Instantaneous regularization of distributions from $(C^0)^\star \times L^2$ in the one-dimensional parabolic Keller-Segel system

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Abstract

The Neumann problem for the Keller-Segel system

$$\begin{cases} u_t = u_{xx} - (uv_x)_x, \\ v_t = v_{xx} - v + u \end{cases} \qquad (\star)$$

is considered in open bounded intervals $\Omega \subset \mathbb{R}$, with a particular focus on the question to which extent supposedly present singularities can be regularized despite the destabilizing cross-diffusive interaction in $(\star)$.

The main results in this direction indicate that in the considered one-dimensional framework, even immediate regularization into spaces of smooth functions occurs for arbitrary distributions no more singular than Radon measures in their first and L^2 functions in their second component. In particular, it is shown that given any nonnegative $\mu_0 \in (C^0(\overline{\Omega}))^\star$ and $v_0 \in L^2(\Omega)$, the corresponding initial-boundary value problem for $(\star)$ admits at least one global weak solution which is smooth in $\overline{\Omega} \times (0,\infty)$ and satisfies $u(\cdot,t) \overset{\star}{\rightharpoonup} \mu_0$ in $(C^0(\overline{\Omega}))^\star$ and $v(\cdot,t) \to v_0$ in $L^2(\Omega)$ as $t \searrow 0$.

This apparently goes somewhat beyond precedent constructions of solutions to $(\star)$ for rough initial data, which inter alia seem to exclusively require the initial signal distribution to satisfy at least $v_0 \in W^{1,q}(\Omega)$ for some $q > 1$.

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1 Introduction

Since its introduction in 1970 ([15]), the Keller-Segel system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ v_t = \Delta v - v + u, \end{cases} \tag{1.1}$$

has been attracting considerable interest in the mathematical literature. Proposed as a macroscopic description for the spatio-temporal evolution in populations of protozoa moving chemotactically toward increasing concentrations of a chemical signal produced by themselves, this system has become one of the most intensely studied biomathematical models during the past decades ([2]). Besides the circumstance that (1.1) furthermore forms the core of numerous more complex models for collective behavior driven by directed motion at various length scales ([11], [22]), a prevailing reason for its prodigious appeal seemingly consists in its ability to quite adequately portray, despite its apparently simple structure, phenomena of aggregation and clustering, known from experiments as the probably most striking effects of chemotactic motion ([12]).

Of accordingly significant importance seems to be a comprehensive understanding of the potential of (1.1) to support locally large population sizes, with singular structures involving infinite densities constituting an idealized limit case widely referred to in analytical studies. Indeed, with regard to the problem of identifying situations in which such singularities may spontaneously arise during evolution, quite a well-developed literature has provided a large variety of criteria for finite-time explosions either to occur or to be ruled out, containing the spatial dimension and properties of the initial data in their main parts ([10], [20], [30], [21], [6], [28]); addressing corresponding questions for related systems, and especially facing challenges linked to issues of optimality of respectively found conditions, has partially led to rather complete pictures in this direction, apart from having stimulated further development of analytical techniques for blow-up detection (cf. e.g. [19], [14], [25], [32], [7], [29], [31], or also [27] and [3], for a non-exhaustive collection of classical and more recent findings).

Somewhat complementary to these developments, the motivation for the present work originates from the question how far the destabilizing action of cross-diffusion in (1.1) endorses singular structures which are presupposed to be already present at some time. With respect to this apparently much less studied problem of deducing conditions for evolutionary smoothing of singularities either to occur or to be absent, precedent findings suggest the conjecture that in any multi-dimensional setting, (1.1) should admit solutions which are biologically meaningful in the sense of representing some finite total population size, but which exhibit persistent singularities. In fact, in n-dimensional cases with $n \geq 3$ this is supported by the simple observation that the Chandrasekhar solutions determined by $u(x,t) := \frac{2(n-2)}{|x|^2}$, $x \neq 0$, $t > 0$, provide explicit examples of even stationary singular solutions to a close relative of (1.1) in which the second equation is slightly simplified so as to become $v_t = \Delta v + u$; corresponding indications addressing two-dimensional frameworks can be found in results on the existence of measure-valued solutions with permanent Dirac-type singularities to certain parabolic-elliptic variants of (1.1) ([18], [8]; cf. also [26] and [1]).

Main results. The purpose of the present study now consists in examining how far in the spatially one-dimensional version of (1.1) the dissipative action of diffusion is able to suppress such phenomena of persistently singular behavior, and our results will reveal that in a corresponding Neumann problem, actually immediate regularization into arbitrarily smooth profiles occurs for virtually any biologically reasonable initial population distribution. More precisely, in a bounded open interval $\Omega \subset \mathbb{R}$ we consider

$$\begin{cases} u_t = u_{xx} - (uv_x)_x, & x \in \Omega,\ t > 0, \\ v_t = v_{xx} - v + u, & x \in \Omega,\ t > 0, \\ u_x = v_x = 0, & x \in \partial\Omega,\ t > 0, \\ u(\cdot, 0) = \mu_0, \quad v(\cdot, 0) = v_0, & x \in \Omega, \end{cases} \tag{1.2}$$

and we shall see that solutions instantaneously becoming smooth exist whenever the initial data herein represent a possibly measure time finite-mass distribution with respect to the population density, and satisfy the technical condition of being square integrable in the chemoattractant concentration. We shall accordingly assume throughout the sequel that

$$\begin{cases} \mu_0 \in \mathcal{M}(\overline{\Omega}), \quad \mu_0 \geq 0, \qquad \text{and} \\ v_0 \in L^2(\Omega), \quad v_0 \geq 0, \end{cases} \tag{1.3}$$

with $\mathcal{M}(\overline{\Omega})$ denoting the space of Radon measures on $\overline{\Omega}$, where here and below we will make use of the identification $\mathcal{M}(\overline{\Omega}) \cong (C^0(\overline{\Omega}))^\star$ in writing $\mu_0(\psi)$ instead of $\int_\Omega \psi d\mu_0$ for $\psi \in C^0(\overline{\Omega})$.

Indeed, in this setting our main results assert global existence of an immediately regularized solution attaining the prescribed initial data in a topological framework optimal in the context of (1.3):

Theorem 1.1 *Let $\Omega \subset \mathbb{R}$ be a bounded open interval, and suppose that $\mu_0 \in \mathcal{M}(\overline{\Omega})$ and $v_0 \in L^2(\Omega)$ are nonnegative. Then there exist nonnegative functions*

$$\begin{cases} u \in C^0([0,\infty); \mathcal{M}(\Omega)) \cap C^\infty(\overline{\Omega} \times (0,\infty)) \qquad \textit{and} \\ v \in C^0([0,\infty); L^2(\Omega)) \cap C^\infty(\overline{\Omega} \times (0,\infty)) \end{cases} \tag{1.4}$$

which have the additional regularity features

$$\begin{cases} u \in L^2_{loc}(\overline{\Omega} \times [0,\infty)) \cap L^1_{loc}([0,\infty); W^{1,1}(\Omega)) \qquad \textit{and} \\ v \in L^2_{loc}([0,\infty); W^{1,2}(\Omega)), \end{cases} \tag{1.5}$$

which satisfy

$$\int_\Omega u(x,t)dx = m := \mu_0(\mathbf{1}) \qquad \textit{for all } t > 0 \tag{1.6}$$

with $\mathbf{1}(x) := 1$ for $x \in \overline{\Omega}$, as well as

$$\sup_{t>\tau} \Big\{ \|u(\cdot,t)\|_{C^2(\overline{\Omega})} + \|v(\cdot,t)\|_{C^2(\overline{\Omega})} \Big\} < \infty \qquad \textit{for all } \tau > 0, \tag{1.7}$$

and which have the following properties: The pair (u,v) solves (1.2) in the sense that the first three lines therein are fulfilled classically in $\overline{\Omega} \times (0,\infty)$, that moreover

$$-\int_0^\infty \int_\Omega u\varphi_t - \mu_0(\varphi(\cdot,0)) = -\int_0^\infty \int_\Omega u_x\varphi_x + \int_0^\infty \int_\Omega uv_x\varphi_x \tag{1.8}$$

as well as
$$-\int_0^\infty \int_\Omega v\varphi_t - \int_\Omega v_0\varphi(\cdot,0) = -\int_0^\infty \int_\Omega v_x\varphi_x - \int_0^\infty \int_\Omega v\varphi + \int_0^\infty \int_\Omega u\varphi \tag{1.9}$$
hold for each $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$, *and that*
$$u(\cdot,t) \overset{\star}{\rightharpoonup} \mu_0 \quad \textit{in } \mathcal{M}(\overline{\Omega}) \qquad \textit{as } t \searrow 0 \tag{1.10}$$
as well as
$$v(\cdot,t) \to v_0 \quad \textit{in } L^2(\Omega) \qquad \textit{as } t \searrow 0. \tag{1.11}$$

Especially by merely requiring an assumption on the initial population distribution that apparently cannot substantially be relaxed without waiving biological relevance, but also by only imposing an integrability condition on v_0 rather than the gradient thereof, Theorem 1.1 seems to go somewhat beyond previous findings on global solutions to either (1.2) or any close relative, even within concepts weaker than the above. In fact, whereas the early work [21] even required (μ_0, v_0) to belong to $L^2(\Omega) \times W^{1,2}(\Omega)$, adapting more recent approaches, as developed in the literature to address two- and higher-dimensional cases ([5], [4], [24]), should after all allow for a relaxation at least to the requirement that μ_0 be merely integrable, but it seems that the methods in all these precedents seem to rely on some suitable regularity features of the derivative v_{0x}.

Due to an evident mass conservation property enjoyed by any reasonably regular solution of (1.2), even when viewed from a purely mathematical perspective it seems widely meaningless to seek for further extensions of Theorem 1.1 to initial data significantly more singular than those from $\mathcal{M}(\overline{\Omega})$ in their first component; appropriate regularization of some very singular initial data, particularly representing infinite total initial mass, hence requires modification of the model itself, including additional dissipative mechanisms; a recent finding even addressing a corresponding higher-dimensional variant of (1.1) confirms that logistic-type growth restrictions provide a positive example therefor ([33]).

2 Approximate problems

In order to conveniently regularize (1.2), throughout the sequel we fix families $(u_{0\varepsilon})_{\varepsilon\in(0,1)}$ and $(v_{0\varepsilon})_{\varepsilon\in(0,1)}$ with the properties that
$$\begin{cases} u_{0\varepsilon} \in C^1(\overline{\Omega}) \text{ is nonnegative with } \int_\Omega u_{0\varepsilon} = m := \mu_0(\mathbf{1}) \text{ for all } \varepsilon \in (0,1), \text{ and such that} \\ u_{0\varepsilon} \overset{\star}{\rightharpoonup} \mu_0 \text{ in } \mathcal{M}(\overline{\Omega}) \text{ as } \varepsilon \searrow 0, \qquad \text{and that} \\ v_{0\varepsilon} \in C^1(\overline{\Omega}) \text{ is nonnegative and such that } v_{0\varepsilon} \to v_0 \text{ in } L^2(\Omega) \text{ as } \varepsilon \searrow 0. \end{cases} \tag{2.1}$$

Then according to well-known theory of Keller-Segel systems with smooth initial data ([21], [13]), for any $\varepsilon \in (0,1)$ the regularized problem
$$\begin{cases} u_{\varepsilon t} = u_{\varepsilon xx} - (u_\varepsilon v_{\varepsilon x})_x, & x \in \Omega,\ t>0, \\ v_{\varepsilon t} = v_{\varepsilon xx} - v_\varepsilon + u_\varepsilon, & x \in \Omega,\ t>0, \\ u_{\varepsilon x} = v_{\varepsilon x} = 0, & x \in \partial\Omega,\ t>0, \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x), \quad v_\varepsilon(x,0) = v_{0\varepsilon}(x), & x \in \Omega, \end{cases} \tag{2.2}$$

possesses global and mass-preserving classical solutions:

Lemma 2.1 *Assume (1.3) and (2.1). Then for each* $\varepsilon \in (0,1)$*, there exist uniquely determined nonnegative functions*

$$\begin{cases} u_\varepsilon \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \qquad \text{and} \\ v_\varepsilon \in \bigcap_{q>1} C^0([0,\infty); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \end{cases}$$

which solve (2.2) in the classical sense in $\overline{\Omega} \times [0,\infty)$*. Moreover,*

$$\int_\Omega u_\varepsilon(x,t)dx = m \qquad \text{for all } t \geq 0, \tag{2.3}$$

with $m \geq 0$ *as determined in (1.6).*

Without further explicit mentioning, in the next sections we shall suppose that (1.3) and (2.1) hold, and that $(u_\varepsilon, v_\varepsilon)$ denotes the corresponding solution of (2.2) for $\varepsilon \in (0,1)$.

3 Estimates for $t \geq 0$

To prepare our construction of a solution to (1.2) as a limit of $(u_\varepsilon, v_\varepsilon)$ along an appropriate sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ fulfilling $\varepsilon_j \searrow 0$, to be achieved in Lemma 5.1, let us first derive some estimates, independent of $\varepsilon \in (0,1)$, which provide information on the entire half-axis $[0,\infty)$ of times, especially including the temporal origin. With regard to the second component, in view of the low regularity requirements on the respective initial data an apparently exhaustive outcome can already be achived by means of a standard L^2 testing procedure.

Lemma 3.1 *For each* $\varepsilon \in (0,1)$*, we have*

$$\frac{1}{2}\int_\Omega v_\varepsilon^2(\cdot,t) + \int_0^t \int_\Omega v_{\varepsilon x}^2 + \int_0^t \int_\Omega v_\varepsilon^2 = \frac{1}{2}\int_\Omega v_{0\varepsilon}^2 + \int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } t > 0, \tag{3.1}$$

and there exists $C > 0$ *such that for any* $\varepsilon \in (0,1)$*,*

$$\int_\Omega v_\varepsilon^2(\cdot,t) \leq C \qquad \text{for all } t \geq 0 \tag{3.2}$$

as well as

$$\int_t^{t+1} \int_\Omega v_{\varepsilon x}^2 \leq C \qquad \text{for all } t \geq 0. \tag{3.3}$$

Proof. On multiplying the second equation in (2.2) by v_ε and integrating by parts, we obtain the identity

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v_\varepsilon^2 + \int_\Omega v_{\varepsilon x}^2 + \int_\Omega v_\varepsilon^2 = \int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } t > 0, \tag{3.4}$$

from which (3.1) immediately results via an integration in time. Furthermore, since due to the fact that $W^{1,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ in the present one-dimensional setting we can find $c_1 > 0$ such that

$\|\psi\|_{L^\infty(\Omega)} \le c_1\|\psi_x\|_{L^2(\Omega)} + c_1\|\psi\|_{L^2(\Omega)}$ for all $\psi \in W^{1,2}(\Omega)$, we can estimate

$$\begin{aligned}
\int_\Omega u_\varepsilon v_\varepsilon &\le \left\{\int_\Omega u_\varepsilon\right\} \cdot \|v_\varepsilon\|_{L^\infty(\Omega)} \\
&= m\|v_\varepsilon\|_{L^\infty(\Omega)} \\
&\le mc_1\|v_{\varepsilon x}\|_{L^2(\Omega)} + mc_1\|v_\varepsilon\|_{L^2(\Omega)} \\
&\le \frac{1}{2}\int_\Omega v_{\varepsilon x}^2 + \frac{1}{2}\int_\Omega v_\varepsilon^2 + c_1^2 m^2,
\end{aligned}$$

for $t > 0$ via (2.3) and Young's inequality. Therefore, (3.4) implies that

$$\frac{d}{dt}\int_\Omega v_\varepsilon^2 + \int_\Omega v_{\varepsilon x}^2 + \int_\Omega v_\varepsilon^2 \le c_2 := 2c_1^2 m^2 \qquad \text{for all } t > 0, \tag{3.5}$$

which upon an ODE comparison firstly entails that

$$\int_\Omega v_\varepsilon^2 \le c_3 := \max\left\{c_2\,, \sup_{\varepsilon\in(0,1)} \int_\Omega v_{0\varepsilon}^2\right\} \qquad \text{for all } t \ge 0$$

with c_3 being finite thanks to (2.1). Secondly, upon direct integration (3.5) thereafter shows that

$$\int_t^{t+1}\int_\Omega v_{\varepsilon x}^2 \le \int_\Omega v_\varepsilon^2(\cdot,t) + c_2 \le c_3 + c_2 \qquad \text{for all } t \ge 0,$$

whence both (3.2) and (3.3) hold if we choose $C > 0$ appropriately large. □

Through a further and meanwhile also quite standard testing procedure, the space-time L^2 integrability property (3.3) of the taxis gradient can be seen to entail a similar feature of $\partial_x \ln(u_\varepsilon + 1)$:

Lemma 3.2 *There exists $C > 0$ such that whenever $\varepsilon \in (0,1)$,*

$$\int_t^{t+1}\int_\Omega \frac{u_{\varepsilon x}^2}{(u_\varepsilon+1)^2} \le C \qquad \textit{for all } t \ge 0. \tag{3.6}$$

PROOF. On the basis of the first equation in (2.2), by using Young's inequality we obtain that

$$\frac{d}{dt}\int_\Omega \ln(u_\varepsilon+1) = \int_\Omega \frac{u_{\varepsilon x}^2}{(u_\varepsilon+1)^2} - \int_\Omega \frac{u_{\varepsilon x}}{u_\varepsilon+1}\cdot v_{\varepsilon x} \ge \frac{1}{2}\int_\Omega \frac{u_{\varepsilon x}^2}{(u_\varepsilon+1)^2} - \frac{1}{2}\int_\Omega v_{\varepsilon x}^2 \qquad \text{for all } t > 0,$$

which after a time integration implies (3.6) due to Lemma 3.1 and the fact that $\int_\Omega \ln(u_\varepsilon+1) \le \int_\Omega u_\varepsilon = m$ for all $t > 0$ according to (2.3). □

Now in the one-dimensional context under consideration, by means of an interpolation argument the above logarithmic estimate can be combined with (2.3) so as to provide some information on u_ε itself and its gradient.

Lemma 3.3 *There exists $C > 0$ such that whenever $\varepsilon \in (0,1)$,*

$$\int_t^{t+1} \int_\Omega u_\varepsilon^2 \le C \qquad \textit{for all } t \ge 0 \tag{3.7}$$

and

$$\int_t^{t+1} |u_{\varepsilon x}| \le C \qquad \textit{for all } t \ge 0 \tag{3.8}$$

as well as

$$\int_t^{t+1} \|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)} ds \le C \qquad \textit{for all } t \ge 0. \tag{3.9}$$

Proof. According to Lemma 3.2. we can find $c_1 > 0$ such that for all $\varepsilon \in (0,1)$,

$$\int_t^{t+1} \int_\Omega \frac{u_{\varepsilon x}^2}{(u_\varepsilon+1)^2} \le c_1 \qquad \text{for all } t \ge 0, \tag{3.10}$$

which we will exploit by making use of the fact that the one-dimensional Gagliardo-Nirenberg inequality provides $c_2 > 0$ such that

$$\|\psi\|_{L^2(\Omega)}^2 \le c_2 \|\psi_x\|_{L^1(\Omega)} \|\psi\|_{L^1(\Omega)} + c_2 \|\psi\|_{L^1(\Omega)}^2 \qquad \text{for all } \psi \in W^{1,1}(\Omega). \tag{3.11}$$

Therefore, namely, for fixed $\varepsilon \in (0,1)$ and $t \ge 0$ writing

$$K_\varepsilon(t) := \int_t^{t+1} \int_\Omega (u_\varepsilon+1)^2,$$

we can firstly relate the expression in (3.8) to $K_\varepsilon(t)$ via the Cauchy-Schwarz inequality by estimating

$$\begin{aligned} \int_t^{t+1} \int_\Omega |u_{\varepsilon x}| &\le \left\{ \int_t^{t+1} \int_\Omega \frac{u_{\varepsilon x}^2}{(u_\varepsilon+1)^2} \right\}^{\frac{1}{2}} \cdot \left\{ \int_t^{t+1} \int_\Omega (u_\varepsilon+1)^2 \right\}^{\frac{1}{2}} \\ &\le \sqrt{c_1} \cdot \sqrt{K_\varepsilon(t)} \end{aligned} \tag{3.12}$$

due to (3.10). Thereupon, an application of (3.11) shows that thanks to (2.3), with $c_3 := m + |\Omega|$ we have

$$\begin{aligned} \int_t^{t+1} \int_\Omega (u_\varepsilon+1)^2 &\le c_2 \int_t^{t+1} \|u_{\varepsilon x}(\cdot,s)\|_{L^1(\Omega)} \|u_\varepsilon(\cdot,s)+1\|_{L^1(\Omega)} ds + c_2 \int_t^{t+1} \|u_\varepsilon(\cdot,s)+1\|_{L^1(\Omega)}^2 ds \\ &\le c_2 c_3 \int_t^{t+1} \|u_{\varepsilon x}(\cdot,s)\|_{L^1(\Omega)} ds + c_2 c_3^2 \\ &\le \sqrt{c_1} c_2 c_3 \cdot \sqrt{K_\varepsilon(t)} + c_2 c_3^2. \end{aligned}$$

Consequently, using Young's inequality we see that

$$\begin{aligned} K_\varepsilon(t) &\le \sqrt{c_1} c_2 c_3 \cdot \sqrt{K_\varepsilon(t)} + c_2 c_3^2 \\ &\le \frac{1}{2} K_\varepsilon(t) + \frac{1}{2} c_1 c_2^2 c_3^2 + c_2 c_3^2 \end{aligned}$$

and that thus

$$\int_t^{t+1}\int_\Omega (u_\varepsilon+1)^2 = K_\varepsilon(t) \le c_1c_2^2c_3^2 + 2c_2c_3^2$$

for any $\varepsilon\in(0,1)$ and $t\ge 0$. In view of (3.12), besides (3.7) this also entails (3.8), whereas (3.9) is an immediate by-product of the latter because of the fact that $W^{1,1}(\Omega)\hookrightarrow L^\infty(\Omega)$. $\Box$

4 Estimates for $t\ge\tau>0$

We next intend to complement the above temporally global regularity features by further estimates which involve more favorable topological information, but which on the other hand are somewhat local in time only, and which in particular can quite easily be seen not to be extensible to time intervals touching $t=0$, at least not in cases of initial data outside e.g. $L^\infty(\Omega)\times W^{1,2}(\Omega)$. Our starting point in this direction makes use of Lemma 3.1 and Lemma 3.3 to derive the following by means of another testing process.

Lemma 4.1 *Let $\tau\in(0,1)$. Then there exists $C(\tau)>0$ such that for all $\varepsilon\in(0,1)$,*

$$\int_\Omega v_{\varepsilon x}^2(\cdot,t)\le C(\tau) \qquad \textit{for all } t\ge\tau. \tag{4.1}$$

PROOF. On multiplying the second equation in (2.2) by $-v_{\varepsilon x}$ and integrating by parts, due to Young's inequality we obtain that

$$\frac12\frac{d}{dt}\int_\Omega v_{\varepsilon x}^2 + \int_\Omega v_{\varepsilon xx}^2 = -\int_\Omega v_{\varepsilon x}^2 - \int_\Omega u_\varepsilon v_{\varepsilon xx} \le \int_\Omega v_{\varepsilon xx}^2 + \frac14\int_\Omega u_\varepsilon^2 \qquad \text{for all } t>0$$

and hence

$$\frac{d}{dt}\int_\Omega v_{\varepsilon x}^2 \le \frac12\int_\Omega u_\varepsilon^2 \qquad \text{for all } t>0. \tag{4.2}$$

To prepare a further integration of this, we observe that as a consequence of Lemma 3.1 and Lemma 3.3, we can find $c_1>0$ and $c_2>0$ such that for all $\varepsilon\in(0,1)$,

$$\int_{t-\tau}^t\int_\Omega v_{\varepsilon x}^2\le c_1 \qquad \text{for all } t\ge\tau \tag{4.3}$$

and

$$\int_{t-\tau}^t\int_\Omega u_\varepsilon^2\le c_2 \qquad \text{for all } t\ge\tau. \tag{4.4}$$

Therefore, namely, given any $t\ge\tau$ we particularly infer from (4.3) that for each $\varepsilon\in(0,1)$ we can fix $t_0(\varepsilon)\in(t-\tau,t)$ such that

$$\int_\Omega v_{\varepsilon x}^2(\cdot,t_0(\varepsilon))\le\frac{c_1}{\tau},$$

and that hence integrating (4.2) over $(t_0(\varepsilon), t)$ yields

$$\begin{aligned}\int_\Omega v_{\varepsilon x}^2(\cdot,t) &\le \int_\Omega v_{\varepsilon x}^2(\cdot,t_0(\varepsilon)) + \frac{1}{2}\int_{t_0(\varepsilon)}^t \int_\Omega u_\varepsilon^2 \\ &\le \frac{c_1}{\tau} + \frac{c_2}{2} \qquad \text{for all } \varepsilon \in (0,1)\end{aligned}$$

because of (4.4) and the fact that $t_0(\varepsilon) > t - \tau$. $\Box$

Together with (3.9), the latter provides sufficient information on the cross-diffusion term in (2.2) so as to imply the following bound through an argument based on smoothing estimates for the heat semigroup.

Lemma 4.2 *For all $\tau \in (0,1)$, one can find $C(\tau) > 0$ such that for all $\varepsilon \in (0,1)$,*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C(\tau) \qquad \textit{for all } t \ge \tau. \tag{4.5}$$

Proof. For fixed $\tau \in (0,1)$, applying (3.9) and Lemma 4.1 we can fix $c_1 > 0$ and $c_2(\tau) > 0$ such that whenever $\varepsilon \in (0,1)$,

$$\int_{\frac{\tau}{2}}^\tau \|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)} ds \le c_1 \qquad \text{for all } t \ge \tau \tag{4.6}$$

and

$$\|v_{\varepsilon x}(\cdot,t)\|_{L^2(\Omega)} \le c_2(\tau) \qquad \text{for all } t \ge \frac{\tau}{2}, \tag{4.7}$$

where (4.6) in particular ensures that for each $\varepsilon \in (0,1)$ it is possible to fix $t_0(\varepsilon,\tau) \in (\frac{\tau}{2}, \tau)$ satisfying

$$\|u_\varepsilon(\cdot, t_0(\varepsilon,\tau))\|_{L^\infty(\Omega)} \le \frac{2c_1}{\tau}. \tag{4.8}$$

Apart from that, let us pick any $q \in (1,2)$ and recall a well-known smoothing property ([9], [28]) of the Neumann heat semigroup over Ω, denoted here by $(e^{-\sigma A})_{\sigma \ge 0}$ with $A = -(\cdot)_{xx}$, in choosing $c_3 > 0$ and $\lambda > 0$ fulfilling

$$\|e^{-\sigma A}\psi_x\|_{L^\infty(\Omega)} \le c_3 \sigma^{-\frac{1}{2}-\frac{1}{2q}} e^{-\lambda\sigma} \|\psi\|_{L^q(\Omega)} \qquad \text{for all } \sigma > 0) \text{ and each } \psi \in C^1(\overline{\Omega}) \text{ such that } \psi|_{\partial\Omega} = 0. \tag{4.9}$$

Along with the maximum principle and the nonnegativity of u_ε, through a variation-of-constants representation associated with the first equation in (2.2) this warrants that for all $\varepsilon \in (0,1)$ and each $t \ge t_0(\varepsilon,\tau)$,

$$\begin{aligned}\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} &= \left\| e^{-(t-t_0(\varepsilon,\tau))A} u_\varepsilon(\cdot,t_0(\varepsilon,\tau)) - \int_{t_0(\varepsilon,\tau)}^t e^{-(t-s)A}\partial_x \Big\{ u_\varepsilon(\cdot,s) v_{\varepsilon x}(\cdot,s)\Big\} ds \right\|_{L^\infty(\Omega)} \\ &\le \left\| e^{-(t-t_0(\varepsilon,\tau))A} u_\varepsilon(\cdot,t_0(\varepsilon,\tau))\right\|_{L^\infty(\Omega)} + \int_{t_0(\varepsilon,\tau)}^t \left\| e^{-(t-s)A}\partial_x \Big\{ u_\varepsilon(\cdot,s) v_{\varepsilon x}(\cdot,s)\Big\}\right\|_{L^\infty(\Omega)} ds \\ &\le \left\| u_\varepsilon(\cdot,t_0(\varepsilon,\tau))\right\|_{L^\infty(\Omega)} \\ &\quad + c_3 \int_{t_0(\varepsilon,\tau)}^t (t-s)^{-\frac{1}{2}-\frac{1}{2q}} e^{-\lambda(t-s)} \|u_\varepsilon(\cdot,s) v_{\varepsilon x}(\cdot,s)\|_{L^q(\Omega)} ds.\end{aligned} \tag{4.10}$$

As our restriction $t_0(\varepsilon,\tau) > \frac{\tau}{2}$ ensures that here we may combine the Hölder inequality and (2.3) with (4.7) to see that

$$\begin{aligned}
\|u_\varepsilon(\cdot,s)v_{\varepsilon x}(\cdot,s)\|_{L^q(\Omega)} &\le \|u_\varepsilon(\cdot,s)\|_{L^{\frac{2q}{2-q}}(\Omega)} \|v_{\varepsilon x}(\cdot,s)\|_{L^2(\Omega)} \\
&\le \|u_\varepsilon(\cdot,s)\|^a_{L^\infty(\Omega)} \|u_\varepsilon(\cdot,s)\|^{1-a}_{L^1(\Omega)} \|v_{\varepsilon x}(\cdot,s)\|_{L^2(\Omega)} \\
&\le m^{1-a} c_2(\tau) \|u_\varepsilon(\cdot,s)\|^a_{L^\infty(\Omega)} \qquad \text{for all } s \ge t_0(\varepsilon,\tau)
\end{aligned}$$

with $a := \frac{3q-2}{2q} \in (0,1)$, from (4.10) and (4.8) it follows that if we let

$$M(\varepsilon,\tau,T) := \max_{t\in[t_0(\varepsilon,\tau),T]} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}, \qquad \tau\in(0,1),\ T>1,\ \varepsilon\in(0,1),$$

then

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{2c_1}{\tau} + m^{1-a}c_2(\tau)c_3M^a(\varepsilon,\tau,T)\int_{t_0(\varepsilon,\tau)}^t (t-s)^{-\frac12-\frac{1}{2q}} e^{-\lambda(t-s)}ds$$
$$\text{for all } t\in[t_0(\varepsilon,\tau),T] \text{ and } \varepsilon\in(0,1).$$

Since herein

$$\begin{aligned}
\int_{t_0(\varepsilon,\tau)}^t (t-s)^{-\frac12-\frac{1}{2q}} e^{-\lambda(t-s)}ds &= \lambda^{-\frac12+\frac{1}{2q}} \int_0^{\lambda(t-t_0(\varepsilon,\tau))} \xi^{-\frac12-\frac{1}{2q}} e^{-\xi}d\xi \\
&\le c_4 := \lambda^{-\frac12+\frac{1}{2q}}\Gamma\Big(\frac12-\frac{1}{2q}\Big) \qquad \text{for all } t\ge t_0(\varepsilon,\tau) \text{ and } \varepsilon\in(0,1)
\end{aligned}$$

thanks to the fact that $q>1$, this implies that if we let $c_5(\tau) := \max\{\frac{2c_1}{\tau}, m^{1-a}c_2(\tau)c_3c_4\}$, then

$$M(\varepsilon,\tau,T) \le c_5(\tau) + c_5(\tau)M^a(\varepsilon,\tau,T) \qquad \text{for all } \varepsilon\in(0,1), \tau\in(0,1) \text{ and } T>1$$

and hence

$$M(\varepsilon,\tau,T) \le \max\Big\{1\,,\,(2c_5(\tau))^{\frac{1}{1-a}}\Big\} \qquad \text{for all } \varepsilon\in(0,1), \tau\in(0,1) \text{ and } T>1.$$

Now using that $t_0(\varepsilon,\tau) < \tau$, we only need to take $T \nearrow \infty$ here to directly derive (4.5) from this. □

We can now perform a standard bootstrap procedure to finally obtain estimates in favorably small spaces.

Lemma 4.3 *Let $\tau\in(0,1)$. Then there exist $\theta=\theta(\tau)\in(0,1)$ and $C(\tau)>0$ such that whenever $\varepsilon\in(0,1)$,*

$$\|u_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} + \|v_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} \le C(\tau) \qquad \textit{for all } t\ge\tau. \tag{4.11}$$

PROOF. Since Lemma 4.2 together with Lemma 4.1 ensures that $(u_\varepsilon v_{\varepsilon x})_{\varepsilon\in(0,1)}$ is bounded in $L^\infty((\tau,\infty);L^2(\Omega))$ for all $\tau\in(0,1)$, relying on the boundedness of $(u_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^\infty(\Omega\times(\tau,\infty))$ for any such τ, as furthermore asserted by Lemma 4.2, we may invoke a well-known result on Hölder regularity in scalar parabolic equations ([23]) to see that for each $\tau\in(0,1)$ we can find $\theta_1=\theta_1(\tau)\in(0,1)$

and $c_1(\tau) > 0$ such that $\|u_\varepsilon\|_{C^{\theta_1,\frac{\theta_1}{2}}(\overline{\Omega}\times[t,t+1])} \le c_1(\tau)$ for all $t \ge \frac{\tau}{4}$ and $\varepsilon \in (0,1)$. Thereupon, standard parabolic Schauder theory ([16]) applies so as to yield $\theta_2 = \theta_2(\tau) \in (0,1)$ and $c_2(\tau) > 0$ fulfilling $\|v_\varepsilon\|_{C^{2+\theta_2,1+\frac{\theta_2}{2}}(\overline{\Omega}\times[t,t+1])} \le c_2(\tau)$ for all $t \ge \frac{\tau}{2}$ and any $\varepsilon \in (0,1)$. Having thus obtained corresponding Hölder bounds for the coefficients $a_\varepsilon := v_{\varepsilon x}$ and $b_\varepsilon := v_{\varepsilon xx}$ in $((u_\varepsilon v_{\varepsilon x})_x)_{\varepsilon\in(0,1)} = (a_\varepsilon u_{\varepsilon x} + b_\varepsilon u_\varepsilon)_{\varepsilon\in(0,1)}$, again from parabolic Schauder theory we may finally infer the existence of $\theta_3 = \theta_3(\tau) \in (0,1)$ and $c_3(\tau) > 0$ such that $\|u_\varepsilon\|_{C^{2+\theta_3,1+\frac{\theta_3}{2}}(\overline{\Omega}\times[t,t+1])} \le c_3(\tau)$ for all $t \ge \tau$ and $\varepsilon \in (0,1)$, whereby the proof is completed. $\Box$

5 Passing to the limit and constructing a smooth solution for $t > 0$

We are thereby prepared to construct a limit pair by extracting a conveniently convergent sequence of the above solutions, and to already assert the intended solution properties thereof outside the initial time.

Lemma 5.1 *There exist $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ as well as nonnegative functions*

$$\begin{cases} u \in C^\infty(\overline{\Omega}\times(0,\infty)) \cap L^\infty((0,\infty);L^1(\Omega)) \cap L^2_{loc}(\overline{\Omega}\times[0,\infty)) \qquad \textit{and} \\ v \in C^\infty(\overline{\Omega}\times(0,\infty)) \cap L^\infty((0,\infty);L^2(\Omega)) \cap L^2_{loc}([0,\infty);W^{1,2}(\Omega)) \end{cases} \tag{5.1}$$

such that $\varepsilon_j \searrow 0$ as $j\to\infty$ and

$$u_\varepsilon \to u \qquad \textit{in } C^{2,1}_{loc}(\overline{\Omega}\times(0,\infty)), \tag{5.2}$$

$$u_\varepsilon \to u \qquad \textit{in } L^1_{loc}(\overline{\Omega}\times[0,\infty)), \tag{5.3}$$

$$u_\varepsilon \rightharpoonup u \qquad \textit{in } L^2_{loc}(\overline{\Omega}\times[0,\infty)), \tag{5.4}$$

$$v_\varepsilon \to v \qquad \textit{in } C^{2,1}_{loc}(\overline{\Omega}\times(0,\infty)), \tag{5.5}$$

$$v_\varepsilon \to v \qquad \textit{in } L^2_{loc}(\overline{\Omega}\times[0,\infty)) \qquad \textit{as well as} \tag{5.6}$$

$$v_{\varepsilon x} \rightharpoonup v_x \qquad \textit{in } L^2_{loc}(\overline{\Omega}\times[0,\infty)) \tag{5.7}$$

as $\varepsilon = \varepsilon_j \searrow 0$. Moreover, (u,v) solves the boundary value problem in (1.2) in the classical sense in $\overline{\Omega}\times(0,\infty)$, and the mass conservation and boundedness properties in (1.6) and (1.7) hold.

Proof. On the basis of Lemma 4.3, the Arzelà-Asoli theorem provides $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ and nonnegative functions u and v from $C^{2,1}(\overline{\Omega}\times(0,\infty))$ such that $\varepsilon_j \searrow 0$ as $j\to\infty$, and that both (5.2) and (5.5) hold as $\varepsilon = \varepsilon_j \searrow 0$. As for each $T > 0$ the family $(u_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^2(\Omega\times(0,T))$ by Lemma 3.3, (5.2) together with Egorov's theorem and the Vitali convergence theorem moreover entails that (5.4) and (5.3) are valid along the same sequence, while (1.6) is an evident consequence of (5.2) and (2.3).

Likewise, (5.7) results from (5.5) and the fact that $(v_{\varepsilon x})_{\varepsilon\in(0,1)}$ is bounded in $L^2(\Omega\times(0,T))$ for all $T > 0$ due to (3.3), while noting that thanks to the Gagliardo-Nirenberg inequality and (3.2) we can find $c_1 > 0$ and $c_2 > 0$ such that

$$\begin{aligned} \int_\Omega v_\varepsilon^6 &\le c_1\|v_{\varepsilon x}\|^2_{L^2(\Omega)}\|v_\varepsilon\|^4_{L^2(\Omega)} + c_1\|v_\varepsilon\|^6_{L^2(\Omega)} \\ &\le c_2\|v_{\varepsilon x}\|^2_{L^2(\Omega)} + c_2 \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1), \end{aligned}$$

from (3.3) we moreover infer boundedness of $(v_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^6(\Omega\times(0,T))$ for all $T>0$. Therefore, the strong convergence property in (5.6) is implied by (5.5) and, again, the Vitali convergence theorem, and hence the proof can be completed on noting that clearly $u\in L^\infty((0,\infty);L^1(\Omega))\cap L^\infty((\tau,\infty);C^2(\overline{\Omega}))$ and $v\in L^\infty((0,\infty);L^2(\Omega))\cap L^\infty((\tau,\infty);C^2(\overline{\Omega}))$ for all $\tau>0$ by (1.6), (3.2), (4.11), (5.2), (5.5) and Fatou's lemma, and on taking $\varepsilon=\varepsilon_j\searrow 0$ in (2.2) to verify validity of the first three lines from (1.2); therefore, namely, repeated application of standard higher-order interior parabolic Schauder theory ([16]) to both parabolic sub-problems of (1.2) finally reveals that actually u and v belong to $C^\infty(\overline{\Omega}\times(0,\infty))$. $\square$

6 Initial behavior of v

It remains to show that u and v satisfy the integral identities (1.8) and (1.9), and moreover attain their respectively expected initial data in the sense specified in (1.10) and (1.11). Firstly concentrating the second component here, on the basis of the approximation properties from Lemma 5.1 we can readily make sure that v indeed solves its sub-problem in (1.2) in the claimed natural weak sense.

Lemma 6.1 *The functions u and v gained in Lemma 5.1 satisfy the integral identity (1.9) for any $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$.*

PROOF. For arbitrary $\varepsilon\in(0,1)$, (2.2) implies that

$$-\int_0^\infty\int_\Omega v_\varepsilon\varphi_t-\int_\Omega v_{0\varepsilon}\varphi(\cdot,0)=-\int_0^\infty\int_\Omega v_{\varepsilon x}\varphi_x-\int_0^\infty\int_\Omega v_\varepsilon\varphi+\int_0^\infty\int_\Omega u_\varepsilon\varphi,$$

where we only need to apply (5.6), (2.1), (5.7) and (5.4) in a straightforward manner to derive (1.9) on taking $\varepsilon=\varepsilon_j\searrow 0$ with $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ as provided by Lemma 5.1. $\square$

Let us next prepare our verification not only of (1.11) but also of (1.8) by the following improvement of the topological information in the convergence statement from (5.7).

Lemma 6.2 *Let $(\varepsilon_j)_{j\in\mathbb{N}}$ and v be as in Lemma 5.1. Then for each $T>0$,*

$$v_{\varepsilon x}\to v_x \quad \textit{in } L^2(\Omega\times(0,T)) \qquad \textit{as } \varepsilon=\varepsilon_j\searrow 0. \tag{6.1}$$

PROOF. For $\varepsilon\in(0,1)$ and $\varepsilon'\in(0,1)$ we use (2.2) to compute

$$\frac12\frac{d}{dt}\int_\Omega(v_\varepsilon-v_{\varepsilon'})^2+\int_\Omega(v_{\varepsilon x}-v_{\varepsilon' x})^2+\int_\Omega(v_\varepsilon-v_{\varepsilon'})^2=\int_\Omega(u_\varepsilon-u_{\varepsilon'})(v_\varepsilon-v_{\varepsilon'}) \qquad \text{for all } t>0,$$

from which it follows that whenever $T>0$,

$$\int_0^T\int_\Omega(v_{\varepsilon x}-v_{\varepsilon' x})^2\le\frac12\int_\Omega(v_{0\varepsilon}-v_{0\varepsilon'})^2+\int_0^T\int_\Omega(u_\varepsilon-u_{\varepsilon'})(v_\varepsilon-v_{\varepsilon'}) \qquad \text{for all } \varepsilon\in(0,1) \text{ and } \varepsilon'\in(0,1). \tag{6.2}$$

Here from Lemma 5.1 we know that for each fixed $\varepsilon\in(0,1)$,

$$u_\varepsilon-u_{\varepsilon'}\rightharpoonup u_\varepsilon-u \text{ in } L^2(\Omega\times(0,T)) \quad \text{and} \quad v_\varepsilon-v_{\varepsilon'}\to v_\varepsilon-v \text{ in } L^2(\Omega\times(0,T)) \qquad \text{as } (\varepsilon_j)_{j\in\mathbb{N}}\ni\varepsilon'\searrow 0,$$

and that hence

$$\int_0^T \int_\Omega (u_\varepsilon - u_{\varepsilon'})(v_\varepsilon - v_{\varepsilon'}) \to \int_0^T \int_\Omega (u_\varepsilon - u)(v_\varepsilon - v) \qquad \text{as } (\varepsilon_j)_{j\in\mathbb{N}} \ni \varepsilon' \searrow 0,$$

so that since furthermore $v_{0\varepsilon} - v_{0\varepsilon'} \to v_{0\varepsilon} - v_0$ in $L^2(\Omega)$ as $\varepsilon' \searrow 0$ by (2.1), by means of (5.5) and Fatou's lemma we see that (6.2) implies the inequality

$$\begin{aligned} \int_0^T \int_\Omega (v_{\varepsilon x} - v_x)^2 &\leq \liminf_{(\varepsilon_j)_{j\in\mathbb{N}}\ni\varepsilon'\searrow 0} \left\{ \frac{1}{2}\int_\Omega (v_{0\varepsilon} - v_{0\varepsilon'})^2 + \int_0^T \int_\Omega (u_\varepsilon - u_{\varepsilon'})(v_\varepsilon - v_{\varepsilon'}) \right\} \\ &= \frac{1}{2}\int_\Omega (v_{0\varepsilon} - v_0)^2 + \int_0^T \int_\Omega (u_\varepsilon - u)(v_\varepsilon - v) \qquad \text{for all } \varepsilon \in (0,1). \end{aligned} \tag{6.3}$$

Since here due to the Cauchy-Schwarz inequality, Lemma 3.3 and Lemma 5.1 we have

$$\begin{aligned} \int_0^T \int_\Omega (u_\varepsilon - u)(v_\varepsilon - v) &\leq \|u_\varepsilon - u\|_{L^2(\Omega\times(0,T))} \|v_\varepsilon - v\|_{L^2(\Omega\times(0,T))} \\ &\leq \Big(\|u_\varepsilon\|_{L^2(\Omega\times(0,T))} + \|u\|_{L^2(\Omega\times(0,T))} \Big) \cdot \|v_\varepsilon - v\|_{L^2(\Omega\times(0,T))} \\ &\to 0 \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0, \end{aligned}$$

again in view of (2.1) we readily infer (6.1) from (6.3). □

In order to next derive (1.11) from this, as a preliminary for the corresponding proof of Lemma 6.3, and moreover also of Lemma 7.2 below, for $t_0 > 0$ and $\delta \in (0,1)$ we introduce

$$\zeta_\delta(t) \equiv \zeta_\delta^{(t_0)}(t) := \begin{cases} 1 & \text{if } t \in [0, t_0], \\ 1 - \frac{t-t_0}{\delta} & \text{if } t \in (t_0, t_0 + \delta), \\ 0 & \text{if } t \geq t_0 + \delta. \end{cases} \tag{6.4}$$

We may then rely on the fact that $\zeta_\delta \in W^{1,\infty}((0,\infty))$ to find $(\zeta_{\delta k})_{k\in\mathbb{N}} \subset C^\infty([0,\infty))$ such that

$$\zeta_{\delta k}(0) = 1 \text{ for all } k \in \mathbb{N} \quad \text{as well as} \quad \zeta_{\delta k} \overset{\star}{\rightharpoonup} \zeta_\delta \text{ and } \zeta'_{\delta k} \overset{\star}{\rightharpoonup} \zeta'_\delta \text{ in } L^\infty((0,\infty)) \quad \text{as } k \to \infty. \tag{6.5}$$

Using these function to achieve an appropriate cut-off in a testing procedure directly acting on the weak identity (1.9), we can complete our considerations concerning v as follows.

Lemma 6.3 *Let v be as in Lemma 5.1. Then*

$$v(\cdot, t) \to v_0 \quad \text{in } L^2(\Omega) \qquad \text{as } t \searrow 0. \tag{6.6}$$

Proof. We fix $t_0 > 0$ and $\delta \in (0,1)$, and take $\zeta_\delta \equiv \zeta_\delta^{(t_0)}$ and $(\zeta_{\delta k})_{k\in\mathbb{N}}$ as accordingly described in (6.4) and (6.5). Given $\psi \in C^\infty(\overline{\Omega})$, we may then apply (1.9) to $\varphi(x,t) := \zeta_{\delta k}(t) \cdot \psi(x)$, $x \in \overline{\Omega}, t \geq 0$, to find that

$$\begin{aligned} &-\int_0^\infty \int_\Omega \zeta'_{\delta k}(t) v(x,t)\psi(x)dxdt - \int_\Omega v_0(x)\psi(x)dx \\ &\quad = -\int_0^\infty \int_\Omega \zeta_{\delta k}(t) v_x(x,t)\psi(x)dxdt - \int_0^\infty \int_\Omega \zeta_{\delta k}(t) v(x,t)\psi(x)dxdt \\ &\qquad + \int_0^\infty \int_\Omega \zeta_{\delta k}(t) u(x,t)\psi(x)dxdt \qquad \text{for all } k \in \mathbb{N}, \end{aligned}$$

where since

$$\{v, v_x, u\} \subset L^2_{loc}(\overline{\Omega} \times [0,\infty)) \tag{6.7}$$

by Lemma 5.1, taking $k \to \infty$ shows that in view of (6.5) and (6.4),

$$\begin{aligned}
&\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\int_\Omega v(x,t)\psi(x)dxdt - \int_\Omega v_0(x)\psi(x)dx \\
&\qquad = -\int_0^\infty\int_\Omega \zeta_\delta(t)v_x(x,t)\psi(x)dxdt - \int_0^\infty\int_\Omega \zeta_\delta(t)v(x,t)\psi(x)dxdt \\
&\qquad\quad +\int_0^\infty\int_\Omega \zeta_\delta(t)u(x,t)\psi(x)dxdt \qquad \text{for all } \delta\in(0,1).
\end{aligned}$$

As v is continuous in $\overline{\Omega}\times(0,\infty)$, we may let $\delta \searrow 0$ here to see that again thanks to (6.7),

$$\begin{aligned}
\int_\Omega v(x,t_0)\psi(x)dx - \int_\Omega v_0(x)\psi(x)dx &= -\int_0^{t_0}\int_\Omega v_x(x,t)\psi(x)dxdt - \int_0^{t_0}\int_\Omega v(x,t)\psi(x)dxdt \\
&\quad +\int_0^{t_0}\int_\Omega u(x,t)\psi(x)dxdt \qquad \text{for all } t_0>0.
\end{aligned}$$

Once more via the inclusion in (6.7), this entails that

$$\int_\Omega v(x,t_0)\psi(x)dx - \int_\Omega v_0(x)\psi(x)dx \to 0 \qquad \text{as } t_0\searrow 0$$

and that hence

$$v(\cdot,t_0) \rightharpoonup v_0 \quad \text{in } L^2(\Omega) \qquad \text{as } t_0 \searrow 0 \tag{6.8}$$

due to the density of $C^\infty(\overline{\Omega})$ in $L^2(\Omega)$ and the boundedness of $(v(\cdot,t_0))_{t_0>0}$ in $L^2(\Omega)$ asserted by Lemma 5.1.

In order to derive the strong convergence statement in (6.6) from this, we only need to recall (3.1), which when combined with (5.5), Lemma 6.2, (5.6), (2.1) and (5.4), namely, ensures that

$$\frac{1}{2}\int_\Omega v^2(\cdot,t_0) - \frac{1}{2}\int_\Omega v_0^2 = \int_0^{t_0}\int_\Omega v_x^2 + \int_0^{t_0}\int_\Omega v^2 + \int_0^{t_0}\int_\Omega uv \qquad \text{for all } t_0>0,$$

and that thus, again by (6.7),

$$\frac{1}{2}\int_\Omega v^2(\cdot,t_0) - \frac{1}{2}\int_\Omega v_0^2 \to 0 \qquad \text{as } t_0 \searrow 0,$$

and thereby entails (6.6) as a consequence of (6.8). □

7 Initial trace of u

Relying on the strong convergence statement from Lemma 6.2, we can next derive (1.8) from (2.2).

Lemma 7.1 *The pair (u,v) from Lemma 5.1 satisfies*

$$u_x \in L^1_{loc}(\overline{\Omega}\times[0,\infty)), \tag{7.1}$$

and the identity (1.8) is valid for each $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$.

Proof. We again let $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ be as in Lemma 5.1, and then especially obtain from (5.2) that $u_{\varepsilon x}(x,t)\to u_x(x,t)$ for all $x\in\Omega$ and $t>0$ as $\varepsilon=\varepsilon_j\searrow 0$. In view of (3.8), the inclusion in (7.1) therefore results from Fatou's lemma.

To verify (1.8), we first fix any $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$ satisfying the additional constraint that $\varphi_x=0$ on $\partial\Omega\times(0,\infty)$, which ensures that integrating by parts in (2.2) yields

$$-\int_0^\infty\int_\Omega u_\varepsilon\varphi_t-\int_\Omega u_{0\varepsilon}\varphi(\cdot,0)=\int_0^\infty\int_\Omega u_\varepsilon\varphi_{xx}+\int_0^\infty\int_\Omega u_\varepsilon v_{\varepsilon x}\varphi_x \qquad \text{for all } \varepsilon\in(0,1).$$

Here we apply (5.4) to the respective first integrals on the left and on the right, and combine (5.4) with the strong convergence statement from Lemma 6.2 in the rightmost summand. Moreover, using (2.1) to treat the contribution involving $u_{0\varepsilon}$, on letting $\varepsilon=\varepsilon_j\searrow 0$ we thus arrive at the identity

$$-\int_0^\infty\int_\Omega u\varphi_t-\mu_0(\varphi(\cdot,0))=\int_0^\infty\int_\Omega u\varphi_{xx}+\int_0^\infty\int_\Omega uv_x\varphi_x.$$

Now relying on (7.1) we may a posteriori integrate by parts in the first summand on the right-hand side herein to infer that

$$-\int_0^\infty\int_\Omega u\varphi_t-\mu_0(\varphi(\cdot,0)) \;=\; -\int_0^\infty\int_\Omega u_x\varphi_x+\int_0^\infty\int_\Omega uv_x\varphi_x$$
$$\text{for all } \varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty)) \text{ such that } \varphi_x=0 \text{ on } \partial\Omega\times(0,\infty). \tag{7.2}$$

Since for arbitrary $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$ we can easily construct a sequence $(\varphi_k)_{k\in\mathbb{N}}\subset C_0^\infty(\overline{\Omega}\times[0,\infty))$ such that $\varphi_{kx}=0$ on $\partial\Omega\times(0,\infty)$ for all $k\in\mathbb{N}$, and that as $k\to\infty$ we have $\varphi_k\overset{\star}{\rightharpoonup}\varphi$, $\varphi_{kt}\overset{\star}{\rightharpoonup}\varphi_t$ and $\varphi_{kx}\overset{\star}{\rightharpoonup}\varphi_x$ in $L^\infty(\Omega\times(0,\infty))$ as well as $\varphi_k(\cdot,0)\to\varphi(\cdot,0)$ in $C^0(\overline{\Omega})$, inserting φ_k into (7.2) and taking $k\to\infty$ readily entails (1.8) in the claimed generality. $\square$

In lastly deriving (1.10), we firstly reduce the set of associated test functions to such an extent that a convenient access to (1.8) becomes possible.

Lemma 7.2 *Let u be as given by Lemma 5.1, and let $\psi\in C^\infty(\overline{\Omega})$. Then*

$$\int_\Omega u(\cdot,t)\psi\to\mu_0(\psi) \qquad \textit{as } t\searrow 0. \tag{7.3}$$

Proof. Similar to the procedure from Lemma 6.3, for fixed $t_0>0$, $\delta\in(0,1)$ and $\psi\in C^\infty(\overline{\Omega})$ we employ (1.8) for $\varphi(x,t):=\zeta_{\delta k}(t)\cdot\psi(x)$, $x\in\overline{\Omega}$, $t\ge 0$, where $(\zeta_{\delta k})_{k\in\mathbb{N}}\subset C^\infty([0,\infty))$ satisfies (6.5) with $\zeta_\delta\equiv\zeta_\delta^{(t_0)}$ as defined in (6.4). As a result, we obtain that

$$-\int_0^\infty\int_\Omega \zeta'_{\delta k}(t)u(x,t)\psi(x)dxdt-\mu_0(\psi) \;=\; \int_0^\infty\int_\Omega \zeta_{\delta k}(t)u_x(x,t)\psi_x(x)dxdt$$
$$+\int_0^\infty\int_\Omega \zeta_{\delta k}(t)u(x,t)v_x(x,t)\psi_x(x)dxdt \qquad \text{for all } k\in\mathbb{N},$$

whence observing that u, u_x and uv_x belong to $L^1_{loc}(\overline{\Omega}\times[0,\infty))$ by Lemma 5.1 and Lemma 7.1 we may rely on (6.5) to firstly conclude on taking $k\to\infty$ that

$$\begin{aligned}\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\int_\Omega u(x,t)\psi(x)dxdt-\mu_0(\psi) &= \int_0^\infty\int_\Omega \zeta_\delta(t)u_x(x,t)\psi_x(x)dxdt\\ &\quad+\int_0^\infty\int_\Omega \zeta_\delta(t)u(x,t)v_x(x,t)\psi_x(x)dxdt \qquad \text{for all } \delta\in(0,1),\end{aligned}$$

and to secondly infer by letting $\delta\searrow 0$ here that

$$\begin{aligned}\int_\Omega u(x,t_0)\psi(x)dx-\mu_0(\psi) &= \int_0^{t_0}\int_\Omega u_x(x,t)\psi_x(x)dxdt\\ &\quad+\int_0^{t_0}\int_\Omega u(x,t)v_x(x,t)\psi_x(x)dxdt \qquad \text{for all } t_0>0,\end{aligned}$$

because u is continuous on $\overline{\Omega}\times\{t_0\}$ for any such t_0. Again by local integrability of u_x and of uv_x in $\overline{\Omega}\times[0,\infty)$, this shows that

$$\int_\Omega u(x,t_0)\psi(x)dx-\mu_0(\psi)\to 0 \qquad \text{as } t_0\searrow 0$$

and hence establishes (7.3). □

By performing a suitable approximation argument, we can finally make sure that the above restriction on the set of test functions can actually be removed.

Lemma 7.3 *The function u constructed in Lemma 5.1 has the property that actually*

$$\int_\Omega u(\cdot,t)\psi\to\mu_0(\psi) \quad \textit{as } t\searrow 0 \qquad \textit{for all } \psi\in C^0(\overline{\Omega}). \tag{7.4}$$

Proof. The proof proceeds by extension of (7.3) through a straightforward approximation argument: For fixed $\psi\in C^0(\overline{\Omega})$ and $\eta>0$, by using Weierstraß' theorem as well as the continuity of the functional $\mu_0\in(C^0(\overline{\Omega}))^\star$ we can find $\psi_\star\in C^\infty(\overline{\Omega})$ such that taking m as defined in (1.6), besides the inequality

$$m\cdot\|\psi-\psi_\star\|_{L^\infty(\Omega)}<\frac{\eta}{3} \tag{7.5}$$

we have

$$\Big|\mu_0(\psi_\star)-\mu_0(\psi)\Big|<\frac{\eta}{3}. \tag{7.6}$$

In accordance with Lemma 7.2 we thereupon choose $t_\star>0$ small enough fulfilling

$$\Big|\int_\Omega u(\cdot,t)\psi_\star-\mu_0(\psi_\star)\Big|<\frac{\eta}{3} \qquad \text{for all } t\in(0,t_\star),$$

and then only need to combine this with (7.5) and (7.6) to see that for each $t\in(0,t_\star)$, thanks to (1.6) we can estimate

$$\begin{aligned}\Big|\int_\Omega u(\cdot,t)\psi-\mu_0(\psi)\Big| &\le \Big|\int_\Omega u(\cdot,t)\cdot(\psi-\psi_\star)\Big|+\Big|\int_\Omega u(\cdot,t)\psi_\star-\mu_0(\psi_\star)\Big|+\Big|\mu_0(\psi_\star)-\mu_0(\psi)\Big|\\ &\le \Big\{\int_\Omega u(\cdot,t)\Big\}\cdot\|\psi-\psi_\star\|_{L^\infty(\Omega)}+\Big|\int_\Omega u(\cdot,t)\psi_\star-\mu_0(\psi_\star)\Big|+\Big|\mu_0(\psi_\star)-\mu_0(\psi)\Big|\\ &< \frac{\eta}{3}+\frac{\eta}{3}+\frac{\eta}{3}=\eta\end{aligned}$$

and conclude as intended. $\Box$

A simple collection of the information pieces gained above now leads us to our main results in the desired flavor:

Proof of Theorem 1.1. Taking u and v from Lemma 5.1, we obtain (1.10) and (1.11) as direct consequences of Lemma 7.3 and Lemma 6.3, respectively, and see that according to Lemma 5.1 and Lemma 7.1, u and v satisfy (1.5) and belong to $C^\infty(\overline{\Omega}\times(0,\infty))$, where the latter in conjunction with (1.10) and (1.11) moreover asserts validity of (1.4). The claimed solution properties, and especially (1.8) and (1.9) for arbitrary $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$, have been verified in Lemma 5.1 as well as in Lemma 7.1 and Lemma 6.1, and the mass identity as well as the boundedness feature in (1.6) and (1.7) have precisely been established in Lemma 5.1 already. $\Box$

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