Global classical solvability and generic infinite-time blow-up in quasilinear Keller-Segel systems with bounded sensitivities

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Abstract

The chemotaxis system

$$\begin{cases} u_t = \nabla \cdot (D(u,v)\nabla u) - \nabla \cdot (S(u,v)\nabla v), \\ v_t = \Delta v - v + u, \end{cases} \qquad (\star)$$

is considered under homogeneous Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \ge 2$, along with initial conditions involving suitably regular and nonnegative data.

It is firstly asserted that if the positive smooth function D decays at most algebraically with respect to u, then for any smooth nonnegative and bounded S fulfilling a further mild assumption especially satisfied when $S \equiv S(u)$ with $S(0) = 0$, $(\star)$ possesses a globally defined classical solution.

If Ω is a ball, then under appropriate assumptions on D and S generalizing the prototypical choices in

$$D(u,v) = (u+1)^{m-1} \quad \text{and} \quad S(u,v) = u(u+1)^{\sigma-1}, \qquad u \ge 0, \ v \ge 0, \qquad (\star\star)$$

with $m \in \mathbb{R}$ and $\sigma \in \mathbb{R}$ such that

$$m - \frac{n-2}{n} < \sigma \le 0, \qquad (\star\star\star)$$

the phenomenon of infinite-time blow-up is next shown to occur for all initial data within a set $\mathcal{B}$ of functions which inter alia is found to be dense in the set of all radially symmetric and suitably regular positive functions on $\overline{\Omega}$.

Up to equality in $(\star\star\star)$ thereby covering the largest possible range of nonpositive σ for the appearance of unbounded solutions, this extends previous findings on blow-up in infinite time which in the context of $(\star\star)$ were limited to a smaller parameter region, and which were restricted to mere existence results without information on the richness of $\mathcal{B}$.

Key words: chemotaxis; nonlinear diffusion; global solutions; infinite-time blow-up
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1 Introduction

Quasilinear chemotaxis problems of the form

$$\begin{cases} u_t = \nabla\cdot(D(u,v)\nabla u) - \nabla\cdot(S(u,v)\nabla v), & x\in\Omega,\ t>0,\\ v_t = \Delta v - v + u, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

have received considerable interest in the mathematical literature of the past two decades. Falling among the category of parabolic systems proposed by Keller and Segel ([22]) already in 1970 as basic macroscopic models for chemotactic movement of individuals in response to a signal produced by themselves, systems of this type appear at the core of numerous theoretical descriptions of taxis-influenced collective behavior in populations at various levels of complexity ([18]).

An apperently predominant part of the analytical research on (1.1) is concerned with questions related to the ability of the cross-diffusive interaction therein to support phenomena reflecting aggregation, known as the probably most important biological effect of chemotaxis in several experimental frameworks. Here besides findings on the existence of structured steady states possibly even forming bubbles near certain critical parameters ([13], [20]), a large literature focuses on either detecting or ruling out the occurrence of concentration phenomena in the extreme sense of dynamical emergence of unbounded cell densities within suitable spatial regions.

For instance, in the most classical version of the Keller-Segel system determined by the choices $D\equiv 1$ and $S(u,v)=u$ in (1.1), some radially symmetric solutions are known to blow up in finite time when $\Omega = B_R(0)\subset\mathbb{R}^n$ with $R>0$ if either $n\ge 3$ ([44]), or $n=2$ and the conserved total mass $\int_\Omega u_0$ of cells exceeds 8π ([17], [29]), while if $\Omega\subset\mathbb{R}^n$ is a smoothly bounded domain and u_0 and v_0 are suitably regular with either $n=1$, or $n=2$ and $\int_\Omega u_0 < 4\pi$, or $n\ge 3$ and $\|u_0\|_{L^{\frac{n}{2}}(\Omega)} + \|v_0\|_{W^{1,n}(\Omega)}$ being sufficiently small, then (1.1) possesses a globally defined classical solution which is bounded ([32], [31], [3]).

The analysis of (1.1) under more general assumptions on the diffusion rate D and the cross-diffusion coefficient function S has been substantially stimulated by refined modeling approaches which, for instance, propose to link $D\equiv D(u)$ and $S\equiv S(u)$ via relations of the form $D(u)=Q(u)-uQ'(u)$ and $S(u)=uQ'(u)$, $u\ge 0$, in cases when the finite volume of individual cells is no longer negligible. Namely, the probability $Q(u)$ for a cell, when located at a spatio-temporal position (x,t) with population density $u=u(x,t)$, to find space in some neighboring site then in general no longer satisfies $Q\equiv 1$, but is rather described by a decreasing function decaying at large densities ([33]). Alternative derivations of (1.1), suggesting various different choices of $D\equiv D(u)$ and $S\equiv S(u)$ not necessarily coupled as above, can e.g. be based on hydrodynamical approaches or on taking macroscopic limits in certain cellular potts models (cf. [46] for a review on related modeling aspects). Apart from that, influences of the signal concentration v on the chemotactic migration, such as saturation effects at large values of v or the presence of an activation threshold for cross-diffusion to occur at all, are reflected in various choices of v-dependent sensitivity functions S in the framework of particular versions of (1.1) (cf. e.g. [41], [18], [36]). Beyond this, more recent developments in the biomathematical literature have identified situations in which accounting for signal dependence even in the diffusion rate D appears to be necessary ([15], [27], [37], [38]).

In such more general quasilinear versions of (1.1), the potential to enforce unboundedness phenomena seems to be essentially dependent on the size of S relative to D, where the corresponding knowledge seems most complete in presence of non-degenerate power-type nonlinearities such as in the prototypical version

$$\begin{cases} u_t = \nabla\cdot\Big((u+1)^{m-1}\nabla u\Big) - \nabla\cdot\Big(u(u+1)^{\sigma-1}\nabla v\Big), & x\in\Omega,\ t>0,\\ v_t = \Delta v - v + u, & x\in\Omega,\ t>0, \end{cases} \tag{1.2}$$

of the parabolic system in (1.1) with parameters $m \in \mathbb{R}$ and $\sigma \in \mathbb{R}$. Namely, for the associated Neumann initial-value problem in bounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$, it is known that

(i) if $\sigma < m - \frac{n-2}{n}$, then for all suitably regular initial data global bounded classical solutions exist ([39]), whereas

(ii) if $\sigma > m - \frac{n-2}{n}$ and $\Omega \subset \mathbb{R}^n$ is a ball, then for each $M > 0$ there exist $T \in (0,\infty]$ and radially symmetric solutions (u,v) in $\Omega \times (0,T)$ for which $\int_\Omega u(\cdot,t) = M$ for all $t \in (0,T)$, but for which u is not bounded in $\Omega \times (0,T)$ ([43]).

We remark that both these results actually extend to considerably larger classes of conveniently smooth functions $D \equiv D(u)$ and $S \equiv S(u)$ in (1.1) in the sense that under appropriate technical assumptions, the conclusion of (i) holds if $\frac{S(u)}{D(u)} \leq Cu^{\frac{2}{n}-\varepsilon}$ for some $\varepsilon > 0$ and $C > 0$ and all $u \geq 1$, while the result from (ii) is valid if there exist $\varepsilon > 0$ and $C > 0$ such that $\frac{S(u)}{D(u)} \geq Cu^{\frac{2}{n}+\varepsilon}$ for all $u \geq 1$ ([39], [43]; cf. also [23], [4], [35], [19]) for some precedents).

The unboundedness phenomenon in (ii) was examined in further detail in [8] and in [9], where it was shown that

(ii.i) if $\sigma > m - \frac{n-2}{n}$ and moreover either $\sigma \geq 1$ or $m \geq 1$, then in (ii) one can achieve that $T < \infty$, that is, the respective solution blows up in finite time

(cf. also a related one-dimensional finite-time blow-up result in [7]). On the other hand, in [9] also examples of *infinite-time blow-up* have been detected in some cases in which unlike in (ii.i) the chemotactic sensitivity decays at large densities at a sufficiently fast rate. More precisely,

(ii.ii) if $\sigma > m - \frac{n-2}{n}$ additionally satisfies $\sigma < \frac{m}{2} - \frac{n-2}{2n}$, then in (ii) we always have $T = \infty$, whence the associated unbounded solution is global in time but satisfies $\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$.

This already indicates that infinite-time blow-up phenomena may in fact play a more significant role in chemotaxis systems than may be expected from findings on Keller-Segel type systems with $D \equiv 1$ and $S(u,v) = u$, in which the occurrence of such large-time explosions seems restricted to solutions at precisely critical mass levels ([21], [30]), or to modified systems e.g. involving certain indirect signal production mechanisms ([40]). Indeed, due to the above statement in (ii) it is asserted in (ii.ii) that for any σ satisfying the requirements there, actually infinitely many global unbounded solutions can be found. Beyond this pure existence statement, however, further questions concerning the distribution of corresponding explosion-enforcing initial data in suitable function spaces seem unaddressed so far; in particular, it appears to be open how far infinite-time blow-up can be regarded as a generic phenomenon in (1.2) under the assumptions in (ii.ii), as known to be the case for finite-time blow-up in the original Keller-Segel system ([44], [29]). Moreover, the gap between the hypotheses in (ii.ii) and those in (ii.i) gives rise to the problem of identifying the maximal region in the (m, σ)-plane within which infinite-time grow-up occurs in (1.2); in light of the statement in (ii), the latter is evidently closely connected to the question under which assumptions on D and S global solutions, possibly unbounded, can be found for widely arbitrary initial data.

Main results I: Global smooth solutions under mild assumptions. The goal of the present study is to provide some further information on the above two questions. First focusing on the latter issue of global solvability, in view of its potential independent relevance to taxis problems involving signal-dependent motilities we shall address this topic in the general context of (1.1) under the assumptions that

$$\begin{cases} D \in C^2([0,\infty)^2) \quad \text{satisfies } D > 0 \text{ in } [0,\infty)^2, \quad \text{and that} \\ S \in C^2([0,\infty)^2) \quad \text{is nonnegative and such that } S(0,v) = 0 \text{ for all } v \ge 0. \end{cases} \tag{1.3}$$

In most parts, we shall moreover suppose that D satisfies a positivity condition stronger than that in (1.3) by assuming that

$$D(u,v) \ge k_D(v)(u+1)^{m-1} \qquad \text{for all } u \ge 0 \text{ and } v \ge 0 \tag{1.4}$$

with some nonincreasing positive function k_D on $[0,\infty)$ and some $m \in \mathbb{R}$, and apart from that we will require that

$$S(u,v) \le K_S \qquad \text{for all } u \ge 0 \text{ and } v \ge 0 \tag{1.5}$$

with a certain number $K_S > 0$, and that

$$\frac{\partial S(u,v)}{\partial v} \ge -k_S u^{-\lambda}(v+1)^{-\mu} \qquad \text{for all } u \ge 0 \text{ and } v \ge 0 \tag{1.6}$$

with some constants $k_S > 0, \lambda > 0$ and $\mu \in \mathbb{R}$. As for the initial data, for simplicity we shall assume that

$$\begin{cases} u_0 \in W^{1,\infty}(\Omega) \text{ with } u_0 > 0 \text{ in } \overline{\Omega} \qquad \text{and} \\ v_0 \in W^{1,\infty}(\Omega) \text{ satisfies } v_0 > 0 \text{ in } \overline{\Omega}, \end{cases} \tag{1.7}$$

In this setting, an argument based on an application of maximal Sobolev regularity theory to the second equation in (1.1) enables us to assert global classical solvability whenever λ in (1.6) is suitably large:

Theorem 1.1 *Let $n \ge 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and suppose that D and S satisfy (1.3) as well as (1.4), (1.5) and (1.6) with some nonincreasing positive function k_D on $[0,\infty)$ and constants $K_S > 0, k_S > 0, m \in \mathbb{R}, \lambda > 0$ and $\mu \ge 0$ fulfilling*

$$\lambda > \frac{n-2}{n} \cdot (1-\mu)_+. \tag{1.8}$$

Then for any choice of u_0 and v_0 satisfying (1.7), the initial-boundary value problem (1.1) possesses a global classical solution (u,v) such that

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ v \in \bigcap_{q>n} C^0([0,\infty); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \end{cases} \tag{1.9}$$

and such that both u and v are positive in $\overline{\Omega} \times (0,\infty)$.

We emphasize that in the case when $S \equiv S(u)$ is independent of v and hence (1.6) trivially fulfilled, the above conclusion thus holds if beyond (1.3) the sensitivity S merely satisfies the boundedness assumption (1.5), which inter alia entails a corresponding global existence statement for (1.2) for arbitrary $m \in \mathbb{R}$ whenever $\sigma \le 0$. But also numerous choices of signal-dependent sensitivities accounting for appropriate saturation effects are covered; simple examples are addressed in the following.

Corollary 1.2 *Suppose that $\Omega \subset \mathbb{R}^n$, $n \ge 2$, is a bounded domain with smooth boundary, that $D \in C^2([0,\infty)^2)$ satisfies (1.4) with some nonincreasing positive function k_D on $[0,\infty)$, and that u_0 and v_0 are such that (1.7) holds. Then in either of the cases*

$$S(u,v) = u(u+1)^{\sigma-1}, \qquad u \ge 0,\ v \ge 0, \qquad \textit{with } \sigma \le 0, \tag{1.10}$$

and

$$S(u,v) = \frac{u}{1+u+v}, \qquad u \ge 0,\ v \ge 0, \tag{1.11}$$

as well as

$$S(u,v) = \frac{u}{(1+u+uv)^\alpha}, \qquad u \ge 0,\ v \ge 0, \qquad \textit{with } \alpha > 1, \tag{1.12}$$

the problem (1.1) admits a global classical solution fulfilling (1.9).

We remark that the assumption (1.4) on at most algebraic decay of D with respect to u is due to the use of a Moser-type iteration appearing in the derivation of L^∞ estimates from L^p bounds (cf. Lemma 2.11). For diffusion mechanisms substantially more strongly degenerate at large cell densities, alternative arguments need to be involved at this stage; reasonings covering certain cases of exponentially decreasing diffusion rates can be found in [11], [12] and [45], for instance.

Main results II: Infinite-time blow-up as a generic phenomenon. Our next objective consists in identifying situations in which the phenomenon of infinite-time blow-up in the spirit of the statement in (ii.ii) above can be viewed generic. Bearing in mind the substantial challenges apparently inherent to the detection of unbounded solutions in chemotaxis systems especially of fully parabolic type, we here resort to the case when in (1.1) we have $D \equiv D(u)$ and $S \equiv S(u)$, and when moreover Ω is a ball and the considered solutions are radially symmetric with respect to the center thereof. In such settings, namely, a well-known contradictory argument ruling out global existence of bounded solutions, as detailed in [43], can be based on an accordingly available natural energy functional associated with (1.1), to be recalled in (3.2) below.

Our remaining task in this direction thereby actually reduces to the problem of making sure that under appropriate assumptions on D and S, the set of all admissible initial data at sufficiently low levels of the corresponding energy is suitably large. In order to thus make the approach from [43] applicable here, slightly strengthening the hypotheses made there we shall assume for simplicity that beyond the above requirements, D and S are such that with some $s_0 \ge 1$ and certain positive constants $L_{DS}, \vartheta, K_{DS}$ and α we have

$$\int_{s_0}^{u} \frac{\xi D(\xi)}{S(\xi)} d\xi \le \begin{cases} L_{DS} \cdot \frac{u}{\ln u} \quad \text{for all } u \ge s_0 & \text{if } n = 2, \\ \frac{n-2-\vartheta}{n} \int_{s_0}^{u} \int_{s_0}^{s} \frac{D(\xi)}{S(\xi)} d\xi ds + L_{DS} \cdot u \quad \text{for all } u \ge s_0 & \text{if } n \ge 3, \end{cases} \tag{1.13}$$

as well as

$$\int_{s_0}^{u} \int_{s_0}^{s} \frac{D(\xi)}{S(\xi)} d\xi ds \le K_{DS} \cdot u^{2-\alpha} \quad \text{for all } u \ge s_0. \tag{1.14}$$

Then the subcriticality condition $\alpha > \frac{2}{n}$, already discovered in [43] as sufficient for the pure existence of *some* unbounded solutions to (1.1), in fact ensures that *infinite-time* blow-up occurs within a considerably *large* set of initial data:

Theorem 1.3 *Let $n \ge 2$ and $\Omega = B_R(0) \subset \mathbb{R}^n$ with some $R > 0$, and suppose that $D \equiv D(u)$ and $S \equiv S(u)$ satisfy (1.3), (1.4) and (1.5) with some nonincreasing $k_D : [0, \infty) \to (0, \infty)$, some $m \in \mathbb{R}$ and some $K_S > 0$. Moreover, assume that there exist $s_0 \ge 1$, $\vartheta > 0$ and $L_{DS} > 0$ such that (1.13) is valid, and that (1.14) holds with some $K_{DS} > 0$ and some $\alpha \in (0, 2)$ fulfilling*

$$\alpha > \frac{2}{n}.$$

Then writing

$$\mathcal{I} := \left\{ (u_0, v_0) \in (W^{1,\infty}(\Omega))^2 \ \middle| \ u_0 \text{ and } v_0 \text{ are positive and radially symmetric in } \overline{\Omega} \right\}$$

and

$$\mathcal{B} := \left\{ (u_0, v_0) \in \mathcal{I} \ \middle| \ \text{The problem (1.1) possesses a global classical solution such that } \limsup_{t \to \infty} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty \right\},$$

we have the following:

i) $\mathcal{B}$ is dense in $\mathcal{I}$ in the following sense: For all $(u_0, v_0) \in \mathcal{I}$ and any choice of $p \ge 1$ and $q \ge 1$ such that

$$p < p_0(\alpha) := \begin{cases} \frac{2n}{n+2} & \text{if } \alpha > \frac{4}{n+2}, \\ \frac{n\alpha}{2} & \text{if } \alpha \le \frac{4}{n+2} \end{cases} \quad \text{and} \quad q < q_0(\alpha) := \begin{cases} 2 & \text{if } \alpha > \frac{4}{n+2}, \\ \frac{n\alpha}{2-\alpha} & \text{if } \alpha \le \frac{4}{n+2}, \end{cases} \tag{1.15}$$

one can find $((u_{0k}, v_{0k}))_{k \in \mathbb{N}} \subset \mathcal{I}$ such that $\int_\Omega u_{0k} = \int_\Omega u_0$ for all $k \in \mathbb{N}$, that

$$u_{0k} \to u_0 \text{ in } L^p(\Omega) \quad \text{and} \quad v_{0k} \to v_0 \text{ in } W^{1,q}(\Omega) \qquad \text{as } k \to \infty,$$

but that $(u_{0k}, v_{0k}) \in \mathcal{B}$ for all $k \in \mathbb{N}$.

ii) $\mathcal{B}$ *contains an open subset in the following sense: For all* $m > 0$ *there exist* $(u_0^{(0)}, v_0^{(0)}) \in \mathcal{I}$ *and* $\varepsilon > 0$ *with the property that* $\int_\Omega u_0^{(0)} = m$ *and that whenever* $(u_0, v_0) \in \mathcal{I}$ *satisfies*

$$\|u_0 - u_0^{(0)}\|_{L^\infty(\Omega)} \le \varepsilon \qquad \text{and} \qquad \|v_0 - v_0^{(0)}\|_{W^{1,2}(\Omega)} \le \varepsilon, \tag{1.16}$$

the pair (u_0, v_0) *belongs to* $\mathcal{B}$.

In particular, this shows that in the prototypical system (1.2), the above statement (ii.ii) actually holds for all supercritical nonpositive σ without the additional assumption $\sigma < \frac{m}{2} - \frac{n-2}{2n}$ which thus turns out to be purely technical. Furthermore, throughout this entire range infinite-time blow-up occurs as a generic phenomenon in the sense of Theorem 1.3:

Corollary 1.4 *Let* $n \ge 2$ *and* $\Omega = B_R(0) \subset \mathbb{R}^n$ *with some* $R > 0$, *and let* $m \in (-\infty, \frac{n-2}{n})$. *Then for any choice of* $\sigma > m - \frac{n-2}{n}$ *satisfying* $\sigma \le 0$, *with* $D(u) := (u+1)^{m-1}$ *and* $S(u) := u(u+1)^{\sigma-1}$ *for* $u \ge 0$ *the conclusion of Theorem 1.3 holds.*

With regard to mere problem of deciding for which pairs (m, σ) solutions of (1.2) remain bounded, for which finite-time blow-up occurs, and for which solutions exist globally but may blow up in inifinite time, beyond the statements in (i) and (ii.i) this provides a range substantially larger than that in (ii.ii) within which the latter phenomenon can be found. We conjecture that this region is essentially maximal with respect to this property, and that accordingly the parameter set where $\sigma > m - \frac{n-2}{n}$ and $0 < \sigma < 1$ and $m < 1$, constituting the only open subset of the (m, σ)-plane yet lacking characterization, belongs to the regime admitting finite-time blow-up. As an indication for this one may refer to corresponding affirmative results on a parabolic-elliptic simplification of (1.2) ([14], [26]; cf. also [5], [6], [10] and [2] for a broader picture concerning this problem); to the best of our knowledge, however, no rigorous result addressing any parameter in this region seems available.

2 Local and global existence. Proof of Theorem 1.1

As a preliminary step toward our results on global solvability, let us state a result on local existence and extensibility which summarizes the outcome of classical reasonings based e.g. on the use of the Schauder fixed point theorem along with standard parabolic regularity theory ([1], [24]).

Lemma 2.1 *Assume that* D *and* S *satisfy (1.3), and that* u_0 *and* v_0 *comply with (1.7). Then there exist* $T_{max} \in (0, \infty]$ *and at least one pair* (u, v) *of functions*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})), \\ v \in \bigcap_{q>n} C^0([0, T_{max}); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})), \end{cases}$$

such that both u *and* v *are positive in* $\overline{\Omega} \times (0, \infty)$, *that* (u, v) *solves (1.1) classically in* $\Omega \times (0, T_{max})$, *and that*

$$\text{if } T_{max} < \infty, \quad \text{then} \quad \limsup_{t \nearrow T_{max}} \Big\{ \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,q}(\Omega)} \Big\} = \infty \quad \text{for all } q > n. \tag{2.1}$$

Moreover,

$$\int_\Omega u(\cdot, t) = \int_\Omega u_0 \qquad \text{for all } t \in (0, T_{max}). \tag{2.2}$$

Proof. A recent and comprehensive proof of a corresponding statement addressing a closely related problem can be found in [25], so that we may restrict ourselves to presenting an outline here. Fixing $q > n$ and abbreviating $M := \|u_0\|_{L^\infty(\Omega)} + 1$ and $N := \|v_0\|_{W^{1,q}(\Omega)}$, by means of standard parabolic regularity theory ([42]) we can find $c_1(M,N) > 0$ with the property that whenever $T \in (0,1]$, $f \in C^0(\overline{\Omega} \times [0,T])$ and $z \in C^0([0,T]; W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ are such that $|f| \le M$ in $\Omega \times (0,T)$ as well as

$$\begin{cases} z_t = \Delta z - z + f(x,t), & x \in \Omega,\ t \in (0,T), \\ \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ z(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{2.3}$$

we have

$$\|z(\cdot,t)\|_{L^\infty(\Omega)} + \|\nabla z(\cdot,t)\|_{L^q(\Omega)} \le c_1(M,N) \qquad \text{for all } t \in (0,T). \tag{2.4}$$

Apart from that, let us invoke well-known results on boundedness and on Hölder regularity in scalar parabolic equations ([28, Theorem 6.40], [34, Theorem 1.3, Remarks 1.3 and 1.4]) to pick $\theta_1 = \theta_1(M,N) \in (0,1)$ and $c_2(M,N) > 0$ such that if $T \in (0,1]$, $a \in L^\infty(\Omega\times(0,T))$, $b \in L^\infty((0,T); L^q(\Omega;\mathbb{R}^n))$ and $z \in L^\infty(\Omega \times (0,T)) \cap L^2((0,T); W^{1,2}(\Omega))$ are such that

$$k_D(c_1(M,N))\cdot\min\Big\{(M+1)^{m-1}\,,\,1\Big\} \le a(x,t) \le \|D\|_{L^\infty((0,M)\times(0,c_1(M,N)))} \qquad \text{for a.e. } (x,t) \in \Omega\times(0,T) \tag{2.5}$$

and

$$\|b(\cdot,t)\|_{L^q(\Omega)} \le K_S c_1(M,N) \qquad \text{for a.e. } t \in (0,T), \tag{2.6}$$

and that z solves

$$\begin{cases} z_t = \nabla\cdot(a(x,t)\nabla z) - \nabla\cdot b(x,t), & x \in \Omega,\ t \in (0,T), \\ \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ z(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.7}$$

in the natural weak sense specified in [28]), then

$$\|z\|_{C^{\theta_1,\frac{\theta_1}{2}}(\overline{\Omega}\times[0,T])} \le c_2(M,N). \tag{2.8}$$

We thereupon set $\theta := \frac{\theta_1}{2}$ and

$$T \equiv T(M,N) := \min\Big\{1\,,\,c_2^{-\frac{2}{\theta_1}}(M,N)\Big\}, \tag{2.9}$$

and introduce the closet subset S of $X := C^\theta(\overline{\Omega} \times [0,T])$ by letting

$$S := \Big\{\varphi \in X \ \Big|\ 0 \le \varphi \le M \text{ in } \overline{\Omega} \times [0,T]\Big\}.$$

Then for $\widehat{u} \in S$ we define $\Phi(\widehat{u}) := u$, where $u \in L^\infty(\Omega \times (0,T)) \cap L^2((0,T); W^{1,2}(\Omega))$ denotes the weak solution of

$$\begin{cases} u_t = \nabla\cdot(D(\widehat{u},v)\nabla u) - \nabla\cdot(S(\widehat{u},v)\nabla v), & x \in \Omega,\ t \in (0,T), \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.10}$$

with $v \in C^0([0,T];W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ representing the classical solution of

$$\begin{cases} v_t = \Delta v - v + \widehat{u}, & x \in \Omega,\ t \in (0,T), \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{2.11}$$

Then given $\widehat{u} \in S$ we know from the comparison principle and (2.4) that the correspondingly defined solution v from (2.11) satisfies $0 \le v(x,t) \le c_1(M,N)$ for all $(x,t) \in \overline{\Omega} \times [0,T]$ and $\|\nabla v(\cdot,t)\|_{L^q(\Omega)} \le c_1(M,N)$ for all $t \in [0,T]$, whence (1.4) and (1.5) ensure that $D(\widehat{u},v) \ge k_D(c_1(M,N)) \cdot \min\{(M+1)^{m-1},1\}$ and $D(\widehat{u},v) \le \|D\|_{L^\infty((0,M)\times(0,c_1(M,N)))}$ in $\Omega\times(0,T)$ as well as $\|S(\widehat{u}(\cdot,t),v(\cdot,t))\nabla v(\cdot,t)\|_{L^q(\Omega)} \le K_S c_1(M,N)$ for all $t \in (0,T)$. Therefore, (2.8) applies so as to warrant that u belongs to X and actually satisfies

$$\|u\|_{C^{\theta_1,\frac{\theta_1}{2}}(\overline{\Omega}\times[0,T])} \le c_2(M,N), \tag{2.12}$$

in view of (2.10) especially implying that $\|u(\cdot,t)-u_0\|_{L^\infty(\Omega)} \le c_2(M,N)t^{\frac{\theta_1}{2}}$ for all $t \in (0,T)$ and hence

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + c_2(M,N)T^{\frac{\theta_1}{2}} \le M \qquad \text{for all } t \in (0,T),$$

because $c_2(M,N)T^{\frac{\theta_1}{2}} \le 1$ by (2.9). As u is nonnegative by a weak comparison principle ([28]), this shows that indeed $u \in S$ and thus $\Phi S \subset S$, while once more relying on (2.12) we may infer using the Arzelà-Asoli theorem and the fact that $\theta < \theta_1$ that $\overline{\Phi S}$ is compact in X. Since furthermore a combination of the latter token with well-known uniqueness properties of (2.3) and (2.7) reveal continuity of Φ ([28]; cf. also the reasoning in [25, Lemma 2.2]), the Schauder fixed point theorem states the existence of $u \in S$ such that $\Phi u = u$. According to (2.10) and (2.11), straightforward bootstrap arguments show that in fact u, along with v as accordingly determined by (2.11), forms a classical solution of (1.1) in $\Omega \times (0,T)$. As our definition (2.9) of $T(M,N)$ involves (u_0,v_0) exclusively through its norm in $L^\infty(\Omega) \times W^{1,q}(\Omega)$, a standard prolongation argument finally yields extensibility of this solution up to a maximal time $T_{max} \in (0,\infty]$ fulfilling (2.1). The positivity properties of u and v can thereafter be obtained by two applications of the classical strong maximum principle, whereas (2.2) results upon an integration in (1.1). $\Box$

Throughout the sequel, without further explicit mentioning we shall assume that (1.3) and (1.7) hold, and that (u,v) denotes the corresponding local solution of (1.1), as obtained in Lemma 2.1 and extended up to its maximal existence time $T_{max} \le \infty$.

2.1 L^p bounds for u via maximal Sobolev regularity estimates

In accordance with Lemma 2.1, verifying the claim from Theorem 1.1 amounts to establishing appropriate bounds for u with respect to the norm in $L^\infty(\Omega)$. A rudimentary preparation for this will be provided by Lemma 2.3 below, which in turn will rely on the following observation that addresses a quantity appearing in the course of an integration by parts during an associated testing procedure.

Lemma 2.2 *Assume that (1.5) and (1.6) are valid with some $K_S > 0, k_S > 0, \lambda > 0$ and $\mu \ge 0$, and for $p > \lambda + 1$ let*

$$\Sigma_p(\widetilde{u},\widetilde{v}) := \int_0^{\widetilde{u}} \xi^{p-2} S(\xi,\widetilde{v})d\xi, \qquad \widetilde{u} \ge 0, \widetilde{v} \ge 0. \tag{2.13}$$

Then whenever $\widetilde{u} \in C^1(\Omega)$ and $\widetilde{v} \in C^1(\Omega)$ are positive, we have

$$\widetilde{u}^{p-2} S(u,v)\nabla\widetilde{u}\cdot\nabla\widetilde{v} \le \nabla\Sigma_p(\widetilde{u},\widetilde{v})\cdot\nabla\widetilde{v} + \frac{k_S}{p-\lambda-1}\widetilde{u}^{p-\lambda-1}(\widetilde{v}+1)^{-\mu}|\nabla\widetilde{v}|^2 \qquad \text{in } \Omega \tag{2.14}$$

and

$$|\Sigma_p(\widetilde{u},\widetilde{v})| \le \frac{K_S}{p-1}\widetilde{u}^{p-1} \qquad \text{in } \Omega. \tag{2.15}$$

PROOF. We compute

$$\begin{aligned}\nabla\Sigma_p(\widetilde{u},\widetilde{v}) &= \frac{\partial\Sigma_p(\widetilde{u},\widetilde{v})}{\partial\widetilde{u}}\nabla\widetilde{u} + \frac{\partial\Sigma_p(\widetilde{u},\widetilde{v})}{\partial\widetilde{v}}\nabla\widetilde{v} \\ &= \widetilde{u}^{p-2}S(\widetilde{u},\widetilde{v})\nabla\widetilde{u} + \left\{\int_0^{\widetilde{u}}\xi^{p-2}\frac{\partial S}{\partial v}(\xi,\widetilde{v})d\xi\right\}\nabla\widetilde{v} \qquad \text{in } \Omega\end{aligned}$$

and thus obtain that

$$\widetilde{u}^{p-2}S(\widetilde{u},\widetilde{v})\nabla\widetilde{u}\cdot\nabla\widetilde{v} = \nabla\Sigma_p(\widetilde{u},\widetilde{v})\cdot\nabla\widetilde{v} - \left\{\int_\Omega\int_0^{\widetilde{u}}\xi^{p-2}\frac{\partial S}{\partial v}(\xi,\widetilde{v})d\xi\right\}|\nabla\widetilde{v}|^2 \qquad \text{in } \Omega.$$

As (1.6) warrants that herein

$$\begin{aligned}-\int_\Omega\int_0^{\widetilde{u}}\xi^{p-2}\frac{\partial S}{\partial v}(\xi,\widetilde{v})d\xi &\le k_S(\widetilde{v}+1)^{-\mu}\int_0^{\widetilde{u}}\xi^{p-2-\lambda}d\xi \\ &= \frac{k_S}{p-\lambda-1}\widetilde{u}^{p-\lambda-1}(\widetilde{v}+1)^{-\mu} \qquad \text{in } \Omega,\end{aligned}$$

this immediately yields (2.14). The inequality (2.15) is a direct consequence of (2.13) and (1.5). □

We can thereby make use of (1.6) to achieve the following basic inequality describing the time evolution of $\|u(\cdot,t)\|_{L^p(\Omega)}$ for arbitrarily large finite p.

Lemma 2.3 *Suppose that (1.6) is valid with some $k_S > 0$, $\lambda > 0$ and $\mu \ge 0$. Then for all $p > \lambda + 1$ there exists $C(p) > 0$ such that*

$$\frac{d}{dt}\int_\Omega u^p \le C(p)\cdot\left\{\int_\Omega u^p + \int_\Omega|\Delta v|^p + \int_\Omega(v+1)^{-\frac{p\mu}{\lambda+1}}|\nabla v|^{\frac{2p}{\lambda+1}}\right\} \qquad \text{for all } t\in(0,T_{max}). \tag{2.16}$$

PROOF. Multiplying the first equation in (1.1) by u^{p-1} and integrating by parts yields

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p &= \int_\Omega u^{p-1}\nabla\cdot\Big\{D(u,v)\nabla u - S(u,v)\nabla v\Big\} \\ &= -(p-1)\int_\Omega u^{p-2}D(u,v)|\nabla u|^2 + (p-1)\int_\Omega u^{p-2}S(u,v)\nabla u\cdot\nabla v \\ &\le (p-1)\int_\Omega u^{p-2}S(u,v)\nabla u\cdot\nabla v \qquad \text{for all } t\in(0,T_{max}),\end{aligned} \tag{2.17}$$

because D is nonnegative and $p \geq 1$. Here in order to make appropriate use of (1.6) in estimating the expression on the right of (2.17), we employ Lemma 2.2. This, namely, enables us to integrate by parts once more, with Σ_p as introduced in (2.13) thus resulting in the inequality

$$\begin{aligned}(p-1)\int_\Omega u^{p-2}S(u,v)\nabla u\cdot\nabla v &\leq (p-1)\int_\Omega \nabla\Sigma_p(u,v)\cdot\nabla v + \frac{(p-1)k_S}{p-\lambda-1}\int_\Omega u^{p-\lambda-1}(v+1)^{-\mu}|\nabla v|^2 \\ &= -(p-1)\int_\Omega \Sigma_p(u,v)\Delta v + \frac{(p-1)k_S}{p-\lambda-1}\int_\Omega u^{p-\lambda-1}(v+1)^{-\mu}|\nabla v|^2 \qquad (2.18)\end{aligned}$$

for all $t \in (0,T_{max})$. Now since by means of Young's inequality we can estimate

$$\begin{aligned}-(p-1)\int_\Omega \Sigma_p(u,v)\Delta v &\leq K_S\int_\Omega u^{p-1}|\Delta v| \\ &\leq K_S\cdot\left\{\int_\Omega u^p + \int_\Omega |\Delta v|^p\right\} \qquad \text{for all } t\in(0,T_{max})\end{aligned}$$

due to (2.15), and moreover

$$\frac{(p-1)k_S}{p-\lambda-1}\int_\Omega u^{p-\lambda-1}(v+1)^{-\mu}|\nabla v|^2 \leq \frac{(p-1)k_S}{p-\lambda-1}\cdot\left\{\int_\Omega u^p + \int_\Omega (v+1)^{-\frac{p\mu}{\lambda+1}}|\nabla v|^{\frac{2p}{\lambda+1}}\right\}$$

for all $t \in (0,T_{max})$, from (2.17) and (2.18) we readily obtain (2.16). $\square$

Here in order to take appropriate advantage of a possibly damping effect at large values of v of the factor $(v+1)^{-\frac{p\mu}{\lambda+1}}$ appearing in the rightmost integral in (2.16), let us briefly derive cred the following weighted embedding inequality. According to this, namely, expressions essentially similar to that in the seecond summand on the right of (2.16) already control certain first-order integrals involving higher summability powers when weakened by suitable weights of this form.

Lemma 2.4 *Let $p > 1$. Then for all $\varphi \in C^2(\overline{\Omega})$ such that $\varphi > 0$ in $\overline{\Omega}$ and $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$, the inequality*

$$\int_\Omega \varphi^{-p}|\nabla\varphi|^{2p} \leq \Big(2+\frac{\sqrt{n}}{p-1}\Big)^p\int_\Omega |D^2\varphi|^p \qquad (2.19)$$

holds.

PROOF. Using that $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$, we integrate by parts to obtain the identity

$$\begin{aligned}\int_\Omega \varphi^{-p}|\nabla\varphi|^{2p} &= -\frac{1}{p-1}\int_\Omega |\nabla\varphi|^{2p-2}\nabla\varphi\cdot\nabla\varphi^{1-p} \\ &= \frac{1}{p-1}\int_\Omega \varphi^{1-p}\nabla\varphi\cdot\nabla|\nabla\varphi|^{2p-2} + \frac{1}{p-1}\int_\Omega \varphi^{1-p}|\nabla\varphi|^{2p-2}\Delta\varphi. \qquad (2.20)\end{aligned}$$

Here since $\nabla|\nabla\varphi|^{2p-2} = (p-1)|\nabla\varphi|^{2p-4}\nabla|\nabla\varphi|^2 = 2(p-1)|\nabla\varphi|^{2p-4}D^2\varphi\cdot\nabla\varphi$, and since $|\Delta\varphi| \leq \sqrt{n}|D^2\varphi|$ according to the Cauchy-Schwarz inequality, we can estimate

$$\begin{aligned}\frac{1}{p-1}\int_\Omega \varphi^{1-p}\nabla\varphi\cdot\nabla|\nabla\varphi|^{2p-2} &= 2\int_\Omega \varphi^{1-p}|\nabla\varphi|^{2p-4}\nabla\varphi\cdot(D^2\varphi\cdot\nabla\varphi) \\ &\leq 2\int_\Omega \varphi^{1-p}|\nabla\varphi|^{2p-2}|D^2\varphi|\end{aligned}$$

as well as

$$\frac{1}{p-1}\int_\Omega \varphi^{1-p}|\nabla\varphi|^{2p-2}\Delta\varphi \le \frac{\sqrt{n}}{p-1}\int_\Omega \varphi^{1-p}|\nabla\varphi|^{2p-2}|D^2\varphi|.$$

From (2.20) we thus infer on using the Hölder inequality that

$$\begin{aligned}
\int_\Omega \varphi^{-p}|\nabla\varphi|^{2p} &\le \Big(2+\frac{\sqrt{n}}{p-1}\Big)\int_\Omega \varphi^{1-p}|\nabla\varphi|^{2p-2}|D^2\varphi| \\
&\le \Big(2+\frac{\sqrt{n}}{p-1}\Big)\cdot\bigg\{\int_\Omega \varphi^{-p}|\nabla\varphi|^{2p}\bigg\}^{\frac{p-1}{p}}\cdot\bigg\{\int_\Omega |D^2\varphi|^p\bigg\}^{\frac{1}{p}},
\end{aligned}$$

which evidently implies (2.19). $\Box$

Now unless $\mu = 1$, Lemma 2.4 apparently does not directly apply to the last summand in (2.16). Especially when $\mu < 1$ and hence the damping influence of the weight function in the latter is weaker than that in (2.19), an additional interpolation argument will be in order, at its core involving the following well-known regularity feature of (1.1) which, apart from that, will also be used in Corollary 2.10 below.

Lemma 2.5 *Let $p \ge 1$ and $q \in [1, \frac{np}{(n-p)_+})$. Then there exists $C(p,q) > 0$ such that*

$$\|v(\cdot,t)\|_{W^{1,q}(\Omega)} \le C(p,q)\cdot\bigg\{\sup_{s\in(0,t)}\|u(\cdot,s)\|_{L^p(\Omega)}+1\bigg\} \qquad \textit{for all } t\in(0,T_{max}). \tag{2.21}$$

PROOF. This can be seen by a standard argument based on known smoothing properties of the Neumann heat semigroup ([19, Lemma 4.1]). $\Box$

A first application of this, relying on a Gagliardo-Nirenberg interpolation, resorts to the choice $p = 1$ in which the right-hand side in (2.21) is bounded thanks to (2.2).

Lemma 2.6 *Let $p > 1$ and $q \in (0, \frac{np}{n-1})$. Then there exists $C(p,q) > 0$ such that*

$$\int_\Omega |\nabla v(\cdot,t)|^q \le C(p,q)\|v(\cdot,t)\|^p_{W^{2,p}(\Omega)} + C(p,q) \qquad \textit{for all } t\in(0,T_{max}). \tag{2.22}$$

PROOF. We first consider the case when $q \ge \frac{(n+1)p}{n}$, in which $r := \frac{n(q-p)}{p}$ satisfies $r \ge 1$. Since $q < \frac{np}{n-1}$ entails that moreover

$$r < \frac{n\cdot(\frac{np}{n-1}-p)}{p} = \frac{n}{n-1},$$

from Lemma 2.5 when combined with (2.2) we obtain $c_1 > 0$ such that

$$\|v(\cdot,t)\|_{W^{1,r}(\Omega)} \le c_1 \qquad \text{for all } t\in(0,T_{max}). \tag{2.23}$$

Now using that $q \ge \frac{(n+1)p}{n}$, we may invoke the Gagliardo-Nirenberg inequality to find $c_2 > 0$ fulfilling

$$\int_\Omega |\nabla v|^q \le c_2\|v\|^{qa}_{W^{2,p}(\Omega)}\|v\|^{q(1-a)}_{W^{1,r}(\Omega)} \qquad \text{for all } t\in(0,T_{max}),$$

with

$$-\frac{n}{q} = \Big(1-\frac{n}{p}\Big)a - \frac{n}{r}(1-a) = \Big(1-\frac{n}{p}\Big)a - \frac{p}{q-p}(1-a) = \frac{pq-nq+np}{p(q-p)} \cdot a - \frac{p}{q-p},$$

that is, with

$$a = \frac{p(q-p)}{pq-nq+np} \cdot \Big(\frac{p}{q-p} - \frac{n}{q}\Big) = \frac{p}{q}.$$

Therefore, in view of (2.23) we have

$$\int_\Omega |\nabla v|^q \le c_1^{q-p} c_2 \|v\|^p_{W^{2,p}(\Omega)} \qquad \text{for all } t \in (0, T_{max})$$

and thus conclude that in fact (2.22) holds for any such q.
If $q \in (0, \frac{(n+1)p}{n})$, however, a simple application of Young's inequality shows that

$$\int_\Omega |\nabla v|^q \le \int_\Omega |\nabla v|^{\frac{(n+1)p}{n}} + |\Omega| \qquad \text{for all } t \in (0, T_{max})$$

and thereby establishes (2.22) also in this case. $\Box$

Now combining Lemma 2.4 with Lemma 2.6 shows that if the number $\lambda > 0$ in (1.6) is large enough in the sense that (1.8) holds, then an expression resembling the second last summand on the right of (2.16) indeed essentially dominates the last integral therein.

Lemma 2.7 *Suppose that $\mu \ge 0$ and that $\lambda > 0$ is such that (1.8) holds, and assume that (1.6) is valid. Then for all $p > 1$ there exists $C(p) > 0$ fulfilling*

$$\int_\Omega (v(\cdot,t)+1)^{-\frac{p\mu}{\lambda+1}} |\nabla v(\cdot,t)|^{\frac{2p}{\lambda+1}} \le C(p)\|v(\cdot,t)\|^p_{W^{2,p}(\Omega)} + C(p) \qquad \textit{for all } t \in (0, T_{max}). \tag{2.24}$$

Proof. If $\mu \le 1$, using that $\lambda > 0$ we can employ Young's inequality to estimate

$$\begin{aligned} \int_\Omega (v+1)^{-\frac{p\mu}{\lambda+1}} |\nabla v|^{\frac{2p}{\lambda+1}} &\le \int_\Omega v^{-\frac{p\mu}{\lambda+1}} |\nabla v|^{\frac{2p}{\lambda+1}} \\ &= \int_\Omega \Big\{ v^{-\frac{p\mu}{\lambda+1}} |\nabla v|^{\frac{2p\mu}{\lambda+1}} \Big\} \cdot |\nabla v|^{\frac{2p(1-\mu)}{\lambda+1}} \\ &\le \int_\Omega v^{-p} |\nabla v|^{2p} + \int_\Omega |\nabla v|^{\frac{2p(1-\mu)}{\lambda+1-\mu}} \qquad \text{for all } t \in (0, T_{max}). \end{aligned} \tag{2.25}$$

Using that herein

$$\frac{2p(1-\mu)}{\lambda+1-\mu} < \frac{2p(1-\mu)}{\frac{(n-2)(1-\mu)}{n} + 1 - \mu} = \frac{2p}{\frac{n-2}{n}+1} = \frac{np}{n-1}$$

according to (1.8), on applying Lemma 2.4 and Lemma 2.6 to the two summands on the right of (2.25) we readily obtain (2.24).

In the case $\mu > 1$, we first use that than $(v+1)^{-\frac{p\mu}{\lambda+1}} \le (v+1)^{-\frac{p}{\lambda+1}} \le v^{-\frac{p}{\lambda+1}}$ in $\Omega \times (0, T_{max})$, so that Young's inequality shows that

$$\int_\Omega (v+1)^{-\frac{p\mu}{\lambda+1}} |\nabla v|^{\frac{2p}{\lambda+1}} \le \int_\Omega v^{-\frac{p}{\lambda+1}} |\nabla v|^{\frac{2p}{\lambda+1}} \le \int_\Omega v^{-p} |\nabla v|^{2p} + |\Omega| \qquad \text{for all } t \in (0, T_{max})$$

and that hence (2.24) becomes a consequence of Lemma 2.4. $\square$

Now in view of the simple structure of the second equation in (1.1), we may invoke maximal Sobolev regularity theory so as to find that when suitably integrated in time, the right-hand side in (2.24) can be estimated in terms of a corresponding time integral involving u.

Lemma 2.8 *Let $p > 1$. Then there exists $C(p) > 0$ such that*

$$\int_\tau^t \|v(\cdot,s)\|_{W^{2,p}(\Omega)}^p ds \le C(p) \cdot \left\{ \int_\tau^t \|u(\cdot,s)\|_{L^p(\Omega)}^p ds + 1 \right\} \qquad \text{for all } t \in (\tau, T_{max}), \tag{2.26}$$

where $\tau := \min\{1, \frac{1}{2} T_{max}\}$.

Proof. According to a well-known result on maximal Sobolev regularity in parabolic equations ([16, Theorem 2.3]), there exists $c_1 > 0$ such that whenever $t_0 \in \mathbb{R}$, $T > t_0$, $f \in C^1(\overline{\Omega} \times [t_0, T])$, $\phi \in C^2(\overline{\Omega})$ and $z \in C^{2,1}(\overline{\Omega} \times [t_0, T])$ are such that $\frac{\partial \phi}{\partial \nu} = 0$ on $\partial\Omega$ and

$$\begin{cases} z_t \;=\; \Delta z - z + f(x,t), & x \in \Omega,\ t \in (t_0, T), \\ \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega,\ t \in (t_0, T), \\ z(x, t_0) = \phi(x), & x \in \Omega, \end{cases}$$

we have

$$\int_{t_0}^T \|z(\cdot,s)\|_{W^{2,p}(\Omega)}^p ds \le c_1 \cdot \left\{ \int_{t_0}^T \|f(\cdot,s)\|_{L^p(\Omega)}^p ds + \|\phi\|_{W^{2,p}(\Omega)}^p \right\}.$$

As $\phi := v(\cdot,\tau)$ belongs to $C^2(\overline{\Omega})$ with $\frac{\partial \phi}{\partial \nu} = 0$ on $\partial\Omega$ due to Lemma 2.1, upon an application to $f := u, z := v,\ t_0 := \tau$ and arbitrary $T \in (\tau, T_{max})$ this immediately yields (2.26) with $C(p) := c_1 \cdot \max\{1, \|v(\cdot,\tau)\|_{W^{2,p}(\Omega)}^p\}$. $\square$

Together with Lemma 2.7 and a Grönwall-type argument, this enables us to turn Lemma 2.3 into the main result of this section:

Lemma 2.9 *Suppose that S satisfies (1.5) and (1.6) with some $K_S > 0, k_S > 0, \lambda > 0$ and $\mu \ge 0$ fulfilling (1.8). Then for all $p > 1$ and any $T > 0$ one can find $C(p,T) > 0$ such that*

$$\int_\Omega u^p(\cdot,t) \le C(p,T) \qquad \text{for all } t \in \Big(0, \min\{T, T_{max}\}\Big). \tag{2.27}$$

Proof. By means of Lemma 2.3 and Lemma 2.7, we see that with some $c_1 > 0$ we have

$$\frac{d}{dt} \int_\Omega u^p \le c_1 \cdot \left\{ \int_\Omega u^p + \|v\|_{W^{2,p}(\Omega)}^p + 1 \right\} \qquad \text{for all } t \in (0, T_{max})$$

and hence, by integration in time,

$$\int_\Omega u^p(\cdot,t) \le \int_\Omega u^p(\cdot,\tau) + c_1\int_\tau^t\int_\Omega u^p(x,s)dxds + c_1\int_\tau^t \|v(\cdot,s)\|^p_{W^{2,p}(\Omega)}ds + c_1 t \qquad \text{for all } t\in(\tau,T_{max}),$$

where again $\tau := \min\{1,\frac{1}{2}T_{max}\}$. In view of Lemma 2.8, this entails the existence of $c_2>0$ and $c_3>0$ such that

$$\int_\Omega u^p(\cdot,t) \le c_2(1+t) + c_3\int_\tau^t\int_\Omega u^p(x,s)dxds \qquad \text{for all } t\in(\tau,T_{max}),$$

so that the Grönwall inequality shows that

$$\int_\Omega u^p(\cdot,t) \le c_2(1+t)e^{c_3(t-\tau)} \qquad \text{for all } t\in(\tau,T_{max}).$$

and thereby establishes (2.27), because u is continuous in $\overline{\Omega}\times[0,\tau]$. □

2.2 An L^∞ estimate for u. Proof of Theorem 1.1 and Corollary 1.2

The following direct consequence of Lemma 2.9 already rules out finite-time blow-up of the second expression appearing in brackets in (2.1).

Corollary 2.10 *Suppose that D has the property (1.4) with some $m\in\mathbb{R}$ and some nonincreasing $k_D:[0,\infty)\to(0,\infty)$, and that there exist $K_S>0, k_S>0, \lambda>0$ and $\mu\ge 0$ fulfilling (1.8) such that S satisfies (1.5) and (1.6). Then for all $q>1$ and any $T>0$ there exists $C(q,T)>0$ such that*

$$\|\nabla v(\cdot,t)\|_{L^q(\Omega)} \le C(q,T) \qquad \textit{for all } t\in\Big(0,\min\{T,T_{max}\}\Big).$$

Proof. In light of Lemma 2.5, this immediately results on applying Lemma 2.9 to $p:=n$, for instance. □

When applied to suitably large q, the latter moreover allows us to invoke a standard iterative argument of Moser type to deduce from Lemma 2.9 and (1.4) that also the quantity $\|u\|_{L^\infty(\Omega)}$ cannot become unbounded within finite time.

Lemma 2.11 *Suppose that there exist $m\in\mathbb{R}$ and a nonincreasing $k_D:[0,\infty)\to(0,\infty)$ such that (1.4) holds, and let S satisfy (1.5) and (1.6) with some $K_S>0, k_S>0, \lambda>0$ and $\mu\ge0$ such that (1.8) holds. Then for all $T>0$ there exists $C(T)>0$ with the property that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C(T) \qquad \textit{for all } t\in\Big(0,\min\{T,T_{max}\}\Big). \tag{2.28}$$

Proof. For each $q>1$, Corollary 2.10 states that $\sup_{t\in(0,\min\{T,T_{max}\})}\|v(\cdot,t)\|_{W^{1,q}(\Omega)}<\infty$, whence for any such q we can find $c_1(q)>0$ such that

$$\|\nabla v(\cdot,t)\|_{L^q(\Omega)} \le c_1(q) \qquad \text{for all } t\in(0,\min\{T,T_{max}\}). \tag{2.29}$$

Choosing any $q>n$ here moreover shows that since in that case $W^{1,q}(\Omega)\hookrightarrow L^\infty(\Omega)$, we have

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le c_2 \qquad \text{for all } t\in(0,\min\{T,T_{max}\})$$

with some $c_2 > 0$. In view of (1.4), the latter entails the inequality

$$D(u,v) \geq k_D(c_2) \cdot (u+1)^{m-1} \qquad \text{in } \Omega \times (0, \min\{T, T_{max}\}),$$

which together with (2.29) enables us to employ a Moser-type recursive argument ([39, Lemma A.1]) so as to assert that (2.28) is a consequence of Lemma 2.9. $\square$

A combination of the latter two lemmata with (2.1) now directly yields our main result on global existence of classical solutions to (1.1).

PROOF of Theorem 1.1. In view of the extensibility criterion from Lemma 2.1, the statement immediately results from Lemma 2.11 and Corollary 2.10. $\square$

The verification of the specific conclusions drawn in Corollary 1.2 is thereupon straightforward:

PROOF of Corollary 1.2. For S as in (1.10), the inequality in (1.5) is obvious due to the assumption that $\sigma \leq 0$, while the condition (1.6) is trivially satisfied for any choice of $k_S > 0, \lambda > 0$ and $\mu \geq 0$.

If S is as in (1.11), then evidently $0 \leq S \leq 1$, and computing

$$\frac{\partial S(u,v)}{\partial v} = -\frac{u}{(1+u+v)^2}, \qquad u \geq 0,\ v \geq 0,$$

we see that if we fix any $\mu \in (0,1)$ and let $\lambda := 1-\mu$, then

$$\begin{aligned} -u^\lambda (v+1)^\mu \cdot \frac{\partial S(u,v)}{\partial v} &= \frac{u^{\lambda+1}(v+1)^\mu}{(1+u+v)^2} \\ &\leq \frac{(1+u+v)^{\lambda+1}(1+u+v)^\mu}{(1+u+v)^2} \\ &= 1 \qquad \text{for all } u \geq 0 \text{ and } v \geq 0, \end{aligned}$$

because $\lambda + 1 + \mu = 2$. Therefore, (1.5) and (1.6) hold with $K_S := 1$ and $k_S := 1$, and since $\mu < 1$ ensures that

$$\lambda = 1 - \mu > \frac{n-2}{n}(1-\mu) = \frac{n-2}{n}(1-\mu)_+,$$

it follows that also (1.8) is satisfied.

Similarly, for S taken from (1.12) we observe that again $0 \leq S \leq 1$, and that since

$$\frac{\partial S(u,v)}{\partial v} = -\alpha \frac{u^2}{(1+u+uv)^{\alpha+1}}, \qquad u \geq 0,\ v \geq 0,$$

with $\lambda := \alpha - 1$ and $\mu := \alpha + 1$ we can estimate

$$\begin{aligned} -u^\lambda (v+1)^\mu \cdot \frac{\partial S(u,v)}{\partial v} &= \alpha \cdot \frac{u^{\lambda+2}(v+1)^\mu}{(1+u+uv)^{\alpha+1}} \\ &= \alpha \cdot \frac{(u+uv)^{\alpha+1}}{(1+u+uv)^{\alpha+1}} \\ &\leq \alpha \qquad \text{for all } u \geq 0 \text{ and } v \geq 0. \end{aligned}$$

Consequently, (1.5) and (1.6) are valid with $K_S := 1$ and $k_S := \alpha$, and our choices of λ and μ moreover warrant that $\lambda > 0 = \frac{n-2}{n}(1-\mu)_+$, because $\alpha > 1$ and hence $\lambda > 0$ and $\mu > 1$.

In all the cases (1.10), (1.11) and (1.12), the claimed global existence results are thus asserted by Theorem 1.1. $\Box$

3 Generic infinite-time blow-up. Proof of Theorem 1.3

We next aim at verifying the occurrence of blow-up in infinite time for all initial data within a set that can be considered generic in the sense specified in the formulation of Theorem 1.3, and under the assumptions on D and S made therein. As apparently all precedent detections of unboundedness phenomena in fully parabolic systems, our argument will substantially rely on a natural gradient structure inherent to (1.1) in the situation when

$$D \equiv D(u) \qquad \text{and} \qquad S \equiv S(u) \tag{3.1}$$

which we will exclusively consider throughout the sequel. Then, namely, letting

$$\mathcal{F}(u,v) := \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega v^2 - \int_\Omega uv + \int_\Omega G(u) \tag{3.2}$$

and

$$\mathcal{D}(u,v) := \int_\Omega v_t^2 + \int_\Omega S(u)\Big|\frac{D(u)}{S(u)}\nabla u - \nabla v\Big|^2 \tag{3.3}$$

for $0 < u \in C^0(\overline{\Omega})$ and $0 \le v \in W^{1,2}(\Omega)$, with

$$G(u) := \int_{s_0}^u \int_{s_0}^s \frac{D(\xi)}{S(\xi)} d\xi ds \qquad \text{for } u > 0 \tag{3.4}$$

and any fixed $s_0 \ge 1$, by straightforward computation one can verify that any global classical solution (u,v) of (1.1) fulfilling (1.9) as well as $u > 0$ and $v \ge 0$ in $\overline{\Omega} \times [0,\infty)$ satisfies

$$\frac{d}{dt}\mathcal{F}(u(\cdot,t),v(\cdot,t)) = -\mathcal{D}(u(\cdot,t),v(\cdot,t)) \qquad \text{for all } t > 0 \tag{3.5}$$

(see e.g. [31] for an early discovery in this direction).

Based on this observation, the reasoning in [43] yields the following basic result on nonexistence of global bounded solutions under the assumption (1.13) which, as we already mention at this point (cf. also the proof of Corollary 1.4 below), in the context of (1.2) is satisfied if and only if the supercriticality condition $\sigma > m - \frac{n-2}{n}$, as appearing in the above statement (ii), is satisfied.

Lemma 3.1 *Let $n \ge 2$ and $\Omega = B_R(0) \subset \mathbb{R}^n$ with some $R > 0$. Assume that D and S satisfy (3.1), (1.3) and (1.13) with some $s_0 \ge 1, \vartheta > 0$ and $L_{DS} > 0$. Then for all $m > 0$ there exists $\mathcal{F}_0(m) > 0$ with the property that if u_0 and v_0 are radially symmetric functions on Ω which besides (1.7) satisfy $\int_\Omega u_0 \le m$ as well as*

$$\mathcal{F}(u_0,v_0) < -\mathcal{F}_0(m),$$

then (1.1) does not possess a global classical solution (u,v) fulfilling (1.9) which is bounded in the sense that

$$u \in L^\infty(\Omega \times (0,\infty)). \tag{3.6}$$

Proof. According to the argument in [43, Lemma 3.3, Lemma 3.4], in both cases $n = 2$ and $n \ge 3$ there exists $c_1 > 0$ with the property that whenever $(u_s, v_s) \in (C^2(\overline{\Omega}))^2$ is a stationary classical solution of the boundary-value problem in (1.1) with $u_s \ge 0$ and $v_s \ge 0$, we have

$$\mathcal{F}(u_s, v_s) \ge -c_1 \cdot \left\{ \left\{ \int_\Omega u_s \right\}^2 + 1 \right\}.$$

On the other hand, whenever (u_0, v_0) satisfies (1.7) and (u,v) is a global classical solution of (1.1) fulfilling (1.9) as well as (3.6), it follows from a straightforward reasoning based on (3.5) and parabolic regularity theory that there exist $(t_k)_{k\in\mathbb{N}} \subset (0,\infty)$ and nonnegative elements u_s and v_s of $C^2(\overline{\Omega})$ such that $t_k \to \infty$ and $(u(\cdot,t_k), v(\cdot,t_k)) \to (u_s, v_s)$ in $C^2(\overline{\Omega})$ as $k \to \infty$, and that $\mathcal{F}(u_s,v_s) \le \mathcal{F}(u_0,v_0)$ (cf. the reasoning in [43, Section 2], for instance). Therefore, the claimed conclusion holds if e.g. we define $\mathcal{F}_0(m) := c_1 \cdot (m^2+1)$ for $m > 0$. $\square$

In view of the latter and our results on global existence gained in Theorem 1.1, for the sets $\mathcal{I}$ and $\mathcal{B}$ introduced in Theorem 1.3 we thus have

$$\mathcal{B} \supset \left\{ (u_0,v_0) \in \mathcal{I} \ \middle| \ \int_\Omega u_0 \le m \text{ and } \mathcal{F}(u_0,v_0) < -\mathcal{F}_0(m) \right\},$$

so that for deriving the statements in Theorem 1.3 it is sufficient to make sure that the latter set is conveniently large.

In order to achieve this, following the basic approximation scheme in [44] we let $u_0 \in W^{1,\infty}(\Omega)$ and $v_0 \in W^{1,\infty}(\Omega)$ be given such that both u_0 and v_0 are positive in $\overline{\Omega}$ and radially symmetric with respect to $x = 0$, and modify these functions essentially only in small regions near the origin in such a way that the negative expression $-\int_\Omega uv$ in (3.2) becomes suitably large in modulus. In sharp contrast to the procedure in the latter reference, however, our approach here will involve modifications of this form which may be substantially more singular in the sense that unlike in [44], also the positive contributions to $\mathcal{F}$ may become unbounded in the respective limit.

This fundament of this further development is formed by the following basic construction involving three essentially fixed parameters and one further approximation index: With numbers $\beta > 0, \gamma > 0$ and $\delta \in (0,1)$ to be specified below, for $\eta \in (0,R)$ we define

$$r_\eta := \eta^{1-\delta} \tag{3.7}$$

and

$$\widetilde{u}_\eta(x) := \begin{cases} a_\eta (|x|^2 + \eta^2)^{-\frac{\beta}{2}} & \text{if } x \in B_{r_\eta}(0), \\ u_0(x) & \text{if } x \in \overline{\Omega} \setminus B_{r_\eta}(0) \end{cases} \tag{3.8}$$

and

$$u_\eta(x) := d_\eta \widetilde{u}_\eta(x), \qquad x \in \overline{\Omega}, \qquad \text{with} \qquad d_\eta := \frac{\|u_0\|_{L^1(\Omega)}}{\|\widetilde{u}_\eta\|_{L^1(\Omega)}} \tag{3.9}$$

as well as

$$v_\eta(x) := \begin{cases} b_\eta(|x|^2+\eta^2)^{-\frac{\gamma}{2}} & \text{if } x \in B_{r_\eta}(0), \\ v_0(x) & \text{if } x \in \Omega \setminus B_{r_\eta}(0), \end{cases} \tag{3.10}$$

where

$$a_\eta := (r_\eta^2+\eta^2)^{\frac{\beta}{2}} u_0(r_\eta e_1) \qquad \textit{and} \qquad b_\eta := (r_\eta^2+\eta^2)^{\frac{\gamma}{2}} v_0(r_\eta e_1) \tag{3.11}$$

with $e_1 := (1,0,...,0) \in \mathbb{R}^n$. Then since u_0 and v_0 are positive in $\overline{\Omega}$, it follows that $\widetilde{u}_\eta, u_\eta$ and v_η are radially symmetric and positive functions from $W^{1,\infty}(\Omega)$, and that hence $(u_\eta, v_\eta) \in \mathcal{I}$ for all $\eta \in (0,R)$. Furthermore, the normalization in (3.9) ensures that $\int_\Omega u_\eta = \int_\Omega u_0$.

Our goal will be to select β, γ and δ in such a manner that $\mathcal{F}(u_\eta, v_\eta) \to -\infty$ as $\eta \searrow 0$. The subsequent arguments in this direction will in several places make use of the following simple result from elementary calculus.

Lemma 3.2 *Let $\delta \in (0,1), N > 0$ and $\theta \in \mathbb{R}$. Then*

$$\begin{aligned} \int_0^{R^{-\delta}} \xi^{N-1}(\xi^2+1)^{-\frac{\theta}{2}} d\xi &\le \eta^{\theta-N} \int_0^{r_\eta} r^{N-1}(r^2+\eta^2)^{-\frac{\theta}{2}} dr \\ &\nearrow I(N,\theta) := \int_0^\infty \xi^{N-1}(\xi^2+1)^{-\frac{\theta}{2}} d\xi \qquad \textit{as } \eta \searrow 0, \end{aligned} \tag{3.12}$$

where

$$I(N,\theta) < \infty \qquad \textit{for all } N > 0 \textit{ and } \theta > N. \tag{3.13}$$

Proof. We only need to observe that since $\frac{r_\eta}{\eta} = \eta^{-\delta}$ for all $\eta \in (0,R)$ by positivity of δ, we have $R^{-\delta} \le \frac{r_\eta}{\eta} \nearrow \infty$ as $\eta \searrow 0$. Therefore, on rewriting by substitution according to

$$\eta^{\theta-N} \int_0^{r_\eta} r^{N-1}(r^2+\eta^2)^{-\frac{\theta}{2}} dr = \int_0^{\frac{r_\eta}{\eta}} \xi^{N-1}(\xi^2+1)^{-\frac{\theta}{2}} d\xi \qquad \text{for all } \eta \in (0,R),$$

we directly obtain (3.12), while (3.13) is obvious. □

Let us next make sure that thanks to the positivity and boundedness of u_0 and v_0, the factors defined in (3.11) essentially behave, up to constant multiples, like a certain positive power of η for small values thereof.

Lemma 3.3 *Let $\beta > 0, \gamma > 0$ and $\delta \in (0,1)$. Then*

$$\eta^{\beta-\beta\delta} \inf_{x\in\Omega} u_0(x) \le a_\eta \le (1+R^{2\delta})^{\frac{\beta}{2}} \eta^{\beta-\beta\delta} \sup_{x\in\Omega} u_0(x) \qquad \textit{for all } \eta \in (0,R) \tag{3.14}$$

and

$$\eta^{\gamma-\gamma\delta} \inf_{x\in\Omega} v_0(x) \le b_\eta \le (1+R^{2\gamma})^{\frac{\gamma}{2}} \eta^{\gamma-\gamma\delta} \sup_{x\in\Omega} v_0(x) \qquad \textit{for all } \eta \in (0,R). \tag{3.15}$$

Proof. Again since the positivity of δ ensures that $\frac{\eta}{r_\eta} = \eta^\delta \le R^\delta$ for all $\eta \in (0,R)$ and hence

$$\eta^{2-2\delta} = r_\eta^2 \le r_\eta^2 + \eta^2 \le (1+R^{2\delta}) r_\eta^2 = (1+R^{2\delta})\eta^{2-2\delta} \qquad \text{for all } \eta \in (0,R),$$

both (3.14) and (3.15) are evident from (3.11). □

3.1 L^p convergence of u_η

A first application of (3.14) enables us to derive the following asymptotic upper bound in $L^p(\Omega)$ for the functions in (3.8).

Lemma 3.4 *Let $\beta > 0, \delta \in (0,1)$ and $p > 0$. Then there exists $C > 0$ such that*

$$\int_\Omega \widetilde{u}_\eta^p \le C \cdot (\eta^{n-p\beta\delta} + 1) \qquad \textit{for all } \eta \in (0,R). \tag{3.16}$$

Proof. We first consider the case when $p\beta > n$, in which using Lemma 3.2 and abbreviating $\omega_n := n|B_1(0)|$ we can estimate

$$\begin{aligned} \int_\Omega \widetilde{u}_\eta^p &= \int_{B_{r_\eta}(0)} \widetilde{u}_\eta^p + \int_{\Omega \setminus B_{r_\eta}(0)} u_0^p \\ &\le \omega_n a_\eta^p \int_0^{r_\eta} r^{n-1} (r^2 + \eta^2)^{-\frac{p\beta}{2}} dr + \int_\Omega u_0^p \\ &\le \omega_n a_\eta^p \cdot \eta^{n-p\beta} I(n, p\beta) + \int_\Omega u_0^p \qquad \text{for all } \eta \in (0,R), \end{aligned} \tag{3.17}$$

where $I(n,p\beta)$ is finite thanks to (3.13). Since Lemma 3.3 says that with $c_1 := (1 + R^{2\delta})^{\frac{\beta}{2}} \|u_0\|_{L^\infty(\Omega)}$ we have

$$a_\eta^p \cdot \eta^{n-p\beta} \le (c_1 \eta^{\beta - \beta\delta})^p \eta^{n-p\beta} = c_1^p \eta^{n-p\beta\delta} \qquad \text{for all } \eta \in (0,R),$$

from (3.17) we obtain (3.16) in this case.

If $p\beta \le n$, however, our assumption $\delta < 1$ warrants that $\widetilde{p} := \frac{n}{\beta\delta}$ satisfies $\widetilde{p}\beta = \frac{n}{\delta} > n$, whence from what we just have shown we infer the existence of $c_2 > 0$ fulfilling

$$\int_\Omega \widetilde{u}_\eta^{\widetilde{p}} \le c_2 \cdot (\eta^{n - \widetilde{p}\beta\delta} + 1) = 2c_2 \qquad \text{for all } \eta \in (0,R).$$

As this definition of $\widetilde{p}$ furthermore guarantees that in this case we have $\frac{p}{\widetilde{p}} = \frac{p\beta\delta}{n} \le \frac{n\delta}{n} = \delta < 1$ and thus $p < \widetilde{p}$, Young's inequality becomes applicable so as to show that

$$\int_\Omega \widetilde{u}_\eta^p \le \int_\Omega (\widetilde{u}_\eta^{\widetilde{p}} + 1) \le 2c_2 + |\Omega| \qquad \text{for all } \eta \in (0,R),$$

from which (3.16) trivially follows also for such p. $\square$

This especially entails that if $p \ge 1$ is fixed and $\beta\delta$ is suitably small, then not only the functions $\widetilde{u}_\eta$ but also their renormalized variants u_η approach u_0 in $L^p(\Omega)$:

Lemma 3.5 *Let $\beta > 0$ and $\delta \in (0,1)$. Then whenever $p \ge 1$ is such that*

$$p\beta\delta < n, \tag{3.18}$$

we have

$$\widetilde{u}_\eta \to u_0 \quad \textit{in } L^p(\Omega) \qquad \textit{as } \eta \searrow 0 \tag{3.19}$$

and

$$u_\eta \to u_0 \quad in\ L^p(\Omega) \qquad as\ \eta \searrow 0. \tag{3.20}$$

In particular, if $\beta > 0$ and $\delta \in (0,1)$ satisfy $\beta\delta < n$, then

$$d_\eta \to 1 \qquad as\ \eta \searrow 0, \tag{3.21}$$

and then there exists $C > 0$ such that

$$u_\eta \ge C \quad in\ \Omega \qquad for\ all\ \eta \in (0,R). \tag{3.22}$$

PROOF. Since $r_\eta \searrow 0$ as $\eta \searrow 0$, (3.8) ensures that $\widetilde{u}_\eta(x) \to u_0(x)$ for all $x \in \Omega \setminus \{0\}$ as $\eta \searrow 0$. Since moreover from Lemma 3.4 we know that $(\widetilde{u}_\eta)_{\eta\in(0,R)}$ is bounded in $L^{\widetilde{p}}(\Omega)$ with $\widetilde{p} := \frac{n}{\beta\delta}$ satisfying $\widetilde{p} > p$ by (3.18), the convergence property (3.19) is a consequence of the Vitali convergence theorem. An application to $p := 1$ readily yields (3.21) whenever $\beta\delta < n$, whereupon the conclusion (3.20) under the assumption (3.18) becomes obvious.
Now writing $c_1 := \inf_{x\in\Omega} u_0(x) > 0$, from Lemma 3.3 we know that $a_\eta \ge c_1\eta^{\beta-\beta\delta}$ for all $\eta \in (0,R)$, so that for any such η and $|x| < r_\eta = \eta^{1-\delta}$, in (3.8) we can estimate

$$\widetilde{u}_\eta(x) \ge c_1\eta^{\beta-\beta\delta} \cdot (|x|^2+\eta^2)^{-\frac{\beta}{2}} \ge c_1\eta^{\beta-\beta\delta} \cdot (\eta^{2-2\delta}+\eta^2)^{-\frac{\beta}{2}} = c_1(1+\eta^{2\delta})^{-\frac{\beta}{2}} \ge c_1(1+R^{2\delta})^{-\frac{\beta}{2}}.$$

Since $\inf_{\eta\in(0,R)} d_\eta$ is evidently positive by (3.9) and (3.21), this readily establishes (3.22). □

3.2 $W^{1,q}$ convergence of v_η

We next proceed similar to derive bounds for v_η in $W^{1,q}(\Omega)$ on the basis of the following estimate which once more makes use of Lemma 3.3.

Lemma 3.6 *Let $\gamma > 0, \delta \in (0,1)$ and $q > 0$. Then one can find $C > 0$ with the property that*

$$\int_\Omega |\nabla v_\eta|^q + \int_\Omega v_\eta^q \le C \cdot (\eta^{n-q-q\gamma\delta} + 1) \qquad for\ all\ \eta \in (0,R). \tag{3.23}$$

PROOF. First concentrating on the case when $q(\gamma+1) > n$ and again writing $\omega_n := n|B_1(0)|$, by means of (3.10) and Lemma 3.2 we see that

$$\begin{aligned}
\int_{B_{r_\eta}(0)} |\nabla v_\eta|^q &= \omega_n \int_0^{r_\eta} r^{n-1} \cdot \Big\{ b_\eta \cdot \gamma r (r^2+\eta)^{-\frac{\gamma}{2}-1} \Big\}^q dr \\
&= \gamma^q \omega_n b_\eta^q \int_0^{r_\eta} r^{n+q-1} (r^2+\eta^2)^{-\frac{q(\gamma+2)}{2}} dr \\
&\le \gamma^q \omega_n b_\eta^q \eta^{n-q(\gamma+1)} I(n+q, q(\gamma+2)) \qquad \text{for all } \eta \in (0,R),
\end{aligned} \tag{3.24}$$

with $I(n+q, q(\gamma+2))$ being finite according to (3.13). Since abbreviating $c_1 := (1+R^{2\delta})^{\frac{\gamma}{2}} \|v_0\|_{L^\infty(\Omega)}$ we obtain from (3.15) that

$$\begin{aligned}
b_\eta^q \eta^{n-q(\gamma+1)} &\le (c_1\eta^{\gamma-\gamma\delta})^q \cdot \eta^{n-q(\gamma+1)} \\
&= c_1^q \eta^{n-q-q\gamma\delta} \qquad \text{for all } \eta \in (0,R),
\end{aligned}$$

and since ∇v_0 is bounded in Ω, from (3.24) we infer that for any such q there exists $c_2(q) > 0$ such that

$$\int_\Omega |\nabla v_\eta|^q \le c_2(q) \cdot (\eta^{n-q-q\gamma\delta} + 1) \qquad \text{for all } \eta \in (0,R). \tag{3.25}$$

For smaller values of q, we once again make use of Young's inequality: Namely, if $q(\gamma+1) \le n$ then we let $\widetilde{q} := \frac{n}{1+\gamma\delta}$, so that $\widetilde{q}(\gamma+1) > \widetilde{q}(1+\gamma\delta) = n$ and $\frac{q}{\widetilde{q}} = \frac{q(1+\gamma\delta)}{n} \le \frac{1+\gamma\delta}{\gamma+1} < 1$, whence combining (3.25) with Young's inequality shows that

$$\int_\Omega |\nabla v_\eta|^q \le \int_\Omega |\nabla v_\eta|^{\widetilde{q}} + |\Omega| \le 2c_2(\widetilde{q}) + |\Omega| \qquad \text{for all } \eta \in (0,R). \tag{3.26}$$

As in quite a similar manner it can be verified that for any $q > 0$ one can find $c_3 > 0$ such that

$$\int_\Omega v_\eta^q \le c_3 \cdot (\eta^{n-q\gamma\delta} + 1) \qquad \text{for all } \eta \in (0,R),$$

upon observing that $\eta^{n-q\gamma\delta} \le R^q \eta^{n-q-q\gamma\delta}$ for all $\eta \in (0,R)$ we conclude from (3.25) and (3.26) that indeed (3.23) can be achieved. $\Box$

Again, this entails favorable convergence properties under appropriate assumptions on our free parameters.

Lemma 3.7 *Let $\gamma > 0$, $\delta \in (0,1)$ and $q \ge 1$ be such that*

$$q(1+\gamma\delta) < n. \tag{3.27}$$

Then

$$v_\eta \to v_0 \quad \text{in } W^{1,q}(\Omega) \qquad \text{as } \eta \searrow 0. \tag{3.28}$$

Proof. Once more due to the Vitali convergence theorem, this follows from the boundedness of $(v_\eta)_{\eta\in(0,R)}$ in $W^{1,\frac{n}{1+\gamma\delta}}(\Omega)$ asserted by Lemma 3.6. $\Box$

3.3 Dominance of $\int_\Omega \widetilde{u}_\eta v_\eta$ over $\int_\Omega |\nabla v_\eta|^2$, $\int_\Omega v_\eta^2$ and $\int_\Omega u_\eta^{2-\alpha}$

We next make essential use of the left inequalities in Lemma 3.3 to derive the following lower bound for the crucial integral $\int_\Omega \widetilde{u}_\eta v_\eta$ which, up to the factor d_η controlled through Lemma 3.5, essentially constitutes the negative part of $\mathcal{F}(u_\eta, v_\eta)$.

Lemma 3.8 *For any choice of $\beta > 0, \gamma > 0$ and $\delta > 0$ one can find $C > 0$ such that*

$$\int_\Omega \widetilde{u}_\eta v_\eta \ge C\eta^{n-(\beta+\gamma)\delta} \qquad \textit{for all } \eta \in (0,R). \tag{3.29}$$

In particular, if

$$(\beta+\gamma)\delta > n, \tag{3.30}$$

then

$$\int_\Omega \widetilde{u}_\eta v_\eta \to \infty \qquad \textit{as } \eta \searrow 0. \tag{3.31}$$

PROOF. Relying on the left inequality in (3.12), once more with $\omega_n := n|B_1(0)|$ we estimate

$$\begin{aligned}\int_\Omega \widetilde{u}_\eta v_\eta &\geq \int_{B_{r_\eta}(0)} \widetilde{u}_\eta v_\eta \\ &= \omega_n a_\eta b_\eta \int_0^{r_\eta} r^{n-1}(r^2+\eta^2)^{-\frac{\beta+\gamma}{2}} dr \\ &\geq c_1 a_\eta b_\eta \eta^{n-\beta-\gamma} \qquad \text{for all } \eta \in (0,R) \end{aligned} \tag{3.32}$$

with $c_1 := \omega_n \int_0^{R^{-\delta}} \xi^{n-1}(\xi^2+1)^{-\frac{\beta+\gamma}{2}} d\xi > 0$. Here by Lemma 3.3, writing $c_2 := \inf_{x\in\Omega} u_0(x)$ and $c_3 := \inf_{x\in\Omega} v_0(x)$ we see that

$$a_\eta b_\eta \eta^{n-\beta-\gamma} \geq (c_2\eta^{\beta-\beta\delta}) \cdot (c_3\eta^{\gamma-\gamma\delta}) \cdot \eta^{n-\beta-\gamma} = c_2c_3\eta^{n-(\beta+\gamma)\delta} \qquad \text{for all } \eta \in (0,R),$$

so that (3.32) implies (3.29), from which in turn (3.31) immediately results whenever (3.30) holds. □

Summarizing the outcomes of Lemma 3.6, Lemma 3.4 and Lemma 3.8, we can now state a set of conditions on the parameters β, γ and δ which ensure dominance of $\int_\Omega \widetilde{u}_\eta v_\eta$ over the first two summands $\frac{1}{2}\int_\Omega |\nabla v_\eta|^2$ and $\frac{1}{2}\int_\Omega v_\eta^2$ in $\mathcal{F}(u_\eta, v_\eta)$, as well as over the expression $\int_\Omega \widetilde{u}_\eta^{2-\alpha}$ which up to multiples will essentially control the rightmost integral therein if (1.14) holds (cf. Lemma 3.10).

Lemma 3.9 *Let $\alpha \in (0,2)$, and suppose that $\beta > 0, \gamma > 0$ and $\delta \in (0,1)$ are such that*

$$\gamma > (1-\alpha)\beta \tag{3.33}$$

and

$$\gamma \geq \frac{n-2}{n+2}\beta \tag{3.34}$$

as well as

$$\gamma < \beta - 2 \tag{3.35}$$

and

$$\delta > \frac{2}{\beta-\gamma}. \tag{3.36}$$

Then

$$\frac{\int_\Omega |\nabla v_\eta|^2 + \int_\Omega v_\eta^2 + \int_\Omega \widetilde{u}_\eta^{2-\alpha}}{\int_\Omega \widetilde{u}_\eta v_\eta} \to 0 \qquad \textit{as } \eta \searrow 0. \tag{3.37}$$

PROOF. An application of Lemma 3.6 to $q := 2$ yields $c_1 > 0$ such that

$$\int_\Omega |\nabla v_\eta|^2 + \int_\Omega v_\eta^2 \leq c_1 \cdot (\eta^{n-2-2\gamma\delta} + 1) \qquad \text{for all } \eta \in (0,R),$$

whereas employing Lemma 3.4 with $p := 2-\alpha$ we find $c_2 > 0$ fulfilling

$$\int_\Omega \widetilde{u}_\eta^{2-\alpha} \leq c_2 \cdot (\eta^{n-(2-\alpha)\beta\delta} + 1) \qquad \text{for all } \eta \in (0,R).$$

As Lemma 3.8 provides $c_3 > 0$ satisfying

$$\int_\Omega \widetilde{u}_\eta v_\eta \geq c_3\eta^{n-(\beta+\gamma)\delta} \qquad \text{for all } \eta \in (0,R),$$

we thereby obtain the inequality

$$\frac{\int_\Omega |\nabla v_\eta|^2 + \int_\Omega v_\eta^2 + \int_\Omega \widetilde{u}_\eta^{2-\alpha}}{\int_\Omega \widetilde{u}_\eta v_\eta} \le \frac{1}{c_3}\Big(c_1\eta^{(\beta-\gamma)\delta-2} + c_2\eta^{[\gamma-(1-\alpha)\beta]\delta} + (c_1+c_2)\eta^{(\beta+\gamma)\delta-n}\Big) \tag{3.38}$$

for all $\eta \in (0,R)$. Here we note that $(\beta-\gamma)\delta-2$ and $[\gamma-(1-\alpha)\beta]\delta$ are positive by (3.36) and (3.33), and that

$$(\beta+\gamma)\delta - n > \frac{2(\beta+\gamma)}{\beta-\gamma} - n = \frac{(n+2)\gamma-(n-2)\beta}{\beta-\gamma} \ge 0$$

according to (3.36) when combined with (3.34). Therefore, (3.37) is a consequence of (3.38). $\Box$

3.4 Achieving divergence of $\mathcal{F}(u_\eta,v_\eta)$ to $-\infty$

The remaining task consists in adjusting $\beta > 0, \gamma > 0$ and $\delta \in (0,1)$ in such a way that all the assumptions from Lemma 3.9 are satisfied, and that moreover Lemma 3.5 and Lemma 3.7 become applicable so as to make sure that (u_η, v_η) indeed approximates (u_0,v_0) in a suitable topology. The following main and concluding step of our construction asserts that all these requirements can indeed be fulfilled under the mere and essentially optimal assumption $\alpha > \frac{2}{n}$ in (1.14).

Lemma 3.10 *Let $n \ge 2$. and suppose that D and S satisfy (1.14) with some $\alpha \in (0,2)$ fulfilling $\alpha > \frac{2}{n}$. Moreover, let $p \ge 1$ and $q \ge 1$ satisfy $p < p_0(\alpha)$ and $q < q_0(\alpha)$ with $p_0(\alpha) > 1$ and $q_0(\alpha) > 1$ given by (1.15). Then there exist $\beta > 0, \gamma > 0$ and $\delta \in (0,1)$ such that the functions u_η and v_η defined in (3.9) and (3.10) have the properties that $\int_\Omega u_\eta = \int_\Omega u_0$ for all $\eta \in (0,R)$, that*

$$u_\eta \to u_0 \quad \text{in } L^p(\Omega) \qquad \text{as well as} \qquad v_\eta \to v \quad \text{in } W^{1,q}(\Omega) \qquad \text{as } \eta \searrow 0, \tag{3.39}$$

and that

$$\mathcal{F}(u_\eta, v_\eta) \to -\infty \qquad \text{as } \eta \searrow 0. \tag{3.40}$$

Proof. We first consider the case when $\alpha > \frac{4}{n+2}$, and according to (1.15) we let $p \ge 1$ and $q \ge 1$ be given such that $p < \frac{2n}{n+2}$ and $q < 2$. We then take any $\beta > \frac{n+2}{2}$ and define

$$\gamma := \frac{n-2}{n+2}\beta, \tag{3.41}$$

which in particular ensures that

$$\gamma = \beta - \frac{4}{n+2}\beta < \beta - 2, \tag{3.42}$$

and thus also $\frac{2}{\beta-\gamma} < 1$. Since

$$\frac{n}{\beta\delta} \to \frac{n}{\beta\cdot\frac{2}{\beta-\gamma}} = \frac{2n}{n+2} > p \quad \text{and} \quad \frac{n}{1+\gamma\delta} \to \frac{n}{1+\gamma\cdot\frac{2}{\beta-\gamma}} = 2 > q \qquad \text{as } \delta \searrow \frac{2}{\beta-\gamma},$$

it is therefore possible to pick $\delta \in (0,1)$ such that

$$\delta > \frac{2}{\beta-\gamma} \tag{3.43}$$

as well as

$$\frac{n}{\beta\delta} > p \qquad \text{and} \qquad \frac{n}{1+\gamma\delta} > q. \tag{3.44}$$

Upon these selections, we let $(\widetilde{u}_\eta)_{\eta\in(0,R)} \subset W^{1,\infty}(\Omega)$, $(d_\eta)_{\eta\in(0,R)} \subset \mathbb{R}$, $(u_\eta)_{\eta\in(0,R)} \subset W^{1,\infty}(\Omega)$ and $(v_\eta)_{\eta\in(0,R)} \subset W^{1,\infty}(\Omega)$ be as defined through (3.8), (3.9) and (3.10), respectively, and then obtain from Lemma 3.5 and Lemma 3.7 that thanks to (3.44) we have

$$d_\eta \to 1 \qquad \text{as } \eta \searrow 0 \tag{3.45}$$

as well as

$$\widetilde{u}_\eta \to u_0 \text{ in } L^p(\Omega), \quad u_\eta \to u_0 \text{ in } L^p(\Omega) \quad \text{and} \quad v_\eta \to v_0 \text{ in } W^{1,q}(\Omega) \qquad \text{as } \eta \searrow 0, \tag{3.46}$$

which inter alia establishes (3.39).
To verify (3.40), we make use of (1.14) and (3.22) to find $c_1 > 0$ such that for all $\eta \in (0,R)$,

$$\begin{aligned}
\mathcal{F}(u_\eta, v_\eta) &\le \frac{1}{2}\int_\Omega |\nabla v_\eta|^2 + \frac{1}{2}\int_\Omega v_\eta^2 - \int_\Omega u_\eta v_\eta + c_1 \int_\Omega u_\eta^{2-\alpha} + c_1 \\
&= -d_\eta \int_\Omega \widetilde{u}_\eta v_\eta \cdot \left\{ 1 - \frac{\frac{1}{2}\int_\Omega |\nabla v_\eta|^2 + \frac{1}{2}\int_\Omega v_\eta^2 + c_1 d_\eta^{2-\alpha}\int_\Omega \widetilde{u}_\eta^{2-\alpha}}{d_\eta \int_\Omega \widetilde{u}_\eta v_\eta} \right\} + c_1.
\end{aligned} \tag{3.47}$$

Here since (3.34), (3.35) and (3.36) are satisfied due to (3.41), (3.42) and (3.43), and since

$$\gamma = \Big(1 - \frac{4}{n+2}\Big)\beta > (1-\alpha)\beta$$

by (3.41) and our restriction on α, it follows from Lemma 3.9 and (3.45) that

$$\frac{\frac{1}{2}\int_\Omega |\nabla v_\eta|^2 + \frac{1}{2}\int_\Omega v_\eta^2 + c_1 d_\eta^{2-\alpha}\int_\Omega \widetilde{u}_\eta^{2-\alpha}}{d_\eta \int_\Omega \widetilde{u}_\eta v_\eta} \to 0 \qquad \text{as } \eta \searrow 0. \tag{3.48}$$

As moreover (3.43) and (3.41) guarantee that

$$(\beta+\gamma)\delta > \frac{2(\beta+\gamma)}{\beta-\gamma} = \frac{2\cdot(1+\frac{n-2}{n+2})}{1-\frac{n-2}{n+2}} = n,$$

from Lemma 3.8 we obtain that

$$\int_\Omega \widetilde{u}_\eta v_\eta \to \infty \qquad \text{as } \eta \searrow 0, \tag{3.49}$$

whence again by means of (3.45) we conclude from (3.47) and (3.48) that indeed (3.40) is valid for these choices of β, γ and δ when $\alpha > \frac{4}{n+2}$.

If, conversely, $\alpha \le \frac{4}{n+2}$, but still $\alpha > \frac{2}{n}$ and hence also $\alpha < 1$, and if $p \ge 1$ and $q \ge 1$ are such that $p < \frac{n\alpha}{2}$ and $q < \frac{n\alpha}{2-\alpha}$, we rather fix any $\beta > \frac{2}{\alpha}$ and note that then

$$\gamma - (\beta - 2) \to -\alpha\beta + 2 < 0 \qquad \text{as } \gamma \searrow (1-\alpha)\beta$$

as well as

$$\frac{n(\beta-\gamma)}{2\beta} \to \frac{n\alpha}{2} > p \quad \text{and} \quad \frac{n(\beta-\gamma)}{\beta+\gamma} \to \frac{n\alpha}{2-\alpha} > q \qquad \text{as } \gamma \searrow (1-\alpha)\beta.$$

This enables us to find

$$\gamma > (1-\alpha)\beta \tag{3.50}$$

such that still

$$\gamma < \beta - 2 \tag{3.51}$$

as well as

$$p < \frac{n(\beta-\gamma)}{2\beta} \qquad \text{and} \qquad q < \frac{n(\beta-\gamma)}{\beta+\gamma},$$

and similarly observing that thus

$$\frac{n}{\beta\delta} \to \frac{n(\beta-\gamma)}{2\beta} > p \quad \text{and} \quad \frac{n}{1+\gamma\delta} \to \frac{n(\beta-\gamma)}{\beta+\gamma} > q \qquad \text{as } \delta \searrow \frac{2}{\beta-\gamma},$$

we can finally choose some $\delta \in (0,1)$ fulfilling (3.43) but suitably close to $\frac{2}{\beta-\gamma}$ such that

$$\frac{n}{\beta\delta} > p \qquad \text{and} \qquad \frac{n}{1+\gamma\delta} > q, \tag{3.52}$$

where (3.51) warrants that indeed $\frac{2}{\beta-\gamma} < 1$.
Then again Lemma 3.5 and Lemma 3.7 assert (3.45) and (3.46) due to (3.52), while (3.49) results from Lemma 3.8, because

$$(\beta+\gamma)\delta > \frac{2(\beta+\gamma)}{\beta-\gamma} > \frac{2\cdot(\beta+(1-\alpha)\beta)}{\beta-(1-\alpha)\beta} = \frac{4}{\alpha} - 2 \ge \frac{4}{\frac{4}{n+2}} - 2 = n$$

according to our requirement on α. Moreover, since the latter together with (3.50) implies that

$$\gamma > (1-\alpha)\beta \ge \Big(1 - \frac{4}{n+2}\Big)\beta = \frac{n-2}{n+2}\beta$$

and that thus (3.34) holds, and since (3.33), (3.35) and (3.36) are precisely ensured by (3.50), (3.51) and the validity of (3.43), we may once more employ Lemma 3.9 to see that (3.48) continues to hold in this case. In consequence, (3.39) and (3.40) result from (3.45)-(3.49) also when $\frac{2}{n} < \alpha \le \frac{4}{n+2}$. $\Box$

3.5 Proofs of Theorem 1.3 and of Corollary 1.4

We now only need to combine Lemma 3.10 with our previously gained knowledge on global existence and on the impossibility of global boundedness to describe infinite-time blow-up as a generic property of (1.1) in the intended flavor.

Proof of Theorem 1.3. i) In view of Theorem 1.1 and Lemma 3.1, the claimed density property immediately results on choosing some suitably small $\eta_\star \in (0,R)$ and applying Lemma 3.10 to $\eta \in$

$(\eta_k)_{k\in\mathbb{N}}$ with any sequence $(\eta_k)_{k\in\mathbb{N}} \subset (0,\eta_\star)$ fulfilling $\eta_k \searrow 0$ as $k\to\infty$.

ii) Given $m>0$, we take $\mathcal{F}_0(m)>0$ from Lemma 3.1 and then apply Lemma 3.10 e.g. to the constant function $(\frac{m}{|\Omega|},\frac{m}{|\Omega|})$ to find $(u_0^{(0)},v_0^{(0)})\in\mathcal{I}$ such that $\int_\Omega u_0^{(0)} = \int_\Omega \frac{m}{|\Omega|} = m$ and $\mathcal{F}(u_0^{(0)},v_0^{(0)}) \le -\mathcal{F}_0(m)-1$. Then since $u_0^{(0)}$ is positive in $\overline{\Omega}$, it readily follows from the definition of G that whenever $((u_{0k},v_{0k}))_{k\in\mathbb{N}}\subset\mathcal{I}$ is such that $u_{0k}\to u_0^{(0)}$ in $L^\infty(\Omega)$ and $v_{0k}\to v_0^{(0)}$ in $W^{1,2}(\Omega)$ as $k\to\infty$, then besides the evident properties

$$\frac{1}{2}\int_\Omega |\nabla v_{0k}|^2 \to \frac{1}{2}\int_\Omega |\nabla v_0^{(0)}|^2, \quad \frac{1}{2}\int_\Omega v_{0k}^2 \to \frac{1}{2}\int_\Omega |v_0^{(0)}|^2 \quad \text{and} \quad \int_\Omega u_{0k}v_{0k} \to \int_\Omega u_0^{(0)}v_0^{(0)}$$

we moreover have $G(u_{0k})\to G(u_0^{(0)})$ in $L^\infty(\Omega)$ and hence also

$$\int_\Omega G(u_{0k}) \to \int_\Omega G(u_0^{(0)})$$

as $k\to\infty$. Accordingly, $\mathcal{F}$ is continuous at $(u_0^{(0)},v_0^{(0)})$ with respect to the topology in $L^\infty(\Omega)\times W^{1,2}(\Omega)$, which immediately entails that if $\varepsilon>0$ is appropriately small and $(u_0,v_0)\in\mathcal{I}$ satisfies (1.16), then $\mathcal{F}(u_0,v_0)<\mathcal{F}(u_0^{(0)},v_0^{(0)})+1$ and thus $\mathcal{F}(u_0,v_0)<\mathcal{F}_0(m)$. Therefore, Lemma 3.1 asserts that for any such (u_0,v_0) the globally existing solution from Theorem 1.1 cannot have its first component bounded in $\Omega\times(0,\infty)$. $\square$

The particular conclusion thereof for the prototypical system (1.2) thereby becomes straightforward.

PROOF of Corollary 1.4. In view of Theorem 1.3 we only need to make sure that the present assumptions warrant the validity of (1.13) and (1.14) with some $s_0\ge 1$, $L_{DS}>0$, $\vartheta>0$, $K_{DS}>0$ and $\alpha>\frac{2}{n}$. This, however, readily results from [43, Corollary 5.2 (i) and (iii)]. $\square$

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