0$ one can find $t_0 > 0$ and $C > 0$ such that the minimal solution of (1.6) *satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{1}{p}} \ln^{\frac{2}{p\beta}+\delta} t \qquad \text{for all } t \geq t_0. \tag{1.16}$$

This refines Theorem 1.2 and we shall see below that the upper bound (1.16) is optimal in an appropriate sense. The second option offered by (1.5) indicates that for initial data with faster decay, also iterated logarithms may occur in the upper bounds:

Corollary 1.6 *Let $p \geq 1$, and assume that $u_0 \in C^0(\mathbb{R}^n)$ is positive and such that there exist positive constants c_0, α, β and γ fulfilling*

$$u_0(x) \leq c_0\, \exp\Big\{ -\alpha \exp\big(\beta|x|^\gamma\big)\Big\} \qquad \text{for all } x \in \mathbb{R}^n. \tag{1.17}$$

Then for all $\delta > 0$ one can find $t_0 > e$ and $C > 0$ such that the minimal solution of (1.6) *satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{1}{p}} \ln^{\frac{2}{p\gamma}+\delta} \ln t \qquad \text{for all } t \geq t_0. \tag{1.18}$$

From a corresponding lower bound below one can see that (1.18) is also sharp. This lower bound will follow from our next result.

Theorem 1.7 *Let $p \geq 1$, and let $\Lambda \in C^0([0,\infty))$ be strictly increasing and such that*

$$\frac{\Lambda(s)}{\ln s} \to +\infty \qquad \text{as } s \to \infty. \tag{1.19}$$

Moreover, assume that u is a positive classical solution of (1.6) *in $\mathbb{R}^n \times (0,\infty)$, with initial data $u_0 \in C^0(\mathbb{R}^n)$ satisfying*

$$u_0(x) \geq e^{-\Lambda(x)} \qquad \text{for all } x \in \mathbb{R}^n. \tag{1.20}$$

Then there exist $t_0 > 0$ and $C > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \geq Ct^{-\frac{1}{p}} \Big\{\Lambda^{-1}\big(C\ln t\big)\Big\}^{\frac{2}{p}} \qquad \text{for all } t \geq t_0, \tag{1.21}$$

where Λ^{-1} denotes the inverse of Λ.

For particular choices of Λ we have the following two consequences:

Corollary 1.8 *Let $p \geq 1$, and suppose that $u_0 \in C^0(\mathbb{R}^n)$ is such that there exist $c_0, \alpha, \beta > 0$ fulfilling*

$$u_0(x) \geq c_0\, e^{-\alpha|x|^\beta} \qquad \text{for all } x \in \mathbb{R}^n. \tag{1.22}$$

Then one can find $t_0 > 1$ and $C > 0$ such that any positive classical solution of (1.6) *satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \geq Ct^{-\frac{1}{p}} \ln^{\frac{2}{p\beta}} t \qquad \text{for all } t \geq t_0. \tag{1.23}$$

Corollary 1.9 *Let $p \geq 1$ and $u_0 \in C^0(\mathbb{R}^n)$ be such that*

$$u_0(x) \geq c_0 \exp\Big\{ -\alpha \exp(\beta|x|^\gamma)\Big\} \qquad \text{for all } x \in \mathbb{R}^n \tag{1.24}$$

with positive constants c_0, α, β and γ. Then there exist $t_0 > e$ and $C > 0$ with the property that any positive classical solution of (1.6) *satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \geq Ct^{-\frac{1}{p}} \ln^{\frac{2}{p\gamma}} \ln t \qquad \text{for all } t \geq t_0. \tag{1.25}$$

These last two corollaries imply that the upper bounds (1.16) and (1.18) cannot hold with $\delta < 0$ which means that the exponent $\frac{1}{q} - \frac{n-2}{2n}$ in (1.4) is sharp.

Let us mention here that for $p > 1$ problem (1.6) can be rewritten using the substitution $v := u^{1-p}$ as a Cauchy problem for the super-fast diffusion equation given by

$$\begin{cases} v_t = \nabla \cdot (v^{m-1}\nabla v), & x \in \mathbb{R}^n,\ t > 0, \\ v(x,0) = v_0(x) := u_0^{1-p}, & x \in \mathbb{R}^n, \end{cases} \tag{1.26}$$

where $m = -\frac{1}{p-1} < 0$ and $v_0 \in C^0(\mathbb{R}^n)$ is such that $v_0(x) \to \infty$ as $|x| \to \infty$. Of course, our results on decay rates for (1.6) can be rephrased as results on growth rates of $\inf_{x\in\mathbb{R}^n} v(x,t)$ for (1.26) in an evident manner. For example, Corollaries 1.5 and 1.8 can be reformulated in the following way:

Corollary 1.10 *Assume that $m < 0$ and let $v_0 \in C^0(\mathbb{R}^n)$ be positive. Let v be the maximal solution of* (1.26)*.*

(i) *If*

$$v_0(x) \geq c_0\, e^{\alpha|x|^\beta} \qquad \text{for all } x \in \mathbb{R}^n$$

for some positive constants c_0, α, β, then for every $\delta > 0$ there exist $t_0, C > 0$ such that

$$\inf_{x\in\mathbb{R}^n} v(x,t) \geq Ct^{\frac{1}{1-m}} \ln^{-\frac{2}{\beta(1-m)}-\delta} t \qquad \text{for all } t \geq t_0.$$

(ii) *If there exist $\varepsilon, c_1, \alpha, \beta > 0$ such that*

$$\varepsilon \leq v_0(x) \leq c_1\, e^{\alpha|x|^\beta} \qquad \text{for all } x \in \mathbb{R}^n,$$

then for some $C, t_0 > 0$ it holds that

$$\inf_{x\in\mathbb{R}^n} v(x,t) \leq Ct^{\frac{1}{1-m}} \ln^{-\frac{2}{\beta(1-m)}} t \qquad \text{for all } t \geq t_0.$$

In general, problem (1.26) is not well-posed for $m < 0$, see for example [18, 19, 46], but under our assumptions on initial data there is a (uniquely determined) maximal solution which exists for all $t > 0$, see Lemma 3.1.

The equation $u_t = u\Delta u$, as corresponding to the borderline case $p = 1$ in (1.6), occurs in the study of nonlinear transport phenomena ([16]), soil freezing processes ([42]) and magma solidification ([43]), for example.

Gagliardo-Nirenberg inequalities were used before to obtain different results on asymptotic behavior of solutions of nonlinear diffusion equations as in (1.26) for various ranges of m, see [7, 21, 24, 25, 26], for instance. Concerning decay of solutions of (1.26) which are above Barenblatt profiles we quote [7, 8, 12]. In particular, in [8, 12] a family of GNI was established and used to show convergence with sharp rates. Caffarelli-Kohn-Nirenberg inequalities (GNI with power weights) are applied in [9, 10] to a weighted fast diffusion equation. For contexts where (1.26) with $m < 0$ arises, as well as for summaries of results on this problem, we refer to [20, 47].

This paper is organized in such a way that Theorem 1.1 will be the objective of Section 2, whereas our study on the decay rates of solutions to (1.6) can be found in Section 3.

2 A Gagliardo-Nirenberg-type inequality

2.1 Properties of functions satisfying (H)

With two exceptions formed by Lemma 3.4 and Lemma 3.5 in which indeed (H) is directly referred to, throughout the sequel we will make use of (H) only through the elementary consequences of (H) stated in the following two lemmata. The first of these, to be applied in Lemma 2.3 but also again in the proof of Theorem 1.1, inter alia entails a property of essentially superalgebraic growth of the function $\mathcal{L}$ therein, for our later purposes formulated by including the derivative $\mathcal{L}'$.

Lemma 2.1 *Assume that $s_0 \in (0,1)$ and that $\mathcal{L} \in C^0([0,s_0)) \cap C^1((0,s_0))$ is positive and nondecreasing on $(0,s_0)$ and such that* (H) *is valid. Then*

$$\frac{s\mathcal{L}'(s)}{\mathcal{L}(s)} \le \frac{a}{\ln\frac{1}{s}} \qquad \text{for all } s \in (0,s_0), \tag{2.1}$$

and in particular,

$$\frac{s\mathcal{L}'(s)}{\mathcal{L}(s)} \to 0 \qquad \text{as } s \searrow 0. \tag{2.2}$$

Proof. Given $s \in (0,s_0)$, we have $s<1$ and thus in particular $s - s^{1+\lambda} > 0$ for all $\lambda > 0$, whence we may apply e.g. l'Hospital's rule to see that

$$\lim_{\lambda\searrow 0} \frac{-\lambda s \ln s}{s - s^{1+\lambda}} = \lim_{\lambda\searrow 0} \frac{s\ln s}{s^{1+\lambda}\ln s} = 1. \tag{2.3}$$

On the other hand, (1.2) implies that

$$\mathcal{L}(s) - \mathcal{L}(s^{1+\lambda}) \le (1+a\lambda)\mathcal{L}(s^{1+\lambda}) - \mathcal{L}(s^{1+\lambda}) = a\lambda\mathcal{L}(s^{1+\lambda}) \qquad \text{for all } \lambda \in (0,\lambda_0),$$

so that

$$\limsup_{\lambda\searrow 0} \frac{\mathcal{L}(s) - \mathcal{L}(s^{1+\lambda})}{-\lambda s \ln s} \le \limsup_{\lambda\searrow 0} \frac{a\lambda\mathcal{L}(s^{1+\lambda})}{-\lambda s\ln s} = \frac{a\mathcal{L}(s)}{s\ln\frac{1}{s}}\,, \tag{2.4}$$

because $\mathcal{L}$ is continuous. As moreover $\mathcal{L}$ is even differentiable at s, combining (2.3) with (2.4) we thus infer that

$$\begin{aligned}
\mathcal{L}'(s) &= \lim_{\lambda\searrow 0} \frac{\mathcal{L}(s)-\mathcal{L}(s^{1+\lambda})}{s-s^{1+\lambda}} = \lim_{\lambda\searrow 0} \left\{ \frac{\mathcal{L}(s)-\mathcal{L}(s^{1+\lambda})}{-\lambda s\ln s} \cdot \frac{-\lambda s \ln s}{s-s^{1+\lambda}} \right\} \\
&\le \left\{ \limsup_{\lambda\searrow 0} \frac{\mathcal{L}(s)-\mathcal{L}(s^{1+\lambda})}{-\lambda s\ln s} \right\} \lim_{\lambda\searrow 0} \frac{-\lambda s\ln s}{s-s^{1+\lambda}} = \frac{a\mathcal{L}(s)}{s\ln\frac{1}{s}},
\end{aligned}$$

which yields (2.1) and thereby also entails (2.2) due to the monotonicity of $\mathcal{L}$. $\square$

Apart from the latter, in Lemma 2.3 we shall also make use of (H) through the following conclusion which is weaker than that in Lemma 2.1 and actually satisfied also by any function $\mathcal{L}$ with precise algebraic behavior near the origin.

Lemma 2.2 *Let $s_0 \in (0,1)$ and $\mathcal{L} \in C^0([0,s_0)) \cap C^1((0,s_0))$ be positive and nondecreasing on $(0,s_0)$ and such that* (H) *holds. Then for any $d \in (0,1)$ there exists $C>0$ such that*

$$\mathcal{L}(ds) \ge C\mathcal{L}(s) \qquad \textit{for all } s \in (0,s_0). \tag{2.5}$$

PROOF. Writing $c_1 := a(\ln\frac{1}{s_0})^{-1}$, from (2.1) we know that

$$\frac{\mathcal{L}'(s)}{\mathcal{L}(s)} \le \frac{c_1}{s} \qquad \text{for all } s\in(0,s_0),$$

which on integration shows that for fixed $d\in(0,1)$ and any $s\in(0,s_0)$ we have

$$\ln\frac{\mathcal{L}(s)}{\mathcal{L}(ds)} \le \ln\frac{s^{c_1}}{(ds)^{c_1}},$$

so that thus (2.5) holds with $C := d^{c_1}$. $\square$

2.2 Interpolation in Lebesgue spaces for rapidly decaying functions

Now a major step toward our derivation of Theorem 1.1 will be accomplished in the next lemma, the outcome of which already anticipates the structure of the desired inequality in (1.4) but yet exclusively contains Lebesgue norms of the considered function itself, rather than its gradient. Accordingly, in the case of algebraic $\mathcal{L}$ given by $\mathcal{L}(s)=s^r$ with $r>0$, the achieved estimate (2.7) essentially reduces to a Hölder-type interpolation.

Lemma 2.3 *Assume that $s_0\in(0,1)$ and $\mathcal{L}\in C^0([0,\infty))\cap C^1((0,s_0))$ is nonnegative, nondecreasing and such that* (H) *holds. Then for any choice of $n\ge 1$, $q_\star>0$, $q\in(0,q_\star)$ and $K>0$ one can find $C=C(n,q,q_\star,K)>0$ with the property that if $0\not\equiv\varphi\in L^{q_\star}(\mathbb{R}^n)$ is nonnegative and such that*

$$\int_{\mathbb{R}^n} \mathcal{L}(\varphi) \le K, \tag{2.6}$$

then the inequality

$$\|\varphi\|_{L^q(\mathbb{R}^n)} \le C\|\varphi\|_{L^{q_\star}(\mathbb{R}^n)} \left\{ \mathcal{L}^{-(\frac{1}{q}-\frac{1}{q_\star})}\Big(\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\Big) + 1 \right\} \tag{2.7}$$

holds.

PROOF. We first recall the outcome of Lemma 2.1 to find $s_1 \in (0,1)$ such that $\frac{s\mathcal{L}'(s)}{\mathcal{L}(s)} \le \min\{q, \frac{q_\star}{2}\}$ for all $s \in (0,s_1)$, which implies that

$$\frac{d}{ds}\Big\{s^{-q}\mathcal{L}(s)\Big\} = s^{-q-1}\Big\{s\mathcal{L}'(s) - q\mathcal{L}(s)\Big\} \le 0 \qquad \text{for all } s \in (0,s_1), \tag{2.8}$$

and that similarly $\frac{d}{ds}\Big(s^{-\frac{q_\star}{2}}\mathcal{L}(s)\Big) \le 0$ for all $s \in (0,s_1)$. On integration, the latter entails that

$$s^{-\frac{q_\star}{2}}\mathcal{L}(s) \ge c_1 := s_1^{-\frac{q_\star}{2}}\mathcal{L}(s_1) \qquad \text{for all } s \in (0,s_1),$$

whence

$$\mathcal{L}(s) \ge c_1 s^{\frac{q_\star}{2}} \qquad \text{for all } s \in (0,s_1). \tag{2.9}$$

We now fix positive numbers D and d such that

$$D^{q_\star} \ge K \tag{2.10}$$

as well as

$$d < 1 \qquad \text{and} \qquad d \le c_1^{\frac{2}{q_\star}} D^{-2}, \tag{2.11}$$

and thereafter apply Lemma 2.2 to choose $c_2 > 0$ satisfying

$$\mathcal{L}(ds) \ge c_2\mathcal{L}(s) \qquad \text{for all } s \in (0,s_1). \tag{2.12}$$

We finally pick $s_2 > 0$ small enough such that $s_2 \le s_1$ and

$$s_2 \le \frac{1}{D} s_1 \mathcal{L}^{\frac{1}{q_\star}}(ds_1^2), \tag{2.13}$$

and suppose that $\varphi \in L^{q_\star}(\mathbb{R}^n)$ is nonnegative and such that $\varphi \not\equiv 0$ and $\int_{\mathbb{R}^n} \mathcal{L}(\varphi) \le K$. Then

$$B := D^{q_\star - q}\|\varphi\|_{L^{q_\star}(\mathbb{R}^n)}^{-(q_\star - q)} \mathcal{L}^{-\frac{q_\star - q}{q_\star}}\Big(d\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\Big) \tag{2.14}$$

is a well-defined positive number, and we first consider the case when

$$B^{-\frac{1}{q_\star - q}} \ge s_2, \tag{2.15}$$

in which we estimate the expression on the left-hand side of (2.7) according to

$$\|\varphi\|_{L^q(\mathbb{R}^n)} \le \|\varphi\|_{L^q(\{\varphi \ge s_2\})} + \|\varphi\|_{L^q(\{\varphi < s_2\})}. \tag{2.16}$$

Here, in view of (2.6) and the monotonicity of $\mathcal{L}$ we see that

$$K \ge \int_{\mathbb{R}^n} \mathcal{L}(\varphi) \ge \mathcal{L}(s_2)\Big|\{\varphi \ge s_2\}\Big|,$$

and hence employing the Hölder inequality we obtain

$$\int_{\{\varphi \ge s_2\}} \varphi^q \;\; \le \;\; \Big(\int_{\{\varphi \ge s_2\}} \varphi^{q_\star}\Big)^{\frac{q}{q_\star}} \Big|\{\varphi \ge s_2\}\Big|^{\frac{q_\star - q}{q_\star}} \le \Big(\int_{\mathbb{R}^n} \varphi^{q_\star}\Big)^{\frac{q}{q_\star}} \Big(\frac{K}{\mathcal{L}(s_2)}\Big)^{\frac{q_\star - q}{q_\star}},$$

that is,

$$\|\varphi\|_{L^q(\{\varphi\geq s_2\})} \leq c_3\|\varphi\|_{L^{q_\star}(\mathbb{R}^n)} \tag{2.17}$$

holds with $c_3 := \left(\frac{K}{\mathcal{L}(s_2)}\right)^{\frac{1}{q}-\frac{1}{q_\star}}$.

To control the second summand on the right of (2.16) we first make use of the monotonicity property expressed in (2.8) to see that since $s_2 \leq s_1$ we have

$$\frac{\varphi^q(x)}{\mathcal{L}(\varphi(x))} \leq c_4 := \frac{s_2^q}{\mathcal{L}(s_2)} \qquad \text{for all } x \in \{\varphi < s_2\}$$

and thus

$$\int_{\{\varphi<s_2\}} \varphi^q \leq c_4 \int_{\{\varphi<s_2\}} \mathcal{L}(\varphi) \leq c_4 K$$

again by (2.6). In conjunction with (2.17) and (2.16), this shows that if (2.15) is valid then

$$\|\varphi\|_{L^q(\mathbb{R}^n)} \leq c_3\|\varphi\|_{L^{q_\star}(\mathbb{R}^n)} + (c_4K)^{\frac{1}{q}}.$$

To derive a suitable estimate for $(c_4K)^{\frac{1}{q}}$ we distinguish two cases.

Let us first assume that $\|\varphi\|_{L^{q_\star}(\mathbb{R}^n)} \leq \sqrt{s_2}$. Then it follows from (2.14) and (2.15) that

$$D^{-1}\|\varphi\|_{L^{q_\star}(\mathbb{R}^n)}\mathcal{L}^{\frac{1}{q_\star}}\Big(d\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\Big) \geq s_2,$$

and

$$\begin{aligned}
D^{-1}\|\varphi\|_{L^{q_\star}(\mathbb{R}^n)}\mathcal{L}^{-\left(\frac{1}{q}-\frac{1}{q_\star}\right)}\Big(\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\Big) &= D^{-1}\|\varphi\|_{L^{q_\star}(\mathbb{R}^n)}\mathcal{L}^{\frac{1}{q_\star}}\Big(\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\Big)\mathcal{L}^{-\frac{1}{q}}\Big(\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\Big) \\
&\geq D^{-1}\|\varphi\|_{L^{q_\star}(\mathbb{R}^n)}\mathcal{L}^{\frac{1}{q_\star}}\Big(d\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\Big)\mathcal{L}^{-\frac{1}{q}}\Big(\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\Big) \\
&\geq s_2\mathcal{L}^{-\frac{1}{q}}\Big(\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\Big) \geq s_2\mathcal{L}^{-\frac{1}{q}}(s_2) = c_4^{\frac{1}{q}}.
\end{aligned}$$

Hence, we obtain that

$$(c_4K)^{\frac{1}{q}} \leq K^{\frac{1}{q}}D^{-1}\|\varphi\|_{L^{q_\star}(\mathbb{R}^n)}\mathcal{L}^{-\left(\frac{1}{q}-\frac{1}{q_\star}\right)}\Big(\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\Big).$$

Obviously, if $\|\varphi\|_{L^{q_\star}(\mathbb{R}^n)} > \sqrt{s_2}$ then

$$(c_4K)^{\frac{1}{q}} < \frac{(c_4K)^{\frac{1}{q}}}{\sqrt{s_2}}\|\varphi\|_{L^{q_\star}(\mathbb{R}^n)},$$

which implies (2.7) in the case (2.15).

Conversely, if instead of (2.15) we have

$$B^{-\frac{1}{q_\star-q}} < s_2, \tag{2.18}$$

then we first note that necessarily

$$\|\varphi\|_{L^{q_\star}(\mathbb{R}^n)} \le s_1, \tag{2.19}$$

because if this was false then by definition (2.14) of B and once more due to the monotonicity of $\mathcal{L}$ we would obtain

$$s_2 > B^{-\frac{1}{q_\star - q}} = \frac{1}{D}\|\varphi\|_{L^{q_\star}(\mathbb{R}^n)} \mathcal{L}^{\frac{1}{q_\star}}\Big(d\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\Big) \ge \frac{1}{D} s_1 \mathcal{L}^{\frac{1}{q_\star}}(ds_1^2),$$

which is absurd in view of (2.13).

We shall next verify that for each $x \in \mathbb{R}^n$ we have

$$\varphi^q(x) \le B\varphi^{q_\star}(x) + \mathcal{B}\mathcal{L}(\varphi(x)), \qquad \mathcal{B} := \frac{1}{B^{\frac{q}{q_\star-q}} \mathcal{L}\Big(B^{-\frac{1}{q_\star-q}}\Big)}, \tag{2.20}$$

which is obvious if $\varphi^q(x) \le B\varphi^{q_\star}(x)$, that is, if $\varphi(x) \ge B^{-\frac{1}{q_\star-q}}$. If $\varphi(x) < B^{-\frac{1}{q_\star-q}}$, however, then by our current assumption (2.18) on B we have $\varphi(x) < s_2 \le s_1$, and hence again the monotonicity property (2.8) implies that

$$\frac{\varphi^q(x)}{\mathcal{L}(\varphi(x))} \le \frac{\Big(B^{-\frac{1}{q_\star-q}}\Big)^q}{\mathcal{L}\Big(B^{-\frac{1}{q_\star-q}}\Big)} = \mathcal{B}$$

which completes the proof of (2.20).

Now integrating (2.20) we find that

$$\int_{\mathbb{R}^n} \varphi^q \quad \le \quad B\int_{\mathbb{R}^n} \varphi^{q_\star} + \mathcal{B}K, \tag{2.21}$$

and we claim that our choice of B ensures that herein

$$\mathcal{B}K \le B\int_{\mathbb{R}^n} \varphi^{q_\star}. \tag{2.22}$$

Indeed, by (2.14) we have

$$\begin{aligned}
\frac{B\int_{\mathbb{R}^n}\varphi^{q_\star}}{\mathcal{B}K} &= \frac{1}{K} B^{\frac{q_\star}{q_\star-q}} \mathcal{L}\Big(B^{-\frac{1}{q_\star-q}}\Big) \int_{\mathbb{R}^n} \varphi^{q_\star} \\
&= \frac{1}{K}\bigg\{D^{q_\star-q}\|\varphi\|^{-(q_\star-q)}_{L^{q_\star}(\mathbb{R}^n)} \mathcal{L}^{-\frac{q_\star-q}{q_\star}}\Big(d\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\Big)\bigg\}^{\frac{q_\star}{q_\star-q}} \times \\
&\qquad \times \mathcal{L}\Bigg(\bigg\{D^{q_\star-q}\|\varphi\|^{-(q_\star-q)}_{L^{q_\star}(\mathbb{R}^n)} \mathcal{L}^{-\frac{q_\star-q}{q_\star}}\Big(d\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\Big)\bigg\}^{-\frac{1}{q_\star-q}}\Bigg) \int_{\mathbb{R}^n}\varphi^{q_\star} \\
&= \frac{D^{q_\star}}{K} \frac{\mathcal{L}\Big(\frac{1}{D}\|\varphi\|_{L^{q_\star}(\mathbb{R}^n)} \mathcal{L}^{\frac{1}{q_\star}}\big(d\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\big)\Big)}{\mathcal{L}\big(d\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\big)},
\end{aligned} \tag{2.23}$$

where thanks to (2.11) and (2.19) we know that

$$d\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)} < \|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)} \le s_1^2 \le s_1, \tag{2.24}$$

so that in particular, by (2.9) and (2.11),

$$\begin{aligned}\frac{1}{D}\|\varphi\|_{L^{q_\star}(\mathbb{R}^n)}\mathcal{L}^{\frac{1}{q_\star}}\Big(d\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\Big) &\ge \frac{1}{D}\|\varphi\|_{L^{q_\star}(\mathbb{R}^n)}c_1^{\frac{1}{q_\star}}\Big(d\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\Big)^{\frac{1}{2}} \\ &= \frac{c_1^{\frac{1}{q_\star}}\sqrt{d}}{D}\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)} \ge d\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}.\end{aligned}$$

Once more by means of the monotonicity of $\mathcal{L}$, from (2.23) and (2.10) we thus obtain that

$$\frac{B\int_{\mathbb{R}^n}\varphi^{q_\star}}{\mathcal{B}K} \ge \frac{D^{q_\star}}{K}\frac{\mathcal{L}\big(d\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\big)}{\mathcal{L}\big(d\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\big)} \ge 1.$$

Having thereby proved (2.22), we may use this to infer from (2.21) that according to our definition (2.14) of B,

$$\begin{aligned}\int_{\mathbb{R}^n}\varphi^q &\le 2B\int_{\mathbb{R}^n}\varphi^{q_\star} = 2\Big\{D^{q_\star-q}\|\varphi\|^{-(q_\star-q)}_{L^{q_\star}(\mathbb{R}^n)}\mathcal{L}^{-\frac{q_\star-q}{q_\star}}\Big(d\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\Big)\Big\}\int_{\mathbb{R}^n}\varphi^{q_\star} \\ &= 2D^{q_\star-q}\|\varphi\|^q_{L^{q_\star}(\mathbb{R}^n)}\mathcal{L}^{-\frac{q_\star-q}{q_\star}}\Big(d\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\Big),\end{aligned}$$

that is,

$$\|\varphi\|_{L^q(\mathbb{R}^n)} \le (2D^{q_\star-q})^{\frac{1}{q}}\|\varphi\|_{L^{q_\star}(\mathbb{R}^n)}\mathcal{L}^{-(\frac{1}{q}-\frac{1}{q_\star})}\Big(d\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\Big).$$

Again making use of (2.24) in estimating

$$\mathcal{L}\Big(d\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\Big) \ge c_2\mathcal{L}\Big(\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\Big)$$

by means of (2.12), from this we readily derive (2.7) also in the case when (2.18) holds. □

2.3 Proof of Theorem 1.1

Combining Lemma 2.3 with the Gagliardo-Nirenberg inequality in its well-known form, by once more making use of Lemma 2.1 we can now establish our main result on interpolation for rapidly decreasing functions.

Proof of Theorem 1.1. Given $0 \not\equiv \varphi \in W^{1,2}(\mathbb{R}^n)$ such that (1.3) holds, we first note that if $\|\varphi\|_{L^q(\mathbb{R}^n)} \le \|\nabla\varphi\|_{L^2(\mathbb{R}^n)}$, then abbreviating $\alpha := \frac{1}{q} - \frac{n-2}{2n}$ we can estimate

$$\frac{\|\varphi\|_{L^q(\mathbb{R}^n)}}{\|\nabla\varphi\|_{L^2(\mathbb{R}^n)}\mathcal{L}^{-\alpha}(\|\nabla\varphi\|_{L^2(\mathbb{R}^n)})} \le \mathcal{L}^\alpha_\infty\frac{\|\varphi\|_{L^q(\mathbb{R}^n)}}{\|\nabla\varphi\|_{L^2(\mathbb{R}^n)}} \le \mathcal{L}^\alpha_\infty \tag{2.25}$$

with $\mathcal{L}_\infty := \|\mathcal{L}\|_{L^\infty((0,\infty))}$ being finite due to the boundedness of $\mathcal{L}$.
We are thus left with the case when

$$\|\varphi\|_{L^q(\mathbb{R}^n)} > \|\nabla\varphi\|_{L^2(\mathbb{R}^n)}, \tag{2.26}$$

in which using that $q \in (0, \frac{2n}{(n-2)_+})$ we can fix a number $q_\star \geq 1$ such that $q_\star > q$ and $q_\star \leq \frac{2n}{(n-2)_+}$, so that an application of Lemma 2.3 yields $c_1 > 0$ fulfilling

$$\|\varphi\|_{L^q(\mathbb{R}^n)} \leq c_1\|\varphi\|_{L^{q_\star}(\mathbb{R}^n)}\Big\{\mathcal{L}^{-\gamma}\Big(\|\varphi\|^2_{L^{q_\star}(\mathbb{R}^n)}\Big) + 1\Big\}, \tag{2.27}$$

where $\gamma := \frac{1}{q} - \frac{1}{q_\star} > 0$. Here by means of the standard Gagliardo-Nirenberg inequality we can find $c_2 \geq 1$ such that

$$\|\varphi\|_{L^{q_\star}(\mathbb{R}^n)} \leq c_2\|\nabla\varphi\|^\theta_{L^2(\mathbb{R}^n)}\|\varphi\|^{1-\theta}_{L^q(\mathbb{R}^n)} \tag{2.28}$$

with $\theta := \frac{\frac{n}{q}-\frac{n}{q_\star}}{1+\frac{n}{q}-\frac{n}{2}} \in (0,1]$, and in order to make appropriate use of this on the right-hand side of (2.27) we recall Lemma 2.1 to pick $s_1 > 0$ satisfying

$$\frac{s\mathcal{L}'(s)}{\mathcal{L}(s)} \leq \frac{1}{2\gamma} \qquad \text{for all } s \in (0, s_1),$$

which, namely, warrants that for

$$\rho(\sigma) := c_1\sigma\Big\{\mathcal{L}^{-\gamma}(\sigma^2) + 1\Big\}, \qquad \sigma > 0, \tag{2.29}$$

we have

$$\rho'(\sigma) = c_1 + c_1\mathcal{L}^{-\gamma-1}(\sigma^2)\Big\{\mathcal{L}(\sigma^2) - 2\gamma\sigma^2\mathcal{L}'(\sigma^2)\Big\} \geq 0 \qquad \text{for all } \sigma \in (0, \sqrt{s_1}).$$

Therefore, if $c_2\|\nabla\varphi\|^\theta_{L^2(\mathbb{R}^n)}\|\varphi\|^{1-\theta}_{L^q(\mathbb{R}^n)} < \sqrt{s_1}$ then we obtain from (2.28) and (2.27) that

$$\begin{aligned}
\|\varphi\|_{L^q(\mathbb{R}^n)} &\leq \rho\Big(c_2\|\nabla\varphi\|^\theta_{L^2(\mathbb{R}^n)}\|\varphi\|^{1-\theta}_{L^q(\mathbb{R}^n)}\Big) \\
&\leq c_1c_2\|\nabla\varphi\|^\theta_{L^2(\mathbb{R}^n)}\|\varphi\|^{1-\theta}_{L^q(\mathbb{R}^n)}\Big\{\mathcal{L}^{-\gamma}\Big(c_2^2\|\nabla\varphi\|^{2\theta}_{L^2(\mathbb{R}^n)}\|\varphi\|^{2(1-\theta)}_{L^q(\mathbb{R}^n)}\Big) + 1\Big\}
\end{aligned}$$

and hence

$$\begin{aligned}
\|\varphi\|_{L^q(\mathbb{R}^n)} &\leq (c_1c_2)^{\frac{1}{\theta}}\|\nabla\varphi\|_{L^2(\mathbb{R}^n)}\Big\{\mathcal{L}^{-\gamma}\Big(c_2^2\|\nabla\varphi\|^{2\theta}_{L^2(\mathbb{R}^n)}\|\varphi\|^{2(1-\theta)}_{L^q(\mathbb{R}^n)}\Big) + 1\Big\}^{\frac{1}{\theta}} \\
&\leq (c_1c_2)^{\frac{1}{\theta}}(1+\mathcal{L}_\infty^\gamma)^{\frac{\gamma}{\theta}}\|\nabla\varphi\|_{L^2(\mathbb{R}^n)}\mathcal{L}^{-\frac{\gamma}{\theta}}\Big(c_2^2\|\nabla\varphi\|^{2\theta}_{L^2(\mathbb{R}^n)}\|\varphi\|^{2(1-\theta)}_{L^q(\mathbb{R}^n)}\Big) \\
&\leq (c_1c_2)^{\frac{1}{\theta}}(1+\mathcal{L}_\infty^\gamma)^{\frac{\gamma}{\theta}}\|\nabla\varphi\|_{L^2(\mathbb{R}^n)}\mathcal{L}^{-\frac{\gamma}{\theta}}\Big(\|\nabla\varphi\|^2_{L^2(\mathbb{R}^n)}\Big),
\end{aligned} \tag{2.30}$$

because by definition of $\mathcal{L}_\infty$ we can estimate

$$1 \leq \mathcal{L}_\infty^\gamma\mathcal{L}^{-\gamma}\Big(c_2^2\|\nabla\varphi\|^{2\theta}_{L^2(\mathbb{R}^n)}\|\varphi\|^{2(1-\theta)}_{L^q(\mathbb{R}^n)}\Big),$$

and because (2.26) along with our restriction $c_2 \ge 1$ implies that

$$\mathcal{L}\Big(c_2^2 \|\nabla\varphi\|_{L^2(\mathbb{R}^n)}^{2\theta} \|\varphi\|_{L^q(\mathbb{R}^n)}^{2(1-\theta)}\Big) \ge \mathcal{L}\Big(\|\nabla\varphi\|_{L^2(\mathbb{R}^n)}^2\Big).$$

If, conversely, $c_2 \|\nabla\varphi\|_{L^2(\mathbb{R}^n)}^{\theta} \|\varphi\|_{L^q(\mathbb{R}^n)}^{1-\theta} \ge \sqrt{s_1}$, then (2.26) entails that

$$\|\varphi\|_{L^q(\mathbb{R}^n)} \ge c_3 := \frac{\sqrt{s_1}}{c_2}. \tag{2.31}$$

In view of the fact that the function ρ from (2.29) satisfies $\rho(\sigma) \to 0$ as $\sigma \to 0$, we can pick $\sigma_1 > 0$ such that $\rho(\sigma) < c_3$ for all $\sigma \in (0, \sigma_1)$, so that using the inequality in (2.27) in the form $\|\varphi\|_{L^q(\mathbb{R}^n)} \le \rho(\|\varphi\|_{L^{q\star}(\mathbb{R}^n)})$, we infer from (2.31) that necessarily $\|\varphi\|_{L^{q\star}(\mathbb{R}^n)} \ge \sigma_1$. In conjunction with (2.27), the monotonicity of $\mathcal{L}$ and (2.28), however, this implies that writing $c_4 := \mathcal{L}^{-\gamma}(\sigma_1^2) + 1$ we have

$$\|\varphi\|_{L^q(\mathbb{R}^n)} \le c_1 c_4 \|\varphi\|_{L^{q\star}(\mathbb{R}^n)} \le c_1 c_2 c_4 \|\nabla\varphi\|_{L^2(\mathbb{R}^n)}^{\theta} \|\varphi\|_{L^q(\mathbb{R}^n)}^{1-\theta}$$

and thus

$$\|\varphi\|_{L^q(\mathbb{R}^n)} \le (c_1 c_2 c_4)^{\frac{1}{\theta}} \|\nabla\varphi\|_{L^2(\mathbb{R}^n)},$$

whence proceeding as in (2.25) we end up with the inequality

$$\frac{\|\varphi\|_{L^q(\mathbb{R}^n)}}{\|\nabla\varphi\|_{L^2(\mathbb{R}^n)} \mathcal{L}^{-\alpha}(\|\nabla\varphi\|_{L^2(\mathbb{R}^n)})} \le (c_1 c_2 c_4)^{\frac{1}{\theta}} \mathcal{L}_\infty^{\alpha}$$

in this case. Together with (2.25), (2.30) and the observation that $\frac{\gamma}{\theta} = \alpha$, this establishes (1.4). $\Box$

3 Decay estimates for solutions of $u_t = u^p \Delta u$

3.1 Preliminaries: Existence and approximation of solutions

Next addressing the degenerate parabolic problem (1.6) for $n \ge 1$ and $p \ge 1$, in order to construct solutions thereof by approximation we follow [29] in considering

$$\begin{cases} u_{Rt} = u_R^p \Delta u_R, & x \in B_R,\ t > 0, \\ u_R(x,t) = 0, & x \in \partial B_R,\ t > 0, \\ u_R(x,0) = u_{0R}(x), & x \in B_R, \end{cases} \tag{3.1}$$

for $R > 0$, where $u_{0R} \in C^3(\bar{B}_R)$ satisfies $0 < u_{0R} < u_0$ in B_R and $u_{0R} = 0$ on ∂B_R as well as

$$u_{0R} \nearrow u_0 \quad \text{in } \mathbb{R}^n \qquad \text{as } R \nearrow \infty. \tag{3.2}$$

Moreover, for $\varepsilon \in (0,1)$ we consider

$$\begin{cases} u_{R\varepsilon t} = u_{R\varepsilon}^p \Delta u_{R\varepsilon}, & x \in B_R,\ t > 0, \\ u_{R\varepsilon}(x,t) = \varepsilon, & x \in \partial B_R,\ t > 0, \\ u_{R\varepsilon}(x,0) = u_{0R\varepsilon}(x) := u_{0R}(x) + \varepsilon, & x \in B_R. \end{cases} \tag{3.3}$$

Then the following basic statement has been shown in [29].

Lemma 3.1 *Let $p \geq 1$, and assume that $u_0 \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is positive. Then with u_{0R} and $(u_{0\varepsilon R})_{\varepsilon\in(0,1)}$ as above, for each $\varepsilon \in (0,1)$ the problem (3.3) possesses a global classical solution $u_{R\varepsilon} \in C^0(\bar{B}_R \times [0,\infty)) \cap C^{2,1}(\bar{B}_R \times (0,\infty))$. As $\varepsilon \searrow 0$, we have $u_{R\varepsilon} \searrow u_R$ with some positive classical solution $u_R \in C^0(\bar{B}_R \times [0,\infty)) \cap C^{2,1}(B_R \times (0,\infty))$ of (3.1). Moreover, there exists a classical solution $u \in C^0(\mathbb{R}^n \times [0,\infty)) \cap C^{2,1}(\mathbb{R}^n \times (0,\infty))$ of (1.6) which is such that*

$$0 < u(x,t) \leq \|u_0\|_{L^\infty(\mathbb{R}^n)} \qquad \textit{for all } x \in \mathbb{R}^n \textit{ and } t \geq 0, \tag{3.4}$$

and that $u_R \nearrow u$ in $\mathbb{R}^n \times (0,\infty)$ as $R \nearrow \infty$. This solution is minimal in the sense that whenever $\widetilde{u} \in C^0(\mathbb{R}^n \times [0,\infty)) \cap C^{2,1}(\mathbb{R}^n \times (0,\infty))$ is positive and solves (1.6) classically, we have $\widetilde{u} \geq u$ in $\mathbb{R}^n \times [0,\infty)$.

We note that in the special case when u_0 is radially symmetric around the origin and nonincreasing with respect to $|x|$, we may and will assume that u_{0R} has the same properties, which then, according to a standard argument involving the comparison principle, are clearly inherited by $u_{R\varepsilon}(\cdot,t)$ and hence also by $u_R(\cdot,t)$ for all $t > 0$.

3.2 A Lyapunov functional ensuring persistence of fast spatial decay

To describe the large time asymptotics in (1.6) using the above interpolation results, let us first make sure that as a particular feature of the strong degeneracy in (1.6) expressed in our hypothesis $p \geq 1$, minimal solutions maintain the initial spatial decay. Our general observation in this direction reads as follows.

Lemma 3.2 *Let $p \geq 1$, $q > 0$ and $s_0 > 0$, and suppose that the nonnegative and nondecreasing function $\mathcal{L} \in C^0([0,s_0]) \cap C^2((0,s_0))$ has the property that*

$$s\mathcal{L}''(s) \geq -\frac{3p+q-2}{p+q}\,\mathcal{L}'(s) \qquad \textit{for all } s \in (0,s_0). \tag{3.5}$$

Then for any positive $u_0 \in C^0(\mathbb{R}^n)$ satisfying $u_0^{\frac{p+q}{2}} < s_0$ in $\mathbb{R}^n$ and all $R > 0$, there exists $\varepsilon_0(R) \in (0,1)$ such that for each $\varepsilon \in (0,\varepsilon_0(R))$ the solution $u_{R\varepsilon}$ of (3.3) satisfies

$$\int_{B_R} \mathcal{L}\Big(u_{R\varepsilon}^{\frac{p+q}{2}}(\cdot,t)\Big) \leq \int_{B_R} \mathcal{L}\Big(u_{0\varepsilon}^{\frac{p+q}{2}}\Big) \qquad \textit{for all } t > 0. \tag{3.6}$$

Proof. Since $u_{0R}^{\frac{p+q}{2}} \leq u_0^{\frac{p+q}{2}} < s_0$ in $\mathbb{R}^n$, for each $R > 0$ we can find $\varepsilon_0(R) \in (0,1)$ such that $u_{0R\varepsilon}^{\frac{p+q}{2}} < s_0$ in $\bar{B}_R$ for all $\varepsilon \in (0,\varepsilon_0(R))$. By comparison, this implies that the solution $u_{R\varepsilon}$ of (3.3) satisfies $u_{R\varepsilon}^{\frac{p+q}{2}} < 0$ in $\bar{B}_R \times [0,\infty)$, so that (3.5) applies to guarantee that

$$u_{R\varepsilon}^{\frac{p+q}{2}}\,\mathcal{L}''\Big(u_{R\varepsilon}^{\frac{p+q}{2}}\Big) \geq -\frac{3p+q-2}{p+q}\,\mathcal{L}'\Big(u_{R\varepsilon}^{\frac{p+q}{2}}\Big) \qquad \text{in } \bar{B}_R \times [0,\infty). \tag{3.7}$$

Now from (3.3) we obtain that for all $R > 0$ and $\varepsilon \in (0, \varepsilon_0(R))$,

$$\begin{aligned}
\frac{2}{p+q}\frac{d}{dt}\int_{B_R} \mathcal{L}\Big(u_{R\varepsilon}^{\frac{p+q}{2}}\Big) &= \int_{B_R} u_{R\varepsilon}^{\frac{p+q-2}{2}} \mathcal{L}'\Big(u_{R\varepsilon}^{\frac{p+q}{2}}\Big) u_{R\varepsilon t} = \int_{B_R} u_{R\varepsilon}^{\frac{3p+q-2}{2}} \mathcal{L}'\Big(u_{R\varepsilon}^{\frac{p+q}{2}}\Big) \Delta u_{R\varepsilon} \\
&= -\int_{B_R} \Big\{ \frac{p+q}{2} u_{R\varepsilon}^{\frac{4p+2q-4}{2}} \mathcal{L}''\Big(u_{R\varepsilon}^{\frac{p+q}{2}}\Big) + \frac{3p+q-2}{2} u_{R\varepsilon}^{\frac{3p+q-4}{2}} \mathcal{L}'\Big(u_{R\varepsilon}^{\frac{p+q}{2}}\Big) \Big\} |\nabla u_{R\varepsilon}|^2 \\
&\quad + \int_{\partial B_R} u_{R\varepsilon}^{\frac{3p+q-2}{2}} \mathcal{L}'\Big(u_{R\varepsilon}^{\frac{p+q}{2}}\Big) \frac{\partial u_{R\varepsilon}}{\partial \nu} \\
&\le -\frac{p+q}{2} \int_{B_R} \Big\{ u_{R\varepsilon}^{\frac{p+q}{2}} \mathcal{L}''\Big(u_{R\varepsilon}^{\frac{p+q}{2}}\Big) + \frac{3p+q-2}{p+q} \mathcal{L}'\Big(u_{R\varepsilon}^{\frac{p+q}{2}}\Big) \Big\} u_{R\varepsilon}^{\frac{3p+q-4}{2}} |\nabla u_{R\varepsilon}|^2
\end{aligned}$$

for all $t > 0$, because $\mathcal{L}' \ge 0$ on $(0, s_0)$ and $\frac{\partial u_{R\varepsilon}}{\partial \nu} \le 0$ on $\partial B_R \times (0, \infty)$ due to the fact that $u_{R\varepsilon} \ge \varepsilon$ in $B_R \times (0, \infty)$ and $u_{R\varepsilon} = \varepsilon$ on $\partial B_R \times (0, \infty)$.
In view of (3.7), however, this shows that $\frac{d}{dt}\int_{B_R} \mathcal{L}\Big(u_{R\varepsilon}^{\frac{p+q}{2}}\Big) \le 0$ for all $t > 0$ and hence indeed

$$\int_{B_R} \mathcal{L}\Big(u_{R\varepsilon}^{\frac{p+q}{2}}\Big) \le \int_{B_R} \mathcal{L}\Big(u_{0R\varepsilon}^{\frac{p+q}{2}}\Big) \qquad \text{for all } t > 0 \tag{3.8}$$

whenever $R > 0$ and $\varepsilon \in (0, \varepsilon_0(R))$. $\square$

When we choose $\mathcal{L}$ as a suitable power-type function, the above in particular implies the control of the spatial L^r quasi-norm in the flavor of (3.6) for any $r \ge 1 - p$, and hence for all positive r whenever $p \ge 1$. As the above reasoning shows, this conclusion actually extends to the not explicitly included cases $p = 0$ and $p \in (0, 1)$ corresponding to the heat equation and the porous medium equation, respectively, thus rediscovering well-known Lyapunov-type properties of $\int_{\mathbb{R}^n} u^r$ for each $r \ge 1 - p$ and any such p. In view of our ambition to study solutions with fast spatial decay, the essential role of our overall assumption $p \ge 1$ is underlined by the observation that the behavior of these functionals drastically changes when $p < 1$ and $r < 1 - p$. Indeed, in the case $p = 0$ it can directly be seen using explicit solution representation through convolution with the Gauss kernel that for all nontrivial nonnegative initial data in $L^1(\mathbb{R}^n)$ the corresponding functional $\int_{\mathbb{R}^n} u^r$ tends to ∞ as $t \to \infty$ for each $r \in (0, 1)$; a similar conclusion can be drawn, e.g. by using comparison from below with Barenblatt-type self-similar solutions, when $p \in (0, 1)$ and $r \in (0, 1 - p)$.

The requirement $p \ge 1$ guarantees that the above can actually be applied to functions $\mathcal{L}$ with a wide class of steepness properties near the origin. Actually, instead of applying Lemma 3.2 directly, in our examples studied in Corollaries 1.5 and 1.6 we will rather refer to the following weaker variant thereof.

Lemma 3.3 *Suppose that for some $s_0 > 0$, $\mathcal{L} \in C^0([0, s_0]) \cap C^2((0, s_0))$ is nonnegative and nondecreasing and such that*

$$\frac{d}{ds}\Big(s\mathcal{L}'(s)\Big) \ge 0 \qquad \textit{for all } s \in (0, s_0). \tag{3.9}$$

Then for all $p \ge 1$ and $q > 0$, the conclusion of Lemma 3.2 holds.

Proof. As (3.9) implies that $s\mathcal{L}''(s) \ge -\mathcal{L}'(s)$ for all $s \in (0, s_0)$, observing that this entails (3.5) due to the fact that

$$\frac{3p+q-2}{p+q} = 1 + \frac{2(p-1)}{p+q} \ge 1 \qquad \text{for all } p \ge 1 \text{ and } q > 0,$$

we only need to apply Lemma 3.2. $\Box$

3.3 Upper bounds in L^q for $q > 0$

Having at hand the above information on conservation of spatial decay, we shall next address a statement resembling that in Theorem 1.3 but involving quasi-norms in $L^q(\mathbb{R}^n)$ for finite $q > 0$. Our result in this direction, to be achieved in Lemma 3.6, will be prepared by two lemmata, the first of them solves some transcendental inequalities involving $\mathcal{L}$ by once more explicitly referring to (H).

Lemma 3.4 *Assume that $\mathcal{L} \in C^0([0,\infty)) \cap C^1((0,\infty))$ is nondecreasing and nonnegative and satisfies* (H)*, and let $\beta > \frac{1}{1+\lambda_0}$, $\gamma > 0$ and $\delta_0 > 0$. Then there exists $C > 0$ such that whenever $\eta > 0$ is such that*

$$\eta^\beta \mathcal{L}^\gamma(\eta) \le \delta \tag{3.10}$$

with some $\delta \in (0, \delta_0]$, then

$$\eta \le C\delta^{\frac{1}{\beta}} \mathcal{L}^{-\frac{\gamma}{\beta}}(\delta). \tag{3.11}$$

Proof. Since $1+\lambda_0 > \frac{1}{\beta}$, it is possible to find $\lambda \in (0, \lambda_0)$ such that still $1+\lambda > \frac{1}{\beta}$, whence invoking (H) provides $s_1 > 0$ and $c_1 > 0$ such that

$$\mathcal{L}(s) \le c_1 \mathcal{L}(s^{1+\lambda}) \qquad \text{for all } s \in (0, s_1). \tag{3.12}$$

We then pick $D > 0$ large such that

$$D^\beta \ge c_1^\gamma \tag{3.13}$$

and

$$c_2 D \ge \delta_0^{1+\lambda-\frac{1}{\beta}}, \tag{3.14}$$

where $c_2 := \mathcal{L}^{-\frac{\gamma}{\beta}}(\delta_0) > 0$.
Now assuming (3.10) to be valid for some $\eta > 0$ and $\delta \in (0, \delta_0]$, we first consider the case when $\delta < s_1$, in which we claim that

$$\eta \le D\delta^{\frac{1}{\beta}} \mathcal{L}^{-\frac{\gamma}{\beta}}(\delta). \tag{3.15}$$

In fact, if on the contrary we had $\eta > D\delta^{\frac{1}{\beta}} \mathcal{L}^{-\frac{\gamma}{\beta}}(\delta)$, then by monotonicity of $\mathcal{L}$ we would have

$$\frac{1}{\delta}\eta^\beta \mathcal{L}^\gamma(\eta) > \frac{1}{\delta}\Big(D\delta^{\frac{1}{\beta}} \mathcal{L}^{-\frac{\gamma}{\beta}}(\delta)\Big)^\beta \mathcal{L}^\gamma\Big(D\delta^{\frac{1}{\beta}} \mathcal{L}^{-\frac{\gamma}{\beta}}(\delta)\Big) = \frac{D^\beta}{\mathcal{L}^\gamma(\delta)} \mathcal{L}^\gamma\Big(D\delta^{\frac{1}{\beta}} \mathcal{L}^{-\frac{\gamma}{\beta}}(\delta)\Big). \tag{3.16}$$

Here since $\delta < s_1$ we may employ (3.12) to estimate

$$\mathcal{L}^\gamma(\delta) \le c_1^\gamma(\delta^{1+\lambda}), \tag{3.17}$$

and again using the monotonicity of $\mathcal{L}$ we see that $\mathcal{L}^{-\frac{\gamma}{\beta}}(\delta) \ge \mathcal{L}^{-\frac{\gamma}{\beta}}(\delta_0) = c_2$ and thus

$$\mathcal{L}^\gamma\Big(D\delta^{\frac{1}{\beta}} \mathcal{L}^{-\frac{\gamma}{\beta}}(\delta)\Big) \ge \mathcal{L}^\gamma\big(c_2 D\delta^{\frac{1}{\beta}}\big). \tag{3.18}$$

As our choice of λ ensures that

$$\frac{\delta^{1+\lambda}}{c_2 D\delta^{\frac{1}{\beta}}} \le \frac{\delta_0^{1+\lambda-\frac{1}{\beta}}}{c_2 D} \le 1$$

by (3.14), inserting (3.17) and (3.18) into (3.16) and recalling (3.13) therefore shows that

$$\frac{\eta^\beta \mathcal{L}^\gamma(\eta)}{\delta} > \frac{D^\beta}{c_1^\gamma} \frac{\mathcal{L}^\gamma(c_2 D\delta^{\frac{1}{\beta}})}{\mathcal{L}^\gamma(c_2 D\delta^{\frac{1}{\beta}})} \geq 1.$$

This contradiction to (3.10) warrants that indeed (3.15) holds if $\delta < s_1$.
However, if $s_1 \leq \delta \leq \delta_0$ then we observe that $\xi^\beta \mathcal{L}^\gamma(\xi) \to +\infty$ as $\xi \to +\infty$ to verify that $\eta_0 := \sup\{\xi > 0 \mid \xi^\beta \mathcal{L}^\gamma(\xi) \leq \delta_0\}$ is well-defined and satisfies $\eta_0 \geq \eta$ according to (3.10). On the other hand, by definition of c_2 we have

$$\delta^{-\frac{1}{\beta}} \mathcal{L}^{-\frac{\gamma}{\beta}}(\delta) \geq \delta^{\frac{1}{\beta}} \mathcal{L}^{-\frac{\gamma}{\beta}}(\delta_0) = c_2 \delta^{\frac{1}{\beta}} \geq c_2 s_1^{\frac{1}{\beta}},$$

because $\delta \leq \delta$ and $\delta \geq s_1$. Consequently, in this case we obtain

$$\frac{\eta}{\delta^{\frac{1}{\beta}} \mathcal{L}^{-\frac{\gamma}{\beta}}(\delta)} \leq \frac{\eta_0}{c_2 s_1^{\frac{1}{\beta}}},$$

and thus we all in all conclude that (3.11) is valid if we let $C := \max\left\{D\,,\, \frac{\eta_0}{c_2 s_1^{1/\beta}}\right\}$. □

Another consequence of (H) used in Lemma 3.6 states that for fixed nonnegative measurable and bounded φ, the family $(\mathcal{L}(\varphi^r))_{r>0}$ either entirely belongs to $L^1(\mathbb{R}^n)$ or lies completely outside, which clearly again reflects a strongly superalgebraic growth of $\mathcal{L}(s)$ near $s = 0$.

Lemma 3.5 *Suppose that $\mathcal{L} \in C^0([0,\infty))$ is nonnegative and nondecreasing and such that* (H) *is valid. Then for any nonnegative $\varphi \in L^\infty(\mathbb{R}^n)$ satisfying*

$$\int_{\mathbb{R}^n} \mathcal{L}(\varphi) < \infty, \tag{3.19}$$

we have

$$\int_{\mathbb{R}^n} \mathcal{L}(\varphi^r) < \infty \qquad \textit{for all } r > 0. \tag{3.20}$$

Proof. Without loss of generality assuming that $s_0 \leq 1$, we first note that since

$$\int_{\mathbb{R}^n} \mathcal{L}(\varphi) \geq \int_{\{\varphi \geq s_0^{\frac{1}{r}}\}} \mathcal{L}(\varphi) \geq \mathcal{L}(s_o^{\frac{1}{r}}) \Big|\{\varphi \geq s_0^{\frac{1}{r}}\}\Big|$$

by monotonicity of $\mathcal{L}$, (3.19) asserts that $c_1 := |\{\varphi \geq s_0^{\frac{1}{r}}\}|$ is finite. Now in the case $r < 1$ it is easy to see that there exist $k \in \mathbb{N}$ and $\lambda \in (0, \lambda_0)$ such that $r(1+\lambda)^k = 1$, whence k applications of (H) yield

$$\mathcal{L}(s^r) \leq (1+a\lambda)^k \mathcal{L}\Big(s^{r(1+\lambda)^k}\Big) = (1+a\lambda)^k \mathcal{L}(s) \qquad \text{for all } s \in [0, s_0^{\frac{1}{r}}),$$

because for any such s and each $j \in \{0, ..., k-1\}$ we have $s^{r(1+\lambda)^j} < s_0^{(1+\lambda)^j} \leq s_0$ due to the fact that $s_0 \leq 1$. Accordingly, again by monotonicity of $\mathcal{L}$ we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{L}(\varphi^r) &= \int_{\{\varphi \geq s_0^{\frac{1}{r}}\}} \mathcal{L}(\varphi^r) + \int_{\{\varphi < s_0^{\frac{1}{r}}\}} \mathcal{L}(\varphi^r) \\
&\leq \mathcal{L}\Big(\|\varphi\|_{L^\infty(\mathbb{R}^n)}^r\Big) \Big|\{\varphi \geq s_0^{\frac{1}{r}}\}\Big| + \int_{\{\varphi < s_0^{\frac{1}{r}}\}} (1+a\lambda)^k \mathcal{L}(\varphi) \\
&\leq c_1 \mathcal{L}\Big(\|\varphi\|_{L^\infty(\mathbb{R}^n)}^r\Big) + (1+a\lambda)^k \int_{\mathbb{R}^n} \mathcal{L}(\varphi),
\end{aligned}$$

so that (3.19) indeed implies (3.20) in this case.
If $r \geq 1$, however, we similarly estimate

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{L}(\varphi^r) &= \int_{\{\varphi \geq 1\}} \mathcal{L}(\varphi^r) + \int_{\{\varphi<1\}} \mathcal{L}(\varphi^r) \\ &\leq \mathcal{L}\Big(\|\varphi\|^r_{L^\infty(\mathbb{R}^n)}\Big)\Big|\{\varphi \geq 1\}\Big| + \int_{\{\varphi<1\}} \mathcal{L}(\varphi^r),\end{aligned}$$

where clearly $s_0 \leq 1$ entails that $|\{\varphi \geq 1\}| \leq c_1$, and where $r \geq 1$ implies that

$$\int_{\{\varphi<1\}} \mathcal{L}(\varphi^r) \leq \int_{\{\varphi<1\}} \mathcal{L}(\varphi) \leq \int_{\mathbb{R}^n} \mathcal{L}(\varphi),$$

whence again (3.20) results from (3.19). $\Box$

Using Theorem 1.1 along with Lemma 3.2, we can now achieve an essential step toward Theorem 1.3 by deriving a corresponding L^q counterpart for solutions to the approximate system (3.1). Indeed, our argument will be based on a refined examination of the time evolution of L^q quasi-norms along trajectories of (3.1), where unlike in Lemma 3.2 we shall here rely on the interpolation property from Theorem 1.1 in gaining a nontrivial estimate from below for the corresponding dissipation rate (cf. (3.28) and (3.29)).

Lemma 3.6 *Let $p \geq 1$, and suppose that $s_0 > 0$ and that $\mathcal{L} \in C^0([0,\infty)) \cap C^2((0,s_0))$ is positive and nondecreasing on $(0,\infty)$ such that* (H) *is satisfied, and such that*

$$s\mathcal{L}''(s) \geq -\frac{3p+q_0-2}{p+q_0}\mathcal{L}'(s) \qquad \textit{for all } s \in (0,s_0) \tag{3.21}$$

with a certain $q_0 > 0$. Moreover, assume that $u_0 \in C^0(\mathbb{R}^n)$ is positive and such that

$$u_0 < \min\Big\{s_0^{\frac{2}{p}}\, ,\, s_0^{\frac{2}{p+q_0}}\Big\} \qquad \textit{in } \mathbb{R}^n \tag{3.22}$$

as well as

$$\int_{\mathbb{R}^n} \mathcal{L}(u_0) < \infty. \tag{3.23}$$

Then there exists $q_1 \in (0,q_0)$ with the property that for all $q \in (0,q_1)$ one can find $t_0 = t_0(q) > 0$ and $C = C(q) > 0$ such that for all $R > 0$, the solution u_R of (3.1) satisfies

$$\|u_R(\cdot,t)\|_{L^q(B_R)} \leq Ct^{-\frac{1}{p}}\mathcal{L}^{-\frac{np+2q}{npq}}\Big(\frac{1}{t}\Big) \qquad \textit{for all } t \geq t_0. \tag{3.24}$$

PROOF. In view of (3.22), upon replacing $\mathcal{L}$ with any nondecreasing bounded function $\widetilde{\mathcal{L}} \in C^0([0,\infty)) \cap C^2((0,s_0))$ such that $\widetilde{\mathcal{L}}(s) = \mathcal{L}(s)$ whenever $0 \leq s \leq \min\big\{s_0^{\frac{2}{p}}\, ,\, s_0^{\frac{2}{p+q_0}}\big\} + 1$, and upon accordingly enlarging t_0 if necessary, we may assume that $\mathcal{L}$ is bounded. We then fix any $q_1 \in (0,q_0)$ such that

$$\frac{2q}{p+q} \leq 1 \quad \text{for all } q \in (0,q_1) \qquad \text{and} \qquad q_1 < p\lambda_0, \tag{3.25}$$

and given $q \in (0, q_1)$ we may combine (3.23) with the outcome of Lemma 3.5 to see that

$$\int_{\mathbb{R}^n} \mathcal{L}\Big(u_0^{\frac{p+q}{2}}\Big) < \infty.$$

As (3.22) implies that moreover $u_0^{\frac{p+q}{2}} < s_0$ in $\mathbb{R}^n$, from Lemma 3.2 we thus infer that for any $R > 0$ one can find $\varepsilon_0(R) \in (0, 1)$ such that whenever $\varepsilon \in (0, \varepsilon_0(R))$, the solution of (3.3) satisfies

$$\int_{B_R} \mathcal{L}\Big(u_{R\varepsilon}^{\frac{p+q}{2}}(\cdot, t)\Big) \le \int_{B_R} \mathcal{L}\Big(u_{0R\varepsilon}^{\frac{p+q}{2}}\Big) \qquad \text{for all } t > 0. \tag{3.26}$$

Since the monotonicity of $\mathcal{L}$ ensures that

$$\int_{B_R} \mathcal{L}\Big(u_{0R\varepsilon}^{\frac{p+q}{2}}\Big) \searrow \int_{B_R} \mathcal{L}\Big(u_{0R}^{\frac{p+q}{2}}\Big) \qquad \text{as } \varepsilon \searrow 0$$

by Beppo Levi's theorem, and that furthermore

$$\int_{B_R} \mathcal{L}\Big(u_{0R}^{\frac{p+q}{2}}\Big) \le c_1 := \int_{\mathbb{R}^n} \mathcal{L}\Big(u_0^{\frac{p+q}{2}}\Big) \qquad \text{for all } R > 0,$$

the inequality (3.26) entails that for each $R > 0$ we can fix $\varepsilon_1(R) \in (0, \varepsilon_0(R))$ such that for any $\varepsilon \in (0, \varepsilon_1(R))$ we have

$$\int_{B_R} \mathcal{L}\Big(u_{R\varepsilon}^{\frac{p+q}{2}}(\cdot, t)\Big) \le 2c_1 \qquad \text{for all } t > 0. \tag{3.27}$$

Now testing (3.3) by the smooth function $u_{R\varepsilon}^{q-1}$ yields

$$\begin{aligned}
\frac{1}{q}\frac{d}{dt}\int_{B_R} u_{R\varepsilon}^q &= \int_{B_R} u_{R\varepsilon}^{p+q-1}\Delta u_{R\varepsilon} \\
&= -(p+q-1)\int_{B_R} u_{R\varepsilon}^{p+q-2}|\nabla u_{R\varepsilon}|^2 + \int_{\partial B_R} u_{R\varepsilon}^{p+q-1}\frac{\partial u_{R\varepsilon}}{\partial\nu} \\
&\le -\frac{4(p+q-1)}{(p+q)^2}\int_{B_R}\Big|\nabla u_{R\varepsilon}^{\frac{p+q}{2}}\Big|^2 \qquad \text{for all } t > 0,
\end{aligned} \tag{3.28}$$

once again because $\frac{\partial u_{R\varepsilon}}{\partial\nu} \le 0$ on $\partial B_R \times (0, \infty)$. In order to estimate the right-hand side herein by means of Theorem 1.1, we observe that for each fixed $t > 0$, the function $u_{R\varepsilon}^{\frac{p+q}{2}}(\cdot, t) - \varepsilon^{\frac{p+q}{2}} \in C^1(\bar{B}_R)$ is positive in B_R and vanishes on ∂B_R, so that its trivial extension to all of $\mathbb{R}^n$ belongs to $W^{1,2}(\mathbb{R}^n)$. As

$$\int_{B_R} \mathcal{L}\Big(u_{R\varepsilon}^{\frac{p+q}{2}}(\cdot, t) - \varepsilon^{\frac{p+q}{2}}\Big) \le \int_{B_R} \mathcal{L}\Big(u_{R\varepsilon}^{\frac{p+q}{2}}(\cdot, t)\Big) \le 2c_1 \qquad \text{for all } t > 0 \text{ and each } \varepsilon \in (0, \varepsilon_1(R)),$$

in view of the boundedness of $\mathcal{L}$, Theorem 1.1 therefore becomes applicable so as to yield $c_2 > 0$ such that

$$\begin{aligned}
&\Big\| u_{R\varepsilon}^{\frac{p+q}{2}} - \varepsilon^{\frac{p+q}{2}} \Big\|_{L^{\frac{2q}{p+q}}(B_R)}^{\frac{2q}{p+q}} \\
&\quad \le c_2 \Bigg\{ \Big\| \nabla (u_{R\varepsilon}^{\frac{p+q}{2}} - \varepsilon^{\frac{p+q}{2}}) \Big\|_{L^2(B_R)} \mathcal{L}^{-(\frac{p+q}{2q} - \frac{n-2}{2n})} \Big(\Big\| \nabla (u_{R\varepsilon}^{\frac{p+q}{2}} - \varepsilon^{\frac{p+q}{2}}) \Big\|_{L^2(B_R)}^2 \Big) \Bigg\}^{\frac{2q}{p+q}} \\
&\quad = c_2 \Big(\Big\| \nabla u_{R\varepsilon}^{\frac{p+q}{2}} \Big\|_{L^2(B_R)}^2 \Big)^{\frac{q}{p+q}} \mathcal{L}^{-\frac{np+2q}{n(p+q)}} \Big(\Big\| \nabla u_{R\varepsilon}^{\frac{p+q}{2}} \Big\|_{L^2(B_R)}^2 \Big) \qquad \text{for all } t > 0, \qquad (3.29)
\end{aligned}$$

where on the left-hand side we may use that thanks to the first restriction in (3.25) we have $(x+y)^{\frac{2q}{p+q}} \le x^{\frac{2q}{p+q}} + y^{\frac{2q}{p+q}}$ for all $x \ge 0$ and $y \ge 0$, so that

$$\begin{aligned}
\int_{B_R} u_{R\varepsilon}^q &= \int_{B_R} \Big\{ \Big(u_{R\varepsilon}^{\frac{p+q}{2}} - \varepsilon^{\frac{p+q}{2}} \Big) + \varepsilon^{\frac{p+q}{2}} \Big\}^{\frac{2q}{p+q}} \\
&\le \Big\| u_{R\varepsilon}^{\frac{p+q}{2}} - \varepsilon^{\frac{p+q}{2}} \Big\|_{L^{\frac{2q}{p+q}}(B_R)}^{\frac{2q}{p+q}} + |B_R|\varepsilon^q \qquad \text{for all } t > 0. \qquad (3.30)
\end{aligned}$$

Now to solve (3.29) with respect to $\|\nabla u_{R\varepsilon}^{\frac{p+q}{2}}\|_{L^2(B_R)}^2$, we abbreviate $\beta := \frac{q}{p+q}$ and $\gamma := \frac{np+2q}{n(p+q)}$ and note that since

$$\frac{d}{ds}\Big\{ s^\beta \mathcal{L}^{-\gamma}(s) \Big\} = s^{\beta-1} \mathcal{L}^{-\gamma-1}(s) \Big\{ \beta \mathcal{L}(s) - \gamma s \mathcal{L}'(s) \Big\} \qquad \text{for all } s \in (0, s_0),$$

invoking Lemma 2.1 provides $s_1 \in (0, s_0)$ such that

$$\frac{d}{ds}\Big\{ s^\beta \mathcal{L}^{-\gamma}(s) \Big\} > 0 \qquad \text{for all } s \in (0, s_1].$$

The function ψ defined on $[0, \infty)$ by letting

$$\psi(s) := c_2 s^\beta \widetilde{\mathcal{L}}^{-\gamma}(s), \qquad s \ge 0,$$

with

$$\widetilde{\mathcal{L}}(s) := \begin{cases} \mathcal{L}(s), & s \in [0, s_1], \\ \mathcal{L}(s_1), & s > s_1, \end{cases}$$

therefore has the properties that $\psi' > 0$ on $(0, \infty) \setminus \{s_1\}$ and $\psi(0) = 0$ as well as $\psi(s) \to +\infty$ as $s \to \infty$, and since $\mathcal{L}$ is nondecreasing we moreover have $\psi(s) \ge c_2 s^\beta \mathcal{L}^{-\gamma}(s)$ for all $s \ge 0$. Accordingly, combining (3.29) with (3.30) shows that for all $R > 0$ and $\varepsilon \in (0, \varepsilon_1(R))$,

$$y_{R\varepsilon}(t) := \int_{B_R} u_{R\varepsilon}^q(\cdot, t) - |B_R|\varepsilon^q, \qquad t \ge 0,$$

satisfies

$$\begin{aligned} y_{R\varepsilon}(t) &\le c_2\bigg(\Big\|\nabla u_{R\varepsilon}^{\frac{p+q}{2}}(\cdot,t)\Big\|_{L^2(B_R)}^2\bigg)^\beta \mathcal{L}^{-\gamma}\bigg(\Big\|\nabla u_{R\varepsilon}^{\frac{p+q}{2}}(\cdot,t)\Big\|_{L^2(B_R)}^2\bigg) \\ &\le \psi\bigg(\Big\|\nabla u_{R\varepsilon}^{\frac{p+q}{2}}(\cdot,t)\Big\|_{L^2(B_R)}^2\bigg) \qquad \text{for all } t>0, \end{aligned}$$

so that since $u_{R\varepsilon} > \varepsilon$ in $B_R \times (0,\infty)$ entails that $y_{R\varepsilon}$ is positive, we may invert this relation so as to achieve that

$$\int_{B_R} \Big|\nabla u_{R\varepsilon}^{\frac{p+q}{2}}(\cdot,t)\Big|^2 \ge \psi^{-1}\big(y_{R\varepsilon}(t)\big) \qquad \text{for all } t>0,$$

where ψ^{-1} denotes the inverse of ψ. Abbreviating $c_3 := \frac{4q(p+q-1)}{(p+q)^2}$, from (3.28) we thus obtain the autonomous ODI

$$y_{R\varepsilon}'(t) \le -c_3\psi^{-1}\big(y_{R\varepsilon}(t)\big) \qquad \text{for all } t>0,$$

which again by positivity of $y_{R\varepsilon}$ can be integrated to see that

$$\int_{y_{R\varepsilon}(0)}^{y_{R\varepsilon}(t)} \frac{dy}{\psi^{-1}(y)} \le -c_3 t \qquad \text{for all } t>0,$$

whence by substituting $s := \psi^{-1}(y)$ we obtain that

$$c_3 t \le \int_{y_{R\varepsilon}(t)}^{y_{R\varepsilon}(0)} \frac{dy}{\psi^{-1}(y)} = \int_{\psi^{-1}(y_{R\varepsilon}(t))}^{\psi^{-1}(y_{R\varepsilon}(0))} \frac{\psi'(s)}{s} ds \qquad \text{for all } t>0. \tag{3.31}$$

Since herein the monotone convergence $u_{R\varepsilon} \searrow u_R$ warrants that for all $t \ge 0$ we have

$$y_{R\varepsilon}(t) \to y_R(t) := \int_{B_R} u_R^q(\cdot,t) \qquad \text{as } \varepsilon \searrow 0,$$

by continuity of ψ^{-1} we infer on taking $(0,\varepsilon_1(R)) \ni \varepsilon \searrow 0$ in (3.31) that

$$c_3 t \le \int_{\psi^{-1}(y_R(t))}^{\psi^{-1}(y_R(0))} \frac{\psi'(s)}{s} ds \qquad \text{for all } t>0. \tag{3.32}$$

Here thanks to the monotonicity of $\widetilde{\mathcal{L}}$, for any $\overline{s} > 0$ we can estimate

$$\begin{aligned} \psi'(s) &= \beta c_2 s^{\beta-1}\widetilde{\mathcal{L}}^{-\gamma}(s) - \gamma c_2 s^\beta \widetilde{\mathcal{L}}^{-\gamma-1}(s)\widetilde{\mathcal{L}}'(s) \le \beta c_2 s^{\beta-1}\widetilde{\mathcal{L}}^{-\gamma}(s) \\ &\le \beta c_2 \widetilde{\mathcal{L}}^{-\gamma}(\overline{s}) s^{\beta-1} \qquad \text{for all } s \in (\overline{s},\infty)\setminus\{s_1\}, \end{aligned}$$

so that (3.32) along with the fact that $\beta = \frac{q}{p+q} < 1$ implies that

$$\begin{aligned} c_3 t &\le \beta c_2 \widetilde{\mathcal{L}}^{-\gamma}\Big(\psi^{-1}(y_R(t))\Big) \int_{\psi^{-1}(y_R(t))}^{\psi^{-1}(y_R(0))} s^{\beta-2} ds \le \beta c_2 \widetilde{\mathcal{L}}^{-\gamma}\Big(\psi^{-1}(y_R(t))\Big) \int_{\psi^{-1}(y_R(t))}^{\infty} s^{\beta-2} ds \\ &= \frac{\beta c_2}{1-\beta} \widetilde{\mathcal{L}}^{-\gamma}\Big(\psi^{-1}(y_R(t))\Big)\Big(\psi^{-1}(y_R(t))\Big)^{\beta-1} \qquad \text{for all } t>0, \end{aligned}$$

that is, we have

$$\Big(\psi^{-1}(y_R(t))\Big)^{1-\beta}\widetilde{\mathcal{L}}^{\gamma}\Big(\psi^{-1}(y_R(t))\Big) \le \frac{\beta c_2}{1-\beta}\frac{1}{c_3 t} \le \frac{c_4}{t} \qquad \text{for all } t>0$$

with $c_4 := \max\{\frac{\beta c_2}{(1-\beta)c_3}\,,1\}$.
Invoking Lemma 3.4, we thus infer the existence of $t_1>0$ and $c_5\ge 1$ satisfying

$$\psi^{-1}(y_R(t)) \le c_5\Big(\frac{c_4}{t}\Big)^{\frac{1}{1-\beta}}\widetilde{\mathcal{L}}^{-\frac{\gamma}{1-\beta}}\Big(\frac{c_4}{t}\Big) \le c_4^{\frac{1}{1-\beta}}c_5 t^{-\frac{1}{1-\beta}}\widetilde{\mathcal{L}}^{-\frac{\gamma}{1-\beta}}\Big(\frac{1}{t}\Big) \qquad \text{for all } t\ge t_1,$$

because $c_4\ge 1$ and $\widetilde{\mathcal{L}}$ is nondecreasing.
Writing $c_6 := c_4^{\frac{1}{1-\beta}}c_5$, upon inversion we thereby obtain that

$$\begin{aligned}
y_R(t) &\le \psi\Big(c_6 t^{-\frac{1}{1-\beta}}\widetilde{\mathcal{L}}^{-\frac{\gamma}{1-\beta}}\Big(\frac{1}{t}\Big)\Big)\\
&= c_2\Big\{c_6 t^{-\frac{1}{1-\beta}}\widetilde{\mathcal{L}}^{-\frac{\gamma}{1-\beta}}\Big(\frac{1}{t}\Big)\Big\}^{\beta}\widetilde{\mathcal{L}}^{-\gamma}\Big(c_6 t^{-\frac{1}{1-\beta}}\widetilde{\mathcal{L}}^{-\frac{\gamma}{1-\beta}}\Big(\frac{1}{t}\Big)\Big)\\
&= c_2 c_6^{\beta} t^{-\frac{\beta}{1-\beta}}\widetilde{\mathcal{L}}^{-\frac{\beta\gamma}{1-\beta}}\Big(\frac{1}{t}\Big)\widetilde{\mathcal{L}}^{-\gamma}\Big(c_6 t^{-\frac{1}{1-\beta}}\widetilde{\mathcal{L}}^{-\frac{\gamma}{1-\beta}}\Big(\frac{1}{t}\Big)\Big) \qquad \text{for all } t\ge t_1.
\end{aligned} \tag{3.33}$$

Here the last factor can be estimated for large t by choosing $t_0\ge t_1$ large enough fulfilling

$$\frac{1}{t_0}\le s_1 \qquad \text{and} \qquad t_0^{-\frac{1}{1-\beta}}\le s_1 \tag{3.34}$$

as well as

$$c_6\widetilde{\mathcal{L}}^{-\frac{\gamma}{1-\beta}}\Big(\frac{1}{t}\Big)\ge 1 \qquad \text{for all } t\ge t_0, \tag{3.35}$$

where the latter is possible since $\widetilde{\mathcal{L}}$ is continuous with $\widetilde{\mathcal{L}}(0)=0$. Using (3.35) and the second restriction in (3.34) we thus infer that

$$\widetilde{\mathcal{L}}^{-\gamma}\Big(c_6 t^{-\frac{1}{1-\beta}}\widetilde{\mathcal{L}}^{-\frac{\gamma}{1-\beta}}\Big(\frac{1}{t}\Big)\Big) \le \widetilde{\mathcal{L}}^{-\gamma}\Big(t^{-\frac{1}{1-\beta}}\Big) = \mathcal{L}^{-\gamma}\Big(t^{-\frac{1}{1-\beta}}\Big) \qquad \text{for all } t\ge t_0, \tag{3.36}$$

where since $\frac{1}{1-\beta}=\frac{p+q}{p}<1+\lambda_0$ by (3.25), and since $\frac{1}{t_0}\le s_1<s_0$ by (3.34), we may invoke (1.2) to find that

$$\mathcal{L}\Big(\frac{1}{t}\Big)\le c_7\mathcal{L}\Big(t^{-\frac{1}{1-\beta}}\Big) \qquad \text{for all } t\ge t_0$$

with $c_7 := 1+\frac{aq}{p}$. Combining this with (3.36), (3.33) and again (3.32), we therefore conclude that

$$\begin{aligned}
y_R(t) &\le c_2 c_6^{\beta} t^{-\frac{\beta}{1-\beta}}\mathcal{L}^{-\frac{\beta\gamma}{1-\beta}}\Big(\frac{1}{t}\Big)c_7^{\gamma}\mathcal{L}^{-\gamma}\Big(\frac{1}{t}\Big)\\
&= c_2 c_6^{\beta} c_7^{\gamma} t^{-\frac{\beta}{1-\beta}}\mathcal{L}^{-\frac{\gamma}{1-\beta}}\Big(\frac{1}{t}\Big) \qquad \text{for all } t\ge t_0.
\end{aligned} \tag{3.37}$$

As $\frac{\beta}{1-\beta}=\frac{q}{p}$ and $\frac{\gamma}{1-\beta}=\frac{np+2q}{np}$, taking q-th roots on both sides of (3.37) readily yields (3.24). □

3.4 Upper bounds in L^∞. Proof of Theorem 1.3

In order to prepare our deduction of spatially uniform estimates from the above inequalities involving L^q seminorms, we recall the following well-known semi-convexity property ([5], [29]).

Lemma 3.7 *Let $p \ge 1$ and $R > 0$, and assume that $u_0 \in C^0(\mathbb{R}^n)$ is positive. Then the solution of (3.1) from Lemma 3.1 satisfies*

$$\frac{u_{Rt}(x,t)}{u_R(x,t)} \ge -\frac{1}{pt} \qquad \textit{for all } x \in B_R \textit{ and } t > 0.$$

For a radially symmetric and radially nondecreasing solution, namely, this entails controllability of its spatial L^∞ norm by its L^q seminorm for arbitrarily small $q > 0$.

Lemma 3.8 *Let $p \ge 1$, and assume that $u_0 \in C^0(\mathbb{R}^n)$ is positive, radially symmetric and nondecreasing with respect to $|x|$. Then for any $q > 0$ and $R > 0$, the solution of (3.1) from Lemma 3.1 satisfies*

$$\|u_R(\cdot,t)\|_{L^\infty(B_R)} \le \left(\frac{2^{q+\frac{n(p-1)}{2}}n}{p^{\frac{n}{2}}\omega_n}\right)^{\frac{2}{np+2q}} t^{-\frac{n}{np+2q}} \|u_R(\cdot,t)\|_{L^q(B_R)}^{\frac{2q}{np+2q}} \qquad \textit{for all } t > 0, \tag{3.38}$$

where $\omega_n := n|B_1|$.

Proof. Without danger of confusion we may write $u(r,t)$ for $r = |x| \ge 0$, and given $t > 0$ we then let

$$r_0 \equiv r_0(t) := \sup\left\{ r \in (0,R) \;\Big|\; u_R(r,t) \ge \frac{1}{2}u_R(0,t)\right\}, \tag{3.39}$$

noting that r_0 is well-defined due to the fact that $u_R(0,t) > u_R(R,t) = 0$. Now from Lemma 3.7 and (3.1) we know that

$$u_R^{p-1}\Delta u_R = \frac{u_{Rt}}{u_R} \ge -\frac{1}{pt} \qquad \text{in } B_R$$

and hence, as $u_R(\cdot,t)$ clearly inherits the symmetry and monotonicity properties of u_{0R} by the maximum principle,

$$\begin{aligned} \partial_r^2 u_R(r,t) &\ge \partial_r^2 u_R(r,t) + \frac{n-1}{r}\partial_r u_R(r,t) \ge -\frac{1}{pt}u_R^{1-p}(r,t) \\ &\ge -\frac{1}{pt}\Big(\frac{1}{2}u_R(0,t)\Big)^{1-p} = \frac{2^{p-1}}{pt}u_R^{1-p}(0,t) \qquad \text{for all } r \in (0,r_0), \end{aligned}$$

because $p \ge 1$. Upon two integrations using that $\partial_r u_R(0,t) = 0$, this first implies that

$$\partial_r u_R(r,t) \ge -\frac{2^{p-1}}{pt}u_R^{1-p}(0,t)r \qquad \text{for all } r \in (0,r_0)$$

and thereafter yields

$$u_R(r,t) \ge u_R(0,t) - \frac{2^{p-1}}{pt}u_R^{1-p}(0,t)\frac{r^2}{2} \qquad \text{for all } r \in [0,r_0].$$

When evaluated at $r=r_0$, this shows that

$$\frac{1}{2}u_R(0,t) \le \frac{2^{p-2}}{pt} u_R^{1-p}(0,t) r_0^2$$

or, equivalently,

$$r_0 \ge \Big(\frac{pt}{2^{p-1}}\Big)^{\frac{1}{2}} u_R^{\frac{p}{2}}(0,t). \tag{3.40}$$

Since from the definition (3.39) of r_0 we see that

$$\int_{B_R} u_R^q(\cdot,t) \ge \int_{B_{r_0}} u_R^q(\cdot,t) \ge 2^{-q} u_R^q(0,t)|B_{r_0}| = 2^{-q} u_R^q(0,t)\frac{\omega_n r_0^n}{n},$$

the inequality (3.40) thus entails that

$$\int_{B_R} u^q(\cdot,t) \ge \frac{\omega_n}{2^q n} u_R^q(0,t)\Big(\frac{pt}{2^{p-1}}\Big)^{\frac{n}{2}} u_R^{\frac{np}{2}}(0,t) = \frac{p^{\frac{n}{2}}\omega_n}{2^{q+\frac{n(p-1)}{2}}n} t^{\frac{n}{2}} u_R^{\frac{np+2q}{2}}(0,t),$$

which precisely yields (3.38), for $u_R(0,t) = \|u_R(\cdot,t)\|_{L^\infty(B_R)}$ again by monotonicity. $\Box$

In conjunction with with Lemma 3.6, this entails our main result concerning upper estimates for decay with respect to the norm in $L^\infty(\mathbb{R}^n)$ of radial and radially nonincreasing solutions emanating from rapidly decreasing initial data.

Proof of Theorem 1.3. We first apply Lemma 3.6 to find $q>0$, $t_0>0$ and $c_1>0$ such that for any $R>0$, the solution of (3.1) from Lemma 3.1 satisfies

$$\|u_R(\cdot,t)\|_{L^q(B_R)} \le c_1 t^{-\frac{1}{p}} \mathcal{L}^{-\frac{np+2q}{npq}}\Big(\frac{1}{t}\Big) \qquad \text{for all } t \ge t_0. \tag{3.41}$$

Thereafter, thanks to the symmetry and monotonicity properties of u_0 we may invoke Lemma 3.8 to obtain $c_2>0$ fulfilling

$$\|u_R(\cdot,t)\|_{L^\infty(B_R)} \le c_2 t^{-\frac{n}{np+2q}} \|u_R(\cdot,t)\|_{L^q(B_R)}^{\frac{2q}{np+2q}} \qquad \text{for all } t>0.$$

Combining this with (3.41) shows that

$$\begin{aligned}
\|u_R(\cdot,t)\|_{L^\infty(B_R)} &\le c_2 t^{-\frac{n}{np+2q}} \Big\{ c_1 t^{-\frac{1}{p}} \mathcal{L}^{-\frac{np+2q}{npq}}\Big(\frac{1}{t}\Big)\Big\}^{\frac{2q}{np+2q}} \\
&= c_1^{\frac{2q}{np+2q}} c_2 t^{-\frac{n}{np+2q}-\frac{2q}{p(np+2q)}} \mathcal{L}^{-\frac{2}{np}}\Big(\frac{1}{t}\Big) \\
&= c_1^{\frac{2q}{np+2q}} c_2 t^{-\frac{1}{p}} \mathcal{L}^{-\frac{2}{np}}\Big(\frac{1}{t}\Big) \qquad \text{for all } t \ge t_0,
\end{aligned}$$

which on an application of Fatou's lemma, relying on the approximation properties asserted by Lemma 3.1, implies (1.12) if we let $C := c_1^{\frac{2q}{np+2q}} c_2$. $\Box$

3.5 Upper bounds in L^∞: examples

We next intend to derive Corollary 1.5 and Corollary 1.6 by applying Theorem 1.3 in the concrete contexts made up by the choices in (1.5).

First concentrating on the former example therein, let us make sure that upon an appropriate and essentially trivial extension, the precise form of the logarithmically fast growth is indeed compatible with both (H) and the requirements from Section 3.2.

Lemma 3.9 *Let $\kappa > 0, M \ge 2$ and*

$$\mathcal{L}(s) := \begin{cases} 0, & s = 0, \\ \ln^{-\kappa} \frac{M}{s}, & s \in (0, \frac{M}{2}), \\ \ln^{-\kappa} 2, & s \ge \frac{M}{2}. \end{cases}$$

Then $\mathcal{L} \in C^0([0,\infty)) \cap C^2((0,\frac{M}{2}))$ is positive and nondecreasing on $(0,\infty)$ with

$$\frac{d}{ds}\Big(s\mathcal{L}'(s)\Big) \ge 0 \qquad \text{for all } s \in \Big(0, \frac{M}{2}\Big). \tag{3.42}$$

Moreover, given any $\lambda_0 > 0$ we have

$$\mathcal{L}(s) \le (1 + a\lambda)\mathcal{L}(s^{1+\lambda}) \qquad \text{for all } s > 0 \text{ and } \lambda \in (0, \lambda_0), \tag{3.43}$$

where

$$a := \begin{cases} \kappa & \text{if } \kappa \le 1, \\ \frac{(1+\lambda_0)^\kappa - 1}{\lambda_0} & \text{if } \kappa > 1. \end{cases} \tag{3.44}$$

Proof. To verify (3.42), we compute

$$\mathcal{L}'(s) = \frac{\kappa}{s} \ln^{-\kappa-1} \frac{M}{s} \quad \text{and} \quad \mathcal{L}''(s) = -\frac{\kappa}{s^2} \ln^{-\kappa-1}\frac{M}{s} + \frac{\kappa(\kappa+1)}{s^2} \ln^{-\kappa-2} \frac{M}{s}, \qquad s \in \Big(0, \frac{M}{2}\Big),$$

whence by positivity of κ we indeed obtain that $\mathcal{L}'(s) > 0$ for $s \in (0, \frac{M}{2})$ and that

$$\frac{s\mathcal{L}''(s)}{\mathcal{L}'(s)} \ge \frac{-s\frac{\kappa}{s^2}\ln^{-\kappa-1}\frac{M}{s}}{\frac{\kappa}{s}\ln^{-\kappa-1}\frac{M}{s}} = -1 \qquad \text{for all } s \in \Big(0, \frac{M}{2}\Big),$$

which readily implies (3.42).
In proving (3.43) we first observe that since $M \ge 2$, for each $s \ge \frac{M}{2} \ge 1$ we trivially have $\mathcal{L}(s^{1+\lambda}) \ge \mathcal{L}(s)$ by monotonicity of $\mathcal{L}$. This implies that we only need to consider the case $s < 1$, in which $s < \frac{M}{2}$ and also $s^{1+\lambda} < \frac{M}{2}$, so that since $\frac{M}{s^{1+\lambda}} \le \big(\frac{M}{s}\big)^{1+\lambda}$ for all $s > 0$ and $\lambda > 0$ due to the fact that $M \ge 1$, we can estimate

$$\frac{\mathcal{L}(s)}{\mathcal{L}(s^{1+\lambda})} = \frac{\ln^\kappa \frac{M}{s^{1+\lambda}}}{\ln^\kappa \frac{M}{s}} \le \frac{\ln^\kappa \big(\frac{M}{s}\big)^{1+\lambda}}{\ln^\kappa \frac{M}{s}} = (1+\lambda)^\kappa \qquad \text{for all } \lambda > 0. \tag{3.45}$$

Here if $\kappa \le 1$ then by convexity of $0 \le \xi \mapsto \xi^{\frac{1}{\kappa}}$ we have $(1 + a\lambda)^{\frac{1}{\kappa}} \ge 1 + \frac{a\lambda}{\kappa} = 1 + \lambda$ according to (3.44), so that (3.45) entails (3.43) in this case.

When $\kappa < 1$, noting that with a as in (3.44), $\psi(\lambda) := 1+a\lambda-(1+\lambda)^\kappa$, $\lambda \ge 0$, satisfies $\psi(0) = \psi(\lambda_0) = 0$ and $\psi''(\lambda) = -\kappa(\kappa-1)(1+\lambda)^{\kappa-2} \le 0$ for all $\lambda \ge 0$, we see that $\psi(\lambda) \ge 0$ for all $\lambda \in (0,\lambda_0)$, which combined with (3.45) completes the proof of (3.43). $\Box$

We now apply Theorem 1.3 to derive a decay result involving a sharp logarithmic correction to the asymptotics described in Theorem 1.2.

PROOF of Corollary 1.4. Since u_0 is bounded, we may choose $M \ge 2$ such that $u_0 < \frac{M}{2}$ in $\mathbb{R}^n$, and let $\mathcal{L}$ be as defined in Lemma 3.9. Then using that $M \ge 1$ and that $\mathcal{L}$ is nondecreasing, we can estimate

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{L}(u_0) &= \int_{\{u_0<\frac{1}{2}\}} \ln^{-\kappa}\frac{M}{u_0} + \int_{\{u_0\ge\frac{1}{2}\}} \mathcal{L}(u_0) \\
&\le \int_{\{u_0<\frac{1}{2}\}} \ln^{-\kappa}\frac{1}{u_0} + \mathcal{L}\Big(\frac{M}{2}\Big)\Big|\Big\{u_0 \ge \frac{1}{2}\Big\}\Big| < \infty
\end{aligned}$$

due to (1.13) and the fact that $\{u_0 \ge \frac{1}{2}\}$ is bounded according to our assumption on asymptotic decay of u_0.
Consequently, Theorem 1.3 provides $t_1 > 0$ and $c_1 > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le c_1 t^{-\frac{1}{p}} \mathcal{L}^{-\frac{2}{np}}\Big(\frac{1}{t}\Big) \qquad \text{for all } t \ge t_1, \tag{3.46}$$

so that if we pick $t_0 > \max\{t_1, M\}$, then in particular $\frac{1}{t_0} < \frac{M}{2}$, so that $\mathcal{L}(\frac{1}{t}) = \ln^{-\kappa}(Mt)$ for all $t \ge t_0$. Since $t_0 > M$ furthermore implies that $\ln(Mt) \le 2\ln t$ for all $t \ge t_0$, (3.46) thus yields

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le c_1 t^{-\frac{1}{p}} \ln^{\frac{2\kappa}{np}}(Mt) \le 2^{\frac{2\kappa}{np}} c_1 t^{-\frac{1}{p}} \ln^{\frac{2\kappa}{np}} t \qquad \text{for all } t \ge t_0$$

and thereby establishes (1.14). $\Box$

For initial data with the pointwise exponential decay behavior assumed in (1.15), on a slight shift in the exponent of the respective logarithmic factor, the above integral condition can be verified, thus yielding temporal decay as claimed.

PROOF of Corollary 1.5. We pick $c_1 \ge c_0$ such that $c_1 > 1$, and let

$$\overline{u}_0(x) := c_1\, e^{-\alpha|x|^\beta}, \qquad x \in \mathbb{R}^n,$$

as well as $r_0 := \big(\frac{2\ln c_1}{\alpha}\big)^{\frac{1}{\beta}}$. Then for $|x| \ge r_0$ we can estimate

$$\frac{\overline{u}_0(x)}{e^{-\frac{\alpha}{2}|x|^\beta}} = c_1\, e^{-\frac{\alpha}{2}|x|^\beta} \le c_1\, e^{-\frac{\alpha}{2}\cdot\frac{\ln c_1}{\alpha}} = 1,$$

so writing $\kappa := \frac{n}{\beta} + \frac{np\delta}{2}$ we have

$$\ln^{-\kappa}\frac{1}{\overline{u}_0(x)} \le \ln^{-\kappa}\Big(e^{\frac{\alpha}{2}|x|^\beta}\Big) = c_2|x|^{-\beta\kappa} \qquad \text{for all } x \in \mathbb{R}^n \setminus B_{r_0},$$

with $c_2 := \left(\frac{2}{\alpha}\right)^\kappa$. Therefore,

$$\int_{\mathbb{R}^n\setminus B_{r_0}} \ln^{-\kappa}\frac{1}{\overline{u}_0(x)}dx \le c_2\int_{\mathbb{R}^n\setminus B_{r_0}} |x|^{-\beta\kappa}dx < \infty$$

thanks to the fact that $\beta\kappa = n + \frac{np\beta\delta}{2} > n$. As clearly also

$$\int_{B_{r_0}\cap\{\overline{u}_0<\frac{1}{2}\}} \ln^{-\kappa}\frac{1}{\overline{u}_0(x)}dx \le \ln^{-\kappa}2\cdot\left|\left\{\overline{u}_0<\frac{1}{2}\right\}\right|$$

is finite, Corollary 1.4 becomes applicable so as to yield $t_0 > 1$ and $c_3 > 0$ such that the minimal solution $\overline{u}$ of (1.6) emanating from $\overline{u}_0$ satisfies

$$\|\overline{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le c_3 t^{-\frac{1}{p}}\ln^{\frac{2\kappa}{np}}t = c_3 t^{-\frac{1}{p}}\ln^{\frac{2}{p\beta}+\delta}t \qquad \text{for all } t\ge t_0 \tag{3.47}$$

according to our definition of κ. Since a comparison argument ([48]) shows that thanks to (1.15) we have $u_R \le \overline{u}$ in $B_R\times(0,\infty)$ for all $R>0$ and hence $u\le\overline{u}$ in $\mathbb{R}^n\times(0,\infty)$, (3.47) entails (1.16). $\square$

Next, in addressing the second example announced in (1.5) we can proceed quite similarly, starting with a corresponding counterpart of Lemma 3.9.

Lemma 3.10 *Let $\kappa>0, M>e$ and $s_0\in[1,\frac{M}{e})$. Then*

$$\mathcal{L}(s) := \begin{cases} 0, & s=0,\\ \ln^{-\kappa}\ln\frac{M}{s}, & s\in(0,s_0),\\ \ln^{-\kappa}\ln\frac{M}{s_0}, & s\ge s_0,\end{cases}$$

defines a function $\mathcal{L}\in C^0([0,\infty))\cap C^2((0,s_0))$ which is positive and nondecreasing on $(0,\infty)$ and satisfies

$$\frac{d}{ds}\Big(s\mathcal{L}'(s)\Big)\ge 0 \qquad \text{for all } s\in(0,s_0). \tag{3.48}$$

Furthermore, given any $\lambda_0>0$ one can find $a>0$ such that

$$\mathcal{L}(s)\le(1+a\lambda)\mathcal{L}(s^{1+\lambda}) \qquad \text{for all } s>0 \text{ and } \lambda\in(0,\lambda_0). \tag{3.49}$$

PROOF. Let us first observe that $\mathcal{L}$ indeed is well-defined, positive and nondecreasing on $(0,\infty)$, because the assumption $s_0<\frac{M}{e}$ warrants that $\ln\ln\frac{M}{s}>\ln\ln\frac{M}{s_0}>0$ for all $s\in(0,s_0)$.
Now (3.48) follows from the fact that for any $s\in(0,s_0)$ we have

$$\mathcal{L}'(s) = \frac{\kappa}{s\ln\frac{M}{s}}\ln^{-\kappa-1}\ln\frac{M}{s}$$

and thus

$$\begin{aligned}\mathcal{L}''(s) &= -\frac{\kappa}{s^2\ln\frac{M}{s}}\ln^{-\kappa-1}\ln\frac{M}{s} + \frac{\kappa}{s^2\ln^2\frac{M}{s}}\ln^{-\kappa-1}\ln\frac{M}{s} + \frac{\kappa(\kappa+1)}{s^2\ln^2\frac{M}{s}}\ln^{-\kappa-2}\ln\frac{M}{s}\\ &\ge -\frac{\kappa}{s^2\ln\frac{M}{s}}\ln^{-\kappa-1}\ln\frac{M}{s} = -\frac{\mathcal{L}'(s)}{s},\end{aligned}$$

and in order to verify (3.49), given $\lambda_0 > 0$ we fix $a > 0$ large enough such that

$$\Big(1 + \frac{\ln(1+\lambda)}{c_1}\Big)^\kappa \le 1 + a\lambda \qquad \text{for all } \lambda \in (0, \lambda_0),$$

where $c_1 := \ln\ln\frac{M}{s_0}$. Then since $M \ge 1$ and hence $M \le M^{1+\lambda}$ for all $\lambda > 0$, in the case when $s < 1$, and hence $\max\{s, s^{1+\lambda}\} < s_0$ for all $\lambda > 0$, we can estimate

$$\begin{aligned} \frac{\mathcal{L}(s)}{\mathcal{L}(s^{1+\lambda})} &= \left\{\frac{\ln\ln\frac{M}{s^{1+\lambda}}}{\ln\ln\frac{M}{s}}\right\}^\kappa \le \left\{\frac{\ln\ln\big(\frac{M}{s}\big)^{1+\lambda}}{\ln\ln\frac{M}{s}}\right\}^\kappa = \left\{1 + \frac{\ln(1+\lambda)}{\ln\ln\frac{M}{s}}\right\}^\kappa \\ &\le \left\{1 + \frac{\ln(1+\lambda)}{c_1}\right\}^\kappa \le 1 + a\lambda \qquad \text{for all } \lambda \in (0, \lambda_0). \end{aligned}$$

As by monotonicity of $\mathcal{L}$ we again have $\mathcal{L}(s) \le \mathcal{L}(s^{1+\lambda})$ for all $s \ge 1$ and $\lambda > 0$, this proves (3.49). $\Box$

By Theorem 1.3, this again implies a decay estimate, now involving a doubly logarithmic factor, under a certain integrability condition requiring adequately fast decay of the data.

Corollary 3.11 *Let $p \ge 1$, and assume that $u_0 \in C^0(\mathbb{R}^n)$ is positive, radially symmetric and nonincreasing with respect to $|x|$ and such that $u_0(x) \to 0$ as $|x| \to \infty$ as well as*

$$\int_{\{u_0<\eta\}} \ln^{-\kappa}\ln\frac{1}{u_0(x)}dx < \infty \tag{3.50}$$

for some $\kappa > 0$ and $\eta > 0$. Then there exist $t_0 > e$ and $C > 0$ such that for the minimal solution of (1.6) *we have*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le Ct^{-\frac{1}{p}}\ln^{\frac{2\kappa}{np}}\ln t \qquad \textit{for all } t \ge t_0. \tag{3.51}$$

Proof. This can be obtained by straightforward adaptation of the argument from Corollary 1.4, relying on Lemma 3.10 rather than Lemma 3.9. $\Box$

For initial data with doubly exponential decay as in (1.17), this can now be seen to imply (1.18).

Proof of Corollary 1.6. Guided by the procedure from Corollary 1.5, we let

$$\overline{u}_0(x) := c_0\exp\Big\{-\alpha\exp\big(\beta|x|^\gamma\big)\Big\}, \qquad x \in \mathbb{R}^n,$$

and note that since

$$\frac{\exp\Big\{-\alpha\exp\big(\beta|x|^\gamma\big)\Big\}}{\exp\Big\{-\exp\big(\frac{\beta}{2}|x|^\gamma\big)\Big\}} \to 0 \qquad \text{as } |x| \to \infty,$$

we can fix $r_0 > 0$ such that

$$\overline{u}_0(x) \le \exp\Big\{-\exp\Big(\frac{\beta}{2}|x|^\gamma\Big)\Big\} \qquad \text{for all } x \in \mathbb{R}^n \setminus B_{r_0}.$$

Therefore, if we let $\kappa := \frac{n}{\gamma} + \frac{np\delta}{2}$ and $c_1 := (\frac{2}{\beta})^\kappa$, then

$$\ln^{-\kappa} \ln \frac{1}{\overline{u}_0(x)} \le \ln^{-\kappa} \left\{ \exp\left(\frac{\beta}{2}|x|^\gamma\right)\right\} = c_1 |x|^{-\gamma\kappa} \qquad \text{for all } x \in \mathbb{R}^n \setminus B_{r_0}$$

and hence

$$\int_{\mathbb{R}^n \setminus B_{r_0}} \ln^{-\kappa} \ln \frac{1}{\overline{u}_0(x)} dx \le c_1 \int_{\mathbb{R}^n \setminus B_{r_0}} |x|^{-\gamma\kappa} dx < \infty,$$

because $\gamma\kappa > n$. Since this entails that e.g.

$$\int_{\{\overline{u}_0 < e^{-2}\}} \ln^{-\kappa} \ln \frac{1}{\overline{u}_0(x)} dx < \infty,$$

Corollary 3.11 provides $t_0 > e$ and $c_2 > 0$ with the property that for the minimal solution $\overline{u}$ of (1.6) with $\overline{u}|_{t=0} = \overline{u}_0$ we have

$$\|\overline{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le c_2 t^{-\frac{1}{p}} \ln^{\frac{2\kappa}{np}} \ln t \qquad \text{for all } t \ge t_0.$$

Since $u \le \overline{u}$ by comparison and (1.17), in view of our definition of κ this establishes (1.18). $\square$

3.6 Lower estimates: Proof of Theorem 1.7

In order to see that the above decay estimates are essentially optimal, and that hence the interpolation inequality from Theorem 1.1 can as well not be substantially improved any further, by means of an independent argument based on comparison with separated solutions we finally derive some lower bounds for arbitrary positive classical solutions to (1.6) which actually hold in a pointwise sense for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$.

To prepare this, let us observe that if u is any positive classical solution of (1.6) in $\mathbb{R}^n \times (0,\infty)$, then the function z defined on $\mathbb{R}^n \times [0,\infty)$ by letting

$$z(x,\tau) := (t+1)^{\frac{1}{p}} u(x,t), \qquad \tau = \ln(t+1), \tag{3.52}$$

is a positive classical solution of

$$\begin{cases} z_\tau = z^p \Delta z + \frac{1}{p} z, & x \in \mathbb{R}^n, \ t > 0, \\ z(x,0) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{3.53}$$

To estimate this function from below, let us first recall the following scaling property of solutions to the associated steady-state problem in a ball with variable radius ([29]).

Lemma 3.12 *Let $p \ge 1$. For $R > 0$, let $w_R \in C^0(\bar{B}_R) \cap C^2(B_R)$ denote the positive solution of*

$$\begin{cases} -\Delta w_R = \frac{1}{p} w_R^{1-p}, & x \in B_R, \\ w_R = 0, & x \in \partial B_R. \end{cases} \tag{3.54}$$

Then for each $R > 0$ we have

$$w_R(x) = R^{\frac{2}{p}} w_1\Big(\frac{x}{R}\Big) \qquad \textit{for all } x \in B_R. \tag{3.55}$$

Selecting appropriate representatives of this family as spatial profiles of separated solutions of the Dirichlet problem for the PDE in (3.53) in suitable balls, we can indeed achieve the announced lower estimate for solutions by comparison.

PROOF of Theorem 1.7. We fix $c_1 > 0$ such that $pc_1 < 1$, and given $\tau > 0$ we let

$$R(\tau) := \Lambda^{-1}(c_1\tau).$$

Then from (1.19) we first obtain that $R(\tau) \to \infty$ as $\tau \to \infty$, whereupon a second application of (1.19) shows that

$$\frac{2}{\tau}\ln R(\tau) = \frac{2c_1}{\Lambda(R(\tau))}\ln R(\tau) \to 0 \qquad \text{as } \tau \to \infty,$$

which since $pc_1 < 1$ entails the existence of $c_2 > 0$ such that

$$p\Lambda(R(\tau)) + 2\ln R(\tau) \le \tau + c_2 \qquad \text{for all } \tau > 0, \tag{3.56}$$

because

$$\frac{p\Lambda(R(\tau)) + 2\ln R(\tau) - \tau}{\tau} = pc_1 + \frac{2}{\tau}\ln R(\tau) - 1 \to pc_1 - 1 \qquad \text{as } \tau \to \infty.$$

We now fix $\tau_0 > 0$ and let $c_3 := 1/\|w_1\|_{L^\infty(B_1)}$, as well as

$$\delta(\tau_0) := c_3 R^{-\frac{2}{p}}(\tau_0)e^{-\Lambda(R(\tau_0))}, \tag{3.57}$$

and

$$y(\tau) := \Big\{\delta^{-p}(\tau_0)e^{-\tau} + 1 - e^{-\tau}\Big\}^{-\frac{1}{p}}, \qquad \tau \ge 0, \tag{3.58}$$

observing that y solves

$$\begin{cases} y'(\tau) = \frac{1}{p}y(\tau) - \frac{1}{p}y^{p+1}(\tau), \qquad \tau > 0, \\ y(0) = \delta(\tau_0). \end{cases} \tag{3.59}$$

Then taking z from (3.52) and

$$\underline{z}(x,\tau) := y(\tau)w_{R(\tau_0)}(x), \qquad x \in \bar{B}_{R(\tau_0)},\ \tau \ge 0,$$

we see that $\underline{z}(x,\tau) = 0 < v(x,\tau)$ for all $x \in \partial B_{R(\tau_0)}$ and $\tau \ge 0$, and that according to the monotonicity of Λ and Lemma 3.12,

$$\begin{aligned} \frac{z(x,0)}{\underline{z}(x,0)} &= \frac{u_0(x)}{\delta(\tau_0)w_{R(\tau_0)}(x)} \ge \frac{e^{-\Lambda(|x|)}}{\delta(\tau_0)R^{\frac{2}{p}}(\tau_0)w_1\big(\frac{x}{R(\tau_0)}\big)} \\ &> \frac{e^{-\Lambda(R(\tau_0))}}{\delta(\tau_0)R^{\frac{2}{p}}\|w_1\|_{L^\infty(B_1)}} = 1 \qquad \text{for all } x \in B_{R(\tau_0)} \end{aligned}$$

thanks to our definition (3.57) of $\delta(\tau_0)$. As furthermore by (3.54) and (3.59),

$$\begin{aligned} \underline{z}_\tau - \underline{z}^p \Delta \underline{z} - \frac{1}{p}\underline{z} &= y' w_{R(\tau_0)} - y^{p+1} w^p_{R(\tau_0)} \Delta w_{R(\tau_0)} - \frac{1}{p} y w_{R(\tau_0)} \\ &= \Big\{ y' - \frac{1}{p} y + \frac{1}{p} y^{p+1} \Big\} w_{R(\tau_0)} = 0 \qquad \text{in } B_{R(\tau_0)} \times (0,\infty), \end{aligned}$$

a comparison argument ([48]) shows that $z \ge \underline{z}$ in $B_{R(\tau_0)} \times (0,\infty)$. When evaluated at $x = 0$ and $\tau = \tau_0$, by (3.58) and again by Lemma 3.12 this in particular implies that

$$z(0,\tau_0) \ge y(\tau_0) w_{R(\tau_0)}(0) = \Big\{ \delta^{-p}(\tau_0) e^{-\tau_0} + 1 - e^{-\tau_0} \Big\}^{-\frac{1}{p}} R^{\frac{2}{p}}(\tau_0) w_1(0),$$

where since (3.57) and (3.56) ensure that

$$\begin{aligned} \delta^{-p}(\tau_0) e^{-\tau_0} + 1 - e^{-\tau_0} &\le \delta^{-p}(\tau_0) e^{-\tau_0} + 1 = c_3^{-p} R^2(\tau_0) e^{p\Lambda(R(\tau_0))} e^{-\tau_0} + 1 \\ &= c_3^{-p} e^{p\Lambda(R(\tau_0)) + 2\ln R(\tau_0) - \tau_0} + 1 \le c_3^{-p} e^{c_2} + 1, \end{aligned}$$

this shows that with $c_4 := (c_3^{-p} e^{c_2} + 1)^{-\frac{1}{p}} w_1(0)$ we have

$$z(0,\tau_0) \ge c_4 R^{\frac{2}{p}}(\tau_0) = c_4 \Big\{ \Lambda^{-1}(c_1 \tau_0) \Big\}^{\frac{2}{p}} \qquad \text{for all } \tau_0 > 0.$$

Transforming back by means of (3.52), we thus obtain that for each $t > 0$, writing $\tau_0 := \ln(t+1)$ we can estimate

$$u(0,t) = (t+1)^{-\frac{1}{p}} z(0,\tau_0) \ge (t+1)^{-\frac{1}{p}} c_4 \Big\{ \Lambda^{-1}\big(c_1 \ln(t+1)\big) \Big\}^{\frac{2}{p}} \qquad \text{for all } t > 0. \tag{3.60}$$

Thus, if we pick $t_0 \ge 1$ large enough such that $c_1 \ln t_0 \in \Lambda^{-1}([0,\infty))$, then since $c_1 \ln(t+1) \ge c_1 \ln t$ and $(t+1)^{-\frac{1}{p}} \ge 2^{-\frac{1}{p}} t^{-\frac{1}{p}}$ for all $t \ge 1$ we can readily derive (1.21) from (3.60). □

3.7 Lower estimates: examples

The application of the latter to the specific frameworks of initial data satisfying (1.22) and (1.24), respectively, is now rather straightforward:

Proof of Corollary 1.8. We let

$$\Lambda(s) := \alpha s^\beta - \ln c_0, \qquad s \ge 0,$$

and then see that $\frac{\Lambda(s)}{\ln s} \to +\infty$ as $s \to \infty$, and that Λ is strictly increasing on $[0,\infty)$ with

$$\Lambda^{-1}(\sigma) = \Big\{ \frac{1}{\alpha}(\sigma + \ln c_0) \Big\}^{\frac{1}{\beta}} \qquad \text{for all } \sigma \ge -\ln c_0.$$

As (1.22) warrants that $u_0(x) \ge e^{-\Lambda(|x|)}$ for all $x \in \mathbb{R}^n$, Theorem 1.7 applies so as to yield $t_1 > 1$ and $c_1 > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \ge c_1 t^{-\frac{1}{p}} \Big\{ \Lambda^{-1}(c_1 \ln t) \Big\}^{\frac{2}{p}} = c_1 t^{-\frac{1}{p}} \Big\{ \frac{1}{\alpha}(c_1 \ln t + \ln c_0) \Big\}^{\frac{2}{p\beta}} \qquad \text{for all } t \ge t_1.$$

Thus, if we pick $t_0 \geq t_1$ large enough fulfilling $\ln c_0 \geq -\frac{c_1}{2}\ln t_0$, from this we infer that

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \geq c_1 t^{-\frac{1}{p}} \Big\{ \frac{c_1}{2\alpha} \ln t \Big\}^{\frac{2}{p\beta}} \qquad \text{for all } t \geq t_0,$$

which immediately establishes (1.23). $\square$

Proof of Corollary 1.9. Proceeding as in the proof of Corollary 1.8, we first observe that

$$\Lambda(s) := \alpha\, e^{\beta|s|^\gamma} - \ln c_0, \qquad s \geq 0,$$

defines a strictly increasing function on $[0,\infty)$ satisfying $\frac{\Lambda(s)}{\ln s} \to +\infty$ as $s \to \infty$ as well as

$$\Lambda^{-1}(\sigma) = \Big\{ \frac{1}{\beta} \ln \Big[\frac{1}{\alpha}(\sigma + \ln c_0) \Big] \Big\}^{\frac{1}{\gamma}} \qquad \text{for all } \sigma \geq \alpha - \ln c_0.$$

According to (1.24), Theorem 1.7 therefore provides $t_1 > e$ and $c_1 > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \geq c_1 t^{-\frac{1}{p}} \Big\{ \Lambda^{-1}(c_1 \ln t) \Big\}^{\frac{2}{p}} = c_1 t^{-\frac{1}{p}} \Big\{ \frac{1}{\beta} \ln \Big[\frac{1}{\alpha}(c_1 \ln t + \ln c_0) \Big] \Big\}^{\frac{2}{p\gamma}} \qquad \text{for all } t \geq t_1.$$

Hence, picking $t_0 \geq t_1$ in such a way that $\ln c_0 \geq -\frac{c_1}{2}\ln t_0$ and $\ln \frac{c_1}{2\alpha} \geq -\frac{1}{2}\ln\ln t_0$, we conclude that

$$\begin{aligned} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} &\geq c_1 t^{-\frac{1}{p}} \Big\{ \frac{1}{\beta} \ln \Big[\frac{c_1}{2\alpha} \ln t \Big] \Big\}^{\frac{2}{p\gamma}} = c_1 t^{-\frac{1}{p}} \Big\{ \frac{1}{\beta} \Big[\ln \frac{c_1}{2\alpha} + \ln\ln t \Big] \Big\}^{\frac{2}{p\gamma}} \\ &\geq c_1 t^{-\frac{1}{p}} \Big\{ \frac{1}{2\beta} \ln\ln t \Big\}^{\frac{2}{p\gamma}} \qquad \text{for all } t \geq t_0, \end{aligned}$$

which clearly entails (1.25). $\square$

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