How unstable is spatial homogeneity in Keller-Segel systems? A new critical mass phenomenon in two- and higher-dimensional parabolic-elliptic cases

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Abstract

The parabolic-elliptic Keller-Segel system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ 0 = \Delta v - \mu + u, \qquad \mu := \frac{1}{|\Omega|} \int_\Omega u, \end{cases} \qquad (\star)$$

is considered under homogeneous Neumann boundary conditions in the ball $\Omega = B_R(0) \subset \mathbb{R}^n$.

The main objective is to reveal that in the context of radially symmetric solutions, this problem exhibits an apparently novel type of critical mass phenomenon: It is shown, namely, that for any choice of $n \ge 2$ and $R > 0$ there exists a positive number $m_c = m_c(n, R)$ with the following properties:

- Whenever $m > m_c$, for any nonconstant nonnegative radial initial data u_0 with $\int_\Omega u_0 = m$ which are, in an appropriately defined sense, more concentrated than the associated spatially homogeneous equilibrium determined by $u \equiv \frac{m}{|\Omega|}$, the corresponding initial-value problem for $(\star)$ admits a solution blowing up in finite time; in particular, this implies that any nonconstant and radially nonincreasing initial data u_0 with $\int_\Omega u_0 > m_c$ enforce blow-up in $(\star)$.
- If $m < m_c$, however, then there exist infinitely many nonnegative radial functions u_0 which satisfy $\int_\Omega u_0 = m$ and which are more concentrated than $u \equiv \frac{m}{|\Omega|}$, but which yet allow for global bounded solutions to $(\star)$ emanating from u_0.

In consequence, precisely at mass levels above m_c the constant steady states of $(\star)$ possess the extreme instability property of repelling arbitrary concentration-increasing perturbations in such a drastic sense that corresponding trajectories collapse in finite time.

Key words: chemotaxis; critical mass; finite-time blow-up
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1 Introduction

As already predicted in the 1970s ([23]), a striking feature of Keller-Segel-type chemotaxis systems consists in their potential to describe spontaneous emergence of cell aggregates in the mathematically extreme sense of singularity formation. Accordingly, the corresponding analytical activities are to a considerable extent concerned with the identification of circumstances under which a respective particular problem of this form is either globally solvable by bounded functions, or admits unbounded solutions. In fact, with regard to these alternatives the literature has revealed various types of criticality, in many cases referring to constitutive system ingredients such as coefficients measuring the strength of chemotactic response (see e.g. [29], [8], [9] and [30], but also [18] and [1] for some recent developments).

Critical mass phenomena in Keller-Segel systems. For models with fixed system parameters, more subtle findings on criticality typically concentrate on a respective role of the total mass of cells as a quantity of immediate biological relevance ([25]). In this direction, the seemingly best understood phenomenon refers to the classical Keller-Segel system ([15]), in its simplified parabolic-elliptic version ([14]) reducing to the initial-boundary value problem,

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \mu + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

posed in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with given nontrivial nonnegative initial data u_0 and $\mu := \frac{1}{|\Omega|} \int_\Omega u_0$. The planar and spatially radial version of this problem obtained when $\Omega = B_R(0) \subset \mathbb{R}^2$ with $R > 0$, namely, is known to exhibit a *critical mass phenomenon* in the sense that whenever

$$u_0 \in C^0_{rad}(\overline{\Omega}) := \left\{ \varphi \in C^0(\overline{\Omega}) \,\middle|\, \varphi \text{ is radially symmetric} \right\} \quad \text{is nonnegative with } u_0 \not\equiv 0 \tag{1.2}$$

satisfies $\int_\Omega u_0 < 8\pi$, then (1.1) possesses a globally defined classical solution (u,v) for which u is bounded in $\Omega \times (0,\infty)$, whereas for each $m > 8\pi$ one can find $u_0 \in C^0_{rad}(\overline{\Omega})$ such that $\int_\Omega u_0 = m$ but that (1.1) admits a solution blowing up in finite time with respect to the spatial L^∞ norm of u ([14]; cf. also [20], [3], [27] and [28] for further information). That this crucial role of the mass functional is limited to the case $n = 2$ is underlined by further observations, explicitly documented for related systems but readily extensible to (1.1), according to which no unbounded solutions exist at all when $n = 1$ ([24]), while whenever $n \geq 3$, for arbitrary $m > 0$ one can fix some $u_0 \in C^0_{rad}$ which enforces an explosion within finite time ([20]; cf. also Lemma 3.3 below). In summary, this means that for $R > 0$, writing

$$\widehat{m} \equiv \widehat{m}(n,R) := \inf \left\{ m > 0 \,\middle|\, \begin{array}{l} \text{There exists } 0 \leq u_0 \in C^0_{rad}(\overline{\Omega}) \text{ such that} \\ \text{(1.1) possesses a solution blowing up in finite time} \end{array} \right\} \tag{1.3}$$

is meaningful if and only if $n \geq 2$, and that $\widehat{m}(n,R)$ is positive, and hence defines a genuine critical mass level, precisely in the case $n = 2$ in which actually $\widehat{m}(2,R) = 8\pi$ for all $R > 0$. Results of a similar flavor, though partially less exhaustive due to more involved system structures and accordingly increased technical challenges, confirm extension of this exclusively two-dimensional critical mass phenomenon to various close relatives of (1.1), either yet with or partly even without assumptions on radial symmetry, e.g. addressing a slightly more complex parabolic-elliptic system ([20], [21]), an analogue posed in the entire plane $\Omega = \mathbb{R}^2$ ([7], [16], [4]), or even the fully parabolic version of the classical Keller-Segel system ([22], [12], [13], [31]). Accordingly, critical mass phenomena in higher-dimensional settings have been detected only for modifications of the classical Keller-Segel system, obtained e.g. on accounting for nonlinear cell diffusion with a particular porous medium exponent ([18]).

The challenge of detecting global large-mass solutions. Going beyond the above, in each case in which blow-up solutions are known to occur a natural next problem appears to consist in characterizing the set of all explosion-enforcing initial data as comprehensively as possible. In this respect, Neumann-type boundary value problems seem to differ substantially from Cauchy problems posed in $\Omega = \mathbb{R}^n$: Unlike in the latter situation, in which homogeneous equilibria with finite mass do not exist, and in which accordingly blow-up may be observed even for widely arbitrary initial data with supercritical mass when e.g. $n = 2$ ([25], [4]), e.g. the boundary value problem (1.1) always possesses the spatially constant steady states $(u, v) \equiv (u_m, v_m) := (\frac{m}{|\Omega|}, \frac{m}{|\Omega|})$ for any choice of the total mass $m > 0$. As far as the global existence of further large-mass solutions is concerned, however, only little seems known; more generally, the detection of bounded solutions at mass levels which are supercritical e.g. in the flavor of (1.3) seems limited to very few exceptional findings so far (see e.g. [4], [5] and the discussions therein). Partial results on generic occurrence of blow-up in a fully parabolic variant of (1.1) in radial frameworks, inter alia revealing considerable instability properties of (u_m, v_m), and actually of any equilibrium therein ([19]), may even be viewed as supporting the conjecture that global existence might in fact be quite a rarely preferred alternative, especially for large-mass data.

Main results: A critical mass phenomenon related to concentration-increasing perturbations of homogeneity. The objective of the present work is to address the latter topic in the context of (1.1), with a particular focus on a certain drastic instability aspect of the homogeneous equilibrium (u_m, v_m) for $m > 0$. In this direction, our main results will reveal an apparently novel type of critical mass phenomenon which at its core is closely linked to the concept of concentration comparison, in the framework of functions defined on $\Omega = B_R(0) \subset \mathbb{R}^n$ based on the following:

Definition 1.1 *Let $\overline{u}_0$ and $\underline{u}_0$ be nonnegative radially symmetric functions belonging to $L^1(\Omega)$. We then say that $\overline{u}_0$* is (strictly) more concentrated than *$\underline{u}_0$, and that $\underline{u}_0$* is (strictly) less concentrated than *$\overline{u}_0$, if*

$$\int_{B_r(0)} \overline{u}_0 \overset{(>)}{\geq} \int_{B_r(0)} \underline{u}_0 \qquad \text{for all } r \in (0, R), \tag{1.4}$$

and then write $\overline{u}_0 \succeq \underline{u}_0$ or, equivalently, $\underline{u}_0 \preceq \underline{u}_0$ (resp., $\overline{u}_0 \prec \underline{u}_0$ or $\underline{u}_0 \succ \overline{u}_0$).

Now the substantial part of our main results asserts that beyond a certain mass level, any arbitrarily small concentration-increasing perturbation of the associated homogeneous equilibrium enforces a finite-time collapse of the corresponding solution. Technically based the well-known fact that when restricted to radially symmetric solutions $(u, v) = (u(r, t), v(r, t))$ with $r = |x| \in [0, R]$, (1.1) reduces to a scalar parabolic problem for the mass accumulation function w given by $w(s, t) := \int_0^{s^{1/n}} r^{n-1} u(r, t) dr$ for $s \in [0, R^n]$ and t within a time interval under consideration ([14]; see also (2.3) and (2.6) below), at several stages its derivation will rely on a key observation, to be made in the course of an explicit construction in Lemma 3.1: According to this, namely, for all suitably large values of the mass the boundary value problem (2.6) for this cumulated density function w possesses a continuous curve in $C^1([0,R^n])$ of stationary subsolutions, connecting the respective homogeneous equilibrium to a distribution corresponding to a Dirac-type profile for (1.1). Through a series of steps involving various types of comparison arguments (cf. Sections 3.4 and 3.5), this will be seen to entail that at such mass levels, in fact for any nonconstant initial data for (1.1) which are more concentrated than their spatial average, the respectively transformed quantity w must lie above a suitable particular solution of (2.6) which is temporally nondecreasing and thus must blow up in finite time, because due to a lack of suitable equilibria (Lemma 3.5) it cannot approach any regular profile in the large time limit (see Lemma 3.6). We will thereby arrive at the following.

Theorem 1.1 *For any $n \ge 2$ and $R > 0$, there exists $m^\star = m^\star(n,R) > 0$ such that if u_0 is such that (1.2) holds and that with $m := \int_\Omega u_0$ we have $m > m^\star$ and*

$$u_0 \succeq \frac{m}{|\Omega|} \qquad \text{but} \qquad u_0 \not\equiv \frac{m}{|\Omega|}, \tag{1.5}$$

then (1.1) admits a solution blowing up in finite time; that is, there exist $T_{max} < \infty$ and a classical solution (u,v) of (1.1) in $\Omega \times (0,T_{max})$ such that

$$\limsup_{t \nearrow T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{1.6}$$

In particular, this conclusion holds whenever beyond satisfying (1.2) and $\int_\Omega u_0 > m^\star$, u_0 is nonconstant and nonincreasing with respect to $|x|$.

In order to appropriately complement the latter finding, let us first recall a simple and essentially well-known observation according to which the constant equilibria of (1.1) provide pointwise upper barriers for w. When combined with a Bernstein-type argument (Lemma 4.1), this namely yields a result on boundedness of $u(r,t) = nw_s(r^n,t)$, and thus on global extensibility, whenever u_0 is less concentrated than $\frac{1}{|\Omega|}\int_\Omega u_0$:

Proposition 1.2 *Let $n \ge 2$, $R > 0$ and $m > 0$, and suppose that u_0 satisfies (1.2) and is such that $\int_\Omega u_0 = m$ and*

$$u_0 \preceq \frac{m}{|\Omega|}. \tag{1.7}$$

Then (1.1) possesses a global classical solution (u,v) for which there exists $C > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \text{for all } t > 0. \tag{1.8}$$

But at sufficiently small mass levels, certain global bounded solutions can also be found for initial data more concentrated than the respective constant states. To see this, let us recall that whenever $n \ge 3$, the corresponding Cauchy problem in $\Omega = \mathbb{R}^n$, as formally obtained on letting $R \to \infty$ in (1.1), possesses a singular equilibrium determined by the explicit relation $u_c(x) := \frac{2(n-2)}{|x|^2}$ for $x \in \mathbb{R}^n \setminus \{0\}$. In the setting of the boundary value problem (1.1), this so-called Chandrasekhar solution, though no longer defining an exact solution, after all retains a certain supersolution property with regard to the respective parabolic equation for the quantity w. Using that this feature is even inherited by a certain family of smooth approximations of u_c (Lemma 4.2), once more relying on the above Bernstein-type result we shall, independently of Proposition 1.2, identify the following condition on mild concentration of u_0 as sufficient to ensure global boundedness:

Theorem 1.3 *Let $n \ge 3$ and $R > 0$, and suppose that u_0 satisfies (1.2) as well as $u_0 \prec \frac{2(n-2)}{|\cdot|^2}$, that is,*

$$\int_{B_r(0)} u_0 < 2\omega_n r^{n-2} \qquad \textit{for all } r \in (0,R). \tag{1.9}$$

Then (1.1) possesses a global classical solution (u,v) which is bounded in the sense that (1.8) holds with some $C > 0$.

Now using that unlike that in (1.7) the quantity on the right-hand side of (1.9) does not explicitly involve the total mass $\int_\Omega u_0$, we will see that in consequence Theorem 1.3 implies that indeed at each sufficiently small mass level, some – and actually even considerably many – initial data can be found which are more concentrated than the corresponding homogeneous equilibria, but which yet allow for global bounded solutions to (1.1):

Corollary 1.4 *Let $n \ge 2$ and $R > 0$. Then there exists $m_\star = m_\star(n,R) > 0$ with the following property: For any choice of $m \in (0, m_\star(n,R))$ one can find a relatively open subset B of $\{0 \le \varphi \in C^0_{rad}(\overline{\Omega}) \mid \int_\Omega \varphi = m\}$ consisting of functions more concentrated than $\overline{\Omega} \ni x \mapsto \frac{m}{|\Omega|} \equiv \frac{nm}{\omega_n R^n}$ such that whenever $u_0 \in B$, the problem (1.1) admits a global classical solution (u,v) which is bounded in the sense that there exists $C > 0$ such that (1.8) is valid.*

In summary, Corollary 1.4 and Theorem 1.1 reveal that for each $n \ge 2$ and $R > 0$, the number

$$\begin{aligned} m_c(n,R) \quad := \quad \inf \Bigg\{ m > 0 \ \Bigg| \ & \text{For all } 0 \le u_0 \in C^0_{rad}(\overline{\Omega}) \text{ with } u_0 \succeq \frac{1}{|\Omega|} \int_\Omega u_0 \text{ but } u_0 \not\equiv \textit{const.}, \\ & \text{(1.1) admits a solution blowing up in finite time} \Bigg\} \end{aligned} \tag{1.10}$$

is well-defined and positive, and by definition it marks a borderline between supercritical mass levels at which any, even arbitrarily small, concentration-increasing perturbation of the homogeneous steady state will lead to finite-time blow-up in (1.1), and a corresponding subcritical range of mass values at which this drastic instability property is absent. In particular, in this sense slightly more subtle than that of $\widehat{m}$ from (1.3), $m_c(n,R)$ plays the role of a critical mass not only in planar settings, but actually also when $n \ge 3$, thus indicating that the total mass of cells does indeed play an important role with regard to blow-up also in higher-dimensional frameworks.

2 Preliminaries: Local existence and transformation to a scalar problem

To begin with, let us state an essentially well-known basic result on local existence and extensibilty of classical solutions to (1.1), referring to a rich literature for details in its derivation (see e.g. [3], [11], [10]).

Lemma 2.1 *Let $n \ge 2$ and $R > 0$, and assume that u_0 satisfies (1.2). Then there exist $T_{max} \in (0, \infty]$ and a pair (u, v) of radially symmetric functions on $\overline{\Omega} \times [0, T_{max})$, uniquely determined by the properties*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})) & \textit{and} \\ v \in \bigcap_{q>n} L^\infty_{loc}([0, T_{max}); W^{1,q}(\Omega)) \cap C^{2,0}(\overline{\Omega} \times (0, T_{max})), \end{cases}$$

as well as

$$\int_\Omega v(\cdot, t) = 0 \qquad \textit{for all } t \in (0, T_{max}),$$

which solve (1.1) classically in $\Omega \times (0, T_{max})$ and are such that $u > 0$ in $\overline{\Omega} \times (0, T_{max})$, that

$$\int_\Omega u(\cdot, t) = \int_\Omega u_0 \qquad \textit{for all } t \in (0, T_{max}), \tag{2.1}$$

and that

$$\textit{if } T_{max} < \infty, \qquad \textit{then} \qquad \limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{2.2}$$

Throughout the sequel, unless otherwise stated, without further explicit mentioning we shall assume that (1.2) holds, and that (u, v) denotes the corresponding local solution of (1.1), as obtained in Lemma 2.1 and extended up to its maximal existence time $T_{max} \le \infty$.

For our further study thereof, following [14] we introduce the corresponding accumulated densities given by

$$w(s, t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho, t) d\rho, \qquad s \in [0, R^n],\ t \in [0, T_{max}), \tag{2.3}$$

and

$$w_0(s) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u_0(\rho) d\rho, \qquad s \in [0, R^n]. \tag{2.4}$$

Then

$$w_s(s, t) = \frac{1}{n} u(s^{\frac{1}{n}}, t) \quad \text{and} \quad w_{ss}(s, t) = \frac{1}{n^2} s^{\frac{1}{n}-1} u_r(s^{\frac{1}{n}}, t), \qquad s \in (0, R^n),\ t \in (0, T_{max}), \tag{2.5}$$

which together with (1.1) and Lemma 2.1 implies that $w_s > 0$ in $[0, R^n] \times [0, T_{max})$, and that w solves the Dirichlet problem

$$\begin{cases} w_t = n^2 s^{2-\frac{2}{n}} w_{ss} + n w w_s - \mu s w_s, & s \in (0, R^n),\ t \in (0, T_{max}), \\ w(0, t) = 0, \quad w(R^n, t) = \frac{m}{\omega_n}, & t \in (0, T_{max}), \\ w(s, 0) = w_0(s), & s \in (0, R^n), \end{cases} \tag{2.6}$$

with $m := \int_\Omega u_0$ and $\mu := \frac{nm}{\omega_n R^n}$.

3 Blow-up enforced by concentration-increasing perturbations of large constant states

The purpose of this section is to prove the blow-up result from Theorem 1.1. Our approach toward this will be launched by an observation to be made already in Lemma 3.1, according to which at suitably large mass levels m the PDE in (2.6) admits a continuous arc $(\underline{W}^{(\theta)})_{\theta\in[0,1)}$ in $C^1([0,R^n])$ of stationary subsolutions, connecting the associated homogeneous equilibrium (u_m,v_m) attained at $\theta=0$ with a corresponding fully concentrated Dirac-type distribution approached in the limit $\theta\nearrow 1$. As this will inter alia rule out the existence of regular steady states more concentrated than (u_m,v_m) (Lemma 3.5), this will imply (Lemma 3.6) that solutions emanating from any such initial data determined by $\underline{W}^{(\theta)}$ for $\theta\in(0,1)$ cannot be global in time, for otherwise they would approach the latter Dirac-type profile in the large time limit, and hence would eventually gather essentially all their mass in arbitrarily small balls, which will be seen to be impossible due to appropriate spatially localized blow-up criteria provided in Section 3.2. Since despite the degeneracy at $s=0$ the equation $z_t=n^2s^{2-\frac{2}{n}}z_{ss}$, as forming the principal part in the linearization of (2.6) about (u_m,v_m) with regard to diffusion, possesses a Hopf-type boundary point property to be derived in Section 3.4, relying on the $C^1([0,R^n])$-valued continuity at $\theta=0$ of the above subsolution curve we see in Section 3.5 that in the considered range of mass values, the latter conclusion on blow-up can in fact be transferred to arbitrary solutions addressed in Theorem 1.1 by means of a suitable comparison argument.

3.1 A curve in $C^1([0,R^n])$ of stationary subsolutions $\underline{W}^{(\theta)}$ to (2.6)

Let us now focus on our key observation with regard to blow-up in (1.1), relying on an explicit construction of a continuous family of stationary subsolutions to (2.6) with the following properties.

Lemma 3.1 *Let $n\ge 2$ and $R>0$. Then there exists $m^\star=m^\star(n,R)>0$ with the following property: For any choice of $m>m^\star$, one can find families $(s_0^{(\theta)})_{\theta\in[0,1)}\subset(0,R^n)$ and $(\underline{W}^{(\theta)})_{\theta\in[0,1)}\subset C^1([0,R^n])$ such that for all $\theta\in(0,1)$, $\underline{W}^{(\theta)}$ moreover belongs to $C^2([0,R^n]\setminus\{s_0^{(\theta)}\})\cap W^{2,\infty}((0,R^n))$ and satisfies $\underline{W}^{(\theta)}(0)=0$, $\underline{W}^{(\theta)}(R^n)=\frac{m}{\omega_n}$ and $\underline{W}^{(\theta)}_s\ge 0$ in $(0,R^n)$ as well as*

$$n^2s^{2-\frac{2}{n}}\underline{W}^{(\theta)}_{ss}(s)+n\underline{W}^{(\theta)}(s)\underline{W}^{(\theta)}_s(s)-\mu s\underline{W}^{(\theta)}_s(s)>0 \qquad \text{for all } s\in(0,R^n)\setminus\{s_0^{(\theta)}\}, \tag{3.1}$$

and such that $[0,1)\ni\theta\mapsto\underline{W}^{(\theta)}$ is continuous as a $C^1([0,R^n])$-valued mapping, where

$$\underline{W}^{(0)}(s)=\frac{\mu}{n}s \qquad \text{for all } s\in(0,R^n) \tag{3.2}$$

and

$$\underline{W}^{(\theta)}(s)\to\frac{m}{\omega_n} \quad \text{for all } s\in(0,R^n) \qquad \text{as } \theta\nearrow 1. \tag{3.3}$$

Proof. Given $R>0$, we write

$$c_1:=\frac{R^{n-2}}{2^{1-\frac{2}{n}}}\cdot\Big(1+\frac{4}{R^n}\Big) \qquad \text{and} \qquad c_2:=\min\Big\{\frac{1}{2}\,,\,\frac{1}{2R^n}\Big\} \tag{3.4}$$

and define

$$m^\star\equiv m^\star(n,R):=4n\omega_n\cdot\frac{c_1}{c_2}. \tag{3.5}$$

For fixed $m > m^\star$ and $\theta \in [0,1)$, we introduce

$$s_0 \equiv s_0^{(\theta)} := (1-\theta) \cdot \frac{R^n}{2} \tag{3.6}$$

and let

$$b \equiv b^{(\theta)} := \Big(\frac{2s_0}{R^n}\Big)^3 \qquad \text{and} \qquad d \equiv d^{(\theta)} := 1 - \frac{2s_0}{R^n} \tag{3.7}$$

as well as

$$a \equiv a^{(\theta)} := \frac{m}{\omega_n} \cdot \frac{(ds_0+b)^2}{ds_0^2 + bR^n} \tag{3.8}$$

and

$$A \equiv A^{(\theta)} := \frac{ab}{(ds_0+b)^2}\, . \tag{3.9}$$

Then with these abbreviations, the function $\underline{W} \equiv \underline{W}^{(\theta)}$ obtained on letting

$$\underline{W}(s) := \begin{cases} \frac{as}{ds+b}, & s \in [0,s_0], \\ \frac{m}{\omega_n} + A \cdot (s - R^n), & s \in (s_0, R^n], \end{cases} \tag{3.10}$$

evidently belongs to $C^2([0,R^n] \setminus \{s_0\})$ and satisfies $\underline{W}(0) = 0$ as well as $\underline{W}(R^n) = \frac{m}{\omega_n}$, and combining (3.9) with (3.8) we see that

$$\begin{aligned} \frac{as_0}{ds_0+b} - \Big\{\frac{m}{\omega_n} + A\cdot(s_0 - R^n)\Big\} &= \frac{as_0}{ds_0+b} - \frac{m}{\omega_n} - \frac{ab(s_0-R^n)}{(ds_0+b)^2} \\ &= \frac{ds_0^2 + bR^n}{(ds_0+b)^2} \cdot a - \frac{m}{\omega_n} = 0 \end{aligned}$$

and that hence $\underline{W}$ is continuous in $[0,R^n]$. Moreover, computing

$$\underline{W}_s(s) = \begin{cases} \frac{ab}{(ds+b)^2}, & s \in [0,s_0), \\ A, & s \in (s_0, R^n], \end{cases} \tag{3.11}$$

and

$$\underline{W}_{ss}(s) = \begin{cases} -\frac{2abd}{(ds+b)^3}, & s \in [0,s_0), \\ 0, & s \in (s_0, R^n], \end{cases} \tag{3.12}$$

from (3.9) we immediately obtain that indeed $\underline{W} \in C^1([0,R^n]) \cap W^{2,\infty}((0,R^n))$ with $\underline{W}_s \ge 0$ in $(0,R^n)$, and that since s_0, b, a and A evidently depend continuously on $\theta \in [0,1)$, the mapping $[0,1) \ni \theta \mapsto \underline{W}^{(\theta)}$ is continuous with values in $C^1([0,R^n])$.

While (3.2) is evident from (3.6)-(3.10), for the verification of (3.3) we only need to observe that as $\theta \nearrow 1$, by (3.6) we have $s_0 \searrow 0$ and therefore

$$\begin{aligned} \underline{W}(s_0) &= \frac{as_0}{ds_0+b} \\ &= \frac{m}{\omega_n} \cdot \frac{s_0(ds_0+b)}{ds_0^2 + bR^n} \end{aligned}$$

$$
\begin{aligned}
&= \frac{m}{\omega_n} \cdot \frac{ds_0^2 + 8R^{-3n}s_0^4}{ds_0^2 + 8R^{-2n}s_0^3} \\
&= \frac{m}{\omega_n} \cdot \frac{d + 8R^{-3n}s_0^2}{d + 8R^{-2n}s_0} \\
&\to \frac{m}{\omega_n}
\end{aligned}
$$

according to (3.8) and the definitions of b and d in (3.7).

It thus remains to derive the differential inequality in (3.1) for each fixed $\theta \in (0,1)$. Indeed, in the outer region where $s > s_0$, through (3.12) this is obvious due to the fact that in the identity

$$
n^2 s^{2-\frac{2}{n}} \underline{W}_{ss}(s) + n\underline{W}(s)\underline{W}_s(s) - \mu s \underline{W}_s(s) = n \cdot \Big(\underline{W}(s) - \frac{\mu}{n}s\Big) \cdot \underline{W}_s(s), \qquad s \in (s_0, R^n),
$$

we can estimate

$$
\underline{W}(s) - \frac{\mu}{n}s = \frac{m}{\omega_n} + A \cdot (s - R^n) - \frac{\mu}{n}s = \Big(\frac{m}{\omega_n R^n} - A\Big) \cdot (R^n - s) > 0 \qquad \text{for all } s \in (s_0, R^n)
$$

thanks to the inequality

$$
A = \frac{ab}{(ds_0 + b)^2} = \frac{m}{\omega_n} \cdot \frac{b}{ds_0^2 + bR^n} < \frac{m}{\omega_n R^n}
$$

asserted by (3.9) and (3.8).

In the corresponding inner part, using (3.12) and (3.11) we first compute

$$
\begin{aligned}
n^2 s^{2-\frac{2}{n}} & \underline{W}_{ss}(s) + n\underline{W}(s)\underline{W}_s(s) - \mu s \underline{W}_s(s) \\
&= -n^2 s^{2-\frac{2}{n}} \cdot \frac{2abd}{(ds+b)^3} + n \cdot \frac{as}{ds+b} \cdot \frac{ab}{(ds+b)^2} - \mu s \cdot \frac{ab}{(ds+b)^2} \\
&= \frac{nabs}{(ds+b)^3} \cdot \Big\{ a - 2nds^{1-\frac{2}{n}} - \frac{\mu}{n} \cdot (ds+b) \Big\}, \qquad s \in (0, s_0), \qquad (3.13)
\end{aligned}
$$

where writing

$$
\eta := \frac{ds_0 \cdot (R^n - s_0)}{R^n \cdot (ds_0 + b)}
$$

we clearly have

$$
\begin{aligned}
\frac{\mu}{n} \cdot (ds+b) - (1-\eta)a &< \frac{\mu}{n} \cdot (ds_0 + b) - (1-\eta)a \\
&= \frac{m}{\omega_n R^n} \cdot (ds_0 + b) - \frac{ds_0^2 + bR^n}{R^n \cdot (ds_0 + b)} \cdot \frac{m}{\omega_n} \cdot \frac{(ds_0+b)^2}{ds_0^2 + bR^n} \\
&= 0 \qquad \text{for all } s \in (0, s_0). \qquad (3.14)
\end{aligned}
$$

Moreover, since $s_0 \le \frac{R^n}{2}$ we can estimate

$$\begin{aligned}
\frac{2nds^{1-\frac{2}{n}}}{\eta a} &\le \frac{2nds_0^{1-\frac{2}{n}}}{\eta a} \\
&= 2nds_0^{1-\frac{2}{n}} \cdot \frac{R^n \cdot (ds_0+b)}{ds_0 \cdot (R^n - s_0)} \cdot \frac{\omega_n}{m} \cdot \frac{ds_0^2 + bR^n}{(ds_0+b)^2} \\
&= \frac{2n\omega_n R^n}{m} \cdot \frac{ds_0^2 + bR^n}{s_0^{\frac{2}{n}} \cdot (R^n - s_0) \cdot (ds_0+b)} \\
&\le \frac{4n\omega_n}{m} \cdot \frac{ds_0^2 + bR^n}{s_0^{\frac{2}{n}} \cdot (ds_0+b)} \qquad \text{for all } s \in (0,s_0),
\end{aligned} \tag{3.15}$$

where by (3.7),

$$\begin{aligned}
\frac{ds_0^2 + bR^n}{s_0^{\frac{2}{n}} \cdot (ds_0+b)} &= \frac{s_0^2 - 2R^{-n}s_0^3 + 8R^{-2n}s_0^3}{s_0^{\frac{2}{n}} \cdot (s_0 - 2R^{-n}s_0^2 + 8R^{-3n}s_0^3)} \\
&= s_0^{1-\frac{2}{n}} \cdot \frac{1 - 2R^{-n}s_0 + 8R^{-2n}s_0}{1 - 2R^{-n}s_0 + 8R^{-3n}s_0^2} \\
&\le \Big(\frac{R^n}{2}\Big)^{1-\frac{2}{n}} \cdot \frac{1 + 8R^{-2n} \cdot \frac{R^n}{2}}{1 - 2R^{-n}s_0 + 8R^{-3n}s_0^2} \\
&= \frac{c_1}{1 - 2R^{-n}s_0 + 8R^{-3n}s_0^2}.
\end{aligned} \tag{3.16}$$

Here if even $s_0 \le \frac{R^n}{4}$, then

$$1 - 2R^{-n}s_0 + 8R^{-3n}s_0^2 \ge 1 - 2R^{-n} \cdot \frac{R^n}{4} = \frac{1}{2},$$

whereas in tha case $s_0 \in (\frac{R^n}{4}, \frac{R^n}{2})$ we have

$$1 - 2R^{-n}s_0 + 8R^{-3n}s_0^2 \ge 1 - 2R^{-n} \cdot \frac{R^n}{2} + 8R^{-3n} \cdot \Big(\frac{R^n}{4}\Big)^2 = \frac{1}{2R^n},$$

so that in view of (3.4), in both these cases from (3.15) and (3.16) we infer that

$$\frac{2nds^{1-\frac{2}{n}}}{\eta a} \le \frac{4n\omega_n}{m} \cdot \frac{c_1}{c_2} < 1$$

due to (3.5) and our assumption that $m > m^\star$.
Together with (3.14) and (3.13) this shows that

$$\begin{aligned}
& n^2 s^{2-\frac{2}{n}} \underline{W}_{ss}(s) + n\underline{W}(s)\underline{W}_s(s) - \mu s \underline{W}_s(s) \\
&\quad = \frac{nabs}{(ds+b)^3} \cdot \Big\{ \Big[\eta a - 2nds^{1-\frac{2}{n}}\Big] + \Big[(1-\eta)a - \frac{\mu}{n} \cdot (ds+b)\Big] \Big\} \\
&\quad > 0 \qquad \text{for all } s \in (0,s_0)
\end{aligned}$$

and thereby completes the proof. □

3.2 Spatially localized criteria for blow-up

In order to study the time evolution of the solutions to (2.6) corresponding to the initial data $\underline{W}^{(\theta)}$ for $\theta \in (0,1)$, let us prepare some sufficient criteria for finite-time blow-up in (2.6) which are essentially local in space by referring to the initial mass distribution exclusively within some suitably small ball around the origin. The first of these criteria addresses the two-dimensional case and may thereby be viewed as a variant of a similar statement on a related Cauchy problem in the whole plane, as recently derived in [4].

Lemma 3.2 *Let $n = 2$, $R > 0$, $m_0 > 8\pi$ and $m \geq m_0$. Then there exists $s_0 = s_0(m_0, m, R) \in (0, R^2)$ with the property that whenever u_0 complies with (1.2) and is such that $\int_\Omega u_0 = m$ and that the corresponding solution w of (2.6) satisfies*

$$w(s_0, t_0) > \frac{m_0}{2\pi} \tag{3.17}$$

for some $t_0 \in [0, T_{max})$, we necessarily have $T_{max} < \infty$.

Proof. In order to prepare our argument, given $m_0 > 8\pi$ and $m > m_0$ let us first fix $\underline{m}_0 > m_0$ sufficiently close to m_0 such that $\underline{m}_0 \leq m$ and $m_0^2 > 8\pi\underline{m}_0$. The latter inequality then enables us to successively pick $\beta \geq 2$ large such that

$$m_0^2 > 8\pi\underline{m}_0 \cdot \frac{(\beta+1)^2}{\beta(\beta+2)},$$

then $\eta \in (0,1)$ small enough fulfilling

$$m_0^2 > 8\pi\underline{m}_0 \cdot \frac{(\beta+1)^2}{(1-\eta)\beta(\beta+2)}$$

and $\kappa \in (0,1)$ small satisfying

$$m_0^2 > 8\pi\underline{m}_0 \cdot \frac{(\beta+1)^2}{(1-\eta)\beta(\beta+2)} \cdot \Big(\frac{1}{1-\kappa}\Big)^{2\beta+2},$$

and thereupon we can finally select some small $s_1 \in (0, R^2)$ such that with $\mu := \frac{m}{\pi R^2}$ we still have

$$c_1 := m_0^2 - 8\pi\underline{m}_0 \cdot \frac{(\beta+1)^2}{(1-\eta)\beta(\beta+2)} \cdot \Big(\frac{1}{1-\kappa}\Big)^{2\beta+2} - \frac{\pi^2\mu^2(\beta+1)^2}{\eta(1-\eta)\beta(\beta+2)} \cdot \Big(\frac{1}{1-\kappa}\Big)^{2\beta+2} \cdot s_1^2 > 0, \tag{3.18}$$

and define

$$s_0 \equiv s_0(m_0, m, R) := \kappa s_1. \tag{3.19}$$

Then assuming that u_0 satisfies (1.2) as well as $\int_\Omega u_0 = m$ and is such that (3.17) holds for some $t_0 \in [0, T_{max})$ but that $T_{max} = \infty$, we construct a minorant of w by introducing an arbitrary nondecreasing $\underline{w}_0 \in C^1([0, R^2])$ such that

$$\underline{w}_0(0) = 0, \qquad \underline{w}_0(R^2) = \frac{m_0}{2\pi} \qquad \text{and} \qquad \underline{w}_0(s) \leq w(s, t_0) \quad \text{for all } s \in (0, R^2), \tag{3.20}$$

but that still

$$\underline{w}_0(s_0) \ge \frac{m_0}{2\pi}. \tag{3.21}$$

Then e.g. the family $(\underline{w}_{0\varepsilon})_{\varepsilon\in(0,R^2)} \subset C^1([0,R^2])$ with $\underline{w}_{0\varepsilon}(s) := \underline{w}_0(\frac{R^2(s-\varepsilon)}{R^2-\varepsilon})$, $s \in [\varepsilon, R^2]$, $\varepsilon \in (0,R^2)$, satisfies $\underline{w}_{0\varepsilon}(\varepsilon) = 0$, $\underline{w}_{0\varepsilon}(R^2) = \frac{m_0}{2\pi}$, $\underline{w}_{0\varepsilon s} \ge 0$ in (ε, R^2) and $\underline{w}_{0\varepsilon} \le \underline{w}_0$ in $[\varepsilon, R^2]$ as well as $\underline{w}_{0\varepsilon} \nearrow \underline{w}_0$ in $(0,R^2]$ and $\underline{w}_{0\varepsilon} \to \underline{w}_0$ in $C^1_{loc}((0,R^2])$ as $\varepsilon \searrow 0$, and for each $\varepsilon \in (0,R^2)$, the non-degenerate problem

$$\begin{cases} \underline{w}_{\varepsilon t} = 4s\underline{w}_{\varepsilon ss} + 2\underline{w}_\varepsilon \underline{w}_{\varepsilon s} - \mu s \underline{w}_{\varepsilon s}, & s \in (\varepsilon, R^2),\ t > t_0, \\ \underline{w}_\varepsilon(\varepsilon, t) = 0, \quad \underline{w}_\varepsilon(R^2, t) = \frac{m_0}{2\pi}, & t > t_0, \\ \underline{w}_\varepsilon(s, t_0) = \underline{w}_{0\varepsilon}(s), & s \in (\varepsilon, R^2), \end{cases} \tag{3.22}$$

can readily be seen to admit a global classical solution $\underline{w}_\varepsilon \in C^0([\varepsilon,R^2]\times[t_0,\infty)) \cap C^{2,1}([\varepsilon,R^2]\times(t_0,\infty))$. Thanks to the ordering property of $(\underline{w}_{0\varepsilon})_{\varepsilon\in(0,R^2)}$, these solutions satisfy $\underline{w}_\varepsilon \nearrow \underline{w}$ in $(0,R^2] \times [t_0,\infty)$ as $\varepsilon \searrow 0$ with some limit function $\underline{w}$ fulfilling $0 \le \underline{w} \le \frac{m_0}{2\pi}$, because $0 \le \underline{w}_\varepsilon \le \frac{m_0}{2\pi}$ in $[\varepsilon, R^2] \times [t_0, \infty)$ due to (3.20) and two evident comparison arguments. Moreover, since $\underline{w}_{\varepsilon s}$ solves a homogeneous linear parabolic equation with continuous coefficients, another application of the classical comparison principle asserts that $\underline{w}_{\varepsilon s} \ge 0$ in $(\varepsilon, R^2) \times (t_0, \infty)$, because $\underline{w}_{0\varepsilon s} \ge 0$, and because clearly $\underline{w}_{\varepsilon s} \ge 0$ both on $\{\varepsilon\} \times (t_0,\infty)$ and on $\{R^2\} \times (t_0, \infty)$. From interior parabolic Schauder estimates ([17]) and the Arzelà-Ascoli theorem, we furthermore obtain that in fact $\underline{w}_\varepsilon \to \underline{w}$ in $C^0_{loc}((0,R^2]\times[0,\infty))$ and in $C^{2,1}_{loc}((0,R^2]\times(t_0,\infty))$ as $\varepsilon \searrow 0$, and since $\underline{w}_\varepsilon \le w$ in $(\varepsilon, R^2) \times (t_0, \infty)$ by another obvious comparison argument, it can readily be verified that actually $\underline{w}$ belongs to $C^0([0,R^n]\times[t_0,\infty))$ and solves the problem

$$\begin{cases} \underline{w}_t = 4s\underline{w}_{ss} + 2\underline{w}\underline{w}_s - \mu s \underline{w}_s, & s \in (0, R^2),\ t > t_0, \\ \underline{w}(0, t) = 0, \quad \underline{w}(R^2, t) = \frac{m_0}{2\pi}, & t > t_0, \\ \underline{w}(s, t_0) = \underline{w}_0(s), & s \in (0, R^2), \end{cases} \tag{3.23}$$

classically.

Now for $\delta \in (0, s_1)$, on the basis of (3.23) and four integrations by parts we compute

$$\begin{aligned} \frac{d}{dt}\int_\delta^{s_1} (s_1-s)^\beta \underline{w}(s,t)ds &= 4\int_\delta^{s_1}(s_1-s)^\beta s\underline{w}_{ss}ds + \int_\delta^{s_1}(s_1-s)^\beta(\underline{w}^2)_s ds - \mu\int_\delta^{s_1}(s_1-s)^\beta s\underline{w}_s ds \\ &= -8\beta\int_\delta^{s_1}(s_1-s)^{\beta-2}\cdot\Big\{s_1 - \frac{\beta+1}{2}s\Big\}\cdot\underline{w}ds + \beta\int_\delta^{s_1}(s_1-s)^{\beta-1}\underline{w}^2 ds \\ &\quad -\mu\int_\delta^{s_1}\Big\{\beta(s_1-s)^{\beta-1}s - (s_1-s)^\beta\Big\}\cdot\underline{w}ds \\ &\quad -4(s_1-\delta)^\beta\delta\underline{w}_s(\delta,t) + 4\Big\{(s_1-\delta)^\beta - \beta(s_1-\delta)^{\beta-1}\delta\Big\}\cdot\underline{w}(\delta,t) \\ &\quad -(s_1-\delta)^\beta\underline{w}^2(\delta,t) + \mu(s_1-\delta)^\beta\delta\underline{w}(\delta,t) \qquad \text{for all } t > t_0, \end{aligned}$$

which on dropping three nonnegative terms and integrating in time shows that

$$\begin{aligned} \int_\delta^{s_1}(s_1-s)^\beta\underline{w}(s,t)ds &\ge \int_\delta^{s_1}(s_1-s)^\beta\underline{w}(s,t_0)ds + \beta\int_{t_0}^t\int_\delta^{s_1}(s_1-s)^{\beta-1}\underline{w}^2(s,\tau)dsd\tau \\ &\quad -8\beta\int_{t_0}^t\int_\delta^{s_1}(s_1-s)^{\beta-2}\cdot\Big\{s_1-\frac{\beta+1}{2}s\Big\}\cdot\underline{w}(s,\tau)dsd\tau \end{aligned}$$

$$
\begin{aligned}
&-\mu\beta\int_{t_0}^{t}\int_{\delta}^{s_1}(s_1-s)^{\beta-1}s\underline{w}(s,\tau)dsd\tau\\
&-4(s_1-\delta)^{\beta}\delta\int_{t_0}^{t}\underline{w}_s(\delta,\tau)d\tau-4\beta(s_1-\delta)^{\beta-1}\delta\int_{t_0}^{t}\underline{w}(\delta,\tau)d\tau\\
&-(s_1-\delta)^{\beta}\int_{t_0}^{t}\underline{w}^2(\delta,\tau)d\tau \qquad \text{for all } t>t_0.
\end{aligned}
\tag{3.24}
$$

Here for fixed $t > t_0$, by continuity of $\underline{w}$ at $s = 0$ it readily follows from (3.23) that $\varphi(s) := \int_{t_0}^{t}\underline{w}(s,\tau)d\tau$, $s \in (0,R^2)$, satisfies $\varphi(s) \to 0$ as $s \searrow 0$ and necessarily also $\liminf_{s\searrow 0} s\varphi_s(s) = 0$, for otherwise there would exist $s_2 \in (0,R^2)$ and $c_2 > 0$ such that $\varphi_s(s) \geq \frac{c_2}{s}$ for all $s \in (0,s_2)$, implying the absurd conclusion that φ would be unbounded on $(0,s_2)$. Hence fixing any $(\delta_j)_{j\in\mathbb{N}} \subset (0,s_1)$ such that $\delta_j \searrow 0$ and $\delta_j\varphi_s(\delta_j) \to 0$ as $j \to \infty$, taking $\delta = \delta_j$ we obtain that in (3.24),

$$
4(s_1-\delta_j)^{\beta}\delta_j\int_{t_0}^{t}\underline{w}_s(\delta_j,\tau)d\tau \leq 4s_1^{\beta}\delta_j\varphi_s(\delta_j) \to 0
$$

as well as

$$
4\beta(s_1-\delta_j)^{\beta-1}\delta_j\int_{t_0}^{t}\underline{w}(\delta_j,\tau)d\tau \leq 4\beta s_1^{\beta-1}\delta_j\varphi(\delta_j) \to 0
$$

and

$$
(s_1-\delta_j)^{\beta}\int_{t_0}^{t}\underline{w}^2(\delta_j,\tau)d\tau \leq s_1^{\beta}\cdot\frac{m_0}{2\pi}\cdot\varphi(\delta_j) \to 0
$$

as $j \to \infty$, whence again due to the continuity of $\underline{w}$ it follows from (3.24) that $y(t) := \int_0^{s_1}(s_1 - s)^{\beta}\underline{w}(s,t)ds$, $t \geq t_0$, satisfies

$$
\begin{aligned}
y(t) \;\geq\; & \int_0^{s_1}(s_1-s)^{\beta}\underline{w}_0(s)ds+\beta\int_{t_0}^{t}\int_0^{s_1}(s_1-s)^{\beta-1}\underline{w}^2(s,\tau)dsd\tau\\
&-8\beta\int_{t_0}^{t}\int_0^{s_1}(s_1-s)^{\beta-2}\cdot\Big\{s_1-\frac{\beta+1}{2}s\Big\}\cdot\underline{w}(s,\tau)dsd\tau\\
&-\mu\beta\int_{t_0}^{t}\int_0^{s_1}(s_1-s)^{\beta-1}s\underline{w}(s,\tau)dsd\tau \qquad \text{for all } t>t_0.
\end{aligned}
\tag{3.25}
$$

Here since $\underline{w}_s \geq 0$ and $s_1 - \frac{\beta+1}{2}s$ is positive if and only if $s \leq \frac{2s_1}{\beta+1}$, we can estimate

$$
\begin{aligned}
&8\beta\int_{t_0}^{t}\int_0^{s_1}(s_1-s)^{\beta-2}\cdot\Big\{s_1-\frac{\beta+1}{2}s\Big\}\cdot\underline{w}(s,\tau)dsd\tau\\
&\quad= 8\beta\int_{t_0}^{t}\bigg\{\int_0^{\frac{2s_1}{\beta+1}}(s_1-s)^{\beta-2}\cdot\Big\{s_1-\frac{\beta+1}{2}s\Big\}\cdot\underline{w}(s,\tau)d\tau\\
&\qquad-\int_{\frac{2s_1}{\beta+1}}^{s_1}(s_1-s)^{\beta-2}\cdot\Big\{\frac{\beta+1}{2}s-s_1\Big\}\cdot\underline{w}(s,\tau)ds\bigg\}d\tau
\end{aligned}
$$

$$
\begin{aligned}
&\le 8\beta\int_{t_0}^t \underline{w}\Big(\frac{2s_1}{\beta+1},\tau\Big)\cdot\bigg\{\int_0^{\frac{2s_1}{\beta+1}}(s_1-s)^{\beta-2}\cdot\Big\{s_1-\frac{\beta+1}{2}s\Big\}ds \\
&\qquad -\int_{\frac{2s_1}{\beta+1}}^{s_1}(s_1-s)^{\beta-2}\cdot\Big\{\frac{\beta+1}{2}s-s_1\Big\}ds\bigg\}d\tau \\
&= 8\beta\cdot\bigg\{\int_0^{s_1}(s_1-s)^{\beta-2}\cdot\Big\{s_1-\frac{\beta+1}{2}s\Big\}ds\bigg\}\cdot\int_{t_0}^t \underline{w}\Big(\frac{2s_1}{\beta+1},\tau\Big)d\tau \\
&= 4s_1^\beta\int_{t_0}^t \underline{w}\Big(\frac{2s_1}{\beta+1},\tau\Big)d\tau \\
&\le \frac{2}{\pi}\underline{m}_0 s_1^\beta\cdot(t-t_0) \qquad \text{for all } t>t_0,
\end{aligned}
$$

and the rightmost summand in (3.25) can be controlled using Young's inequality according to

$$
\begin{aligned}
&\mu\beta\int_{t_0}^t\int_0^{s_1}(s_1-s)^{\beta-1}s\underline{w}(s,\tau)dsd\tau \\
&\qquad\le \eta\beta\int_{t_0}^t\int_0^{s_1}(s_1-s)^{\beta-1}\underline{w}^2(s,\tau)dsd\tau+\frac{\mu^2\beta}{4\eta}\int_{t_0}^t\int_0^{s_1}(s_1-s)^{\beta-1}s^2dsd\tau \\
&\qquad\le \eta\beta\int_{t_0}^t\int_0^{s_1}(s_1-s)^{\beta-1}\underline{w}^2(s,\tau)dsd\tau+\frac{\mu^2\beta s_1^2}{4\eta}\int_{t_0}^t\int_0^{s_1}(s_1-s)^{\beta-1}dsd\tau \\
&\qquad= \eta\beta\int_{t_0}^t\int_0^{s_1}(s_1-s)^{\beta-1}\underline{w}^2(s,\tau)dsd\tau+\frac{\mu^2 s_1^{\beta+2}}{4\eta}\cdot(t-t_0) \qquad \text{for all } t>t_0.
\end{aligned}
$$

As furthermore

$$
\begin{aligned}
y(t) &\le \bigg\{\int_0^{s_1}(s_1-s)^{\beta-1}\underline{w}^2(s,\tau)ds\bigg\}^{\frac12}\cdot\bigg\{\int_0^{s_1}(s_1-s)^{\beta+1}ds\bigg\}^{\frac12} \\
&= \bigg\{\int_0^{s_1}(s_1-s)^{\beta-1}\underline{w}^2(s,\tau)ds\bigg\}^{\frac12}\cdot\Big\{\frac{s_1^{\beta+2}}{\beta+2}\Big\}^{\frac12} \qquad \text{for all } t>t_0
\end{aligned}
$$

by the Cauchy-Schwarz inequality, (3.25) entails that

$$
y(t)\ge y_0+c_3\int_{t_0}^t y^2(\tau)d\tau-c_4\cdot(t-t_0) \qquad \text{for all } t>t_0
$$

with

$$
y_0:=\int_0^{s_1}(s_1-s)^\beta\underline{w}_0(s)ds, \quad c_3:=(1-\eta)\beta(\beta+2)s_1^{-\beta-2} \quad\text{and}\quad c_4:=\frac{2}{\pi}\underline{m}_0 s_1^\beta+\frac{\mu^2 s_1^{\beta+2}}{4\eta}.
$$

Now from a straightforward comparison argument it follows that therefore $y(t)\ge\underline{y}(t)$ for all $t\in(t_0,T)$, where $\underline{y}\in C^1([t_0,T))$ denotes the solution of the initial-value problem

$$
\begin{cases}
\underline{y}'(t)=c_3\underline{y}^2(t)-c_4, & t\in(t_0,T),\\
\underline{y}(t_0)=y_0,
\end{cases}
\tag{3.26}
$$

extended up to its maximal existence time $T \in (t_0, \infty]$. In view of the fact that by (3.21),

$$y_0 \geq \int_{s_0}^{s_1} (s_1 - s)^\beta \cdot \frac{m_0}{2\pi} ds = \frac{m_0}{2\pi} \cdot \frac{(s_1 - s_0)^{\beta+1}}{\beta + 1}$$

and that hence

$$\begin{aligned}
c_3 y_0^2 - c_4 &\geq (1-\eta)\beta(\beta+2) s_1^{-\beta-2} \cdot \frac{m_0^2}{4\pi^2} \frac{(s_1 - s_0)^{2\beta+2}}{(\beta+1)^2} - \left\{ \frac{2}{\pi} \underline{m}_0 s_1^\beta + \frac{\mu^2 s_1^{\beta+2}}{4\eta} \right\} \\
&= \frac{(1-\eta)\beta(\beta+2)}{4\pi^2(\beta+1)^2} \cdot \frac{(s_1 - s_0)^{2\beta+2}}{s_1^{\beta+2}} \cdot \left\{ m_0^2 - 8\pi \underline{m}_0 \cdot \frac{(\beta+1)^2}{(1-\eta)\beta(\beta+2)} \cdot \left(\frac{s_1}{s_1 - s_0} \right)^{2\beta+2} \right. \\
&\qquad \left. - \frac{\pi^2 \mu^2 (\beta+1)^2}{\eta(1-\eta)\beta(\beta+2)} \cdot \left(\frac{s_1}{s_1 - s_0} \right)^{2\beta+2} s_1^2 \right\} \\
&= \frac{(1-\eta)\beta(\beta+2)}{4\pi^2(\beta+1)^2} \cdot \frac{(s_1 - s_0)^{2\beta+2}}{s_1^{\beta+2}} \cdot c_1 \\
&> 0
\end{aligned}$$

by (3.19) and (3.18), however, an elementary ODE argument shows that in (3.26) we actually must have $T < \infty$ and $\underline{y}(t) \nearrow +\infty$ as $t \nearrow T$, in particular meaning that $y(T) = +\infty$ and hence clearly contradicting our hypothesis that w be global. $\square$

The next lemma provides an analogue in higher-dimensional cases, in contrast to the above now no longer involving any restriction on the size of the level m_0 of mass concentration.

Lemma 3.3 *Let $n \geq 3$, $R > 0$, $m_0 > 0$ and $m \geq m_0$. Then there exists $s_0 = s_0(m_0, m, R) \in (0, R^n)$ such that if u_0 is such that (1.2) holds, that $\int_\Omega u_0 = m$, and that for the function w defined in (2.3) we have*

$$w(s_0, t_0) > \frac{m_0}{\omega_n} \qquad \textit{for some } t_0 \in [0, T_{max}), \tag{3.27}$$

then $T_{max} < \infty$.

Proof. We abbreviate

$$c_1 := \frac{4n^2}{n-1}, \qquad c_2 := \frac{n^2}{(n+1)\omega_n^2} \qquad \text{and} \qquad c_3 := \frac{n}{2(n+2)\omega_n} \cdot \left(n - \frac{n+1}{2^{\frac{2}{n}}} \right), \tag{3.28}$$

observing that c_3 is positive because $n \cdot 2^{\frac{2}{n}} = n \cdot e^{\frac{2}{n}\ln 2} \geq n \cdot (1 + \frac{2}{n}\ln 2) = n + 2\ln 2 > n + 1$ by convexity of $\mathbb{R} \ni \xi \mapsto e^\xi$.

Then given $R > 0$, $m_0 > 0$ and $m > 0$ we can fix $s_0 = s_0(m_0, m, R) \in (0, \frac{R^n}{2})$ such that $s_1 := 2s_0$ satisfies

$$s_1^{2-\frac{4}{n}} \leq \frac{c_3^2}{8c_1} m_0^2 \tag{3.29}$$

and

$$s_1^2 \leq \frac{c_3^2}{8c_2} \cdot \frac{m_0^2 R^{2n}}{m^2}, \tag{3.30}$$

and henceforth we assume (1.2) with $\int_\Omega u_0 = m$, that (3.27) is valid for some $t_0 \in [0, T_{max})$, but that $T_{max} = \infty$. For $\delta \in (0, \frac{s_1}{2})$, we then use (2.6) to compute

$$\begin{aligned}
&\frac{d}{dt}\int_\delta^{s_1} s^{-1+\frac{2}{n}}(s_1-s)w(s,t)ds \\
&= n^2\int_\delta^{s_1} s(s_1-s)w_{ss}(s,t)ds + \frac{n}{2}\int_\delta^{s_1} s^{-1+\frac{2}{n}}(s_1-s)(w^2)_s(s,t)ds - \mu\int_\delta^{s_1} s^{\frac{2}{n}}(s_1-s)w_s(s,t)ds \\
&= -2n^2\int_\delta^{s_1} w(s,t)ds + \frac{n-2}{2}\int_\delta^{s_1} s^{-2+\frac{2}{n}}(s_1-s)w^2(s,t)ds + \frac{n}{2}\int_\delta^{s_1} s^{-1+\frac{2}{n}}w^2(s,t)ds \\
&\quad +\frac{2\mu}{n}\int_\delta^{s_1} s^{-1+\frac{2}{n}}(s_1-s)w(s,t)ds - \mu\int_\delta^{s_1} s^{\frac{2}{n}}w(s,t)ds \\
&\quad -n^2\delta(s_1-\delta)w_s(\delta,t) + n^2 s_1 w(s_1,t) - n^2(s_1-2\delta)w(\delta,t) \\
&\quad -\frac{n}{2}\delta^{-1+\frac{2}{n}}(s_1-\delta)w^2(\delta,t) + \mu\delta^{\frac{2}{n}}(s_1-\delta)w(\delta,t) \qquad \text{for all } t > t_0
\end{aligned}$$

with $\mu = \frac{nm}{\omega_n R^n}$, which after neglecting some nonnegative summands and integrating in time shows that

$$\begin{aligned}
\int_\delta^{s_1} s^{-1+\frac{2}{n}}(s_1-s)w(s,t)ds \geq\ & \int_\delta^{s_1} s^{-1+\frac{2}{n}}(s_1-s)w(s,t_0)ds \\
& -2n^2\int_{t_0}^t\int_\delta^{s_1} w(s,\tau)dsd\tau + \frac{n}{2}\int_{t_0}^t\int_\delta^{s_1} s^{-1+\frac{2}{n}}w^2(s,\tau)dsd\tau \\
& -\mu\int_{t_0}^t\int_\delta^{s_1} s^{\frac{2}{n}}w(s,\tau)dsd\tau \\
& -n^2\delta(s_1-\delta)\int_{t_0}^t w_s(\delta,\tau)d\tau - n^2(s_1-2\delta)\int_{t_0}^t w(s,\tau)d\tau \\
& -\frac{n}{2}\delta^{-1+\frac{2}{n}}(s_1-\delta)\int_{t_0}^t w^2(\delta,\tau)d\tau \qquad \text{for all } t > t_0. \qquad (3.31)
\end{aligned}$$

Here since the boundedness of w_s in $(0, R^n)\times(t_0,t)$ warrants that $\sup_{(s,\tau)\in(0,R^n)\times(t_0,t)} \frac{w(s,\tau)}{s}$ is finite, we readily infer that

$$n^2\delta(s_1-\delta)\int_{t_0}^t w_s(\delta,\tau)d\tau + n^2(s_1-2\delta)\int_{t_0}^t w(s,\tau)d\tau + \frac{n}{2}\delta^{-1+\frac{2}{n}}(s_1-\delta)\int_{t_0}^t w^2(\delta,\tau)d\tau \to 0 \quad \text{as } \delta \searrow 0,$$

whence on several applications of the monotone convergence theorem we infer from (3.31) that $y(t) := \int_0^{s_1} s^{-1+\frac{2}{n}}(s_1-s)w(s,t)ds$, $t \geq t_0$, satisfies

$$\begin{aligned}
y(t) \geq\ & y(t_0) - 2n^2\int_{t_0}^t\int_0^{s_1} w(s,\tau)dsd\tau + \frac{n}{2}\int_{t_0}^t s^{-1+\frac{2}{n}}w^2(s,\tau)dsd\tau \\
& -\mu\int_{t_0}^t\int_0^{s_1} s^{\frac{2}{n}}w(s,\tau)dsd\tau \qquad \text{for all } t > t_0.
\end{aligned}$$

Since by Young's inequality and the Cauchy-Schwarz inequality,

$$2n^2\int_0^{s_1} w(s,\tau)ds \leq \frac{n}{8}\int_0^{s_1} s^{-1+\frac{2}{n}}w^2(s,\tau)ds + 8n^3\int_0^{s_1} s^{1-\frac{2}{n}}ds$$

$$= \frac{n}{8}\int_0^{s_1} s^{-1+\frac{2}{n}} w^2(s,\tau)ds + c_1 s_1^{2-\frac{2}{n}} \qquad \text{for all } \tau > t_0$$

and

$$\begin{aligned}
\mu\int_0^{s_1} s^{\frac{2}{n}} w(s,\tau)ds &\le \frac{n}{8}\int_0^{s_1} s^{-1+\frac{2}{n}} w^2(s,\tau)ds + \frac{2\mu^2}{n}\int_0^{s_1} s^{1+\frac{2}{n}}ds \\
&= \frac{n}{8}\int_0^{s_1} s^{-1+\frac{2}{n}} w^2(s,\tau)ds + c_2\frac{m^2}{R^{2n}} s_1^{2+\frac{2}{n}} \qquad \text{for all } \tau > t_0
\end{aligned}$$

as well as

$$\begin{aligned}
y(\tau) &\le \left\{\int_0^{s_1} s^{-1+\frac{2}{n}} w^2(s,\tau)ds\right\}^{\frac{1}{2}} \cdot \left\{\int_0^{s_1} s^{-1+\frac{2}{n}} (s_1-s)^2 ds\right\}^{\frac{1}{2}} \\
&\le \left\{\int_0^{s_1} s^{-1+\frac{2}{n}} w^2(s,\tau)ds\right\}^{\frac{1}{2}} \cdot \left\{s_1^2\int_0^{s_1} s^{-1+\frac{2}{n}} ds\right\}^{\frac{1}{2}} \\
&= \left\{\int_0^{s_1} s^{-1+\frac{2}{n}} w^2(s,\tau)ds\right\}^{\frac{1}{2}} \cdot \left\{\frac{n}{2} s_1^{2+\frac{2}{n}}\right\}^{\frac{1}{2}} \qquad \text{for all } \tau > t_0,
\end{aligned}$$

this entails that

$$y(t) \ge y(t_0) + \frac{1}{2} s_1^{-2-\frac{2}{n}} \int_{t_0}^t y^2(\tau)d\tau - \left\{c_1 s_1^{2-\frac{2}{n}} + c_2\frac{m^2}{R^{2n}} s_1^{2+\frac{2}{n}}\right\}\cdot(t-t_0) \qquad \text{for all } t > t_0. \tag{3.32}$$

Now since (3.27) along with our selections of s_0 and c_3 guarantees that

$$y(t_0) \ge \frac{m_0}{\omega_n}\cdot\int_{\frac{s_1}{2}}^{s_1} s^{-1+\frac{2}{n}}(s_1-s)ds = c_3 m_0 s_1^{1+\frac{2}{n}}$$

and that hence, by (3.29) and (3.30),

$$\frac{c_1 s_1^{2-\frac{2}{n}} + c_2\frac{m^2}{R^{2n}} s_1^{2+\frac{2}{n}}}{\frac{1}{4} s_1^{-2-\frac{2}{n}} y^2(t_0)} \le \frac{4c_1}{c_3^2 m_0^2} s_1^{2-\frac{4}{n}} + \frac{4c_2 m^2}{c_3^2 m_0^2 R^{2n}} s_1^2 \le \frac{1}{2} + \frac{1}{2} = 1,$$

it follows that there exists $T > t_0$ such that the problem

$$\begin{cases} \underline{y}'(t) = \frac{1}{2} s_1^{2-\frac{2}{n}} \underline{y}^2(t) - \left\{c_1 s_1^{2-\frac{2}{n}} + c_2\frac{m^2}{R^{2n}} s_1^{2+\frac{2}{n}}\right\}, & t\in(t_0,T), \\ \underline{y}(t_0) = y(t_0), \end{cases}$$

admits a solution $\underline{y} \in C^1([t_0,T))$ fulfilling $\underline{y}(t) \nearrow +\infty$ as $t \nearrow T$. But a comparison argument based on (3.32) ensures that $y(t) \ge \underline{y}(t)$ for all $t \in (t_0,T)$, which is incompatible with our hypothesis that (u,v) be global. $\square$

3.3 Finite-time collapse into singular profiles for solutions emanating from $\underline{W}^{(\theta)}$

Our analysis of the solutions to (2.6) evolving from $\underline{W}^{(\theta)}$ will, secondly, be prepared by the observation that as a consequence of Lemma 3.1 an associated steady state problem, formulated here as the one-point boundary value problem

$$\begin{cases} n^2 s^{2-\frac{2}{n}} W_{ss} + nWW_s - \mu s W_s = 0, & s \in (0, R^n), \\ W(R^n) = \frac{m}{\omega_n}, \end{cases} \tag{3.33}$$

does not admit any solution above $s \mapsto \frac{\mu}{n} s$, other than the trivial solution $s \mapsto \frac{m}{\omega_n}$. This will be seen in Lemma 3.5 on the basis of the following auxiliary statement on a certain strong maximum principle property of the degenerate elliptic equation therein.

Lemma 3.4 *Let $n \ge 2, R > 0$ and $m > 0$, and let $\overline{W}$ and $\underline{W}$ be two functions from $W^{2,\infty}_{loc}((0, R^n])$ such that $\overline{W}_s \ge 0$ and $\underline{W}_s \ge 0$ on $(0, R^n)$, that $\overline{W}(R^n) = \underline{W}(R^n)$, and that*

$$n^2 s^{2-\frac{2}{n}} \overline{W}_{ss} + n\overline{W}\,\overline{W}_s - \mu s \overline{W}_s \le n^2 s^{2-\frac{2}{n}} \underline{W}_{ss} + n\underline{W}\,\underline{W}_s - \mu s \underline{W}_s \qquad \text{a.e. in } (0, R^n). \tag{3.34}$$

Then if moreover

$$\overline{W}(s) \ge \underline{W}(s) \quad \text{for all } s \in (0, R^n), \qquad \text{but} \qquad \overline{W} \not\equiv \underline{W}, \tag{3.35}$$

there exists $C > 0$ such that

$$\overline{W}(s) \ge \underline{W}(s) + Cs(R^n - s) \qquad \text{for all } s \in (0, R^n). \tag{3.36}$$

Proof. We let $z := \overline{W} - \underline{W} \in W^{2,\infty}_{loc}((0, R^n])$ and first observe that $z \ge 0$ by (3.35), whence (3.34) implies that

$$\begin{aligned} n^2 s^{2-\frac{2}{n}} z_{ss} &\le \Big\{ - n\overline{W}\,\overline{W}_s + \mu s \overline{W}_s \Big\} + \Big\{ n\underline{W}\,\underline{W}_s - \mu s \underline{W}_s \Big\} \\ &= -nz\underline{W}_s - n\overline{W} z_s + \mu s z_s \\ &\le -(n\overline{W} - \mu s) \cdot z_s \qquad \text{a.e. in } (0, R^n), \end{aligned}$$

because $\underline{W}_s \ge 0$ according to our hypotheses. On integration, this yields

$$z_s(s_2) \le z_s(s_1) \cdot e^{-\frac{1}{n^2} \int_{s_1}^{s_2} s^{-2+\frac{2}{n}} (n\overline{W}(s) - \mu s)} ds \qquad \text{for all } s_1 \in (0, R^n) \text{ and any } s_2 \in (s_1, R^n], \tag{3.37}$$

which as a first consequence immediately implies that

$$z(s) > 0 \qquad \text{for all } s \in (0, R^n). \tag{3.38}$$

Indeed, if this was false then by nonnegativity of z we could find $s_0 \in (0, R^n)$ such that $z(s_0) = z_s(s_0) = 0$, due to (3.37) implying that both $z_s(s) \le 0$ for all $s \in (s_0, R^n)$ and $z_s(s) \ge 0$ for all $s \in (0, s_0)$. Again since $z \ge 0$, these two properties would entail that necessarily $z \equiv 0$ which is incompatible with (3.35).

Now (3.38) ensures the existence of $s_1 \in (0, R^n)$ such that $z_s(s_1) < 0$, for otherwise we would have $z_s \ge 0$ in $(0, R^n)$ and hence $z(R^n) > 0$ by (3.38), contrary to our assumption that $\overline{W}(R^n) = \underline{W}(R^n)$.

As a second implication of (3.37), from this we could infer that in fact $z_s(s) < 0$ for all $s \in [s_1, R^n]$ and that hence

$$z(s) \geq c_1(R^n - s) \qquad \text{for all } s \in (s_1, R^n) \tag{3.39}$$

with $c_1 := -\max_{s\in[s_1,R^n]} z_s(s)$ being positive by continuity of z_s in $(0, R^n]$.
Near the point $s = 0$ of degeneracy where our hypotheses essentially reduce to the mere nonnegativity of z, we may argue similarly when $\liminf_{s\searrow 0} z(s) = 0$. Then, namely, once more using (3.38) we obtain $s_2 \in (0, s_1)$ such that $c_2 := z_s(s_2) > 0$, due to (3.37) meaning that

$$z_s(s) \geq c_2\, e^{\frac{1}{n^2}\int_s^{s_2} \sigma^{-2+\frac{2}{n}}(n\overline{W}(\sigma)-\mu\sigma)d\sigma} \geq c_3 \qquad \text{for all } s \in \Big(0, \frac{s_2}{2}\Big)$$

if we let $c_3 := c_2 \cdot \exp\{\frac{1}{n^2}\int_{s_2/2}^{s_2} \sigma^{-2+\frac{2}{n}}(n\overline{W}(\sigma) - \mu\sigma)d\sigma\} > 0$. When supplemented by an evident reasoning in the opposite situation when $\liminf_{s\searrow 0} z(s) > 0$, in any case we infer the existence of $s_3 \in (0, s_1)$ and $c_4 > 0$ such that

$$z(s) \geq c_4 s \qquad \text{for all } s \in (0, s_3).$$

Along with (3.39) and (3.38), this readily yields (3.36) with $C := \min\{\frac{c_1}{R^n}, \frac{c_4}{R^n}, \min_{s\in[s_2,s_3]} \frac{z(s)}{s(R^n-s)}\}$.
$\square$

As announced, thanks to the latter we may infer from Lemma 3.1 the following nonexistence result.

Lemma 3.5 *Let $n \geq 2, R > 0$ and $m > m^\star$ with $m^\star$ taken from Lemma 3.1, and suppose that $W \in C^2((0, R^n])$ is a nonnegative solution of (3.33) such that $W_s \geq 0$ in $(0, R^n)$, and that moreover*

$$W(s) \geq \frac{\mu}{n} s \quad \textit{for all } s \in (0, R^n) \tag{3.40}$$

but

$$W \not\equiv \frac{\mu}{n}(\cdot). \tag{3.41}$$

Then

$$W(s) = \frac{m}{\omega_n} \qquad \textit{for all } s \in (0, R^n). \tag{3.42}$$

Proof. With $(\underline{W}^{(\theta)})_{\theta\in[0,1)}$ taken from Lemma 3.1, we let

$$S := \Big\{\theta \in [0, 1) \;\Big|\; W(s) \geq \underline{W}^{(\theta)}(s) \text{ for all } s \in (0, R^n)\Big\}$$

and then note that S is not empty, because $0 \in S$ according to (3.40) and the fact that $\underline{W}^{(0)}(s) = \frac{\mu}{n}s$ for all $s \in (0, R^n)$ by Lemma 3.1. Moreover, from the continuity of $[0, 1) \ni \theta \mapsto \underline{W}^{(\theta)}(s)$ for all $s \in (0, R^n)$, as forming a by-product of Lemma 3.1, we immediately obtain that S is closed in the relative topology of $[0, 1)$. In order to make sure that S is also relatively open in $[0, 1)$, we fix $\theta_0 \in S$ and then observe that in both cases $\theta_0 = 0$ and $\theta_0 \in (0, 1)$ we have

$$W \not\equiv \underline{W}^{(\theta_0)} \qquad \text{in } (0, R^n), \tag{3.43}$$

which, indeed, when $\theta_0 = 0$ is directly implied by our hypothesis (3.41), and which when $\theta \in (0, 1)$ results from the strictness of the differential inequality in (3.1), namely ruling out that $\underline{W}^{(\theta_0)}$ coincides with a solution of (3.33). Now thanks to (3.43), we may invoke Lemma 3.4 to infer that in fact there exists $c_1 > 0$ such that

$$W(s) \geq \underline{W}^{(\theta_0)}(s) + c_1 s(R^n - s) \qquad \text{for all } s \in (0, R^n), \tag{3.44}$$

and using that $[0,1) \ni \theta \mapsto \underline{W}^{(\theta)}$ is continuous with values in $C^1([0,R^n])$ by Lemma 3.1, we can thereupon find $\varepsilon > 0$ such that whenever $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon) \cap [0,1)$, we have

$$\left| \underline{W}_s^{(\theta)}(s) - \underline{W}_s^{(\theta_0)}(s) \right| \leq \frac{c_1 R^n}{2} \qquad \text{for all } s \in (0, R^n). \tag{3.45}$$

Since $\underline{W}^{(\theta)}(0) = 0$ and $\underline{W}^{(\theta)}(R^n) = \frac{m}{\omega_n}$ for all $\theta \in [0,1)$ by Lemma 3.1, this guarantees that for all $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon) \cap [0,1)$,

$$\begin{aligned}
\underline{W}^{(\theta)}(s) - \underline{W}^{(\theta_0)}(s) &= \int_0^s \Big\{ \underline{W}_s^{(\theta)}(\sigma) - \underline{W}_s^{(\theta_0)}(\sigma) \Big\} d\sigma \\
&\leq \frac{c_1 R^n}{2} \cdot s \\
&\leq c_1 s(R^n - s) \qquad \text{for all } s \in \Big(0, \frac{R^n}{2}\Big],
\end{aligned}$$

and that, similarly,

$$\begin{aligned}
\underline{W}^{(\theta)}(s) - \underline{W}^{(\theta_0)}(s) &= - \int_s^{R^n} \Big\{ \underline{W}_s^{(\theta)}(\sigma) - \underline{W}_s^{(\theta_0)}(\sigma) \Big\} d\sigma \\
&\leq \frac{c_1 R^n}{2} \cdot (R^n - s) \\
&\leq c_1 s(R^n - s) \qquad \text{for all } s \in \Big(\frac{R^n}{2}, R^n\Big).
\end{aligned}$$

Together with (3.44), these inequalities show that indeed $(\theta_0 - \varepsilon, \theta_0 + \varepsilon) \cap [0,1) \subset S$ and that consequently, S as a nonempty relatively closed and relatively open subset of $[0,1)$ must actually coincide with $[0,1)$. Thus, $W \geq \underline{W}^{(\theta)}$ in $(0,R^n)$ for all $\theta \in [0,1)$, which implies (3.42) due to the fact that from Lemma 3.1 we moreover know that $\underline{W}^{(\theta)}(s) \to \frac{m}{\omega_n}$ for all $s \in (0,R^n)$ as $\theta \nearrow 1$. $\square$

We are now in the position to make sure that as a consequence of Lemma 3.5, the global nonexistence criteria from Section 3.2 warrant finite-time collapse of any solution to (2.6) corresponding to initial data $\underline{W}^{(\theta)}$ with $\theta \in (0,1)$. Our verification thereof will make essential use of a comparison principle for problems of the form (2.6); as the latter will independently be applied in several subsequent arguments (cf. the proofs of Lemmata 3.12, 4.1 and 4.3 as well as Proposition 1.2 and Theorem 1.1), for convenient reference we postpone its formulation so as to become Lemma 5.1 from the appendix below.

Lemma 3.6 *Given $n \geq 2$ and $R > 0$, let $m^\star > 0$ and, for $m > m^\star$, the family $(\underline{W}^{(\theta)})_{\theta \in [0,1)} \subset C^1([0,R^n]) \cap W^{2,\infty}((0,R^n))$ be as in Lemma 3.1. For $\theta \in [0,1)$, let $w^{(\theta)} \in C^0([0,T_{max}^{(\theta)}); C^1([0,R^n])) \cap C^{2,1}((0,R^n] \times (0,T_{max}^{(\theta)}))$ denote the local solution of (2.6), as obtained through Lemma 2.1 and (2.3) when applied to the initial data given by $u_0(r) := n\underline{W}_s^{(\theta)}(r^n)$, $r \in [0,R]$, and extended up to its maximal* *existence time $T^{(\theta)}_{max} \in (0,\infty]$. Then actually $T^{(\theta)}_{max} < \infty$ for all $\theta \in (0,1)$, and for any such θ there exists a nondecreasing $W^{(\theta)}: \ (0,R^n] \to (0,\frac{m}{\omega_n}]$ such that*

$$w^{(\theta)}(s,t) \to W^{(\theta)}(s) \quad \text{for all } s \in (0,R^n) \qquad \text{as } t \nearrow T^{(\theta)}_{max}, \tag{3.46}$$

and that

$$\limsup_{s\searrow 0} \frac{W^{(\theta)}(s)}{s} = +\infty. \tag{3.47}$$

Proof. Let us first make sure that for $w := w^{(\theta)}$ we have

$$w_t \ge 0 \qquad \text{in } (0,R^n) \times (0,T^{(\theta)}_{max}). \tag{3.48}$$

Indeed, this can be seen by adaptation of a standard argument ([26, Section 52.6]) to the present setting: Given $T \in (0,T^{(\theta)}_{max})$ and $h \in (0,T)$ we let $\kappa := n\|w_s\|_{L^\infty((0,R^n)\times(0,T))}+1$ and $z(s,t) := w(s,t+h) - w(s,t) + \varepsilon e^{\kappa t}$ for $s \in [0,R^n]$, $t \in [0,T-h)$ and $\varepsilon > 0$, and observe that then $z(s,0) \ge \varepsilon > 0$ for all $s \in (0,R^n)$, because $w(s,h) \ge w(s,0) = \underline{W}^{(\theta)}(s)$ due to the subsolution property of $\underline{W}^{(\theta)}$ expressed in (3.1) and the comparison principle from Lemma 5.1. Moreover, $z(0,t) = z(R^n,t) = \varepsilon e^{\kappa t} > 0$ for all $t \in (0,T-h)$, and using (2.6) we compute

$$z_t(s,t) = n^2 s^{2-\frac{2}{n}} z_{ss}(s,t) + nw(s,t+h)z_s(s,t) + nw_s(s,t)z(s,t) - \mu s z_s(s,t) - n\varepsilon w_s(s,t)e^{\kappa t} + \kappa\varepsilon e^{\kappa t} \tag{3.49}$$

for $s \in (0,R^n)$ and $t \in (0,T-h)$. Therefore,

$$t_0 := \sup\Big\{\hat t \in (0,T-h) \ \Big| \ z(s,t) > 0 \text{ for all } s \in [0,R^n] \text{ and } t \in [0,\hat t]\Big\}$$

is well-defined with $t_0 \in (0,T-h]$, and if $t_0 < T-h$ then necessarily $z(s_0,t_0) = 0$ for some $s_0 \in [0,R^n]$ which in fact clearly must satisfy $s_0 \in (0,R^n)$, whence we actually know that $z_s(s_0,t_0) = 0$ and $z_{ss}(s_0,t_0) \ge 0$ as well as $z_t(s_0,t_0) \le 0$. From (3.49) we thus infer that in this case

$$\begin{aligned} 0 \ge z_t(s_0,t_0) &\ge -n\varepsilon w_s(s_0,t_0)e^{\kappa t_0} + \kappa\varepsilon e^{\kappa t_0} \\ &\ge \Big\{\kappa - n\|w_s\|_{L^\infty((0,R^n)\times(0,T))}\Big\} \cdot \varepsilon e^{\kappa t_0} \\ &> 0, \end{aligned}$$

which is absurd and hence proves that actually $t_0 = T-h$, implying that $z > 0$ in $(0,R^n) \times (0,T-h)$. On taking $\varepsilon \searrow 0$, then $h \searrow 0$ and finally $T \nearrow T^{(\theta)}_{max}$, from this we readily obtain (3.48).

As a first consequence thereof, we can now verify that w cannot exist globally: In fact, assuming for contradiction that $T^{(\theta)}_{max} = \infty$, using (3.48) we could find a nondecreasing function W defined on $(0,R^n]$, fulfilling $0 \le W \le \frac{m}{\omega_n}$, such that

$$\underline{W}^{(\theta)}(s) \le w(s,t) \nearrow W(s) \quad \text{for all } s \in (0,R^n) \qquad \text{as } t \to \infty. \tag{3.50}$$

Here due to the boundedness of w on $(0,R^n) \times (0,\infty)$, interior parabolic Schauder theory ([17]) ensures that $(w(\cdot,t))_{t>1}$ is bounded in $C^3_{loc}((0,R^n])$ and that thus, by the Arzelà-Ascoli theorem, W actually belongs to $C^2((0,R^n])$ with $W_s \ge 0$ in $(0,R^n)$ and $W(R^n)=\frac{m}{\omega_n}$. As clearly $\int_0^\infty \int_0^{R^n} |w_t(s,t)|dsdt \le \frac{mR^n}{\omega_n} < \infty$ and thus $\liminf_{t\to\infty} \|w_t(\cdot,t)\|_{L^1((0,R^n))}=0$, from (3.50) and (2.6) we moreover infer that $n^2 s^{2-\frac{2}{n}} W_s s + nWW_s - \mu s W_s = 0$ in $(0,R^n)$, so that Lemma 3.5 applies so as to assert that in fact $W \equiv \frac{m}{\omega_n}$ throughout $(0,R^n)$, by (3.50) meaning that

$$w(s,t) \to \frac{m}{\omega_n} \quad \text{for all } s \in (0,R^n) \qquad \text{as } t \to \infty. \tag{3.51}$$

Relying on the mere positivity of m when $n \ge 3$, and on the fact that $m > m^\star \ge 8\pi$ if $n=2$, in view of Lemma 3.2 and Lemma 3.3 we see that in both cases (3.51) is impossible and that thus indeed $T_{max}^{(\theta)}$ must be finite.

Therefore, again by means of (3.48) we infer the existence of $W^{(\theta)}: (0,R^n] \to (0,\frac{m}{\omega_n}]$ such that

$$\underline{W}^{(\theta)}(s) \le w(s,t) \nearrow W^{(\theta)}(s) \quad \text{for all } s \in (0,R^n) \qquad \text{as } t \nearrow T_{max}^{(\theta)}, \tag{3.52}$$

which directly establishes (3.46) and can be seen to furthermore entail (3.47) as follows: If (3.47) was false and thus

$$W^{(\theta)}(s) \le c_1 s \qquad \text{for all } s \in (0,R^n)$$

with some $c_1 > 0$, then clearly

$$w(s,t) \le c_1 s \qquad \text{for all } s \in (0,R^n) \text{ and } t \in (0,T_{max}^{(\theta)}) \tag{3.53}$$

by (3.52). Once more using (3.48), now rewritten through (2.6) in the form

$$n^2 s^{2-\frac{2}{n}} w_{ss}(s,t) + nw(s,t)w_s(s,t) - \mu s w_s(s,t) \ge 0 \qquad \text{for all } s \in (0,R^n) \text{ and } t \in (0,T_{max}^{(\theta)}),$$

from (3.53) we would obtain that

$$\begin{aligned} n^2 s^{2-\frac{2}{n}} w_{ss}(s,t) &\ge -nc_1 s w_s(s,t) + \mu s w_s(s,t) \\ &\ge -nc_1 s w_s(s,t) \qquad \text{for all } s \in (0,R^n) \text{ and } t \in (0,T_{max}^{(\theta)}) \end{aligned}$$

and hence, after integration,

$$\begin{aligned} w_s(R^n,t) &\ge w_s(s,t) \cdot e^{-\frac{c_1}{n}\int_s^{R^n} \sigma^{-1+\frac{2}{n}} d\sigma} \\ &= w_s(s,t) \cdot e^{-\frac{c_1}{2}\cdot(R^2 - s^{\frac{2}{n}})} \qquad \text{for all } s \in (0,R^n) \text{ and } t \in (0,T_{max}^{(\theta)}). \end{aligned}$$

This, however, would imply that

$$w_s(s,t) \le e^{\frac{c_1 R^2}{2}} \cdot w_s(R^n,t) \qquad \text{for all } s \in (0,R^n) \text{ and } t \in (0,T_{max}^{(\theta)}),$$

and that thus, since

$$w_s(R^n,t) = \lim_{s \nearrow R^n} \frac{\frac{m}{\omega_n} - w(s,t)}{R^n - s} \le \lim_{s \nearrow R^n} \frac{\frac{m}{\omega_n} - \underline{W}^{(\theta)}(s)}{R^n - s} = c_2 := \underline{W}_s^{(\theta)}(R^n) \qquad \text{for all } t \in (0,T_{max}^{(\theta)})$$

by (3.52), we would have

$$\|w_s(\cdot,t)\|_{L^\infty((0,R^n))} \le c_2 e^{\frac{c_1 R^2}{2}} \qquad \text{for all } t \in (0,T_{max}^{(\theta)}),$$

which is incompatible with the blow-up criterion (2.2) and thereby shows that in fact (3.47) must be valid. □

3.4 Positivity and boundary behavior in an auxiliary degenerate problem

The goal of this section is to prepare a comparison argument, to be performed in Lemma 3.12 below, according to which it will be possible to relate the latter findings to solutions evolving from arbitrary initial data merely fulfilling the requirements from Theorem 1.1. For this purpose, we shall briefly consider a pure degenerate diffusion problem which arises as the principal part in the linearization of (2.6) about the steady state $s \mapsto \frac{\mu}{n}s$ (see (3.76)), and which is specified in the following basic observation.

Lemma 3.7 *Let $n \geq 2$, $R > 0$, $\underline{z}_0 \in C_0^\infty((0,R^n))$ be nonnegative and $\delta_0 \in (0,R^n)$ be such that $\underline{z}_0 \equiv 0$ in $[0,\delta_0]$. Then for each $\delta \in (0,\delta_0)$, the problem*

$$\begin{cases} \underline{z}_t^{(\delta)} = n^2 s^{2-\frac{2}{n}} \underline{z}_{ss}^{(\delta)}, & s \in (\delta, R^n),\ t > 0, \\ \underline{z}^{(\delta)}(\delta,t) = \underline{z}^{(\delta)}(R^n,t) = 0, & t > 0, \\ \underline{z}^{(\delta)}(s,0) = \underline{z}_0(s), & s \in (\delta, R^n), \end{cases} \tag{3.54}$$

admits a unique classical solution $\underline{z}^{(\delta)} \in C^\infty([\delta,R^n] \times [0,\infty))$.

Proof. Since each of the parabolic problems (3.54) is non-degenerate, and since our assumption on $\underline{z}_0$ along with our choice of δ_0 warrants that the associated compatibility conditions of arbitrary order are fulfilled, this directly follows from standard parabolic theory ([17]). $\Box$

A first comparison argument asserts temporal decay of this solution, along with some temporally uniform smallness property near the origin, in the following sense.

Lemma 3.8 *Let $n \geq 2$, $R > 0$, $0 \leq \underline{z}_0 \in C_0^\infty((0,R^n))$ and $\delta_0 \in (0,R^n)$ be as in Lemma 3.7. Then for any $\gamma \in (0,1)$ one can find $\kappa > 0$ and $C > 0$ such that for all $\delta \in (0,\delta_0)$, the solution $\underline{z}^{(\delta)}$ of (3.54) satisfies*

$$\underline{z}^{(\delta)}(s,t) \leq C s^\gamma e^{-\kappa t} \qquad \text{for all } s \in (\delta,R^n) \text{ and } t > 0. \tag{3.55}$$

Proof. Given $\gamma \in (0,1)$, we can pick $\kappa > 0$ such that

$$\kappa \leq \frac{n^2\gamma(1-\gamma)}{R^2}, \tag{3.56}$$

and since $\underline{z}_0 \in C_0^\infty((0,R^n))$ we can moreover fix $y_0 > 0$ in such a way that

$$y_0 \geq s^{-\gamma} \underline{z}_0(s) \qquad \text{for all } s \in (0,R^n). \tag{3.57}$$

Then writing

$$y(t) := y_0\, e^{-\kappa t}, \qquad t \geq 0, \tag{3.58}$$

and

$$\widehat{z}(s,t) := y(t) s^\gamma, \qquad s \in [0,R^n],\ t \geq 0,$$

from (3.57) we obtain that

$$\widehat{z}(s,0) = y_0 s^\gamma \geq \underline{z}_0(s) \qquad \text{for all } s \in (0,R^n),$$

whereas (3.56) warrants that

$$\begin{aligned}
\widehat{z}_t - n^2 s^{2-\frac{2}{n}} \widehat{z}_{ss} &= y' s^\gamma + n^2\gamma(1-\gamma) s^{\gamma-\frac{2}{n}} y \\
&\ge s^\gamma \cdot \Big\{ y' + \frac{n^2\gamma(1-\gamma)}{R^2} \cdot y \Big\} \\
&= 0 \qquad \text{in } (0,R^n) \times (0,\infty),
\end{aligned}$$

because $y' = -\kappa y$ by (3.58). As for $\delta \in (0,\delta_0)$ we clearly have $\widehat{z} \ge 0 = \underline{z}^{(\delta)}$ on $\{\delta, R^n\} \times (0,\infty)$, we may thus invoke a standard comparison principle to see that for any such δ, $\widehat{z} \ge \underline{z}^{(\delta)}$ in $(\delta, R^n) \times (0,\infty)$, which precisely yields (3.55) if we let $C := y_0$. $\square$

Apart from that, a second comparison argument enables us to conveniently control some derivatives of this solution in a pointwise sense.

Lemma 3.9 *Let $n \ge 2$, $R > 0$, $0 \le \underline{z}_0 \in C_0^\infty((0,R^n))$ and $\delta_0 \in (0,R^n)$ be as in Lemma 3.7. Then there exists $C > 0$ such that for any $\delta \in (0,\delta_0)$,*

$$|\underline{z}_{ss}^{(\delta)}(s,t)| \le C s^{-1+\frac{2}{n}} \qquad \textit{for all } s \in (\delta,R^n) \textit{ and each } t > 0 \tag{3.59}$$

as well as

$$|\underline{z}_t^{(\delta)}(s,t)| \le C \qquad \textit{for all } s \in (\delta,R^n) \textit{ and } t > 0. \tag{3.60}$$

Proof. Using that $\underline{z}_0$ belongs to $C^2([0,R^n])$, we can find $c_1 > 0$ such that

$$s^{1-\frac{2}{n}} |\underline{z}_{0ss}(s)| \le c_1 \qquad \text{for all } s \in (0,R^n), \tag{3.61}$$

and let

$$\overline{\psi}(s,t) := c_1 s^{-1+\frac{2}{n}}, \qquad s \in (0,R^n],\ t \ge 0. \tag{3.62}$$

Then for each $\delta \in (0,\delta_0)$,

$$\overline{\psi}_t - n^2 \cdot (s^{2-\frac{2}{n}} \overline{\psi})_{ss} = -n^2 \partial_s^2 s = 0 \qquad \text{in } (\delta,R^n) \times (0,\infty)$$

and $\overline{\psi} > 0$ on $\{\delta, R^n\} \times (0,\infty)$, while by (3.54), for any such δ the function $\psi^{(\delta)} := \underline{z}_{ss}^{(\delta)}$ satisfies

$$\psi_t^{(\delta)} - n^2 \cdot (s^{2-\frac{2}{n}} \psi^{(\delta)})_{ss} = 0 \qquad \text{in } (\delta,R^n) \times (0,\infty)$$

with $\psi^{(\delta)}(\delta,t) = \psi^{(\delta)}(R^n,t) = 0$ for all $t > 0$, because $\underline{z}_t^{(\delta)}(\delta,t) = \underline{z}_t^{(\delta)}(R^n,t) = 0$ for all $t > 0$ due to the constant Dirichlet data in (3.54). Since (3.61) guarantees that furthermore

$$\psi^{(\delta)}(s,0) \le c_1 s^{-1+\frac{2}{n}} = \overline{\psi}(s,0) \qquad \text{for all } s \in (\delta,R^n),$$

by comparison we thus infer that $\psi^{(\delta)} \le \overline{\psi}$ in $(\delta,R^n) \times (0,\infty)$, and in quite a similar manner one can derive the lower estimate $\psi^{(\delta)} \ge -\overline{\psi}$ in $(\delta,R^n) \times (0,\infty)$ for all $\delta \in (0,\delta_0)$. In view of (3.62), this hence establishes (3.59), from which in turn (3.60) immediately results in view of (3.54). $\square$

Combining the latter two lemmata with parabolic Schauder theory, we can readily pass to the limit $\delta \searrow 0$ along a suitable subsequence so as to obtain a solution to the corresponding limit problem in the following sense.

Lemma 3.10 *Let $n \geq 2$, $R > 0$, $0 \leq \underline{z}_0 \in C_0^\infty((0, R^n))$ and $\delta_0 \in (0, R^n)$ be as in Lemma 3.7. Then there exist $(\delta_j)_{j\in\mathbb{N}} \subset (0, \delta_0)$ and a nonnegative function $\underline{z} \in C^0([0, R^n] \times [0, \infty)) \cap C^{2,1}((0, R^n] \times [0, \infty))$ such that $\underline{z}_s \in C^0([0, \mathbb{R}^n] \times [0, \infty))$, that $\delta_j \searrow 0$ as $j \to \infty$, that*

$$\underline{z}^{(\delta)} \to \underline{z} \text{ in } C^0_{loc}([0, R^n] \times [0, \infty)) \cap C^{2,1}_{loc}((0, R^n] \times [0, \infty)) \qquad \text{as } \delta = \delta_j \searrow 0, \tag{3.63}$$

and that $\underline{z}$ is a classical solution of

$$\begin{cases} \underline{z}_t = n^2 s^{2-\frac{2}{n}} \underline{z}_{ss}, & s \in (0, R^n),\ t > 0, \\ \underline{z}(0, t) = \underline{z}(R^n, t) = 0, & t > 0, \\ \underline{z}(s, 0) = \underline{z}_0(s), & s \in (0, R^n). \end{cases} \tag{3.64}$$

Proof. Since for each $\delta_\star \in (0, \delta_0)$ we know from Lemma 3.8 that $(\underline{z}^{(\delta)})_{\delta\in(0,\delta_\star)}$ is bounded in $C^0_{loc}([\delta_\star, R^n] \times [0, \infty))$, parabolic Schauder theory ([17]) combined with the Arzelà-Ascoli theorem asserts that for any such $\delta_\star$,

$$(\underline{z}^{(\delta)})_{\delta\in(0,\delta_\star)} \quad \text{is relatively compact in } C^{2,1}_{loc}([\delta_\star, R^n] \times [0, \infty)). \tag{3.65}$$

Hence, along an appropriate sequence $(\delta_j)_{j\in\mathbb{N}} \subset (0, \delta_0)$ fulfilling $\delta_j \searrow 0$ as $j \to \infty$, for some nonnegative $\underline{z} \in C^{2,1}((0, R^n] \times [0, \infty))$ we have

$$\underline{z}^{(\delta)} \to \underline{z} \text{ in } C^{2,1}_{loc}((0, R^n] \times [0, \infty)) \qquad \text{as } \delta = \delta_j \searrow 0, \tag{3.66}$$

which due to (3.54) also implies that $\underline{z}_t = n^2 s^{2-\frac{2}{n}} \underline{z}_{ss}$ in $(0, R^n) \times (0, \infty)$ with $\underline{z}(R^n, t) = 0$ for all $t > 0$ and $\underline{z}(s, 0) = \underline{z}_0(s)$ for each $s \in (0, R^n)$. In order to show that furthermore $\underline{z}$ and $\underline{z}_s$ belong to $C^0([0, R^n] \times [0, \infty))$ with $\underline{z}(0, t) = 0$ for all $t > 0$, we note that for each $\delta \in (0, \delta_0)$, $\phi_\delta(\sigma) := \delta + \frac{R^n - \delta}{R^n} \cdot \sigma$, $\sigma \in [0, R^n]$, defines an affine bijection of $[0, R^n]$ onto $[\delta, R^n]$, and letting

$$\psi^{(\delta)}(\sigma, t) := \underline{z}^{(\delta)}(\phi_\delta(\sigma), t), \qquad \sigma \in [0, R^n],\ t \geq 0,$$

we readily infer from Lemma 3.9 and evident properties of $(\phi_\delta)_{\delta\in(0,\delta_0)}$ that there exist $c_1 > 0$ and $c_2 > 0$ such that for any choice of $\delta \in (0, \delta_0)$ we have

$$|\psi^{(\delta)}_{\sigma\sigma}(\sigma, t)| \leq c_1 \sigma^{-1+\frac{2}{n}} \qquad \text{for all } \sigma \in (0, R^n) \text{ and } t > 0 \tag{3.67}$$

as well as

$$|\psi^{(\delta)}_t \sigma, t)| \leq c_2 \qquad \text{for all } \sigma \in (0, R^n) \text{ and } t > 0. \tag{3.68}$$

In particular, (3.67) warrants that if $\delta \in (0, \delta_0)$, $t > 0$ and $0 < \sigma_1 < \sigma_2 < R^n$, then

$$\begin{aligned} \left|\psi^{(\delta)}_\sigma(\sigma_2, t) - \psi^{(\delta)}_\sigma(\sigma_1, t)\right| &= \left|\int_{\sigma_1}^{\sigma_2} \psi^{(\delta)}_{\sigma\sigma}(\sigma, t) d\sigma\right| \\ &\leq c_1 \int_{\sigma_1}^{\sigma_2} \sigma^{-1+\frac{2}{n}} d\sigma \\ &= \frac{nc_1}{2} \cdot (\sigma_2^{\frac{2}{n}} - \sigma_1^{\frac{2}{n}}) \\ &\leq \frac{nc_1}{2} \cdot |\sigma_2 - \sigma_1|^{\frac{2}{n}}, \end{aligned}$$

together with e.g. (3.65) implying the existence of $c_3 > 0$ such that

$$\|\psi^{(\delta)}(\cdot,t)\|_{C^{1+\frac{2}{n}}([0,R^n])} \le c_3 \qquad \text{for all } t>0 \text{ and each } \delta \in (0,\delta_0). \tag{3.69}$$

Now due to an Ehrling-type inequality associated with the the embeddings $C^{1+\frac{2}{n}}([0,R^n]) \hookrightarrow\hookrightarrow C^1([0,R^n]) \hookrightarrow C^0([0,R^n])$, given $\varepsilon > 0$ we can find $c_4(\varepsilon) > 0$ such that

$$\|\varphi\|_{C^1([0,R^n])} \le \frac{\varepsilon}{4c_3}\|\varphi\|_{C^{1+\frac{2}{n}}([0,R^n])} + c_4(\varepsilon)\|\varphi\|_{C^0([0,R^n])} \qquad \text{for all } \varphi \in C^{1+\frac{2}{n}}([0,R^n]).$$

Therefore, if we let $\eta = \eta(\varepsilon) := \frac{\varepsilon}{2c_2c_4}$, then for arbitrary $\delta \in (0,\delta_0)$ and any $t_1 > 0$ and $t_2 \in (t_1, t_1+\eta)$ we can combine (3.69) with (3.68) to estimate

$$\begin{aligned}
\Big\|\psi^{(\delta)}(\cdot,t_2) - \psi^{(\delta)}(\cdot,t_1)\Big\|_{C^1([0,R^n])} &\le \frac{\varepsilon}{2c_3}\Big\|\psi^{(\delta)}(\cdot,t_2) - \psi^{(\delta)}(\cdot,t_1)\Big\|_{C^{1+\frac{2}{n}}([0,R^n])} \\
&\quad + c_4\Big\|\psi^{(\delta)}(\cdot,t_2) - \psi^{(\delta)}(\cdot,t_1)\Big\|_{C^0([0,R^n])} \\
&\le \frac{\varepsilon}{2c_3}\cdot\Big\{\|\psi^{(\delta)}(\cdot,t_2)\|_{C^{1+\frac{2}{n}}([0,R^n])} + \|\psi^{(\delta)}(\cdot,t_1)\|_{C^{1+\frac{2}{n}}([0,R^n])}\Big\} \\
&\quad + c_4\int_{t_1}^{t_2}\|\psi_t^{(\delta)}(\cdot,t)\|_{C^0([0,R^n])}dt \\
&\le \frac{\varepsilon}{2c_3}\cdot(c_3+c_3) + c_4\cdot c_2(t_2-t_1) \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

and to thereby obtain precompactness of both $(\psi^{(\delta)})_{\delta\in(0,\delta_0)}$ and $(\psi_\sigma^{(\delta)})_{\delta\in(0,\delta_0)}$ in $C^0_{loc}([0,R^n]\times[0,\infty))$ as a consequence of the Arzelà-Ascoli theorem. Since again due to the properties of $(\phi_\delta)_{\delta\in(0,\delta_0)}$ it follows from (3.66) that for each fixed $\sigma \in (0,R^n)$ and $t>0$ we have $\psi^{(\delta)}(\sigma,t) \to \underline{z}(\sigma,t)$ as $\delta = \delta_j \searrow 0$, this readily implies that both $\underline{z}$ and $\underline{z}_s$ must indeed be continuous on $[0,R^n]\times[0,\infty)$, and that since $\psi^{(\delta)}(0,t) = 0$ for all $t>0$ we thus also have $\underline{z}(0,t) = 0$ for any such t. $\Box$

Now the main outcome of this section states that not only this limit solution but also essentially all its approximates enjoy a Hopf-type boundary point property near the degeneracy point $s=0$ of (3.64) in the following temporally averaged form.

Lemma 3.11 *Let $n \ge 2$, $R>0$, $0 \le \underline{z}_0 \in C_0^\infty((0,R^n))$ and $\delta_0 \in (0,R^n)$ be as in Lemma 3.7. Then there exist $t_\star > 0$, $s_\star \in (0,\delta_0)$ and $C>0$ such that whenever $\delta \in (0,s_\star)$, for the solution $\underline{z}^{(\delta)}$ of (3.54) we have*

$$\int_0^{t_\star}\underline{z}_s^{(\delta)}(s,t)dt \ge C \qquad \textit{for all } s \in (\delta,s_\star). \tag{3.70}$$

Proof. We first fix $\gamma \in (0,1)$ sufficiently close to 1 such that $\gamma > 1-\frac{2}{n}$, and then obtain from Lemma 3.8 that there exist $\kappa > 0$ and $c_1 > 0$ such that for all $\delta \in (0,\delta_0)$,

$$\underline{z}^{(\delta)}(s,t) \le c_1 s^\gamma e^{-\kappa t} \qquad \text{for all } s \in (\delta,R^n) \text{ and } t>0. \tag{3.71}$$

Moreover abbreviating $c_2 := \frac{R^{n\gamma+2}}{\gamma-1+\frac{2}{n}}c_1$ and $c_3 := \int_0^{R^n} s^{-2+\frac{2}{n}}(R^n-s)\underline{z}_0(s)ds$, from our assumptions on $\underline{z}_0$ we know that c_3 is positive and that hence we can pick $t_\star > 0$ such that

$$t_\star \ge -\frac{1}{\kappa}\ln\frac{c_3}{4c_2}. \tag{3.72}$$

Finally choosing $s_\star \in (0,\delta_0)$ small enough fulfilling

$$s_\star \le \Big(\frac{\kappa c_3}{4c_1}\Big)^{\frac{1}{\gamma}}, \tag{3.73}$$

for $\delta \in (0,\delta_0)$ and $s_0 \in (\delta, s_\star)$ we integrate by parts in (3.54) to compute

$$\begin{aligned}
\frac{d}{dt}\int_{s_0}^{R^n} s^{-2+\frac{2}{n}}(R^n-s)\underline{z}^{(\delta)}(s,t)ds &= \int_{s_0}^{R^n}(R^n-s)\underline{z}^{(\delta)}_{ss}(s,t)ds \\
&= \int_{s_0}^{R^n}\underline{z}^{(\delta)}_s(s,t)ds-(R^n-s_0)\underline{z}^{(\delta)}_s(s_0,t) \\
&= -\underline{z}^{(\delta)}(s_0,t)-(R^n-s_0)\underline{z}^{(\delta)}_s(s_0,t) \qquad \text{for all } t>0,
\end{aligned}$$

which on further integration yields

$$\begin{aligned}
R^n\int_0^{t_\star}\underline{z}^{(\delta)}_s(s_0,t)dt &\ge (R^n-s_0)\int_0^{t_\star}\underline{z}^{(\delta)}_s(s_0,t)dt \\
&\ge \int_{s_0}^{R^n}s^{-2+\frac{2}{n}}(R^n-s)\underline{z}_0(s)ds-\int_{s_0}^{R^n}s^{-2+\frac{2}{n}}(R^n-s)\underline{z}^{(\delta)}(s,t_\star)ds \\
&\quad -\int_0^{t_\star}\underline{z}^{(\delta)}(s_0,t)dt.
\end{aligned} \tag{3.74}$$

Here we observe that thanks to (3.71) and (3.72),

$$\begin{aligned}
\int_{s_0}^{R^n}s^{-2+\frac{2}{n}}(R^n-s)\underline{z}^{(\delta)}(s,t_\star)ds &\le c_1R^ne^{-\kappa t_\star}\int_{s_0}^{R^n}s^{\gamma-2+\frac{2}{n}}ds \\
&= c_1R^ne^{-\kappa t_\star}\cdot\frac{R^{n\gamma-n+2}-s_0^{\gamma-1+\frac{2}{n}}}{\gamma-1+\frac{2}{n}} \\
&\le c_2e^{-\kappa t_\star} \\
&\le \frac{c_3}{4}
\end{aligned}$$

and that by (3.71) and (3.73),

$$\int_0^{t_\star}\underline{z}^{(\delta)}(s_0,t)dt \le c_1s_0^\gamma\int_0^{t_\star}e^{-\kappa t}dt = c_1s_0^\gamma\cdot\frac{1-e^{-\kappa t_\star}}{\kappa} \le \frac{c_1}{\kappa}s_0^\gamma \le \frac{c_3}{4}.$$

As $\operatorname{supp} z_0 \subset (\delta_0, R^n)$, in view of our definition of c_3 the inequality (3.74) therefore entails that for any $\delta \in (0,\delta_0)$,

$$R^n\int_0^{t_\star}\underline{z}^{(\delta)}_s(s_0,t)dt \ge c_3-\frac{c_3}{4}-\frac{c_3}{4}=\frac{c_3}{2} \qquad \text{for all } s\in(\delta,s_\star),$$

which evidently implies (3.70). □

3.5 Proof of Theorem 1.1

In order to show that in the situation of Theorem 1.1 the corresponding solution w cannot be global in time, in view of the results from Section 3.3 it thus remains to make sure that possibly after some waiting time we may assume that any such solution, trivially remaining above the equilibrium $(s,t) \mapsto \frac{\mu}{n}s$ by comparison, actually deviates from the latter by such a substantial amount that it can actually compared from below by one of the particular solutions $w^{(\theta)}$ addressed in Lemma 3.6. This will be achieved on the basis of the boundary point property from Lemma 3.11 in the course of the following argument.

Lemma 3.12 *Let $n \ge 2$ and $R > 0$, and suppose that u_0 satisfies (1.2) as well as (1.5) with $m := \int_\Omega u_0$, but that the corresponding solution (u,v) of (1.1) from Lemma 2.1 is global in time. Then one can find $t_0 > 0$ and $C > 0$ such that for w as in (2.3) we have*

$$w(s,t_0) \ge \frac{\mu}{n}s + Cs(R^n - s) \qquad \textit{for all } s \in (0,R^n). \tag{3.75}$$

PROOF. We let $z(s,t) := w(s,t) - \frac{\mu}{n}s$ for $s \in [0,R^n]$ and $t \ge 0$, and then from (1.5), (2.6) and the comparison principle in Lemma 5.1 we obtain that $z \in C^0([0,R^n]\times[0,\infty)) \cap C^{2,1}((0,R^n]\times(0,\infty))$ is a nonnegative function fulfilling $z(R^n,t) = 0$ for all $t > 0$ as well as

$$\begin{aligned} z_t &= n^2 s^{2-\frac{2}{n}} w_{ss} + (nw - \mu s) w_s \\ &= n^2 s^{2-\frac{2}{n}} z_{ss} + nzw_s \\ &\ge n^2 s^{2-\frac{2}{n}} z_{ss} \qquad \text{in } (0,R^n)\times(0,\infty), \end{aligned} \tag{3.76}$$

because $w_s \ge 0$. Since our hypothesis (1.5) moreover says with w_0 as defined in (2.4) we have $w_0 \not\equiv \frac{\mu}{n}(\cdot)$, we can find a nontrivial $\underline{z}_0 \in C_0^\infty((0,R^n))$ such that $0 \le \underline{z}_0(s) \le w_0(s) - \frac{\mu}{n}s$ for all $s \in (0,R^n)$, whence Lemma 3.7 and Lemma 3.10 apply so as to yield $\delta_0 \in (0,R^n)$ as well as solutions $\underline{z}^{(\delta)}$, $\delta \in (0,\delta_0)$, and $\underline{z}$ of (3.54) and (3.64), respectively. About these solutions, Lemma 3.11 says that there exist $t_\star > 0$, $s_\star \in (0,\delta_0)$ and $c_1 > 0$ such that

$$\int_0^{t_\star} \underline{z}_s^{(\delta)}(s,t)dt \ge c_1 \qquad \text{whenever } 0 < \delta < s < s_\star,$$

which on taking $\delta = \delta_j \searrow 0$ with $(\delta_j)_{j\in\mathbb{N}} \subset (0,\delta_0)$ as provided by Lemma 3.10 shows that

$$\int_0^{t\star} \underline{z}_s(s,t)dt \ge c_1 \qquad \text{for all } s \in (0,s_\star)$$

and hence

$$\int_0^{t_\star} \underline{z}_s(0,t)dt \ge c_1.$$

We can therefore find $t_0 \in (0,t_\star)$ such that $\underline{z}_s(0,t_0) \ge c_2 := \frac{c_1}{t_\star}$, which means that

$$\frac{\underline{z}(s,t_0)}{s} = \frac{\underline{z}(s,t_0) - \underline{z}(0,t_0)}{s} \to c_2 \qquad \text{as } s \searrow 0$$

and that hence we can fix $s_1 \in (0, R^n)$ such that

$$\underline{z}(s, t_0) \geq \frac{c_2}{2} s \qquad \text{for all } s \in (0, s_1). \tag{3.77}$$

As clearly $c_3 := -\underline{z}_s(R^n, t_0)$ is positive due to the Hopf boundary point lemma applied to the point $s = R^n$ near which (3.64) is uniformly parabolic, we can similarly fix $s_2 \in (s_1, R^n)$ fulfilling

$$\underline{z}(s, t_0) \geq \frac{c_3}{2}(R^n - s) \qquad \text{for all } s \in (s_2, R^n). \tag{3.78}$$

Since finally, by uniform parabolicity of (3.64) in an appropriate open neighborhood of $[s_1, s_2] \times \{t_0\}$, the classical strong maximum principle applies so as to assert positivity of $c_4 := \min_{s \in [s_1, s_2]} \underline{z}(s, t_0)$, on combining this with (3.77) and (3.78) we obtain that

$$\underline{z}(s, t_0) \geq c_5 s(R^n - s) \qquad \text{for all } s \in (0, R^n)$$

with $c_5 := \min\{\frac{c_2}{2R^n}, \frac{c_3}{2R^n}, \frac{4c_4}{R^{2n}}\}$. Therefore, (3.75) is a consequence of the observation that for all $\delta \in (0, \delta_0)$, the classical comparison principle ensures that $z \geq \underline{z}^{(\delta)}$ in $(\delta, R^n) \times (0, \infty)$, by Lemma 3.10 namely implying that $z(s, t_0) \geq \underline{z}(s, t_0)$ for all $s \in (0, R^n)$. □

As a last preparation for our verification of Theorem 1.1, without proof let us state the following elementary observation.

Lemma 3.13 *Let $n \geq 1$ and $R > 0$, and let $(\varphi_j)_{j \in \mathbb{N}} \subset C^1([0, R^n])$ be such that $\varphi_j(0) = \varphi_j(R^n) = 0$ for all $j \in \mathbb{N}$, and that $\varphi_j \to 0$ in $C^1([0, R^n])$ as $j \to \infty$. Then for all $\varepsilon > 0$ there exists $j_0 \in \mathbb{N}$ such that*

$$|\varphi_j(s)| \leq \varepsilon s(R^n - s) \qquad \textit{for all } s \in (0, R^n) \textit{ and any } j \geq j_0.$$

Now again making substantial use of the continuity property of $(\underline{W}^{(\theta)})_{\theta \in [0,1)}$ asserted by Lemma 3.1, we can finally establish our main result on blow-up in (1.1) for arbitrary large-mass initial data more concentrated than the respective constant equilibria:

Proof of Theorem 1.1. We let $m^\star = m^\star(n, R) > 0$ be as provided by Lemma 3.1, and given any u_0 complying with (1.2) and satisfying (1.5) let us assume that the corresponding solution (u, v) of (1.1) from Lemma 2.1 be global, and that hence $T_{max} = \infty$ and that for w taken from (2.3) we have $w_s \in L^\infty_{loc}([0, R^n] \times [0, \infty))$. We then first invoke Lemma 3.12 to make sure that as a consequence of (1.5), we can find $t_0 > 0$ and $c_1 > 0$ such that

$$w(s, t_0) \geq \frac{\mu}{n} s + c_1 s(R^n - s) \qquad \text{for all } s \in (0, R^n). \tag{3.79}$$

Once more relying on the $C^1([0, R^n])$-valued continuity of $\underline{W}^{(\theta)}$ as a function of $\theta \in [0, 1)$, we particularly obtain that $\underline{W}^{(\theta)}_s \to \frac{\mu}{n}$ in $C^0([0, R^n])$ as $\theta \searrow 0$, so that from (3.79) and Lemma 3.13 we readily infer the existence of $\theta \in (0, 1)$ fulfilling

$$w(s, t_0) \geq \underline{W}^{(\theta)}(s) \qquad \text{for all } s \in (0, R^n). \tag{3.80}$$

Keeping this value of θ fixed henceforth, with $T^{(\theta)}_{max} \in (0, \infty)$ and $w^{(\theta)} \in C^0([0, T^{(\theta)}_{max}); C^1([0, R^n])) \cap C^{2,1}((0, R^n) \times (0, T^{(\theta)}_{max}))$ taken from Lemma 3.6 we let $T := t_0 + T^{(\theta)}_{max}$ and $\underline{w}(s, t) := w^{(\theta)}(s, t - t_0)$ for

$s \in [0, R^n]$ and $t \in [t_0, T)$. Then thanks to (3.80), the comparison priciple from Lemma 5.1 warrants that with the function $W^{(\theta)}$ introduced there we have

$$w(s,T) \geq W^{(\theta)}(s) \qquad \text{for all } s \in (0, R^n).$$

As a consequence of (3.47), we thus obtain that

$$\limsup_{s \searrow 0} \frac{w(s,T) - w(0,T)}{s} = \limsup_{s \searrow 0} \frac{w(s,T)}{s} \geq \limsup_{s \searrow 0} \frac{W^{(\theta)}(s)}{s} = \infty$$

and that hence $w(\cdot, T)$ cannot belong to $C^1([0, R^n])$. This contradiction to Lemma 2.1 completes the proof. □

4 Boundedness for mildly concentrated initial data

The goal of this section is to complement the outcome of Theorem 1.1 by deriving the boundedness results claimed in Proposition 1.2, Theorem 1.3 and Corollary 1.4. Our arguments in this direction will rely on comparison of the mass accumulation function w from above by certain explicitly constructed barriers vanishing at $s = 0$ and having their spatial derivatives globally bounded. That this is already sufficient for global extensibility and boundedness of the derivative w_s, as determining u through (2.5), is asserted by a general statement which we prepend for repeated use in the sequel.

4.1 How mass accumulation can control densities: A Bernstein-type argument

In fact, all our boundedness arguments below will at their final stage rely on the following outcome of a Bernstein-type regularity procedure following well-known precedents in the analysis of scalar parabolic equations ([26]).

Lemma 4.1 *Let $n \geq 2$ and $R > 0$, and suppose that u_0 satisfies (1.2) and is such that the function w defined in (2.3) has the property that*

$$\sup_{(s,t) \in (0,R^n) \times (0,T_{max})} \frac{w(s,t)}{s} < \infty. \tag{4.1}$$

Then there exists $C > 0$ such that

$$w_s(s,t) \leq C \qquad \textit{for all } s \in (0, R^n) \textit{ and } t \in (0, T_{max}). \tag{4.2}$$

Proof. According to (4.1), let us fix $c_1 > 0$ such that

$$w(s,t) \leq c_1 s \qquad \text{for all } s \in (0, R^n) \text{ and } t \in (0, T_{max}), \tag{4.3}$$

which clearly entails that

$$w_s(0,t) \leq c_1 \qquad \text{for all } t \in (0, T_{max}). \tag{4.4}$$

Moreover, picking $\tau := \min\{1, \frac{1}{2}T_{max}\}$ and going back to the original variables, from Lemma 2.1 we infer that u is bounded in $\overline{\Omega} \times [0,\tau]$ and that $u(\cdot,\tau)$ is positive in $\overline{\Omega}$, through (2.3) implying the existence of $c_2 > 0$ and $c_3 > 0$ such that

$$\|w_s(\cdot,t)\|_{L^\infty((0,R^n))} \le c_2 \qquad \text{for all } t \in (0,\tau] \tag{4.5}$$

and

$$w_s(s,\tau) \ge c_3 \qquad \text{for all } s \in [0,R^n], \tag{4.6}$$

where the latter, upon integration, clearly entails that

$$w(s,\tau) \ge c_3 s \qquad \text{for all } s \in [0,R^n].$$

Now if we fix any $c_4 > \frac{\mu}{2n}$ and then take $c_5 > 0$ small such that $c_5 \le c_3 e^{-c_4 R^2}$, then

$$\underline{w}(s,t) := c_5 \int_0^s e^{c_4 \sigma^{\frac{2}{n}}} d\sigma, \qquad s \in [0,R^n],\ t \ge \tau,$$

satisfies

$$\underline{w}(s,t) \le c_5 e^{c_4 R^2} \cdot s \le c_3 s \qquad \text{for all } s \in [0,R^n] \text{ and } t \ge \tau,$$

whence (4.6) implies that both $\underline{w}(s,\tau) \le w(s,\tau)$ for all $s \in (0,R^n)$ and $\underline{w}(R^n,t) \le c_3 R^n \le w(R^n,\tau) = w(R^n,t)$ for all $t \in (\tau, T_{max})$. As for all $s \in (0,R^n)$ and $t > \tau$ we have

$$\underline{w}_s(s,t) = c_5 e^{c_4 s^{\frac{2}{n}}} \ge 0 \qquad \text{and} \qquad \underline{w}_{ss}(s,t) = \frac{2c_4 c_5}{n} s^{-1+\frac{2}{n}} e^{c_4 s^{\frac{2}{n}}},$$

it furthermore follows that

$$\begin{aligned}
\underline{w}_t - n^2 s^{2-\frac{2}{n}} \underline{w}_{ss} - n\underline{w}\underline{w}_s + \mu s \underline{w}_s &\le \underline{w}_t - n^2 s^{2-\frac{2}{n}} \underline{w}_{ss} + \mu s \underline{w}_s \\
&= -2nc_5 \cdot \Big(c_4 - \frac{\mu}{2n}\Big) \cdot s e^{c_4 s^{\frac{2}{n}}} \\
&< 0 \qquad \text{for all } s \in (0,R^n) \text{ and } t > \tau
\end{aligned}$$

according to our choice of c_4. Therefore, the comparison principle from Lemma 5.1 below asserts that $w(s,t) \ge \underline{w}(s,t)$ for all $s \in (0,R^n)$ and $t \in [\tau, T_{max})$, so that in particular, by nonnegativity of c_4,

$$w(s,t) \ge c_5 s \qquad \text{for all } s \in (0,R^n) \text{ and } t \in [\tau, T_{max}). \tag{4.7}$$

Apart from that, as w is bounded in $(0,R^n) \times (0,T_{max})$ and (2.6) is non-degenerate e.g. in the region $(\frac{R^n}{2}, R^n) \times (0,T_{max})$, from standard parabolic Schauder theory ([17]) we readily obtain $c_6 > 0$ fulfilling

$$w_s(R^n,t) \le c_6 \qquad \text{for all } t \in (\tau, T_{max}). \tag{4.8}$$

According to this, (4.5) and (4.7), for each $\varepsilon \in (0,1)$ the function z_ε defined on $(0,R^n] \times [\tau, T_{max})$ by letting

$$z_\varepsilon(s,t) := s^{1+\varepsilon} \cdot \frac{w_s^2(s,t)}{w(s,t)}, \qquad s \in (0,R^n],\ t \in [\tau, T_{max}), \tag{4.9}$$

actually belongs to $C^0([0,R^n]\times[\tau,T_{max}))$ with

$$z_\varepsilon(0,t)=0 \qquad \text{for all } t\in(\tau,T_{max}) \tag{4.10}$$

and

$$z_\varepsilon(s,\tau)\le \frac{c_2^2}{c_5}s^\varepsilon \le c_7:=\frac{c_2^2}{c_5}\cdot\max\{R^n,1\} \qquad \text{for all } s\in(0,R^n) \tag{4.11}$$

as well as

$$z_\varepsilon(R^n,t)\le \frac{c_6^2}{c_5}R^{n\varepsilon}\le c_8:=\frac{c_6^2}{c_5}\cdot\max\{R^n,1\} \qquad \text{for all } t\in(\tau,T_{max}). \tag{4.12}$$

In order to derive and upper bound for z_ε inside $(0,R^n)\times(\tau,T_{max})$, we compute

$$z_{\varepsilon s}=2s^{1+\varepsilon}\frac{w_sw_{ss}}{w}-s^{1+\varepsilon}\frac{w_s^3}{w^2}+(1+\varepsilon)s^\varepsilon\frac{w_s^2}{w} \tag{4.13}$$

and

$$\begin{aligned} z_{\varepsilon ss} &= 2s^{1+\varepsilon}\frac{w_sw_{sss}}{w}+2s^{1+\varepsilon}\frac{w_{ss}^2}{w}-5s^{1+\varepsilon}\frac{w_s^2w_{ss}}{w^2}\\ &\quad +4(1+\varepsilon)s^\varepsilon\frac{w_sw_{ss}}{w}+2s^{1+\varepsilon}\frac{w_s^4}{w^3}-2(1+\varepsilon)s^\varepsilon\frac{w_s^3}{w^2}+\varepsilon(1+\varepsilon)s^{\varepsilon-1}\frac{w_s^2}{w} \end{aligned} \tag{4.14}$$

as well as

$$\begin{aligned} z_{\varepsilon t} &= 2s^{1+\varepsilon}\frac{w_s}{w}\cdot\Big\{n^2s^{2-\frac{2}{n}}w_{sss}+(2n^2-2n)s^{1-\frac{2}{n}}w_{ss}+nww_{ss}+nw_s^2-\mu sw_{ss}-\mu w_s\Big\}\\ &\quad -s^{1+\varepsilon}\frac{w_s^2}{w^2}\cdot\Big\{n^2s^{2-\frac{2}{n}}w_{ss}+nww_s-\mu sw_s\Big\}\\ &= 2n^2s^{3-\frac{2}{n}+\varepsilon}\frac{w_sw_{sss}}{w}+(4n^2-4n)s^{2-\frac{2}{n}+\varepsilon}\frac{w_sw_{ss}}{w}+2ns^{1+\varepsilon}w_sw_{ss}+ns^{1+\varepsilon}\frac{w_s^3}{w}\\ &\quad -2\mu s^{2+\varepsilon}\frac{w_sw_{ss}}{w}-2\mu s^{1+\varepsilon}\frac{w_s^2}{w}\\ &\quad -n^2s^{3-\frac{2}{n}+\varepsilon}\frac{w_s^2w_{ss}}{w^2}+\mu s^{2+\varepsilon}\frac{w_s^3}{w^2} \end{aligned} \tag{4.15}$$

for $s\in(0,R^n)$ and $t\in(\tau,T_{max})$.

Now if for some $T\in(\tau,T_{max})$ we have $\max_{(s,t)\in[0,R^n]\times[\tau,T]}z_\varepsilon(s,t)>\max\{c_7,c_8\}$, then according to (4.10), (4.11) and (4.12) we can find $s_0\in(0,R^n)$ and $t_0\in(\tau,T]$ such that $z_{\varepsilon s}(s_0,t_0)=0$, $z_{\varepsilon ss}(s_0,t_0)\le 0$ and $z_{\varepsilon t}(s_0,t_0)\ge 0$. As thus necessarily $w_s(s_0,t_0)>0$, (4.13) and (4.14) warrant that at this point (s_0,t_0),

$$w_{ss}=\frac{w_s^2}{2w}-\frac{(1+\varepsilon)w_s}{2s} \tag{4.16}$$

and, as a consequence thereof,

$$2n^2s^{3-\frac{2}{n}+\varepsilon}\frac{w_sw_{sss}}{w} \le -2n^2s^{3-\frac{2}{n}+\varepsilon}\frac{w_{ss}^2}{w}+5n^2s^{3-\frac{2}{n}+\varepsilon}\frac{w_s^2w_{ss}}{w^2}-4n^2(1+\varepsilon)s^{2-\frac{2}{n}+\varepsilon}\frac{w_sw_{ss}}{w}$$

$$
\begin{aligned}
& -2n^2 s^{3-\frac{2}{n}+\varepsilon}\frac{w_s^4}{w^3} + 2n^2(1+\varepsilon)s^{2-\frac{2}{n}+\varepsilon}\frac{w_s^3}{w^2} - n^2\varepsilon(1+\varepsilon)s^{1-\frac{2}{n}+\varepsilon}\frac{w_s^2}{w} \\
= \;& -2n^2 s^{3-\frac{2}{n}+\varepsilon}\cdot\frac{1}{w}\cdot\left\{\frac{w_s^4}{4w^2} - \frac{(1+\varepsilon)w_s^3}{2sw} + \frac{(1+\varepsilon)^2 w_s^2}{4s^2}\right\} \\
& +5n^2 s^{3-\frac{2}{n}+\varepsilon}\cdot\frac{w_s^2}{w^2}\cdot\left\{\frac{w_s^2}{2w} - \frac{(1+\varepsilon)w_s}{2s}\right\} \\
& -4n^2(1+\varepsilon)s^{2-\frac{2}{n}+\varepsilon}\cdot\frac{w_s}{w}\cdot\left\{\frac{w_s^2}{2w} - \frac{(1+\varepsilon)w_s}{2s}\right\} \\
& -2n^2 s^{3-\frac{2}{n}+\varepsilon}\frac{w_s^4}{w^3} + 2n^2(1+\varepsilon)s^{2-\frac{2}{n}+\varepsilon}\frac{w_s^3}{w^2} - n^2\varepsilon(1+\varepsilon)s^{1-\frac{2}{n}+\varepsilon}\frac{w_s^2}{w} \\
= \;& -\frac{1}{2}n^2 s^{3-\frac{2}{n}+\varepsilon}\frac{w_s^4}{w^3} + n^2(1+\varepsilon)s^{2-\frac{2}{n}+\varepsilon}\frac{w_s^3}{w^2} - \frac{1}{2}n^2(1+\varepsilon)^2 s^{1-\frac{2}{n}+\varepsilon}\frac{w_s^2}{w} \\
& +\frac{5}{2}n^2 s^{3-\frac{2}{n}+\varepsilon}\frac{w_s^4}{w^3} - \frac{5}{2}n^2(1+\varepsilon)s^{2-\frac{2}{n}+\varepsilon}\frac{w_s^3}{w^2} \\
& -2n^2(1+\varepsilon)s^{2-\frac{2}{n}+\varepsilon}\frac{w_s^3}{w^2} + 2n^2(1+\varepsilon)^2 s^{1-\frac{2}{n}+\varepsilon}\frac{w_s^2}{w} \\
& -2n^2 s^{3-\frac{2}{n}+\varepsilon}\frac{w_s^4}{w^3} + 2n^2(1+\varepsilon)s^{2-\frac{2}{n}+\varepsilon}\frac{w_s^3}{w^2} - n^2\varepsilon(1+\varepsilon)s^{1-\frac{2}{n}+\varepsilon}\frac{w_s^2}{w} \\
= \;& -\frac{3}{2}n^2(1+\varepsilon)s^{2-\frac{2}{n}+\varepsilon}\frac{w_s^3}{w^2} + \frac{1}{2}n^2(1+\varepsilon)(3+\varepsilon)s^{1-\frac{2}{n}+\varepsilon}\frac{w_s^2}{w}.
\end{aligned}
$$

Therefore, (4.15) shows that at (s_0, t_0), once more due to (4.16) we must have

$$
\begin{aligned}
0 \;\leq\; & -\frac{3}{2}n^2(1+\varepsilon)s^{2-\frac{2}{n}+\varepsilon}\frac{w_s^3}{w^2} + \frac{1}{2}n^2(1+\varepsilon)(3+\varepsilon)s^{1-\frac{2}{n}+\varepsilon}\frac{w_s^2}{w} \\
& +(4n^2-4n)s^{2-\frac{2}{n}+\varepsilon}\frac{w_s}{w}\cdot\left\{\frac{w_s^2}{2w} - \frac{(1+\varepsilon)w_s}{2s}\right\} \\
& +2ns^{1+\varepsilon}w_s\cdot\left\{\frac{w_s^2}{2w} - \frac{(1+\varepsilon)w_s}{2s}\right\} + ns^{1+\varepsilon}\frac{w_s^3}{w} \\
& -2\mu s^{2+\varepsilon}\frac{w_s}{w}\cdot\left\{\frac{w_s^2}{2w} - \frac{(1+\varepsilon)w_s}{2s}\right\} - 2\mu s^{1+\varepsilon}\frac{w_s^2}{w} \\
& -n^2 s^{3-\frac{2}{n}+\varepsilon}\frac{w_s^2}{w^2}\cdot\left\{\frac{w_s^2}{2w} - \frac{(1+\varepsilon)w_s}{2s}\right\} + \mu s^{2+\varepsilon}\frac{w_s^3}{w^2} \\
= \;& -\frac{3}{2}n^2(1+\varepsilon)s^{2-\frac{2}{n}+\varepsilon}\frac{w_s^3}{w^2} + \frac{1}{2}n^2(1+\varepsilon)(3+\varepsilon)s^{1-\frac{2}{n}+\varepsilon}\frac{w_s^2}{w} \\
& +(2n^2-2n)s^{2-\frac{2}{n}+\varepsilon}\frac{w_s^3}{w^2} - (2n^2-2n)(1+\varepsilon)s^{1-\frac{2}{n}+\varepsilon}\frac{w_s^2}{w} \\
& +ns^{1+\varepsilon}\frac{w_s^3}{w} - n(1+\varepsilon)s^{\varepsilon}w_s^2 + ns^{1+\varepsilon}\frac{w_s^3}{w} \\
& -\mu s^{2+\varepsilon}\frac{w_s^3}{w^2} + \mu(1+\varepsilon)s^{1+\varepsilon}\frac{w_s^2}{w} - 2\mu s^{1+\varepsilon}\frac{w_s^2}{w} \\
& -\frac{1}{2}n^2 s^{3-\frac{2}{n}+\varepsilon}\frac{w_s^4}{w^3} + \frac{1}{2}n^2(1+\varepsilon)s^{2-\frac{2}{n}+\varepsilon}\frac{w_s^3}{w^2} + \mu s^{2+\varepsilon}\frac{w_s^3}{w^2}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & -\frac{1}{2}n^2 s^{3-\frac{2}{n}+\varepsilon}\frac{w_s^4}{w^3} + [(1-\varepsilon)n^2 - 2n]s^{2-\frac{2}{n}+\varepsilon}\frac{w_s^3}{w^2} \\
& +2ns^{1+\varepsilon}\frac{w_s^3}{w} \\
& +\Big\{\frac{\varepsilon-1}{2}n^2 + 2n\Big\} \cdot (1+\varepsilon)s^{1-\frac{2}{n}+\varepsilon}\frac{w_s^2}{w} \\
& -n(1+\varepsilon)s^\varepsilon w_s^2 \\
& -\mu(1-\varepsilon)s^{1+\varepsilon}\frac{w_s^2}{w}.
\end{aligned}
\tag{4.17}
$$

Here the rightmost two summands are nonpositive, whereas by Young's inequality and (4.3),

$$
[(1-\varepsilon)n^2 - 2n]s^{2-\frac{2}{n}+\varepsilon}\frac{w_s^3}{w^2} \le \frac{1}{8}n^2 s^{3-\frac{2}{n}+\varepsilon}\frac{w_s^4}{w^3} + c_9 s^{1-\frac{2}{n}+\varepsilon}\frac{w_s^2}{w}
$$

and

$$
\begin{aligned}
2ns^{1+\varepsilon}\frac{w_s^3}{w} \quad & \le \quad \frac{1}{8}n^2 s^{3-\frac{2}{n}+\varepsilon}\frac{w_s^4}{w^3} + 8s^{-1+\frac{2}{n}+\varepsilon}ww_s^2 \\
& \le \quad \frac{1}{8}n^2 s^{3-\frac{2}{n}+\varepsilon}\frac{w_s^4}{w^3} + 8c_1 s^{\frac{2}{n}+\varepsilon}w_s^2
\end{aligned}
$$

with $c_9 := \frac{2}{n^2}[(1-\varepsilon)n^2 - 2n]^2$. As furthermore our restriction $\varepsilon \in (0,1)$ ensures that

$$
\Big\{\frac{\varepsilon-1}{2}n^2 + 2n\Big\} \cdot (1+\varepsilon)s^{1-\frac{2}{n}+\varepsilon}\frac{w_s^2}{w} \le 4ns^{1-\frac{2}{n}+\varepsilon}\frac{w_s^2}{w},
$$

from (4.17) we infer that at this maximum point,

$$
\begin{aligned}
z_\varepsilon \quad & = \quad \Big\{\frac{4}{n^2}s^{-2+\frac{2}{n}}\frac{w^2}{w_s^2}\Big\} \cdot \frac{1}{4}n^2 s^{3-\frac{2}{n}+\varepsilon}\frac{w_s^4}{w^3} \\
& \le \quad \Big\{\frac{4c_9}{n^2} + \frac{16}{n}\Big\} \cdot \frac{w}{s} + \frac{32c_1}{n^2}s^{-2+\frac{2}{n}+\varepsilon}w^2.
\end{aligned}
$$

Once more using (4.3), we obtain that at (s_0, t_0),

$$
z_\varepsilon \le c_{10} := \Big\{\frac{4c_9}{n^2} + \frac{16}{n}\Big\} \cdot c_1 + \frac{32c_1^3}{n^2} \cdot \max\Big\{R^{\frac{2}{n}+1}, 1\Big\},
$$

in light of our premises meaning that

$$
z_\varepsilon \le c_{11} := \max\{c_7, c_8, c_{10}\} \quad \text{in } (0, R^n) \times (\tau, T) \qquad \text{for all } \varepsilon \in (0,1).
$$

Taking $\varepsilon \searrow 0$ and then $T \nearrow T_{max}$, upon a final application of (4.3) we conclude that

$$
w_s^2(s,t) \le c_{11} \cdot \frac{w(s,t)}{s} \le c_1 c_{11} \qquad \text{for all } s \in (0, R^n) \text{ and any } t \in (\tau, T_{max}),
$$

which together with (4.5) establishes (4.2). $\Box$

4.2 Boundedness for data less concentrated than constants. Proof of Proposition 1.2

In light of the latter, the boundedness statement from Proposition 1.2 becomes evident immediately:

Proof of Proposition 1.2. Rephrased in terms of the function w_0 introduced in (2.4), (1.7) says that

$$w_0(s) \le \frac{\mu}{n}s \qquad \text{for all } s \in (0, R^n)$$

with $\mu := \frac{nm}{\omega_n R^n}$. Since $[0,R^n]\times[0,\infty) \ni (s,t) \mapsto \overline{w}(s,t) := \frac{\mu}{n}s$ forms a classical solution of the parabolic equation in (2.6) which, clearly, moreover satisfies $\overline{w}(0,t) = 0 = w(s,t)$ and $\overline{w}(R^n,t) = \frac{m}{\omega_n} = w(R^n,t)$ for all $t \in (0,T_{max})$ with $T_{max} \in (0,\infty]$ and w taken from Lemma 2.1 and (2.3), respectively, the comparison principle in Lemma 5.1 states that

$$w(s,t) \le \frac{\mu}{n}s \qquad \text{for all } s \in (0,R^n) \text{ and } t \in (0,T_{max}).$$

Therefore, Lemma 4.1 applies to ensure that w_s is bounded in $(0,R^n)\times(0,T_{max})$, via (2.5) and Lemma 2.1 meaning that in fact $T_{max} = \infty$ and that (1.8) holds with some $C > 0$. $\square$

4.3 Boundedness for data below a singular steady state. Proofs of Theorem 1.3 and Corollary 1.4

For initial mass distributions possibly exceeding those determined by constant densities, we will instead attempt to bound w from above by certain members of the family of regular stationary supersolutions to (2.6) which in a slightly more general form were already utilized in [6], and which are described in the following.

Lemma 4.2 *Let $n \ge 2$ and $b > 0$. Then the function $\overline{w}$ defined by*

$$\overline{w}(s) := \frac{2s}{s^{\frac{2}{n}} + b}, \qquad s \ge 0, \tag{4.18}$$

belongs to $C^1([0,\infty)) \cap C^2((0,\infty))$ and satisfies $\overline{w}_s(s) \ge 0$ for all $s > 0$ as well as

$$n^2 s^{2-\frac{2}{n}}\overline{w}_{ss}(s) + n\overline{w}(s)\overline{w}_s(s) - \mu s\overline{w}_s(s) < 0 \qquad \textit{for all } s > 0 \tag{4.19}$$

whenever $\mu \ge 0$.

Proof. It is evident that $\overline{w} \in C^2((0,\infty))$, and computing

$$\overline{w}_s(s) = \frac{(2-\frac{4}{n})s^{\frac{2}{n}} + 2b}{(s^{\frac{2}{n}}+b)^2}, \qquad s > 0,$$

and

$$\overline{w}_{ss}(s) = \frac{-(\frac{4}{n}-\frac{8}{n^2})s^{\frac{4}{n}-1} - (\frac{4}{n}+\frac{8}{n^2})bs^{\frac{2}{n}-1}}{(s^{\frac{2}{n}}+b)^3}, \qquad s > 0,$$

we see that $\overline{w}_s$ actually belongs to $C^0([0,\infty))$ and is nonnegative, and that

$$\begin{aligned}
&n^2 s^{2-\frac{2}{n}}\overline{w}_{ss}(s) + n\overline{w}(s)\overline{w}_s(s) \\
&= \frac{1}{(s^{\frac{2}{n}}+b)^3}\cdot\Big\{n^2 s^{2-\frac{2}{n}}\cdot\Big[-\Big(\frac{4}{n}-\frac{8}{n^2}\Big)s^{\frac{4}{n}-1} - \Big(\frac{4}{n}+\frac{8}{n^2}\Big)bs^{\frac{2}{n}-1}\Big] + n\cdot 2s\cdot\Big[\Big(2-\frac{4}{n}\Big)s^{\frac{2}{n}}+2b\Big]\Big\} \\
&= \frac{1}{(s^{\frac{2}{n}}+b)^3}\cdot\Big\{-(4n-8)s^{1+\frac{2}{n}} - (4n+8)bs + (4n-8)s^{1+\frac{2}{n}} + 4nbs\Big\} \\
&= -\frac{8bs}{(s^{\frac{2}{n}}+b)^3} \qquad \text{for all } s>0.
\end{aligned}$$

As b was assumed to be positive, this implies (4.19) for any choice of $\mu \ge 0$, again because $\overline{w}_s \ge 0$. $\Box$

Indeed, using that the above functions $\overline{w}$ conveniently approximate the singular supersolution $0 \le s \mapsto 2s^{1-\frac{2}{n}}$ of the first equation in (2.6), by means of a comparison argument applied to (2.6), and again Lemma 4.1, we can achieve the following main step toward Theorem 1.3.

Lemma 4.3 *Let $n \ge 2$ and $R>0$, and let u_0 be such that (1.2) holds, and that the function w_0 from (2.4) satisfies*

$$w_0(s) < 2s^{1-\frac{2}{n}} \qquad \textit{for all } s \in (0,R^n]. \tag{4.20}$$

Then there exists $C>0$ such that for w as in (2.3) we have

$$\|w_s(\cdot,t)\|_{L^\infty((0,R^n))} \le C \qquad \textit{for all } t \in (0,T_{max}). \tag{4.21}$$

Proof. Due to (1.2), there exists $c_1>0$ such that beyond (4.20) w_0 satisfies

$$w_0(s) \le c_1 s \qquad \text{for all } s \in (0,R^n), \tag{4.22}$$

and fixing this value of c_1 we let $s_0 \in (0,R^n)$ be suitably small such that

$$s_0 \le \Big(\frac{1}{c_1}\Big)^{\frac{n}{2}}. \tag{4.23}$$

Thereupon, (4.20) warrants that

$$a := \max_{s\in[s_0,R^n]}\Big\{s^{-1+\frac{2}{n}}w_0(s)\Big\} < 2, \tag{4.24}$$

which enables us to pick $b>0$ small enough fulfilling

$$b \le \frac{1}{c_1} \qquad \text{and} \qquad b \le \frac{2-a}{a}s_0^{\frac{2}{n}}. \tag{4.25}$$

Then (4.22), (4.23) and the first inequality in (4.25) guarantee that for small s,

$$\frac{s^{\frac{2}{n}}+b}{2s}\cdot w_0(s) \le \frac{c_1}{2}s^{\frac{2}{n}} + \frac{c_1}{2}b \le \frac{c_1}{2}s_0^{\frac{2}{n}} + \frac{c_1}{2}b \le \frac{c_1}{2}\cdot\frac{1}{c_1} + \frac{c_1}{2}\cdot\frac{1}{c_1} = 1 \qquad \text{for all } s \in (0,s_0),$$

whereas for larger s, (4.24) and the second inequality in (4.25) ensure that

$$\begin{aligned} \frac{s^{\frac{2}{n}}+b}{2s}\cdot w_0(s) &\le \frac{s^{\frac{2}{n}}+b}{2s}\cdot as^{1-\frac{2}{n}} \\ &= \frac{a}{2}+\frac{ab}{2}s^{-\frac{2}{n}} \\ &\le \frac{a}{2}+\frac{ab}{2}s_0^{-\frac{2}{n}} \\ &\le \frac{a}{2}+\frac{a}{2}\cdot\frac{2-a}{a} \\ &= 1 \qquad \text{for all } s\in[s_0,R^n]. \end{aligned}$$

Therefore,

$$w_0(s) \le \frac{2s}{s^{\frac{2}{n}}+b} \qquad \text{for all } s\in[0,R^n],$$

so that an application of Lemma 4.2 to $\mu := \frac{nm}{\omega_n R^n}$ shows that due to the comparison principle from Lemma 5.1,

$$w(s,t) \le \frac{2s}{s^{\frac{2}{n}}+b} \qquad \text{for all } s\in(0,R^n) \text{ and } t\in(0,T_{max}).$$

As this implies that

$$\frac{w(s,t)}{s} \le \frac{2}{b} \qquad \text{for all } s\in(0,R^n) \text{ and } t\in(0,T_{max}),$$

we may in turn invoke Lemma 4.1 to verify (4.21) for some appropriately large $C>0$. $\square$

In fact, this immediately implies the following.

Proof of Theorem 1.3. In view of (1.9), the function w_0 defined in (2.4) satisfies $w_0(s) < 2s^{1-\frac{2}{n}}$ for all $s\in(0,R^n)$. Therefore, Lemma 4.3 in conjunction with (2.5) and Lemma 2.1 readily implies both the statement on global solvability and the claimed boundedness feature. $\square$

At suitably small mass levels, through quite an elementary argument the latter entails that global bounded solutions exist for all initial data even within some considerably large set of functions more concentrated than the respective homogeneous equilibrium:

Proof of Corollary 1.4. As for the case $n=2$, we only need to recall the well-known fact that whenever u_0 satisfies (1.2) as well as $\int_\Omega u_0 < 8\pi$, the problem (1.1) is globally classically solvable by a pair (u,v) for which u is bounded ([20]), so that in this planar setting the claim is obvious if we let $m_\star(2,R) := 8\pi$ for any $R>0$.

When $n\ge 3$, we let $m_\star(n,R) := 2\omega_n R^{n-2}$ for $R>0$, and given $m\in(0,m_\star(n,R))$ we define

$$\begin{aligned} B \;:=\; \Bigg\{ u_0\in C^0_{rad}(\overline{\Omega}) \;\Bigg|\; \int_\Omega u_0 = m, \quad \frac{mr^n}{R^n} < \int_{B_r(0)} u_0 < 2\omega_n r^{n-2} \text{ for all } r\in(0,R) \\ \text{and } u_0(0) > \frac{nm}{\omega_n R^n} > u_0|_{\partial\Omega} \Bigg\}. \end{aligned}$$

Then given $u_0 \in B$ we have

$$\int_{B_r(0)} u_0 - \int_{B_r(0)} \frac{nm}{\omega_n R^n} \ge \frac{mr^n}{R^n} - |B_r(0)| \cdot \frac{nm}{\omega_n R^n} = 0 \qquad \text{for all } r \in (0,R],$$

so that indeed $u_0 \succeq \frac{m}{|\Omega|}$. Moreover, combining the inequality $\int_{B_r(0)} u_0 < 2\omega_n r^{n-2}$ for $r \in (0,R)$ with the fact that $\int_\Omega u_0 = m < m_\star(n,R) = 2\omega_n R^n$ shows that (1.9) is satisfied, whence Theorem 1.3 warrants global existence of a bounded classical solution for any such u_0. To finally verify that B is relatively open in $\{0 \le \varphi \in C^0_{rad}(\overline{\Omega}) \mid \int_\Omega \varphi = m\}$, for fixed $u_0 \in B$ we may pick $r_1 \in (0,R)$ and $r_2 \in (r_1, R)$ such that with $c_1 := u_0(0)$ and $\mu := \frac{nm}{\omega_n R^n}$ we have $c_1 + 1 \ge u_0 > \mu$ in $\overline{B}_{r_1}(0)$ and $u_0 < \mu$ in $\overline{\Omega} \setminus B_{r_2}(0)$, and that moreover

$$r_1 \le \sqrt{\frac{2n}{c_1+2}} \qquad \text{and} \qquad 2\omega_n r_2^{n-2} > m, \tag{4.26}$$

where the latter can indeed be achieved due to the fact that $m < 2\omega_n R^{n-2}$.
Thereafter, we may choose $\delta \in (0,1)$ such that still $u_0 \ge \mu + \delta$ in $\overline{B}_{r_1}(0)$ and $u_0 \le \mu - \delta$ in $\overline{\Omega} \setminus B_{r_2}(0)$, and that furthermore

$$\frac{(m+\delta\omega_n)r^n}{n} \le \int_{B_r(0)} u_0 \le 2\omega_n r^{n-2} - \frac{\delta\omega_n r^n}{n} \qquad \text{for all } r \in [r_1, r_2], \tag{4.27}$$

which is clearly possible by continuity of $[0,R] \ni r \mapsto \int_{B_r(0)} u_0$.

Then whenever $\widetilde{u}_0 \in C^0_{rad}(\overline{\Omega})$ is nonnegative and such that $\int_\Omega \widetilde{u}_0 = m$ and $\|\widetilde{u}_0 - u_0\|_{L^\infty(\Omega)} < \delta$, we have $\widetilde{u}_0 > u_0 - \delta \ge (\mu+\delta) - \delta = \mu$ in $\overline{B}_{r_1}(0)$ and $\widetilde{u}_0 < u_0 + \delta \le (\mu - \delta) + \delta = \mu$ in $\overline{\Omega} \setminus B_{r_2}(0)$, whence in particular also $\widetilde{u}_0(0) > \mu > \widetilde{u}_0|_{\partial\Omega}$ as well as

$$\int_{B_r(0)} \widetilde{u}_0 > \mu |B_r(0)| = \frac{mr^n}{R^n} \qquad \text{for all } r \in (0, r_1) \tag{4.28}$$

and

$$\int_{B_r(0)} \widetilde{u}_0 = \int_\Omega \widetilde{u}_0 - \int_{\Omega \setminus B_r(0)} \widetilde{u}_0 = m - \int_{\Omega \setminus B_r(0)} \widetilde{u}_0 > m - \mu \cdot |\Omega \setminus B_r(0)| = \frac{mr^n}{R^n} \qquad \text{for all } r \in (r_2, R]. \tag{4.29}$$

Apart from that, (4.26) warrants that for all $r \in (0, r_1)$,

$$\frac{\int_{B_r(0)} \widetilde{u}_0}{2\omega_n r^{n-2}} < \frac{\int_{B_r(0)} (u_0 + \delta)}{2\omega_n r^{n-2}} \le \frac{(c_1+2)|B_r(0)|}{2\omega_n r^{n-2}} = \frac{(c_1+2)r^2}{2n} \le \frac{(c_1+2)r_1^2}{2n} \le 1, \tag{4.30}$$

and that

$$\frac{\int_{B_r(0)} \widetilde{u}_0}{2\omega_n r^{n-2}} \le \frac{\int_\Omega \widetilde{u}_0}{2\omega_n r^{n-2}} = \frac{m}{2\omega_n r^{n-2}} < 1 \qquad \text{for all } r \in (r_2, R]. \tag{4.31}$$

For intermediate values of r, we use (4.27) to see that

$$\int_{B_r(0)} \widetilde{u}_0 \quad > \quad \int_{B_r(0)} (u_0 - \delta)$$

$$
\begin{aligned}
&= \int_{B_r(0)} u_0 - \delta|B_r(0)| \\
&\geq \frac{(m+\delta\omega_n)r^n}{n} - \delta|B_r(0)| \\
&= \frac{mr^n}{n} \qquad \text{for all } r \in [r_1, r_2]
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{B_r(0)} \widetilde{u}_0 &< \int_{B_r(0)} (u_0+\delta) \\
&= \int_{B_r(0)} u_0 + \delta|B_r(0)| \\
&\leq 2\omega_n r^{n-2} - \frac{\delta\omega_n r^n}{n} + \delta|B_r(0)| \\
&= 2\omega_n r^{n-2} \qquad \text{for all } r \in [r_1, r_2],
\end{aligned}
$$

which together with (4.28)-(4.31) shows that indeed any such $\widetilde{u}_0$ belongs to B, as intended. $\Box$

5 Appendix: A comparison principle for (2.6)

Lemma 5.1 *Let $L > 0$ and $T > 0$, and suppose that $\underline{w}$ and $\overline{w}$ are two functions which belong to $C^1([0,L]\times[0,T))$ and satisfy*

$$
\underline{w}_s(s,t) > 0 \quad \textit{and} \quad \overline{w}(s,t) > 0 \qquad \textit{for all } s \in (0,L) \textit{ and } t \in (0,T)
$$

as well as

$$
\underline{w}(\cdot,t) \in W^{2,\infty}_{loc}((0,L)) \quad \textit{and} \quad \overline{w}(\cdot,t) \in W^{2,\infty}_{loc}((0,L)) \qquad \textit{for all } t \in (0,T).
$$

If for some constants $a \geq 0, \alpha \in \mathbb{R}, b \in \mathbb{R}$ and $c \in \mathbb{R}$ we have

$$
\underline{w}_t \leq as^\alpha \underline{w}_{ss} + b\underline{w}\underline{w}_s + c\underline{w}_s \quad \textit{and} \quad \overline{w}_t \geq as^\alpha \overline{w}_{ss} + b\overline{w}\overline{w}_s + c\overline{w}_s \qquad \textit{for all } t \in (0,T) \textit{ and a.e. } s \in (0,L),
$$

and if moreover

$$
\underline{w}(s,0) \leq \overline{w}(s,0) \qquad \textit{for all } s \in (0,L)
$$

as well as

$$
\underline{w}(0,t) \leq \overline{w}(0,t) \quad \textit{and} \quad \underline{w}(L,t) \leq \overline{w}(L,t) \qquad \textit{for all } t \in (0,T),
$$

then

$$
\underline{w}(s,t) \leq \overline{w}(s,t) \qquad \textit{for all } s \in [0,L] \textit{ and } t \in [0,T).
$$

Proof. This directly follows upon application of [1, Lemma 5.1]. □

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References

[1] Bellomo, N., Winkler, M.: *Finite-time blow-up in a degenerate chemotaxis system with flux limitation.* Trans. Amer. Math. Soc. Ser. B **4**, 31-67 (2017)

[2] Bellomo, N., Winkler, M.: *A degenerate chemotaxis system with flux limitation: maximally extended solutions and absence of gradient blow-up.* Comm. Part. Differential Eq. **42**, 436-473 (2017)

[3] Biler, P.: *Local and global solvability of some parabolic systems modelling chemotaxis.* Adv. Math. Sci. Appl. **8**, 715-743 (1998)

[4] Biler, P., Cieślak, T., Karch, G., Zienkiewicz, J.: *Local criteria for blowup in two-dimensional chemotaxis models.* Discr. Cont. Dyn. Syst. **37**, 1841-1856 (2017)

[5] Biler, P., Corrias, L., Dolbeault, J.: *Large mass self-similar solutions of the parabolic-parabolic Keller-Segel model of chemotaxis.* J. Math. Biol. **63**, 1-32 (2011)

[6] Biler, P., Nadzieja, T.: *Growth and accretion of mass in an astrophysical model II.* Applicationes Math. (Warsaw) **23**, 351-361 (1995)

[7] Blanchet, A., Dolbeault, J., Perthame, B.: *Two-dimensional Keller-Segel model: Optimal critical mass and qualitative properties of the solutions.* Electron. J. Differential Eq. **44**, 32 pp. (2006)

[8] Cieślak, T., Stinner, C.: *Finite-time blowup and global-in-time unbounded solutions to a parabolic-parabolic quasilinear Keller-Segel system in higher dimensions.* J. Differ. Eq. **252** (10), 5832-5851 (2012)

[9] Cieślak, T., Stinner, C.: *New critical exponents in a fully parabolic quasilinear Keller-Segel system and applications to volume filling models.* J. Differ. Eq. **258** (6), 2080-2113 (2015)

[10] Cieślak, T., Winkler, M.: *Finite-time blow-up in a quasilinear system of chemotaxis.* Nonlinearity **21**, 1057-1076 (2008)

[11] Djie, K., Winkler, M.: *Boundedness and finite-time collapse in a chemotaxis system with volume-filling effect.* Nonlinear Anal. **72**, 1044-1064 (2010)

[12] Herrero, M. A., Velázquez, J. J. L.: *A blow-up mechanism for a chemotaxis model.* Ann. Scuola Normale Superiore Pisa Cl. Sci. **24**, 633-683 (1997)

[13] Horstmann, D., Wang, G.: *Blow-up in a chemotaxis model without symmetry assumptions.* European J. Appl. Math. **12**, 159-177 (2001)

[14] Jäger, W., Luckhaus, S.: *On explosions of solutions to a system of partial differential equations modelling chemotaxis.* Trans. Am. Math. Soc. **329**, 819-824 (1992)

[15] Keller, E.F., Segel, L.A.: *Initiation of slime mold aggregation viewed as an instability.* J. Theoret. Biol. **26** 399-415 (1970)

[16] Kozono, H., Sugiyama, Y.: *Local existence and finite time blow-up of solutions in the 2-D Keller-Segel system.* J. Evol. Eq. **8**, 353-378 (2008)

[17] Ladyzenskaja, O. A., Solonnikov, V. A., Ural'ceva, N. N.: *Linear and Quasi-Linear Equations of Parabolic Type.* Amer. Math. Soc. Transl., Vol. **23**, Providence, RI, 1968

[18] Laurençot, Ph., Mizoguchi, N.: *Finite time blowup for the parabolicparabolic KellerSegel system with critical diffusion.* Ann. Inst. H. Poincaré Anal. Non Linéaire **34**, 197-220 (2017)

[19] Mizoguchi, N., Winkler, M.: *Finite-time blow-up in the two-dimensional parabolic Keller-Segel system.* Preprint

[20] Nagai, T.: *Blow-up of radially symmetric solutions to a chemotaxis system.* Adv. Math. Sci. Appl. **5**, 581-601 (1995)

[21] Nagai, T.: *Blowup of Nonradial Solutions to Parabolic-Elliptic Systems Modeling Chemotaxis in Two-Dimensional Domains.* J. Inequal. Appl. **6**, 37-55 (2001)

[22] Nagai, T., Senba, T., Yoshida, K.: *Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis.* Funkc. Ekvacioj, Ser. Int. **40**, 411-433 (1997)

[23] Nanjundiah, V.: *Chemotaxis, signal relaying and aggregation morphology.* J. Theor. Biol. **42**, 63-105 (1973)

[24] Osaki, K., Yagi, A.: *Finite dimensional attractor for one-dimensional Keller-Segel equations,* Funkcialaj Ekvacioj **44**, 441 - 469 (2001)

[25] Perthame, B.: *Transport Equations in Biology.* Birkhäuser, Basel, 2007

[26] Quittner, P., Souplet, Ph.: *Superlinear Parabolic Problems. Blow-up, Global Existence and Steady States.* Birkhäuser, Basel/Boston/Berlin, 2007

[27] Senba, T., Suzuki, T.: *Chemotactic collapse in a parabolic-elliptic system of mathematical biology.* Adv. Differential Eq. **6**, 21-50 (2001)

[28] Senba, T., Suzuki, T.: *Weak solutions to a parabolic-elliptic system of chemotaxis.* J. Funct. Anal. **191**, 17-51 (2002)

[29] Tao, Y., Winkler, M.: *Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity.* J. Differential Eq. **252** (1), 692-715 (2012)

[30] Winkler, M.: *Does a 'volume-filling effect' always prevent chemotactic collapse?* Math. Meth. Appl. Sci. **33**, 12-24 (2010)

[31] Winkler, M: *Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system.* J. Math. Pures Appl. **100**, 748-767 (2013), `arXiv:1112.4156v1`