Global solvability and stabilization in a two-dimensional cross-diffusion system modeling urban crime propagation

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Abstract

The system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot \left(\frac{u}{v} \nabla v \right) - uv + B_1(x,t), \\ v_t = \Delta v + uv - v + B_2(x,t), \end{cases} \qquad (\star)$$

is considered in a disk $\Omega \subset \mathbb{R}^2$, with a positive parameter χ and given nonnegative and suitably regular functions B_1 and B_2 defined on $\Omega \times (0,\infty)$. In the particular version obtained when $\chi = 2$, $(\star)$ was proposed in [33] as a model for crime propagation in urban regions.

Within a suitable generalized framework, it is shown that under mild assumptions on the parameter functions and the initial data the no-flux initial-boundary value problem for $(\star)$ possesses at least one global solution in the case when all model ingredients are radially sysmmetric with respect to the center of Ω. Moreover, under an additional hypothesis on stabilization of the given external source terms in both equations, these solutions are shown to approach the solution of an elliptic boundary value problem in an appropriate sense.

The analysis is based on deriving a priori estimates for a family of approximate problems, in a first step achieving some spatially global but weak initial regularity information which in a series of spatially localized arguments is thereafter successively improved.

To the best of our knowledge, this is the first result on global existence of solutions to the two-dimensional version of the full original system $(\star)$ for arbitrarily large values of χ.

Keywords: urban crime; renormalized solutions; global existence; a priori estimates
MSC (2010): 35Q91 (primary); 35B40, 35K55, 35D30 (secondary)

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1 Introduction

Describing complex social systems by means of continuous models based on evolutionary partial differential equations has received considerable interest in the past few years, where a particular focus is on systems exhibiting certain tendencies toward spontaneous development of heterogeneous structures ([2], [9]). A phenomenon of this type which is of evident relevance consists in the emergence of high burglary activity within certain regions, that is, the formation of so-called crime hotspots. Going back to [33], a seminal modeling approach for such phenomena assumes that besides the density of criminal agents, the only relevant unknown is the so-called attractiveness value as an abstract quantity. Letting $u = u(x,t)$ denote the former and $v = v(x,t)$ the latter, based on a series of sociological hypotheses relying on corresponding statistical observations the authors therein are led to a two-component parabolic system of the form

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot\left(\frac{u}{v}\nabla v\right) - uv + B_1(x,t), & x\in\Omega,\ t>0, \\ v_t = \Delta v + uv - v + B_2(x,t), & x\in\Omega,\ t>0, \end{cases} \tag{1.1}$$

in a planar urban region Ω. Here, the given function B_1 represents the density of additional criminal agents entering the system, and B_2 reflects sources of attractiveness of certain sites to criminal activity which are possibly present even in the absence of any criminal agent. We emphasize that (1.1) moreover accounts for an advective movement of criminals toward increasing attractiveness values through the taxis-type term $-\chi\nabla\cdot(\frac{u}{v}\nabla v)$ in which the particular model derivation in [33] suggests to choose $\chi = 2$. Apart from that, it is assumed that individuals tend to refrain from repeatedly committing crimes, and that criminal activity increases attractiveness of a given neighborhood ([20], [32]), as becoming manifest in the summands $-uv$ and $+uv$ in the first and the second equation in (1.1), respectively. For details on the sociological background and further modeling aspects related to (1.1), we refer to [21], [7] and [11], to [33] and [29] and also to the references in the latter.

From a mathematical point of view, through the cross-diffusive interaction expressed in its first equation the system (1.1) shares its probably most characteristic model ingredients with the celebrated Keller-Segel system from mathematical biology which in its original form, corresponding to the choice $S \equiv const.$ in

$$\begin{cases} u_t = \Delta u - \nabla\cdot(uS(v)\nabla v), & x\in\Omega,\ t>0, \\ v_t = \Delta v - v + u, & x\in\Omega,\ t>0, \end{cases} \tag{1.2}$$

is able to describe structure generation phenomena even in the extreme sense of finite-time blow-up, as known to occur for some solutions in bounded domains $\Omega\subset\mathbb{R}^n$ with either $n=2$ or also $n\geq 3$ ([17], [39]). Even in the presence of an additional dampening effect on cross-diffusion at large signal densities such as induced by choosing $S(v) = \frac{\chi}{v}$ like in (1.1), it is not clear up to now how far the corresponding version of (1.2) possesses globally defined solutions; indeed, partial results suggest that with regard to this question, in the latter system the size of the value $\chi > 0$ plays an important role: If $\chi < \sqrt{\frac{2}{n}}$, then for all reasonably regular initial data global smooth solutions exist ([38]), where in the case $n=2$ it could recently be shown that actually $\chi < \chi_0$ with some $\chi_0 > 1.015$ is sufficient for this conclusion ([24]). For larger ranges of χ including the value $\chi = 2$ relevant to (1.1), however, global existence results are available only for certain types of weak solutions ([38], [34], [25]), or restricted to parabolic-elliptic simplifications of (1.2) ([14]). An additional caveat is provided by a known result on the existence of blow-up solutions to the latter variants of (1.2) in higher-dimensional settings ([14]).

As for the crime model (1.1), both numerical simulations ([16]) as well as analytical results on existence and linear stability of spatially inhomogeneous solutions to the associated steady-state system ([4], [6], [16], [22], [36]) indicate that this system indeed is able to adequately describe the spontaneous emergence of localized patterns representing hotspots. However, a rigorous theory on global solvability in initial-value problems associated with (1.1), even in generalized frameworks, is apparently lacking yet, especially for solutions far from spatial homogeneity. This may reflect the additionally complicating circumstance that as a further feature which marks a fundamental difference to (1.2), instead of the linear source $+u$ the system (1.1) contains the nonlinear signal production term $+uv$ in its second equation, which is of comparable size only as long as v is known to be bounded. The additional mathematical challenges going along with this more complex and potentially more destabilizing type of coupling evidently require new analytical approaches to be dealt with adequately, and accordingly, beyond a result on local existence and uniqueness obtained in [30], only quite little is known for the full parabolic system (1.1) with regard to issues from basic existence theory.

After all, in the spatial one-dimensional case the no-flux initial-boundary value problem associated with (1.1) in bounded intervals $\Omega \subset \mathbb{R}$ has recently been shown to possess globally defined classical solutions for suitably smooth initial data which are positive in their second component ([31]). In the corresponding two-dimensional situation, however, global smooth solutions so far have been found exclusively under appropriate restrictions either requiring that $\chi < 1$ ([12]), or that the coefficient functions B_1 and B_2 and the initial data satisfy certain further assumptions, inter alia fulfilled if the quadruple $(B_1, B_2, u(\cdot,0), v(\cdot,0))$ is suitably close to the spatially homogeneous steady-state constellation $(0,a,0,a)$ for some $a>0$ ([35]). Only for a certain variant of (1.1), including an additional dissipative effect in the second equation that a priori ensures boundedness of the quantity v, global existence could be achieved for a slightly larger class of initial data, though yet far from covering arbitrarily large initial values ([26]).

Main results. The first objective of the present work consists in developing a basic theory of global existence for solutions to (1.1) without imposing any smallness conditions on the critical parameter χ nor on the source terms B_1 and B_2 and the initial data. Specifically, we shall be concerned with the initial-boundary value problem

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot \left(\frac{u}{v} \nabla v \right) - uv + B_1(x,t), & x \in \Omega,\ t>0, \\ v_t = \Delta v + uv - v + B_2(x,t), & x \in \Omega,\ t>0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t>0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{1.3}$$

in the disk $\Omega := B_R(0) \subset \mathbb{R}^2$, where $R>0$, where B_1 and B_2 are sufficiently regular nonnegative functions on $\Omega \times (0,\infty)$, and where $\chi > 0$ is an arbitrary positive parameter.

Then resorting to suitably generalized solution concepts, and to situations when all ingredients are radially symmetric, with regard to global solvability we shall obtain the following comprehensive result.

Theorem 1.1 *Let $R>0$ and $\Omega = B_R(0) \subset \mathbb{R}^2$, and suppose that with some $\vartheta \in (0,1)$, B_1 and B_2 are nonnegative functions from $C^{\vartheta}_{loc}(\overline{\Omega} \times [0,\infty))$ which are radially symmetric with respect to $x \in \Omega$. Then for each pair of radially symmetric functions $u_0 \in C^{\vartheta}(\overline{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ which are such that $u_0 \ge 0$ and $v_0 > 0$ in $\overline{\Omega}$, the problem (1.3) possesses at least one global renormalized solution (u,v) in the sense* *of Definition 6.1 below. This solution has the additional properties that*

$$\text{both } u \text{ and } v \text{ belong to } C^0((\overline{\Omega}\setminus\{0\})\times[0,\infty))\cap C^{2,1}((\overline{\Omega}\setminus\{0\})\times(0,\infty)), \tag{1.4}$$

that

$$v>0 \quad \text{in } (\overline{\Omega}\setminus\{0\})\times[0,\infty), \tag{1.5}$$

that for each $T>0$ one can find $C(T)>0$ fulfilling

$$\int_\Omega u(\cdot,t)\le C(T) \quad \text{and} \quad \int_\Omega v(\cdot,t)\le C(T) \qquad \text{for all } t\in(0,T), \tag{1.6}$$

and that for arbitrary $T>0$ we furthermore have

$$\int_0^T\int_\Omega uv<\infty \tag{1.7}$$

and

$$\begin{aligned}
&\int_0^T\int_\Omega u^p<\infty \quad \text{for all } p\in\Big[1,1+\min\Big\{1,\frac{1}{\chi^2}\Big\}\Big) \qquad \text{and}\\
&\int_0^T\int_\Omega u^{p-2}|\nabla u|^2<\infty \quad \text{for all } p\in\Big(0,\min\Big\{1,\frac{1}{\chi^2}\Big\}\Big)
\end{aligned} \tag{1.8}$$

as well as

$$\begin{aligned}
&\int_0^T\int_\Omega v^p<\infty \quad \text{for all } p\in(-\infty,2) \qquad \text{and}\\
&\int_0^T\int_\Omega v^{p-2}|\nabla v|^2<\infty \quad \text{for all } p\in(-\infty,1).
\end{aligned} \tag{1.9}$$

Since we are not aware of any regularity properties beyond those in (1.4) and (1.6)-(1.9), in this general context we presently cannot exclude possibly irregular solution behavior. After all, from (1.8) we can infer that unlike in the two-dimensional classical Keller-Segel system, possibly occurring blow-up phenomena involving unboundedness of the solution component u cannot result in the emergence of Dirac-type singularities at some finite time beyond which they persistently remain present; namely, for arbitrary $p\in[1,1+\min\{1,\frac{1}{\chi^2}\})$ – that is, for $p\in[1,\frac{5}{4})$ in the relevant case $\chi=2$ – the first property in (1.8) in fact rules out even any collapse into temporally stable singularities not belonging to $L^p(\Omega)$.

Next addressing the problem of describing the qualitative behavior of these solutions, we first observe that under the above general assumptions on the source terms in (1.3), inter alia allowing for unbounded functions B_1 and B_2, we evidently cannot expect solutions to stabilize in the large time limit, and not even to remain bounded with respect to the spatial L^1 norms of their components. In order to derive further information on the asymptotic behavior of the solutions found above, we will accordingly rely on additional requirements on B_2 and B_2; in particular, throughout our analysis in this direction we shall suppose that

$$\begin{cases}
B_1\in L^\infty(\Omega\times(0,\infty))\cap L^1(\Omega\times(0,\infty)) \qquad \text{and}\\
B_2(\cdot,t)\to B_{2,\infty} \text{ in } L^\infty(\Omega) \ \text{ as } t\to\infty \text{ with some } 0\not\equiv B_{2,\infty}\in C^0(\overline{\Omega})
\end{cases} \tag{H}$$

Then, formally, in the large time limit the boundary value problem in (1.3) approaches the corresponding system associated with the temporally constant sources given by $B_1 \equiv 0$ and $B_2 \equiv B_{2,\infty}$ which possesses the nontrivial steady state $(0, v_\infty)$, where $v_\infty \in \bigcap_{p>1} W^{2,p}(\Omega)$ is the unique (weak) solution of the Helmholtz problem

$$\begin{cases} -\Delta v_\infty + v_\infty = B_{2,\infty}, & x \in \Omega, \\ \frac{\partial v_\infty}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{1.10}$$

and since $B_{2,\infty} \not\equiv 0$, it follows that v_∞ is positive in $\overline{\Omega}$.

In fact, we shall see that under these extra assumptions, this equilibrium attracts any of the solutions constructed above in an appropriate sense:

Theorem 1.2 *Suppose that beyond the assumptions from Theorem 1.1, the hypothesis (H) holds. Then the global renormalized solution (u, v) of (1.3) found in Theorem 1.1 has the properties that as $t \to \infty$ we have*

$$u(\cdot, t) \to 0 \qquad \textit{in } L^1(\Omega) \textit{ and in } C^2_{loc}(\overline{\Omega} \setminus \{0\}) \tag{1.11}$$

and

$$v(\cdot, t) \to v_\infty \qquad \textit{in } L^1(\Omega) \textit{ and in } C^2_{loc}(\overline{\Omega} \setminus \{0\}) \tag{1.12}$$

as well as

$$\int_t^{t+1} \|u(\cdot, s)\|^q_{L^p(\Omega)} ds \to 0 \qquad \textit{for all } p > 1 \textit{ and } q > 0 \textit{ such that } q < \frac{p}{p-1} \cdot \min\left\{1, \frac{1}{\chi^2}\right\} \tag{1.13}$$

and

$$\int_t^{t+1} \|v(\cdot, s) - v_\infty\|^q_{L^p(\Omega)} ds \to 0 \qquad \textit{for all } p > 1 \textit{ and } q > 0 \textit{ such that } q < \frac{p}{p-1}, \tag{1.14}$$

where v_∞ denotes the solution of (1.10). In particular, for each $p > 1$ and any $k \in \mathbb{N}$ one can find $t_k \in (k, k+1)$ such that

$$u(\cdot, t_k) \to 0 \qquad \textit{in } L^p(\Omega) \tag{1.15}$$

and

$$v(\cdot, t_k) \to v_\infty \qquad \textit{in } L^p(\Omega) \tag{1.16}$$

as $k \to \infty$.

Main ideas. As a consequence of the delicate coupling of cross-diffusion and nonlinear signal production in (1.3), standard methods from the regularity theories of reaction-diffusion systems and of cross-diffusive systems seem limited; for instance, functionals which additively combine integrals of the form $\int_\Omega \phi(u)$ with corresponding expressions merely containing either v or its derivatives, while known to play essential roles in large bodies of the analysis of taxis-type parabolic systems (cf. e.g. the survey [3] for examples and references), apparently do no longer enjoy meaningful entropy-like features in (1.3) for any strictly convex ϕ. Beyond easily available bounds on the mass functionals for both solution components (Lemma 2.3) and for space-time L^2 estimates for $\nabla v^{\frac{p}{2}}$ for arbitrary $p < 1$ (Lemma 3.1), our approach will accordingly need to be based, at the spatially global level, on the rather poor knowledge obtained from an entropy-type property of integrals of the form

$$\int_\Omega u^p v^q$$

with suitably small $p \in (0,1)$ and certain $q \in (0,1-p)$, thus exhibiting only sublinear growth with respect to (u,v) (Lemma 3.3). After all, exploiting the regularity information provided by the corresponding dissipation rate, inter alia containing a weighted L^2 norm of ∇u, will yield space-time estimates for u in some reflexive Lebesgue spaces (Lemma 3.4). However, the integrability powers therein may fail to be substantially superlinear when χ is large – for instance, bounds for $\int_0^T \int_\Omega u^p$ will thereby be guaranteed only when $p < 1 + \frac{1}{\chi^2}$, thus e.g. requiring $p < \frac{5}{4}$ when $\chi = 2$. This seems not only inappropriate as a starting point for standard iterative procedures yielding successively improved regularity properties, but moreover also insufficient for the construction of generalized solutions, e.g. through adequate approximation procedures and compactness arguments, within usual concepts of weak solvability for (1.3).

Accordingly, in our subsequent analysis we shall make use of our assumption on radial symmetry which, by enforcing (1.3) to be of essentially one-dimensional structure, will enable us to utilize the spatially global regularity information gathered before in order to derive further estimates which now will be local in the sense of remaining restricted to annuli away from the origin. Since our initial knowledge is yet quite limited here especially for large χ, we will need to proceed along quite a number of technical steps, each of which concentrates on one of the equations in (1.3) and controls the respective crucial nonlinearity therein by making essential use of one-dimensional interpolation arguments based on previously gained estimates. We thereby obtain a series of integral inequalities firstly for v (Section 4.3) and its derivatives (Section 4.4) and then for u (Section 4.5), whereupon Section 5 will provide local estimates in Hölder spaces, firstly for v and u and and then for their derivatives.

Thereafter, in Section 6 we shall introduce a generalized solution concept which, in the style of the notion of renormalized solutions from the celebrated work [8], in weakly formulating the equations from (1.3) for both u and v refers to values of the respective component from finite intervals only (Definition 6.1). This framework will allow us to combine our obtained spatially global and local estimates to verify that indeed a global renormalized solution can be constructed through a limit procedure involving approximation by global smooth solutions of the regularized problems (2.1).

Finally, relying on a careful tracking of the behavior of all essential constants in the presence of (H) throughout our analysis, Section 7 will establish Theorem 1.2.

2 A family of approximate problems and their basic properties

In order to construct a solution of (1.3) through an appropriate approximation procedure, let us modify the crucial signal production term $+uv$ in the second equation therein by introducing an artificial small saturation mechanism. Accordingly, for $\varepsilon \in (0,1)$ we shall consider the approximate problem

$$\begin{cases} u_{\varepsilon t} = \Delta u_\varepsilon - \chi \nabla \cdot \left(\frac{u_\varepsilon}{v_\varepsilon} \nabla v_\varepsilon \right) - u_\varepsilon v_\varepsilon + B_1(x,t), & x \in \Omega,\ t > 0, \\ v_{\varepsilon t} = \Delta v_\varepsilon + \frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} - v_\varepsilon + B_2(x,t), & x \in \Omega,\ t > 0, \\ \frac{\partial u_\varepsilon}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x,0) = u_0(x), \quad v_\varepsilon(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{2.1}$$

which in fact allows for classical local-in-time solvability and extensibility in the following sense.

Lemma 2.1 *Under the assumptions of Theorem 1.1, for each $\varepsilon \in (0,1)$ there exist $T_{max,\varepsilon} \in (0,\infty]$ and a uniquely determined pair $(u_\varepsilon, v_\varepsilon)$ of radially symmetric functions*

$$\begin{cases} u_\varepsilon \in C^0(\overline{\Omega} \times [0,T_{max,\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{max,\varepsilon})), \\ v_\varepsilon \in \bigcap_{p>2} C^0([0,T_{max,\varepsilon}); W^{1,p}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,T_{max,\varepsilon})), \end{cases}$$

which solve (2.1) classically in $\overline{\Omega} \times [0,T_{max,\varepsilon})$, and which are such that $u_\varepsilon > 0$ in $\overline{\Omega} \times (0,T_{max,\varepsilon})$ and $v_\varepsilon > 0$ in $\overline{\Omega} \times [0,T_{max,\varepsilon})$, and that

$$\begin{aligned} &\text{if } T_{max,\varepsilon} < \infty \qquad \text{then} \\ &\qquad \limsup_{t \nearrow T_{max,\varepsilon}} \Big\{ \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \Big\| \frac{1}{v_\varepsilon(\cdot,t)} \Big\|_{L^\infty(\Omega)} + \|\nabla v_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \Big\} = \infty \quad \text{for all } p > 2. \end{aligned} \tag{2.2}$$

PROOF. This can be seen in a straightforward manner by application of well-established techniques from the theory of cross-diffusive systems, especially of chemotaxis type (see e.g. [1] or also [19]). □

A first and elementary but nevertheless quite important property of (1.3) and also of (2.1) consists in the fact that the interplay of diffusion and the nonnegative source B_2 in the second equations therein prevents the second solution component to approach the value zero, and thus rules out the emergence of singularities in the factor $\frac{1}{v_\varepsilon}$ appearing in the cross-diffusive term in (2.1).

Lemma 2.2 *For all $T > 0$ there exists $C(T) > 0$ such that for all $\varepsilon \in (0,1)$, writing $\widehat{T}_{max,\varepsilon} := \min\{T, T_{max,\varepsilon}\}$ we have*

$$v_\varepsilon(x,t) \ge C(T) \qquad \text{for all } x \in \Omega \text{ and } t \in (0,T) \tag{2.3}$$

and

$$\inf_{T>0} C(T) > 0 \qquad \text{if (H) holds.} \tag{2.4}$$

PROOF. By nonnegativity of $u_\varepsilon, v_\varepsilon$ and B_2, v_ε satisfies

$$v_{\varepsilon t} \ge \Delta v_\varepsilon - v_\varepsilon \qquad \text{in } \Omega \times (0,T_{max,\varepsilon}).$$

Writing

$$\underline{v}(x,t) := y(t), \qquad x \in \overline{\Omega},\ t \ge 0,$$

where y solves

$$\begin{cases} y'(t) = -y(t), \qquad t > 0, \\ y(0) = \inf_{x\in\Omega} v_0(x), \end{cases}$$

we thus infer that since $\frac{\partial \underline{v}}{\partial \nu} = 0$ on $\partial\Omega \times (0,\infty)$, the comparison principle asserts that $v_\varepsilon \ge \underline{v}$ and hence

$$v_\varepsilon(x,t) \ge e^{-t} \cdot \inf_{y\in\Omega} v_0(y) \qquad \text{for all } x \in \Omega \text{ and } t \in (0,T_{max,\varepsilon}). \tag{2.5}$$

For general B_1 and B_2 satisfying the assumptions from Theorem 1.1, due to our assumption that v_0 be positive in $\overline{\Omega}$ this already establishes (2.3) with $C(T) := e^{-T} \cdot \inf_{y\in\Omega} v_0(y)$, and to see that under the additional hypotheses in (H) actually (2.4) can be achieved, we note that the stabilization property of B_2 required in (H) entails the existence of $t_0 > 0$ and $c_1 > 0$ fulfilling

$$\int_\Omega B_2(\cdot,t) \ge c_1 \qquad \text{for all } t > t_0.$$

Since due to the convexity of Ω it is possible to find $c_2 > 0$ such that the Neumann heat semigroup $(e^{t\Delta})_{t\ge 0}$ on Ω satisfies

$$e^{t\Delta}\varphi \ge c_2 \int_\Omega \varphi \qquad \text{for all nonnegative } \varphi \in C^0(\overline{\Omega})$$

([13], [18]), this implies that

$$\begin{aligned} \int_0^t e^{-(t-s)} e^{(t-s)\Delta} B_2(\cdot,s)ds &\ge c_1c_2 \int_{t_0}^t e^{-(t-s)}ds \\ &= c_1c_2(1-e^{-(t-t_0)}) \\ &\ge c_1c_2(1-e^{-1}) \quad \text{in } \Omega \qquad \text{for all } t > t_0+1. \end{aligned}$$

As $(e^t v_\varepsilon)_t \ge \Delta(e^t v_\varepsilon) + e^t B_2(x,t)$ in $\Omega \times (0, T_{max,\varepsilon})$ and hence

$$v_\varepsilon(\cdot,t) \ge \int_0^t e^{-(t-s)} e^{(t-s)\Delta} B_2(\cdot,s)ds \qquad \text{in } \Omega \qquad \text{for all } t \in (0, T_{max,\varepsilon}),$$

together with (2.5) this indeed verifies (2.4). □

A second fundamental feature of (2.1) concerns the evolution of the mass functionals $\int_\Omega u_\varepsilon$ and $\int_\Omega v_\varepsilon$, but also space-time integrability of the nonlinear production term $u_\varepsilon v_\varepsilon$ appearing in (2.1).

Lemma 2.3 *For any $T > 0$ one can find $C(T) > 0$ such that for all $\varepsilon \in (0,1)$, writing $\widehat{T}_{max,\varepsilon} := \min\{T, T_{max,\varepsilon}\}$ we have*

$$\int_\Omega u_\varepsilon(\cdot,t) \le C(T) \qquad \textit{for all } t \in (0, \widehat{T}_{max,\varepsilon}) \tag{2.6}$$

and

$$\int_\Omega v_\varepsilon(\cdot,t) \le C(T) \qquad \textit{for all } t \in (0, \widehat{T}_{max,\varepsilon}) \tag{2.7}$$

as well as

$$\int_0^t \int_\Omega u_\varepsilon v_\varepsilon \le C(T) \qquad \textit{for all } t \in (0, \widehat{T}_{max,\varepsilon}), \tag{2.8}$$

and such that moreover

$$\sup_{T>0} C(T) < \infty \quad \textit{if (H) holds.} \tag{2.9}$$

PROOF. We integrate the first equation in (2.1) over $\Omega \times (0,t)$ to see that

$$\int_\Omega u_\varepsilon(\cdot,t) + \int_0^t \int_\Omega u_\varepsilon v_\varepsilon \le \int_\Omega u_0 + \int_0^t \int_\Omega B_1 \qquad \text{for all } t \in (0,\widehat{T}_{max,\varepsilon}). \tag{2.10}$$

Since $\frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} \le u_\varepsilon v_\varepsilon$, by integrating the second equation in (2.1) over Ω we obtain that

$$\begin{aligned} \frac{d}{dt}\int_\Omega v_\varepsilon + \int_\Omega v_\varepsilon &= \int_\Omega \frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} + \int_\Omega B_2 \\ &\le \int_\Omega u_\varepsilon v_\varepsilon + \int_\Omega B_2 \qquad \text{for all } t \in (0,T_{max,\varepsilon}), \end{aligned}$$

and that hence

$$\begin{aligned} \int_\Omega v_\varepsilon(\cdot,t) &\le e^{-t}\int_\Omega v_0 + \int_0^t \int_\Omega e^{-(t-s)} u_\varepsilon(\cdot,s) v_\varepsilon(\cdot,s)ds + \int_0^t \int_\Omega e^{-(t-s)} B_2(\cdot,s)ds \\ &\le \int_\Omega v_0 + \int_0^t \int_\Omega u_\varepsilon v_\varepsilon + |\Omega| \cdot \|B_2\|_{L^\infty(\Omega\times(0,T))} \qquad \text{for all } t \in (0,\widehat{T}_{max,\varepsilon}). \end{aligned}$$

Together with (2.10), this establishes (2.6)-(2.8) and (2.10). $\square$

2.1 Global existence in the approximate problems

Thanks to the efficient dampening of signal production achieved by introducing the factor $\frac{1}{1+\varepsilon u_\varepsilon v_\varepsilon}$ therein, the L^1 boundedness information on u_ε from Lemma 2.3 is already sufficient to guarantee that each of our approximate solutions in fact exists globally.

Lemma 2.4 *Let $\varepsilon \in (0,1)$. Then the solution of (2.1) is global in time; that is, in Lemma 2.1 we have $T_{max,\varepsilon} = \infty$.*

PROOF. Assuming on the contrary that $T_{max,\varepsilon} < \infty$ for some $\varepsilon \in (0,1)$, we write the second equation in (2.1) in the form $v_{\varepsilon t} = \Delta v_\varepsilon - v_\varepsilon + h_\varepsilon$ in $\Omega \times (0,T_{max,\varepsilon})$, where $h_\varepsilon := \frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} + B_2$ satisfies $|h_\varepsilon| \le \frac{1}{\varepsilon} + \|B_2\|_{L^\infty(\Omega\times(0,T_{max,\varepsilon}))}$ in $\Omega \times (0,T_{max,\varepsilon})$. Therefore, standard arguments from parabolic regularity theory (see e.g. [19, Lemma 4.1]) warrant the existence of $c_1(\varepsilon) > 0$ such that with $\tau := \frac{1}{4}T_{max,\varepsilon}$ we have

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^8(\Omega)} \le c_1(\varepsilon) \qquad \text{for all } t \in (\tau,T_{max,\varepsilon}), \tag{2.11}$$

which along with Lemma 2.2 shows that

$$\Big\|\frac{\nabla v_\varepsilon(\cdot,t)}{v_\varepsilon(\cdot,t)}\Big\|_{L^8(\Omega)} \le c_2(\varepsilon) \qquad \text{for all } t \in (\tau,T_{max,\varepsilon}) \tag{2.12}$$

with some $c_2(\varepsilon) > 0$. Now letting $M_\varepsilon(T) := \sup_{t\in(\tau,T)} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$ for $T \in (\tau,T_{max,\varepsilon})$, in the variation-of-constants representation

$$\begin{aligned} u_\varepsilon(\cdot,t) &= e^{t\Delta}u_\varepsilon(\cdot,\tau) - \chi\int_\tau^t e^{(t-s)\Delta}\nabla\cdot\Big(u_\varepsilon(\cdot,s)\frac{\nabla v_\varepsilon(\cdot,s)}{v_\varepsilon(\cdot,s)}\Big)ds \\ &\quad - \int_\tau^t e^{(t-s)\Delta}u_\varepsilon(\cdot,s)v_\varepsilon(\cdot,s)ds + \int_\tau^t e^{(t-s)\Delta}B_1(\cdot,s)ds, \qquad t \in (\tau,T_{max,\varepsilon}), \end{aligned} \tag{2.13}$$

we may combine standard estimates for the Neumann heat semigroup (cf. e.g. [37]) with the Hölder inequality and (2.12) to see that there exists $c_3 > 0$ such that

$$\begin{aligned}
&\Big\| -\chi \int_\tau^t e^{(t-s)\Delta} \nabla \cdot \Big(u_\varepsilon(\cdot,s) \frac{\nabla v_\varepsilon(\cdot,s)}{v_\varepsilon(\cdot,s)} \Big) ds \Big\|_{L^\infty(\Omega)} \\
&\qquad \le c_3 \int_\tau^t (t-s)^{-\frac{3}{4}} \Big\| u_\varepsilon(\cdot,s) \frac{\nabla v_\varepsilon(\cdot,s)}{v_\varepsilon(\cdot,s)} \Big\|_{L^4(\Omega)} ds \\
&\qquad \le c_3 \int_\tau^t (t-s)^{-\frac{3}{4}} \| u_\varepsilon(\cdot,s)\|_{L^8(\Omega)} \Big\| \frac{\nabla v_\varepsilon(\cdot,s)}{v_\varepsilon(\cdot,s)} \Big\|_{L^8(\Omega)} ds \\
&\qquad \le c_3 \int_\tau^t (t-s)^{-\frac{3}{4}} \| u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}^{\frac{7}{8}} \| u_\varepsilon(\cdot,s)\|_{L^1(\Omega)}^{\frac{1}{8}} \Big\| \frac{\nabla v_\varepsilon(\cdot,s)}{v_\varepsilon(\cdot,s)} \Big\|_{L^8(\Omega)} ds \\
&\qquad \le c_3 c_2(\varepsilon) c_4 c_5(\varepsilon) M_\varepsilon^{\frac{7}{8}}(T) \qquad \text{for all } t \in (\tau, T)
\end{aligned}$$

with $c_4 := \int_\tau^{T_{max,\varepsilon}} (t-s)^{-\frac{3}{4}} ds$ and $c_5(\varepsilon) := \sup_{t\in(\tau,T_{max,\varepsilon})} \|u_\varepsilon(\cdot,t)\|_{L^1(\Omega)}$ being finite thanks to Lemma 2.3 and our hypothesis that $T_{max,\varepsilon} < \infty$. Since we may moreover apply the maximum principle to find that

$$\|e^{t\Delta} u_\varepsilon(\cdot,\tau)\|_{L^\infty(\Omega)} \le \|u_\varepsilon(\cdot,\tau)\|_{L^\infty(\Omega)} \qquad \text{for all } t > \tau,$$

that

$$\Big\| \int_\tau^t e^{(t-s)\Delta} B_1(\cdot,s) ds \Big\|_{L^\infty(\Omega)} \le \int_\tau^t \|B_1(\cdot,s)\|_{L^\infty(\Omega)} ds \qquad \text{for all } t > \tau,$$

and that the second last summand on the right of (2.13) is nonpositive, from the latter we conclude that with some $c_6(\varepsilon) > 0$ we have

$$M_\varepsilon(T) \le c_6(\varepsilon) M_\varepsilon^{\frac{7}{8}}(T) + c_6(\varepsilon) \qquad \text{for all } T \in (\tau, T_{max,\varepsilon}),$$

which implies that

$$\limsup_{t \nearrow T_{max,\varepsilon}} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} < \infty.$$

Together with (2.11) and again Lemma 2.2, this establishes a contradiction to (2.2) and thereby shows that actually we must have $T_{max,\varepsilon} = \infty$. □

3 Further spatially global estimates

We proceed to derive some further estimates valid on the entire spatial domain including the origin. Here first considering the second solution component, in essential difference to the situation in Keller-Segel type systems with linear signal production terms of the form $+u$ in their second equation, we need to adequately cope with the circumstance that the L^1 boundedness property (2.6) does not a priori guarantee boundedness of ∇v_ε, locally uniform in time, with respect to the norm in $L^p(\Omega)$ for some $p > 1$, as known to be true in the two-dimensional version of (1.2) for arbitrary $p \in (1,2)$, for instance ([19]). Indeed, in the present case the only regularity information (2.8) on the part $+\frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon}$ of the inhomogeneity f_ε in the linear heat equation $v_{\varepsilon t} = \Delta v_\varepsilon + f_\varepsilon$ in (2.1) merely refers to spatio-*temporal* L^1 norm, so that our immediate conclusion on the regularity of ∇v_ε accordingly becomes sparser:

Lemma 3.1 *Let $p \in (0,1)$. Then for all $T>0$ there exists $C(T)>0$ such that for each $\varepsilon \in (0,1)$,*

$$\int_t^{t+1} \int_\Omega v_\varepsilon^{p-2} |\nabla v_\varepsilon|^2 \le C(T) \qquad \textit{for all } t>0, \tag{3.1}$$

where

$$\sup_{T>0} C(T) < \infty \quad \textit{if (H) holds.} \tag{3.2}$$

Proof. Testing the second equation in (2.1) by the smooth positive function v_ε^{p-1} we obtain

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_\Omega v_\varepsilon^p &= (1-p) \int_\Omega v_\varepsilon^{p-2} |\nabla v_\varepsilon|^2 + \int_\Omega \frac{u_\varepsilon v_\varepsilon^p}{1+\varepsilon u_\varepsilon v_\varepsilon} - \int_\Omega v_\varepsilon^p + \int_\Omega B_2 v_\varepsilon^{p-1} \\ &\ge (1-p) \int_\Omega v_\varepsilon^{p-2} |\nabla v_\varepsilon|^2 - \int_\Omega v_\varepsilon^p \qquad \text{for all } t>0. \end{aligned} \tag{3.3}$$

Since Lemma 2.3 together with the Hölder inequality shows that with some $c_1(T)>0$ we have

$$\int_\Omega v_\varepsilon^p \le |\Omega|^{1-p} \bigg\{ \int_\Omega v_\varepsilon \bigg\}^p \le c_1(T) \qquad \text{for all } t \in (0,T+1),$$

a time integration in (3.3) yields

$$\begin{aligned} (1-p) \int_t^{t+1} \int_\Omega v_\varepsilon^{p-2} |\nabla v_\varepsilon|^2 &\le \frac{1}{p} \int_\Omega v_\varepsilon^p(\cdot, t+1) + \int_t^{t+1} \int_\Omega v_\varepsilon^p \\ &\le \frac{1}{p} \cdot c_1(T) + c_1(T) \qquad \text{for all } t \in (0,T) \end{aligned}$$

and thereby entails (3.1) and (3.2). □

Through interpolation using (2.7), the latter implies space-time integrability in certain superlinear Lebesgue spaces:

Lemma 3.2 *Let $p \in (0,1)$ and $q>1$. Then for all $\delta \in (0,R)$ and $T>0$ there exists $C(\delta,T)>0$ such that*

$$\int_t^{t+1} \|v_\varepsilon(\cdot,s)\|_{L^q(\Omega)}^{\frac{pq}{q-1}} ds \le C(\delta,T) \qquad \textit{for all } t \in (0,T) \tag{3.4}$$

and that

$$\sup_{T>0} C(T) < \infty \quad \textit{if (H) holds.} \tag{3.5}$$

Proof. As with some $c_1>0$, for all $t>0$ we have

$$\begin{aligned} \int_t^{t+1} \|v_\varepsilon(\cdot,s)\|_{L^q(\Omega)}^{\frac{pq}{q-1}} ds &= \int_t^{t+1} \|v_\varepsilon^{\frac{p}{2}}(\cdot,s)\|_{L^{\frac{2q}{p}}(\Omega)}^{\frac{2q}{q-1}} ds \\ &\le c_1 \int_t^{t+1} \|\nabla v_\varepsilon^{\frac{p}{2}}(\cdot,s)\|_{L^2(\Omega)}^2 \|v_\varepsilon^{\frac{p}{2}}(\cdot,s)\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{q-1}} ds + c_1 \int_t^{t+1} \|v_\varepsilon^{\frac{p}{2}}(\cdot,s)\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2q}{q-1}} ds \end{aligned}$$

by the Gagliardo-Nirenberg inequality, both (3.4) and (3.5) result on combining Lemma 3.1 with Lemma 2.3. □

We next plan to achieve some estimates for u_ε which are basically of the same flavor as those in Lemma 3.1 and Lemma 3.2. Here in order to appropriately respect the particular structure of the nonlinear cross-diffusion term in the first equation of (2.1), we follow an approach which apparently goes back to the analysis of Keller-Segel type systems with singular sensitivities ([38]), and which at its core relies on certain quasi-entropy properties enjoyed by functionals of the form

$$\int_\Omega u_\varepsilon^p v_\varepsilon^q$$

for suitably small positive p and appropriately chosen $q \in (0, 1-p)$. Indeed, the corresponding bounds for the associated dissipation rate functionals, together with the lower estimate for v_ε from Lemma 2.2, will imply the following.

Lemma 3.3 *Let $p \in (0,1)$ be such that $p < \frac{1}{\chi^2}$. Then for all $T > 0$ there exists $C(T) > 0$ such that*

$$\int_t^{t+1} \int_\Omega u_\varepsilon^{p-2} |\nabla u_\varepsilon|^2 \le C(T) \qquad \textit{for all } t > 0 \tag{3.6}$$

as well as

$$\sup_{T>0} C(T) < \infty \quad \textit{if (H) holds.} \tag{3.7}$$

PROOF. We abbreviate $q := \frac{1-p}{2}$ and use (2.1) to compute

$$\begin{aligned}
\frac{d}{dt} \int_\Omega u_\varepsilon^p v_\varepsilon^q &= p(1-p) \int_\Omega u_\varepsilon^{p-2} v_\varepsilon^q |\nabla u_\varepsilon|^2 + q(p\chi + 1 - q) \int_\Omega u_\varepsilon^p v_\varepsilon^{q-2} |\nabla v_\varepsilon|^2 \\
&\quad -p(\chi - p\chi + 2q) \int_\Omega u_\varepsilon^{p-1} v_\varepsilon^{q-1} \nabla u_\varepsilon \cdot \nabla v_\varepsilon \\
&\quad -p \int_\Omega u_\varepsilon^p v_\varepsilon^{q+1} + p \int_\Omega B_1 u_\varepsilon^{p-1} v_\varepsilon^q \\
&\quad +q \int_\Omega \frac{u_\varepsilon^{p+1} v_\varepsilon^q}{1 + \varepsilon u_\varepsilon v_\varepsilon} - q \int_\Omega u_\varepsilon^p v_\varepsilon^q + q \int_\Omega B_2 u_\varepsilon^p v_\varepsilon^{q-1} \qquad \text{for all } t > 0,
\end{aligned} \tag{3.8}$$

where since q is positive due to our hypothesis that $p < 1$, we clearly have

$$p \int_\Omega B_1 u_\varepsilon^{p-1} v_\varepsilon^q + q \int_\Omega \frac{u_\varepsilon^{p+1} v_\varepsilon^q}{1 + \varepsilon u_\varepsilon v_\varepsilon} + q \int_\Omega B_2 u_\varepsilon^p v_\varepsilon^{q-1} \ge 0 \qquad \text{for all } t > 0, \tag{3.9}$$

and where combining the Hölder inequality with Lemma 2.3 and the fact that $\frac{q}{1-p} = \frac{1}{2}$, we see that there exists $c_1(T) > 0$ such that

$$\begin{aligned}
\int_\Omega u_\varepsilon^p v_\varepsilon^q &\le \left\{ \int_\Omega u_\varepsilon \right\}^p \cdot \left\{ \int_\Omega v_\varepsilon^{\frac{1}{2}} \right\}^{1-p} \\
&\le \left\{ \int_\Omega u_\varepsilon \right\}^p \cdot \left\{ \int_\Omega v_\varepsilon \right\}^{\frac{1-p}{2}} |\Omega|^{\frac{1-p}{2}} \\
&\le c_1(T) \qquad \text{for all } t \in (0, T+1)
\end{aligned} \tag{3.10}$$

with

$$\sup_{T>0} c_1(T) < \infty \quad \text{if (H) holds.} \tag{3.11}$$

In order to control the fourth summand on the right of (3.8), we apply Lemma 3.2 to the exponent $q := \frac{3}{2}$, which in conjunction with the Hölder inequality and Lemma 2.3 shows that with some $c_2(T) > 0$ we have

$$\begin{aligned} p\int_t^{t+1}\int_\Omega u_\varepsilon^p v_\varepsilon^{q+1} &= p\int_t^{t+1}\int_\Omega (u_\varepsilon v_\varepsilon)^p \cdot v_\varepsilon^{\frac{3(1-p)}{2}} \\ &\le p\cdot\left\{\int_t^{t+1}\int_\Omega u_\varepsilon v_\varepsilon\right\}^p \cdot \left\{\int_t^{t+1}\int_\Omega v_\varepsilon^{\frac{3}{2}}\right\}^{1-p} \\ &\le c_2(T) \qquad \text{for all } t\in(0,T) \end{aligned} \tag{3.12}$$

and

$$\sup_{T>0} c_2(T) < \infty \quad \text{if (H) holds.} \tag{3.13}$$

Finally, the crucial integral in (3.8) which involves both ∇u_ε and ∇v_ε can be estimated in modulus by means of Young's inequality, which namely implies that

$$\begin{aligned} \left| -p(\chi - p\chi + 2q)\int_\Omega u_\varepsilon^{p-1} v_\varepsilon^{q-1}\nabla u_\varepsilon \cdot \nabla v_\varepsilon\right| &\le q(p\chi+1-q)\int_\Omega u_\varepsilon^p v_\varepsilon^{q-2}|\nabla|v_\varepsilon|^2 \\ &\quad + \frac{p^2(\chi-p\chi+2q)^2}{4q(p\chi+1-q)}\int_\Omega u_\varepsilon^{p-2}v_\varepsilon^q|\nabla u_\varepsilon|^2 \end{aligned} \tag{3.14}$$

for all $t > 0$, where we have used that the definition of q in particular entails that $q \in (0,1)$ and hence $q(p\chi+1-q)$ is positive. Since this definition along with our assumption that $p < \frac{1}{\chi^2}$ moreover guarantees that

$$c_3 := p(1-p) - \frac{p^2(\chi-p\chi+2q)^2}{4q(p\chi+1-q)} = p(1-p) - \frac{p^2(1-p)(\chi+1)^2}{2p\chi+p+1} = \frac{p(1-p)(1-p\chi^2)}{2p\chi+p+1}$$

is positive, inserting (3.9)-(3.14) into (3.8) and integrating in time we infer that the resulting inequality

$$\begin{aligned} c_3\int_t^{t+1}\int_\Omega u_\varepsilon^{p-2}v_\varepsilon^q|\nabla u_\varepsilon|^2 &\le \int_\Omega u_\varepsilon^p(\cdot,t+1)v_\varepsilon^q(\cdot,t+1) + q\int_t^{t+1}\int_\Omega u_\varepsilon^p v_\varepsilon^q + p\int_t^{t+1}\int_\Omega u_\varepsilon^p v_\varepsilon^{q+1} \\ &\le c_4(T) := c_1(T) + qc_1(T) + c_2(T), \qquad t\in(0,T), \end{aligned} \tag{3.15}$$

implies that

$$\int_t^{t+1}\int_\Omega u_\varepsilon^{p-2}v_\varepsilon^q|\nabla u_\varepsilon|^2 \le \frac{c_4(T)}{c_3} \qquad \text{for all } t\in(0,T). \tag{3.16}$$

As herein the positivity of q allows for an application of Lemma 2.2 which provides $c_5(T) > 0$ fulfilling

$$v_\varepsilon(x,t) \ge c_5(T) \qquad \text{for all } x\in\Omega \text{ and } t\in(0,T)$$

with

$$\inf_{T>0} c_5(T) > 0 \quad \text{if (H) is valid,}$$

from (3.16), (3.15), (3.11) and (3.13) we directly obtain (3.6) and (3.7) if we let $C(T) := \frac{c_4(T)}{c_3 c_5(T)}$, for instance. $\Box$

Again by interpolation using Lemma 2.3, this has immediate consequences for the integrability properties of u_ε itself.

Lemma 3.4 *Let $p \in (0,1)$ be such that $p < \frac{1}{\chi^2}$, and let $q > 1$. Then for any $T > 0$ one can find $C(T) > 0$ satisfying*

$$\int_t^{t+1} \|u_\varepsilon(\cdot,s)\|_{L^q(\Omega)}^{\frac{pq}{q-1}} ds \le C(T) \qquad \textit{for all } t \in (0,T) \tag{3.17}$$

as well as

$$\sup_{T>0} C(T) < \infty \quad \textit{if (H) holds.} \tag{3.18}$$

PROOF. In much the same manner as in Lemma 3.2, this can be derived from Lemma 3.3 in conjunction with Lemma 2.3 upon an interpolation argument. $\Box$

4 Integral estimates outside the origin

In the present section we shall make essential use of the assumed radial symmetry of the problem, which namely entails that inside any annular region around – but not containing – the origin, the PDE system in (2.1) actually exhibits a spatially one-dimensional structure. Thanks to accordingly improved embedding properties when compared with the full two-dimensional case, the gradient integrability estimates from Lemma 3.1 and Lemma 3.3 therefeore give rise to alternative classes of basic regularity estimates that turn out to be an appropriate staring point for a bootstrap procedure yielding locally uniform bounds, outside the spatial origin, for u_ε and v_ε as well as their derivatives up to conveniently high order.

Here and throughout the sequel, without further explicit mentioning we shall tacitly switch to the standard notation for radially symmetric functions by writing e.g. $u_\varepsilon(r,t)$ instead of $u(x,t)$ for $r = |x| \in [0,R]$ whenever this appears suitable.

4.1 Immediate consequences of Lemma 2.3, Lemma 3.1 and Lemma 3.3

We first state three direct implications of the estimates from Section 3.

Lemma 4.1 *Let $\delta \in (0,R)$ and $T > 0$. Then there exists $C(\delta,T) > 0$ such that whenever $\varepsilon \in (0,1)$,*

$$\int_\delta^R u_\varepsilon(r,t)dr \le C(\delta,T) \quad \textit{and} \quad \int_\delta^R v_\varepsilon(r,t)dr \le C(\delta,T) \qquad \textit{for all } t \in (0,T),$$

and such that

$$\sup_{T>0} C(\delta,T) < \infty \quad \textit{for all } \delta \in (0,R) \quad \textit{if (H) holds.} \tag{4.1}$$

PROOF. Since Lemma 2.3 yields $c_1(T) > 0$ such that

$$\int_0^R r u_\varepsilon(r,t)dr + \int_0^R r v_\varepsilon(r,t)dr \le c_1(T) \qquad \text{for all } t \in (0,T)$$

with $\sup_{T>0} c_1(T) < \infty$ if (H) is in force, both claimed inequalities as well as (4.1) immediately result if we define $C(\delta, T) := \frac{c_1(T)}{\delta}$. $\square$

Lemma 4.2 *Let $p \in (0,1)$ satisfy $p < \frac{1}{\chi^2}$. Then for any $\delta \in (0,R)$ and $T > 0$ one can pick $C(\delta, T) > 0$ such that*

$$\int_t^{t+1} \int_\delta^R u_\varepsilon^{p-2}(r,s) u_{\varepsilon r}^2(r,s) dr ds \le C(\delta, T) \qquad \text{for all } t \in (0,T)$$

and

$$\sup_{T>0} C(\delta, T) < \infty \quad \text{for all } \delta \in (0,R) \quad \text{if (H) holds.}$$

PROOF. This can be verified by applying the reasoning from Lemma 4.1 to the outcome of Lemma 3.3. $\square$

Lemma 4.3 *Let $p \in (0,1)$. Then for all $\delta \in (0,R)$ and any $T > 0$ there exists $C(\delta, T) > 0$ such that*

$$\int_t^{t+1} \int_\delta^R v_\varepsilon^{p-2}(r,s) v_{\varepsilon r}^2(r,s) dr ds \le C(\delta, T) \qquad \text{for all } t \in (0,T),$$

and that

$$\sup_{T>0} C(\delta, T) < \infty \quad \text{for all } \delta \in (0,R) \quad \text{if (H) holds.}$$

PROOF. This is an evident consequence of Lemma 3.1. $\square$

4.2 Results of one-dimensional interpolation

Now using one-dimensional rather than two-dimensional interpolation we infer that outside the origin, both functions u_ε and v_ε actually enjoy better regularity properties than those expressed in the spatially global estimates from Lemma 3.2 and Lemma 3.4. Indeed, we firstly have the following.

Lemma 4.4 *Let $p \in (0,1)$ be such that $p < \frac{1}{\chi^2}$. Then for all $\delta \in (0,R)$ and each $T > 0$ one can find $C(\delta, T) > 0$ such that for all $\varepsilon \in (0,1)$,*

$$\int_t^{t+1} \int_\delta^R u_\varepsilon^{p+2}(r,s) dr ds \le C(\delta, T) \qquad \text{for all } t \in (0,T), \tag{4.2}$$

and such that

$$\sup_{T>0} C(\delta, T) < \infty \quad \text{for all } \delta \in (0,R) \quad \text{if (H) holds.} \tag{4.3}$$

PROOF. According to the one-dimensional version of the Gagliardo-Nirenberg inequality, there exists $c_1(\delta) > 0$ such that

$$\|\varphi\|_{L^{\frac{2(p+2)}{p}}((\delta,R))}^{\frac{2(p+2)}{p}} \le c_1(\delta) \|\varphi_r\|_{L^2((\delta,R))}^2 \|\varphi\|_{L^{\frac{2}{p}}((\delta,R))}^{\frac{4}{p}} + c_1(\delta) \|\varphi\|_{L^{\frac{2}{p}}((\delta,R))}^{\frac{2(p+2)}{p}} \qquad \text{for all } \varphi \in W^{1,2}((\delta,R)),$$

which we apply to $\varphi := u_\varepsilon^{\frac{p}{2}}(\cdot,t)$ for $t>0$ and $\varepsilon\in(0,1)$. Since Lemma 4.1 provides $c_2(\delta,T)>0$ such that

$$\|u_\varepsilon^{\frac{p}{2}}(\cdot,t)\|_{L^{\frac{2}{p}}((\delta,R))} \le c_2(\delta,T) \qquad \text{for all } t\in(0,T+1)$$

and

$$\sup_{T>0} c_2(\delta,T)<\infty \quad \text{for all } \delta\in(0,R) \quad \text{if (H) holds,} \tag{4.4}$$

upon an integration in time we thereby obtain that

$$\begin{aligned}
\int_t^{t+1}\int_\delta^R u_\varepsilon^{p+2}(r,s)drds &= \int_t^{t+1} \|u_\varepsilon^{\frac{p}{2}}(\cdot,s)\|_{L^{\frac{2(p+2)}{p}}((\delta,R))}^{\frac{2(p+2)}{p}} ds \\
&\le c_1(\delta)\int_t^{t+1} \Big\|(u_\varepsilon^{\frac{p}{2}})_r(\cdot,s)\Big\|_{L^2((\delta,R))}^2 \|u_\varepsilon^{\frac{p}{2}}(\cdot,s)\|_{L^{\frac{2}{p}}((\delta,R))}^{\frac{4}{p}} ds \\
&\quad + c_1(\delta)\int_t^{t+1} \|u_\varepsilon^{\frac{p}{2}}(\cdot,s)\|_{L^{\frac{2}{p}}((\delta,R))}^{\frac{2(p+2)}{p}} ds \\
&\le c_1(\delta)c_2^{\frac{4}{p}}(\delta,T)\cdot\frac{p^2}{4}\int_t^{t+1}\int_\delta^R u_\varepsilon^{p-2}(r,s)u_{\varepsilon r}^2(r,s)drds + c_1(\delta)c_2^{\frac{2(p+2)}{p}}(\delta,T)
\end{aligned}$$

for all $t\in(0,T)$. In view of Lemma 4.2 and (4.4), this directly leads to (4.2) and (4.3). $\square$

From the same sources we can even derive an estimate involving spatial L^∞ norms.

Lemma 4.5 *Let $p\in(0,1)$ satisfy $p<\frac{1}{\chi^2}$. Then for all $\delta\in(0,R)$ and $T>0$ one can find $C(\delta,T)>0$ fulfilling*

$$\int_t^{t+1}\|u_\varepsilon(\cdot,s)\|_{L^\infty((\delta,R))}^{p+1}ds \le C(\delta,T) \qquad \text{for all } t\in(0,T) \tag{4.5}$$

and any $\varepsilon\in(0,1)$, and such that

$$\sup_{T>0} C(\delta,T)<\infty \quad \text{for all } \delta\in(0,R) \quad \text{if (H) holds.} \tag{4.6}$$

PROOF. Since from the Gagliardo-Nirenberg inequality we know that with some $c_1(\delta)>0$ we have

$$\begin{aligned}
\int_t^{t+1}\|u_\varepsilon(\cdot,s)\|_{L^\infty((\delta,R))}^{p+1}ds &= \int_t^{t+1}\|u_\varepsilon^{\frac{p}{2}}(\cdot,s)\|_{L^\infty((\delta,R))}^{\frac{2(p+1)}{p}}ds \\
&\le c_1(\delta)\int_t^{t+1}\|(u_\varepsilon^{\frac{p}{2}})_r(\cdot,s)\|_{L^2((\delta,R))}^2\|u_\varepsilon^{\frac{p}{2}}(\cdot,s)\|_{L^{\frac{2}{p}}((\delta,R))}^{\frac{2}{p}}ds \\
&\quad + c_1(\delta)\int_t^{t+1}\|u_\varepsilon^{\frac{p}{2}}(\cdot,s)\|_{L^{\frac{2}{p}}((\delta,R))}^{\frac{2(p+1)}{p}}ds \qquad \text{for all } t>0,
\end{aligned}$$

in much the same manner as in Lemma 4.4 we can derive (4.5) and (4.6) from Lemma 4.2 and Lemma 4.1. $\square$

In much the same manner, Lemma 2.3 and Lemma 3.1 entail an estimate for v_ε.

Lemma 4.6 *let $p \in [1,2)$. Then for all $\delta \in (0,R)$ and $T>0$ there exists $C(\delta,T)>0$ such that*

$$\int_t^{t+1} \|v_\varepsilon(\cdot,s)\|^p_{L^\infty((\delta,R))} ds \le C(\delta,T) \qquad \textit{for all } t \in (0,T) \tag{4.7}$$

and $\varepsilon \in (0,1)$, and that

$$\sup_{T>0} C(\delta,T) < \infty \quad \textit{for all } \delta \in (0,R) \quad \textit{ if (H) holds.} \tag{4.8}$$

PROOF. Without loss of generality assuming that $p>1$ and that hence $p-1 \in (0,1)$, from Lemma 4.3 we know that there exists $c_1(\delta,T)>0$ fulfilling

$$\int_t^{t+1} \|(v_\varepsilon^{\frac{p-1}{2}})_r(\cdot,s)\|^2_{L^2((\delta,R))} ds \le c_1(\delta,T) \qquad \text{for all } t \in (0,T)$$

and

$$\sup_{T>0} c_1(\delta,T) < \infty \quad \text{for all } \delta \in (0,R) \quad \text{ if (H) holds,}$$

whereas from Lemma 4.1 we obtain $c_2(\delta,T)>0$ such that

$$\|v_\varepsilon^{\frac{p-1}{2}}(\cdot,s)\|_{L^{\frac{2}{p-1}}((\delta,R))} \le c_2(\delta,T) \qquad \text{for all } s \in (0,T+1)$$

and

$$\sup_{T>0} c_2(\delta,T) < \infty \quad \text{for all } \delta \in (0,R), \quad \text{ provided that (H) is satisfied.}$$

Therefore, invoking the Gagliardo-Nirenberg inequality in the interval (δ,R), we find $c_3(\delta)>0$ such that

$$\begin{aligned}
\int_t^{t+1} \|v_\varepsilon(\cdot,s)\|^p_{L^\infty((\delta,R))} ds &= \int_t^{t+1} \|v_\varepsilon^{\frac{p-1}{2}}(\cdot,s)\|^{\frac{2p}{p-1}}_{L^\infty((\delta,R))} ds \\
&\le c_3(\delta) \int_t^{t+1} \|(v_\varepsilon^{\frac{p-1}{2}})_r(\cdot,s)\|^2_{L^2((\delta,R))} \|v_\varepsilon^{\frac{p-1}{2}}(\cdot,s)\|^{\frac{2}{p-1}}_{L^{\frac{2}{p-1}}((\delta,R))} ds \\
&\quad + c_3(\delta) \int_t^{t+1} \|v_\varepsilon^{\frac{p-1}{2}}(\cdot,s)\|^{\frac{2p}{p-1}}_{L^{\frac{2}{p-1}}((\delta,R))} ds \\
&\le c_3(\delta) c_1(\delta,T) c_2^{\frac{2}{p-1}}(\delta,T) + c_3(\delta) c_2^{\frac{2p}{p-1}}(\delta,T) \qquad \text{for all } t \in (0,T)
\end{aligned}$$

whenever $\varepsilon \in (0,1)$. □

4.3 Localized testing procedures I: Estimates for v_ε

We next intend to improve our knowledge on regularity in such annular regions by going back to (2.1) and performing appropriate localized testing procedures. To this end, in what follows we shall fix an arbitrary nondecreasing function $\zeta \in C^\infty([0,\infty))$ such that $\zeta \equiv 0$ in $[0,\frac{1}{2}]$ and $\zeta \equiv 1$ in $[1,\infty)$, and for $\delta \in (0,R)$ we let

$$\zeta_\delta(r) := \zeta\Big(\frac{r}{\delta}\Big), \qquad r \in [0,R]. \tag{4.9}$$

Then an accordingly localized analysis of the second equation in (2.1) yields the following result which will form the core of an inductive argument in Corollary 4.8 below.

Lemma 4.7 *Let $p > \frac{1}{3}$. Then for all $K > 0, \delta \in (0,R)$ and $T > 0$ there exists $C(K,\delta,T) > 0$ such that if for some $\varepsilon \in (0,1)$ we have*

$$\int_{\frac{\delta}{2}}^R v_\varepsilon^p(r,t)dr \le K \qquad \textit{for all } t \in (0,T+1) \tag{4.10}$$

and

$$\int_t^{t+1}\int_{\frac{\delta}{2}}^R v_\varepsilon^{p-2}(r,s)v_{\varepsilon r}^2(r,s)drds \le K \qquad \textit{for all } t \in (0,T), \tag{4.11}$$

then

$$\int_\delta^R v_\varepsilon^{3p}(r,t)dr \le C(K,\delta,T) \qquad \textit{for all } t \in (0,T+1) \tag{4.12}$$

and

$$\int_t^{t+1}\int_\delta^R v_\varepsilon^{3p-2}(r,s)v_{\varepsilon r}^2(r,s)drds \le C(K,\delta,T) \qquad \textit{for all } t \in (0,T), \tag{4.13}$$

and such that moreover

$$\sup_{T>0} C(K,\delta,T) < \infty \quad \textit{for all } K > 0 \textit{ and } \delta \in (0,R) \quad \textit{if (H) holds.} \tag{4.14}$$

Proof. We first apply the Gagliardo-Nirenberg inequality to find $c_1 > 0$ such that

$$\begin{aligned}
\int_t^{t+1}\int_{\frac{\delta}{2}}^R v_\varepsilon^{3p}(r,s)drds &= \int_t^{t+1} \|v_\varepsilon^{\frac{p}{2}}(\cdot,s)\|_{L^6((\frac{\delta}{2},R))}^6 ds \\
&\le c_1\int_t^{t+1} \|(v_\varepsilon^{\frac{p}{2}})_r(\cdot,s)\|_{L^2((\frac{\delta}{2},R))}^2 \|v_\varepsilon^{\frac{p}{2}}(\cdot,s)\|_{L^2((\frac{\delta}{2},R))}^4 ds \\
&\quad + c_1\int_t^{t+1} \|v_\varepsilon^{\frac{p}{2}}(\cdot,s)\|_{L^2((\frac{\delta}{2},R))}^6 ds \qquad \text{for all } t > 0,
\end{aligned}$$

whence it follows from (4.10) and (4.11) that

$$\int_t^{t+1}\int_{\frac{\delta}{2}}^R v_\varepsilon^{3p}(r,s)drds \le \frac{p^2}{4}c_1K^3 + c_1K^3 \qquad \text{for all } t \in (0,T). \tag{4.15}$$

Now with $\xi(x) := \zeta_\delta(|x|)$ and ζ_δ taken from (4.9), for convenience temporarily returning to the original spatial variable x we test the second equation in (2.1) by $\xi^2 v_\varepsilon^{3p-1}$ and use Young's inequality to see that

$$\begin{aligned}
\frac{1}{3p}\frac{d}{dt}\int_\Omega \xi^2 v_\varepsilon^{3p} &= \int_\Omega \xi^2 v_\varepsilon^{3p-1}\Big\{\Delta v_\varepsilon + \frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} - v_\varepsilon + B_2\Big\} \\
&= -(3p-1)\int_\Omega \xi^2 v_\varepsilon^{3p-2}|\nabla v_\varepsilon|^2 - 2\int_\Omega \xi\nabla\xi\cdot v_\varepsilon^{3p-1}\nabla v_\varepsilon \\
&\quad + \int_\Omega \xi^2 \frac{u_\varepsilon v_\varepsilon^{3p}}{1+\varepsilon u_\varepsilon v_\varepsilon} - \int_\Omega \xi^2 v_\varepsilon^{3p} + \int_\Omega \xi^2 B_2 v_\varepsilon^{3p-1} \\
&\le -\frac{3p-1}{2}\int_\Omega \xi^2 v_\varepsilon^{3p-2}|\nabla v_\varepsilon|^2 + \frac{2}{3p-1}\int_\Omega |\nabla\xi|^2 v_\varepsilon^{3p} \\
&\quad + \int_\Omega \xi^2 u_\varepsilon v_\varepsilon^{3p} + \int_\Omega \xi^2 B_2^{3p} \\
&\le -\frac{3p-1}{2}\int_\Omega \xi^2 v_\varepsilon^{3p-2}|\nabla v_\varepsilon|^2 + \frac{2}{3p-1}\|\nabla\xi\|_{L^\infty(\Omega)}^2 \int_{\Omega\setminus B_{\frac{\delta}{2}}} v_\varepsilon^{3p} \\
&\quad + \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega\setminus B_{\frac{\delta}{2}})}\int_\Omega \xi^2 v_\varepsilon^{3p} + \int_\Omega B_2^{3p} \qquad \text{for all } t>0,
\end{aligned}$$

meaning that $y_\varepsilon(t) := \int_\Omega \xi^2 v_\varepsilon^{3p}(\cdot,t)$, $t\ge 0$, as well as $g_\varepsilon(t) := \frac{3p(3p-1)}{2}\int_\Omega \xi^2 v_\varepsilon^{3p-2}(\cdot,t)|\nabla v_\varepsilon(\cdot,t)|^2$, $a_\varepsilon(t) := 3p\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega\setminus B_{\frac{\delta}{2}})}$ and $h_\varepsilon(t) := \frac{6p}{3p-1}\|\nabla\xi\|_{L^\infty(\Omega)}^2\int_{\Omega\setminus B_{\frac{\delta}{2}}} v_\varepsilon^{3p}(\cdot,t) + 3p\int_\Omega B_2^{3p}(\cdot,t)$, $t>0$, satisfy

$$y_\varepsilon'(t) + g_\varepsilon(t) \le a_\varepsilon y_\varepsilon(t) + h_\varepsilon(t) \qquad \text{for all } t>0. \tag{4.16}$$

In order to proceed from this, we note that Lemma 4.5 in particular provides $c_2(\delta,T)>0$ such that for all $\varepsilon\in(0,1)$,

$$\int_{(t-1)_+}^{t+1} a_\varepsilon(s)ds \le c_2(\delta,T) \qquad \text{for all } t\in(0,T),$$

and that

$$\sup_{T>0} c_2(\delta,T) < \infty \quad \text{for all } \delta\in(0,R) \quad \text{if (H) holds,} \tag{4.17}$$

whereas (4.15) says that with some $c_3(K,\delta)>0$ we have

$$\int_{(t-1)_+}^{t+1} h_\varepsilon(s)ds \le c_3(K,\delta) \qquad \text{for all } t\in(0,T).$$

Moreover, an application of Lemma 4.6 yields $c_4(\delta,T)>0$ with the property that for each $t\in(0,T+1)$ and any $\varepsilon\in(0,1)$ we can find $t_\varepsilon(t)\in[(t-1)_+,t)$ fulfilling

$$y_\varepsilon(t_\varepsilon(t)) \le c_4(\delta,T)$$

and

$$\sup_{T>0} c_4(\delta,T) < \infty \quad \text{for all } \delta\in(0,R) \quad \text{if (H) holds.} \tag{4.18}$$

Therefore, from (4.16) we infer that firstly

$$\begin{aligned} y_\varepsilon(t) &\le y_\varepsilon(t_\varepsilon(t))e^{\int_{t_\varepsilon(t)}^t a_\varepsilon(s)ds} + \int_{t_\varepsilon(t)}^t e^{\int_s^t a_\varepsilon(\sigma)d\sigma} h_\varepsilon(s)ds \\ &\le c_5(K,\delta,T) := c_4(\delta,T)e^{c_2(\delta,T)} + c_3(K,\delta)e^{c_2(\delta,T)} \qquad \text{for all } t \in (0,T+1), \end{aligned}$$

and that thus, secondly,

$$\begin{aligned} \int_t^{t+1} g_\varepsilon(s)ds &\le y_\varepsilon(t) + \int_t^{t+1} a_\varepsilon(s)y_\varepsilon(s)ds + \int_t^{t+1} h_\varepsilon(s)ds \\ &\le c_5(K,\delta,T) + c_2(\delta,T)c_5(K,\delta,T) + c_3(K,\delta) \qquad \text{for all } t \in (0,T), \end{aligned}$$

whence in view of (4.17) and (4.18) the proof is complete. □

Using Lemma 2.3 and Lemma 4.3 as a starting point herein, by means of a straightforward recursion we inter alia obtain bounds for v_ε in $L^p((\delta,R))$ for arbitrary $p > 1$ and $\delta \in (0,R)$, and for $v_{\varepsilon r}$ in corresponding spatio-temporal L^2 norms.

Corollary 4.8 *Let $p > 1$. Then for all $\delta \in (0,R)$ and $T > 0$ there exists $C(\delta,T) > 0$ such that for all $\varepsilon \in (0,1)$ we have*

$$\int_\delta^R v_\varepsilon^p(r,t)dr \le C(\delta,T) \qquad \textit{for all } t \in (0,T) \tag{4.19}$$

as well as

$$\int_t^{t+1}\int_\delta^R v_\varepsilon^{p-2}(r,s)v_{\varepsilon r}^2(r,s)drds \le C(\delta,T) \qquad \textit{for all } t \in (0,T), \tag{4.20}$$

and such that moreover

$$\sup_{T>0} C(\delta,T) < \infty \quad \textit{for all } \delta \in (0,R) \quad \textit{if (H) holds.} \tag{4.21}$$

Proof. Given $p > 1$, we fix $p_0 \in (\frac{1}{3},1)$ and $k \in \mathbb{N}$ such that $p \le 3^k p_0$. Then since Lemma 2.3 and Lemma 4.3 provide $c_1(\delta,T) > 0$ and $c_2(\delta,T) > 0$ such that

$$\int_{2^{-k}\delta}^R v_\varepsilon^{p_0}(r,t)dr \le c_1(\delta,T) \qquad \text{for all } t \in (0,T+1) \text{ and } \varepsilon \in (0,1)$$

and

$$\int_t^{t+1}\int_{2^{-k}\delta}^R v_\varepsilon^{p_0-2}(r,s)v_{\varepsilon r}^2(r,s)drds \le c_2(\delta,T) \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1)$$

with

$$\sup_{T>0} c_1(\delta,T) < \infty \quad \text{and} \quad \sup_{T>0} c_2(\delta,T) < \infty \quad \text{for all } \delta \in (0,R) \text{ whenever (H) is valid,}$$

repeated application of Lemma 4.7 shows that for each $j \in \{0,...,k\}$ we can find $c_3(j,\delta,T) > 0$ and $c_4(j,\delta,T) > 0$ fulfilling

$$\int_{2^{j-k}\delta}^R v^{3^j p_0}(r,t)dr \le c_3(j,\delta,T) \qquad \text{for all } t \in (0,T+1) \text{ and } \varepsilon \in (0,1)$$

and

$$\int_t^{t+1}\int_{2^{j-k}\delta}^R v_\varepsilon^{3^jp_0-2}(r,s)v_{\varepsilon r}^2(r,s)drds \le c_4(j,\delta,T) \qquad \text{for all } t\in(0,T) \text{ and any } \varepsilon\in(0,1)$$

as well as

$$\sup_{T>0} c_3(j,\delta,T)<\infty \quad \text{and} \quad \sup_{T>0} c_4(j,\delta,T)<\infty \quad \text{for all } \delta\in(0,R) \text{ if (H) holds.}$$

When evaluated at $j=k$, this proves the claim, because in view of Lemma 2.2, our restriction $p\le 3^kp_0$ warrants that $v_\varepsilon^p \le c_5(T)v_\varepsilon^{3^kp_0}$ in $\Omega\times(0,T+1)$ with some $c_5(T)>0$ which is such that $\sup_{T>0}c_5(T)<\infty$ if (H) is satisfied. $\Box$

4.4 Localized testing procedures II: Higher-order integral estimates for v_ε

To further improve our regularity information on v_ε, we shall next pursue the time evolution of correspondingly localized variants of the Dirichlet integral $\int_\Omega |\nabla v_\varepsilon|^2$ which thanks to the outcome of Corollary 4.8 leads to the following.

Lemma 4.9 *Let $\delta\in(0,R)$ and $T>0$. Then there exists $C(\delta,T)>0$ such that*

$$\int_\delta^R v_{\varepsilon r}^2(r,t)\le C(\delta,T) \qquad \text{for all } t\in(0,T) \tag{4.22}$$

and

$$\int_t^{t+1}\int_\delta^R v_{\varepsilon rr}^2(r,s)drds \le C(\delta,T) \qquad \text{for all } t\in(0,T) \tag{4.23}$$

as well as

$$\int_t^{t+1}\int_{\Omega\setminus B_\delta}|\Delta v_\varepsilon(x,s)|^2dxds \le C(\delta,T) \qquad \text{for all } t\in(0,T), \tag{4.24}$$

and such that

$$\sup_{T>0}C(\delta,T)<\infty \quad \text{for all } \delta\in(0,R) \text{ if (H) holds.} \tag{4.25}$$

Proof. Once more letting $\xi(x):=\zeta_\delta(|x|)$, $x\in\Omega$, with ζ_δ taken from (4.9), on the basis of the second equation in (2.1) we compute

$$\begin{aligned}
\frac12\frac{d}{dt}\int_\Omega \xi^2|\nabla v_\varepsilon|^2 &= -\int_\Omega \xi^2\Delta v_\varepsilon\cdot v_{\varepsilon t} - 2\int_\Omega \xi(\nabla\xi\cdot\nabla v_\varepsilon)v_{\varepsilon t} \\
&= -\int_\Omega \xi^2|\Delta v_\varepsilon|^2 - \int_\Omega \xi^2\frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon}\Delta v_\varepsilon + \int_\Omega \xi^2 v_\varepsilon\Delta v_\varepsilon - \int_\Omega \xi^2 B_2\Delta v_\varepsilon \\
&\quad -2\int_\Omega \xi(\nabla\xi\cdot\nabla v_\varepsilon)\Delta v_\varepsilon - 2\int_\Omega \xi\nabla\xi\cdot\frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon}\nabla v_\varepsilon \\
&\quad +2\int_\Omega \xi\nabla\xi\cdot v_\varepsilon\nabla v_\varepsilon - 2\int_\Omega \xi\nabla\xi\cdot B_2\nabla v_\varepsilon \qquad \text{for all } t>0.
\end{aligned} \tag{4.26}$$

Here we pick any $p \in (0,1)$ such that $p < \frac{1}{\chi^2}$ and invoke Young's inequality to see that as a consequence of Corollary 4.8, with

$$c_1(\delta, T) := \sup_{\varepsilon \in (0,1)} \sup_{t \in (0,T+1)} \int_{\Omega \setminus B_{\frac{\delta}{2}}} v_\varepsilon^{\frac{2(p+2)}{p}}(\cdot, t)$$

we have

$$\begin{aligned}
-\int_\Omega \xi^2 \frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} \Delta v_\varepsilon &\le \frac{1}{2}\int_\Omega \xi^2 |\Delta v_\varepsilon|^2 + \frac{1}{2}\int_\Omega \xi^2 \Big(\frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon}\Big)^2 \\
&\le \frac{1}{2}\int_\Omega \xi^2 |\Delta v_\varepsilon|^2 + \frac{1}{2}\int_\Omega \xi^2 u_\varepsilon^2 v_\varepsilon^2 \\
&\le \frac{1}{2}\int_\Omega \xi^2 |\Delta v_\varepsilon|^2 + \frac{1}{2}\int_{\Omega \setminus B_{\frac{\delta}{2}}} u_\varepsilon^{p+2} + \frac{1}{2}\int_{\Omega \setminus B_{\frac{\delta}{2}}} v_\varepsilon^{\frac{2(p+2)}{p}} \\
&\le \frac{1}{2}\int_\Omega \xi^2 |\Delta v_\varepsilon|^2 + \frac{1}{2}\int_{\Omega \setminus B_{\frac{\delta}{2}}} u_\varepsilon^{p+2} + \frac{1}{2} c_1(\delta, T) \qquad \text{for all } t \in (0, T+1) \qquad (4.27)
\end{aligned}$$

and

$$\sup_{T>0} c_1(\delta, T) < \infty \quad \text{for all } \delta \in (0, R) \text{ if (H) holds.} \qquad (4.28)$$

Similarly, Young's inequality along with Corollary 4.8 yields $c_2(\delta, T) > 0$ and $c_3(T) > 0$ such that

$$\begin{aligned}
\int_\Omega \xi^2 v_\varepsilon \Delta v_\varepsilon &\le \frac{1}{4}\int_\Omega \xi^2 |\Delta v_\varepsilon|^2 + \int_\Omega \xi^2 v_\varepsilon^2 \\
&\le \frac{1}{4}\int_\Omega \xi^2 |\Delta v_\varepsilon|^2 + c_2(\delta, T) \qquad \text{for all } t \in (0, T+1) \qquad (4.29)
\end{aligned}$$

and

$$\begin{aligned}
-\int_\Omega \xi^2 B_2 \Delta v_\varepsilon &\le \frac{1}{8}\int_\Omega \xi^2 |\Delta v_\varepsilon|^2 + 2\int_\Omega \xi^2 B_2^2 \\
&\le \frac{1}{8}\int_\Omega \xi^2 |\Delta v_\varepsilon|^2 + c_3(T) \qquad \text{for all } t \in (0, T+1), \qquad (4.30)
\end{aligned}$$

and that

$$\sup_{T>0} c_2(\delta, T) < \infty \quad \text{for all } \delta \in (0, R) \quad \text{and} \quad \sup_{T>0} c_3(T) < \infty \quad \text{if (H) holds.} \qquad (4.31)$$

Also using Young's inequality, we see that writing $c_4(\delta) := \|\nabla \xi\|_{L^\infty(\Omega)}^2$ we have

$$\begin{aligned}
-2\int_\Omega \xi(\nabla \xi \cdot \nabla v_\varepsilon) \Delta v_\varepsilon &\le \frac{1}{16}\int_\Omega \xi^2 |\Delta v_\varepsilon|^2 + 16\int_\Omega |\nabla \xi|^2 |\nabla v_\varepsilon|^2 \\
&\le \frac{1}{16}\int_\Omega \xi^2 |\Delta v_\varepsilon|^2 + 16 c_4(\delta) \int_{\Omega \setminus B_{\frac{\delta}{2}}} |\nabla v_\varepsilon|^2 \qquad \text{for all } t \in (0, T+1) \qquad (4.32)
\end{aligned}$$

and

$$
\begin{aligned}
2\int_\Omega \xi\nabla\xi \cdot \frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon}\nabla v_\varepsilon &\le \int_\Omega \xi^2|\nabla v_\varepsilon|^2 + \int_\Omega |\nabla\xi|^2\Big(\frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon}\Big)^2 \\
&\le \int_\Omega \xi^2|\nabla v_\varepsilon|^2 + c_4(\delta)\int_{\Omega\setminus B_{\frac{\delta}{2}}} u_\varepsilon^2 v_\varepsilon^2 \\
&\le \int_\Omega \xi^2|\nabla v_\varepsilon|^2 + c_4(\delta)\int_{\Omega\setminus B_{\frac{\delta}{2}}} u_\varepsilon^{p+2} + c_4(\delta)\int_{\Omega\setminus B_{\frac{\delta}{2}}} v_\varepsilon^{\frac{2(p+2)}{p}} \\
&\le \int_\Omega \xi^2|\nabla v_\varepsilon|^2 + c_4(\delta)\int_{\Omega\setminus B_{\frac{\delta}{2}}} u_\varepsilon^{p+2} + c_4(\delta)c_1(\delta,T) \qquad (4.33)
\end{aligned}
$$

for all $t\in(0,T+1)$. By two more applications of Young's inequality, again because of Corollary 4.8 we obtain $c_5(\delta,T)>0$ and $c_6(\delta,T)>0$ such that

$$
\begin{aligned}
2\int_\Omega \xi\nabla\xi\cdot v_\varepsilon\nabla v_\varepsilon &\le \int_\Omega \xi^2|\nabla v_\varepsilon|^2 + \int_\Omega |\nabla\xi|^2 v_\varepsilon^2 \\
&\le \int_\Omega \xi^2|\nabla v_\varepsilon|^2 + c_5(\delta,T) \qquad \text{for all } t\in(0,T+1) \qquad (4.34)
\end{aligned}
$$

and

$$
\begin{aligned}
-2\int_\Omega \xi\nabla\xi\cdot B_2\nabla v_\varepsilon &\le \int_\Omega \xi^2|\nabla v_\varepsilon|^2 + \int_\Omega |\nabla\xi|^2 B_2^2 \\
&\le \int_\Omega \xi^2|\nabla v_\varepsilon|^2 + c_6(\delta,T) \qquad \text{for all } t\in(0,T+1) \qquad (4.35)
\end{aligned}
$$

as well as

$$
\sup_{T>0} c_5(\delta,T)<\infty \quad \text{and} \quad \sup_{T>0} c_6(\delta,T)<\infty \quad \text{for all } \delta\in(0,R) \text{ if (H) holds.} \qquad (4.36)
$$

From (4.26) we thus infer upon collecting (4.22)-(4.36) that with some $c_7(\delta,T)>0$, for

$$
\begin{aligned}
&y_\varepsilon(t) := \int_\Omega \xi^2|\nabla v_\varepsilon(\cdot,t)|^2, \quad t\ge 0, \\
&g_\varepsilon := \frac{1}{8}\int_\Omega \xi^2|\Delta v_\varepsilon(\cdot,t)|^2, \quad t>0, \qquad \text{and} \\
&h_\varepsilon(t) := c_7(\delta,T)\int_{\Omega\setminus B_{\frac{\delta}{2}}} u_\varepsilon^{p+2}(\cdot,t) + c_7(\delta,T)\int_{\Omega\setminus B_{\frac{\delta}{2}}} |\nabla v_\varepsilon(\cdot,t)|^2 + c_7(\delta,T), \quad t>0,
\end{aligned}
$$

we have

$$
y_\varepsilon'(t) + g_\varepsilon(t) \le c_7(\delta,T)y_\varepsilon(t) + h_\varepsilon(t) \qquad \text{for all } t\in(0,T+1) \qquad (4.37)
$$

and

$$
\sup_{T>0} c_7(\delta,T)<\infty \quad \text{for all } \delta\in(0,R) \text{ if (H) is valid.} \qquad (4.38)
$$

Here we note that combining Lemma 4.4 with Corollary 4.8 shows that there exists $c_8(\delta,T)>0$ satisfying

$$\int_{(t-1)_+}^{t+1} h_\varepsilon(s)ds \le c_8(\delta,T) \qquad \text{for all } t\in(0,T),$$

where

$$\sup_{T>0} c_8(\delta,T)<\infty \quad \text{for all } \delta\in(0,R) \text{ if (H) holds,} \tag{4.39}$$

and and that Corollary 4.8 moreover provides $c_9(\delta,T)>0$ such that also

$$\sup_{T>0} c_9(\delta,T)<\infty \quad \text{for all } \delta\in(0,R) \text{ if (H) holds,} \tag{4.40}$$

and that for all $t\in(0,T+1)$ we can pick $t_\varepsilon(t)\in[(t-1)_+,t)$ fulfilling

$$y_\varepsilon(t_\varepsilon(t)) \le c_9(\delta,T).$$

On integrating (4.37) we thus infer that

$$\begin{aligned} y_\varepsilon(t) &\le y_\varepsilon(t_\varepsilon(t))e^{c_7(\delta,T)(t-t_\varepsilon(t))} + \int_{t_\varepsilon(t)}^t e^{c_7(\delta,T)(t-s)} h_\varepsilon(s)ds \\ &\le c_{10}(\delta,T) := c_9(\delta,T)e^{c_7(\delta,T)} + c_8(\delta,T)e^{c_7(\delta,T)} \qquad \text{for all } t\in(0,T+1) \end{aligned} \tag{4.41}$$

and that hence

$$\begin{aligned} \int_t^{t+1} g_\varepsilon(s)ds &\le y_\varepsilon(t) + c_7(\delta,T)\int_t^{t+1} y_\varepsilon(s)ds + \int_t^{t+1} h_\varepsilon(s)ds \\ &\le c_{10}(\delta,T) + c_7(\delta,T)c_{10}(\delta,T) + c_8(\delta,T) \qquad \text{for all } t\in(0,T). \end{aligned}$$

By definition of ξ, these inequalities immediately yield (4.22) and (4.24) and also (4.23) due to the observation that by Young's inequality,

$$\begin{aligned} \int_t^{t+1}\int_\delta^R v_{\varepsilon rr}^2(r,s)drds &\le 2\int_t^{t+1}\int_\delta^R \Big(v_{\varepsilon rr}(r,s)+\frac{1}{r}v_{\varepsilon r}(r,s)\Big)^2 drds + 2\int_t^{t+1}\int_\delta^R \frac{1}{r^2}v_{\varepsilon r}^2(r,s)drds \\ &\le \frac{1}{\pi\delta}\int_t^{t+1}\int_{\Omega\setminus B_\delta} |\Delta v_\varepsilon|^2 + \frac{2}{\delta^2}\sup_{s\in(0,T+1)}\int_\delta^R v_{\varepsilon r}^2(r,s)dr \qquad \text{for all } t\in(0,T), \end{aligned}$$

while (4.38), (4.39), (4.40) and (4.41) show that in fact (4.25) can be achieved. □

By means of another interpolation again relying on the one-dimensional problem structure, from this we immediately obtain a convenient estimate for the crucial taxis gradient in (2.1).

Corollary 4.10 *Let $p\in[1,4]$. Then for all $\delta\in(0,R)$ and $T>0$ one can find $C(\delta,T)>0$ such that*

$$\int_t^{t+1} \|v_{\varepsilon r}(\cdot,s)\|_{L^\infty((\delta,R))}^p ds \le C(\delta,T) \qquad \textit{for all } t\in(0,T),$$

where

$$\sup_{T>0} C(\delta,T)<\infty \quad \textit{for all } \delta\in(0,R) \textit{ if (H) holds.}$$

Proof. Since with some $c_1 = c_1(\delta) > 0$, for all $t > 0$ and $\varepsilon \in (0,1)$ we have

$$\int_t^{t+1} \|v_{\varepsilon r}(\cdot,s)\|^4_{L^\infty((\delta,R))} ds \le c_1 \int_t^{t+1} \|v_{\varepsilon rr}(\cdot,s)\|^2_{L^2((\delta,R))} \|v_{\varepsilon r}(\cdot,s)\|^2_{L^2((\delta,R))} ds + c_1 \int_t^{t+1} \|v_{\varepsilon r}(\cdot,s)\|^4_{L^2((\delta,R))} ds$$

by the Gagliardo-Nirenberg inequality, this is an immediate consequence of Lemma 4.9. □

4.5 Localized testing procedures III: Estimates for u_ε

Now with the information from Corollary 4.10 at hand, we can turn our attention to the first equation in (2.1) and perform an L^p testing procedure yielding a conditional result in the style of Lemma 4.7, to be used within an inductive step in Corollary 4.12.

Lemma 4.11 *Let $p > 1$. Then for all $K > 0, \delta \in (0,R)$ and $T > 0$ there exists $C(K,\delta,T) > 0$ with the property that if $\varepsilon \in (0,1)$ is such that*

$$\int_t^{t+1} \int_{\frac{\delta}{2}}^R u_\varepsilon^p(r,s)drds \le K \qquad \text{for all } t \in (0,T), \tag{4.42}$$

then

$$\int_\delta^R u_\varepsilon^p(r,t)dr \le C(K,\delta,T) \qquad \text{for all } t \in (0,T+1) \tag{4.43}$$

as well as

$$\int_t^{t+1} \int_\delta^R u_\varepsilon^{p-2}(r,s) u_{\varepsilon r}^2(r,s) drds \le C(K,\delta,T) \qquad \text{for all } t \in (0,T) \tag{4.44}$$

and

$$\int_t^{t+1} \int_\delta^R u_\varepsilon^{3p}(r,s)drds \le C(K,\delta,T) \qquad \text{for all } t \in (0,T), \tag{4.45}$$

and that furthermore

$$\sup_{T>0} C(K,\delta,T) < \infty \quad \text{for all } K > 0 \text{ and } \delta \in (0,R), \text{ provided that (H) holds.} \tag{4.46}$$

Proof. Similar to the procedure in Lemma 4.7, in the original variables we let $\xi(x) := \zeta_\delta(|x|)$, $x \in \Omega$, with ζ_δ taken from (4.9), and use the first equation in (2.1) to compute

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega \xi^2 u_\varepsilon^p + (p-1)\int_\Omega \xi^2 u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 &= -2\int_\Omega \xi\nabla\xi \cdot u_\varepsilon^{p-1}\nabla u_\varepsilon \\
&\quad +(p-1)\chi\int_\Omega \xi^2 \frac{u_\varepsilon^{p-1}}{v_\varepsilon}\nabla u_\varepsilon \cdot \nabla v_\varepsilon + 2\chi\int_\Omega \xi\nabla\xi \cdot \frac{u_\varepsilon^p}{v_\varepsilon}\nabla v_\varepsilon \\
&\quad -\int_\Omega \xi^2 u_\varepsilon^p v_\varepsilon + \int_\Omega \xi^2 B_1 u_\varepsilon^{p-1} \qquad \text{for all } t > 0,
\end{aligned} \tag{4.47}$$

where by Young's inequality,

$$\begin{aligned}
-2\int_\Omega \xi\nabla\xi \cdot u_\varepsilon^{p-1}\nabla u_\varepsilon &\le \frac{p-1}{4}\int_\Omega \xi^2 u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 + \frac{4}{p-1}\int_\Omega |\nabla\xi|^2 u_\varepsilon^p \\
&\le \frac{p-1}{4}\int_\Omega \xi^2 u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 + \frac{4}{p-1}\|\nabla\xi\|^2_{L^\infty(\Omega)} \int_{\Omega\setminus B_{\frac{\delta}{2}}} u_\varepsilon^p \qquad \text{for all } t > 0.
\end{aligned} \tag{4.48}$$

Likewise,

$$
\begin{aligned}
(p-1)\chi\int_\Omega \xi^2\frac{u_\varepsilon^{p-1}}{v_\varepsilon}\nabla u_\varepsilon\cdot\nabla v_\varepsilon
&\le \frac{p-1}{4}\int_\Omega \xi^2 u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2+(p-1)\chi^2\int_\Omega \xi^2\frac{u_\varepsilon^p}{v_\varepsilon^2}|\nabla v_\varepsilon|^2\\
&\le \frac{p-1}{4}\int_\Omega \xi^2 u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2+(p-1)\chi^2 c_1(T)\|\nabla v_\varepsilon(\cdot,t)\|^2_{L^\infty(\Omega\setminus B_{\frac{\delta}{2}})}\int_\Omega \xi^2 u_\varepsilon^p \qquad (4.49)
\end{aligned}
$$

and

$$
\begin{aligned}
2\chi\int_\Omega \xi\nabla\xi\cdot\frac{u_\varepsilon^p}{v_\varepsilon}\nabla v_\varepsilon
&\le \int_\Omega \xi^2\frac{u_\varepsilon^p}{v_\varepsilon^2}|\nabla v_\varepsilon|^2+\chi^2\int_\Omega|\nabla\xi|^2u_\varepsilon^p\\
&\le c_1\|\nabla v_\varepsilon(\cdot,t)\|^2_{L^\infty(\Omega\setminus B_{\frac{\delta}{2}})}\int_\Omega \xi^2u_\varepsilon^p\\
&\quad +\chi^2\|\nabla\xi\|^2_{L^\infty(\Omega)}\int_{\Omega\setminus B_{\frac{\delta}{2}}}u_\varepsilon^p \qquad (4.50)
\end{aligned}
$$

for all $t\in(0,T+1)$, with $c_1(T):=\sup_{\varepsilon\in(0,1)}\|\frac{1}{v_\varepsilon}\|^2_{L^\infty(\Omega\times(0,T+1))}$ satisfying

$$\sup_{T>0}c_1(T)<\infty \quad \text{if (H) holds.}$$

Since moreover

$$
\begin{aligned}
\int_\Omega \xi^2B_1u_\varepsilon^{p-1} &\le \int_\Omega \xi^2u_\varepsilon^p+\int_\Omega\xi^2B_1^p\\
&\le \int_\Omega\xi^2u_\varepsilon^p+c_2(T) \qquad \text{for all } t\in(0,T+1)
\end{aligned}
$$

with $c_2(T):=|\Omega|\cdot\|B_1\|^p_{L^\infty(\Omega\times(0,T+1))}$ fulfilling

$$\sup_{T>0}2_1(T)<\infty \quad \text{if (H) is valid,}$$

from (4.47)-(4.50) it follows that there exists $c_3(\delta,T)>0$ such that

$$\sup_{T>0}c_3(\delta,T)<\infty \quad \text{for all } \delta\in(0,R) \text{ if (H) holds,}$$

and that

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega\xi^2u_\varepsilon^p+\frac{p(p-1)}{2}\int_\Omega\xi^2u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 &\le c_3(\delta,T)\Big\{\|\nabla v_\varepsilon(\cdot,t)\|^2_{L^\infty(\Omega\setminus B_{\frac{\delta}{2}})}+1\Big\}\int_\Omega\xi^2u_\varepsilon^p\\
&\quad +c_3(\delta,T)\int_{\Omega\setminus B_{\frac{\delta}{2}}}u_\varepsilon^p+c_3(\delta,T) \qquad \text{for all } t\in(0,T+1),
\end{aligned}
$$

that is, for all $t\in(0,T+1)$ we see that

$$y_\varepsilon'(t)+g_\varepsilon(t)\le a_\varepsilon(t)y_\varepsilon(t)+h_\varepsilon(t) \qquad (4.51)$$

holds for $y_\varepsilon(t) := \int_\Omega \xi^2 u_\varepsilon^p(\cdot,t)$, $t \ge 0$, as well as $g_\varepsilon(t) := \frac{p(p-1)}{2}\int_\Omega \xi^2 u_\varepsilon^{p-2}(\cdot,t)|\nabla u_\varepsilon(\cdot,t)|^2$, $a_\varepsilon(t) := c_3(\delta,T)\Big\{\|\nabla v_\varepsilon(\cdot,t)\|^2_{L^\infty(\Omega\setminus B_{\frac{\delta}{2}})}+1\Big\}$ and $h_\varepsilon(t) := c_3(\delta,T)\int_{\Omega\setminus B_{\frac{\delta}{2}}} u_\varepsilon^p(\cdot,t) + c_3(\delta,T)$, $t>0$. Here, Corollary 4.10 yields $c_4(\delta,T)>0$ such that

$$\int_{(t-1)_+}^{t+1} a_\varepsilon(s)ds \le c_4(\delta,T) \qquad \text{for all } t\in(0,T)$$

and

$$\sup_{T>0} c_4(\delta,T) < \infty \quad \text{for all } \delta\in(0,R) \text{ if (H) holds,}$$

and from the hypothesis (4.42) we obtain $c_5(K,T)>0$ such that

$$\int_{(t-1)_+}^{t+1} h_\varepsilon(s)ds \le c_5(K,\delta,T) \qquad \text{for all } t\in(0,T),$$

with

$$\sup_{T>0} c_5(K,\delta,T) < \infty \quad \text{for all } K>0 \text{ and } \delta\in(0,R) \text{ if (H) holds.}$$

Upon integration, from (4.51) it thus follows in a straightforward manner that

$$y_\varepsilon(t) \le c_6(K,\delta,T) := e^{c_4(\delta,T)}\int_\Omega u_0^p + c_5(K,\delta,T)e^{c_4(\delta,T)} \qquad \text{for all } t\in(0,T+1)$$

and

$$\int_t^{t+1} g_\varepsilon(s)ds \le c_6(K,\delta,T) + c_6(K,\delta,T)c_4(\delta,T) + c_5(K,\delta,T) \qquad \text{for all } t\in(0,T),$$

which immediately entails (4.43) and (4.44). As an application of the one-dimensional Gagliardo-Nirenberg inequality shows that with some $c_7(\delta)>0$ we have

$$\begin{aligned}\int_t^{t+1}\int_\delta^R u_\varepsilon^{3p}(r,sdrds &= \int_t^{t+1}\|u_\varepsilon^{\frac{p}{2}}(\cdot,s)\|^6_{L^6((\delta,R))}ds \\ &\le c_7(\delta)\int_t^{t+1}\|(u_\varepsilon^{\frac{p}{2}})_r(\cdot,s)\|^2_{L^2((\delta,R))}\|u_\varepsilon^{\frac{p}{2}}(\cdot,s)\|^4_{L^2((\delta,R))}ds \\ &\quad + c_7(\delta)\int_t^{t+1}\|u_\varepsilon^{\frac{p}{2}}(\cdot,s)\|^6_{L^2((\delta,R))}ds\end{aligned}$$

for all $t>0$, this moreover shows that also (4.45) can be achieved with some suitably large $C(K,\delta,T)>0$ fulfilling (4.46). □

In fact, starting from the basic information in Lemma 4.4 we can make sure that this entails essentially the same conclusion for u_ε as that drawn for v_ε in Corollary 4.8.

Corollary 4.12 *Let $p > 1$. Then for all $\delta \in (0,R)$ and $T > 0$ we can pick $C(\delta,T) > 0$ such that for each $\varepsilon \in (0,1)$ we have*

$$\int_\delta^R u_\varepsilon^p(r,t)dr \le C(\delta,T) \qquad \textit{for all } t \in (0,T) \tag{4.52}$$

and

$$\int_t^{t+1}\int_\delta^R u_\varepsilon^{p-2}(r,s)u_{\varepsilon r}^2(r,s)drds \le C(\delta,T) \qquad \textit{for all } t \in (0,T), \tag{4.53}$$

and that moreover

$$\sup_{T>0} C(\delta,T) < \infty \quad \textit{for all } \delta \in (0,R), \textit{ provided that (H) is satisfied.} \tag{4.54}$$

Proof. Proceeding in a way similar to that in Corollary 4.8, for fixed $p > 1$ we take $p_0 \in (1,2]$ and an integer $k \ge 0$ such that $p \le 3^k p_0$, and observe that then from Lemma 4.4 we know that there exists $c_1(\delta,T) > 0$ such that

$$\int_t^{t+1}\int_{2^{-k}\delta}^R u_\varepsilon^{p_0}(r,s)drds \le c_1(\delta,T) \qquad \text{for all } t \in (0,T),$$

and that

$$\sup_{T>0} c_1(\delta,T) < \infty \quad \text{for all } \delta \in (0,R) \text{ if (H) holds.}$$

Upon $k+1$ applications of Lemma 4.11 we thus infer that for each $j \in \{0,...,k\}$ we can find $c_2(j,\delta,T) > 0$ such that

$$\sup_{T>0} c_2(j,\delta,T) < \infty \quad \text{for all } \delta \in (0,R) \text{ if (H) holds,} \tag{4.55}$$

that

$$\int_{2^{j-k}\delta}^R u_\varepsilon^{3^j p_0}(r,t)dr \le c_2(j,\delta,T) \qquad \text{for all } t \in (0,T) \tag{4.56}$$

and

$$\int_t^{t+1}\int_{2^{j-k}\delta}^R u_\varepsilon^{3^{j+1}p_0}(r,s)drds \le c_2(j,\delta,T) \qquad \text{for all } t \in (0,T),$$

and that

$$\int_t^{t+1}\int_{3^{j-k}\delta}^R u_\varepsilon^{3^j p_0-2}(r,s)u_{\varepsilon r}^2(r,s)drds \le c_2(j,\delta,T) \qquad \text{for all } t \in (0,T) \tag{4.57}$$

whenever $\varepsilon \in (0,1)$. Since the interval (δ,R) is bounded and $p \le 3^k p_0$, in view of the Hölder inequality we particularly infer from (4.56) and (4.57) that (4.52) and (4.53) are satisfied with some $C(\delta,T) > 0$ fulfilling (4.54) due to (4.55). $\square$

5 Local estimates in Hölder spaces

Unlike those underlying the essential steps in Lemma 4.7, Lemma 4.9 and Lemma 4.11, our subsequent regularity arguments will explicitly address the corresponding one-dimensional versions of the first two equations in (2.1). To transform these into identities focusing on appropriate spatio-temporal regions only, let us announce that when localizing exclusively only with respect to the spatial variable we will make use of the cut-off function defined by

$$\xi(r,t) \equiv \xi_\delta(r,t) := \zeta_\delta(r), \qquad r \in [0,R],\ t \ge 0, \tag{5.1}$$

for $\delta \in (0,R)$, while in situations when also the temporal origin needs to be faded out we will rather employ

$$\xi(r,t) \equiv \xi_{\delta,\tau}(r,t) := \zeta_\delta(r)\zeta_\tau(t), \qquad r \in [0,R],\ t \ge 0, \tag{5.2}$$

for $\delta \in (0,R)$ and $\tau > 0$.

In both these cases, for the solution of (2.1) we then have

$$(\xi u_\varepsilon)_t = (\xi u_\varepsilon)_{rr} + a_{1r}(r,t) + a_2(r,t), \qquad r \in (0,R),\ t > 0, \tag{5.3}$$

with

$$a_1 \equiv a_1(r,t;\xi,\varepsilon) := -\chi\xi\frac{u_\varepsilon}{v_\varepsilon}v_{\varepsilon r} - 2\xi_r u_\varepsilon + \frac{1}{r}\xi u_\varepsilon \tag{5.4}$$

and

$$a_2 \equiv a_2(r,t;\xi,\varepsilon) := \chi\xi_r\frac{u_\varepsilon}{v_\varepsilon}v_{\varepsilon r} - \chi\frac{1}{r}\xi\frac{u_\varepsilon}{v_\varepsilon}v_{\varepsilon r} + \xi_{rr}u_\varepsilon - \frac{1}{r}\xi_r u_\varepsilon + \frac{1}{r^2}\xi u_\varepsilon - \xi u_\varepsilon v_\varepsilon + \xi B_1 + \xi_t u_\varepsilon \tag{5.5}$$

for $r \in (0,R)$ and $t > 0$. Similarly,

$$(\xi v_\varepsilon)_t = (\xi v_\varepsilon)_{rr} + b(r,t), \qquad r \in (0,R),\ t > 0, \tag{5.6}$$

where

$$b \equiv b(r,t;\xi,\varepsilon) := -2\xi_r v_{\varepsilon r} + \frac{1}{r}\xi v_{\varepsilon r} - \xi_{rr}v_\varepsilon + \xi\frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} - \xi v_\varepsilon + \xi B_2 + \xi_t v_\varepsilon \tag{5.7}$$

for $r \in (0,R)$ and $t > 0$.

In some places below we will use the abbreviation $(e^{-tA})_{t\ge0}$ for the Neumann heat semigroup over the one-dimensional interval $(0,R)$.

Now by means of well-known results on Hölder regularity and of maximal Sobolev regularity in the linear heat equation (5.6), the integrability properties of b implied by Lemma 4.9, Corollary 4.12 and Corollary 4.8 entail the following.

Lemma 5.1 *Let $p > \frac{4}{3}$ and $\delta \in (0,R)$. Then there exists $\theta = \theta(\delta) \in (0,1)$ with the property that for all $T > 0$ one can find $C(\delta,T) > 0$ such that*

$$\|v_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}([\delta,R]\times[t,t+1])} \le C(\delta,T) \qquad \text{for all } t \in (0,T), \tag{5.8}$$

that moreover

$$\int_t^{t+1} \int_\delta^R |v_{\varepsilon r}(r,s)|^p drds \le C(\delta,T) \qquad \textit{for all } t \in (0,T), \tag{5.9}$$

and that

$$\sup_{T>0} C(\delta,T) < \infty \quad \textit{for all } \delta \in (0,R) \textit{ if (H) holds.} \tag{5.10}$$

PROOF. Choosing ξ as in (5.1) and using that $W^{1,2}((0,R)) \hookrightarrow L^\infty((0,R))$, from Lemma 4.9, Corollary 4.8 and Corollary 4.12 we infer that

$$\|\xi v_\varepsilon(\cdot,t)\|_{L^\infty((0,R))} \le c_1(\delta,T) \qquad \text{for all } t \in (0,T)$$

and that b as in (5.7) satisfies

$$\int_t^{t+1} \|b(\cdot,s)\|^p_{L^2((0,R))} ds \le c_1(\delta,T) \qquad \text{for all } t \in (0,T), \tag{5.11}$$

where

$$\sup_{T>0} c_1(\delta,T) < \infty \quad \text{for all } \delta \in (0,R) \text{ if (H) holds.} \tag{5.12}$$

Therefore, in view of our hypothesis $p > \frac{4}{3}$ and our overall assumption that $v_0 \in W^{1,\infty}(\Omega)$, a standard result on Hölder regularity in the Neumann problem for the inhomogeneous linear heat equation (5.6) (see e.g. [28, Theorem 1.3, Remark 1.4, Remark 1.2]) applies so as to yield $\theta \in (0,1)$ and $c_2(\delta,T) > 0$ such that

$$\sup_{T>0} c_2(\delta,T) < \infty \quad \text{for all } \delta \in (0,R) \text{ if (H) holds,} \tag{5.13}$$

and that

$$\|\xi v_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}([0,R]\times[t,t+1])} \le c_2(\delta,T) \qquad \text{for all } t \in (0,T). \tag{5.14}$$

Moreover, a well-known maximal Sobolev regularity property in the Neumann problem

$$\begin{cases} z_t = z_{rr} + b(r,t), & r \in (0,R),\ t > 0, \\ z_r(0,t) = z_r(R,t) = 0, & t > 0, \\ z(r,0) = 0, & r \in (0,R), \end{cases}$$

([15]) asserts that due to (5.11) and (5.12), with some $c_3(\delta,T) > 0$ the function z_ε defined by $z_\varepsilon(\cdot,t) := \xi v_\varepsilon(\cdot,t) - e^{-tA}(\xi v_0)$, $t \ge 0$, satisfies

$$\int_t^{t+1} \|z_\varepsilon(\cdot,s)\|^p_{W^{2,2}((0,R))} ds \le c_3(\delta,T) \qquad \text{for all } t \in (0,T),$$

and hence, as $W^{2,2}((0,R)) \hookrightarrow W^{1,p}((0,R))$, there exists $c_4(\delta,T) > 0$ fulfilling

$$\int_t^{t+1} \int_0^R |z_{\varepsilon r}(r,s)|^p drds \le c_4(\delta,T) \qquad \text{for all } t \in (0,T),$$

where

$$\sup_{T>0} c_3(\delta,T) < \infty \quad \text{and} \quad \sup_{T>0} c_4(\delta,T) < \infty \quad \text{for all } \delta \in (0,R) \text{ if (H) holds.} \tag{5.15}$$

Since

$$\|\partial_r e^{-tA}(\xi v_0)\|_{L^p((0,R))} \le c_5 \|\xi v_0\|_{W^{1,p}((0,R))} \le c_6(\delta)\|v_0\|_{W^{1,\infty}(\Omega)} \qquad \text{for all } t>0$$

and hence

$$\int_t^{t+1}\int_0^R \Big|\partial_r e^{-sA}(\xi v_0)(r)\Big|^p drds \le c_7(\delta) \qquad \text{for all } t>0$$

with some appropriately large constants $c_5, c_6(\delta)$ and $c_7(\delta)$, we readily obtain that

$$\int_t^{t+1}\int_\delta^R |v_{\varepsilon r}(r,s)|^p drds \le \int_t^{t+1}\int_0^R |(\xi v_\varepsilon)_r(r,s)|^p drds \le 2^{p-1}(c_4(\delta,T)+c_7(\delta)) \qquad \text{for all } t\in(0,T).$$

Together with (5.14), (5.13) and (5.15), this establishes (5.8)-(5.10). $\Box$

If we moreover exclude the temporal origin, then without any further requirements on the initial data we may once more invoke results on maximal Sobolev regularity to infer from the latter that in fact both $v_{\varepsilon rr}$ and $v_{\varepsilon t}$ are bounded in corresponding space-time L^p spaces for arbitrary finite $p>1$.

Lemma 5.2 *Let $p>1$. Then for all $\tau\in(0,1)$, $\delta\in(0,R)$ and $T>1$ there exists $C(\tau,\delta,T)>0$ such that*

$$\int_t^{t+1}\int_\delta^R \Big\{|v_{\varepsilon rr}(r,s)|^p + |v_{\varepsilon t}(r,s)|^p\Big\} drds \le C(\tau,\delta,T) \qquad \textit{for all } t\in(\tau,T), \tag{5.16}$$

and that

$$\sup_{T>1} C(\tau,\delta,T)<\infty \quad \textit{for all } \tau\in(0,1) \textit{ and } \delta\in(0,R) \textit{ if (H) holds.} \tag{5.17}$$

PROOF. This time defining ξ as in (5.2), we infer from Lemma 5.1 in conjunction with Corollary 4.12 and Corollary 4.8 that there exists $c_1(\tau,\delta,T)>0$ such that the function in (5.7) satisfies

$$\int_t^{t+1}\int_0^R |b(r,s)|^p drds \le c_1(\tau,\delta,T) \qquad \text{for all } t\in(0,T),$$

and such that

$$\sup_{T>1} c_1(\tau,\delta,T)<\infty \quad \text{for all } \tau\in(0,1) \text{ and } \delta\in(0,R) \text{ if (H) holds.}$$

In view of (5.6), a straightforward application of maximal Sobolev regularity estimates thus yields $c_2(\tau,\delta,T)>0$ fulfilling

$$\int_t^{t+1}\int_0^R \Big\{|(\xi v_\varepsilon)_{rr}(r,s)|^p + |(\xi v_\varepsilon)_t(r,s)|^p\Big\} drds \le c_2(\tau,\delta,T) \qquad \text{for all } t\in(0,T)$$

as well as

$$\sup_{T>1} c_2(\tau,\delta,T)<\infty \quad \text{for all } \tau\in(0,1) \text{ and } \delta\in(0,R) \text{ if (H) is valid.}$$

By definition of ξ, this immediately implies (5.16) and (5.17). $\Box$

In order to draw similar conclusions for the first solution component, we first concentrate on a corresponding Hölder bound.

Lemma 5.3 *There exists $\theta \in (0,1)$ such that for each $\delta \in (0,R)$ and any $T>0$ one can find $C(\delta,T)>0$ satisfying*

$$\|u_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}([\delta,R]\times[t,t+1])} \le C(\delta,T) \qquad \textit{for all } t\in(0,T), \tag{5.18}$$

and that moreover

$$\sup_{T>0} C(\delta,T) < \infty \quad \textit{for all } \delta\in(0,R) \textit{ whenever (H) holds.} \tag{5.19}$$

PROOF. Again with ξ as in (5.1), fixing any $p>3$ we know from Lemma 5.1, Corollary 4.12, Corollary 4.8 and Lemma 2.2 that there exists $c_1(\delta,T)>0$ such that for the functions a_1 and a_2 defined by (5.4) and (5.5) we have

$$\int_t^{t+1}\int_0^R \Big\{|a_1(r,s)|^p + |a_2(r,s)|^p\Big\} drds \le c_1 \qquad \text{for all } t\in(0,T), \tag{5.20}$$

and such that

$$\sup_{T>0} c_1(\delta,T) < \infty \quad \text{for all } \delta\in(0,R) \text{ if (H) holds.} \tag{5.21}$$

Now on the basis of a variation-of-constants representation associated with (5.3), applying known smoothing properties of the Neumann heat semigroup $(e^{-tA})_{t\ge0}$ (see e.g. [37]) we can find positive constants c_2, c_3 and c_4 such that

$$\begin{aligned}
&\|\xi u_\varepsilon(\cdot,t)\|_{L^\infty((0,R))} \\
&\quad= \left\| e^{-(t-(t-1)_+)A}(\xi u_0) + \int_{(t-1)_+}^t e^{-(t-s)A} a_{1r}(\cdot,s)ds + \int_{(t-1)_+}^t e^{-(t-s)A} a_2(\cdot,s)ds \right\|_{L^\infty((0,R))} \\
&\quad\le c_2\cdot \max\Big\{ \|\xi u_0\|_{L^\infty((0,R))}\,,\, \|\xi u(\cdot,(t-1)_+)\|_{L^1((0,R))}\Big\} \\
&\qquad + c_3 \int_{(t-1)_+}^t (t-s)^{-\frac12-\frac{1}{2p}} \|a_1(\cdot,s)\|_{L^p((0,R))} ds \\
&\qquad + c_4 \int_{(t-1)_+}^t (t-s)^{-\frac{1}{2p}} \|a_2(\cdot,s)\|_{L^p((0,R))} ds \qquad \text{for all } t>0.
\end{aligned} \tag{5.22}$$

Since herein by Young's inequality we have

$$c_3\int_{(t-1)_+}^t (t-s)^{-\frac12-\frac{1}{2p}}\|a_1(\cdot,s)\|_{L^p((0,R))}ds \le c_3\int_{(t-1)_+}^t \|a_1(\cdot,s)\|^p_{L^p((0,R))}ds + c_3\int_{(t-1)_+}^t (t-s)^{-\frac{p+1}{2(p-1)}}ds$$

and

$$c_4\int_{(t-1)_+}^t (t-s)^{-\frac{1}{2p}}\|a_2(\cdot,s)\|_{L^p((0,R))}ds \le c_4\int_{(t-1)_+}^t \|a_2(\cdot,s)\|^p_{L^p((0,R))}ds + c_4\int_{(t-1)_+}^t (t-s)^{-\frac{1}{2(p-1)}}ds$$

for all $t>0$, and since our assumption $p>3$ warrants that $\frac{1}{2(p-1)} < \frac{p+1}{2(p-1)} < 1$, from (5.22), (5.20) and (5.21) together with Lemma 4.1 it follows that for some $c_5(\delta,T)>0$ we have

$$\|\xi u_\varepsilon(\cdot,t)\|_{L^\infty((0,R))} \le c_5(\delta,T) \qquad \text{for all } t\in(0,T)$$

and

$$\sup_{T>0} c_5(\delta,T) < \infty \quad \text{for all } \delta \in (0,R) \text{ if (H) holds.}$$

Relying on this boundedness information and again (5.20) and (5.21), using that in fact u_0 was assumed to be Hölder continuous in $\overline{\Omega}$, we may now once more apply [28, Theorem 1.3, Remark 1.4, Remark 1.2] to obtain $\theta \in (0,1)$ and $c_6(\delta,T) > 0$ such that

$$\sup_{T>0} c_6(\delta,T) < \infty \quad \text{for all } \delta \in (0,R) \text{ if (H) holds,}$$

and that

$$\|\xi u_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}([0,R]\times[t,t+1])} \le c_6(\delta,T) \qquad \text{for all } t \in (0,T),$$

thus implying (5.18) and (5.19). □

Now an estimate for u_ε analogous to that for v_ε from Lemma 5.2 will again require an iteration, the core of which is prepared as follows.

Lemma 5.4 *Let $p \ge 2$. Then for all $K > 0, \tau \in (0,1), \delta \in (0,R)$ and $T > 1$ there exists $C(K,\tau,\delta,T) > 0$ such that if for some $\varepsilon \in (0,1)$ we have*

$$\int_t^{t+1} \int_{\frac{\delta}{2}}^R |u_{\varepsilon r}(r,s)|^p dr ds \le K \qquad \textit{for all } t \in \Big(\frac{\tau}{2}, T\Big), \tag{5.23}$$

then

$$\int_t^{t+1} \int_\delta^R \Big\{ |u_{\varepsilon rr}(r,s)|^p + |u_{\varepsilon t}(r,s)|^p \Big\} dr ds \le C(K,\tau,\delta,T) \qquad \textit{for all } t \in (\tau,T) \tag{5.24}$$

and

$$\int_t^{t+1} \int_\delta^R |u_{\varepsilon r}(r,t)|^{2p} dr \le C(K,\tau,\delta,T) \qquad \textit{for all } t \in (\tau,T), \tag{5.25}$$

and such that furthermore

$$\sup_{T>1} C(K,\tau,\delta,T) < \infty \quad \textit{for all } K>0, \tau \in (0,1) \textit{ and } \delta \in (0,R), \textit{ provided that (H) holds.} \tag{5.26}$$

Proof. We let ξ be as in (5.2), and taking a_1 and a_2 from (5.4) and (5.5) we expand a_{1r} so as to obtain

$$\begin{aligned} a_{1r} + a_2 &= -\chi\xi\frac{u_\varepsilon}{v_\varepsilon} v_{\varepsilon rr} - \chi\xi\frac{1}{v_\varepsilon} u_{\varepsilon r} v_{\varepsilon r} + \chi\xi\frac{u_\varepsilon}{v_\varepsilon^2} v_{\varepsilon r}^2 - \chi\frac{1}{r}\xi\frac{u_\varepsilon}{v_\varepsilon} v_{\varepsilon r} \\ &\quad -\xi u_\varepsilon v_\varepsilon + \xi B_1 + \xi_t u_\varepsilon, \qquad r \in (0,R),\ t>0. \end{aligned}$$

In conjunction with Lemma 4.9, Lemma 5.3, Lemma 5.2 and Lemma 2.2, our assumption (5.23) thus arrants the existence of $c_1(K,\tau,\delta,T) > 0$ fulfilling

$$\int_t^{t+1} \int_0^R \Big| a_{1r}(r,s) + a_2(r,s)|^p dr ds \le c_1(K,\tau,\delta,T) \qquad \text{for all } t \in (0,T)$$

as well as

$$\sup_{T>0} c_1(K,\tau,\delta,T) < \infty \quad \text{for all } K>0, \tau\in(0,1) \text{ and } \delta\in(0,R) \text{ if holds,}$$

so that maximal Sobolev regularity estimates applied to (5.3) yield $c_2(K,\tau,\delta,T)>0$ such that

$$\sup_{T>0} c_1(K,\tau,\delta,T) < \infty \quad \text{for all } K>0, \tau\in(0,1) \text{ and } \delta\in(0,R) \text{ if holds,}$$

and that

$$\int_t^{t+1}\int_0^R \Big\{ |(\xi u_\varepsilon)_{rr}(r,s)|^p + |(\xi u_\varepsilon)_t(r,s)|^p \Big\} drds \le c_2(K,\tau,\delta,T) \qquad \text{for all } t\in(0,T),$$

and that thus (5.24) holds with some $C(K,\tau,\delta,T)>0$ satisfying (5.26). Since the Gagliardo-Nirenberg inequality says that there exists $c_3(\delta)>0$ such that

$$\begin{aligned}\int_t^{t+1} \|u_{\varepsilon r}(\cdot,s)\|_{L^{2p}((\delta,R))}^{2p} ds &\le c_3(\delta)\int_t^{t+1} \|u_{\varepsilon rr}(\cdot,s)\|_{L^p((\delta,R))}^p \|u_\varepsilon(\cdot,t)\|_{L^\infty((\delta,R))}^p ds \\ &\quad + c_3(\delta)\int_t^{t+1} \|u_\varepsilon(\cdot,s)\|_{L^\infty((\delta,R))}^{2p} ds\end{aligned}$$

for all $t>0$, in view of the boundedness property implied by Lemma 5.3 we see that this also entails (5.25) if $C(K,\tau,\delta,T)$ is enlarged appropriately. □

Along with a known embedding result, in conjunction with Lemma 5.2 a repeated application of Lemma 5.4 yields bounds in some intermediate Hölder spaces.

Corollary 5.5 *There exists $\theta\in(0,1)$ with the property that for each $\tau\in(0,1)$, any $\delta\in(0,R)$ and all $T>1$ one can find $C(\tau,\delta,T)>0$ such that*

$$\|u_\varepsilon\|_{C^{1+\theta,\theta}([\delta,R]\times[t,t+1])} \le C(\tau,\delta,T) \qquad \text{for all } t\in(\tau,T) \tag{5.27}$$

and

$$\|v_\varepsilon\|_{C^{1+\theta,\theta}([\delta,R]\times[t,t+1])} \le C(\tau,\delta,T) \qquad \text{for all } t\in(\tau,T) \tag{5.28}$$

as well as

$$\sup_{T>1} C(\tau,\delta,T) < \infty \quad \text{for all } \tau\in(0,1) \text{ and } \delta\in(0,R) \text{ if (H) holds.} \tag{5.29}$$

PROOF. As a starting point using the spatio-temporal L^2 boundedness property (4.53)-(4.54) of $(u_{\varepsilon r})_{\varepsilon\in(0,1)}$ asserted by Corollary 4.12 , we may iterate Lemma 5.4 in a style similar to that e.g. in the proof of Corollary 4.12 so as to conclude that given any $p\ge 2$ we can find $c_1(\tau,\delta,T)>0$ such that

$$\int_t^{t+1}\int_\delta^R \Big\{ |u_{\varepsilon rr}(r,s)|^p + |u_{\varepsilon t}(r,s)|^p \Big\} drds \le c_1(\tau,\delta,T) \qquad \text{for all } t\in(\tau,T),$$

and that

$$\sup_{T>1} c_1(\tau,\delta,T) < \infty \quad \text{for all } \tau\in(0,1) \text{ and } \delta\in(0,R) \text{ if (H) holds.}$$

Applying this to some suitably large $p \geq 2$, in view of a known embedding result ([1]) we immediately obtain (5.27) and (5.29) as a consequence thereof. Likewise, using Lemma 5.2 we see that on suitably enlarging $C(\tau, \delta, T)$ we may also achieve (5.28). $\Box$

Now the latter provides sufficient regularity for the inhomogeneity in (5.6) so as to allow for the application of standard parabolic Schauder theory.

Lemma 5.6 *There exists $\theta \in (0,1)$ such that to all $\tau \in (0,1)$, $\delta \in (0,R)$ and $T > 1$ there corresponds some $C(\tau, \delta, T) > 0$ such that*

$$\|v_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}([\delta,R]\times[t,t+1])} \leq C(\tau, \delta, T) \qquad \text{for all } t \in (\tau, T), \tag{5.30}$$

and such that

$$\sup_{T>1} C(\tau, \delta, T) < \infty \quad \text{for all } \tau \in (0,1) \text{ and } \delta \in (0,R) \text{ if (H) holds.} \tag{5.31}$$

Proof. Using Lemma 5.3 and Corollary 5.5, we infer that with some $\theta \in (0,1)$, taking ξ as in (5.2) we can find $c_1(\tau, \delta, T) > 0$ such that for the function b from (5.7) we have

$$\|b\|_{C^{\theta,\frac{\theta}{2}}([0,R]\times[t,t+1])} \leq c_1(\tau, \delta, T) \qquad \text{for all } t \in (0,T),$$

and such that

$$\sup_{T>1} c_1(\tau, \delta, T) < \infty \quad \text{for all } \tau \in (0,1) \text{ and } \delta \in (0,R) \text{ if (H) holds.}$$

Therefore, classical parabolic Schauder estimates for the inhomogeneous linear heat equation (5.6) ([23]) provides $c_2(\tau, \delta, T) > 0$ such that

$$\|\xi v_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}([0,R]\times[t,t+1])} \leq c_2(\tau, \delta, T) \qquad \text{for all } t \in (0,T)$$

and

$$\sup_{T>1} c_2(\tau, \delta, T) < \infty \quad \text{for all } \tau \in (0,1) \text{ and } \delta \in (0,R) \text{ if (H) holds,}$$

which establishes both (5.30) and (5.31). $\Box$

This in turn warrants that indeed the full cross-diffusive term in (5.3) can be viewed as part of a corresponding inhomogeneity in a linear heat equation which satisfies bounds in some Hölder space and thus enables us to conclude, again from classical Schauder theory, that also u_ε enjoys a regularity property comparable to that of v_ε from Lemma 5.6.

Lemma 5.7 *There exists $\theta \in (0,1)$ such that to all $\tau \in (0,1)$, $\delta \in (0,R)$ and $T > 1$ one can pick $C(\tau, \delta, T) > 0$ such that*

$$\|u_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}([\delta,R]\times[t,t+1])} \leq C(\tau, \delta, T) \qquad \text{for all } t \in (\tau, T), \tag{5.32}$$

and such that

$$\sup_{T>1} C(\tau, \delta, T) < \infty \quad \text{for all } \tau \in (0,1) \text{ and } \delta \in (0,R) \text{ if (H) holds.} \tag{5.33}$$

PROOF. We once more employ Corollary 5.5 which in conjunction with Lemma 5.6 and Lemma 2.2 shows that with some $\theta \in (0,1)$, if we fix ξ as in (5.2) then for a_1 and a_2 taken from (5.4) and (5.5) we have

$$\|a_{1r} + a_2\|_{C^{\theta,\frac{\theta}{2}}([0,R]\times[t,t+1])} \le c_1(\tau,\delta,T) \qquad \text{for all } t \in (0,T)$$

with some $c_1(\tau,\delta,T) > 0$ fulfilling

$$\sup_{T>1} c_1(\tau,\delta,T) < \infty \quad \text{for all } \tau \in (0,1) \text{ and } \delta \in (0,R) \text{ if (H) holds.}$$

In view of (5.3), parabolic Schauder theory thus readily leads to (5.32) and (5.33). $\square$

6 Construction of global solutions to (1.3). Proof of Theorem 1.1

6.1 The concept of renormalized solutions

Let us now specify a generalized solution framework within which the regularity information collected above is sufficient for appropriately passing to the limit $\varepsilon \searrow 0$ in (2.1). Since at the spatially global level including the origin, our available information seems insufficient to ensure any L^1 compactness property of the crucial contribution $\frac{u_\varepsilon}{v_\varepsilon}\nabla v_\varepsilon$ to the taxis term in (2.1), in reminiscence of an invention going back to [8] we will here resort to a concept which seems well-adapted to a priori estimates which involve solution-dependent weights in the flavor of Lemma 3.1 and Lemma 3.3,

Definition 6.1 *Let $u_0 \in L^1(\Omega)$ and $v_0 \in L^1(\Omega)$ be nonnegative. Then a pair of nonnegative functions*

$$(u,v) \in (L^1_{loc}(\overline{\Omega}\times[0,\infty)))^2 \tag{6.1}$$

which are such that

$$v > 0 \qquad \text{a.e. in } \Omega\times(0,\infty), \tag{6.2}$$

that

$$\chi_{\{u<M\}}\nabla u \text{ and } \chi_{\{v<M\}}\nabla v \quad \text{belong to } L^2_{loc}(\overline{\Omega}\times[0,\infty)) \text{ for all } M>0, \tag{6.3}$$

and that

$$\frac{1}{v}\nabla v \in L^2_{loc}(\overline{\Omega}\times[0,\infty)), \tag{6.4}$$

will be called a global renormalized solution *of (1.3) if the identities*

$$\begin{aligned} -\int_0^\infty\int_\Omega \phi(u)\varphi_t - \int_\Omega \phi(u_0)\varphi(\cdot,0) &= -\int_0^\infty\int_\Omega \phi'(u)\nabla u\cdot\nabla\varphi - \int_0^\infty\int_\Omega \phi''(u)|\nabla u|^2\varphi \\ &\quad +\chi\int_0^\infty\int_\Omega \frac{u}{v}\phi''(u)(\nabla u\cdot\nabla v)\varphi + \chi\int_0^\infty\int_\Omega \frac{u}{v}\phi'(u)\nabla v\cdot\nabla\varphi \\ &\quad -\int_0^\infty\int_\Omega uv\phi'(u)\varphi + \int_0^\infty\int_\Omega B_1\phi'(u)\varphi \end{aligned} \tag{6.5}$$

and

$$
\begin{aligned}
-\int_0^\infty\int_\Omega \phi(v)\varphi_t - \int_\Omega \phi(v_0)\varphi(\cdot,0) \;=\;& -\int_0^\infty\int_\Omega \phi'(v)\nabla v\cdot\nabla\varphi - \int_0^\infty\int_\Omega \phi''(v)|\nabla v|^2\varphi \\
&+\int_0^\infty\int_\Omega uv\phi'(v)\varphi - \int_0^\infty\int_\Omega v\phi'(v)\varphi + \int_0^\infty\int_\Omega B_2\phi'(v)\varphi \qquad (6.6)
\end{aligned}
$$

are valid for each $\phi\in C^\infty([0,\infty))$ with $\phi'\in C_0^\infty([0,\infty))$, and for any $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$.

Remark. As can be verified in a straightforward manner, the regularity and positivity requirements in (6.1)-(6.4) ensure that indeed each of the integrals appearing in (6.5) and (6.6) are well-defined. Moreover, it can easily be checked that any sufficiently smooth pair (u,v) of nonnegative functions which form a global renormalized solution of (1.3) in the above sense must actually be classical.

6.2 Global existence. Proof of Theorem 1.1

Now the following lemma collects the essential among our estimates gained above so as to assert actually slightly more than claimed in Theorem 1.1.

Lemma 6.2 *There exist $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ and radial functions u and v belonging to $C^0((\overline{\Omega}\setminus\{0\})\times[0,\infty))\cap C^{2,1}((\overline{\Omega}\setminus\{0\})\times(0,\infty))$ such that $\varepsilon_j\searrow 0$ as $j\to\infty$, that*

$$
\begin{aligned}
&u_\varepsilon\to u \quad \textit{in } C^0_{loc}((\overline{\Omega}\setminus\{0\})\times[0,\infty))\cap C^{2,1}_{loc}((\overline{\Omega}\setminus\{0\})\times(0,\infty)) \qquad \textit{and}\\
&v_\varepsilon\to v \quad \textit{in } C^0_{loc}((\overline{\Omega}\setminus\{0\})\times[0,\infty))\cap C^{2,1}_{loc}((\overline{\Omega}\setminus\{0\})\times(0,\infty))
\end{aligned}
\qquad (6.7)
$$

as $\varepsilon=\varepsilon_j\searrow 0$, and that (u,v) is a global renormalized solution of (1.3) in the sense of Definition 6.1. Moreover, this solution enjoys the additional properties specified in (1.6)-(1.9).

Proof. In view of Lemma 5.3, Lemma 5.1, Lemma 5.7 and Lemma 5.6, a standard extraction procedure on the basis of the Arzelà-Ascoli theorem enables us to find $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ such that $\varepsilon_j\searrow 0$ as $j\to\infty$, and such that (6.7) holds as $\varepsilon=\varepsilon_j\searrow 0$ with some radial functions u and v which clearly satisfy $u\ge 0$ and $v>0$ in $(\overline{\Omega}\setminus\{0\})\times[0,\infty)$ due to Lemma 2.1, Lemma 2.4 and Lemma 2.2. By means of Fatou's lemma, it can moreover easily be derived from (6.7) that Lemma 2.3, Lemma 3.2, Lemma 3.1 and Lemma 2.2 as well as Lemma 3.4 and Lemma 3.3 imply the inequalities in (1.6)-(1.9), which in particular entail that the regularity requirements in Definition 6.1 are met.

To verify the integral identities in (6.5) and (6.6), we let $\phi\in C^\infty([0,\infty))$ with $\phi'\in C_0^\infty([0,\infty))$ and $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$ be given, and choose $T>1$ large such that $\varphi\equiv 0$ in $\overline{\Omega}\times[T,\infty)$. Then for arbitrary $\delta\in(0,R)$ and $\tau\in(0,1)$, taking $(\zeta_\lambda)_{\lambda>0}$ as introduced in (4.9) we let $\xi_{\delta,\tau}(x,t):=\zeta_\delta(|x|)\cdot\zeta_\tau(t)$, $(x,t)\in\overline{\Omega}\times[0,\infty)$, and test the first equation in (2.1) by $\phi'(u_\varepsilon)\varphi\xi_{\delta,\tau}$ to see that since $\xi_{\delta,\tau}(\cdot,0)\equiv 0$,

$$
\begin{aligned}
-\int_0^T\int_\Omega \phi(u_\varepsilon)(\varphi\xi_{\delta,\tau})_t \;=\;& -\int_0^T\int_\Omega \phi'(u_\varepsilon)\nabla u_\varepsilon\cdot\nabla(\varphi\xi_{\delta,\tau}) - \int_0^T\int_\Omega \phi''(u_\varepsilon)|\nabla u_\varepsilon|^2\varphi\xi_{\delta,\tau}\\
&+\chi\int_0^T\int_\Omega \frac{u_\varepsilon}{v_\varepsilon}\phi''(u_\varepsilon)(\nabla u_\varepsilon\cdot\nabla v_\varepsilon)\cdot\varphi\xi_{\delta,\tau} + \chi\int_0^T\int_\Omega \frac{u_\varepsilon}{v_\varepsilon}\phi'(u_\varepsilon)\nabla v_\varepsilon\cdot\nabla(\varphi\xi_{\delta,\tau})\\
&-\int_0^T\int_\Omega u_\varepsilon v_\varepsilon\phi'(u_\varepsilon)\cdot\varphi\xi_{\delta,\tau} + \int_0^T\int_\Omega B_1\phi'(u_\varepsilon)\cdot\varphi\xi_{\delta,\tau}
\end{aligned}
$$

for all $\varepsilon \in (0,1)$. Now since $\xi_{\delta,\tau} \equiv 0$ in $B_{\frac{\delta}{2}} \times (0,\infty)$, we may use (6.7) to see that in the limit $\varepsilon = \varepsilon_j \searrow 0$, this turns into the identity

$$\begin{aligned}
-\int_0^T \int_\Omega \phi(u) \cdot (\varphi\xi_{\delta,\tau})_t &= -\int_0^T \int_\Omega \phi'(u) \nabla u \cdot \nabla(\varphi\xi_{\delta,\tau}) - \int_0^T \int_\Omega \phi''(u)|\nabla u|^2 \varphi\xi_{\delta,\tau} \\
&\quad + \chi \int_0^T \int_\Omega \frac{u}{v} \phi''(u)(\nabla u \cdot \nabla v) \cdot \varphi\xi_{\delta,\tau} + \chi \int_0^T \int_\Omega \frac{u}{v} \phi'(u) \nabla v \cdot \nabla(\varphi\xi_{\delta,\tau}) \\
&\quad - \int_0^T \int_\Omega uv\phi'(u) \cdot \varphi\xi_{\delta,\tau} + \int_0^T \int_\Omega B_1 \phi'(u) \cdot \varphi\xi_{\delta,\tau},
\end{aligned} \tag{6.8}$$

where expanding the left-hand side and using the dominated convergence theorem along with the observation that

$$0 \le \xi_{\delta,\tau}(x,t) \nearrow \xi_\delta(x) := \zeta_\delta(|x|) \quad \text{for all } x \in \Omega \text{ and } t > 0 \qquad \text{as } \tau \searrow 0, \tag{6.9}$$

we obtain

$$\begin{aligned}
-\int_0^T \int_\Omega \phi(u) \cdot (\varphi\xi_{\delta,\tau})_t &= -\int_0^T \int_\Omega \phi(u)\varphi_t \cdot \xi_{\delta,\tau} - \int_0^T \int_\Omega \phi(u)\varphi \cdot \partial_t \xi_{\delta,\tau} \\
&\to -\int_0^T \int_\Omega \phi(u)\varphi_t \cdot \xi_\delta - \int_\Omega \phi(u_0)\varphi(\cdot,0) \cdot \xi_\delta \qquad \text{as } \tau \searrow 0,
\end{aligned} \tag{6.10}$$

because $\phi(u)\varphi_t\xi_\delta$ belongs to $L^1(\Omega \times (0,T))$ by e.g. (6.7), and because by nonnegativity of $\partial_t\xi_{\delta,\tau}$, for each $\psi \in C^0(\overline{\Omega} \times [0,\infty))$ we have

$$\begin{aligned}
\left| \int_0^T \int_\Omega \psi \cdot \partial_t \xi_{\delta,\tau} - \int_\Omega \psi(\cdot,0) \cdot \xi_\delta \right| &= \left| \int_0^T \int_\Omega \Big\{ \psi(x,t) - \psi(x,0) \Big\} \cdot \partial_t \xi_{\delta,\tau}(x,t)\,dxdt \right| \\
&\le \sup_{(x,t) \in \Omega \times (0,\tau)} \Big| \psi(x,t) - \psi(x,0) \Big| \cdot \int_0^T \int_\Omega \partial_t \xi_{\delta,\tau}(x,t)\,dxdt \\
&= \sup_{(x,t) \in \Omega \times (0,\tau)} \Big| \psi(x,t) - \psi(x,0) \Big| \cdot \int_\Omega \xi_\delta \\
&\to 0 \qquad \text{as } \tau \searrow 0.
\end{aligned}$$

In view of (6.9) and the dominated convergence theorem, we may therefore let $\tau \searrow 0$ in (6.8) to infer that

$$\begin{aligned}
&-\int_0^T \int_\Omega \phi(u)\varphi_t \cdot \xi_\delta - \int_\Omega \phi(u_0)\varphi(\cdot,0) \cdot \xi_\delta \\
&\quad = -\int_0^T \int_\Omega \phi'(u) \nabla u \cdot \nabla(\varphi\xi_\delta) - \int_0^T \int_\Omega \phi''(u)|\nabla u|^2 \varphi \cdot \xi_\delta \\
&\qquad + \chi \int_0^T \int_\Omega \frac{u}{v} \phi''(u)(\nabla u \cdot \nabla v)\varphi \cdot \xi_\delta + \chi \int_0^T \int_\Omega \frac{u}{v} \phi'(u) \nabla v \cdot \nabla(\varphi\xi_\delta) \\
&\qquad - \int_0^T \int_\Omega uv\phi'(u)\varphi \cdot \xi_\delta + \int_0^T \int_\Omega B_1 \phi'(u)\varphi \cdot \xi_\delta
\end{aligned} \tag{6.11}$$

for all $\delta \in (0, R)$. Here since

$$0 \le \xi_\delta(x) \nearrow 1 \quad \text{for all } x \in \Omega \setminus \{0\} \qquad \text{as } \delta \searrow 0, \tag{6.12}$$

and since (1.6) along with the fact that $\phi' \equiv 0$ on (M, ∞) for some $M > 0$ implies that $\phi(u)\varphi_t, uv\phi'(u)\varphi$ and $B_1\phi'(u)\varphi$ belong to $L^1(\Omega \times (0,T))$, again by the dominated convergence theorem we see that

$$-\int_0^T \int_\Omega \phi(u)\varphi_t \cdot \xi_\delta \to -\int_0^T \int_\Omega \phi(u)\varphi_t \qquad \text{as } \delta \searrow 0 \tag{6.13}$$

and

$$-\int_0^T \int_\Omega uv\phi'(u)\varphi \cdot \xi_\delta \to -\int_0^T \int_\Omega uv\phi'(u)\varphi \qquad \text{as } \delta \searrow 0 \tag{6.14}$$

as well as

$$\int_0^T \int_\Omega B_1\phi'(u)\varphi \cdot \xi_\delta \to \int_0^T \int_\Omega B_1\phi'(u)\varphi \qquad \text{as } \delta \searrow 0, \tag{6.15}$$

and clearly we also have

$$-\int_\Omega \phi(u_0)\varphi(\cdot,0) \cdot \xi_\delta \to -\int_\Omega \phi(u_0)\varphi(\cdot,0) \qquad \text{as } \delta \searrow 0. \tag{6.16}$$

Likewise, the above support property of ϕ' together with (6.3) and (6.4) warrants that

$$\phi''(u)|\nabla u|^2\varphi,\ \phi'(u)\nabla u \cdot \nabla\varphi,\ \frac{u}{v}\phi''(u)(\nabla u \cdot \nabla v)\varphi \text{ and } \frac{u}{v}\phi'(u)\nabla v \cdot \nabla\varphi \quad \text{belong to } L^1(\Omega \times (0,T)).$$

Therefore, again thanks to (6.12) and the dominated convergence theorem we have

$$-\int_0^T \int_\Omega \phi''(u)|\nabla u|^2\varphi \cdot \xi_\delta \to -\int_0^T \int_\Omega \phi''(u)|\nabla u|^2\varphi \qquad \text{as } \delta \searrow 0 \tag{6.17}$$

and

$$\chi\int_0^T \int_\Omega \frac{u}{v}\phi''(u)(\nabla u \cdot \nabla v)\varphi \cdot \xi_\delta \to \chi\int_0^T \int_\Omega \frac{u}{v}\phi''(u)(\nabla u \cdot \nabla v)\varphi \qquad \text{as } \delta \searrow 0, \tag{6.18}$$

and also in the first and fourth summands on the right of (6.11), expanded according to

$$-\int_0^T \int_\Omega \phi'(u)\nabla u \cdot \nabla(\varphi\xi_\delta) = -\int_0^T \int_\Omega \phi'(u)(\nabla u \cdot \nabla\varphi)\xi_\delta - \int_0^T \int_\Omega \phi'(u)(\nabla u \cdot \nabla\xi_\delta)\varphi \tag{6.19}$$

and

$$\chi\int_0^T \int_\Omega \frac{u}{v}\phi'(u)\nabla v \cdot \nabla(\varphi\xi_\delta) = \chi\int_0^T \int_\Omega \frac{u}{v}\phi'(u)(\nabla v \cdot \nabla\varphi)\xi_\delta + \chi\int_0^T \int_\Omega \frac{u}{v}\phi'(u)(\nabla v \cdot \nabla\xi_\delta)\varphi, \tag{6.20}$$

we obtain convergence in the respective first integrals in the sense that

$$-\int_0^T \int_\Omega \phi'(u)(\nabla u \cdot \nabla\varphi)\xi_\delta \to -\int_0^T \int_\Omega \phi'(u)\nabla u \cdot \nabla\varphi \qquad \text{as } \delta \searrow 0, \tag{6.21}$$

and

$$\chi\int_0^T\int_\Omega \frac{u}{v}\phi'(u)(\nabla v\cdot\nabla\varphi)\xi_\delta \to \chi\int_0^T\int_\Omega \frac{u}{v}\phi'(u)\nabla v\cdot\nabla\varphi \qquad \text{as } \delta\searrow 0. \tag{6.22}$$

In order to show decay of the remaining rightmost terms in (6.19) and (6.20) in the limit $\delta\searrow 0$, we first invoke the Cauchy-Schwarz inequality to estimate

$$\left|-\int_0^T\int_\Omega \phi'(u)(\nabla u\cdot\nabla\xi_\delta)\varphi\right| \le \sqrt{T}\|\varphi\|_{L^\infty(\Omega\times(0,T))}\left\{\int_0^T\int_{B_\delta}\phi'^2(u)|\nabla u|^2\right\}^{\frac{1}{2}}\cdot\left\{\int_{B_\delta}|\nabla\xi_\delta|^2\right\}^{\frac{1}{2}}$$

for all $\delta\in(0,R)$, where we note that in the present two-dimensional setting we know that

$$\begin{aligned}
\int_{B_\delta}|\nabla\xi_\delta|^2 &= 2\pi\int_0^\delta r(\partial_r\zeta_\delta)^2(r)dr \\
&= \frac{2\pi}{\delta^2}\int_0^\delta r\zeta'^2\Big(\frac{r}{\delta}\Big)dr \\
&\le \frac{2\pi}{\delta^2}\|\zeta'\|^2_{L^\infty((0,\infty))}\int_0^\delta rdr \\
&= \pi\|\zeta'\|^2_{L^\infty((0,\infty))} \qquad \text{for all } \delta\in(0,R),
\end{aligned}$$

whence the inclusion $\phi'^2(u)|\nabla u|^2\in L^1(\Omega\times(0,T))$ guaranteed by (6.3) entails that in fact

$$-\int_0^T\int_\Omega \phi'(u)(\nabla u\cdot\nabla\xi_\delta)\varphi \to 0 \qquad \text{as } \delta\searrow 0. \tag{6.23}$$

Similarly,

$$\begin{aligned}
&\left|\chi\int_0^T\int_\Omega \frac{u}{v}\phi'(u)(\nabla v\cdot\nabla\xi_\delta)\varphi\right| \\
&\quad\le \chi\sqrt{T}\|\varphi\|_{L^\infty(\Omega\times(0,T))}\cdot\sup_{z\in(0,M)}|z\phi'(z)|\cdot\left\{\int_0^T\int_{B_\delta}\frac{|\nabla v|^2}{v^2}\right\}^{\frac{1}{2}}\cdot\left\{\int_{B_\delta}|\nabla\xi_\delta|^2\right\}^{\frac{1}{2}} \\
&\quad\to 0 \qquad \text{as } \delta\searrow 0
\end{aligned} \tag{6.24}$$

due to (1.9), whence from (6.11) we infer on letting $\delta\searrow 0$ and taking into account (6.13)-(6.24) that indeed (6.5) holds for any such ϕ and φ.

Finally the derivation of (6.6) can be achieved in quite a similar manner: Given ϕ and φ as above and defining $T, M, \xi_{\delta,\tau}$ and ξ_δ as before, on multiplying the second equation in (2.1) by $\phi'(v_\varepsilon)\varphi\xi_{\delta,\tau}$ we obtain after taking $\varepsilon=\varepsilon_j\searrow 0$ and $\tau\searrow 0$ that

$$\begin{aligned}
-\int_0^T\int_\Omega\phi(v)\varphi_t\cdot\xi_\delta - \int_\Omega\phi(v_0)\varphi(\cdot,0)\cdot\xi_\delta &= -\int_0^T\int_\Omega\phi''(v)|\nabla v|^2\varphi\xi_\delta \\
&\quad -\int_0^T\int_\Omega\phi'(v)(\nabla v\cdot\nabla\varphi)\xi_\delta - \int_0^T\int_\Omega\phi'(v)(\nabla v\cdot\nabla\xi_\delta)\varphi \\
&\quad +\int_0^T\int_\Omega uv\phi'(v)\varphi\cdot\xi_\delta - \int_0^T\int_\Omega v\phi'(v)\varphi\cdot\xi_\delta \\
&\quad +\int_0^T\int_\Omega B_2\phi'(v)\varphi\cdot\xi_\delta \qquad \text{for all } \delta\in(0,R).
\end{aligned} \tag{6.25}$$

Here from (1.6) and (6.3) we know that

$$\phi(v)\varphi_t,\ \phi'(v)\nabla v\cdot\nabla\varphi,\ \phi''(v)|\nabla v|^2\varphi,\ uv\phi'(v)\varphi,\ v\phi'(v)\varphi \text{ and } B_2\phi'(v)\varphi \quad \text{belong to } L^1((\Omega\times(0,T)),$$

so that the dominated convergence theorem along with (6.12) allows for taking $\delta\searrow 0$ in both integrals on the left, as well as in the first, third, fourth, fifth, sixth and seventh summand on the right, with the respectively expected limits. In the second integral on the right-hand side of (6.25) we once more invoke the Cauchy-Schwarz inequality to see that

$$\begin{aligned} \left| -\int_0^T\int_\Omega \phi'(v)(\nabla v\cdot\nabla\xi_\delta)\varphi\right| &\le \sqrt{T}\|\varphi\|_{L^\infty(\Omega\times(0,T))}\left\{\int_0^T\int_{B_\delta}\phi'^2(v)|\nabla v|^2\right\}^{\frac{1}{2}}\cdot\left\{\int_{B_\delta}|\nabla\xi_\delta|^2\right\}^{\frac{1}{2}} \\ &\to 0 \qquad \text{as } \delta\searrow 0, \end{aligned}$$

because also $\phi'^2(v)|\nabla v|^2\in L^1(\Omega\times(0,T))$ due to (6.3). In summary, (6.25) therefore implies the validity of (6.6) and thereby completes the proof. □

In fact, our main result on global solvability in (1.3) now needs no further comment.

PROOF of Theorem 1.1. All statements have been asserted by Lemma 6.2. □

7 Large time behavior. Proof of Theorem 1.2

Let us finally discuss the large time behavior of the solutions obtained above under the additional assumption that the hypotheses in (H) are satisfied. Indeed, a first implication thereof is that the dampening effect of the absorptive contribution $-u_\varepsilon v_\varepsilon$ to the first equation in (2.1) is substantial enough so as to warrant decay of $\int_\Omega u_\varepsilon$ and of $\int_\Omega |v_\varepsilon - v_\infty$ in the large time limit. Our verification of this is based on the following.

Lemma 7.1 *Assume (H). Then there exists $C>0$ such that for all $\varepsilon\in(0,1)$,*

$$\frac{d}{dt}\int_\Omega u_\varepsilon + C\int_\Omega u_\varepsilon \le -\frac{1}{2}\int_\Omega u_\varepsilon v_\varepsilon + \int_\Omega B_1 \qquad \textit{for all } t>0. \tag{7.1}$$

PROOF. According to (H), Lemma 2.2 says that with some $c_1>0$ we have $v_\varepsilon\ge c_1$ in $\Omega\times(0,\infty)$ whenever $\varepsilon\in(0,1)$. Therefore, once more integrating the first equation in (2.1) over Ω we can estimate

$$\begin{aligned} \frac{d}{dt}\int_\Omega u_\varepsilon &= -\frac{1}{2}\int_\Omega u_\varepsilon v_\varepsilon - \frac{1}{2}\int_\Omega u_\varepsilon v_\varepsilon + \int_\Omega B_1 \\ &\le -\frac{c_1}{2}\int_\Omega u_\varepsilon - \frac{1}{2}\int_\Omega u_\varepsilon v_\varepsilon + \int_\Omega B_1 \qquad \text{for all } t>0, \end{aligned}$$

which yields (7.1). □

Since the expression $v_\varepsilon - v_\infty$ is not necessarily nonnegative, our reasoning for the corresponding integral involving the second solution component requires a slightly more careful argument that is prepared by the following elementary observation on an approximation of the function $\mathbb{R}\ni\xi\mapsto|\xi|$.

Lemma 7.2 *For $\eta > 0$, let*

$$\psi_\eta(z) := (z^2+\eta)^{\frac{1}{2}} - \eta^{\frac{1}{2}}, \qquad z \in \mathbb{R}. \tag{7.2}$$

Then

$$\psi_\eta(z) \ge 0, \quad |\psi'_\eta(z)| \le 1, \quad z\psi'_\eta(z) \ge \psi_\eta(z) \quad \textit{and} \quad z\psi''_\eta(z) \ge 0 \qquad \textit{for all } z \in \mathbb{R}. \tag{7.3}$$

Proof. Nonnegativity of ψ_η is obvious from (7.2), and computing

$$\psi'_\eta(z) = z(z^2+\eta)^{-\frac{1}{2}} \quad \text{and} \quad \psi''_\eta(z) = \eta(z^2+\eta)^{-\frac{3}{2}} \qquad \text{for } z \in \mathbb{R},$$

we also immediately obtain that $|\psi'_\eta| \le 1$ and $\psi''_\eta \ge 0$ on $\mathbb{R}$. As thus also

$$z\psi'_\eta(z) = (z^2+\eta)^{\frac{1}{2}} - \eta(z^2+\eta)^{-\frac{1}{2}} \ge (z^2+\eta)^{\frac{1}{2}} - \eta^{\frac{1}{2}} = \psi_\eta(z) \qquad \text{for all } z \in \mathbb{R},$$

it follows that indeed all inequalities in (7.3) hold. □

As all these functions ψ_η are smooth, we may apply a straightforward testing procedure to derive the following counterpart of Lemma 7.1 from the second equation in (2.1).

Lemma 7.3 *Let $\eta > 0$. Then with ψ_η as in (7.2) and v_∞ denoting the solution of (1.10),*

$$\frac{d}{dt}\int_\Omega \psi_\eta(v_\varepsilon - v_\infty) + \int_\Omega \psi_\eta(v_\varepsilon - v_\infty) \le \int_\Omega u_\varepsilon v_\varepsilon + \int_\Omega |B_2 - B_{2,\infty}| \qquad \textit{for all } t > 0 \tag{7.4}$$

whenever $\varepsilon \in (0,1)$.

Proof. By using the second equation in (2.1) together with (1.10), we see that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \psi_\eta(v_\varepsilon - v_\infty) &= \int_\Omega \psi'_\eta(v_\varepsilon - v_\infty) v_{\varepsilon t} \\
&= \int_\Omega \psi'_\eta(v_\varepsilon - v_\infty)\cdot\Big\{\Delta(v_\varepsilon - v_\infty) - (v_\varepsilon - v_\infty) + \frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} + (B_2 - B_{2,\infty})\Big\} \\
&= -\int_\Omega \psi''_\eta(v_\varepsilon - v_\infty)\cdot|\nabla(v_\varepsilon - v_\infty)|^2 - \int_\Omega \psi'_\eta(v_\varepsilon - v_\infty)\cdot(v_\varepsilon - v_\infty) \\
&\quad + \int_\Omega \psi'_\eta(v_\varepsilon - v_\infty)\cdot\frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} + \int_\Omega \psi'_\eta(v_\varepsilon - v_\infty)\cdot(B_2 - B_{2,\infty})
\end{aligned} \tag{7.5}$$

for all $t > 0$. Since $\psi''_\eta(z) \ge 0$ and $z\psi'_\eta(z) \ge \psi_\eta(z)$ for all $z \in \mathbb{R}$ by Lemma 7.2, herein we have

$$-\int_\Omega \psi''_\eta(v_\varepsilon - v_\infty)\cdot|\nabla(v_\varepsilon - v_\infty)|^2 \le 0 \qquad \text{for all } t > 0$$

and

$$-\int_\Omega \psi'_\eta(v_\varepsilon - v_\infty)\cdot(v_\varepsilon - v_\infty) \le -\int_\Omega \psi_\eta(v_\varepsilon - v_\infty) \qquad \text{for all } t > 0.$$

As Lemma 7.2 moreover warrants that $|\psi'_\eta| \le 1$, we also obtain that

$$\int_\Omega \psi'_\eta(v_\varepsilon - v_\infty)\cdot\frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} \le \int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } t > 0$$

and

$$\int_\Omega \psi_\eta'(v_\varepsilon - v_\infty)\cdot(B_2 - B_{2,\infty}) \le \int_\Omega |B_2 - B_{2,\infty}| \qquad \text{for all } t>0,$$

so that (7.5) entails (7.4). $\square$

Now combing the latter with Lemma 7.1 indeed yields an estimate on the large time behavior of $\int_\Omega u_\varepsilon + \frac{1}{2}\psi_\eta(v_\varepsilon - v_\infty)$ which on its right-hand side is independent of both $\eta>0$ and $\varepsilon\in(0,1)$.

Lemma 7.4 *Suppose that (H) holds. Then there exist $\alpha>0$ and $C>0$ such that for each $\eta>0$ and any $\varepsilon\in(0,1)$ we have*

$$\int_\Omega u_\varepsilon(\cdot,t)+\frac{1}{2}\int_\Omega \psi_\eta(v_\varepsilon(\cdot,t)-v_\infty) \le Ce^{-\alpha t}+\int_0^t e^{-\alpha(t-s)}\cdot\Big\{\int_\Omega B_1(\cdot,s)+\frac{1}{2}\int_\Omega |B_2(\cdot,s)-B_{2,\infty}|\Big\}ds \qquad \text{for all } t>0, \tag{7.6}$$

where ψ_η has been taken from (7.2), and where v_∞ solves (1.10).

Proof. On combining Lemma 7.1 with Lemma 7.3, we see that there exists $c_1>0$ such that $y(t) := \int_\Omega u_\varepsilon(\cdot,t) + \frac{1}{2}\int_\Omega \psi_\eta(v_\varepsilon(\cdot,t)-v_\infty)$, $t\ge 0$, satisfies

$$y'(t) + c_1\int_\Omega u_\varepsilon + \frac{1}{2}\int_\Omega \psi_\eta(v_\varepsilon - v_\infty) \le h(t) := \int_\Omega B_1 + \frac{1}{2}\int_\Omega |B_2 - B_{2,\infty}| \qquad \text{for all } t>0.$$

Writing $\alpha := \min\{c_1, 1\}$, by nonnegativity of ψ_η we thus infer that

$$y'(t) + \alpha y(t) \le h(t) \qquad \text{for all } t>0$$

and hence

$$y(t) \le e^{-\alpha t}\cdot\Big\{\int_\Omega u_0 + \frac{1}{2}\int_\Omega \psi_\eta(v_0 - v_\infty)\Big\} + \int_0^t e^{-\alpha(t-s)}h(s)ds \qquad \text{for all } t>0.$$

As

$$\int_\Omega u_0 + \frac{1}{2}\int_\Omega \psi_\eta(v_0 - v_\infty) \le C := \int_\Omega u_0 + \frac{1}{2}\int_\Omega |v_0 - v_\infty|$$

according to (7.3), this implies (7.6). $\square$

Now to ensure that (7.6) indeed entails decay of the quantities appearing on its right-hand side, let us recall an elementary decay feature of convolutive integrals involving functions with a certain averaged decay property.

Lemma 7.5 *Let $h\in L^1_{loc}([0,\infty))$ be nonnegative and such that*

$$\int_t^{t+1} h(s)ds \to 0 \qquad \text{as } t\to\infty.$$

Then for any $\alpha>0$,

$$\int_0^t e^{-\alpha(t-s)}h(s)ds \to 0 \qquad \text{as } t\to\infty.$$

PROOF. A verification of this elementary statement can be found e.g. in [10, Lemma 4.6]. $\square$

We can thereby derive the desired statement on stabilization with respect to the spatial L^1 norm from Lemma 7.4.

Lemma 7.6 *Assume that (H) holds, and let (u,v) denote the global renormalized solution of (1.3) from Theorem 1.1. Then*

$$u(\cdot,t) \to 0 \quad \text{in } L^1(\Omega) \qquad \text{as } t \to \infty \tag{7.7}$$

and

$$v(\cdot,t) \to v_\infty \quad \text{in } L^1(\Omega) \qquad \text{as } t \to \infty, \tag{7.8}$$

where v_∞ is the solution of (1.10).

PROOF. Using that with $(\psi_\eta)_{\eta>0}$ from (7.2) we have $\psi_\eta(z) \to |z|$ as $\eta \searrow 0$ for all $z \in \mathbb{R}$, from Lemma 7.4 we infer on applying Fatou's lemma that with some $c_1 > 0$ and $\alpha > 0$, for all $\varepsilon \in (0,1)$ we have

$$\int_\Omega u_\varepsilon(\cdot,t) + \frac{1}{2}\int_\Omega |v_\varepsilon(\cdot,t) - v_\infty| \le c_1 e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} h(s)ds \qquad \text{for all } t > 0, \tag{7.9}$$

where again $h(t) := \int_\Omega B_1(\cdot,t) + \frac{1}{2}\int_\Omega |B_2(\cdot,t) - B_{2,\infty}|$ for $t \ge 0$. Since for each $t > 0$ we know from Lemma 6.2 that $u_\varepsilon(x,t) \to u(x,t)$ and $v_\varepsilon(x,t) \to v(x,t)$ for all $x \in \overline{\Omega} \setminus \{0\}$ as $\varepsilon = \varepsilon_j \searrow 0$, we may once more invoke Fatou's lemma to conclude from (7.9) that also

$$\int_\Omega u(\cdot,t) + \frac{1}{2}\int_\Omega |v(\cdot,t) - v_\infty| \le c_1 e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} h(s)ds \qquad \text{for all } t > 0.$$

As our hypothesis (H) guarantees that

$$\int_t^{t+1} h(s)ds \to 0 \qquad \text{as } t \to \infty,$$

in view of Lemma 7.5 this entails both (7.7) and (7.8). $\square$

In order to finally achieve the claimed results on convergence with respect to higher Lebesgue norms, we combine Lemma 7.6 with the boundedness information contained in Lemma 3.4 by means of a simple interpolation.

Lemma 7.7 *Assume (H), and let $p > 1$ and $q > 0$ be such that $q < \frac{p}{p-1} \cdot \min\{1, \frac{1}{\chi^2}\}$. Then the global renormalized solution of (1.3) from Theorem 1.1 satisfies*

$$\int_t^{t+1} \|u(\cdot,s)\|_{L^p(\Omega)}^q ds \to 0 \qquad \text{as } t \to \infty. \tag{7.10}$$

PROOF. Abbreviating $\lambda := \min\{1, \frac{1}{\chi^2}\}$, from our assumption $q < \frac{p}{p-1}\lambda$ we know that

$$\frac{2(p-1)q}{2p-1} < \frac{2p}{2p-1} \cdot \lambda.$$

Therefore, an application of Lemma 3.4 yields $c_1 > 0$ such that whenever $\varepsilon \in (0,1)$,

$$\int_t^{t+1} \|u_\varepsilon(\cdot,s)\|_{L^{2p}(\Omega)}^{\frac{2(p-1)q}{2p-1}} ds \le c_1 \qquad \text{for all } t > 0, \tag{7.11}$$

so that by means of the Hölder inequality, for any such ε and arbitrary $\delta \in (0,R)$ we can estimate

$$\begin{aligned} \int_t^{t+1} \|u_\varepsilon(\cdot,s)\|_{L^p(\Omega\setminus B_\delta)}^q ds &\le \int_t^{t+1} \|u_\varepsilon(\cdot,s)\|_{L^{2p}(\Omega\setminus B_\delta)}^{\frac{2(p-1)q}{2p-1}} \|u_\varepsilon(\cdot,s)\|_{L^1(\Omega\setminus B_\delta)}^{\frac{q}{2p-1}} ds \\ &\le c_1 \cdot \sup_{s\in(t,t+1)} \|u_\varepsilon(\cdot,s)\|_{L^1(\Omega\setminus B_\delta)}^{\frac{q}{2p-1}} \qquad \text{for all } t > 0. \end{aligned}$$

Here since Lemma 6.2 asserts that $u_\varepsilon \to u$ in $L^\infty_{loc}((\overline{\Omega}\setminus B_\delta)\times[0,\infty))$ as $\varepsilon = \varepsilon_j \searrow 0$, we may let $\varepsilon = \varepsilon_j \searrow 0$ on both sides to infer that for all $\delta \in (0,R)$,

$$\begin{aligned} \int_t^{t+1} \|u(\cdot,s)\|_{L^p(\Omega\setminus B_\delta)}^q ds &\le c_1 \cdot \sup_{s\in(t,t+1)} \|u(\cdot,s)\|_{L^1(\Omega\setminus B_\delta)}^{\frac{q}{2p-1}} \\ &\le c_1 \cdot \sup_{s>t} \|u(\cdot,s)\|_{L^1(\Omega)}^{\frac{q}{2p-1}} \qquad \text{for all } t > 0. \end{aligned}$$

We may now invoke Fatou's lemma to see that this implies the inequality

$$\int_t^{t+1} \|u(\cdot,s)\|_{L^p(\Omega)}^q ds \le c_1 \cdot \sup_{s>t} \|u(\cdot,s)\|_{L^1(\Omega)}^{\frac{q}{2p-1}} \qquad \text{for all } t > 0,$$

whereupon (7.10) becomes a consequence of Lemma 7.6. □

Similarly, Lemma 7.6 in conjunction with the bound from Lemma 3.2 yields an analogous result for v_ε.

Lemma 7.8 *Suppose that (H) is valid. Then for all $p > 1$ and each $q \in (0, \frac{p}{p-1})$, the global renormalized solution of (1.3) found in Theorem 1.1 has the property that*

$$\int_t^{t+1} \|v(\cdot,s) - v_\infty\|_{L^p(\Omega)}^q ds \to 0 \qquad \text{as } t \to \infty, \tag{7.12}$$

where v_∞ denotes the solution of (1.10).

Proof. Since $q < \frac{p}{p-1}$ implies that $\frac{2(p-1)q}{2p-1} < \frac{2p}{2p-1}$, from Lemma 3.2 we obtain $c_1 > 0$ such that

$$\int_t^{t+1} \|v_\varepsilon(\cdot,s)\|_{L^{2p}(\Omega)}^{\frac{2(p-1)q}{2p-1}} ds \le c_1 \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1).$$

As thus

$$\int_t^{t+1} \|v_\varepsilon(\cdot,s)\|_{L^p(\Omega\setminus B_\delta)}^q ds \le c_1 \cdot \sup_{s\in(t,t+1)} \|v_\varepsilon(\cdot,s)\|_{L^1(\Omega\setminus B_\delta)}^{\frac{q}{2p-1}} \qquad \text{for all } t > 0, \varepsilon \in (0,1) \text{ and } \delta \in (0,R),$$

on taking $\varepsilon = \varepsilon_j \searrow 0$ and then $\delta \searrow 0$ we can derive (7.12) in a way similar to that in Lemma 7.7. □

We hence immediately arrive at our main results concerning the large time behavior in (1.3).

PROOF of Theorem 1.2. Since $(u(\cdot,t))_{t>1}$ and $(v(\cdot,t))_{t>1}$ are relatively compact in $C^2_{loc}(\overline{\Omega}\setminus\{0\})$ according to Lemma 5.7, Lemma 6.2 and the Arzelà-Ascoli theorem, both (1.11) and (1.12) result from Lemma 7.6. The statements (1.13) and (1.14) have precisely been asserted by Lemma 7.7 and Lemma 7.8, whereas (1.15) and (1.16) are evident consequences thereof. $\square$

Acknowledgement. The author warmly thanks the anonymous reviewers for their helpful remarks which led to significant improvement of the exposition. The author furthermore acknowledges support of the *Deutsche Forschungsgemeinschaft* in the context of the project *Analysis of chemotactic cross-diffusion in complex frameworks.*

References

[1] H. AMANN: *Dynamic theory of quasilinear parabolic systems III. Global existence.* Math. Z. **202**, 219-250 (1989)

[2] BELLOMO, N., BERESTYCKI, H., BREZZI, F., NADAL, J.-P.: *Mathematics and complexity in life and human sciences.* Math. Mod. Meth. Appl. Sci. **20**, 1391-1395 (2010)

[3] BELLOMO, N., BELLOUQUID, A., TAO, Y., WINKLER, M.: *Toward a Mathematical Theory of Keller-Segel Models of Pattern Formation in Biological Tissues.* Math. Mod. Meth. Appl. Sci. **25**, 1663-1763 (2015)

[4] BERESTYCKI, H., WEI, J., WINTER, M.: *Existence of symmetric and asymmetric spikes for a crime hotspot model.* SIAM J. Math. Anal. **46**, 691-719 (2014)

[5] BILER, P.: *Global solutions to some parabolic-elliptic systems of chemotaxis.* Adv. Math. Sci. Appl. **9**, 347-359 (1999)

[6] CANTRELL, R.S., COSNER, C., MANÁSEVICH, R.: *Global bifurcation of solutions for crime modeling equations.* SIAM J. Appl. Math. **44**, 1340-1358 (2012)

[7] COHEN, L.E., FELSON, M.: *Social change and crime rate trends: a routine activity approach.* Amer. Sociological Review **44**, 588-608 (1979)

[8] DI PERNA, R.-J., LIONS, P.-L.: *On the Cauchy problem for Boltzmann equations: Global existence and weak stability.* Ann. Math. **130**, 321-366 (1989)

[9] D'ORSOGNA, M.R., PERC, M.: *Statistical physics of crime: A review.* Physics of Life Reviews **12**, 1-21 (2015)

[10] ESPEJO, E., WINKLER, M.: *Global classical solvability and stabilization in a two-dimensional chemotaxis-Navier-Stokes system modeling coral fertilization.* Nonlinearity **31**, 1227-1259 (2018)

[11] FELSON, M.: *Routine activities and crime prevention in the developing metropolis.* Criminology **25**, 911-932 (1987)

[12] Freitag, M.: *Global solutions to a higher-dimensional system related to crime modeling.* Preprint

[13] Fujie, K.: *Study of reaction-diffusion systems modeling chemotaxis.* PhD thesis, Tokyo University of Science, 2016

[14] Fujie, K., Senba, T.: *Global existence and boundedness in a parabolic-elliptic Keller-Segel system with general sensitivity.* Discrete Contin. Dyn. Syst. Ser. B **21**, 81-102 (2016)

[15] Giga, Y., Sohr, H.: *Abstract L^p Estimates for the Cauchy Problem with Applications to the Navier-Stokes Equations in Exterior Domains.* J. Functional Anal. **102**, 72-94 (1991)

[16] Gu, Y., Wang, Q., Guangzeng, Y.: *Stationary patterns and their selection mechanism of Urban crime models with heterogeneous near-repeat victimization effect.* Arxiv preprint:1409.0835v2, 2016

[17] Herrero, M. A., Velázquez, J. J. L.: *A blow-up mechanism for a chemotaxis model.* Ann. Scuola Normale Superiore Pisa Cl. Sci. **24**, 633-683 (1997)

[18] Hillen, T., Painter, K.J., Winkler, M.: *Convergence of a cancer invasion model to a logistic chemotaxis model.* Math. Models Methods Appl. Sci. **23**, 165-198 (2013)

[19] Horstmann, D., Winkler, M.: *Boundedness vs. blow-up in a chemotaxis system.* J. Differential Eq. **215** (1), 52-107 (2005)

[20] Johnson, S.D., Bowers, K., Hirschfeld, A.: *New insights into the spatial and temporal distribution of repeat victimisation.* British Journal of Criminology **37**, 224-241 (1997)

[21] Kelling, G.L., Wilson, J.Q.: *Broken windows.* 1982

[22] Kolokolnikov, T., Ward, M.J., Wei, J.: *The stability of hotspot patterns for reaction-diffusion models of urban crime.* Discrete Cont. Dyn. Syst. Ser. B **19**, 1373-1401 (2014)

[23] Ladyzenskaja, O. A., Solonnikov, V. A., Ural'ceva, N. N.: *Linear and Quasi-Linear Equations of Parabolic Type.* Amer. Math. Soc. Transl., Vol. **23**, Providence, RI, 1968

[24] Lankeit, J.: *A new approach toward boundedness in a two-dimensional parabolic chemotaxis system with singular sensitivity.* Math. Meth. Appl. Sci. **39**, 394-404 (2016)

[25] Lankeit, J., Winkler, M.: *A generalized solution concept for the Keller-Segel system with logarithmic sensitivity: global solvability for large nonradial data.* Nonlin. Differential Eq. Appl. **24**, Art. 49, 33 pp. (2017)

[26] Manásevich, R., Phan, Q. H., Souplet, Ph.: *Global existence of solutions for a chemotaxis-type system arising in crime modelling.* European J. Appl. Math. **24**, 273-296 (2013)

[27] Nagai, T., Senba, T.: *Global existence and blow-up of radial solutions to a parabolic-elliptic system of chemotaxis.* Adv. Math. Sci. Appl. **8**, 145-156 (1998)

[28] Porzio, M.M., Vespri, V.: *Holder Estimates for Local Solutions of Some Doubly Nonlinear Degenerate Parabolic Equations,* J. Differential Eq. **103**, 146-178 (1993)

[29] Rodriguez, N.: *On the global well-posedness theory for a class of PDE models for criminal activity.* Physica D: Nonlinear Phenomena **260**, 191-200 (2013)

[30] Rodriguez, N., Bertozzi, A.: *Local existence and uniqueness of solutions to a PDE model for criminal behavior.* Math. Mod. Meth. Appl. Sci. **20** (suppl.), 1425-1457 (2010)

[31] Rodriguez, N., Winkler, M.: *On the global existence and qualitative behavior of one-dimensional solutions to a model for urban crime.* Preprint

[32] Short, M.B., D'Orsogna, M.R., Brantingham, P.J., Tita, G.E.: *Measuring and modeling repeat and near-repeat burglary effects.* Journal of Quantitative Criminology **25**, 325-339 (2009)

[33] Short, M.B., D'Orsogna, M.R., Pasour, V.B., Tita, G.E., Brantingham, P.J., Bertozzi, A.L., Chayes, L.B.: *A statistical model of criminal behavior.* Math. Mod. Meth. Appl. Sci. **18** (suppl.), 1249-1267 (2008)

[34] Stinner, C., Winkler, M.: *Global weak solutions in a chemotaxis system with large singular sensitivity.* Nonlin. Anal. Real World Appl. **12**, 3727-3740 (2011)

[35] Tao, Y., Winkler, M.: *Global smooth solutions in a two-dimensional cross-diffusion system modeling urban crime.* Preprint

[36] Tse, W.H., Ward, M.J.: *Hotspot formation and dynamics for a continuum model of urban crime.* European J. Appl. Math. **27**, 583-624 (2016)

[37] Winkler, M.: *Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model.* J. Differential Eq. **248**, 2889-2905 (2010)

[38] Winkler, M.: *Global solutions in a fully parabolic chemotaxis system with singular sensitivity.* Math. Meth. Appl. Sci. **34**, 176-190 (2011)

[39] Winkler, M: *Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system.* J. Math. Pures Appl. **100**, 748-767 (2013), `arXiv:1112.4156v1`