Facing low regularity in chemotaxis systems

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Abstract

PDE systems describing chemotaxis, the directed motion of organisms in response to a chemical signal, contain a cross-diffusive term which in many cases causes the unavailability of strong regularity information. An important part of their mathematical analysis is thus concerned with their behavior in situations where solutions are known to blow-up or where singularities cannot be excluded a priori. In this note we review some results, as well as some underlying fundamental analytical ideas, from the context of rigorous blow-up detection, and discuss some approaches addressing the design of solution theories which are able to adequately cope with the possible destabilizing effects of chemotactic cross-diffusion.

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1 Introduction

Chemotaxis systems have been fascinating mathematicians for decades. This is, on the one hand, due to their outstanding relevance in biomathematical modeling at various levels of complexity. Indeed, experimental findings have provided a considerable fund of scenarios in which the ability of individuals to orient their motion in response to chemical gradients goes along with quite colorful collective behavior of the respective population as a whole, with some typical examples ranging from the paradigmatic processes of slime mold formation in *Dictyostelium discoideum* ([75]) and of pattern generation in colonies of e.g. *Salmonella typhimurium* or also *Bacillus subtilis* ([153], [53, 95]), over invasion of tumor cells into healthy tissue ([31]), to the emergence of plume-like aggregates in populations of *Bacillus subtilis* suspended to sessile water drops ([44]), and to self-organization during embryonic development ([111]). The pursuit of understanding corresponding causal nexus, as in most biological contexts not as unquestionably clear as in many situations e.g. in physics, has motivated substantial efforts not only in the modeling literature, but in close connection to this also in associated analytical research.

On the other hand, however, significant part of the interest in chemotaxis models apparently originates from various remarkable mathematical features that some of such systems either have rigorously been proved to possess, or at least, e.g. by means of formal asymptotic analysis or also simulations, have been conjectured to exhibit ([33], [25], [61], [134]). Maintaining evident links to aspects concerning the structure-supporting potential of the respective particular model, representative findings in this

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direction report on taxis-driven instability of spatial homogeneity, partially even in the extreme sense of spontaneous singularity formation, and on threshold effects relating the occurrence of such blow-up phenomena to various system ingredients such as total population sizes or also model parameters, but also on solution behavior near and even beyond singularities.

On this note, it seems worth remarking that the comprehension of explosion-generating features has been forming essential parts in the analysis of further important evolution systems, such as the Navier–Stokes, the Ginzburg–Landau or also nonlinear wave or Schrödinger equations (see e.g. [126], [30], [98], [96] and the references therein).

The classical Keller-Segel system ([75]), widely accepted as a prototypical macroscopic model for self-enhanced chemotaxis, in its fully parabolic version is given by

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ v_t = \Delta v - v + u. \end{cases} \tag{1.1}$$

Here, the population density of some chemotactically active species is denoted by u, and by v the concentration of a signal substance towards higher concentrations of which the former directs its motion. In close neighbourhoods of this system, a noticeable knowledge was available already in the early 2000s, as exhaustively documented in the comprehensive surveys [65] and [66]. More recent advances in the modeling literature, however, have been embedding taxis-type cross-diffusive interaction of the above flavor, as forming the most characteristic ingredient in (1.1), into more involved evolution equations capable of adequately describing more complex biological systems ([64], [110]). Partially in parallel with this, considerable efforts in the field of corresponding analytical research have led to further substantial enrichments with regard to methodological aspects, and thereby especially augmented the knowledge on mathematical features of chemotactic cross-diffusion, both in more complex systems and also in (1.1) itself.

The purpose of the present note consists in retracing some selected among these developments, where in view of a meanwhile abundant literature concerned with the analysis of chemotaxis systems, we concentrate on aspects related to the behavior of such systems in the mathematically most delicate situation in which singularities are either known to arise, or at least cannot be a priori excluded. Even in this reduced scope far from making any claim to be complete, we thereby intend to create some basic impression about mathematical challenges linked to, but also about possible approaches toward the understanding of chemotactic cross-diffusion, with one focus on describing some methods which potentially allow for bridging gaps between an analysis of prototypical systems such as (1.1), and that of models for taxis in more complex frameworks.

2 Taxis-driven singularity formation

The mathematically probably most striking implication of the particular interplay between nonlinear cross-diffusion and chemoattractant production in systems of type (1.1) appears to consist in a resulting ability to describe spontaneous generation of structures in the utmost sense of finite-time blow-up of some solutions in two- and higher-dimensional settings. Results in this direction on the one hand underline the appropriateness of Keller-Segel systems as models for taxis-driven aggregation ([1]); on the other hand, they serve as a perpetual caveat inter alia indicating that cross-diffusion mechanisms of the considered class may not simply destabilize spatial homogeneity in a sense frequently strived for in attempts to model pattern formation ([101]), but that in fact they may do so to quite an extreme extent not suitable for each application context ([64], [119]).

All the more, it thus seems favorable to create an adequately rich set of tools capable of deciding whether or not blow-up may occur in chemotaxis systems, where we note that a huge majority of the existing analytical literature is concerned with the development of techniques for asserting the *absence* of such explosions under appropriate assumptions on the model ingredients such as the respective system parameters or initial data. This may be viewed as reflecting the circumstance that while classical and mainly functional analysis-based methods can well be applied in many cases in which the cross-diffusive action is overbalanced by dissipative mechanisms and especially diffusion, the detection of blow-up seems to require more subtle approaches due to the fact that in sharp contrast to classical examples of explosion-enforcing reaction-diffusion interplay such as in the scalar equation $u_t = \Delta u + u^p$, $p > 1$, no explicit directional effect of the driving nonlinearity seems obvious in chemotaxis systems.

We thus refrain from attempting to report on the numerous relevant contributions addressing issues of global classical solvability, boundedness and large time behavior in various types of chemotaxis systems here (cf. [106], [49], as well as the survey [5] and references therein), and rather concentrate on reviewing some techniques for rigorous blow-up detection.

2.1 Methods of detecting blow-up

2.1.1 Analysis via transformation of parabolic-elliptic chemotaxis systems to scalar parabolic equations

Only more than two decades after the introduction of the model (1.1) by Keller and Segel in 1970, a substantial breakthrough with regard to blow-up detection could be achieved in [72] in the spatially two- and later on in [102] for the three-dimensional case, at least for the simplified parabolic-elliptic variant given by

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ 0 = \Delta v - \mu + u, \\ u(\cdot, 0) = u_0, \end{cases} \tag{2.1}$$

in a ball $\Omega := B_R(0) \subset \mathbb{R}^n$, $R > 0$, $n \geq 2$, where $\mu := \frac{m}{|\Omega|}$ with $m := \int_\Omega u_0$ denotes the conserved spatial average of the population density function $u = u(x,t)$, and where both u and the concentration $v = v(x,t)$ of the attractive signal are supposed to satisfy homogeneous Neumann boundary conditions on $\partial\Omega$. For radially symmetric solutions $(u,v) = (u(r,t), v(r,t))$ in $\Omega \times (0,T)$, namely, the substitution

$$w(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho,t) d\rho, \qquad s \in [0, R^n], \ t \in [0,T), \tag{2.2}$$

can easily be seen to transform (2.1) into the scalar parabolic Dirichlet problem given by

$$\begin{cases} w_t = n^2 s^{2-\frac{2}{n}} w_{ss} + n w w_s - \mu s w_s, & s \in (0, R^n), \ t \in (0,T), \\ w(0,t) = 0, \quad w(R^n, t) = \frac{m}{\omega_n}, & t \in (0,T). \end{cases} \tag{2.3}$$

In this reformulation, the advective character of the cross-diffusion term from (2.1) is reflected in the Burgers-type nonlinearity $w w_s$ which, now carrying a comparatively clear directional information, counteracts the linear second-order contribution $s^{2-\frac{2}{n}}w_{ss}$ which is diffusive in its nature, but degenerate at $s = 0$.

Now a first and quite evident option to analyze the resulting interplay consists in applying methods based on parabolic comparison. In fact, in the case $n = 2$ it was shown in [72] by means of an explicit construction that for suitably large values of μ one can find nonnegative initial data w_0 and $T_\star > 0$ in such a way that (2.3) does not admit any classical solution w in $(0, R^2) \times (0, T_\star)$ with $w(\cdot, 0) = w_0$ for which w_s is bounded, hence implying nonexistence of a classical solution to the associated initial-boundary value problem for (2.1) in $\Omega \times (0, T_\star)$.

Alternatively, a contradictory argument can be designed by tracing the time evolution of suitably chosen, possibly weighted, linear functionals of w. In said planar setting, for instance, the quantity

$$\Phi(t) := \int_0^{R^2} w(s,t)ds, \qquad t \in [0,T), \tag{2.4}$$

satisfies ([13], [9])

$$\begin{aligned}
\Phi'(t) &= 4\int_0^{R^2} sw_{ss} + 2\int_0^{R_2} ww_s - \mu\int_0^{R^2} sw_s \\
&= -4w(R^2,t) + 4R^2 w_s(R^2,t) + w^2(R^2,t) + \mu\int_0^{R^2} w - \mu R^2 w(R^2,t) \\
&\geq -4w(R^2,t) + w^2(R^2,t) + \mu\int_0^{R^2} w - \mu R^2 w(R^2,t) \\
&= \frac{m}{\pi R^2}\Phi(t) - \frac{2m}{\pi} - \frac{m^2}{4\pi^2} \qquad \text{for all } t \in (0,T),
\end{aligned}$$

because $w_s \geq 0$ by nonnegativity of u. Here we note that

$$\frac{m}{\pi R^2} \cdot \int_0^{R^2} \frac{m}{2\pi} ds - \frac{2m}{\pi} - \frac{m^2}{4\pi^2} = \frac{2m}{\pi} \cdot \Big(\frac{m}{8\pi} - 1\Big) > 0 \qquad \text{whenever } m > 8\pi,$$

so that if, conversely, $m > 8\pi$ is any prescribed number and $u_0 = u(\cdot, 0)$ is such that $\int_\Omega u_0 = m$ and that its mass is sufficiently concentrated near the origin in the sense that

$$\frac{m}{\pi R^2}\Phi(0) - \frac{2m}{\pi} - \frac{m^2}{4\pi^2} > 0,$$

then assuming that (u, v) be global leads to the conclusion that Φ should grow to $+\infty$ exponentially fast as $t \to \infty$, which is impossible since $\Phi \leq \frac{mR^2}{2\pi}$.

This second way of exploiting (2.3) does not only sharpen the result from [72] so as to detect blow-up at all mass levels m belonging to the range $(8\pi, \infty)$ known to be optimal in this respect ([102]); it moreover allows for various extensions, e.g. to higher-dimensional versions of (2.1) ([13]). Along with corresponding knowledge on local existence and extensibility ([102]), this implies the following result which has counterparts also in the limit case when Ω coincides e.g. with the whole plane $\mathbb{R}^2$ ([10]).

Theorem 2.1 *([72], [102], [9])* *Let $R > 0$ and $\Omega = B_R \subset \mathbb{R}^n$, $n \geq 2$.*
i) If $n = 2$, then for any choice of $m > 8\pi$ one can find radially symmetric $u_0 \in C^0(\overline{\Omega})$ with $\int_\Omega u_0 = m$

and $T > 0$ such that (2.1) possesses a classical solution (u, v) in $\Omega \times (0, T)$ with $(u(\cdot, t), v(\cdot, t))$ being radially symmetric for each $t \in (0, T)$, and with

$$\limsup_{t \nearrow T} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{2.5}$$

ii) If $n \geq 3$, then for each $m > 0$ there exist radial initial data enforcing blow-up in the sense specified in i).

Even though the results of Theorem 2.1 have later on not only been re-discovered but actually even been considerably extended by means of the method to be described below, a substantial merit of the above approach is due to its potential to allow for addressing yet more subtle questions e.g. related to the question which initial data enforce blow-up, and apart from that due to its applicability to a considerable variety of models in which chemotactic cross-diffusion interacts with diffusion, and possibly even further mechanisms, in a possibly more intricate manner.

For instance, a refined analysis of the blow-up enforcing potential in (2.1) on the basis of (2.3) can be found in [149], where the occurrence of blow-up in (2.1) was related to the extent to which the initial data are concentrated near the origin, where for nonnegative radial functions $\overline{u}_0$ and $\underline{u}_0$ from $L^1(\Omega)$ we say that $\overline{u}_0$ is more concentrated than $\underline{u}_0$ if

$$\int_{B_r(0)} \overline{u}_0 \geq \int_{B_r(0)} \underline{u}_0 \qquad \text{for all } r \in (0, R), \tag{2.6}$$

and then write $\overline{u}_0 \succeq \underline{u}_0$. Then the following result reveals the existence of a critical mass level, unlike the number 8π in Theorem 2.1 also present in higher-dimensional cases, that distinguishes between regions of stability of spatially homogeneous steady states in (2.1) on the one hand, and regions within which these equilibria exhibit quite a drastic instability feature:

Theorem 2.2 *([149]) Let $n \geq 2$, $R > 0$ and $\Omega = B_R(0)$. Then*

$$m_c(n, R) := \inf \left\{ m > 0 \;\middle|\; \begin{array}{l} \textit{For all radial } 0 \leq u_0 \in C^0(\overline{\Omega}) \textit{ with } u_0 \succeq \frac{1}{|\Omega|} \int_\Omega u_0 = \frac{m}{|\Omega|} \textit{ but } u_0 \not\equiv \textit{const.}, \\ \textit{(2.1) admits a solution blowing up in finite time} \end{array} \right\}$$

is well-defined and positive.

By its mere definition, this critical mass thus marks a genuine borderline between supercritical mass levels at which any, even arbitrarily small, concentration-increasing perturbation of the homogeneous steady state will lead to finite-time blow-up in (2.1), and a corresponding subcritical range of mass values at which this extreme instability property is absent.

Apart from that, detections of singularity formation through equivalent scalar reformulations based on (2.2) have been achieved in several chemotaxis systems more complex than (2.1). Examples in this direction include the occurrence of finite-time blow-up in variants of (2.1) involving growth restrictions of logistic type ([51], [142], [89], [147]), the possibility of attaining infinite or also certain finite but singular population density values in quasilinear modifications of (2.1) accounting for nonlinear diffusion and cross-diffusion ([36], [42], [152], [139]), and also the discovery of infinite-time blow-up in a three-component extension of (2.1) in which signal production occurs according to a certain more indirect mechanism than in (2.1) ([132]).

To illustrate a possible flavor of increase in mathematical complexity arising in the course of such generalizations of (2.1), let us briefly recall a recent development concerning the so-called flux-limited Keller-Segel system ([3])

$$\begin{cases} u_t = \nabla \cdot \left(\frac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}} \right) - \chi\nabla \cdot \left(\frac{u\nabla v}{\sqrt{1+|\nabla v|^2}} \right), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \mu + u, \end{cases} \tag{2.7}$$

once more with $\mu := \frac{1}{|\Omega|}\int_\Omega u(\cdot,0)$, and with $\chi > 0$. When again considered together with no-flux boundary conditions in balls $\Omega = B_R(0) \subset \mathbb{R}^n$, $n \geq 1$, along radial trajectories this system transforms to a Dirichlet problem, in the flavor of (2.3), for the scalar equation

$$w_t = n^2 \cdot \frac{s^{2-\frac{2}{n}} w_s w_{ss}}{\sqrt{w_s^2 + n^2 s^{2-\frac{2}{n}} w_{ss}^2}} + n\chi \cdot \frac{(w - \frac{\mu}{n}s)\cdot w_s}{\sqrt{1 + s^{\frac{2}{n}-2}\left(w - \frac{\mu}{n}s\right)^2}}, \tag{2.8}$$

which now contains a diffusion operator that is multiply degenerate, namely not only at $s = 0$, but also near zeroes of w_s and large values of w_{ss}. Despite this, the corresponding parabolic operator can be shown to firstly admit a comparison principle, and to secondly possess suitably rich families of exploding subsolutions within appropriate ranges. Thereby quite a complete picture with regard to the occurrence of blow-up can be drawn:

Theorem 2.3 *([6], [7])* *Let $\Omega := B_R(0) \subset \mathbb{R}^n$ with some $R > 0$.*
i) Assume that $0 \not\equiv u_0 \in C^0(\overline{\Omega})$ is radial, and that either

$$n \geq 2 \quad \textit{and} \quad \chi < 1, \qquad \textit{or} \qquad n = 1, \quad \chi > 0 \quad \textit{and} \quad \int_\Omega u_0 < m_c, \tag{2.9}$$

where in the case $n = 1$ we have set

$$m_c := \begin{cases} \frac{1}{\sqrt{\chi^2-1}} & \textit{if } \chi > 1, \\ +\infty & \textit{if } \chi \leq 1. \end{cases} \tag{2.10}$$

Then the no-flux initial-boundary value problem for (2.7) with $u|_{t=0} = u_0$ possesses a unique global bounded classical solution.

ii) Suppose that $\chi > 1$, and that

$$\begin{cases} m > m_c & \textit{if } n = 1, \\ m > 0 \textit{ is arbitrary} & \textit{if } n \geq 2, \end{cases} \tag{2.11}$$

where m_c is as in (2.10). Then there exists a radial nonnegative function $u_0 \in C^0(\overline{\Omega})$ such that the corresponding solution blows up in finite time in the sense that (2.5) holds.

2.1.2 Moment-based blow-up detection

With regard to blow-up detection in parabolic-elliptic chemotaxis systems, significant progress could be achieved through a second technique, due to [16] and [102], which for radial solutions can be regarded as a relative of the approach sketched above in the context of (2.4), but which has the striking advantage that within certain classes of systems, including (2.1) and also its variant given by

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ 0 = \Delta v - \kappa v + u, \end{cases} \tag{2.12}$$

for $\kappa \geq 0$, also nonradial solutions can be dealt with ([103]). The underlying idea can most plainly be seen in the Cauchy problem for (2.12) with $\kappa = 0$ when posed in all of $\mathbb{R}^2$ as the spatial domain, where for uniqueness purposes we define v, in dependence on u, through convolution with the Newtonian kernel according to

$$v(x,t) := -\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln |x-y| u(y,t) dy, \qquad x \in \mathbb{R}^2, t \in (0,T), \tag{2.13}$$

within the maximal time interval $(0,T)$ of existence. Then

$$\nabla v(x,t) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} u(y,t) dy, \qquad x \in \mathbb{R}^2, t \in (0,T),$$

and hence testing the first equation in (2.12) by $|x|^2$, and assuming suitable spatial decay properties of u, formally leads to the identity

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 u dx &= \int_{\mathbb{R}^2} u \Delta |x|^2 dx + \int_{\mathbb{R}^2} u \nabla v \cdot \nabla |x|^2 dx \\ &= 4 \int_{\mathbb{R}^2} u dx - \frac{1}{\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{x \cdot (x-y)}{|x-y|^2} u(x,t) u(y,t) dy dx \qquad \text{for } t \in (0,T). \end{aligned} \tag{2.14}$$

Here a simple but essential observation confirms that due to an evident symmetry property in the rightmost integral we have

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{x \cdot (x-y)}{|x-y|^2} u(x,t) u(y,t) dy dx = \frac{1}{2} \cdot \left\{ \int_{\mathbb{R}^2} u(x,t) dx \right\}^2 \qquad \text{for } t \in (0,T), \tag{2.15}$$

whence abbreviating $m := \int_{\mathbb{R}^2} u(x,0)dx$, due to mass conservation we obtain that

$$\frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 u dx = 4m - \frac{m^2}{2\pi} \qquad \text{for } t \in (0,T),$$

and thus, by nonnegativity of u, assuming finiteness of the second moment $\int_{\mathbb{R}^2} |x|^2 u(x,0)dx$ we conclude that (u,v) cannot exist globally whenever $m > 8\pi$. In fact, this argument is at the core of corresponding statements on unboundedness, thereby leading to quite a comprehensive picture on the dichotomy between global solvability and blow-up in this simplified Keller-Segel system on the entire plane ([23], [22]), and on $\mathbb{R}^n$ with $n \geq 3$ ([27]), and also in (2.12) with $\kappa > 0$ on $\mathbb{R}^2$ ([76]).

When (2.12) is posed along with e.g. homogeneous Neumann boundary conditions for both u and v

in smoothly bounded domains, and hence necessarily with $\kappa > 0$, (2.13) needs to be replaced with a respectively modified representation formula involving Green's function for the associated Helmholtz operator. Together with further adaptations due to suitable localization procedures, this implies the appearance of correction terms in (2.14) which, however, in the seminal work [103] could adequately be coped with; together with a complementing statement on global solvability implicitly obtained in [106] as a by-product of the analysis of the fully parabolic analogue (1.1), this provides essentially complete knowledge also in this case:

Theorem 2.4 *([103], [106], [9], [124])* *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and let $\kappa = 1$.*
i) For any choice of $m > 4\pi$ there exists some nonnegative $u_0 \in C^0(\overline{\Omega})$ such that $\int_\Omega u_0 = m$ and that the Neumann problem for (2.12) admits a classical solution (u, v) in $\Omega \times (0, T)$ with $u|_{t=0} = u_0$ which blows up at $T \in (0, \infty)$ in the sense that (2.5) holds.

ii) If $u_0 \in C^0(\overline{\Omega})$ is nonnegative and such that $\int_\Omega u_0 < 4\pi$, then the Neumann problem for (2.12) possesses a global bounded classical solution with $u|_{t=0} = u_0$.

In view of the central role of the symmetry argument around (2.15) in the above reasoning, it is not surprising that the potential for extensions of this method to further, and especially to more complex, chemotaxis systems appears to be much more limited than that of the approaches in Section 2.1.1. Nevertheless, not only versions of (2.12) either posed in higher-dimensional domains, or together with different types of boundary conditions are accessible to techniques of this flavor ([123], [8]), but also extensions of (2.12) either involving two species attracted by a jointly produced chemoattractant ([12], [48], [159]), or a second chemical that repels cells ([77], [128], [47], [87]), or also some systems accounting for two-species chemotactic interaction with two chemicals ([157], [90]), as well as even some parabolic-elliptic chemotaxis systems containing variants in their cross-diffusion terms, such as the Keller-Segel system with logarithmic sensitivity in which $-\nabla\cdot(u\nabla v)$ is replaced with $-\chi\nabla\cdot(u\nabla \ln v)$ for suitably large $\chi > 0$ in three- or higher-dimensional domains ([104]). As shown in [15] and [17], for the simple version of (2.12) with $\Omega = \mathbb{R}^n$, $n \ge 2$, and $\kappa = 0$, even some more general functionals of the form $\int \psi(x,t)u(x,t)dx$ can be used to derive sufficient criteria, in particular involving only local concentration properties of the initial data, for blow-up.

2.1.3 Energy-based blow-up arguments for parabolic systems I: Detecting unboundedness

Fully parabolic chemotaxis systems substantially differ from their parabolic-elliptic variants due to the circumstance that the cross-diffusive interaction therein involves a certain memory, that is, a *temporally* nonlocal influence of u on v in addition to spatially nonlocal mechanisms such as that expressed through (2.13); already in the prototypical system (1.1) with its comparatively simple structure of cross-diffusion and signal evolution, representing v in terms of u via Duhamel formulae for the heat equation leads to a considerably complex and hence doubly nonlocal parabolic equation for u which, to the best of our knowledge, has nowhere successfully been made accessible to methods in the flavor of those discussed in the previous two sections.

It can thus be viewed as a fortunate circumstance that independently from the above, (1.1) and some of its close relatives possess a certain global dissipative structure in that they enjoy meaningful

Lyapunov-type properties. In the context of (1.1), this becomes manifest in the energy inequality

$$\frac{d}{dt}\mathcal{F}(u,v) = -\mathcal{D}(u,v) \le 0, \tag{2.16}$$

with

$$\mathcal{F}(u,v) := \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega v^2 - \int_\Omega uv + \int_\Omega u\ln u \tag{2.17}$$

and

$$\mathcal{D}(u,v) := \int_\Omega v_t^2 + \int_\Omega \Big|\frac{\nabla u}{\sqrt{u}} - \sqrt{u}\nabla v\Big|^2, \tag{2.18}$$

in which the nonincreasing functional $\mathcal{F}$ may potentially be unbounded from below due to the appearance of the negative contribution $-\int_\Omega uv$ therein. In fact, this observation can be viewed as a motivation to pursue the following basic strategy for discovering unboundedness phenomena:

Step 1. Show that *bounded solutions approach equilibria* in the sense that whenever (u,v) is a global solution which is bounded, there exist $(t_j)_{j\in\mathbb{N}} \subset (0,\infty)$ and a solution (u_∞, v_∞) of the associated steady-state system such that $t_j \to \infty$ and $(u(\cdot,t_j), v(\cdot,t_j)) \to (u_\infty, v_\infty)$ in some appropriate topology as $j \to \infty$.

Step 2. Derive a *lower bound for all conceivable steady-state energies* by finding $K > 0$ with the property that every steady state (u_∞, v_∞) compatible with Step 1 satisfies $\mathcal{F}(u_\infty, v_\infty) \ge -K$.

Step 3. Verify the existence of *low-energy initial data* by constructing functions u_0 and v_0 admissible for the argument in Step 1 such that $\mathcal{F}(u_0,v_0) < -K$. Then due to (2.16), the solution emanating from (u_0,v_0) cannot be global and bounded.

Here it should be noted that if (1.1) is considered along with no-flux boundary conditions for u and v in bounded domains in $\mathbb{R}^n$, then already the the presence of the family $((a,a))_{a\ge 0}$ of constant equilibria implies that the constant K in Step 2 cannot be chosen in a way completely independent of the initial data. Fortunately, however, in various important cases one can at least achieve a dependence of K on the initial data which, e.g. by involving $u(\cdot,0)$ only through it total mass $\int_\Omega u(\cdot,0)$, is mild enough so as to indeed allow a consistent application of all three of the above steps.

For instance, a reasoning of this type indeed reveals the occurrence of unbounded solutions to the two-dimensional version of (1.1) under an essentially optimal condition on the level of the total population mass:

Theorem 2.5 *([67], [106]) i) Suppose that $\Omega \subset \mathbb{R}^2$ is simply connected and $m \in (4\pi,\infty) \setminus \{4k\pi \mid k \in \mathbb{N}\}$. Then there exist initial data $0 \le u_0 \in C^\infty(\overline{\Omega})$ and $0 \le v_0 \in C^\infty(\overline{\Omega})$ with $\int_\Omega u_0 = m$, such that for some $T \in (0,\infty]$, the no-flux initial-boundary value problem for (1.1) with $(u,v)|_{t=0} = (u_0,v_0)$ possesses a classical solution on $\Omega \times (0,T)$ whose component u is unbounded.*

ii) Whenever $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary and $(u_0,v_0) \in (W^{1,\infty}(\Omega))^2$ is such that $\int_\Omega u_0 < 4\pi$, the Neumann problem for (1.1) admits a global classical solution with $(u,v)|_{t=0} = (u_0,v_0)$ for which both u and v are bounded in $\Omega \times (0,\infty)$.

A further refinement of this approach can be employed to reveal unboundedness phenomena also in the quasilinear extension of (1.1) given by

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v) & \text{in } \Omega \times (0,T), \\ v_t = \Delta v - v + u & \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial \nu} = 0, \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0, \ v(\cdot,0) = v_0 & \text{in } \Omega, \end{cases} \tag{2.19}$$

which for various particular choices of the diffusion and cross-diffusion rates appears in refined models for chemotactic migration, e.g. in the presence of saturation effects due to the finite volume of cells ([64], [4], [125]).

Indeed, (2.19) shares with (1.1) an energy-type property similar to that in (2.16), (2.17) and (2.18), and it turns out that the above overall strategy can successfully be pursued in essentially all cases in which the corresponding energy functional $\mathcal{F}$ is unbounded from below. This becomes most transparent in the particular framework determined by the prototypical choices

$$D(s) := (s+1)^{p-1} \quad \text{and} \quad S(s) := s(s+1)^{q-1}, \qquad s \ge 0, \tag{2.20}$$

in which a corresponding analysis leads to a picture that with regard to the occurrence of unboundedness phenomena, in dependence on the parameters $p \in \mathbb{R}$ and $q \in \mathbb{R}$ therein, is already quite complete at least in spatially radial settings:

Theorem 2.6 *([68], [141], [129]) i) If $\Omega = B_R(0) \subset \mathbb{R}^n$ with $n \ge 2$ and $R > 0$, and if (2.20) holds with some $p \in \mathbb{R}$ and $q \in \mathbb{R}$ such that $p < q + 1 - \frac{2}{n}$, then for each $m > 0$ there exist radially symmetric initial data $0 \le u_0 \in C^\infty(\overline{\Omega})$ with $\int_\Omega u_0 = m$ and $0 \le v_0 \in C^\infty(\overline{\Omega})$ such that with some $T \in (0,\infty]$, (2.19)-(2.20) has a classical solution (u,v) which satisfies (2.5).*

ii) Let $n \ge 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and assume (2.20) with $p \in \mathbb{R}$ and $q \in \mathbb{R}$ fulfilling $p > q + 1 - \frac{2}{n}$. Then for any choice of nonnegative $u_0 \in W^{1,\infty}(\Omega)$ and $v_0 \in W^{1,\infty}(\Omega)$, the problem (2.19)-(2.20) with $T = \infty$ admits aclassical solution for which u and v are bounded.

While through its nature this approach is not capable of allowing for a decision whether the respective blow-up time is finite or infinite, a particular strength becomes salient in cases in which global solutions are known to exist, and thus unboundedness, if at all, can only occur in the sense of infinite-time blow-up.

Theorem 2.7 *([39], [41], [151]) Let $\Omega = B_R(0) \subset \mathbb{R}^n$ with $n \ge 2$ and $R > 0$, and suppose that (2.20) holds with some $p \in \mathbb{R}$ and $q \le 0$ such that $p < q + 1 - \frac{2}{n}$. Then there exist radially symmetric nonnegative initial data $(u_0, v_0) \in (W^{1,\infty}(\Omega))^2$ such that (2.19)-(2.20) admits a global classical solution (u,v) on $\Omega \times (0,\infty)$ for which u is unbounded.*

On the basis of this strategy, unbounded solutions have actually been found under much more general assumptions on D and S in (2.19), in particular including diffusion rates which decay exponentially or even more rapidly at large population densities ([141]); in [146], this led to the detection of global unbounded solutions in the flavor of Theorem 2.7 for (2.19) with $D(s) = e^{-\beta s}$ and $S(s) = se^{-\alpha s}$, $s \ge 0$, with arbitrary $\beta > 0$ and suitably chosen $\alpha > 0$. Extensions e.g. address Keller-Segel systems involving two chemical stimuli ([52], [73]) or degenerate diffusion operators of porous medium type ([70]), and also the occurrence of infinite-time blow-up in a parabolic-elliptic analogue of (2.19)-(2.20) in which Step 3 from the above list is considerably complicated due to the lack of freedom in the choice of the second component of low-energy initial data ([82]).

2.1.4 Energy-based blow-up arguments for parabolic systems II: Finite-time blow-up

A second option to make use of (2.16), in frameworks of energy functionals $\mathcal{F}$ which are unbounded from below, consists in relating $\mathcal{F}$ to the dissipation rate $\mathcal{D}$ via appropriate functional inequalities which are such that their application to (2.16) leads to superlinearly forced ordinary differential inequalities for $-\mathcal{F}$. In the simplest case of (1.1), this would, for instance, be accomplished if the negative part of $\mathcal{F}$ could be estimated in terms of an essentially sublinear function of $\mathcal{D}$, e.g. in that

$$\int_\Omega uv \le C \cdot \left\{ \left\| \Delta v - v + u \right\|_{L^2(\Omega)}^\theta + \left\| \frac{\nabla u}{\sqrt{u}} - \sqrt{u}\nabla v \right\|_{L^2(\Omega)}^\theta + 1 \right\} \tag{2.21}$$

with some $\theta \in (0,1)$, and with some $C > 0$ only depending on u and v through quantities which are well controllable through the initial conditions, such as $\int_\Omega u(\cdot,0)$, for instance.

In the particular context of radial solutions to (1.1) and, more generally, for suitable supercritical versions of (2.19)-(2.20), in three- and higher-dimensional cases the corresponding and quite evident challenges, mainly due to the coupling of u and v in both expressions on the right of (2.21), this strategy can indeed be pursued successfully. In fact, at their core relying on (2.21) and an accordingly modified variant adapted to the nonlinearities in (2.19)-(2.20), three findings from the past few years in summary revealed the following.

Theorem 2.8 *([143], [39], [41], [40])* *Let $\Omega = B_R(0) \subset \mathbb{R}^n$ with $n \ge 2$ and $R > 0$, and suppose that (2.20) holds with some $p \in \mathbb{R}$ and $q \ge 1$ such that $p < q + 1 - \frac{2}{n}$. Then there exist $T > 0$ and nonnegative initial data $u_0 \in W^{1,\infty}(\Omega)$ and $v_0 \in W^{1,\infty}(\Omega)$ with the property that (2.19)-(2.20) possesses a classical solution (u,v) which blows up at $t = T$ in the sense that (2.5) holds.*

This approach even extends to the critical case of $n = 2$, $p = q = 1$, i.e. (1.1), although the necessary refinement of (2.21) becomes more technical ([99]). Furthermore, applications of methods of this type to systems involving degenerate diffusion operators can be found in [84] and in [58], and even extensions to some two-species chemotaxis systems ([86]) and to certain systems systems simultaneously involving an attractive and repulsive taxis mechanisms ([88]) are possible. In the particular one-dimensional case in which favorable Sobolev embedding inequalities can be relied on, certain versions of (2.19), with $S \equiv id$ and diffusion rates decaying suitably fast at large population densities, are accessible to even slightly more direct arguments revealing finite-time blow-up under essentially optimal assumptions ([37], [26]).

An interesting problem, apparently left open by all precedents in this direction, consists in determining how far methods of this flavor can be applied to – and, in particular, how far functional inequalities of the form in (2.21) can continue to hold for – nonradial solutions of chemotaxis systems.

2.2 Qualitative analysis near blow-up

As, according to the above discussions, already the mere detection of exploding solutions goes along with significant challenges that could so far be overcome only in a moderate number of cases, it may not be too surprising that knowledge beyond this, and especially on qualitative aspects of blow-up mechanisms, is yet limited to few particular situations. A fortiori, it is remarkable that quite a precise description of a possible blow-up mechanism in a fully parabolic system of the form (1.1) could be achieved in [63] at least for one particular solution that could be constructed by means of a method based on matching asymptotic expansions through a topological argument; an analogue for (2.1) can be found in [62].

A certain stability property of this mechanism has been asserted in [113] (cf. also [54] for a related result) the outcome of which we state here in a form slightly weaker than the one actually proved there, in order to avoid abundant notation. We also remark that similar to this, also a certain mechanism of infinite-time concentration in a the mass-critical version of (2.1) in $\mathbb{R}^2$ with $\mu = 0$ can be shown to enjoy some stability property of a comparable flavor ([54]).

Theorem 2.9 *Let $\varepsilon > 0$. Then there exists an uncountable family $\mathcal{S} \subset (C^0(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)) \times (C^1(\mathbb{R}^2) \cap W^{1,2}(\mathbb{R}^2))$ with the property that each $(u_0, v_0) \in \mathcal{S}$ is such that $u_0 > 0$ and $v_0 < 0$ in $\mathbb{R}^2$, that $\int_{\mathbb{R}^2} u_0 < 8\pi + \varepsilon$, and that with some $T > 0$, the problem*

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) & \text{in } \mathbb{R}^2 \times (0,T), \\ v_t = \Delta v + u & \text{in } \mathbb{R}^2 \times (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \mathbb{R}^2, \end{cases} \tag{2.22}$$

possesses a classical solution (u,v) which blows up at time T in the sense that

$$u(x,t) = \frac{1}{\lambda^2(t)} \cdot (Q + \delta)\Big(\frac{x}{\lambda(t)}, t\Big) \qquad \text{for all } x \in \mathbb{R}^2 \text{ and } t \in (0,T)$$

with some function $\delta \in L^\infty((0,T); X)$ fulfilling $\|\delta(\cdot,t)\|_X \to 0$ as $t \nearrow T$, where

$$Q(z) := \frac{8}{(1+|z|^2)^2}, \qquad z \in \mathbb{R}^2,$$

and

$$\lambda(t) := \sqrt{T-t}\, e^{-\sqrt{\frac{|\ln(T-t)|}{2}} + \ell(t)}, \qquad t \in (0,T),$$

with some $\ell \in L^\infty((0,T))$. Here, X denotes the space of measurable functions φ on $\mathbb{R}^2$ with finite norm $\|\varphi\|_X := \Big\{ \int_{\mathbb{R}^2} (1+|x|^2)^2 |\Delta\varphi|^2 + \int_{\mathbb{R}^2} (1+|x|)^2 |\nabla\varphi|^2 + \int_{\mathbb{R}^2} \varphi^2 \Big\}^{\frac{1}{2}}$.

Results on spatial blow-up asymptotics for more general classes of initial data, and actually addressing the genuinely original system (1.1) in bounded planar domains, can be found in [105] and in [57], where regardless of the question of their existence, certain classes of solutions blowing up in finite time are analyzed, inter alia leading to the conclusion that such solutions approach finite linear combinations of Dirac distributions near their explosion times (cf. also [117]). A more comprehensive picture was obtained in [116] for the parabolic-elliptic variant (2.12), where a similar result on Dirac mass formation, as well as on finiteness of blow-up points, was derived without substantial restrictions. Even in the mass critical case, in which solutions to a parabolic-elliptic relative of (2.22) exist globally but blow up in infinite time either in the context of an associated Cauchy problem in $\mathbb{R}^2$, or also in the radially symmetric framework of a no-flux-Dirichlet problem in a disk, it is known that the spatial profile near the corresponding blow-up time $T = \infty$ is essentially dictated by Dirac distributions ([14], [108], [74], [22]).

In sharp contrast to this, in higher-dimensional domains the spatial behavior near blow-up rather seems determined by integrable profiles, as the following result concerned with radial solutions to (2.1) indicates:

Theorem 2.10 *([118]) Let $n \geq 3, R > 0$ and $\Omega = B_R(0) \subset \mathbb{R}^n$, and suppose that $0 \leq u_0 \in C^0(\overline{\Omega})$ is radially symmetric and such that the solution (u, v) of the Neumann problem for (2.1) blows up at $T \in (0, \infty)$ in the sense that (2.5) holds. Then there exist $C > 0$ and $U \in L^1(\Omega)$ such that*

$$u(x,t) \leq \frac{C}{|x|^2} \qquad \text{for all } x \in \Omega \text{ and } t \in (0,T), \tag{2.23}$$

and that

$$u(\cdot, t) \to U \quad \text{in } L^1(\Omega) \qquad \text{as } t \nearrow T.$$

If, furthermore, $u_0 = u_0(r)$ belongs to $C^1(\overline{\Omega})$ with $r^{n-1}u_{0,r}(r) + u_0(r) \int_0^r \big(u_0(s) - \mu\big)s^{n-1}ds \geq 0$ for all $r \in (0, R)$, then there exist $c > 0$ and $\eta \in (0, R)$ such that

$$U(x) \geq \frac{c}{|x|^2} \qquad \text{for all } x \in B_\eta \setminus \{0\}.$$

Apart from asserting integrability of blow-up profiles, this substantiates a noticeable influence of the convective term $-\nabla u \cdot \nabla v$ marking the apparently essential difference between (2.1), when rewritten in the form

$$u_t = \Delta u + u^2 - \nabla u \cdot \nabla v - \mu u \tag{2.24}$$

together with the second equation therein, and the corresponding semilinear heat equation

$$u_t = \Delta u + u^2. \tag{2.25}$$

Indeed, it is known from [60], [133] and [97] that if $n \leq 6$, then for any radial and radially decreasing solution of (2.25) the associated blowup profile satisfies

$$U(x) \sim 16 \, \frac{|\log|x||}{|x|^2}, \qquad x \to 0,$$

and that some radial decreasing solutions with this behavior even exist for all $n \geq 1$ ([2], [92]). Theorem 2.10 thus quantitatively confirms a mathematically somewhat subtle spreading effect of said convective term in (2.24) in the sense of enforcing blow-up profiles less singular than those for (2.25). For the fully parabolic problem (1.1), the respective knowledge seems yet much less developed, apparently reducing to the availability of an upper estimate of the form in (2.23) for radial solutions in three- and higher-dimensional balls, but with a significantly more singular expression on the corresponding right hand side ([150]), and to a statement on the lack of certain uniform integrability properties of general unbounded trajectories ([28]; cf. also [50] for a related result on the quasilinear system (2.19)).

Beyond the understanding of spatial behavior, also aspects related to the speed at which blow-up occurs have aroused interest among mathematicians. For instance, the Cauchy problem in $\mathbb{R}^n$, $n \geq 3$, for (2.1) is known to possess exploding solutions which are self-similar (in contrast to the solutions in $n = 2$ from [62]) and hence their blow-up is of "type I" in that it occurs at a rate essentially determined by that arising in the ordinary differential equation $y'(t) = y^2(t)$ associated with (2.24) ([59], [114], [56], [107], [122], [55]). When $n \geq 11$, however, also blow-up at faster rates, and hence of "type II", is possible ([115]). For a variant of (1.1) involving porous medium-type diffusion in bounded domains, it is known that finite-time blow-up must be of type II ([71]).

3 Analysis of taxis systems in situations of low regularity information

In light of the previous sections, the analysis of any chemotaxis system apparently needs to adequately cope with potentially substantial destabilizing effects of chemotactic cross-diffusion. In particular, this implies restrictions to solution theories especially in frameworks more complex than those particular ones that allow for approaches of the above types. The purpose of this section is to report on some recent developments in this direction.

3.1 Natural weak solution frameworks

In several application-relevant chemotaxis systems some basic a priori regularity information can be obtained as consequences of certain global, and in many cases quite elementary, dissipative properties. For example, passing over from (1.1) to the Neumann problem in bounded domains $\Omega \subset \mathbb{R}^n$ for the logistic Keller-Segel system given by

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + \kappa u - \lambda u^2, \\ v_t = \Delta v - v + u, \end{cases} \qquad (3.1)$$

with $\kappa \in \mathbb{R}$ and $\lambda > 0$, on the one hand destroys the gradient-like feature of (1.1) expressed through (2.16), but on the other hand, through a simple integration of the first equation in (3.1), this model modification brings about a spatio-temporal L^2 estimate for u. Unlike in favorable situations, however, such fundamental properties may be insufficient to launch appropriate bootstrap procedures finally leading to estimates in suitable spaces of, say, classical solutions. In the particular problem (3.1), for instance, such improvements of regularity information seem possible only when either $n \leq 2$, or $n \geq 3$ and $\lambda > 0$ is adequately large; accordingly, available results on global solvability, and hence of blow-up suppression, have so far been restricted to such constellations ([109], [154], [140]).

In order to establish some theory also in less favorable settings not covered by such approaches, an increasing number of studies resorts to certain generalized solution concepts involving regularity requirements below those needed for classical solvability, which are mild enough so as to be satisfied due to the respectively available basic solution properties. In fortunate cases, a subsequent analysis may reveal further properties of accordingly obtained generalized solutions, e.g. in the sense of statements on dominance of dissipation at least in their large time behavior.

In the particular framework of (3.1) with arbitrarily small $\lambda > 0$, applying said basic L^2 information to globally existing solutions of suitably regularized approximate systems, along with an implication thereof for the regularity of the respective second solution components, allows for corresponding conclusions already in quite a natural framework of weak solvability:

Theorem 3.1 *([79]) Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let $\kappa \in \mathbb{R}$ and $\lambda > 0$. Then for any choice of $0 \leq u_0 \in C^0(\overline{\Omega})$ and $0 \leq v_0 \in W^{1,\infty}(\Omega)$ one can find nonnegative functions*

$$\begin{cases} u \in L^\infty((0,\infty); L^1(\Omega)) \cap L^2_{loc}(\overline{\Omega} \times [0,\infty)) & \text{and} \\ v \in L^2_{loc}([0,\infty); W^{1,2}(\Omega)) \end{cases}$$

such that for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$ fulfilling $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega \times (0,\infty)$,

$$-\int_0^\infty \int_\Omega u\varphi_t - \int_\Omega u_0\varphi(\cdot,0) = \int_0^\infty \int_\Omega u\Delta\varphi + \int_0^\infty \int_\Omega u\nabla v \cdot \nabla\varphi + \int_0^\infty \int_\Omega (\kappa u - \lambda u^2)\varphi, \tag{3.2}$$

and that for each $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$,

$$-\int_0^\infty \int_\Omega v\varphi_t - \int_\Omega v_0\varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla v \cdot \nabla\varphi - \int_0^\infty \int_\Omega v\varphi + \int_0^\infty \int_\Omega u\varphi. \tag{3.3}$$

Furthermore, if $n = 3$ and Ω is convex, then for every $\lambda > 0$ there exists $\kappa_0 > 0$ such that if $\kappa < \kappa_0$, then for some $T > 0$, u and v belong to $C^{2,1}(\overline{\Omega} \times (T,\infty))$ and solve the Neumann problem for (3.1) in $\Omega \times (T,\infty)$ in the classical sense.

In some relatives of (1.1), even nontrivial Lyapunov functionals are available. Simple examples for such situations are given by the Neumann problems in bounded domains $\Omega \subset \mathbb{R}^n$ for the chemorepulsion system

$$\begin{cases} u_t = \Delta u + \nabla \cdot (u\nabla v), \\ v_t = \Delta v - v + u, \end{cases} \tag{3.4}$$

describing populations of individuals driven off by a substance secreted by themselves, and for the nutrient taxis system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v), \\ v_t = \Delta v - uv, \end{cases} \tag{3.5}$$

forming a prototypical model for attractive chemotactic motion toward a chemical signal which, unlike in all previously considered systems, is consumed, rather than produced, by individuals.

The system (3.4) formally allows for an energy inequality of the form

$$\frac{d}{dt}\Big\{ \int_\Omega u \ln u + \frac{1}{2}\int_\Omega |\nabla v|^2 \Big\} = -\int_\Omega \frac{|\nabla u|^2}{u} - \int_\Omega |\Delta v|^2 - \int_\Omega |\nabla v|^2 \leq 0, \tag{3.6}$$

whereas for suitably smooth solutions of (3.5) one can similarly derive the identity

$$\frac{d}{dt}\Big\{ \int_\Omega u \ln u + \frac{1}{2}\int_\Omega \frac{|\nabla v|^2}{v} \Big\} + \int_\Omega \frac{|\nabla u|^2}{u} + \int_\Omega v|D^2 \ln v|^2 + \frac{1}{2}\int_\Omega \frac{u}{v}|\nabla v|^2 = \frac{1}{2}\int_{\partial\Omega} \frac{1}{v}\frac{\partial|\nabla v|^2}{\partial\nu} \tag{3.7}$$

in which the integral on the right-hand side can be seen to be nonpositive when Ω is convex. In parallel to the situation in (3.1), in both these problems correspondingly obtained basic a priori estimates can indeed be used as a starting point for iterative regularity arguments in the case $n = 2$, thus leading to statements on global existence of bounded solutions for all reasonably regular initial data in planar domains, at least if additionally Ω is assumed to be convex when (3.5) is considered ([38], [130]). In three-dimensional cases, however, resorting to weak solution concepts seems in order:

Theorem 3.2 *([38], [130])* *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, and let $0 \le u_0 \in W^{1,\infty}(\Omega)$ and $0 \le v_0 \in W^{1,\infty}(\Omega)$.*

i) There exist nonnegative functions

$$\begin{cases} u \in L^\infty((0,\infty); L^1(\Omega)) \cap L^1_{loc}([0,\infty); L^2(\Omega)) \qquad \text{and} \\ v \in L^\infty([0,\infty); W^{1,2}(\Omega)) \end{cases} \tag{3.8}$$

such that (u,v) forms a weak solution of the Neumann problem for (3.4) with $(u,v)|_{t=0} = (u_0,v_0)$ in the sense that (3.3) *and*

$$-\int_0^\infty \int_\Omega u\varphi_t - \int_\Omega u_0\varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla u \cdot \nabla\varphi - \int_0^\infty \int_\Omega u\nabla v \cdot \nabla\varphi$$

hold for each $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$.

ii) If Ω is convex and $v_0 > 0$ in $\overline{\Omega}$, then one can find nonnegative functions fulfilling (3.8) as well as $v \in L^\infty(\Omega \times (0,\infty))$, which are such that (u,v) solves the corresponding no-flux initial-boundary value problem for (3.5) in the sense that for every $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$ we have

$$-\int_0^\infty \int_\Omega u\varphi_t - \int_\Omega u_0\varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla u \cdot \nabla\varphi + \int_0^\infty \int_\Omega u\nabla v \cdot \nabla\varphi$$

and

$$-\int_0^\infty \int_\Omega v\varphi_t - \int_\Omega v_0\varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla v \cdot \nabla\varphi - \int_0^\infty \int_\Omega uv\varphi. \tag{3.9}$$

Moreover, one can find $T > 0$ such that $(u,v) \in (C^{2,1}(\overline{\Omega} \times (T,\infty)))^2$ and that (u,v) solves the respective boundary value problem for (3.5) classically in $\Omega \times (T,\infty)$.

Approaches of this flavor have successfully been applied in the derivation of results on global weak solvability, and partially even on eventual smoothness, also for certain classes of considerably more complex models in which chemotaxis systems, both of type (1.1) and of the form in (3.5), are coupled to the Stokes or the Navier-Stokes system from fluid mechanics in order to account for interplay with liquid environments (e.g. [93]), or also to further mechanisms such as haptotactic interaction (e.g. [127]); some recent results on such chemotaxis-fluid or chemotaxis-haptotaxis systems can be found in [18], [80], [100], [20], [160], [24], [91], [120], [5].

Apart from that, weak solution concepts form natural analytical frameworks in contexts in which regularity properties of solutions are limited due to degeneracies such as present in systems involving porous medium type cell diffusion ([81], [69], [46], [158], [29], [35], [34], [138], [131], [121], [85], [112], [32]).

3.2 Solution concepts based on renormalization. Analysis beyond singularities

It can broadly be observed that especially when further developing (1.1) and its relatives toward more complex models, situations in which energy-like structures such as those expressed in (2.16), (3.6) or also (3.7) are present should actually be viewed as fortunate exceptions. In fact, in most among the more realistic macroscopic models for tactic migration, beyond evident mass identities further fundamental global properties, witnessing some conservation or dissipation in a nontrivial sense comparable to those in said examples, seem to be lacking. In order to nevertheless open perspectives for the advancement of appropriate solution theories in such cases of particularly poor regularity information, some part of the more recent literature considers solutions within frameworks yet weaker than those natural ones underlying the studies mentioned in the previous section.

In order to substantiate the basic ideas forming the core of some approaches toward such further generalizations, let us consider the variant of the nutrient taxis system (3.5) given by

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\chi(x,u,v)\cdot\nabla v), \\ v_t = \Delta v - uv, \end{cases} \tag{3.10}$$

in which according to refined modeling approaches, the sensitivity function χ is allowed to be matrix-valued, and to hence possibly include rotational flux components which, in line with experimental observations, seems to significantly influence bacterial motion near boundaries of their, usually liquid, environment ([155]). Now the fragility of energy structures becomes manifest through the observation that even when χ is assumed to be constant, as soon as this matrix contains off-diagonal entries it seems that neither (3.7) nor any no meaningful analogue continues to be available. In consequence, any analysis of (3.10) apparently needs to be based on the very poor remaining a priori information that at a formal level is expressed in the three inequalities

$$v \le \|v_0\|_{L^\infty(\Omega)}, \qquad \int_0^\infty \int_\Omega |\nabla v|^2 \le \frac{1}{2}\int_\Omega v_0^2 \qquad \text{and} \qquad \int_0^\infty \int_\Omega \frac{|\nabla u|^2}{(u+1)^2} \le C \tag{3.11}$$

with $C := 2\int_\Omega u_0 + \frac{\|\chi\|_{L^\infty}}{2}\int_\Omega v_0^2$, $(u_0,v_0) = (u,v)|_{t=0}$, which with regard to u, in particular, does not go substantially beyond the L^1 boundedness features already known from mass conservation.

One conceivable strategy for a possible way out consists in resorting to a concept involving suitable renormalization of solutions in the style of classical precedents, e.g. from the context of Boltzmann equations ([43]), but augmented by an additional idea to overcome possibly lacking compactness features. To make this more precise, let us observe that the crucial first component of a supposedly existing smooth solution to (3.10), when posed e.g. along with no-flux boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^n$, satisfies

$$\begin{aligned} -\int_0^\infty \int_\Omega \phi(u)\varphi_t - \int_\Omega \phi(u_0)\varphi(\cdot,0) \;\;=\;\; & \int_0^\infty \int_\Omega \phi(u)\Delta\varphi - \int_0^\infty \int_\Omega \phi''(u)|\nabla u|^2\varphi \\ & + \int_0^\infty \int_\Omega u\phi''(u)\nabla u \cdot \Big(\chi(x,u,v)\cdot\nabla v\Big)\varphi \\ & + \int_0^\infty \int_\Omega u\phi'(u)\Big(\chi(x,u,v)\cdot\nabla v\Big)\cdot\nabla\varphi \end{aligned} \tag{3.12}$$

for any $\phi \in C^2([0,\infty))$ and each $\varphi \in C_0^\infty(\overline{\Omega}\times[0,\infty))$ fulfilling $\frac{\partial\varphi}{\partial\nu}|_{\partial\Omega\times(0,\infty)} = 0$. Under appropriate assumptions on the decay of $\phi'(s)$ and $\phi''(s)$ as $s \to \infty$, here satisfied if $\phi(s) = \ln(s+1)$, $s \ge 0$, still assuming χ to be bounded one can verify that at least the existence of each of the appearing integrals is asserted by (3.11) together with mass conservation. If genuine equality was ultimately strived for in (3.12), however, the need to construct (u,v) e.g. through approximation by solutions of suitably regularized variants of (3.10) would apparently give rise to the further requirement that each of the corresponding expressions should approach its expected limit. Thus particularly demanding a certain strong, though possibly weighted, spatio-temporal L^2 compactness property of ∇u, this seems to go beyond the information available on the basis of (3.11), and hence suggests to resort to a concept requiring $-\phi''$ and φ to be nonnegative, and demanding u to merely require the inequality obtained from (3.12) upon replacing "=" with "≥". According to lower semicontinuity of L^2 norms with respect to weak convergence, this supersolution property of u can now in fact be seen to be achievable by using the compactness properties implied by (3.11). Fortunately, it turns out that as a complementing subsolution feature, an essentially trivial consequence of Fatou's lemma and mass conservation in suitable approximate systems, is already sufficient to complete the design of a generalized solvability concept consistent with that of classical solutions:

Theorem 3.3 *([144]) Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let $\chi \in C^2(\overline{\Omega} \times [0,\infty) \times [0,\infty); \mathbb{R}^{n\times n})$ be bounded. Then for any choice of nonnegative $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$, one can find nonnegative functions*

$$\begin{cases} u \in L^\infty((0,\infty); L^1(\Omega)) \qquad \textit{and} \\ v \in L^\infty(\Omega \times (0,\infty)) \cap L^2((0,\infty); W^{1,2}(\Omega)) \end{cases}$$

such that $\ln(u+1) \in L^2_{loc}([0,\infty); W^{1,2}(\Omega))$, and that (u,v) forms a generalized solution of the no-flux initial-boundary value problem for (3.10) with $(u,v)|_{t=0} = (u_0, v_0)$ in the sense that

$$\int_\Omega u(\cdot,t) \leq \int_\Omega u_0 \qquad \textit{for a.e. } t>0, \tag{3.13}$$

that

$$\begin{aligned} -\int_0^\infty \int_\Omega \ln(u+1)\varphi_t - &\int_\Omega \ln(u_0+1)\varphi(\cdot,0) \\ &\geq \int_0^\infty \int_\Omega \ln(u+1)\Delta\varphi + \int_0^\infty \int_\Omega |\nabla \ln(u+1)|^2 \varphi \\ &\quad - \int_0^\infty \int_\Omega \frac{u}{u+1} \nabla \ln(u+1) \cdot \Big(\chi(x,u,v)\cdot\nabla v\Big)\varphi \\ &\quad + \int_0^\infty \int_\Omega \frac{u}{u+1} \Big(\chi(x,u,v)\cdot\nabla v\Big)\cdot\nabla\varphi \end{aligned} \tag{3.14}$$

holds for each nonnegative $\varphi \in C_0^\infty(\bar{\Omega} \times [0,\infty))$ with $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega \times (0,\infty)$, and that (3.9) holds for each $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$.

That even fruit beyond statements on mere existence can be harvested in the aftermath of such efforts, at least in some particular cases, is witnessed by the observation that in two-dimensional domains, (3.11) actually implies eventual smoothness of the obtained solution, as well as its stabilization toward the spatially homogeneous equilibrium determined by the mass level of the initial data ([148]).

Approaches of this form have been applied to further and partially even more complex situations, e.g. involving singular chemotactic sensitivities or also couplings to Stokes and Navier-Stokes equations; some examples can be found in [156], [45], [136], [145], [19] and [137], and also in [135]. Also when the presence of logistic source terms threatens the validity of (3.13), a related solution concept can be employed; then (3.13) is replaced by (3.2) with "≤" instead of "=" ([78]). A yet further generalization, addressing a chemotaxis system with logarithmic sensitivity, instead of relying on the corresponding analogue of (3.14) rests on the weak formulation of a parabolic inequality that simultaneously involves the equations for both solution components, and thereby achieves a result on global solvability under quite mild assumptions on the system ingredients ([83]; for an extension involving a fluid-coupling see also [21]).

An interesting feature of such generalizations seems to be that in some cases a further development thereof can be undertaken in such a manner that the construction of global solutions becomes possible also in some situations in which finite-time blow-up is known to occur. While precedent approaches toward establishing existence of solutions beyond blow-up, based on certain concepts of measure-valued solutions, required substantial efforts and have apparently been limited to the Cauchy problem in $\mathbb{R}^2$ for the simple system (2.1) with $\mu = 0$ ([94]), a recent refinement of the idea to combine subsolution properties with mass conservation features, as reported in [162] and thereby extending a method developed in [161], allows to get along without any substantial restriction of this type. The essential difference between the notion of solution pursued there on the one hand, and that underlying Theorem 3.3 and the study [83] on the other, consists in the introduction of a measure-valued part to the contribution of the first solution component to the second equation, which inter alia entails the necessity to furthermore include additional requirements concerning the attainment of boundary data; for details in the precise formulation of this concept, for reasons of compactness in presentation we may refer the reader to [162, Definition 3.1].

Theorem 3.4 *([162]) Let $n \ge 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and suppose that $\chi \in C^0([0,\infty)^2)$, and that $u_0 \in L^1(\Omega)$ and $v_0 \in L^1(\Omega)$ are nonnegative. Then there exist nonnegative functions*

$$\begin{cases} u \in L^\infty((0,\infty); L^1(\Omega)) \qquad \textit{and} \\ v \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)) \end{cases}$$

such that (u,v) solves the no-flux initial-boundary value problem with $(u,v)|_{t=0} = (u_0,v_0)$ for

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\chi(u,v) \cdot \nabla v), \\ v_t = \Delta v - v + u, \end{cases} \tag{3.15}$$

in the generalized sense specified in [162, Definition 3.1].

Thereby especially having at hand some globally defined solution in any of the cases in which finite-time blow-up is known to occur in (3.15), one may wonder how far the behavior of such exploding solutions can be described past their blow-up time. Whereas the results from [94] rather precisely describe an unfavorable nonuniqueness property in this regard for general nonradial solutions (2.1) with $\mu = 0$ on $\mathbb{R}^2$, for the fully parabolic Keller-Segel system (1.1) only very limited knowledge seems available, referring to certain global renormalized radial solutions in bounded balls in $\mathbb{R}^n$, $n \ge 2$, and asserting their smoothness outside the spatial original as well as a singular pointwise upper bound ([150]).

In contrast to this, quite a comprehensive result concerning radial solutions of (2.1) with $\mu = 0$ on the whole plane, in particular stating persistence of Dirac-type singularities and asymptotically complete mass concentration at the origin, could be derived by appropriately analyzing the respective variant of (2.3):

Theorem 3.5 *([11])* *Suppose that $u_0 \in C^0(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ is nonnegative and radially symmetric with respect to $x = 0$, with $m := \int_{\mathbb{R}^2} u_0 > 8\pi$. Then there exist $T > 0$ and a uniquely determined function*

$$w \in C^0([0,\infty); L^2_{loc}([0,\infty)) \cap C^{2,1}((0,\infty) \times (0,\infty))$$

such that

$$0 \le w(s,t) \le \frac{m}{2\pi} \quad \text{and} \quad w_s(s,t) \ge 0 \qquad \text{for all } s > 0 \text{ and } t > 0,$$

that w solves the Dirichlet problem

$$\begin{cases} w_t = 4sw_{ss} + 2ww_s, & s > 0,\ t > 0, \\ w(s,t) \to \frac{m}{2\pi} \quad \text{as } s \to \infty, & t > 0, \\ w(s,0) = \int_0^{\sqrt{s}} \rho u_0(\rho) d\rho, & s > 0, \end{cases} \tag{3.16}$$

and that, additionally,

$$w(0,t) := \lim_{s \searrow 0} w(s,t), \qquad t > 0,$$

has the properties that $0 < t \mapsto w(0,t)$ is nondecreasing with

$$w(0,t) = 0 \qquad \text{for all } t \in (0,T)$$

and

$$w(0,t) \ge 4 \qquad \text{for all } t > T$$

as well as

$$w(0,t) \to \frac{m}{2\pi} \qquad \text{as } t \to \infty.$$

And with this observation let us finish the journey of this article that has led us from blow-up detection to weak solutions and to their behavior after blow-up. There is no question that, especially in complex settings, much more remains to be discovered and while in light of the rich literature it is unavoidable that even a note with limited scope such as this one has to remain incomplete, we nevertheless hope to have given an impression about chemotactic cross-diffusion, and some mathematical challenges and possible approaches toward their solution.

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