Global generalized solutions to a multi-dimensional doubly tactic resource consumption model accounting for social interactions

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Abstract

This work is concerned with a prototypical model for the spatio-temporal evolution of a forager-exploiter system, consisting of two species which simultaneously consume a common nutrient, and which interact through a taxis-type mechanism according to which individuals from the the exploiter subpopulation move upward density gradients of the forager subgroup.

Specifically, the model

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla w), \\ v_t = \Delta v - \chi_2 \nabla \cdot (v \nabla u), \\ w_t = d\Delta v - \lambda(u+v)w - \mu w + r(x,t), \end{cases} \qquad (\star)$$

for the population densities u and v of foragers and exploiters, as well as the nutrient concentration w, is considered in smoothly bounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 1$.

It is firstly shown that under an explicit condition linking the sizes of the resource production rate r and of the initial nutrient concentration, an associated Neumann-type initial-boundary value problem admits a global solution within an appropriate generalized concept. The second of the main results asserts stabilization of these solutions toward spatially homogeneous equilibria in the large time limit, provided that r satisfies a mild assumption on temporal decay.

To the best of our knowledge, these are the first rigorous analytical results addressing taxis-type cross-diffusion mechanisms coupled in a cascade-like manner as in $(\star)$.

Key words: social interaction; chemotaxis; generalized solutions; asymptotic behavior
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1 Introduction

Social interactions have been known to play an essential role in the dynamics of complex systems in various experimental contexts, with typical application areas in biology such as in flocking, swarming or aggregation processes, but also touching a variety of other fields such as economy or crime evolution ([6], [11], [13], [15], [17], [24]). Here in order to understand fundamental principles and strategies at a theoretical level, besides relying on simulations of individual-based models ([15]) the theoretical analysis is more and more built on density-based models using partial differential equations ([10], [23]). Indeed, as paradigmatically illustrated by the celebrated Keller-Segel model for chemotactic aggregation, the latter type of approach brings about the advantage of potentially making questions of collective behavior accessible to methods from qualitative PDE analysis ([18], [30], see also [3]).

Specifically, the interacting dynamics of swarms, when exhibiting collective behavior to search food, have been the objective of [15] and [17], where following the seminal paper [8] the focus is mainly on microscopic scales at which both mechanical and social influences are modeled by taking advantage of general principles of social foraging theory ([13]); beyond this, recent approaches have developed modeling approaches at mesoscopic scales ([6], cf. also [5]), and have shown how the derivation of kinetic models can be obtained as mean field limits from individual based models ([16]). An example involving a fully macroscopic final outcome is constituted in [24] which addresses the evolution of criminality in the search of preys, in a given and usual urban territory; here, namely, the decisive interaction is eventually modeled at macroscopic scales and hence leads to a parabolic system including an attractive cross-diffusive term. A more general survey on Fokker-Planck type methods to model social dynamics can be found in [11], where in accordance with usual features of kinetic theory approaches, the lower scale is that of individual based interactions. Apart from that, some recent developments in the mathematical description of crowd dynamics have taken into account social interactions in the evolution of crowds by modeling consensus towards a commonly shared emotional state ([28]), or accounting for stress onset and propagation in crisis situations ([4]).

The particular objective of the present work is a model for resource consumption in populations consisting of two fractions, where the first of these consists of individuals that directly orient their movement toward increasing food concentrations, and where in contrast to this, the members of the second group arrange their search for food by rather moving upward density gradients of the first subpopulation. In fact, numerical experiments as well as some considerations based on formal analysis ([25]) indicate that the interplay even of such simple mechanisms of "foraging" and "scrounging", conjectured as relevant e.g. for the formation of certain shearwater flocks through attraction to kittiwake foragers observed in Alaska ([17]), may already lead to considerably more complex dynamical behavior than the corresponding single-species taxis-consumption model for which rigorous results have asserted eventual dominance of spatial homogeneity ([26]; cf. also the discussion around (1.2) below).

Specifically, we shall be concerned with a parabolic problem proposed in [25] for the spatio-temporal evolution in such forager-scrounger systems, which after non-dimensionalization amounts to studying the initial-boundary value problem

$$\begin{cases} u_t &= \Delta u - \chi_1 \nabla \cdot (u \nabla w), & x \in \Omega,\ t > 0, \\ v_t &= \Delta v - \chi_2 \nabla \cdot (v \nabla u), & x \in \Omega,\ t > 0, \\ w_t &= d\Delta v - \lambda(u+v)w - \mu w + r(x,t), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), & & x \in \Omega, \end{cases} \tag{1.1}$$

in a smoothly bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, for the unknown population densities $u = u(x,t)$ and $v = v(x,t)$ of foragers and scroungers, respectively, and the resource concentration $w = w(x,t)$. Here in line with the above, the standing assumptions that χ_1, χ_2 and λ be positive reflect the modeling hypotheses that foragers are attracted by food, whereas scroungers follow foragers, with both groups moreover diffusing randomly and consuming the nutrient upon contact. Apart from that, the food resources, supposed to be diffusible by the requirement that $d > 0$, are allowed to spontaneously decay through the assumption that $\mu \geq 0$, and to be renewed from an external repository at a rate $r = r(x,t) \geq 0$.

In accordance with the numerical experiments from [25], but also from a purely mathematical-technical perspective, a substantial increase of complexity seems likely to be expected when passing to (1.1) from the corresponding exploiter-free problem associated with the one-species chemotaxis-consumption system ([22])

$$\begin{cases} u_t &= \Delta u - \chi_1 \nabla \cdot (u \nabla w), & x \in \Omega,\ t > 0, \\ w_t &= d\Delta v - \lambda u w - \mu w + r(x,t), & x \in \Omega,\ t > 0: \end{cases} \tag{1.2}$$

While in (1.2) the attractant within the single taxis mechanism is subject to the significantly dissipative process of consumption through individuals, no such relaxation-enhancing effect influences the evolution of the scrounger population in (1.1). In fact, this partially becomes manifest in some rigorous analytical findings, according to which the prototypical version of (1.2), as obtained on letting $\mu = 0$ and $r \equiv 0$, for all reasonably regular initial data (u_0, w_0) admits global smooth solutions when $n \leq 2$ and global weak solutions when $n = 3$, with all these solutions at least eventually becoming smooth and and approaching the homogeneous equilibrium $(\frac{1}{|\Omega|}\int_\Omega u_0, 0)$ in the large time limit ([26]); in quite drastic contrast to this, systems which such as (1.1) involve sequential taxis mechanisms seem to lack any rigorous theory yet already at the basic level of questions from mere existence theory.

Main results: Global existence and a qualitative description. Accordingly, the goal of the present work consists in providing an apparently first step toward a theoretical understanding of cascade-type taxis interplay in general, and in particular of the specific coupling in the forager-exploiter model from [25]. Here in order to at least partially exceed the scope of fundamental existence theory, our objectives as well include the ambition to yield information on qualitative aspects in some cases of apparent relevance for applications.

In fact, we will firstly derive a result on global existence of certain generalized solutions under an explicit smallness assumption linking the initial nutrient distribution to the food reproduction rate; secondly, we shall thereafter see that similar to the behavior in (1.2), the absence of substantially persistent resource renewal enforces asymptotic homogenization of these solutions. In particular, this will imply that in the latter case of suitably fast nutrient decay, any phenomena related to pattern formation must necessarily be restricted to intermediate time scales; after all, in view of the poor regularity information gathered below for our solutions, it is well conceivable, and thus forming an interesting open topic for further analysis, that such transient structure formation may occur in the extreme sense of finite-time blow-up of some solutions.

Thus subsequently concentrating on (1.1), our standing assumptions on the ingredients therein will be that

$$r \in C^1(\overline{\Omega} \times [0,\infty)) \text{ is nonnegative,} \tag{1.3}$$

and that

$$\begin{cases} u_0 \in W^{1,\infty}(\Omega) \text{ is nonnegative with } u_0 \not\equiv 0, \\ v_0 \in W^{1,\infty}(\Omega) \text{ is nonnegative with } v_0 \not\equiv 0 \qquad \text{and} \\ w_0 \in W^{1,\infty}(\Omega) \text{ is nonnegative.} \end{cases} \tag{1.4}$$

In this setting, the first of our main results indeed asserts global solvability, within a generalized concept extending that introduced in [32] for a single-species chemotaxis system, provided that w_0 and r comply with a fully explicit smallness hypothesis.

Theorem 1.1 *Let $n \ge 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, let $d > 0, \chi_1 > 0, \chi_2 > 0, \lambda > 0$ and $\mu \ge 0$, and let (u_0, v_0, w_0) as well as r be such that apart from (1.3) and (1.4) we have*

$$\sup_{t>0} \left\{ \|w_0\|_{L^\infty(\Omega)} e^{-\mu t} + \int_0^t e^{-\mu(t-s)} \|r(\cdot,s)\|_{L^\infty(\Omega)} ds \right\} < w^\star, \tag{1.5}$$

where the positive constant $w^\star$ is defined by

$$w^\star := \frac{1}{\chi_1} \cdot \begin{cases} \frac{d-1+\sqrt{d^2-d+1}}{2} & \text{if } d \le \frac{7+\sqrt{13}}{6}, \\ \frac{2d(d-1)}{d^2-d+1} & \text{if } \frac{7+\sqrt{13}}{6} < d \le \frac{3+\sqrt{5}}{2}, \\ \sqrt{d} & \text{if } d > \frac{3+\sqrt{5}}{2}. \end{cases} \tag{1.6}$$

Then there exist nonnegative functions

$$\begin{cases} u \in L^\infty((0,\infty); L^4(\Omega)) \cap L^2_{loc}([0,\infty); W^{1,2}(\Omega)), \\ v \in L^\infty((0,\infty); L^1(\Omega)) \qquad \text{and} \\ w \in L^\infty(\Omega \times (0,\infty)) \cap L^2_{loc}([0,\infty); W^{1,2}(\Omega)) \end{cases} \tag{1.7}$$

such that

$$\int_\Omega u(\cdot,t) = \int_\Omega u_0 \quad \text{and} \quad \int_\Omega v(\cdot,t) \le \int_\Omega v_0 \qquad \text{for a.e. } t > 0, \tag{1.8}$$

and that (u,v,w) forms a global generalized solution of (1.1) in the sense of Definition 2.1 below.

Here the hypotheses (1.5) is formulated in such a way that in both cases $\mu = 0$ and $\mu > 0$, some conveniently verifiable criteria on w_0 and r can be identified as sufficient for the above conclusion:

Proposition 1.2 *i) In the case $\mu = 0$, (1.5) holds if and only if*

$$\|w_0\|_{L^\infty(\Omega)} + \int_0^\infty \|r(\cdot,t)\|_{L^\infty(\Omega)} dt < w^\star. \tag{1.9}$$

ii) If $\mu > 0$, then (1.5) is valid whenever

$$\|w_0\|_{L^\infty(\Omega)} < w^\star \qquad \text{and} \qquad \frac{1}{\mu} \cdot \sup_{t>0} \|r(\cdot,t)\|_{L^\infty(\Omega)} < w^\star. \tag{1.10}$$

Proof. While the statement in i) is obvious, the claim in ii) can be seen by observing that if $\mu > 0$ and (1.10) holds, then taking $c_1 \in (0, w^\star)$ such that $\|w_0\|_{L^\infty(\Omega)} < c_1$ and $\frac{1}{\mu}\|r(\cdot,t)\|_{L^\infty(\Omega)} \le c_1$ for all $t > 0$, we can estimate

$$\|w_0\|_{L^\infty(\Omega)} e^{-\mu t} + \int_0^t e^{-\mu(t-s)} \|r(\cdot,s)\|_{L^\infty(\Omega)} ds \le c_1 e^{-\mu t} + c_1 \mu \int_0^t e^{-\mu(t-s)} ds = c_1 \qquad \text{for all } t > 0$$

and conclude as intended. □

Now if r decays suitably fast in time, then all of these solutions approach spatially homogeneous profiles in the large time limit, where in view of the low regularity information on v asserted by Theorem 1.1 it is may not be too surprising that stabilization of this crucial solution component will be asserted only with regard to some quite coarse topology. Here and below, we use the abbreviation $\overline{\varphi} := \frac{1}{|\Omega|} \int_\Omega \varphi$ for $\varphi \in L^1(\Omega)$.

Theorem 1.3 *Suppose that beyond the hypotheses from Theorem 1.1,*

$$\int_0^\infty \|r(\cdot,t)\|_{L^\infty(\Omega)} dt < \infty. \tag{1.11}$$

Then there exist a null set $N \subset (0,\infty)$ and a positive constant v_∞ such that

$$\|u(\cdot,t) - \overline{u}_0\|_{L^4(\Omega)} \to 0 \qquad \text{as } (0,\infty) \setminus N \ni t \to \infty \tag{1.12}$$

as well as

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } (0,\infty) \setminus N \ni t \to \infty, \tag{1.13}$$

and that for each $p \in (0,1)$,

$$\int_t^{t+1} \int_\Omega |v(x,s) - v_\infty|^p dx ds \to 0 \qquad \text{as } t \to \infty. \tag{1.14}$$

An interesting question that has to be left open here is whether the number v_∞ in (1.14), beyond merely known to be positive, can actually be identified with the average $\frac{1}{|\Omega|}\int_\Omega v_0$; indeed, due to lacking information on respective L^1 compactness of corresponding solution components to the regularized systems (2.7) below, our analysis will be unable to exclude the case that for some of the obtained solutions the inequality in (1.8) is strict within a significantly large set of times.

2 Specifying the solution concept and regularizing (1.1)

To begin with, let us specify the solution concept that we plan to pursue throughout the sequel. By requiring v to simultaneously possess a certain weak supersolution property (see (2.5)) as well as an upper limitation of its mass functional formally campatible with (1.1) (see (2.6)), with regard to this crucial second component our concept on the one hand becomes modest enough so as to favorably cooperate with the poor regularity information to be collected therefor below, but on the other hand yet remains consistent with that of classical solvability. In moreover involving the strictly monotone function $\ln(v+1)$ instead of v itself, our notion can be viewed as a far relative of the concept of renormalized solutions coined by Di Perna and Lions ([9]), and precedents in the literature of simpler taxis-type systems can be found in [32], [33] and also in [35], for instance.

Definition 2.1 *A triple of functions*

$$\begin{cases} u \in L^2_{loc}([0,\infty); W^{1,2}(\Omega)), \\ v \in L^1_{loc}(\overline{\Omega}\times[0,\infty)) \qquad \textit{and} \\ w \in L^\infty_{loc}(\overline{\Omega}\times[0,\infty)) \cap L^2_{loc}([0,\infty); W^{1,2}(\Omega)) \end{cases} \tag{2.1}$$

such that

$$\nabla \ln(v+1) \textit{ and } u\nabla w \quad \textit{belong to } L^2_{loc}(\overline{\Omega}\times[0,\infty); R^n) \tag{2.2}$$

will be called a global generalized solution *of (1.1) if for all* $\varphi \in C_0^\infty(\overline{\Omega}\times[0,\infty))$ *the identities*

$$-\int_0^\infty\int_\Omega u\varphi_t - \int_\Omega u_0\varphi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla u\cdot\nabla\varphi + \chi_1\int_0^\infty\int_\Omega u\nabla w\cdot\nabla\varphi \tag{2.3}$$

and

$$\begin{aligned} -\int_0^\infty\int_\Omega w\varphi_t - \int_\Omega w_0\varphi(\cdot,0) \quad &= \quad -d\int_0^\infty\int_\Omega \nabla w\cdot\nabla\varphi - \lambda\int_0^\infty\int_\Omega (u+v)w\varphi \\ &\quad -\mu\int_0^\infty\int_\Omega w\varphi + \int_0^\infty\int_\Omega r\varphi \end{aligned} \tag{2.4}$$

hold, if for each nonnegative $\varphi \in C_0^\infty(\overline{\Omega}\times[0,\infty))$ *the inequality*

$$\begin{aligned} -\int_0^\infty\int_\Omega \ln(v+1)\varphi_t - \int_\Omega \ln(v_0+1)\varphi(\cdot,0) \\ \geq \int_0^\infty\int_\Omega |\nabla\ln(v+1)|^2\varphi - \int_0^\infty\int_\Omega \nabla\ln(v+1)\cdot\nabla\varphi \\ -\chi_2\int_0^\infty\int_\Omega \frac{v}{v+1}\Big(\nabla u\cdot\nabla\ln(v+1)\Big)\varphi + \chi_2\int_0^\infty\int_\Omega \frac{v}{v+1}\nabla u\cdot\nabla\varphi \end{aligned} \tag{2.5}$$

is valid, and if finally

$$\int_\Omega v(\cdot,t) \leq \int_\Omega v_0 \qquad \textit{for a.e. } t>0. \tag{2.6}$$

Remark. It can be verified by straightforward adaptation of the reasoning in [33] that the above notion is consistent with that of classical solvability in the sense that if (u,v,w) is a global generalized solution in the above sense which additionally satisfies $(u,v,w) \in (C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty)))^3$, then (u,v,w) already solves (1.1) classically in $\Omega\times(0,\infty)$.

In order to construct such a solution by means of a suitable approximation procedure, let us conveniently regularize (1.1) by considering the problems

$$\begin{cases} u_{\varepsilon t} &= \Delta u_\varepsilon - \chi_1 \nabla \cdot (u_\varepsilon \nabla w_\varepsilon), & x \in \Omega,\ t>0,\\ v_{\varepsilon t} &= \Delta v_\varepsilon - \chi_2 \nabla \cdot (v_\varepsilon \nabla u_\varepsilon), & x \in \Omega,\ t>0,\\ w_{\varepsilon t} &= d\Delta w_\varepsilon - \lambda \frac{(u_\varepsilon+v_\varepsilon)w_\varepsilon}{1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon} - \mu w_\varepsilon + r(x,t), & x \in \Omega,\ t>0,\\ \frac{\partial u_\varepsilon}{\partial\nu} = \frac{\partial v_\varepsilon}{\partial\nu} = \frac{\partial w_\varepsilon}{\partial\nu} = 0, & & x \in \partial\Omega,\ t>0,\\ u_\varepsilon(x,0)=u_0(x), \quad v_\varepsilon(x,0)=v_0(x), \quad w_\varepsilon(x,0)=w_0(x), & & x\in\Omega, \end{cases} \tag{2.7}$$

for $\varepsilon \in (0,1)$. These are all globally solvable in the classical sense:

Lemma 2.1 *Let $n \ge 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, let $d>0, \chi_1>0, \chi_2>0, \lambda>0$ and $\mu \ge 0$, and suppose that (1.4) and (1.3) hold. Then for each $\varepsilon \in (0,1)$ there exist functions*

$$\begin{cases} u_\varepsilon \in \bigcap_{q>n} C^0([0,\infty); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)),\\ v_\varepsilon \in \bigcap_{q>n} C^0([0,\infty); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)) \qquad \textit{and}\\ w_\varepsilon \in \bigcap_{q>n} C^0([0,\infty); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)) \end{cases} \tag{2.8}$$

which are such that $u_\varepsilon>0, v_\varepsilon>0$ and $w_\varepsilon>0$ in $\overline{\Omega}\times(0,\infty)$, and that $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ solves (2.7) in the classical sense. Moreover,

$$\int_\Omega u_\varepsilon(\cdot,t) = \int_\Omega u_0 \quad \textit{and} \quad \int_\Omega v_\varepsilon(\cdot,t) = \int_\Omega v_0 \qquad \textit{for all } t>0. \tag{2.9}$$

Proof. Straightforward adaptation of Amann's theory ([1]) yields local existence of a classical solution within a regularity class corresponding to that in (2.8), maximally extensible up to some $T_{max,\varepsilon} \in (0,\infty]$ which is such that if $T_{max,\varepsilon} < \infty$, then

$$\limsup_{t\nearrow T_{max,\varepsilon}} \Big\{ \|u_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} + \|v_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} + \|w_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} \Big\} = \infty \qquad \text{for all } q>n. \tag{2.10}$$

Moreover, successive applications of the strong maximum principle ensure positivity of all solution components in $\overline{\Omega}\times(0,T_{max,\varepsilon})$, and integrating the first two equations from (2.7) shows that $\frac{d}{dt}\int_\Omega u_\varepsilon = \frac{d}{dt}\int_\Omega v_\varepsilon = 0$ for all $t \in (0,T_{max,\varepsilon})$, hence implying the identities in (2.9) throughout $(0,T_{max,\varepsilon})$.

Now if $T_{max,\varepsilon}$ was finite, then thanks to the boundedness of r in $\Omega\times(0,T_{max,\varepsilon})$ and of $[0,\infty) \ni \xi \mapsto \lambda\frac{\xi}{1+\varepsilon\xi}$ we could apply a well-known parabolic gradient estimate ([19]) as well as standard maximal Sobolev regularity theory ([12]) to the third equation in (2.7) to see that w_ε would belong to $X_p := L^\infty((0,T_{max,\varepsilon}); W^{1,\infty}(\Omega)) \cap L^p((0,T_{max,\varepsilon}); W^{2,p}(\Omega))$ for all $p\in(1,\infty)$. By the same token, we could conclude from the regularity properties of the coefficient functions $a(x,t) := -\chi_1\nabla w_\varepsilon$ and $b(x,t) := -\chi_1\Delta w_\varepsilon$ in the equation $u_{\varepsilon t} = \Delta u_\varepsilon + a(x,t)\cdot\nabla u_\varepsilon + b(x,t)u_\varepsilon$, as thereby implied, that also $u_\varepsilon \in X_p$ for each $p\in(1,\infty)$. This enables us to repeat this argument in the second equation from (2.7) so as to conclude that also $v_\varepsilon \in X_p$ for any such p, thus contradicting (2.10) and hence showing that $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ actually must be global. □

3 Bounds for $u_\varepsilon, \nabla u_\varepsilon, u_\varepsilon \nabla u_\varepsilon$ and $u_\varepsilon \nabla w_\varepsilon$ implied by smallness of w_ε

The goal of this section is to reveal some favorable implications of our overall assumption (1.5)-(1.6) on regularity properties of u_ε and w_ε. This will be achieved in Lemma 3.3 and Lemma 3.4 on the basis of the following essentially immediate consequence of a comparison argument.

Lemma 3.1 *Assume (1.5) with some $w^\star > 0$. Then*

$$\sup_{\varepsilon\in(0,1)} \sup_{t>0} \|w_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} < w^\star. \tag{3.1}$$

Proof. We let $\overline{w}(x,t) := y(t)$ for $x \in \overline{\Omega}$ and $t \ge 0$, where

$$y(t) := \|w_0\|_{L^\infty(\Omega)} e^{-\mu t} + \int_0^t e^{-\mu(t-s)} \|r(\cdot,s)\|_{L^\infty(\Omega)} ds, \qquad t \ge 0.$$

Then for all $\varepsilon \in (0,1)$,

$$\begin{aligned} \overline{w}_t - d\Delta\overline{w} + \lambda \cdot \frac{(u_\varepsilon+v_\varepsilon)w_\varepsilon}{1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon} + \mu\overline{w} - r(x,t) &= y'(t) + \mu y(t) - r(x,t) \\ &\ge y'(t) + \mu y(t) - \|r(\cdot,t)\|_{L^\infty(\Omega)} \\ &= 0 \qquad \text{in } \Omega \times (0,\infty), \end{aligned}$$

so that since clearly $\frac{\partial \overline{w}}{\partial \nu}|_{\partial\Omega\times(0,\infty)} = 0$, an application of the comparison principle asserts that $\overline{w} \ge w_\varepsilon$ in $\Omega \times (0,\infty)$. In particular,

$$\|w_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le y(t) \le c_1 := \sup_{\tilde{t}>0} y(\tilde{t}) \qquad \text{for all } t>0 \text{ and } \varepsilon \in (0,1),$$

with c_1 satisfying $c_1 < w^\star$ according to (1.5). □

As can be seen by relying on a series of preparatory elementary arguments detailed in the appendix, namely, the latter entails a pointwise estimate that will play a crucial role in our derivation of Lemma 3.3:

Lemma 3.2 *Suppose that (1.5) holds with $w^\star > 0$ determined through (1.6). Then there exists $p_0 > 4$ such that for each $p \in [2,p_0]$ one can find $\kappa > 0, \delta > w^\star, \eta \in (0,1)$ and $C > 0$ such that for all $\varepsilon \in (0,1)$,*

$$\kappa(\kappa+1)d - p\kappa\chi_1(\delta - w_\varepsilon) - \frac{\Big\{ -p\kappa(d+1) + p(p-1)\chi_1(\delta-w_\varepsilon)\Big\}^2}{4p(p-1)\eta} \ge C \qquad \text{in } \Omega\times(0,\infty). \tag{3.2}$$

In fact, the above parameter selections enable us to conclude from (1.5)-(1.6) that for any $p \in [2,p_0]$ a certain expression of the form $\int_\Omega u_\varepsilon^p (\delta - w_\varepsilon)^{-\kappa}$, as already utilized in various related studies ([21], [26], [31], [34]), plays the role of a quasi-energy functional also in the present and somewhat more complex context. Here we state the essence of this observation in such a manner that besides implying bounds useful for our existence theory through Lemma 3.4, it can later on once more be recalled so as to provide some basic information on decay of the associated dissipation rate under the assumptions from Theorem 1.3 (see Lemma 6.2).

Lemma 3.3 *Assume (1.5) with $w^\star > 0$ as given by (1.6). Then there exists $p_0 > 4$ such that for all $p \in [2, p_0]$ it is possible to find $\kappa > 0, \delta > w^\star$ and $C > 0$ such that whenever $\varepsilon \in (0,1)$,*

$$\frac{d}{dt}\int_\Omega u_\varepsilon^p(\delta-w_\varepsilon)^{-\kappa}+\frac{1}{C}\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2+\frac{1}{C}\int_\Omega u_\varepsilon^2|\nabla w_\varepsilon|^2 \le C\|r\|_{L^\infty(\Omega)}\int_\Omega u_\varepsilon^p(\delta-w_\varepsilon)^{-\kappa} \qquad \text{for all } t>0. \tag{3.3}$$

Proof. In accordance with Lemma 3.2, we fix $p_0 > 4$ with the property that for all $p \in [2, p_0]$ we can choose $\kappa > 0, \delta > w^\star, \eta \in (0,1)$ and $c_1 > 0$ fulfilling

$$\kappa(\kappa+1)d-p\kappa\chi_1(\delta-w_\varepsilon)-\frac{\Big\{-p\kappa(d+1)+p(p-1)\chi_1(\delta-w_\varepsilon)\Big\}^2}{4p(p-1)\eta} \ge c_1 \quad \text{in } \Omega\times(0,\infty) \quad \text{for all } \varepsilon\in(0,1). \tag{3.4}$$

Now given $p \in [2, p_0]$, we take κ, δ and η as above and use the first and third equations in (2.7) to see that since κ, λ and μ are nonnegative,

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u_\varepsilon^p(\delta-w_\varepsilon)^{-\kappa} &= p\int_\Omega u_\varepsilon^{p-1}(\delta-w_\varepsilon)^{-\kappa}\nabla\cdot\Big\{\nabla u_\varepsilon-\chi_1 u_\varepsilon\nabla w_\varepsilon\Big\} \\
&\quad +\kappa\int_\Omega u_\varepsilon^p(\delta-w_\varepsilon)^{-\kappa-1}\cdot\Big\{d\Delta w_\varepsilon-\lambda\cdot\frac{(u_\varepsilon+v_\varepsilon)w_\varepsilon}{1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon}-\mu w_\varepsilon+r\Big\} \\
&\le -p(p-1)\int_\Omega u_\varepsilon^{p-2}(\delta-w_\varepsilon)^{-\kappa}|\nabla u_\varepsilon|^2-p\kappa\int_\Omega u_\varepsilon^{p-1}(\delta-w_\varepsilon)^{-\kappa-1}\nabla u_\varepsilon\cdot\nabla w_\varepsilon \\
&\quad +p(p-1)\chi_1\int_\Omega u_\varepsilon^{p-1}(\delta-w_\varepsilon)^{-\kappa}\nabla u_\varepsilon\cdot\nabla w_\varepsilon+p\kappa\chi_1\int_\Omega u_\varepsilon^p(\delta-w_\varepsilon)^{-\kappa-1}|\nabla w_\varepsilon|^2 \\
&\quad -p\kappa d\int_\Omega u_\varepsilon^{p-1}\nabla u_\varepsilon\cdot\nabla w_\varepsilon-\kappa(\kappa+1)d\int_\Omega u_\varepsilon^p(\delta-w_\varepsilon)^{-\kappa-2}|\nabla w_\varepsilon|^2 \\
&\quad +\kappa\int_\Omega u_\varepsilon^p(\delta-w_\varepsilon)^{-\kappa-1}r \\
&= -p(p-1)\int_\Omega u_\varepsilon^{p-2}(\delta-w_\varepsilon)^{-\kappa}|\nabla u_\varepsilon|^2 \\
&\quad -\int_\Omega\Big\{\kappa(\kappa+1)d-p\kappa\chi_1(\delta-w_\varepsilon)\Big\}u_\varepsilon^p(\delta-w_\varepsilon)^{-\kappa-2}|\nabla w_\varepsilon|^2 \\
&\quad +\int_\Omega\Big\{-p\kappa(d+1)+p(p-1)\chi_1(\delta-w_\varepsilon)\Big\}u_\varepsilon^{p-1}(\delta-w_\varepsilon)^{-\kappa-1}\nabla u_\varepsilon\cdot\nabla w_\varepsilon \\
&\quad +\kappa\int_\Omega u_\varepsilon^p(\delta-w_\varepsilon)^{-\kappa-1}r \qquad \text{for all } t>0.
\end{aligned} \tag{3.5}$$

Here employing Young's inequality we find that

$$\begin{aligned}
&\int_\Omega\Big\{-p\kappa(d+1)+p(p-1)\chi_1(\delta-w_\varepsilon)\Big\}u_\varepsilon^{p-1}(\delta-w_\varepsilon)^{-\kappa-1}\nabla u_\varepsilon\cdot\nabla w_\varepsilon \\
&\qquad \le p(p-1)\eta\int_\Omega u_\varepsilon^{p-2}(\delta-w_\varepsilon)^{-\kappa}|\nabla u_\varepsilon|^2 \\
&\qquad\quad +\int_\Omega\frac{\Big\{-p\kappa(d+1)+p(p-1)\chi_1(\delta-w_\varepsilon)\Big\}^2}{4p(p-1)\eta}u_\varepsilon^p(\delta-w_\varepsilon)^{-\kappa-2}|\nabla w_\varepsilon|^2 \qquad \text{for all } t>0,
\end{aligned}$$

whence due to (3.4) we infer from (3.5) that

$$\frac{d}{dt}\int_\Omega u_\varepsilon^p(\delta-w_\varepsilon)^{-\kappa}+(1-\eta)p(p-1)\int_\Omega u_\varepsilon^{p-2}(\delta-w_\varepsilon)^{-\kappa}|\nabla u_\varepsilon|^2+c_1\int_\Omega u_\varepsilon^p(\delta-w_\varepsilon)^{-\kappa-2}|\nabla w_\varepsilon|^2$$
$$\le \kappa\int_\Omega u_\varepsilon^p(\delta-w_\varepsilon)^{-\kappa-1}r \qquad \text{for all } t>0.$$

We therefore directly obtain (3.3) upon observing that

$$(1-\eta)p(p-1)\int_\Omega u_\varepsilon^{p-2}(\delta-w_\varepsilon)^{-\kappa}|\nabla u_\varepsilon|^2 \ge (1-\eta)p(p-1)\delta^{-\kappa}\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 \qquad \text{for all } t>0$$

and

$$c_1\int_\Omega u_\varepsilon^p(\delta-w_\varepsilon)^{-\kappa-2}|\nabla w_\varepsilon|^2 \ge c_1\delta^{-\kappa-2}\int_\Omega u_\varepsilon^p|\nabla w_\varepsilon|^2 \qquad \text{for all } t>0,$$

and that

$$\kappa\int_\Omega u_\varepsilon^p(\delta-w_\varepsilon)^{-\kappa-1}r \le \frac{\kappa}{\delta-w^\star}\|r\|_{L^\infty(\Omega)}\int_\Omega u_\varepsilon^p(\delta-w_\varepsilon)^{-\kappa} \qquad \text{for all } t>0,$$

because $\eta<1$ and $\delta>w^\star$. $\square$

Among all possible implications achievable through appropriate integration of the inequality in (3.3), those of interest for us in the construction of global solutions will be the following.

Lemma 3.4 *Assume (1.5) with $w^\star>0$ as given by (1.6). Then there exists $p_0>4$ with the property that for all $T>0$ one can find $C(T)>0$ fulfilling*

$$\int_\Omega u_\varepsilon^{p_0}(\cdot,t)\le C(T) \qquad \textit{for all } t>0 \textit{ and } \varepsilon\in(0,1) \tag{3.6}$$

and

$$\int_0^T\int_\Omega|\nabla u_\varepsilon|^2\le C(T) \qquad \textit{for all } \varepsilon\in(0,1) \tag{3.7}$$

as well as

$$\int_0^T\int_\Omega u_\varepsilon^2|\nabla u_\varepsilon|^2\le C(T) \qquad \textit{for all } \varepsilon\in(0,1) \tag{3.8}$$

and

$$\int_0^T\int_\Omega u_\varepsilon^2|\nabla w_\varepsilon|^2\le C(T) \qquad \textit{for all } \varepsilon\in(0,1). \tag{3.9}$$

Proof. We take $p_0>4$ as provided by Lemma 3.3 and then infer from the latter that for each $p\in[2,p_0]$ we can pick $\kappa=\kappa(p)>0, \delta=\delta(p)>w^\star, c_1(p)>0$ and $c_2(p)>0$ such that whenever $\varepsilon\in(0,1)$,

$$\frac{d}{dt}\int_\Omega u_\varepsilon^p(\delta-w_\varepsilon)^{-\kappa}+c_1(p)\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2+c_1(p)\int_\Omega u_\varepsilon^p|\nabla w_\varepsilon|^2$$
$$\le c_2(p)\|r\|_{L^\infty(\Omega)}\int_\Omega u_\varepsilon^p(\delta-w_\varepsilon)^{-\kappa} \qquad \text{for all } t>0. \tag{3.10}$$

Since for fixed $T > 0$ we may rely on (1.3) in choosing $c_3(p,T) > 0$ fulfilling

$$c_2(p)\|r(\cdot,t)\|_{L^\infty(\Omega)} \le c_3(p,T) \qquad \text{for all } t \in (0,T),$$

integrating (3.10) firstly shows that

$$\int_\Omega u_\varepsilon^p(\delta - w_\varepsilon)^{-\kappa} \le c_4(p,T) := \left\{ \int_\Omega u_0^p(\delta - w_0)^{-\kappa} \right\} \cdot e^{c_3(p,T)T} \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1), \quad (3.11)$$

and, as a consequence thereof, secondly implies that

$$\begin{aligned} c_1(p)\int_0^T \int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 + c_1(p)\int_0^T \int_\Omega u_\varepsilon^p |\nabla w_\varepsilon|^2 &\le c_3(p,T)\int_0^T \int_\Omega u_\varepsilon^p(\delta - w_\varepsilon)^{-\kappa} \\ &\le c_3(p,T)c_4(p,T)\cdot T \qquad \text{for all } \varepsilon \in (0,1). \quad (3.12) \end{aligned}$$

When restricted to $p = p_0$, (3.11) yields (3.6) due to the fact that $(\delta - w_\varepsilon)^{-\kappa} \ge \delta^{-\kappa}$ in $\Omega \times (0,\infty)$, whereas evaluating (3.12) for $p = 2$ and $p = 4$, respectively, shows that (3.7) as well as (3.8) and (3.9) are valid. □

4 Further integrability properties. Construction of a limit (u, v, w)

4.1 Estimates for ∇w_ε

The following observation, though rather straightforward, will be of importance both in our existence theory and in our asymptotic analysis later on (see Lemma 6.1).

Lemma 4.1 *Assume (1.5) with some $w^\star > 0$. Then*

$$\int_0^T \int_\Omega |\nabla w_\varepsilon|^2 \le \frac{1}{2d}\int_\Omega w_0^2 + \frac{|\Omega| w^\star}{d}\int_0^T \|r(\cdot,t)\|_{L^\infty(\Omega)}dt \qquad \textit{for all } T > 0 \textit{ and } \varepsilon \in (0,1). \qquad (4.1)$$

Proof. We test the third equation in (2.7) by w_ε and use the nonnegativity of λ and μ as well as Lemma 3.1 to estimate

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega w_\varepsilon^2 + d\int_\Omega |\nabla w_\varepsilon|^2 &= -\lambda \int_\Omega \frac{(u_\varepsilon + v_\varepsilon)w_\varepsilon}{1 + \varepsilon(u_\varepsilon + v_\varepsilon)w_\varepsilon} - \mu\int_\Omega w_\varepsilon^2 + \int_\Omega r w_\varepsilon \\ &\le |\Omega| \cdot \|w_\varepsilon\|_{L^\infty(\Omega)}\|r\|_{L^\infty(\Omega)} \\ &\le |\Omega| w^\star \|r\|_{L^\infty(\Omega)} \qquad \text{for all } t > 0, \end{aligned}$$

from which (4.1) readily follows by integration. □

Of immediate relevance for the limit procedure in (2.7) will be the following direct implication of the latter.

Corollary 4.2 *Assume (1.5) with some $w^\star > 0$. Then for all $T > 0$ there exists $C(T) > 0$ such that*

$$\int_0^T \int_\Omega |\nabla w_\varepsilon|^2 \le C(T) \qquad \textit{for all } \varepsilon \in (0,1). \qquad (4.2)$$

Proof. Since r is bounded in $\Omega \times (0,T)$ by (1.3), this is obvious from Lemma 4.1. □

4.2 Bounds for ∇v_ε

In comparison to this, for the crucial second solution component much less regularity information seems available. Precisely reflecting the additional complexification due to the presence of cascade-type taxis coupling, namely, the regularity of v_ε is apparently linked in quite a close manner to that of ∇u_ε, about which, in turn, our knowledge is limited to the outcome of Lemma 3.4. As in any multi-dimensional setting the spatio-temporal L^2 bound therefor provided by the latter seems far from sufficient to ensure e.g. any time-independent L^p bounds for v_ε through standard testing procedures, however, our considerations in this direction will essentially be restricted to drawing appropriate conclusions from the following basic observation. The parameter therein will be specified by setting $a = 1$ throughout our existence analysis, whereas further application in the course of our convergence argument will, inter alia, involve the choice $a = 0$ (Lemma 6.11).

Lemma 4.3 *Assume (1.5) with some $w^\star > 0$. Then for any choice of $a \ge 0$,*

$$\frac{d}{dt}\int_\Omega \ln(v_\varepsilon + a) \ge \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+a)^2} - \frac{\chi_2^2}{2}\int_\Omega |\nabla u_\varepsilon|^2 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1). \tag{4.3}$$

Proof. Using the second equation in (2.7) along with Young's inequality, we see that indeed

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \ln(v_\varepsilon + a) &= \int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+a)^2} - \chi_2 \int_\Omega \frac{v_\varepsilon}{(v_\varepsilon+a)^2}\nabla u_\varepsilon\cdot\nabla v_\varepsilon \\
&\ge \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+a)^2} - \frac{\chi_2^2}{2}\int_\Omega \frac{v_\varepsilon^2}{(v_\varepsilon+a)^2}|\nabla u_\varepsilon|^2 \\
&\ge \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+a)^2} - \frac{\chi_2^2}{2}\int_\Omega |\nabla u_\varepsilon|^2 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),
\end{aligned}$$

as claimed. □

When integrated upon letting $a = 1$, due to (2.9) the latter implies the following inequality which, beyond preparing a bound favorable for our limit procedure, will also be used in Lemma 6.9 to assert some spatial homogenization property of the second solution component.

Lemma 4.4 *Assume (1.5) with some $w^\star > 0$. Then*

$$\int_0^T\int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2} \le 2\int_\Omega v_0 + 2|\Omega| + \chi_2^2\int_0^T\int_\Omega |\nabla u_\varepsilon|^2 \qquad \text{for all } T>0 \text{ and } \varepsilon\in(0,1). \tag{4.4}$$

Proof. We apply Lemma 4.3 to $a = 1$ and integrate the corresponding version of (4.3) to obtain that for arbitrary $T > 0$,

$$\frac{1}{2}\int_0^T\int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2} \le \int_\Omega \ln\Big(v_\varepsilon(\cdot,T)+1\Big) - \int_\Omega \ln(v_0+1) + \frac{\chi_2^2}{2}\int_0^T\int_\Omega |\nabla u_\varepsilon|^2 \qquad \text{for all } \varepsilon\in(0,1).$$

Since $0 \le \ln(z+1) \le z$ for all $z \ge 0$, and since thus

$$\int_\Omega \ln\Big(v_\varepsilon(\cdot,T)+1\Big) - \int_\Omega \ln(v_0+1) \le \int_\Omega \Big(v_\varepsilon(\cdot,T)+1\Big) = \int_\Omega v_0 + |\Omega| \qquad \text{for all } \varepsilon\in(0,1)$$

according to (2.9), this clearly yields (4.4). □

Through Lemma 3.4, this especially entails the following quite directly.

Lemma 4.5 *Assume (1.5) with $w^\star > 0$ as given by (1.6). Then for all $T > 0$ there exists $C(T) > 0$ such that*

$$\int_0^T \int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2} \le C(T) \qquad \text{for all } \varepsilon \in (0,1). \tag{4.5}$$

Proof. In view of (3.7), this is an immediate consequence of Lemma 4.4. □

4.3 Uniform integrability of the nonlinear absorption in (2.7)

A next crucial step toward our solution construction consists in asserting appropriate compactness properties of the nonlinear absorption term in the third equation from (2.7). This will be achieved on the basis of a further testing procedure, examining the time evolution of $\int_\Omega w_\varepsilon \ln(v_\varepsilon+1)$ and thereby following an idea apparently going back to [33], which thanks to the estimates provided by Lemma 3.4, Lemma 4.4 and Corollary 4.2 will yield the following.

Lemma 4.6 *Assume (1.5) with $w^\star > 0$ as given by (1.6). Then for all $T > 0$ there exists $C(T) > 0$ such that*

$$\int_0^T \int_\Omega \frac{(u_\varepsilon+v_\varepsilon)w_\varepsilon \ln(v_\varepsilon+1)}{1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon} \le C(T) \qquad \text{for all } \varepsilon \in (0,1). \tag{4.6}$$

Proof. By means of the second and third equations from (2.7), we compute

$$\begin{aligned}
\frac{d}{dt}\int_\Omega w_\varepsilon \ln(v_\varepsilon+1) &= \int_\Omega \Big\{ d\Delta w_\varepsilon - \lambda \cdot \frac{(u_\varepsilon+v_\varepsilon)w_\varepsilon}{1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon} - \mu w_\varepsilon + r \Big\} \cdot \ln(v_\varepsilon+1) \\
&\quad + \int_\Omega \frac{w_\varepsilon}{v_\varepsilon+1} \nabla \cdot \Big\{ \nabla v_\varepsilon - \chi_2 v_\varepsilon \nabla u_\varepsilon \Big\} \\
&= -d \int_\Omega \frac{\nabla v_\varepsilon}{v_\varepsilon+1} \cdot \nabla w_\varepsilon - \lambda \int_\Omega \frac{(u_\varepsilon+v_\varepsilon)w_\varepsilon)\ln(v_\varepsilon+1)}{1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon} \\
&\quad -\mu \int_\Omega w_\varepsilon \ln(v_\varepsilon+1) + \int_\Omega r \ln(v_\varepsilon+1) \\
&\quad - \int_\Omega \frac{\nabla v_\varepsilon}{v_\varepsilon+1} \cdot \nabla w_\varepsilon + \int_\Omega w_\varepsilon \frac{|\nabla v_\varepsilon}{(v_\varepsilon+1)^2} \\
&\quad + \chi_2 \int_\Omega \frac{v_\varepsilon}{v_\varepsilon+1} \nabla u_\varepsilon \cdot \nabla w_\varepsilon - \chi_2 \int_\Omega \frac{v_\varepsilon w_\varepsilon}{v_\varepsilon+1} \nabla u_\varepsilon \cdot \frac{\nabla v_\varepsilon}{v_\varepsilon+1} \qquad \text{for all } t>0.
\end{aligned} \tag{4.7}$$

Here by Young's inequality,

$$-d \int_\Omega \frac{\nabla v_\varepsilon}{v_\varepsilon+1} \cdot \nabla w_\varepsilon - \int_\Omega \frac{\nabla v_\varepsilon}{v_\varepsilon+1} \cdot \nabla w_\varepsilon \le \int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2} + \frac{(d+1)^2}{4} \int_\Omega |\nabla w_\varepsilon|^2 \qquad \text{for all } t>0 \tag{4.8}$$

and

$$\begin{aligned}
\chi_2 \int_\Omega \frac{v_\varepsilon}{v_\varepsilon+1} \nabla u_\varepsilon \cdot \nabla w_\varepsilon &\le \chi_2 \int_\Omega |\nabla u_\varepsilon| \cdot |\nabla w_\varepsilon| \\
&\le \int_\Omega |\nabla u_\varepsilon|^2 + \frac{\chi_2^2}{4} \int_\Omega |\nabla w_\varepsilon|^2 \qquad \text{for all } t>0
\end{aligned} \tag{4.9}$$

as well as

$$
\begin{aligned}
-\chi_2 \int_\Omega \frac{v_\varepsilon w_\varepsilon}{v_\varepsilon+1} \nabla u_\varepsilon \cdot \frac{\nabla v_\varepsilon}{v_\varepsilon+1} &\le \chi_2 w^\star \int_\Omega |\nabla u_\varepsilon| \cdot \frac{|\nabla v_\varepsilon|}{v_\varepsilon+1} \\
&\le \int_\Omega |\nabla u_\varepsilon|^2 + \frac{\chi_2^2 (w^\star)^2}{4} \int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2} \qquad \text{for all } t>0, \qquad (4.10)
\end{aligned}
$$

because $w_\varepsilon \le w^\star$ in $\Omega \times (0,\infty)$ by Lemma 3.1. For the same reason,

$$
\int_\Omega w_\varepsilon \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2} \le w^\star \int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2} \qquad \text{for all } t>0, \qquad (4.11)
$$

and clearly

$$
-\mu \int_\Omega w_\varepsilon \ln(v_\varepsilon+1) \le 0 \qquad \text{for all } t>0, \qquad (4.12)
$$

while moreover

$$
\begin{aligned}
\int_\Omega r \ln(v_\varepsilon+1) &\le \|r\|_{L^\infty(\Omega)} \int_\Omega \ln(v_\varepsilon+1) \\
&\le \|r\|_{L^\infty(\Omega)} \int_\Omega (v_\varepsilon+1) \\
&\le \|r\|_{L^\infty(\Omega)} \cdot \Big\{ \int_\Omega v_0 + |\Omega| \Big\} \qquad \text{for all } t>0 \qquad (4.13)
\end{aligned}
$$

due to (2.9) and, again, the validity of $\ln(z+1) \le z$ for all $z \ge 0$.
From (4.7)-(4.13) we therefore obtain after an integration that

$$
\begin{aligned}
\lambda \int_0^T \int_\Omega \frac{(u_\varepsilon+v_\varepsilon) w_\varepsilon \ln(v_\varepsilon+1)}{1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon} &\le \int_\Omega w_0 \ln(v_0+1) \\
&\quad +2 \int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \\
&\quad + \Big\{ 1 + \frac{\chi_2^2 (w^\star)^2}{4} + w^\star \Big\} \cdot \int_0^T \int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2} \\
&\quad + \Big\{ \frac{(d+1)^2}{4} + \frac{\chi_2^2}{4} \Big\} \cdot \int_0^T \int_\Omega |\nabla w_\varepsilon|^2 \\
&\quad + \Big\{ \int_\Omega v_0 + |\Omega| \Big\} \cdot \int_0^T \|r(\cdot,t)\|_{L^\infty(\Omega)} dt \qquad \text{for all } T>0,
\end{aligned}
$$

which by Lemma 3.4, Lemma 4.4 and Corollary 4.2, and once more by boundedness of r in $\Omega \times (0,T)$, results in (4.6). □

Now a careful analysis reveals that the above estimate, together with some the superlinear integrability properties of u_ε implied Lemma 3.4, is sufficient to ensure equi-integrability of the expressions in question.

Lemma 4.7 *Assume (1.5) with $w^\star > 0$ as given by (1.6), and let $T > 0$. Then*

$$\left(\frac{(u_\varepsilon v_\varepsilon) w_\varepsilon}{1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon} \right)_{\varepsilon\in(0,1)} \qquad \text{is uniformly integrable over } \Omega\times(0,T). \tag{4.14}$$

PROOF. For fixed $T > 0$, according to Lemma 4.6 and Lemma 3.4 let us pick $c_1 > 0$ and $c_2 > 0$ such that

$$\int_0^T \int_\Omega \frac{(u_\varepsilon+v_\varepsilon)w_\varepsilon \ln(v_\varepsilon+1)}{1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon} \le c_1 \qquad \text{for all } \varepsilon\in(0,1), \tag{4.15}$$

and that

$$\int_0^T \int_\Omega u_\varepsilon^2 \le c_2 \qquad \text{for all } \varepsilon\in(0,1). \tag{4.16}$$

Given $\eta > 0$, we then take $M > 0$ large enough satisfying

$$\frac{c_1}{\ln(M+1)} \le \frac{\eta}{3}, \tag{4.17}$$

and thereafter we choose $\delta > 0$ suitably small such that

$$w^\star M\delta \le \frac{\eta}{3} \tag{4.18}$$

as well as

$$c_2^{\frac{1}{2}} w^\star \delta^{\frac{1}{2}} \le \frac{\eta}{3}. \tag{4.19}$$

Now if $E \subset \Omega\times(0,T)$ is an arbitrary measurable set fulfilling $|E| \le \delta$, we split

$$\begin{aligned} \int\int_E \frac{(u_\varepsilon+v_\varepsilon)w_\varepsilon}{1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon} &= \int\int_{E\cap\{v_\varepsilon\ge M\}} \frac{(u_\varepsilon+v_\varepsilon)w_\varepsilon}{1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon} \\ &\quad + \int\int_{E\cap\{v_\varepsilon< M\}} \frac{v_\varepsilon w_\varepsilon}{1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon} \\ &\quad + \int\int_{E\cap\{v_\varepsilon< M\}} \frac{u_\varepsilon w_\varepsilon}{1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon} \qquad \text{for } \varepsilon\in(0,1), \end{aligned} \tag{4.20}$$

where using the monotonicity of $0 \le z \mapsto \ln(z+1)$ along with (4.15) and (4.17) we may estimate

$$\begin{aligned} \int\int_{E\cap\{v_\varepsilon\ge M\}} \frac{(u_\varepsilon+v_\varepsilon)w_\varepsilon}{1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon} &\le \frac{1}{\ln(M+1)} \int\int_{E\cap\{v_\varepsilon\ge M\}} \frac{(u_\varepsilon+v_\varepsilon)w_\varepsilon \ln(v_\varepsilon+1)}{1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon} \\ &\le \frac{1}{\ln(M+1)}\cdot c_1 \\ &\le \frac{\eta}{3} \qquad \text{for all } \varepsilon\in(0,1). \end{aligned} \tag{4.21}$$

In the second summand on the right of (4.20), we rather rely on (4.18) to see that thanks to Lemma 3.1 and the trivial inequality $1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon \ge 1$,

$$\begin{aligned} \int\int_{E\cap\{v_\varepsilon> M\}} \frac{v_\varepsilon w_\varepsilon}{1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon} &\le Mw^\star \cdot \Big| E\cap\{v_\varepsilon<M\}\Big| \\ &\le Mw^\star\delta \\ &\le \frac{\eta}{3} \qquad \text{for all } \varepsilon\in(0,1), \end{aligned} \tag{4.22}$$

while, finally, combining Lemma 3.1 with (4.16) and (4.19) shows that due to the Cauchy-Schwarz inequality,

$$
\begin{aligned}
\int\int_{E\cap\{v_\varepsilon<M\}} \frac{u_\varepsilon w_\varepsilon}{1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon} &\le w^\star \int\int_E u_\varepsilon \\
&\le w^\star \cdot \left\{ \int_0^T \int_\Omega u_\varepsilon^2 \right\}^{\frac{1}{2}} \cdot |E|^{\frac{1}{2}} \\
&\le w^\star c_2^{\frac{1}{2}} \delta^{\frac{1}{2}} \\
&\le \frac{\eta}{3} \qquad \text{for all } \varepsilon \in (0,1). \qquad (4.23)
\end{aligned}
$$

Inserting (4.21)-(4.23) into (4.20) shows that for any such set E we have

$$
\int\int_E \frac{(u_\varepsilon+v_\varepsilon)w_\varepsilon}{1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon} \le \frac{\eta}{3}+\frac{\eta}{3}+\frac{\eta}{3}=\eta \qquad \text{for all } \varepsilon\in(0,1),
$$

and that thus (4.14) follows from the fact that $\eta>0$ was arbitrary. □

4.4 Regularity of time derivatives

The following implications of our above estimates on regularity of time derivatives in (2.7) can be obtained in a rather straightforward manner.

Lemma 4.8 *Suppose that (1.5) holds with $w^\star>0$ as in (1.6), and let $m\in\mathbb{N}$ be such that $m>\frac{n}{2}$. Then for all $T>0$ there exists $C(T)>0$ such that*

$$
\int_0^T \|u_{\varepsilon t}(\cdot,t)\|^2_{(W^{1,2}(\Omega))^\star} dt \le C(T) \qquad \textit{for all } \varepsilon\in(0,1) \qquad (4.24)
$$

and

$$
\int_0^T \Big\|\partial_t \ln\Big(v_\varepsilon(\cdot,t)+1\Big)\Big\|_{(W^{m,2}(\Omega))^\star} dt \le C(T) \qquad \textit{for all } \varepsilon\in(0,1) \qquad (4.25)
$$

as well as

$$
\int_0^T \|w_{\varepsilon t}(\cdot,t)\|_{(W^{m,2}(\Omega))^\star} dt \le C(T) \qquad \textit{for all } \varepsilon\in(0,1). \qquad (4.26)
$$

Proof. For fixed $t>0$ and $\psi\in C^\infty(\overline{\Omega})$ we obtain from (2.7) and the Cauchy-Schwarz inequality that

$$
\begin{aligned}
\left|\int_\Omega u_{\varepsilon t}(\cdot,t)\cdot\psi\right| &= \left| -\int_\Omega \nabla u_\varepsilon\cdot\nabla\psi + \chi_1\int_\Omega u_\varepsilon \nabla w_\varepsilon\cdot\nabla\psi\right| \\
&\le \left\{ \left\{\int_\Omega |\nabla u_\varepsilon|^2\right\}^{\frac{1}{2}} + \chi_1 \left\{\int_\Omega u_\varepsilon^2|\nabla w_\varepsilon|^2\right\}^{\frac{1}{2}} \right\} \cdot \|\nabla\psi\|_{L^2(\Omega)} \qquad \text{for all } \varepsilon\in(0,1),
\end{aligned}
$$

which by Young's inequality shows that

$$
\|u_{\varepsilon t}(\cdot,t)\|^2_{(W^{1,2}(\Omega))^\star} \le 2\int_\Omega |\nabla u_\varepsilon|^2 + 2\chi_1^2 \int_\Omega u_\varepsilon^2|\nabla w_\varepsilon|^2 \qquad \text{for all } \varepsilon\in(0,1)
$$

and thereby implies (4.24) upon integrating and using Lemma 3.4.

Similarly, by means of the second equation in (2.7) an Young's inequality we see that for $t > 0$ and $\psi \in C^\infty(\overline{\Omega})$,

$$\begin{aligned}
\left|\int_\Omega \partial_t \ln\Big(v_\varepsilon(\cdot,t)+1\Big)\cdot\psi\right| &= \left|\int_\Omega \frac{\psi}{v_\varepsilon+1}\nabla\cdot\Big\{\nabla v_\varepsilon - \chi_2 v_\varepsilon \nabla u_\varepsilon\Big\}\right| \\
&= \bigg|\int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2}\psi - \int_\Omega \frac{\nabla v_\varepsilon}{v_\varepsilon+1}\cdot\nabla\psi \\
&\quad -\chi_2\int_\Omega \frac{v_\varepsilon}{v_\varepsilon+1}\Big(\nabla u_\varepsilon\cdot\frac{\nabla v_\varepsilon}{v_\varepsilon+1}\Big)\psi + \chi_2\int_\Omega \frac{v_\varepsilon}{v_\varepsilon+1}\nabla u_\varepsilon\cdot\nabla\psi\bigg| \\
&\le \bigg\{\int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2}\bigg\}\cdot\|\psi\|_{L^\infty(\Omega)} \\
&\quad +\bigg\{\int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2} + \frac14\bigg\}\cdot\|\nabla\psi\|_{L^2(\Omega)} \\
&\quad +\chi_2\cdot\bigg\{\int_\Omega |\nabla u_\varepsilon|^2 + \frac14\int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2}\bigg\}\cdot\|\psi\|_{L^\infty(\Omega)} \\
&\quad +\chi_2\cdot\bigg\{\int_\Omega |\nabla u_\varepsilon|^2 + \frac14\bigg\}\cdot\|\nabla\psi\|_{L^2(\Omega)} \qquad \text{for all } \varepsilon\in(0,1).
\end{aligned}$$

Since our assumption $m > \frac{n}{2}$ ensures that $W^{m,2}(\Omega) \hookrightarrow L^\infty(\Omega)$, this entails the existence of $c_1 > 0$ such that

$$\Big\|\partial_t \ln\Big(v_\varepsilon(\cdot,t)+1\Big)\Big\|_{(W^{m,2}(\Omega))^\star} \le c_1\cdot\bigg\{\int_\Omega |\nabla u_\varepsilon|^2 + \int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2} + 1\bigg\} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),$$

so that (4.24) becomes a consequence of Lemma 3.4 and Lemma 4.5.

Finally, combining the third equation in (2.7) with Lemma 3.1, (2.9) and Young's inequality we find that for $t > 0$ and $\psi \in C^\infty(\overline{\Omega})$,

$$\begin{aligned}
\left|\int_\Omega w_{\varepsilon t}(\cdot,t)\cdot\psi\right| &= \left|-d\int_\Omega \nabla w_\varepsilon\cdot\nabla\psi - \lambda\int_\Omega \frac{(u_\varepsilon+v_\varepsilon)w_\varepsilon}{1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon}\psi - \mu\int_\Omega w_\varepsilon\psi + \int_\Omega r\psi\right| \\
&\le d\cdot\bigg\{\int_\Omega |\nabla w_\varepsilon|^2 + \frac14\bigg\}\cdot\|\nabla\psi\|_{L^2(\Omega)} + \lambda w^\star\cdot\bigg\{\int_\Omega u_0 + \int_\Omega v_0\bigg\}\cdot\|\psi\|_{L^\infty(\Omega)} \\
&\quad +\mu|\Omega|w^\star\cdot\|\psi\|_{L^\infty(\Omega)} + |\Omega|\cdot\|r\|_{L^\infty(\Omega)}\cdot\|\psi\|_{L^\infty(\Omega)} \qquad \text{for all } \varepsilon\in(0,1),
\end{aligned}$$

whence again by continuity of the embedding $W^{m,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ we infer that with some $c_2 > 0$ we have

$$\|w_{\varepsilon t}(\cdot,t)\|_{(W^{m,2}(\Omega))^\star} \le c_2\cdot\bigg\{\int_\Omega |\nabla w_\varepsilon|^2 + \|r\|_{L^\infty(\Omega)} + 1\bigg\} \qquad \text{for all } t>0 \text{ and any } \varepsilon\in(0,1),$$

and that thus (4.26) results from Corollary 4.2 and the local boundedness of r in $\overline{\Omega}\times[0,\infty)$. □

4.5 Constructing (u, v, w) by passing to the limit. Solution properties of u and w

An essentially straightforward exploitation of the integrability featues collected so far enables us to extract appropriately convergent subsequences and a corresponding limit triple (u, v, w) which in its first and third component can already at this stage be seen to comply with the requirements from Definition 2.1.

Lemma 4.9 *Let (1.5) be satisfied with $w^\star > 0$ as given by (1.6). Then there exist $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ and nonnegative functions u, v and w defined a.e. in $\Omega \times (0,\infty)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$, that with some $p_0 > 0$ we have*

$$\begin{cases} u \in L^\infty((0,\infty); L^{p_0}(\Omega)) \cap L^2_{loc}([0,\infty); W^{1,2}(\Omega)), \\ v \in L^\infty((0,\infty); L^1(\Omega)) \qquad \textit{and} \\ w \in L^\infty(\Omega \times (0,\infty)) \cap L^2_{loc}([0,\infty); W^{1,2}(\Omega)), \end{cases} \tag{4.27}$$

that

$$\nabla \ln(v+1) \quad \textit{and} \quad u\nabla w \quad \textit{belong to } L^2_{loc}([0,\infty); W^{1,2}(\Omega)), \tag{4.28}$$

and that

$$u_\varepsilon \to u \qquad \textit{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)) \quad \textit{and a.e. in } \Omega \times (0,\infty), \tag{4.29}$$
$$u_\varepsilon(\cdot,t) \to u(\cdot,t) \qquad \textit{in } L^2(\Omega) \quad \textit{for a.e. } t > 0, \tag{4.30}$$
$$\nabla u_\varepsilon \rightharpoonup \nabla u \qquad \textit{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)), \tag{4.31}$$
$$u_\varepsilon \nabla u_\varepsilon \rightharpoonup u\nabla u \qquad \textit{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)), \tag{4.32}$$
$$v_\varepsilon \to v \qquad \textit{a.e. in } \Omega \times (0,\infty), \tag{4.33}$$
$$\ln(v_\varepsilon + 1) \rightharpoonup \ln(v+1) \qquad \textit{in } L^2_{loc}([0,\infty); W^{1,2}(\Omega)), \tag{4.34}$$
$$w_\varepsilon \to w \qquad \textit{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)) \quad \textit{and a.e. in } \Omega \times (0,\infty), \tag{4.35}$$
$$w_\varepsilon(\cdot,t) \to w(\cdot,t) \qquad \textit{in } L^2(\Omega) \quad \textit{for a.e. } t > 0, \tag{4.36}$$
$$w_\varepsilon \overset{\star}{\rightharpoonup} w \qquad \textit{in } L^\infty(\Omega \times (0,\infty)), \tag{4.37}$$
$$\nabla w_\varepsilon \rightharpoonup \nabla w \qquad \textit{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)), \tag{4.38}$$
$$\frac{(u_\varepsilon + v_\varepsilon) w_\varepsilon}{1 + \varepsilon(u_\varepsilon + v_\varepsilon) w_\varepsilon} \to (u+v)w \qquad \textit{in } L^1_{loc}(\overline{\Omega} \times [0,\infty)) \qquad \textit{and} \tag{4.39}$$
$$u_\varepsilon \nabla w_\varepsilon \rightharpoonup u\nabla w \qquad \textit{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)) \tag{4.40}$$

as $\varepsilon = \varepsilon_j \searrow 0$. Moreover, (2.6) holds as well as

$$\int_\Omega u(\cdot,t) = \int_\Omega u_0 \qquad \textit{for a.e. } t > 0, \tag{4.41}$$

and the identities (2.3) and (2.4) in Definition 2.1 are satisfied for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$.

Proof. According to Lemma 3.4 and Lemma 4.8,

$(u_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^2(0,T); W^{1,2}(\Omega))$ and $(u_{\varepsilon t})_{\varepsilon\in(0,1)}$ is bounded in $L^2((0,T); (W^{1,2}(\Omega))^\star)$

for all $T > 0$, whence employing an Aubin-Lions lemma ([27]) we obtain $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ and a nonnegative function $u \in L^2_{loc}([0,\infty); W^{1,2}(\Omega))$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$ and that (4.29), (4.30) and (4.31) hold. Apart from that, for all $T > 0$ it follows from Lemma 3.4 that $(u_\varepsilon \nabla u_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^2(\Omega \times (0,T))$, so that (4.32) results upon observing that $u_\varepsilon \nabla u_\varepsilon \rightharpoonup u\nabla u$ in $L^1(\Omega \times (0,T))$ by (4.29) and (4.31). Due to (3.6) and Fatou's lemma, it moreover follows that with $p_0 > 4$ taken from Lemma 3.4 we have $u \in L^\infty((0,\infty); L^{p_0}(\Omega))$, and (4.41) is an abvious consequence of (2.9) when combined with (4.29).

Likewise, choosing any integer $m > \frac{n}{2}$ we obtain from Lemma 4.5 together with (2.9) and Lemma 4.8 that

$$\Big(\ln(v_\varepsilon + 1)\Big)_{\varepsilon\in(0,1)} \text{ is bounded in } L^2(0,T); W^{1,2}(\Omega)) \quad \text{and}$$
$$\Big(\partial_t \ln(v_\varepsilon + 1)\Big)_{\varepsilon\in(0,1)} \text{ is bounded in } L^2((0,T); (W^{m,2}(\Omega))^\star) \qquad \text{for all } T > 0,$$

so that again an appropriate Aubin-Lions lemma ([27]) applies so as to assert that along a suitable subsequence we can furthermore achieve (4.33) and (4.34), in particular implying (2.6) through (2.9) and Fatou's lemma, and validity of the first claim in (4.28).

Next, Corollary 4.2 in conjunction with Lemma 3.1 and Lemma 4.8 says that

$$(w_\varepsilon)_{\varepsilon\in(0,1)} \text{ is bounded in } L^2(0,T); W^{1,2}(\Omega)) \quad \text{and} \quad (w_{\varepsilon t})_{\varepsilon\in(0,1)} \text{ is bounded in } L^2((0,T); (W^{m,2}(\Omega))^\star)$$

for all $T > 0$, and that thus another application of an Aubin-Lions lemma enables us to establish, possibly after passing to a further subsequence, (4.35), (4.36) and (4.38), whereupon in view of (3.1), the Banach-Alaoglu theorem facilitates (4.37).

According to the pointwise convergence features asserted in (4.29), (4.33) and (4.35), thanks to the Vitali convergence theorem the claim in (4.39) thereafter results from the uniform integrability property derived in Lemma 4.7, and (4.40) as well as the second inclusion claimed in (4.28) can be seen by making use of (3.8), which for all $T > 0$ namely warrants relative compactness of $(u_\varepsilon \nabla w_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^2(\Omega \times (0,T); \mathbb{R}^n)$ with respect to the weak topology therein, and thus shows that as $\varepsilon = \varepsilon_j \searrow 0$ we indeed must have $u_\varepsilon \nabla w_\varepsilon \rightharpoonup u\nabla w$ in $L^2(\Omega \times (0,T))$, for from (4.29) and (4.38) we already know that $u_\varepsilon \nabla w_\varepsilon \rightharpoonup u\nabla w$ in $L^1(\Omega \times (0,T))$.

Finally, a verification of (2.3) and (2.4) can be performed on the basis of (4.31), (4.35), (4.38), (4.39) and (4.40) in a straightforward manner, so that we may omit giving details on this here. □

5 Strong L^2 convergence of ∇w_{ε_j} and ∇u_{ε_j}. Solution properties of v

Now for a verification of the weak inequality in (2.5), the weak convergence features in (4.31) and (4.34) are apparently insufficient for an appropriate limit procedure targeting at the second last summand therein. Of essential importance for our approach in this direction will thus be a corresponding strong convergence property of ∇u_{ε_j} our derivation of which will, in turn, rely on an associated statement on strong L^2 convergence of the signal gradient. Both these results, to be successively achieved in Lemma 5.1 and Lemma 5.2, will later play useful roles in our asymptotic analysis as well (see Lemma 6.3).

As a preparation for both Lemma 5.1 and Lemma 5.2, given $t_0 > 0$ and $\delta \in (0,1)$ let us set

$$\zeta_\delta(t) \equiv \zeta_\delta^{(t_0)}(t) := \begin{cases} 1 & \text{if } t \in [0,t_0], \\ 1 - \frac{t-t_0}{\delta} & \text{if } t \in (t_0, t_0+\delta), \\ 0 & \text{if } t \geq t_0+\delta, \end{cases} \tag{5.1}$$

and recall the well-known fact that for $T > 0$, $p \in [1,\infty]$, $N \geq 1$ and $\psi \in L^p_{loc}(\overline{\Omega} \times \mathbb{R}; \mathbb{R}^N)$, the Steklov averages $S_h\psi \in L^p(\Omega \times (0,T); \mathbb{R}^N)$, $h \in (0,1)$, given by

$$(S_h\psi)(x,t) := \frac{1}{h} \int_{t-h}^t \psi(x,s)ds, \qquad x \in \Omega,\ t \in (0,T),\ h \in (0,1), \tag{5.2}$$

have the properties that as $h \searrow 0$, $S_h\psi \to \psi$ in $L^p(\Omega \times (0,T))$ if $p \in [1,\infty)$ and $S_h\psi \overset{\star}{\rightharpoonup} \psi$ in $L^\infty(\Omega \times (0,T))$ if $p = \infty$.

Now using these ingredients in constructing appropriate test functions for the weak identity satisfied by w, inspired by a related procedure performed in [33] we can indeed achieve the following.

Lemma 5.1 *Assume (1.5) with $w^\star > 0$ taken from (1.6), and let $(\varepsilon_j)_{j\in\mathbb{N}}$ and w be as in Lemma 4.9. Then for all $T > 0$,*

$$\nabla w_\varepsilon \to \nabla w \quad \textit{in } L^2(\Omega \times (0,T)) \qquad \textit{as } \varepsilon = \varepsilon_j \searrow 0. \tag{5.3}$$

PROOF. Following [33, Section 8], given $T > 0$ we rely on Lemma 4.9 in choosing $t_0 > T$ such that t_0 is a Lebesgue point of $0 < t \mapsto \int_\Omega w^2(\cdot,t)$, and that moreover

$$\int_\Omega w_\varepsilon^2(\cdot,t_0) \to \int_\Omega w^2(\cdot,t_0) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0. \tag{5.4}$$

Apart from that, we take any sequence $(w_{0k})_{k\in\mathbb{N}} \subset C^1(\overline{\Omega})$ such that $w_{0k} \to w_0$ in $L^2(\Omega)$ as $k \to \infty$, and for $\delta \in (0,1)$, $k \in \mathbb{N}$ and $h \in (0,1)$ we let

$$\varphi(x,t) := \zeta_\delta(t) \cdot (S_h\widehat{w}_k)(x,t), \qquad x \in \Omega,\ t > 0,$$

where ζ_δ and S_h are as in (5.1) and (5.2), and where

$$\widehat{w}_k(x,t) := \begin{cases} w(x,t) & \text{if } x \in \Omega \text{ and } t > 0, \\ w_{0k}(x) & \text{if } x \in \Omega \text{ and } t \leq 0. \end{cases}$$

Then $\varphi \in L^\infty(\Omega \times (0,\infty)) \cap L^2((0,\infty); W^{1,2}(\Omega))$ with $\varphi_t \in L^\infty(\Omega \times (0,\infty))$ and $\varphi \equiv 0$ in $\Omega \times (t_0+1,\infty)$, whence a standard completion argument shows that (2.4) extends so as to remain valid for this particular choice of φ, hence resulting in the identity

$$\begin{aligned} &-\int_0^\infty \int_\Omega \zeta_\delta'(t) w(x,t) (S_h\widehat{w}_k)(x,t) dxdt \\ &-\int_0^\infty \int_\Omega \zeta_\delta(t) w(x,t) \cdot \frac{\widehat{w}_k(x,t) - \widehat{w}_k(x,t-h)}{h} dxdt - \int_\Omega w_0(x) w_{0k}(x) dx \end{aligned}$$

$$
\begin{aligned}
= \quad & -d\int_0^\infty\int_\Omega \zeta_\delta(t)\nabla w(x,t)\cdot\nabla(S_h\widehat{w}_k)(x,t)dxdt \\
& -\lambda\int_0^\infty\int_\Omega \zeta_\delta(t)\Big(u(x,t)+v(x,t)\Big)w(x,t)(S_h\widehat{w}_k)(x,t)dxdt \\
& -\mu\int_0^\infty\int_\Omega \zeta_\delta(t)w(x,t)(S_h\widehat{w}_k)(x,t)dxdt \\
& +\int_0^\infty\int_\Omega \zeta_\delta(t)r(x,t)(S_h\widehat{w}_k)(x,t)dxdt \\
& \qquad\qquad\qquad\qquad \text{for all } \delta\in(0,1), k\in\mathbb{N} \text{ and } h\in(0,1).
\end{aligned} \tag{5.5}
$$

Here the first summand on the left as well as each of the integrals on the right-hand side approach their expected limits as $h\searrow 0$, because the inclusions $\nabla\widehat{w}_k\in L^2(\Omega\times(0,t_0+1);\mathbb{R}^n)$ and $\widehat{w}_k\in L^\infty(\Omega\times(0,t_0+1))$ ensure that $\nabla(S_h\widehat{w}_k)=S_h(\nabla\widehat{w}_k)\rightharpoonup\nabla\widehat{w}_k=\nabla w$ in $L^2(\Omega\times(0,t_0+1))$ and $S_h\widehat{w}_k\stackrel{\star}{\rightharpoonup}\widehat{w}_k=w$ in $L^\infty(\Omega\times(0,t_0+1))$ as $h\searrow 0$.
Moreover, by Young's inequality and a linear substitution we obtain that

$$
\begin{aligned}
& -\int_0^\infty\int_\Omega \zeta_\delta(t)w(x,t)\cdot\frac{\widehat{w}_k(x,t)-\widehat{w}_k(x,t-h)}{h}dxdt \\
& = \quad -\frac{1}{h}\int_0^\infty\int_\Omega \zeta_\delta(t)\widehat{w}_k^2(x,t)dxdt+\frac{1}{h}\zeta_\delta(t)\widehat{w}_k(x,t)\widehat{w}_k(x,t-h)dxdt \\
& \le \quad -\frac{1}{2h}\int_0^\infty\int_\Omega \zeta_\delta(t)\widehat{w}_k^2(x,t)dxdt+\frac{1}{2h}\int_0^\infty\int_\Omega \zeta_\delta(t)\widehat{w}_k^2(x,t-h)dxdt \\
& = \quad \frac{1}{2}\int_0^\infty\int_\Omega \frac{\zeta_\delta(s+h)-\zeta_\delta(s)}{h}w^2(x,s)dxds+\frac{1}{2}\int_\Omega w_{0k}^2(x)dx \qquad \text{for all } \delta\in(0,1), k\in\mathbb{N} \text{ and } h\in(0,1),
\end{aligned}
$$

where by the dominated convergence theorem,

$$
\frac{1}{2}\int_0^\infty\int_\Omega \frac{\zeta_\delta(s+h)-\zeta_\delta(s)}{h}w^2(x,s)dxds \to \frac{1}{2}\zeta_\delta'(s)w^2(x,s)dxds \qquad \text{as } h\searrow 0.
$$

Since $\zeta_\delta'\equiv-\frac{1}{\delta}$ in $(t_0,t_0+\delta)$ and $\zeta_\delta'\equiv 0$ in $(0,t_0)\cup(t_0+\delta,\infty)$, in the limit $h\searrow 0$ from (5.5) we therefore obtain the inequality

$$
\begin{aligned}
& \frac{1}{2\delta}\int_{t_0}^{t_0+\delta}\int_\Omega w^2(x,t)dxdt+\frac{1}{2}\int_\Omega w_{0k}^2(xdx-\int_\Omega w_0(x)w_{0k}(x)dx \\
& \ge \quad -d\int_0^\infty\int_\Omega \zeta_\delta(t)|\nabla w(x,t)|^2dxdt-\lambda\int_0^\infty\int_\Omega \zeta_\delta(t)\Big(u(x,t)+v(x,t)\Big)w^2(x,t)dxdt \\
& \qquad -\mu\int_0^\infty\int_\Omega \zeta_\delta(t)w^2(x,t)dxdt+\int_0^\infty\int_\Omega \zeta_\delta(t)r(x,t)w(x,t)dxdt \qquad \text{for all } \delta\in(0,1) \text{ and } k\in\mathbb{N},
\end{aligned}
$$

which on taking $k\to\infty$ implies that since $w_{0k}\to w_0$ in $L^2(\Omega)$,

$$
d\int_0^\infty\int_\Omega \zeta_\delta(t)|\nabla w(x,t)|^2dxdt \quad \ge \quad -\frac{1}{2\delta}\int_{t_0}^{t_0+\delta}\int_\Omega w^2(x,t)dxdt+\frac{1}{2}\int_\Omega w_0^2(x)dx
$$

$$
\begin{aligned}
&-\lambda\int_0^\infty\int_\Omega \zeta_\delta(t)\Big(u(x,t)+v(x,t)\Big)w^2(x,t)dxdt\\
&-\mu\int_0^\infty\int_\Omega \zeta_\delta(t)w^2(x,t)dxdt\\
&+\int_0^\infty\int_\Omega \zeta_\delta(t)r(x,t)w(x,t)dxdt \qquad \text{for all } \delta\in(0,1).
\end{aligned}
$$

Here the Lebesgue point property of t_0 enters so as to warrant, when combined with several applications of the dominated convergence theorem, that

$$
\begin{aligned}
d\int_0^{t_0}\int_\Omega |\nabla w|^2 \ \geq\ & -\frac{1}{2}\int_\Omega w^2(\cdot,t_0)+\frac{1}{2}\int_\Omega w_0^2\\
&-\lambda\int_0^{t_0}\int_\Omega (u+v)w^2-\mu\int_0^{t_0}\int_\Omega w^2+\int_0^{t_0}\int_\Omega rw.
\end{aligned}
\tag{5.6}
$$

Now since (5.4) along with (4.39), (4.37) and (4.35) implies that

$$
\begin{aligned}
&-\frac{1}{2}\int_\Omega w^2(\cdot,t_0)+\frac{1}{2}\int_\Omega w_0^2-\lambda\int_0^{t_0}\int_\Omega (u+v)w^2-\mu\int_0^{t_0}\int_\Omega w^2+\int_0^{t_0}\int_\Omega rw\\
&= \lim_{\varepsilon=\varepsilon_j\searrow 0}\left\{-\frac{1}{2}\int_\Omega w_\varepsilon^2(\cdot,t_0)+\frac{1}{2}\int_\Omega w_0^2-\lambda\int_0^{t_0}\int_\Omega \frac{(u_\varepsilon+v_\varepsilon)w_\varepsilon}{1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon}\cdot w_\varepsilon-\mu\int_0^{t_0}\int_\Omega w_\varepsilon^2+\int_0^{t_0}\int_\Omega rw_\varepsilon\right\}\\
&= \lim_{\varepsilon=\varepsilon_j\searrow 0}\left\{d\int_0^{t_0}\int_\Omega |\nabla w_\varepsilon|^2\right\}
\end{aligned}
$$

according to (2.7), from (5.6) we infer that $\int_0^{t_0}\int_\Omega |\nabla w|^2 \ge \liminf_{\varepsilon=\varepsilon_j\searrow 0}\int_0^{t_0}\int_\Omega |\nabla w_\varepsilon|^2$ and that thus, by (4.38), $\nabla w_\varepsilon\to\nabla w$ in $L^2(\Omega\times(0,t_0))$ as $\varepsilon=\varepsilon_j\searrow 0$, which entails (5.3) due to our restriction that $t_0>T$. $\square$

Building on the latter, through a procedure of a similar flavor we can moreover assert the desired strong convergence property of the forager gradient.

Lemma 5.2 *Suppose that (1.5) holds with $w^\star>0$ as given by (1.6), and let $(\varepsilon_j)_{j\in\mathbb{N}}$ and u be as provided by Lemma 4.9. Then for all $T>0$,*

$$
\nabla u_\varepsilon\to\nabla u \quad \text{in } L^2(\Omega\times(0,T)) \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0. \tag{5.7}
$$

PROOF. In order to prepare a procedure similar to that from Lemma 5.1, according to Lemma 4.9 let us fix, given $T>0$, a number $t_0>T$ such that t_0 is a Lebesgue point of $0<t\mapsto\int_\Omega u^2(\cdot,t)$, and that furthermore

$$
\int_\Omega u_\varepsilon^2(\cdot,t_0)\to\int_\Omega u^2(\cdot,t_0) \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0. \tag{5.8}
$$

Then choosing $(u_{0k})_{k\in\mathbb{N}}\subset C^1(\overline{\Omega})$ such that $u_{0k}\to u_0$ as $k\to\infty$, for $\delta\in(0,1), k\in\mathbb{N}$ and $h\in(0,1)$ we apply (2.3) to

$$
\varphi(x,t):=\zeta_\delta(t)\cdot(S_h\widehat{u}_k)(x,t), \qquad (x,t)\in\Omega\times(0,\infty),
$$

where

$$\widehat{u}_k(x,t) := \begin{cases} u(x,t) & \text{if } (x,t) \in \Omega \times (0,\infty), \\ u_{0k}(x) & \text{if } (x,t) \in \Omega \times (-\infty,0), \end{cases}$$

which can readily be seen to be admissible by a completion argument based on the observation that $\varphi \in L^\infty((0,\infty); W^{1,2}(\Omega))$ with $\varphi_t \in L^2(\Omega \times (0,\infty))$. We thereby see that

$$\begin{aligned}
& - \int_0^\infty \int_\Omega \zeta_\delta'(t) u(x,t) (S_h \widehat{u}_k)(x,t) dx dt \\
& - \int_0^\infty \int_\Omega \zeta_\delta(t) u(x,t) \cdot \frac{\widehat{u}_k(x,t) - \widehat{u}_k(x,t-h)}{h} dx dt - \int_\Omega u_0(x) u_{0k}(x) dx \\
& \qquad = - \int_0^\infty \int_\Omega \zeta_\delta(t) \nabla u(x,t) \cdot \nabla (S_h \widehat{u}_k)(x,t) dx dt \\
& \qquad \quad + \chi_1 \int_0^\infty \int_\Omega \zeta_\delta(t) u(x,t) \nabla w(x,t) \cdot \nabla (S_h \widehat{u}_k)(x,t) dx dt \\
& \qquad \qquad \text{for all } \delta \in (0,1), \text{ each } k \in \mathbb{N} \text{ and any } h \in (0,1),
\end{aligned}$$

where based on the fact that both ∇u and $u\nabla w$ belong to $L^2(\Omega \times (0, t_0+1); \mathbb{R}^n)$ by Lemma 4.9, and again on an argument relying on Young's inequality to estimate the second summand on the left, we may conclude on letting $h \searrow 0$ and then $k \to \infty$ that

$$\begin{aligned}
\int_0^\infty \int_\Omega \zeta_\delta(t) |\nabla u(x,t)|^2 dx dt \ \geq \ & -\frac{1}{2\delta} \int_{t_0}^{t_0+\delta} u^2(x,t) dx dt + \frac{1}{2} \int_\Omega u_0^2(x) dx \\
& + \chi_1 \int_0^\infty \int_\Omega \zeta_\delta(t) u(x,t) \nabla u(x,t) \cdot \nabla w(x,t) dx dt \qquad \text{for all } \delta \in (0,1).
\end{aligned}$$

Here as $\delta \searrow 0$, using that $|\nabla u|^2$ and $u\nabla u \cdot \nabla w$ lie in $L^1(\Omega \times (0, t_0+1))$ due to Lemma 4.9 we can employ the dominated convergence theorem to infer that

$$\int_0^{t_0} \int_\Omega |\nabla u|^2 \geq -\frac{1}{2} \int_\Omega u^2(\cdot, t_0) + \frac{1}{2} \int_\Omega u_0^2 + \chi_1 \int_0^{t_0} \int_\Omega u \nabla u \cdot \nabla w \tag{5.9}$$

thanks to the Lebesgue point feature of t_0.
In order to derive (5.7) from this, we now make substantial use of the strong convergence property asserted by Lemma 5.1, which together with (4.32), namely, ensures that

$$\chi_1 \int_0^{t_0} \int_\Omega u_\varepsilon \nabla u_\varepsilon \cdot \nabla w_\varepsilon \to \chi_1 \int_0^{t_0} \int_\Omega u \nabla u \cdot \nabla w \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0.$$

Therefore, relying on (5.8) as well as on (2.7) we can turn (5.9) into the inequality

$$\begin{aligned}
\int_0^{t_0} \int_\Omega |\nabla u|^2 \ & \geq \ \lim_{\varepsilon = \varepsilon_j \searrow 0} \left\{ -\frac{1}{2} \int_\Omega u_\varepsilon^2(\cdot, t_0) + \frac{1}{2} \int_\Omega u_0^2 + \chi_1 \int_0^{t_0} u_\varepsilon \nabla u_\varepsilon \cdot \nabla w_\varepsilon \right\} \\
& = \ \lim_{\varepsilon = \varepsilon_j \searrow 0} \int_0^{t_0} \int_\Omega |\nabla u_\varepsilon|^2
\end{aligned}$$

and hence conclude as intended, because $\nabla u_\varepsilon \rightharpoonup \nabla u$ in $L^2(\Omega\times(0,t_0))$ as $\varepsilon=\varepsilon_j\searrow 0$ by Lemma 4.9, and because $t_0>T$. $\Box$

With this preparation at hand, we can finalize our verification of the requirements from Definition 2.1).

Lemma 5.3 *Assume (1.5) with $w^\star>0$ taken from (1.6), and let u and v be as in Lemma 4.9. Then (2.5) holds for all nonnegative $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$.*

PROOF. For fixed nonnegative $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$, integrating by parts in the second equation from (2.7) shows that

$$\begin{aligned}\int_0^\infty\int_\Omega |\nabla\ln(v_\varepsilon+1)|^2\varphi &= -\int_0^\infty\int_\Omega \ln(v_\varepsilon+1)\varphi_t - \int_\Omega \ln(v_0+1)\varphi(\cdot,0)\\ &\quad+\int_0^\infty\int_\Omega \nabla\ln(v_\varepsilon+1)\cdot\nabla\varphi\\ &\quad+\chi_2\int_0^\infty\int_\Omega \frac{v_\varepsilon}{v_\varepsilon+1}\Big(\nabla u_\varepsilon\cdot\nabla\ln(v_\varepsilon+1)\Big)\varphi\\ &\quad-\chi_2\int_0^\infty\int_\Omega \frac{v_\varepsilon}{v_\varepsilon+1}\nabla u_\varepsilon\cdot\nabla\varphi \qquad \text{for all } \varepsilon\in(0,1), \end{aligned} \tag{5.10}$$

where by (4.34), taking $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ from Lemma 4.9 we have

$$-\int_0^\infty\int_\Omega \ln(v_\varepsilon+1)\varphi_t \to -\int_0^\infty\int_\Omega \ln(v+1)\varphi_t \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0 \tag{5.11}$$

and

$$\int_0^\infty\int_\Omega \nabla\ln(v_\varepsilon+1)\cdot\nabla\varphi \to \int_0^\infty\int_\Omega \nabla\ln(v+1)\cdot\nabla\varphi \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0, \tag{5.12}$$

and where since $\frac{v_\varepsilon}{v_\varepsilon+1}\to\frac{v}{v+1}$ in $L^2_{loc}(\overline{\Omega}\times[0,\infty))$ as $\varepsilon=\varepsilon_j\searrow 0$ by e.g. (4.33) and the dominated convergence theorem, the weak convergence property in (4.31) is sufficient to ensure that

$$-\chi_2\int_0^\infty\int_\Omega \frac{v_\varepsilon}{v_\varepsilon+1}\nabla u_\varepsilon\cdot\nabla\varphi \to -\chi_2\int_0^\infty\int_\Omega \frac{v}{v+1}\nabla u\cdot\nabla\varphi \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0. \tag{5.13}$$

Now in the second last summand in (5.10) we rather rely on the statement on strong L^2 convergence from Lemma 5.2, which, along with the two-sided inequality $0\le\frac{v_\varepsilon}{v_\varepsilon+1}\le 1$ and the fact that $\frac{v_\varepsilon}{v_\varepsilon+1}\to\frac{v}{v+1}$ a.e. in $\Omega\times(0,\infty)$ as $\varepsilon=\varepsilon_j\searrow 0$, guarantees that $\frac{v_\varepsilon}{v_\varepsilon+1}\nabla u_\varepsilon\to\frac{v}{v+1}\nabla u$ in $L^2_{loc}(\overline{\Omega}\times[0,\infty))$ as $\varepsilon=\varepsilon_j\searrow 0$ ([33, Lemma 10.4]) and hence

$$\chi_2\int_0^\infty\int_\Omega \frac{v_\varepsilon}{v_\varepsilon+1}\Big(\nabla u_\varepsilon\cdot\nabla\ln(v_\varepsilon+1)\Big)\varphi \to \chi_2\int_0^\infty\int_\Omega \frac{v}{v+1}\Big(\nabla u\cdot\nabla\ln(v+1)\Big)\varphi \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0 \tag{5.14}$$

due to (4.34). By lower semicontinuity of the norm in $L^2(\Omega\times(0,\infty);\mathbb{R}^n)$ with respect to weak convergence, in view of (4.31) we readily infer the validity of (2.5) from (5.10)-(5.14). $\Box$

Our main existence result thereby becomes complete.

PROOF of Theorem 1.1. All statements have already been verified in Lemma 4.9 and Lemma 5.3. $\Box$

6 Stabilization. Proof of Theorem 1.3

6.1 Two basic homogenization features

Let us launch our qualitative analysis by stating two quite straightforward consequences of the decay assumption (1.11) when combined with our previously gained estimates. The first of these concludes the following from Lemma 4.1.

Lemma 6.1 *In addition to the assumptions from Theorem 1.1, suppose that (1.11) holds. Then there exists $C > 0$ such that*

$$\int_0^\infty \int_\Omega |\nabla w_\varepsilon|^2 \le C \qquad \text{for all } \varepsilon \in (0,1). \tag{6.1}$$

In particular,

$$\int_0^\infty \int_\Omega |\nabla w|^2 < \infty.$$

PROOF. Since from Lemma 4.1 we know that

$$\int_0^T \int_\Omega |\nabla w_\varepsilon|^2 \le \frac{1}{2d} \int_\Omega w_0^2 + \frac{|\Omega| w^\star}{d} \int_0^\infty \|r(\cdot,t)\|_{L^\infty(\Omega)} dt \qquad \text{for all } \varepsilon \in (0,1) \text{ and } T > 0,$$

in view of Lemma 4.9 both claims immediately follow. □

By going back to Lemma 3.3, we can derive a similar property of the first solution components.

Lemma 6.2 *Suppose that the assumptions from Theorem 1.1 as well as (1.11) hold. Then there exists $C > 0$ such that*

$$\int_0^\infty \int_\Omega |\nabla u_\varepsilon|^2 \le C \qquad \text{for all } \varepsilon \in (0,1), \tag{6.2}$$

whence in particular

$$\int_0^\infty \int_\Omega |\nabla u|^2 < \infty.$$

PROOF. An application of Lemma 3.3 to $p := 2$ provides $\delta > w^\star, \kappa > 0, c_1 > 0$ and $c_2 > 0$ such that for all $\varepsilon \in (0,1)$,

$$\frac{d}{dt} \int_\Omega u_\varepsilon^2 (\delta - w_\varepsilon)^{-\kappa} + c_1 \int_\Omega |\nabla u_\varepsilon|^2 \le c_2 \|r\|_{L^\infty(\Omega)} \int_\Omega u_\varepsilon^2 (\delta - w_\varepsilon)^{-\kappa} \qquad \text{for all } t > 0, \tag{6.3}$$

which upon a comparison argument firstly implies that

$$\begin{aligned} \int_\Omega u_\varepsilon^2 (\delta - w_\varepsilon)^{-\kappa} &\le \left\{ \int_\Omega u_0^2 (\delta - w_0)^{-\kappa} \right\} \cdot e^{c_2 \int_0^t \|r(\cdot,s)\|_{L^\infty(\Omega)} ds} \\ &\le c_3 := \left\{ \int_\Omega u_0^2 (\delta - w_0)^{-\kappa} \right\} \cdot e^{c_2 \int_0^\infty \|r(\cdot,s)\|_{L^\infty(\Omega)} ds} \qquad \text{for all } t > 0 \end{aligned}$$

with c_3 being finite according to (1.11). Direct integration of (6.3) thereafter shows that

$$c_1 \int_0^t \int_\Omega |\nabla u_\varepsilon|^2 \le c_2 c_3 \int_0^t \|r(\cdot,s)\|_{L^\infty(\Omega)} ds \qquad \text{for all } t > 0,$$

which by again using (1.11) implies (6.2) on taking $t \nearrow \infty$ and then relying on Lemma 4.9 in letting $\varepsilon = \varepsilon_j \searrow 0$. □

6.2 Stabilization of u

Along with the strong convergence features asserted by Lemma 5.1 and Lemma 5.2, the previous two lemmata entail sufficient decay of the nutrient taxis mechanism so as to allow for the following conclusion on stabilization of u, though yet in a sense weaker than that claimed in Theorem 1.3.

Lemma 6.3 *Beyond assuming the hypotheses from Theorem 1.1, assume (1.11). Then there exists a null set $N \subset (0,\infty)$ such that*

$$\int_\Omega u(\cdot,t)\ln\frac{u(\cdot,t)}{\overline{u}_0} \to 0 \qquad \textit{as } (0,\infty)\setminus N \ni t\to\infty. \tag{6.4}$$

Proof. Thanks to (2.9), on testing the first equation in (2.7) by the smooth function $\ln u_\varepsilon$ we see that due to Young's inequality,

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_\varepsilon \ln\frac{u_\varepsilon}{\overline{u}_0} + \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} &= \chi_1\int_\Omega \nabla u_\varepsilon\cdot\nabla w_\varepsilon \\ &\le h_\varepsilon(t) := \int_\Omega |\nabla u_\varepsilon|^2 + \frac{\chi_1^2}{4}\int_\Omega|\nabla w_\varepsilon|^2 \qquad \text{for all } t>0,\end{aligned}$$

where according to a logarithmic Sobolev inequality ([2], [14]) we can find $c_1>0$ such that for all $\varepsilon\in(0,1)$ we have

$$\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} \ge c_1\int_\Omega u_\varepsilon\ln\frac{u_\varepsilon}{\overline{u}_0} \qquad \text{for all } t>0.$$

Therefore,

$$\frac{d}{dt}\int_\Omega u_\varepsilon\ln\frac{u_\varepsilon}{\overline{u}_0} + c_1\int_\Omega u_\varepsilon\ln\frac{u_\varepsilon}{\overline{u}_0} \le h_\varepsilon(t) \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),$$

which after an integration shows that

$$\int_\Omega u_\varepsilon(\cdot,t)\ln\frac{u_\varepsilon(\cdot,t)}{\overline{u}_0} \le c_2 e^{-c_1 t} + \int_0^t e^{-c_1(t-s)}h_\varepsilon(s)ds \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1) \tag{6.5}$$

with $c_2 := \int_\Omega u_0\ln\frac{u_0}{\overline{u}_0} > 0$ by (1.4).
We now go back to Lemma 4.9 and use that $|z\ln z| \le \frac{2}{e}z^{\frac{3}{2}} + \frac{1}{e}$ for all $z>0$ to infer that there exist a null set $N\subset(0,\infty)$, as well as a subsequence $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ of the sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ provided by Lemma 4.9, such that

$$\int_\Omega u_\varepsilon(\cdot,t)\ln\frac{u_\varepsilon(\cdot,t)}{\overline{u}_0} \to \int_\Omega u(\cdot,t)\ln\frac{u(\cdot,t)}{\overline{u}_0} \quad \text{for all } t\in(0,\infty)\setminus N \qquad \text{as } \varepsilon=\varepsilon_{j_k}\searrow 0. \tag{6.6}$$

Since furthermore Lemma 5.1 together with Lemma 5.2 asserts that

$$h_\varepsilon \to h := \int_\Omega |\nabla u|^2 + \frac{\chi_1^2}{4}\int_\Omega|\nabla w|^2 \quad \text{in } L^1_{loc}([0,\infty)) \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0,$$

on letting $\varepsilon = \varepsilon_{j_k} \searrow 0$ we infer from (6.5) that

$$\int_\Omega u(\cdot,t) \ln \frac{u(\cdot,t)}{\overline{u}_0} \le c_2 e^{-c_1 t} + \int_0^t e^{-c_1(t-s)} h(s) ds \qquad \text{for all } t \in (0,\infty) \setminus N. \tag{6.7}$$

But as a combination of Lemma 6.1 with Lemma 6.2 shows that $c_3 := \int_0^\infty h(s)ds$ is finite, given $\eta > 0$ we can fix $t_0 > 0$ large enough fulfilling

$$c_2 e^{c_1 t} \le \frac{\eta}{3}, \quad c_3 e^{-\frac{c_1 t}{2}} \le \frac{\eta}{3} \quad \text{and} \quad \int_{\frac{t}{2}}^\infty h(s)ds \le \frac{\eta}{3} \qquad \text{for all } t > t_0,$$

whence for any $t > t_0$ fulfilling $t_0 \notin N$ we infer from (6.7) that

$$\begin{aligned} \int_\Omega u(\cdot,t) \ln \frac{u(\cdot,t)}{\overline{u}_0} &\le c_2 e^{-c_1 t} + \int_0^{\frac{t}{2}} e^{-\frac{c_1 t}{2}} h(s)ds + \int_{\frac{t}{2}}^t h(s)ds \\ &\le \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta. \end{aligned}$$

Since from (4.41) and (6.6) we know that $\int_\Omega u(\cdot,t) \ln \frac{u(\cdot,t)}{\overline{u}_0} \ge 0$ for all $t \in (0,\infty) \setminus N$, this entails (6.4) due to the fact that $\eta > 0$ was arbitrary. □

By means of a Csiszár-Kullback inequality and appropriate interpolation, however, the above can readily be turned into a convergence statement of the intended flavor.

Lemma 6.4 *In addition to the assumptions from Theorem 1.1, suppose that (1.11) holds. Then one can find a null set $N \subset (0,\infty)$ with the property that*

$$\|u(\cdot,t) - \overline{u}_0\|_{L^4(\Omega)} \to 0 \qquad \text{as } (0,\infty) \setminus N \ni t \to \infty. \tag{6.8}$$

Proof. According to (4.41) and a Csiszár-Kullback inequality ([7]), there exists $c_1 > 0$ such that

$$\|u(\cdot,t) - \overline{u}_0\|_{L^1(\Omega)}^2 \le c_1 \int_\Omega u(\cdot,t) \ln \frac{u(\cdot,t)}{\overline{u}_0} \qquad \text{for a.e. } t > 0,$$

so that Lemma 6.3 entails the existence of a null set $N_1 \subset (0,\infty)$ such that

$$\|u(\cdot,t) - \overline{u}_0\|_{L^1(\Omega)} \to 0 \qquad \text{as } (0,\infty) \setminus N_1 \ni t \to \infty. \tag{6.9}$$

As furthermore with $p_0 > 4$ taken from Lemma 4.9 we can use the Hölder inequality to interpolate

$$\begin{aligned} \|u(\cdot,t) - \overline{u}_0\|_{L^4(\Omega)} &\le \|u(\cdot,t) - \overline{u}_0\|_{L^{p_0}(\Omega)}^a \|u(\cdot,t) - \overline{u}_0\|_{L^1(\Omega)}^{1-a} \\ &\le \Big\{ \|u(\cdot,t)\|_{L^{p_0}(\Omega)} + \overline{u}_0 |\Omega|^{\frac{1}{p_0}} \Big\}^a \cdot \|u(\cdot,t) - \overline{u}_0\|_{L^1(\Omega)}^{1-a} \qquad \text{for all } t > 0 \end{aligned}$$

with $a := \frac{3p_0}{4(p_0-1)} \in (0,1)$, the claim therefore results upon combining (6.8) with the inclusion $u \in L^\infty((0,\infty); L^{p_0}(\Omega))$ asserted by Lemma 4.9. □

6.3 Decay of w

Next addressing the claimed temporal asymptotics of w, we again build our argument in this direction on a first fundamental though yet quite weak decay information:

Lemma 6.5 *Assume the hypotheses from Theorem 1.1, and suppose that (1.11) holds. Then*

$$\int_t^{t+1}\int_\Omega uw \to 0 \qquad \text{as } t\to\infty. \tag{6.10}$$

PROOF. By integrating the third equation in (2.7), we see that since μ is nonnegative,

$$\frac{d}{dt}\int_\Omega w_\varepsilon + \lambda\int_\Omega \frac{(u_\varepsilon+v_\varepsilon)w_\varepsilon}{1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon} = -\mu\int_\Omega w_\varepsilon + \int_\Omega r \le |\Omega|\cdot\|r\|_{L^\infty(\Omega)} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),$$

so that

$$\int_0^t\int_\Omega \frac{(u_\varepsilon+v_\varepsilon)w_\varepsilon}{1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon} \le c_1 := \frac{|\Omega|}{\lambda}\int_0^\infty \|r(\cdot,s)\|_{L^\infty(\Omega)}ds \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),$$

where $c_1<\infty$ by (1.11). Thanks to Lemma 4.9 and Fatou's lemma, this implies that with $(\varepsilon_j)_{j\in\mathbb{N}}$ as in Lemma 4.9 we have

$$\int_0^t\int_\Omega (u+v)w \le \liminf_{\varepsilon=\varepsilon_j\searrow 0}\int_0^t\int_\Omega \frac{(u_\varepsilon+v_\varepsilon)w_\varepsilon}{1+\varepsilon(u_\varepsilon+v_\varepsilon)w_\varepsilon} \le c_1 \qquad \text{for all } t>0,$$

which in particular shows that $\int_0^\infty\int_\Omega uw<\infty$ and hence entails (6.10). □

Already knowing that u stabilizes at the nontrivial constant level $\overline{u}_0$, we can indeed turn the latter into a statement on decay of w itself.

Lemma 6.6 *Suppose that apart from the assumptions from Theorem 1.1, the condition (1.11) is satisfied. Then*

$$\int_t^{t+1}\int_\Omega w \to 0 \qquad \text{as } t\to\infty. \tag{6.11}$$

PROOF. We fix $\eta>0$ and then can employ Lemma 6.4 to readily find $t_1>0$ such that

$$\int_t^{t+1}\int_\Omega |u(\cdot,s)-\overline{u}_0| \le \frac{\overline{u}_0\eta}{2w^\star} \qquad \text{for all } t>t_1. \tag{6.12}$$

Since, apart from that, Lemma 6.5 provides $t_2>0$ fulfilling

$$\int_t^{t+1}\int_\Omega uw \le \frac{\overline{u}_0\eta}{2} \qquad \text{for all } t>t_2,$$

by combining this with (6.12) and relying on Lemma 3.1 we can estimate

$$\begin{aligned}
\int_t^{t+1}\int_\Omega w &= \frac{1}{\overline{u}_0}\int_t^{t+1}\int_\Omega uw - \frac{1}{\overline{u}_0}\int_t^{t+1}\int_\Omega (u-\overline{u}_0)w \\
&\le \frac{1}{\overline{u}_0}\int_t^{t+1}\int_\Omega uw + \frac{w^\star}{\overline{u}_0}\int_t^{t+1}\int_\Omega |u-\overline{u}_0| \\
&\le \frac{\eta}{2}+\frac{\eta}{2}=\eta \qquad \text{for all } t>\max\{t_1,t_2\},
\end{aligned}$$

so that (6.11) follows. □

Finally, the topological setup herein can be adjusted in the desired manner by applying appropriate parabolic smoothing arguments to the third equation in (2.7).

Lemma 6.7 *In addition to the requirements from Theorem 1.1, assume that (1.11) holds. Then there exists a null set $N \subset (0,\infty)$ satisfying*

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \qquad as\ (0,\infty)\setminus N \ni t \to \infty. \tag{6.13}$$

PROOF. Due to well-known smoothing properties of the Neumann heat semigroup $(e^{\sigma\Delta})_{\sigma\ge 0}$ on Ω ([29, Lemma 1.4]), we can choose $c_1>0$ such that

$$\|e^{d\sigma\Delta}\varphi\|_{L^\infty(\Omega)} \le c_1\sigma^{-\frac{n}{2}}\|\varphi\|_{L^1(\Omega)} \qquad \text{for all } \sigma\in(0,1) \text{ and each } \varphi\in C^0(\overline{\Omega}). \tag{6.14}$$

With this value of c_1 fixed, given $\eta>0$ we employ Lemma 6.6 to pick $t_1>1$ large enough satisfying

$$2^{\frac{n}{2}}c_1\int_{t-1}^{t-\frac12}\|w(\cdot,s)\|_{L^1(\Omega)}ds \le \frac{\eta}{8} \qquad \text{for all } t>t_1, \tag{6.15}$$

while according to (1.11) we may take $t_2>1$ suitably large such that

$$\int_{t-1}^{t}\|r(\cdot,s)\|_{L^\infty(\Omega)}ds \le \frac{\eta}{2} \qquad \text{for all } t>t_2. \tag{6.16}$$

Furthermore relying on Lemma 4.9 in selecting a null set $N\subset(0,\infty)$ such that with $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,\infty)$ as introduced there we have

$$w_\varepsilon(\cdot,t)\to w(\cdot,t) \quad \text{a.e. in } \Omega \quad \text{for all } t\in(0,\infty)\setminus N \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0, \tag{6.17}$$

for arbitrary $t>t_0:=\max\{t_1,t_2\}$ fulfilling $tnot\in N$ we may once more invoke Lemma 4.9 to see that in light of (6.15) we can find $\varepsilon_\star=\varepsilon_\star(t)\in(0,1)$ such that

$$2^{\frac{n}{2}}c_1\int_{t-1}^{t-\frac12}\|w_\varepsilon(\cdot,s)\|_{L^1(\Omega)}ds \le \frac{\eta}{4} \qquad \text{for all } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}} \text{ such that } \varepsilon<\varepsilon_\star,$$

which for any such ε entails the existence of $t_\star=t_\star(t,\varepsilon)\in(t-1,t-\frac12)$ fulfilling

$$2^{\frac{n}{2}}c_1\|w_\varepsilon(\cdot,t_\star)\|_{L^1(\Omega)} \le \frac{\eta}{2}. \tag{6.18}$$

Now since due to the comparison principle we have

$$\begin{aligned}
w_\varepsilon(\cdot,t) &= e^{d(t-t_\star)\Delta}w_\varepsilon(\cdot,t_\star) - \int_{t_\star}^t e^{d(t-s)\Delta}\Big\{\lambda\frac{(u_\varepsilon(\cdot,s)+v_\varepsilon(\cdot,s))w_\varepsilon(\cdot,s)}{1+\varepsilon(u_\varepsilon(\cdot,s)+v_\varepsilon(\cdot,s))w_\varepsilon(\cdot,s)} + \mu w_\varepsilon(\cdot,s)\Big\}ds \\
&\quad + \int_{t_\star}^t e^{d(t-s)\Delta}r(\cdot,s)ds \\
&\le e^{d(t-t_\star)\Delta}w_\varepsilon(\cdot,t_\star) + \int_{t_\star}^t\|r(\cdot,s)\|_{L^\infty(\Omega)}ds \qquad \text{in } \Omega,
\end{aligned}$$

by combining (6.14) with (6.18) and using (6.16) we infer that

$$\begin{aligned}\|w_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} &\le c_1(t-t_\star)^{-\frac{n}{2}}\|w_\varepsilon(\cdot,t_\star)\|_{L^1(\Omega)}+\int_{t_\star}^t\|r(\cdot,s)\|_{L^\infty(\Omega)}ds\\ &\le 2^{\frac{n}{2}}c_1\|w_\varepsilon(\cdot,t_\star)\|_{L^1(\Omega)}+\int_{t_\star}^t\|r(\cdot,s)\|_{L^\infty(\Omega)}ds\\ &\le \frac{\eta}{2}+\frac{\eta}{2}=\eta \qquad \text{for all } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}} \text{ fulfilling } \varepsilon<\varepsilon_\star,\end{aligned}$$

which in view of (6.17) and Fatou's lemma implies that

$$\|w(\cdot,t)\|_{L^\infty(\Omega)}\le\eta \qquad \text{for all } t\in(t_0,\infty)\setminus N$$

and thus establishes (6.13). □

6.4 Stabilization of v

In view of the poor regularity information available for v, it may seem not very suprising that our argument toward convergence in this component requires somewhat more subtle efforts; in fact, as we particularly only know (2.6) instead of a genuine mass conservation property, already identifying an appropriate limit for the corresponding total mass functional appears to be nontrivial. After all, once more resorting to Lemma 4.3 enables us to identify a certain stabilization property of the functional $\int_\Omega \ln(v+1)$ that can, in a later step, be favorably related to spatio-temporal L^2 norms of $\frac{\nabla v}{v+1}$, as having formed the core of virtually all our previous access to regularity of v.

Lemma 6.8 *Let the conditions from Theorem 1.1 be fulfilled, and assume (1.11). Then there exist a null set $N\subset(0,\infty)$ and $b\ge 0$ such that*

$$\frac{1}{|\Omega|}\int_\Omega \ln\Big(v(\cdot,t)+1\Big)\to b \qquad \text{as } (0,\infty)\setminus N\ni t\to\infty. \tag{6.19}$$

Proof. For $\varepsilon\in(0,1)$, we let

$$y_\varepsilon(t):=\int_\Omega \ln\Big(v_\varepsilon(x,t)+1\Big)dx+\frac{\chi_2^2}{2}\int_0^t\int_\Omega|\nabla u_\varepsilon(x,s)|^2dxds, \qquad t\ge 0,$$

and first observe that thanks to (2.6), Lemma 4.9 and Lemma 5.2 we can fix a null set $N\subset(0,\infty)$ such that

$$\int_\Omega v(\cdot,t)\le\int_\Omega v_0 \qquad \text{for all } t\in(0,\infty)\setminus N, \tag{6.20}$$

that

$$v_\varepsilon(\cdot,t)\to v(\cdot,t) \quad \text{a.e. in } \Omega \quad \text{for all } t\in(0,\infty)\setminus N \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0, \tag{6.21}$$

and that

$$y_\varepsilon(t)\to y(t):=\int_\Omega \ln\Big(v(x,t)+1\Big)dx+\frac{\chi_1^2}{2}\int_0^t\int_\Omega|\nabla u(x,s)|^2dxds \quad \text{for all } t\in(0,\infty)\setminus N \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0, \tag{6.22}$$

where $(\varepsilon_j)_{j\in\mathbb{N}}$ is as given by Lemma 4.9.
Now when applied to $a := 1$, Lemma 4.3 in particular says that for each $\varepsilon \in (0,1)$ we have

$$y_\varepsilon'(t) = \frac{d}{dt}\int_\Omega \ln(v_\varepsilon+1) + \frac{\chi_2^2}{2}\int_\Omega |\nabla u_\varepsilon|^2 \ge \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon+1)^2} \ge 0 \qquad \text{for all } t>0,$$

and that thus

$$y_\varepsilon(t) \ge y_\varepsilon(t_0) \qquad \text{for all } t_0 \ge 0 \text{ and any } t>t_0.$$

In view of (6.22), on letting $\varepsilon = \varepsilon_j \searrow 0$ this implies that

$$y(t) \ge y(t_0) \qquad \text{for all } t_0 \in (0,\infty)\setminus N \text{ and each } t\in (t_0,\infty)\setminus N, \tag{6.23}$$

and that, correspondingly, y is nondecreasing on $(0,\infty)\setminus N$. Since, on the other hand, $c_1 := \frac{\chi_2^2}{2}\int_0^\infty\int_\Omega |\nabla u|^2$ is finite by Lemma 6.2, and since thus

$$y(t) \le \int_\Omega \Big(v(\cdot,t)+1\Big) + c_1 \le \int_\Omega v_0 + |\Omega| + c_1 \qquad \text{for all } t\in(0,\infty)\setminus N,$$

from (6.23) we infer the existence of a finite number $c_2 \ge 0$ such that

$$y(t) \nearrow c_2 \qquad \text{as } (0,\infty)\setminus N \ni t\to\infty.$$

But by definition of y, this means that

$$\int_\Omega \ln\Big(v(x,t)+1\Big)dx = y(t) - \frac{\chi_2^2}{2}\int_0^t\int_\Omega |\nabla u(x,s)|^2 dxds \to c_2 - c_1 \qquad \text{as } (0,\infty)\setminus N \ni t\to\infty,$$

which directly yields (6.19) with $b := \frac{c_2-c_1}{|\Omega|}$ necessarily being nonnegative due to the fact that $\ln(v(\cdot,t)+1) \ge 0$ a.e. in Ω for each $t\in(0,\infty)\setminus N$ by (6.21). $\Box$

Indeed, through the logarithmic gradient estimate from Lemma 4.4 the latter entails the following, still quite weak, spatial homogenization feature of $\ln(v+1)$.

Lemma 6.9 *In addition to the hypotheses from Theorem 1.1, assume that (1.11) holds. Then taking $b\ge 0$ as in Lemma 6.8, we have*

$$\int_t^{t+1}\int_\Omega \Big|\ln\Big(v(x,s)+1\Big) - b\Big| dxds \to 0 \qquad \textit{as } t\to\infty. \tag{6.24}$$

Proof. Going back to Lemma 4.4, we recall that for each $T>0$,

$$\int_0^T\int_\Omega |\nabla \ln(v_\varepsilon+1)|^2 \le 2\int_\Omega v_0 + 2|\Omega| + \chi_2^2\int_0^T\int_\Omega |\nabla u_\varepsilon|^2 \qquad \text{for all } \varepsilon\in(0,1),$$

which in light of Lemma 4.9 and the strong convergence statement from Lemma 5.2 implies that

$$\int_0^T\int_\Omega |\nabla\ln(v+1)|^2 \le 2\int_\Omega v_0 + 2|\Omega| + \chi_2^2\int_0^T\int_\Omega|\nabla u|^2.$$

Thanks to Lemma 6.2, we therefore know that

$$\int_0^\infty \left\| \nabla \ln \Big(v(\cdot,s)+1 \Big) \right\|_{L^2(\Omega)}^2 ds < \infty,$$

whence choosing $c_1 > 0$ such that in accordance with a Poincaré inequality we have

$$\|\varphi - \overline{\varphi}\|_{L^1(\Omega)}^2 \le c_1 \|\nabla\varphi\|_{L^2(\Omega)}^2 \qquad \text{for all } \varphi \in W^{1,2}(\Omega),$$

by means of a Cauchy-Schwarz inequality we see that

$$\begin{aligned}
\int_t^{t+1} & \left\| \ln\Big(v(\cdot,s)+1\Big) - \overline{\ln\Big(v(\cdot,s)+1\Big)} \right\|_{L^1(\Omega)} ds \\
&\le \left\{ \int_t^{t+1} \left\| \ln\Big(v(\cdot,s)+1\Big) - \overline{\ln\Big(v(\cdot,s)+1\Big)} \right\|_{L^1(\Omega)}^2 ds \right\}^{\frac{1}{2}} \\
&\le c_1^{\frac{1}{2}} \cdot \left\{ \int_t^{t+1} \left\| \nabla \ln\Big(v(\cdot,s)+1\Big) \right\|_{L^2(\Omega)}^2 ds \right\}^{\frac{1}{2}} \\
&\to 0 \qquad \text{as } t \to \infty.
\end{aligned} \tag{6.25}$$

But Lemma 6.8 provides a null set $N \subset (0,\infty)$ such that

$$\overline{\ln\Big(v(\cdot,s)+1\Big)} \to b \qquad \text{as } (0,\infty)\setminus N \ni s \to \infty,$$

which clearly implies that

$$\int_t^{t+1} \left\| \overline{\ln\Big(v(\cdot,s)+1\Big)} - b \right\|_{L^1(\Omega)} ds \to 0 \qquad \text{as } t \to \infty.$$

In conjunction with (6.25), this establishes (6.24). □

By means of a contradictory argument, this can be turned into a statement on convergence of v itself, which with regard to its topological framework is already precisely of the form claimed in Theorem 1.3.

Lemma 6.10 *Suppose that the requirements from Theorem 1.1 are met, and that (1.11) is valid. Then with $b \ge 0$ taken from Lemma 6.8, for any choice of $p \in (0,1)$ we have*

$$\int_t^{t+1} \int_\Omega |v(x,s) - (e^b - 1)|^p dx ds \to 0 \qquad \textit{as } t \to \infty. \tag{6.26}$$

PROOF. Assuming (6.26) to be false for some $p \in (0,1)$, we could find $(t_k)_{k\in\mathbb{N}} \subset (0,\infty)$ and $c_1 > 0$ such that $t_k \to \infty$ as $k \to \infty$ and

$$\int_{t_k}^{t_k+1} |v(x,s) - (e^b - 1)|^p dx ds \ge c_1 \qquad \text{for all } k \in \mathbb{N}. \tag{6.27}$$

On the other hand, from Lemma 6.8 we know that

$$\int_{t_k}^{t_k+1} \int_\Omega \Big| \ln \Big(v(x,s)+1 \Big) - b \Big| dxds \to 0 \qquad \text{as } k\to\infty,$$

which, on letting $z_k(x,s) := v(x,t_k+s)$ for $x\in\Omega$ $s\in(0,1)$ and $k\in\mathbb{N}$, becomes equivalent to saying that

$$\int_0^1 \int_\Omega \Big| \ln \Big(z_k(x,s)+1 \Big) - b \Big| dxds \to 0 \qquad \text{as } k\to\infty.$$

We could therefore extract a suitable subsequence $(z_{k_l})_{l\in\mathbb{N}}$ along which

$$\ln(z_{k_l}+1) - b \to 0 \quad \text{a.e. in } \Omega\times(0,1) \qquad \text{as } l\to\infty,$$

that is, for which we would have

$$\Big| z_{k_l} - (e^b-1) \Big|^p \to 0 \quad \text{a.e. in } \Omega\times(0,1) \qquad \text{as } l\to\infty. \tag{6.28}$$

Now as a consequence of (2.6), using the Hölder inequality we can estimate

$$\begin{aligned} \int_0^1 \int_\Omega \Big| \Big| z_k - (e^b-1) \Big|^p \Big|^{\frac{1}{p}} &\le 2^{\frac{1}{p}} \int_0^1 \int_\Omega |z_k| + 2^{\frac{1}{p}} (e^b-1)|\Omega| \\ &\le 2^{\frac{1}{p}} \int_\Omega v_0 + 2^{\frac{1}{p}} (e^b-1)|\Omega| \qquad \text{for all } k\in\mathbb{N}, \end{aligned}$$

which thanks to the fact that $\frac{1}{p} > 1$ warrants uniform integrability of $(|z_k-(e^b-1)|^p)_{k\in\mathbb{N}}$ over $\Omega\times(0,1)$. When combined with (6.28), through an application of the Vitali convergence theorem this implies that we would have

$$\int_0^1 \int_\Omega \Big| z_{k_l} - (e^b-1) \Big|^p \to 0 \qquad \text{as } l\to\infty,$$

which is incompatible with (6.27) and thus proves that actually (6.26) must have been true. □

Thus left with the verification that the constant e^b-1 in (6.26), and hence b itself, is positive, one last time we go back to Lemma 4.3. In fact, on applying the latter to $a=0$ now we can make sure that the singular value $v=0$ thus appearing therein cannot be attained, nor be approximated, within sets of positive measure, as resulting from the following estimate.

Lemma 6.11 *In addition to the assumptions from Theorem 1.1, suppose that (1.11) holds. Then there exist a null set $N\subset(0,\infty)$ and $C>0$ such that*

$$\int_\Omega \ln_+ \frac{1}{v(x,t)} dx \le C \qquad \textit{for all } t\in(0,\infty)\setminus N. \tag{6.29}$$

Proof. We once more apply Lemma 4.3, but this time to $a := 0$, to see that for all $\varepsilon \in (0,1)$,

$$\frac{d}{dt}\int_\Omega \ln\frac{1}{v_\varepsilon} \le -\frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} + \frac{\chi_2^2}{2}\int_\Omega |\nabla u_\varepsilon|^2 \le \frac{\chi_2^2}{2}\int_\Omega |\nabla u_\varepsilon|^2 \qquad \text{for all } t>0,$$

so that

$$\int_\Omega \ln\frac{1}{v_\varepsilon(\cdot,t)} \le \int_\Omega \ln\frac{1}{v_0} + \frac{\chi_2^2}{2}\int_0^t\int_\Omega |\nabla u_\varepsilon|^2 \qquad \text{for all } t>0.$$

Since due to the fact that $\ln z \le z$ for all $z>0$ we can again use (2.9) to see that

$$\begin{aligned}\int_\Omega \ln_+\frac{1}{v_\varepsilon(\cdot,t)} &= \int_\Omega \ln\frac{1}{v_\varepsilon(\cdot,t)} + \int_{\{v_\varepsilon(\cdot,t)>1\}} \ln v_\varepsilon(\cdot,t)\\ &\le \int_\Omega \ln\frac{1}{v_\varepsilon(\cdot,t)} + \int_\Omega v_\varepsilon(\cdot,t)\\ &= \int_\Omega \ln\frac{1}{v_\varepsilon(\cdot,t)} + \int_\Omega v_0 \qquad \text{for all } t>0,\end{aligned}$$

this shows that

$$\int_\Omega \ln_+\frac{1}{v_\varepsilon(\cdot,t)} \le h_\varepsilon(t) := c_1 + \frac{\chi_2^2}{2}\int_0^t\int_\Omega |\nabla u_\varepsilon|^2 \qquad \text{for all } t>0, \tag{6.30}$$

where $c_1 := \int_\Omega \ln\frac{1}{v_0} + \int_\Omega v_0$.
We now return to Lemma 4.9 to pick a null set $N\subset(0,\infty)$ such that with $(\varepsilon_j)_{j\in\mathbb{N}}$ as given there we have $v_\varepsilon(\cdot,t)\to v(\cdot,t)$ a.e. in Ω for all $t\in(0,\infty)\setminus N$ as $\varepsilon=\varepsilon_j\searrow 0$. Since furthermore Lemma 5.2 implies that for all $t>0$,

$$h_\varepsilon(t)\to h(t) := c_1 + \frac{\chi_2^2}{2}\int_0^t\int_\Omega |\nabla u|^2 \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0,$$

we readily infer from Fatou's lemma that (6.30) entails the inequality

$$\int_\Omega \ln_+\frac{1}{v(\cdot,t)} \le h(t) \qquad \text{for all } t\in(0,\infty)\setminus N.$$

As Lemma 6.2 warrants boundedness of h throughout $(0,\infty)$, this already yields (6.29). □

Indeed, the latter entails positivity of the constant approached by v in the large time limit:

Lemma 6.12 *Let the hypotheses from Theorem 1.1 be satisfied, and assume (1.11). Then the number b from Lemma 6.8 satisfies $b>0$.*

Proof. In line with Lemma 6.11, we fix a null set $N\subset(0,\infty)$ and a positive constant c_1 such that

$$\int_\Omega \ln_+\frac{1}{v(\cdot,t)} \le c_1 \qquad \text{for all } t\in(0,\infty)\setminus N. \tag{6.31}$$

Then since Lemma 6.10 inter alia says that

$$\eta_k := \int_k^{k+1} \int_\Omega \Big| v(x,t) - (e^b - 1) \Big|^{\frac{1}{2}} dxdt, \qquad k \in \mathbb{N},$$

satisfies $\eta_k \to 0$ as $k \to \infty$, we can find $(t_k)_{k\in\mathbb{N}} \subset (0,\infty) \setminus N$ such that $t_k \in (k, k+1)$ and $\int_\Omega |v(\cdot, t_k - (e^b - 1)|^{\frac{1}{2}} \le \eta_k$ for all $k \in \mathbb{N}$, whence in particular $|v(\cdot, t_k) - (e^b-1)|^{\frac{1}{2}} to 0$ in $L^1(\Omega)$ as $k \to \infty$. For a suitably chosen subsequence $(t_{k_l})_{l\in\mathbb{N}}$ of $(t_k)_{k\in\mathbb{N}}$, we can thus achieve that $v(\cdot, t_{k_l} - (e^b - 1) \to 0$ a.e. in Ω and hence, by $[0,\infty]$-valued continuity of $0 \le z \mapsto \ln_+ \frac{1}{z}$, that $\ln_+ \frac{1}{v(\cdot,t_{k_l})} \to \ln_+ \frac{1}{e^b-1} \in [0,\infty]$ a.e. in Ω as $l \to \infty$. In light of (6.31), however, due to Fatou's lemma this implies that

$$\int_\Omega \ln_+ \frac{1}{e^b - 1} \le \liminf_{l\to\infty} \int_\Omega \ln_+ \frac{1}{v(\cdot, t_{k_l})} \le c_1$$

and thereby shows that indeed $e^b - 1$ must be positive. □

6.5 Proof of Theorem 1.3

Summarizing the above, we obtain the claimed results on stabilization in all three solution components:

Proof of Theorem 1.3. The statements on convergence of u and w have precisely been established in Lemma 6.4 and Lemma 6.7. To verify the coresponding stabilization property of v, we take the constant b as provided by Lemma 6.8, and let $v_\infty := e^b - 1$ to indeed obtain (1.14) as a direct consequence of Lemma 6.10, whereas positivity of b, and hence of v_∞, is asserted by Lemma 6.12. □

7 Appendix: Details in preparing Lemma 3.2

Mainly in order to enlighten the origin of our assumption (1.5)-(1.6), and especially the particular value of $w^\star$ appearing therein, let us finally provide some technical but exclusively elementary details necessary for our derivation of the fundamental Lemma 3.3, as having become manifest in Lemma 3.2. Our first three observations in this direction are concerned with expressions resembling that from (3.2), but in place of the yet free parameter κ involving a number $\theta \in [0,1]$ which will be subject to an optimization procedure below.

Lemma 7.1 *For $d > 0$ and $p > 1$, let*

$$\xi^\pm \equiv \xi^\pm_{d,p}(\theta) := (d-1)\theta \pm 2\sqrt{\frac{d\theta(1-\theta)}{p}}, \qquad \theta \in [0,1]. \tag{7.1}$$

Then for fixed $d > 0, p > 1$ and $\theta \in (0,1)$,

$$I_{d,p,\theta}(\xi) := p\xi^2 - 2p(d-1)\theta\xi + [p(d-1)^2 + 4d]\theta^2 - 4d\theta, \qquad \theta \in \mathbb{R}, \tag{7.2}$$

satisfies

$$I_{d,p,\theta}(\xi) < 0 \quad \textit{if and only if} \quad \xi \in (\xi^-, \xi^+). \tag{7.3}$$

PROOF. As simple computation shows that ξ^- and ξ^+ precisely coincide with the two zeros of $I_{d,p,\theta}$, this is evident from (7.2). □

Lemma 7.2 *Let $d>0, p>1$ and $\theta\in[0,1]$. Then the number $\xi^-_{d,p}(\theta)$ from (7.1) satisfies*

$$\xi^-_{d,p}(\theta)\le 0 \quad \textit{if and only if} \quad \begin{cases} d\le 1,\ p>1 \ \textit{and}\ \theta\in[0,1], & \textit{or} \\ d>1,\ p>1 \ \textit{and}\ \theta\in[0,\theta^\star_{p,d}], \end{cases} \tag{7.4}$$

where for $d>1$,

$$\theta^\star_{p,d} := \frac{4d}{p(d-1)^2+4d}\in(0,1). \tag{7.5}$$

PROOF. This can be verified by elementary calculation. □

Lemma 7.3 *Let $\chi_1>0, d>0, p>1$ and $\theta\in[0,1]$, and suppose that $s^\star>0$ and $\delta>s^\star$. Then*

$$J_{d,p,\theta}(s) := p\chi_1^2(\delta-s)^2-2p(d-1)\theta\chi_1(\delta-s)+[p(d-1)^2+4d]\theta^2-4d\theta, \qquad s\in[0,s^\star], \tag{7.6}$$

defines a function which is negative throughout $[0,s^\star]$ if and only if with $\xi^\pm_{d,p}$ taken from (7.1) we have

$$s^\star+\frac{\xi^-_{d,p}(\theta)}{\chi_1}<\delta<\frac{\xi^+_{d,p}(\theta)}{\chi_1}. \tag{7.7}$$

PROOF. For $s\in[0,s^\star]$ abbreviating $\xi\equiv\xi(s):=\chi_1\cdot(\delta-s)$, we see that the inequality $J_{d,p,\theta}(s)<0$ is precisely equivalent to saying that the expression defined in (7.2) satisfies $I_{d,p,\theta}(\xi)<0$. Therefore, (7.7) implies negativity of $J_{d,p,\theta}$ throughout $[0,s^\star]$, because the right inequality therein warrants that $\xi=\chi_1\cdot(\delta-s)\le\xi_1\delta<\xi^+_{d,p}(\theta)$, and because according to the first condition implied by (7.7) we then know that $\xi\ge\chi_1\cdot(\delta-s^\star)>\xi^-_{d,p}(\theta)$. The necessity of (7.7) for negativity of $\max_{[s\in[0,s^\star]}J_{d,p,\theta}(s)$ can be seen similarly. □

Indeed, the need for a maximization process is indicated by the following.

Lemma 7.4 *For $d>0$ an $p>1$, let*

$$\rho^{(1)}_{d,p}(\theta) := (d-1)\theta+2\sqrt{\frac{d\theta(1-\theta)}{p}}, \qquad \theta\in[0,1], \tag{7.8}$$

and

$$\rho^{(2)}_{d,p}(\theta) := 4\sqrt{\frac{d\theta(1-\theta)}{p}}, \qquad \theta\in[0,1], \tag{7.9}$$

as well as

$$\rho_{d,p}(\theta) := \begin{cases} \rho^{(1)}_{d,p}(\theta) & \textit{if either } d\le 1 \textit{ and } \theta\in[0,1], \\ & \textit{or } d>1 \textit{ and } \theta\in[0,\theta^\star_{d,p}], \\ \rho^{(2)}_{d,p}(\theta) & \textit{if } d>1 \textit{ and } \theta\in(\theta^\star_{d,p},1], \end{cases} \tag{7.10}$$

with $\theta^\star_{d,p}$ taken from (7.5). Then given $s^\star>0$, one can find $\delta>s^\star$ and $\theta\in[0,1]$ fulfilling (7.7) if and only if

$$s^\star<\frac{1}{\chi_1}\cdot\max_{\tilde\theta\in[0,1]}\rho_{d,p}(\tilde\theta). \tag{7.11}$$

Proof. Since by (7.1) we have

$$\rho^{(1)}_{d,p}(\theta) = \xi^+_{d,p}(\theta) \quad \text{and} \quad \rho^{(2)}_{d,p}(\theta) = \xi^+_{d,p}(\theta) - \xi^-_{d,p}(\theta) \qquad \text{for } d > 1, p > 1 \text{ and } \theta \in [0,1],$$

we only need to observe that if $\xi^-_{d,p}(\theta) > 0$, and hence equivalently $d > 1$ and $\theta > \theta^\star_{p,d}$ by Lemma 7.2, then in (7.7) we have

$$\frac{\xi^+_{d,p}(\theta)}{\chi_1} - \left\{ s^\star + \frac{\xi^-_{d,p}(\theta)}{\chi_1} \right\} = \frac{\xi^+_{d,p}(\theta) - \xi^-_{d,p}(\theta)}{\chi_1} - s^\star = \frac{\rho^{(2)}_{d,p}(\theta)}{\chi_1} - s^\star = \frac{\rho_{d,p}(\theta)}{\chi_1} - s^\star$$

by (7.10), whereas in $\xi^-_{d,p}(\theta) \le 0$, then, similarly,

$$\frac{\xi^+_{d,p}(\theta)}{\chi_1} - s^\star = \frac{\rho^{(1)}_{d,p}(\theta)}{\chi_1} - s^\star = \frac{\rho_{d,p}(\theta)}{\chi_1} - s^\star$$

according to Lemma 7.2 and (7.10). In both these cases, namely, choosing $\delta > s^\star$ such that (7.7) holds with some $\theta \in [0,1]$ is thus equivalent to finding $\theta \in [0,1]$ such that $\frac{\rho_{d,p}(\theta)}{\chi_1} - s^\star > 0$, as claimed. □

We next separately address the corresponding maximization problems for $\rho^{(1)}_{d,p}$ and $\rho^{(2)}_{d,p}$.

Lemma 7.5 *Let $d > 0$ and $p > 1$. Then with*

$$\theta^+_{d,p} := \frac{1}{2} + \frac{d-1}{2\sqrt{(d-1)^2 + \frac{4d}{p}}} \in (0,1), \tag{7.12}$$

$\rho^{(1)}_{d,p}$ is increasing on $(0, \theta^+_{d,p})$ and decreasing on $(\theta^+_{d,p}, 1)$, and

$$\max_{\theta \in [0,1]} \rho^{(1)}_{d,p}(\theta) = \rho^{(1)}_{d,p}(\theta^+_{d,p}) = \frac{d-1+\sqrt{(d-1)^2 + \frac{4d}{p}}}{2}. \tag{7.13}$$

Proof. By differentiation in (7.8), we see that

$$\frac{d}{d\theta}\rho^{(1)}_{d,p}(\theta) = d - 1 + \sqrt{\frac{d}{p\theta(1-\theta)}} \cdot (1-2\theta), \qquad \theta \in (0,1),$$

vanishes precisely for $\theta = \theta^+_{d,p}$. Computing

$$\rho^{(1)}_{d,p}(\theta^+_{d,p}) = \frac{d-1+\sqrt{(d-1)^2 + \frac{4d}{p}}}{2}$$

we thereby obtain (7.13), whereupon the claimed monotonicity properties become evident. □

Lemma 7.6 *Let $d > 0$ and $p > 1$, and let $\theta^\star_{d,p} \in (0,1)$ and $\rho^{(1)}_{d,p}$ be as in (7.5) and (7.8). Then*

$$\max_{\theta \in [0,\theta^\star_{d,p}]} \rho^{(1)}_{d,p}(\theta) = \begin{cases} \frac{d-1+\sqrt{(d-1)^2+\frac{4d}{p}}}{2} & \text{if } p(d-1)^2 \le \frac{4d}{3}, \\ \frac{8d(d-1)}{p(d-1)^2+4d} & \text{if } p(d-1)^2 > \frac{4d}{3}. \end{cases} \tag{7.14}$$

Proof. With $\theta^+_{d,p}$ taken from (7.12), we note that

$$2(\theta^+_{d,p} - \theta^\star_{d,p}) = \frac{p(d-1)\sqrt{(d-1)^2 + \frac{4d}{p}} - 4d + p(d-1)^2}{p(d-1)^2 + 4d}$$

is nonpositive if and only if $z := p(d-1)^2$ satisfies $z < 4d$ and $pz \cdot \{(d-1)^2 + \frac{4d}{p}\} \le (4d - z)^2$, that is, if and only if

$$z^2 + 4dz \le 16d^2 - 8dz + z^2,$$

which is precisely equivalent to the inequality $p(d-1)^2 \le \frac{4d}{3}$. For any such d and p, we thus have $\theta^+_{d,p} \le \theta^\star_{d,p}$, so that (7.14) then results from Lemma 7.5.

If, conversely, $p(d-1)^2 > \frac{4d}{3}$, then $\theta^+_{d,p} > \theta^\star_{d,p}$ and hence the upward monotonicity of $\rho^{(1)}_{d,p}$ on $(0, \theta^+_{d,p})$, as asserted by Lemma 7.5, upon a straightforward computation implies that

$$\max_{\theta \in [0, \theta^\star_{d,p}]} \rho^{(1)}_{d,p}(\theta) = \rho^{(1)}_{d,p}(\theta^\star_{d,p}) = \frac{8d(d-1)}{p(d-1)^2 + 4d}$$

in this case. □

Lemma 7.7 *Let $d > 0$ and $p > 1$, and let $\theta^\star_{d,p} \in (0,1)$ and $\rho^{(2)}_{d,p}$ be as defined in (7.5) and (7.9). Then*

$$\max_{\theta \in [\theta^\star_{d,p}, 1]} \rho^{(2)}_{d,p}(\theta) = \begin{cases} \frac{8d(d-1)}{p(d-1)^2+4d} & \text{if } p(d-1)^2 \le 4d, \\ 2\sqrt{\frac{d}{p}} & \text{if } p(d-1)^2 > 4d. \end{cases} \tag{7.15}$$

Proof. It can readily be verified that $\rho^{(2)}_{d,p}$ is increasing on $(0, \frac{1}{2})$ and decreasing on $(\frac{1}{2}, 1)$ with

$$\max_{\theta \in [0,1]} \rho^{(2)}_{d,p}(\theta) = \rho^{(2)}_{d,p}\Big(\frac{1}{2}\Big) = 2\sqrt{\frac{d}{p}}.$$

Since $\theta^\star_{d,p} < \frac{1}{2}$ if and only if again writing $z := p(d-1)^2$ we have $\frac{4d}{z+4d} < \frac{1}{2}$, that is, if and only if $z > 4d$, by computing $\rho^{(2)}_{d,p}(\theta^\star_{d,p}) = \frac{8d(d-1)}{p(d-1)^2+4d}$ we immediately obtain (7.15). □

In summary, we can precisely identify the maximum of $\rho_{d,p}$ as follows.

Lemma 7.8 *Let $d > 0$ and $p > 1$. Then*

$$\max_{\theta \in [0,1]} \rho_{d,p}(\theta) = P(d,p) := \begin{cases} \frac{d-1+\sqrt{(d-1)^2+\frac{4d}{p}}}{2} & \text{if } p(d-1)_+^2 \le \frac{4d}{3}, \\ \frac{8d(d-1)}{p(d-1)^2+4d} & \text{if } \frac{4d}{3} < p(d-1)_+^2 \le 4d, \\ 2\sqrt{\frac{d}{p}} & \text{if } 4d < p(d-1)_+^2 > . \end{cases} \tag{7.16}$$

PROOF. In view of (7.10), this directly results on combining Lemma 7.5 with Lemma 7.6) and Lemma 7.7. □

In order to allow for a definition of $w^\star$ which is independent of our particular choice of $p \in [2, p_0]$ in Lemma 3.3, let us finally state a rather immediate monotonicity property of the above maximizer with respect to p.

Lemma 7.9 *The function P defined through (7.16) belongs to $W^{1,\infty}_{loc}((0,\infty)\times(1,\infty))$ and satisfies $\frac{\partial P}{\partial p} \le 0$ a.e. in $(0,\infty)\times(1,\infty)$.*

PROOF. By direct computation on the basis of (7.16), it can easily be verified that P is continuous in $(0,\infty)\times(1,\infty)$ and hence, according to its evident smoothness properties outside the regions $\{p(d-1)^2_+ = \frac{4d}{3}\}$ and $\{p(d-1)^2 = 4d\}$, indeed is locally Lipschitz continuous in $(0,\infty)\times(1,\infty)$. Nonpositivity of $\frac{\partial P}{\partial p}$ thereafter immediately follows from (7.16). □

We can thereby verify the main outcome of this appendix, as already referred to in the main body of our above analysis:

PROOF of Lemma 3.2. According to (1.5) and Lemma 3.1, we can find $s^\star \in (0, w^\star)$ such that

$$w_\varepsilon(x,t) \le s^\star \qquad \text{for all } x \in \Omega, t > 0 \text{ and } \varepsilon \in (0,1), \tag{7.17}$$

and using that with P as in (7.16) we have $w^\star = \frac{P(d,4)}{\chi_1}$, by continuity of $P(d,\cdot)$ we can fix $p_0 > 4$ such that still

$$\frac{P(d,p_0)}{\chi_1} > s^\star.$$

Then given any $p \in [2, p_0]$, according to the monotonicity property of P from Lemma 7.9 we know that

$$\frac{P(d,p)}{\chi_1} \ge \frac{P(d,p_0)}{\chi_1} > s^\star,$$

whence by definition (7.16) of $P(d,p)$ we can find $\theta \in [0,1]$ such that $\rho_{d,p}$ defined in (7.10) satisfies

$$\frac{\rho_{d,p}(\theta)}{\chi_1} = \frac{P(d,p)}{\chi_1} > s^\star.$$

Now as a consequence of Lemma 7.4 and Lemma 7.3, this in turn enables us to pick $\delta > s^\star$ and $c_1 > 0$ such that

$$p\chi_1^2(\delta-s)^2 - 2p(d-1)\theta\chi_1(\delta-s) + [p(d-1)^2 + 4d]\theta^2 - 4d\theta \le -c_1 \qquad \text{for all } [0, s^\star].$$

By means of a simple argument based on continuous dependence, this guarantees that with some $\eta \in (0,1)$ suitably close to 1 we have

$$p\chi_1^2(\delta-s)^2 - 2p(d+1-2\eta)\theta\chi_1(\delta-s) + [p(d+1)^2 - 4(p-1)d\eta]\theta^2 - 4d\eta\theta \le -\frac{c_1}{2} \qquad \text{for all } [0, s^\star].$$

Upon multiplication by $(p-1)^2$, due to (7.17) this shows if we let

$$\kappa := (p-1)\theta,$$

then for all $\varepsilon \in (0,1)$,

$$p(p-1)^2\chi_1^2(\delta-w_\varepsilon)^2 - 2p(p-1)(d+1-2\eta)\theta\chi_1(\delta-w_\varepsilon) + [p(d+1)^2 - 4(p-1)d\eta]\kappa^2 - 4(p-1)d\eta\kappa \le -\frac{c_1(p-1)^2}{2} \qquad \text{in } \Omega\times(0,\infty),$$

which can readily be seen to be equivalent to (3.2) with some suitably some $C>0$. □

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