

Blow-up profiles for the parabolic-elliptic Keller-Segel system in dimensions $n \geq 3$

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Abstract

We study the blow-up asymptotics of radially decreasing solutions of the parabolic-elliptic Keller-Segel-Patlak system in space dimensions $n \geq 3$. In view of the biological background of this system and of its mass conservation property, blowup is usually interpreted as a phenomenon of concentration or aggregation of the bacterial population. Understanding the asymptotic behavior of solutions at the blowup time is thus meaningful for the interpretation of the model.

Under mild assumptions on the initial data, for $n \geq 3$, we show that the final profile satisfies $C_1|x|^{-2} \leq u(x, T) \leq C_2|x|^{-2}$, with convergence in L^1 as $t \rightarrow T$. This is in sharp contrast with the two-dimensional case, where solutions are known to concentrate to a Dirac mass at the origin (plus an integrable part). We also obtain refined space-time estimates of the form $u(x, t) \leq C(T - t + |x|^2)^{-1}$ for type I blowup solutions. Previous work had shown that radial, self-similar blowup solutions (which satisfy the above estimates) exist in dimensions $n \geq 3$ and do not exist in dimension 2. Our results thus reveal that the final profile displayed by these special solutions actually corresponds to a much more general phenomenon.

Key words: parabolic-elliptic Keller-Segel system, chemotaxis, blowup profile, space-time estimates

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1 Introduction and main results

Let $\Omega = B_R \subset \mathbb{R}^n$ or $\Omega = \mathbb{R}^n$, with $R > 0$ and $n \geq 1$. In this article we consider radially symmetric solutions of the well-known parabolic-elliptic Keller-Segel-Patlak system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega, t > 0, \\ 0 = \Delta v + u - \mu, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (1.1)$$

where

$$\mu := \begin{cases} \frac{1}{|\Omega|} \int_{\Omega} u_0 & \text{if } \Omega = B_R, \\ 0 & \text{if } \Omega = \mathbb{R}^n \end{cases} \quad (1.2)$$

(and the boundary conditions in (1.1) are understood to be empty if $\Omega = \mathbb{R}^n$). As for the initial data, unless explicitly stated otherwise, we shall always assume that

$$\begin{aligned} u_0 \in L^\infty(\Omega), \quad u_0 \geq 0, \quad u_0 \text{ is radially symmetric} \\ \text{and nonincreasing with respect to } |x|, \text{ with } u_0 \not\equiv \text{const.} \end{aligned} \quad (1.3)$$

This system (with or without radial symmetry assumptions) arises as a simplified model of chemotaxis, where u and v respectively stand for the density of the bacterial population and of the secreted chemoattractant. System (1.1) is also involved in a model of gravitational interaction of particles. It has received considerable attention from the mathematical point of view (see e.g. the surveys [18], [19] and the book [38] for references).

Problem (1.1) is locally well posed (see Proposition 3.1 for a precise statement) and we denote by (u, v) its unique, maximal classical solution, and by T its maximal existence time. We note that u is nonnegative and radial decreasing. It is known that if $n \geq 2$ and u_0 is suitably large, then $T < \infty$ and the solution blows up in the following sense:

$$\lim_{t \rightarrow T} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$$

(cf. [20], [3], [4], [30], [10], [33], [5]). For instance this is the case whenever $\|u_0\|_1 > c(n)R^{n-2}$ if $\Omega = B_R$ (see Proposition 3.4).

Our concern in this article is about the asymptotic behavior of u at and near the blowup time. Recalling the well-known mass conservation property $\|u(t)\|_1 = \|u_0\|_1$, and keeping in mind the biological background of system (1.1), blowup is usually interpreted as a phenomenon of concentration or aggregation of the bacterial population. Understanding the asymptotic behavior of u at the blowup time is thus meaningful for the interpretation of the model. We focus on the properties of u , as the quantity v can be recovered from u by the linear second equation in (1.1).

For $n = 2$, this behavior is rather well understood: the solution exhibits a quantized concentration of mass at the origin. More precisely, for any blowing up solution, there exist $m \geq 8\pi$ and $f \in L^1(\Omega)$ such that

$$u(\cdot, t) \rightharpoonup m\delta_0 + f \quad \text{in } \mathcal{M}(\Omega), \quad \text{as } t \rightarrow T \quad (1.4)$$

(see [17], [34], [35], [38]). For $n \geq 3$, the situation may be quite different. Indeed, in the case $\Omega = \mathbb{R}^n$, there exist radial, positive, backward self-similar solutions of the form

$$u(x, t) = (T - t)^{-1} V(x/\sqrt{T - t}), \quad x \in \mathbb{R}^n, \quad 0 < t < T, \quad (1.5)$$

where the radially decreasing profile function V satisfies $\lim_{y \rightarrow \infty} y^2 V(y) = L \in (0, \infty)$ (see [15], [33], [13]). This leads to the final blowup profile

$$U(x) := \lim_{t \rightarrow T} u(x, t) = L|x|^{-2}, \quad x \neq 0, \quad (1.6)$$

where the convergence also takes place in $L^1_{loc}(\mathbb{R}^n)$. Therefore, these self-similar solutions have an integrable singularity and exhibit no mass concentration. More generally, for $\Omega = B_R$ and $n \geq 2$, it is known (see, e.g., [14]) that under assumption (1.3), blowup can occur only at $x = 0$. However the final blowup profile of general radial decreasing solutions does not seem to have been studied so far.

The purpose of this article is to show that the $|x|^{-2}$ final profile, displayed by the special self-similar solutions in (1.5), actually corresponds to a much more general phenomenon, which marks a strong difference between dimensions $n \geq 3$ and $n = 2$. In what follows, the blowup set $B(u_0)$ of (u, v) is defined by

$$B(u_0) := \{x_0 \in \bar{\Omega}; u(x_j, t_j) + |\nabla v(x_j, t_j)| \rightarrow \infty \text{ for some sequence } (x_j, t_j) \rightarrow (x_0, T)\}. \quad (1.7)$$

Our first main result is the following:

Theorem 1.1 *Let $n \geq 3$. Consider problem (1.1), where u_0 satisfies (1.3) and $T < \infty$.*

(i) *Let $\Omega = B_R \subset \mathbb{R}^n$. Then there exists $C > 0$ such that*

$$u(x, t) \leq \frac{C}{|x|^2}, \quad 0 < |x| \leq R, \quad 0 < t < T. \quad (1.8)$$

Moreover, we have $B(u_0) = \{0\}$, the final blowup profile $U(x) := \lim_{t \rightarrow T} u(x, t)$ exists for all $x \in \bar{\Omega} \setminus \{0\}$, where convergence also takes place in $L^1(B_R)$, and U satisfies

$$U(x) \leq \frac{C}{|x|^2}, \quad 0 < |x| \leq R. \quad (1.9)$$

(ii) *Let $\Omega = \mathbb{R}^n$ and assume $B(u_0) \neq \mathbb{R}^n$. Then assertion (i) remains valid, with $R = 1$ in (1.8) and (1.9). The assumption $B(u_0) \neq \mathbb{R}^n$ is satisfied in particular whenever $u_0 \in L^1(\mathbb{R}^n)$.*

Our second main result shows that (1.9) is sharp. Namely, the final blowup profile satisfies the corresponding lower estimate, assuming the following additional hypothesis on u_0 :

$$u_0 \in C^1(\bar{\Omega}), \quad r^{n-1} u_{0,r}(r) + u_0(r) \int_0^r (u_0(s) - \mu) s^{n-1} ds \geq 0, \quad \text{for all } r \in (0, R). \quad (1.10)$$

We note that, for $\Omega = B_R$ and any $u_0 \in C^2(\bar{\Omega})$ verifying (1.3), $u_{0,r}(R) = 0$ and $u_0(R) > 0$, property (1.10) is in particular satisfied if we take a sufficiently large multiple of u_0 as initial data (see Proposition 3.5(ii)).

Theorem 1.2 *Let $n \geq 3$. Consider problem (1.1), where u_0 satisfies (1.3), (1.10) and $T < \infty$.*

(i) Let $\Omega = B_R \subset \mathbb{R}^n$. Then there exist $c, \eta > 0$, such that

$$U(x) \geq \frac{c}{|x|^2}, \quad \text{for all } x \in B_\eta \setminus \{0\}.$$

(ii) Let $\Omega = \mathbb{R}^n$ and assume $B(u_0) \neq \mathbb{R}^n$. Then assertion (i) remains valid.

Our last main result gives a more precise upper estimate which provides information on the space-time blowup behavior. To this end, let us recall that any blowup solution of problem (1.1) satisfies $\liminf_{t \rightarrow T} (T - t) \|u(t)\|_\infty > 0$ and that blowup is said to be type I if

$$\limsup_{t \rightarrow T} (T - t) \|u(t)\|_\infty < \infty$$

and type II otherwise. This classification is motivated by scale invariance considerations and the underlying ODE. Indeed, substituting the equation for v into the equation for u in (1.1), we obtain (in the case $\mu = 0$)

$$u_t = \Delta u + u^2 - \nabla v \cdot \nabla u, \tag{1.11}$$

whose spatially homogeneous solutions are given by $u(t) = (T - t)^{-1}$. Examples of type II blowup are known for $n = 2$ (see [17], [32]) and $n \geq 11$ (see [28]). In dimensions $3 \leq n \leq 9$, for $\Omega = \mathbb{R}^n$, it was proved in [29] that blowup is type I unless $B(u_0) = \mathbb{R}^n$. See also [14] for a sufficient condition for type I blowup in the case $\Omega = B_R$ with $n \geq 3$ (where $\mu = 0$ and Dirichlet instead of Neumann conditions are taken for v in (1.1)).

Theorem 1.3 *Under the assumptions of Theorem 1.1, there exists $K > 0$ such that*

$$u(x, t) \leq \left(\frac{1}{u(0, t)} + K|x|^2 \right)^{-1}, \quad 0 < |x| \leq R, \quad 0 < t < T. \tag{1.12}$$

In particular, if blowup is type I, then there exists $C > 0$ such that

$$u(x, t) \leq C(T - t + |x|^2)^{-1}, \quad 0 < |x| \leq R, \quad 0 < t < T. \tag{1.13}$$

This is true for instance if $3 \leq n \leq 9$ with $\Omega = \mathbb{R}^n$ and $u_0 \in L^1(\mathbb{R}^n)$.

2 Remarks and discussion

(i) As noted by several authors (see, e.g., [4], [17]), there is a natural parallel between system (1.1) and the classical semilinear heat equation with quadratic nonlinearity

$$u_t = \Delta u + u^2, \tag{2.1}$$

since the equation (1.11), obtained from (1.1), is precisely (2.1) with an added convection term. The latter is expected to produce spreading effects and, interestingly, Theorems 1.1 and 1.2 show that

the final blowup profile is different and less singular than for (2.1). Indeed, if $n \leq 6$, for any radial decreasing solution of (2.1), the blowup profile satisfies

$$U(x) \sim 16 |x|^{-2} |\log |x||, \quad x \rightarrow 0 \quad (2.2)$$

(see [16], [39], [25], [37]), and some radial decreasing solutions with this behavior exist for all $n \geq 1$ (see [1], [24], [37]). On the other hand, for $7 \leq n \leq 15$, beside solutions with profile (2.2), equation (2.1) also possesses (radial decreasing) self-similar solutions of the form (1.5), leading to the homogeneous profile (1.6) (see [12], [22], [7], [23]).

(ii) In [14] and [13], it is shown, respectively in the case $\Omega = B_R$ (where $\mu = 0$ and Dirichlet instead of Neumann conditions are taken for v) and $\Omega = \mathbb{R}^n$, that some classes of radial decreasing solutions of (1.1) (satisfying suitable zero number properties) are attracted by self-similar solutions, in the sense that

$$\lim_{t \rightarrow T} (T - t) u(y\sqrt{T-t}) = V(y), \quad \text{uniformly for } y \text{ bounded}$$

(where V is the profile of one of the self-similar solutions mentioned above). However, this convergence in the “microscopic” scale y bounded does not provide information on the final blowup profile in the original variable x .

(iii) It remains an open question whether the limit $\lim_{x \rightarrow 0} |x|^2 U(x)$ exists under the assumptions of Theorems 1.1 and 1.2. Note that these results show that the liminf is positive and the limsup is finite. In any case, this limit, if it exists, cannot be universal (i.e., independent of the solution, like in property (2.2) for equation (2.1)). Indeed, it is shown in [13] that there exists a one-parameter family u_α of self-similar solutions of the form (1.5), such that the limit $L(\alpha) := \lim_{y \rightarrow \infty} y^2 V_\alpha(y) = \lim_{x \rightarrow 0} |x|^2 U_\alpha(x) \in (0, \infty)$ and is different for each α .

(iv) The self-similar solutions in (1.5) satisfy

$$c_1 (T - t + |x|^2)^{-1} \leq u(x, t) \leq c_2 (T - t + |x|^2)^{-1}, \quad x \in \mathbb{R}^n, \quad 0 < t < T$$

(due to $\lim_{y \rightarrow \infty} y^2 V(y) \in (0, \infty)$). We expect that, for general radial nondecreasing solutions, the lower space-time estimate corresponding to (1.13), i.e.:

$$u(x, t) \geq c (T - t + |x|^2)^{-1},$$

should be true as $(x, t) \rightarrow (0, T)$ if blowup is type I. However we are presently unable to show this. Note that for radially decreasing solutions of the semilinear heat equation (2.1) with $n \leq 6$, the refined space-time blowup behavior is completely known, given by

$$u(x, t) = (1 + o(1)) \left[T - t + \frac{|x|^2}{8 \min\{|\log(T-t)|, 2|\log|x||\}} \right]^{-1}, \quad \text{as } (x, t) \rightarrow (0, T)$$

(see [16], [39], [25], [37]). This relies on powerful techniques based, among other things, on suitable linearization arguments, which require a variational structure that seems to be absent for system (1.1) (even under the reduced scalar form (3.5) below).

(v) The upper estimates in Theorem 1.1(i) remain true for $n = 2$, but they are then immediate consequences of the mass conservation property. Indeed, the latter (cf. (3.3) below) yields

$$r^n u(r, t) \leq c \int_0^r s^{n-1} u(s, t) ds \leq c \|u(t)\|_{L^1(B_R)} = c \|u_0\|_{L^1(B_R)}, \quad 0 < r \leq R, \quad 0 < t < T. \quad (2.3)$$

Moreover, in view of (1.4), the lower estimate in Theorem 1.2 fails for $n = 2$, and the upper estimate (1.9) is no longer optimal at $t = T$. As for the space-time estimate (1.12) in Theorem 1.3, it also remains true for $n = 2$ and may however be new in this case.

On the other hand, for $n \geq 3$, property (2.3) yields $u(r, t) \leq Cr^{-n}$ (hence single-point blow-up for u , as noted in [14, p.2140]), but this estimate is not optimal in view of Theorem 1.1.

(vi) The conclusions of Theorems 1.1-1.3 remain valid, with same proofs, for the modified problem (1.1) (considered for instance in [3], [14]), where $\Omega = B_R$, $\mu = 0$ and the boundary conditions are replaced with $\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} = 0$ and $v = 0$ on $\partial\Omega$.

(vii) We do not know if $B(u_0) = \mathbb{R}^n$ can occur for radial nonincreasing solutions, unless u_0 is constant. The known proof (see [8]) for the nonlinear heat equation (2.1) does not seem to apply to this case. Also, in Theorem 1.2, the assumption $B(u_0) \neq \mathbb{R}^n$ is actually not necessary since, in case $B(u_0) = \mathbb{R}^n$, then $U(x) \equiv \infty$ (recalling $u_t \geq 0$ and $u_r \leq 0$).

The outline of the rest of paper is as follows. In Section 3 we collect a number of preliminary results about local existence-uniqueness, transformed equation, blow-up criterion, and monotonicity properties. Theorems 1.1 and 1.3 are then proved in Section 4, and Theorem 1.2 in Section 5.

3 Preliminary results

We rewrite problem (1.1) in the form

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega, t > 0, \\ v = \int_{\Omega} G(x, y)(u(y, t) - \mu) dy, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (3.1)$$

where μ is defined in (1.2) and G is the Neumann Green kernel of $-\Delta$ in $\Omega = B_R$ (or the Newtonian potential in case $\Omega = \mathbb{R}^n$). Note that problems (1.1) and (3.1) are equivalent up to the addition of a constant to $v(\cdot, t)$, and considering (3.1) instead of (1.1) enables one to avoid the related uniqueness issues. The solution of (1.1) considered in Sections 1 and 2 is the solution of (3.1) given by the following local existence-uniqueness result. We shall not give the proof, which is standard and follows from arguments in, e.g., [9]. Here we denote by $S(t)_{t \geq 0}$ the heat semigroup on $L^\infty(\Omega)$ (with Neumann boundary conditions if $\Omega = B_R$).

Proposition 3.1 *Let $\Omega = B_R \subset \mathbb{R}^n$ or $\Omega = \mathbb{R}^n$, with $R > 0$ and $n \geq 2$, and let u_0 satisfy (1.3). There exists $\tau > 0$ and a unique, classical solution (u, v) of (3.1) such that*

$$\begin{cases} (u, v) \in BC^{2,1}(\overline{\Omega} \times (0, \tau)) \times BC^{2,0}(\overline{\Omega} \times (0, \tau)) \\ u - S(t)u_0 \in BC(\overline{\Omega} \times [0, \tau]). \end{cases}$$

Moreover, (u, v) can be extended to a unique maximal solution, whose existence time $T = T(u_0, v_0) \in (0, \infty]$ satisfies

$$\text{either } T < \infty \text{ or } \lim_{t \rightarrow T} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \quad (3.2)$$

The couple (u, v) also solves (1.1) and, for each $t \in (0, T)$, the function $u(\cdot, t)$ is nonnegative and radially symmetric nonincreasing. Furthermore, if $\Omega = B_R$, then u enjoys the mass conservation property

$$\|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}, \quad 0 < t < T, \quad (3.3)$$

and (3.3) remains true for $\Omega = \mathbb{R}^n$ if we assume in addition $u_0 \in L^1(\mathbb{R}^n)$.

Following, e.g., [6], [14], we shall rely on a transformed scalar equation involving the averaged mass of u over balls:

Proposition 3.2 *For any given solution of (3.1) provided by Proposition 3.1, let*

$$w(r, t) := r^{-n} \int_0^r s^{n-1} u(s, t) ds, \quad r \in (0, R), \quad t \in [0, T] \quad (3.4)$$

(with $R = \infty$ in case $\Omega = \mathbb{R}^n$). Then w is a classical solution of

$$w_t - w_{rr} - \frac{n+1}{r} w_r = n(w + brw_r)(w - \tilde{\mu}), \quad r \in (0, R), \quad t \in (0, T), \quad (3.5)$$

where $b = 1/n$ and $\tilde{\mu} = \mu/n$. Moreover, for each $t \in (0, T)$, $w(\cdot, t)$ can be extended to a C^2 function up to $r = 0$ and w satisfies the boundary conditions

$$w_r(0, t) = 0, \quad w(R, t) = \tilde{\mu} \quad (3.6)$$

(dropping the second condition in (3.6) if $\Omega = \mathbb{R}^n$). Furthermore, we have

$$w_r \leq 0, \quad r \in (0, R), \quad t \in (0, T). \quad (3.7)$$

PROOF. Using $\Delta = r^{1-n} \partial_r (r^{n-1} \partial_r)$, multiplying the second equation in (1.1) with r^{n-1} , integrating over $(0, r)$ and using $v_r(0, t) = 0$, we get

$$-v_r = r^{1-n} \int_0^r s^{n-1} (u - \mu) ds = r(w - \tilde{\mu}). \quad (3.8)$$

On the other hand, from the first equation in (1.1), we obtain

$$r^{n-1} u_t - (r^{n-1} u_r)_r = -r^{n-1} u_r v_r - u(r^{n-1} v_r)_r = -(r^{n-1} u v_r)_r.$$

Integrating over $(0, r)$ and using $u_r(0, t) = v_r(0, t) = 0$, we get

$$w_t - \frac{u_r}{r} = -\frac{u v_r}{r}. \quad (3.9)$$

Differentiating (3.4), we obtain $w_r = -\frac{n}{r} w + \frac{u}{r}$, hence

$$u = r w_r + n w \quad (3.10)$$

and

$$u_r = r w_{rr} + (n+1) w_r. \quad (3.11)$$

Substituting (3.8), (3.10) and (3.11) in (3.9), we finally obtain (3.5).

Next, from (3.4), we easily see that, for each $t \in (0, T)$, $w(\cdot, t)$ can be extended to a C^2 function up to $r = 0$. The boundary condition (3.6) at $r = 0$ then follows from the radial symmetry of u and (3.11). As for the boundary condition at $r = R$, it follows from the mass conservation property (3.3).

Finally let us prove (3.7). Since $u_r \leq 0$ in $[0, R] \times (0, T)$ due to Proposition 3.1, we have $nw(r, t) \geq r^{-n}u(r, t) \int_0^r ns^{n-1} ds = u(r, t)$. Property (3.7) then follows from (3.10). \square

Remark 3.3 (i) For future reference we observe that, as a consequence of Proposition 3.2, w can be viewed as a solution of

$$w_t - \tilde{\Delta}w = n(w + bx \cdot \tilde{\nabla}w)(w - \tilde{\mu}) \quad \text{in } \tilde{B} \times (0, T), \quad (3.12)$$

where \tilde{B} is the centered ball of radius R in \mathbb{R}^{n+2} (or $\tilde{B} = \mathbb{R}^{n+2}$) and $\tilde{\Delta}, \tilde{\nabla}$ are respectively the Laplacian and the spatial gradient in $n+2$ space variables. Also, we note that the function $z := w - \tilde{\mu} \geq 0$ satisfies (cf. (3.10)):

$$\begin{cases} z_t - \tilde{\Delta}z = uz \geq 0, & \text{in } \tilde{B} \times (0, T), \\ z = 0, & \text{on } \partial\tilde{B} \times (0, T). \end{cases} \quad (3.13)$$

(ii) When $\Omega = B_R$, the mass conservation property (3.3) ensures that

$$w(r, t) \leq C(n)\|u(t)\|_1 r^{-n} \leq Cr^{-n} \quad \text{in } (0, R] \times (0, T). \quad (3.14)$$

For convenience, we give a short proof of blowup for large initial mass in the framework of radial decreasing solutions in any dimension $n \geq 2$. A similar result (with a different proof) can be found in [3] for a variant of system (1.3) with Dirichlet boundary conditions on v . We leave apart the case $\Omega = \mathbb{R}^n$ with $n = 2$, for which we refer to [10], [5] (and we recall that more precise results are also available for $\Omega = B_R$ when $n = 2$).

Proposition 3.4 (i) Let $\Omega = B_R \subset \mathbb{R}^n$, with $R > 0$ and $n \geq 2$, and let u_0 satisfy (1.3). There exists $c_1(n) > 0$ such that if $\|u_0\|_1 > c_1(n)R^{n-2}$, then the solution of (3.1) blows up in a finite time $T < \infty$.

(ii) Let $\Omega = \mathbb{R}^n$, $n \geq 3$, and let u_0 satisfy (1.3). There exists $c_2(n) > 0$ such that if $\int_{\mathbb{R}^n} u_0(x)e^{-|x|^2} dx > c(n)$, then the solution of (3.1) blows up in a finite time $T < \infty$.

PROOF. (i) Let w be the corresponding solution of (3.5)-(3.6) given by Proposition 3.2 and set $z := w - \tilde{\mu}$. We have $z \geq 0$ due to (3.6), (3.7). By Remark 3.3, z satisfies

$$z_t - \tilde{\Delta}z = n(z + \tilde{\mu} + bx \cdot \tilde{\nabla}z)z = n\left[z^2 + \tilde{\mu}z + \frac{b}{2}x \cdot \tilde{\nabla}(z^2)\right] = n\left[az^2 + \tilde{\mu}z + \frac{b}{2}\tilde{\nabla} \cdot (xz^2)\right]$$

in $\tilde{B} \times (0, T)$, where \tilde{B} is the centered ball of radius R in \mathbb{R}^{n+2} and $a = (n-2)/2n$. Let ϕ be the first positive eigenfunction of $-\tilde{\Delta}$ in $H_0^1(\tilde{B})$, normalized in $L^1(\tilde{B})$, and $\lambda_1 > 0$ be the corresponding first eigenvalue, by scaling satisfying $\lambda_1 = C_1(n)R^{-2} > 0$ with some $C_1(n) > 0$. Multiplying with ϕ , integrating by parts over \tilde{B} and using $z = 0$ on $\partial\tilde{B}$, we get

$$\frac{d}{dt} \int_{\tilde{B}} z\phi dx = \frac{n-2}{2} \int_{\tilde{B}} z^2\phi dx + (\mu - \lambda_1) \int_{\tilde{B}} z\phi dx - \frac{1}{2} \int_{\tilde{B}} (x \cdot \nabla\phi)z^2 dx, \quad 0 < t < T.$$

Note that $x \cdot \tilde{\nabla} \phi \leq 0$ (indeed, by uniqueness, ϕ is radially symmetric, and it is radially decreasing due to $(r^{n+1} \phi_r)_r = -\lambda_1 r^{n+1} \phi \leq 0$ and $\phi_r(0) = 0$). Letting $y(t) = \int_{\tilde{B}} z \phi \, dx > 0$ and using Jensen's inequality, we then obtain

$$y'(t) \geq \frac{n-2}{2} y^2 + (\mu - \lambda_1) y, \quad 0 < t < T.$$

If $n \geq 3$ and $\mu \equiv |B_R|^{-1} \|u_0\|_1 \geq \lambda_1$, then no positive solution of this differential inequality can exist globally.

If $n = 2$ and $\mu > \lambda_1$, then $T = \infty$ would imply exponential growth of $y(t)$ as $t \rightarrow \infty$. However, by (3.3) and (3.4), with some $C_2 > 0$ we have

$$w(r, t) \leq C_2 r^{-n} \quad \text{in } (0, R] \times (0, T), \quad (3.15)$$

hence $\int_{\tilde{B}} z \phi \, dx \leq C \int_0^R r^{-n} r^{n+1} \, dr = CR^2$ for all $t \in (0, T)$, a contradiction. Assertion (i) follows.

(ii) We now have $\mu = 0$, hence $z = w$. We use the above argument with \tilde{B} and ϕ respectively replaced with \mathbb{R}^{n+2} and $\phi = C_3 e^{-|x|^2}$, where $C_3 = C_3(n) > 0$ is chosen so that $\int_{\mathbb{R}^{n+2}} \phi \, dx = 1$. A straightforward calculation yields $\tilde{\Delta} \phi \geq -2(n+2)\phi$ and we arrive at

$$y'(t) \geq \frac{n-2}{2} y^2 - 2(n+2)y, \quad 0 < t < T \quad (3.16)$$

(note that all the calculations can be justified by the fast decay of ϕ). We thus infer $T < \infty$ whenever $n \geq 3$ and the right-hand side of (3.16) is positive at $t = 0$. In view of (3.4), this can be written as $I > C_4(n)$ with some $C_4(n) > 0$, where

$$\begin{aligned} I &= \int_0^\infty r^{-n} \left(\int_0^r u_0(s) s^{n-1} \, ds \right) e^{-r^2} r^{n+1} \, dr \\ &= \int_0^\infty \left(\int_s^\infty r e^{-r^2} \, dr \right) u_0(s) s^{n-1} \, ds = \frac{1}{2} \int_0^\infty u_0(s) e^{-s^2} s^{n-1} \, ds. \end{aligned}$$

Assertion (ii) follows. □

We finish this preliminary section with the following proposition, which recalls a standard time monotonicity property, and also shows that large multiples of rather general initial data satisfy the assumptions of Theorem 1.2.

Proposition 3.5 (i) Let $n \geq 3$. Consider problem (3.1) with $\Omega = B_R \subset \mathbb{R}^n$ or $\Omega = \mathbb{R}^n$. Assume that u_0 satisfies (1.3) and (1.10). Then the corresponding solution w of (3.5)-(3.6) satisfies $w_t \geq 0$.

(ii) Let $\Omega = B_R$ and let $\phi \in C^2(\bar{\Omega})$ be radially symmetric and nonincreasing with respect to $|x|$, with $\phi(R) > 0$, $\phi_r(R) = 0$ and $\phi \not\equiv \text{const}$. Then, for all $\lambda > 1$ sufficiently large, $u_0 = \lambda \phi$ satisfies property (1.10).

PROOF. (i) Let $w_0 = w(\cdot, 0)$. Using (3.10), (3.11), we obtain, for all $r \in (0, R)$,

$$\begin{aligned} w_{0,rr} + \frac{n+1}{r} w_{0,r} + n(w_0 + b r w_{0,r})(w_0 - \tilde{\mu}) \\ = r^{-1} u_{0,r} + u_0 r^{-n} \int_0^r s^{n-1} (u_0(s) - \mu) \, ds \geq 0. \end{aligned} \quad (3.17)$$

In the case $\Omega = B_R$, the assertion then follows from a standard argument (see e.g. [31, Proposition 52.19] or [14, Lemma 4.4]), that we give for completeness. In view of (3.17), we see that $\underline{w}(x, t) := w_0(|x|)$ is a subsolution of (3.12) with $\underline{w}(\cdot, t) = \tilde{\mu} = w(\cdot, t)$ on $\partial\tilde{B}$. Therefore, by the comparison principle, $w(r, \tau) \geq \underline{w}(r, \tau) = w_0(r)$ in $[0, R]$ for each $\tau \in (0, T)$. Applying the comparison principle again, it follows that $w(r, \tau + t) \geq w(r, t)$ in $[0, R]$ for all $t \in (0, T - \tau)$ and we conclude that $w_t \geq 0$. In the case $\Omega = \mathbb{R}^n$, the above argument still works provided we can apply the comparison principle, which might cause some difficulties due to the unboundedness of the coefficient of the gradient term in the right-hand side $nw^2 + (x \cdot \nabla w)w$ of (3.12). However, owing to $w_r \leq 0$, this term has a favorable sign, since, at a possible positive maximum of the difference of two solutions w_1, w_2 :

$$(x \cdot \nabla w_1)w_1 - (x \cdot \nabla w_2)w_2 = (x \cdot \nabla w_1)w_1 - (x \cdot \nabla w_1)w_2 = rw_{1,r}(w_1 - w_2) \leq 0$$

As a consequence, the required comparison principle can be deduced from the proof of [31, Proposition 52.6].

(ii) Set $\psi(r) = r^{-n} \int_0^r \phi(s)s^{n-1} ds$ and $\hat{\mu} = R^{-n} \int_0^R \phi(s)s^{n-1} ds = \lambda^{-1}\tilde{\mu}$. We have

$$z(r) := \lambda^{-1}r^{1-n} [r^{n-1}u_{0,r} + u_0 \int_0^r s^{n-1}(u_0(s) - \mu) ds] = \phi_r + \lambda r \phi(\psi - \hat{\mu}), \quad 0 < r \leq R. \quad (3.18)$$

Since $\phi(r)$ is nonincreasing and nonconstant, we have

$$\psi'(R) = -nR^{-n-1} \int_0^R \phi(s)s^{n-1} ds + R^{-1}\phi(R) = -nR^{-n-1} \int_0^R (\phi(s) - \phi(R))s^{n-1} ds > 0.$$

By the proof of (3.7), $\psi(r)$ is nonincreasing. Since $\psi(R) = \hat{\mu}$, we deduce that, for some $C_1 > 0$,

$$\psi(r) - \hat{\mu} \geq C_1(R - r), \quad 0 \leq r \leq R. \quad (3.19)$$

Also, since $\phi_r(0) = \phi_r(R) = 0$ and $\phi \in C^2([0, R])$, we have, for some $C_2 > 0$,

$$\phi_r(r) \geq -C_2r(R - r), \quad 0 \leq r \leq R.$$

Combining this with (3.18), (3.19) and $\phi \geq \phi(R) > 0$, we obtain, assuming $\lambda \geq C_2(C_1\phi(R))^{-1}$,

$$z(r) \geq [-C_2 + \lambda C_1 \phi(R)]r(R - r) \geq 0, \quad 0 < r \leq R,$$

hence (1.10). □

4 Upper estimates: proof of Theorems 1.1 and 1.3

Remark 4.1 *We shall show (see (4.12)) that any radially nonincreasing solution $w \geq 0$ of (3.5) which blows up only at $r = 0$ satisfies*

$$w(r, t) \leq \left(\frac{1}{w(0, t)} + Kr^2 \right)^{-1}, \quad 0 \leq r < R, \quad 0 < t < T \quad (4.1)$$

(replacing R by 1 in (4.1) in case $R = \infty$). The proof works for any $b \in (0, 1/2]$ (not just $b = 1/n$), or any $b > 0$ if $\mu = 0$. We stress that this estimate is no longer true for $b = 0$ in general, since equation (3.5) for $b = \mu = 0$ (scaling out the factor n on the right-hand side) corresponds to (2.1) in dimension $n + 2$ (cf. Section 2, Remarks (i) and (iv)). Indeed, the proof crucially uses the fact that $b > 0$ (see (4.7) below).

PROOF OF THEOREMS 1.1 AND 1.3. The proof is based on nontrivial modifications of the idea in [11]. We shall use the maximum principle to show that

$$J := w_r + d(r)F(w) \leq 0$$

for appropriate functions d and F . In this proof we set $R := 1$ in case $\Omega = \mathbb{R}^n$.

Step 1. *Single-point blowup.* Note that $0 \in B(u_0)$ since otherwise u is uniformly bounded due to $u_r \leq 0$, contradicting (3.2).

First assume $\Omega = B_R$. Recalling the definition (1.7), we have $B(u_0) = \{0\}$ as a direct consequence of (2.3), (3.14) and (3.8).

Next consider the case $\Omega = \mathbb{R}^n$ and $u_0 \in L^1(\mathbb{R}^n)$. Then (2.3), (3.14) and (3.8) remain valid and we conclude as before.

Finally we consider the case $\Omega = \mathbb{R}^n$ and $B(u_0) \neq \mathbb{R}^n$. Then there exists $a > 0$ such that $u(a, t) + |v_r(a, t)| \leq C$ for all $t \in (T/2, T)$, hence, for any such t and some $C_1 > 0$, $w(a, t) \leq C_1$ by (3.8) with $\tilde{\mu} = 0$. Using (3.7), (3.10) and (3.4), it follows that

$$u(r, t) \leq nw(r, t) \leq n(a/r)^n w(a, t) \leq C_2 r^{-n}, \quad 0 < r \leq a, \quad T/2 < t < T$$

hence, owing to $u_r \leq 0$,

$$u(r, t) \leq C_3 \max[r^{-n}, a^{-n}], \quad r > 0, \quad T/2 < t < T, \quad (4.2)$$

for some $C_2 > 0$ and $C_3 > 0$. Now combining this with (3.8) and (3.9), we obtain $C_4 > 0$ such that $w_t \leq uw \leq C_4 \max[r^{-n}, a^{-n}]w$. Integrating in time and going back to (3.8), with some $C_5 > 0$ we deduce

$$|v_r| = rw(r, t) \leq \|w(T/2)\|_\infty r \exp\{C_5 T \max[r^{-n}, a^{-n}]\}, \quad r > 0, \quad T/2 < t < T. \quad (4.3)$$

It follows from (4.2) and (4.3) that $B(u_0) = \{0\}$.

Step 2. *Uniform negativity of w_r on the parabolic boundary.* First consider the case $\Omega = B_R$. Setting $V = \partial_{x_1} w$, we have $V \leq 0$ in $\tilde{B}_+ = \{x \in \mathbb{R}^{n+2}; |x| < R, x_1 > 0\}$, due to $w_r \leq 0$. Differentiating (3.12), we see that v solves

$$V_t - \tilde{\Delta} V = n(w + bx \cdot \nabla w)V + n(w - \tilde{\mu})((b+1)V + bx \cdot \nabla V) \quad \text{in } \tilde{B}_+ \times (0, T). \quad (4.4)$$

It then follows from the strong maximum principle and the Hopf lemma that

$$w_r(r, T/2) = V(r, 0, \dots, 0, T/2) < 0 \quad \text{in } (0, R]$$

and $w_{rr}(0, T/2) = V_{x_1}(0, T/2) < 0$, hence there exists $C_6 > 0$ such that

$$w_r(r, T/2) \leq -C_6 r \quad \text{in } [0, R]. \quad (4.5)$$

On the other hand, owing to (3.13) and again the Hopf Lemma, we can find $C_7 > 0$ fulfilling

$$w_r(R, t) = z_r(R, t) \leq -C_7 \quad \text{in } [T/2, T). \quad (4.6)$$

Next consider the case $\Omega = \mathbb{R}^n$. Property (4.5) with $R = 1$ follows from the above argument (where now $\tilde{B}_+ = \{x \in \mathbb{R}^{n+2}; x_1 > 0\}$). On the other hand, since $B(u_0) = \{0\}$ by Step 1, it follows from (3.8) and (3.10) that w and w_r are bounded on $[T/2, T]$ near $r = 1$. Property (4.6) with $R = 1$ is then a consequence of the strong maximum principle applied to equation (4.4).

Step 3. *Local parabolic inequality for J .* Setting $N := (w + brw_r)(w - \tilde{\mu})$, where $b > 0$, in $(0, R) \times (0, T)$ we compute

$$(\partial_t - \partial_r^2)(dF(w)) = dF'(w)(w_t - w_{rr}) - dF''(w)w_r^2 - 2d'F'(w)w_r - d''F(w)$$

and, differentiating (3.5),

$$(\partial_t - \partial_r^2)w_r = \frac{n+1}{r}w_{rr} - \frac{n+1}{r^2}w_r + nN_r.$$

Omitting the variables r, t, w when no confusion arises, it follows that

$$J_t - J_{rr} = \frac{n+1}{r}w_{rr} - \frac{n+1}{r^2}w_r + nN_r + dF'\left(\frac{n+1}{r}w_r + nN\right) - dF''w_r^2 - 2d'F'w_r - d''F.$$

Substituting $w_r = J - dF$ and $w_{rr} = J_r - d'F - dF'w_r = J_r - dF'J + d^2FF' - d'F$, we obtain

$$\begin{aligned} J_t - J_{rr} &= \frac{n+1}{r}(J_r - dF'J + d^2FF' - d'F) - \frac{n+1}{r^2}(J - dF) + nN_r \\ &\quad + dF'\left(\frac{n+1}{r}(J - dF) + nN\right) - dF''(J - dF)^2 - 2d'F'(J - dF) - d''F. \end{aligned}$$

Setting

$$\mathcal{P}J := J_t - J_{rr} - \frac{n+1}{r}J_r + \left(\frac{n+1}{r^2} + dF''(J - 2dF) + 2d'F'\right)J,$$

it follows that

$$\begin{aligned} \mathcal{P}J &= \frac{n+1}{r}(d^2FF' - d'F) + \frac{n+1}{r^2}dF + nN_r \\ &\quad + dF'\left(-\frac{n+1}{r}dF + nN\right) - d^3F''F^2 + 2dd'FF' - d''F \\ &= \frac{n+1}{r^2}(d - rd')F + n(N_r + dF'N) - d^3F''F^2 + 2dd'FF' - d''F. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} N_r &= (w + brw_r)w_r + ((b+1)w_r + brw_{rr})(w - \tilde{\mu}) \\ &= (w + br(J - dF))(J - dF) + [(b+1)(J - dF) + br(J_r - dF'J + d^2FF' - d'F)](w - \tilde{\mu}) \\ &= \mathcal{L}_1J + (w - bdrF)(-dF) + [-(b+1)d + br(d^2F' - d')](w - \tilde{\mu})F \\ &= \mathcal{L}_1J - 2dFw + b[r d^2F + (r d^2F' - r d' - d)w]F - \tilde{\mu}[-(b+1)d + br(d^2F' - d')]F, \end{aligned}$$

where

$$\mathcal{L}_1J := \left[w + br(J - dF) - dFbr + (b+1 - bdrF')(w - \tilde{\mu}) \right]J + br(w - \tilde{\mu})J_r,$$

and

$$dF'N = dF'(w + br(J - dF))(w - \tilde{\mu}) = \mathcal{L}_2J + dF'w^2 - brd^2wFF' - \tilde{\mu}(w - brdF)dF',$$

where $\mathcal{L}_2J := dF'br(w - \tilde{\mu})J$. Therefore, setting $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, we obtain

$$\begin{aligned} N_r + dF'N &= \mathcal{L}J - 2dFw + dF'w^2 + b[rd^2F + (rd^2F' - rd' - d)w - rd^2wF']F \\ &\quad - \tilde{\mu}[-(b+1)dF + br(d^2FF' - d'F) + (w - brdF)dF'] \\ &= \mathcal{L}J + dw(wF' - 2F) + bd[rdF - (1 + \frac{rd'}{d})w]F - \tilde{\mu}d[wF' - (1 + b + b\frac{rd'}{d})F]. \end{aligned}$$

Now choose $d = \varepsilon r$, with $\varepsilon > 0$ to be fixed, and $F(w) = w^2$. Setting $\tilde{\mathcal{P}} = \mathcal{P} - n\mathcal{L}$, we get

$$\begin{aligned} \tilde{\mathcal{P}}J &= nbd[rdF - (1 + \frac{rd'}{d})w]F - n\tilde{\mu}d[wF' - (1 + b(1 + \frac{rd'}{d}))F] - d^3F''F^2 + 2dd'FF' \\ &= nb\varepsilon rw^3[\varepsilon r^2w - 2] - \mu\varepsilon(1 - 2b)rw^2 - 2\varepsilon^3r^3w^4 + 4\varepsilon^2rw^3, \end{aligned}$$

hence

$$\tilde{\mathcal{P}}J = (nb - 2\varepsilon)w^3\varepsilon r[\varepsilon r^2w - 2] - \mu\varepsilon(1 - 2b)rw^2 \quad \text{in } (0, R) \times (0, T). \quad (4.7)$$

Step 4. Nonlocal parabolic inequality for J and conclusion. To relate to J the term $\varepsilon r^2w - 2$ on the right-hand side of (4.7), we next observe that

$$\begin{aligned} \varepsilon r^2w - 2 &= 2w\left(\frac{\varepsilon}{2}r^2 - \frac{1}{w}\right) = 2w\left(-\frac{1}{w(0, t)} + \int_0^r \left(\varepsilon s + \frac{w_r}{w^2}(s, t)\right) ds\right) \\ &= 2w\left(-\frac{1}{w(0, t)} + \int_0^r w^{-2}J(s, t) ds\right). \end{aligned} \quad (4.8)$$

We note that with some $C_8 > 0$ we have $w \geq C_8$ on $[0, R] \times [T/2, T)$. Indeed, if $\Omega = B_R$, then this is a consequence of (3.6), (3.7), whereas if $\Omega = \mathbb{R}^n$ (and $R = 1$), this follows from the fact that w is a supersolution of the heat equation by (3.13). Taking $0 < \varepsilon < nb/2$ and using (4.7), $\mu \geq 0$, and $b \leq 1/2$ if $\mu > 0$, it follows that

$$\tilde{\mathcal{P}}J \leq 2\varepsilon r(nb - 2\varepsilon)w^4\left(-\frac{1}{w(0, t)} + C_9 \int_0^r J_+(s, t) ds\right) \quad (4.9)$$

for some constant $C_9 > 0$, where $\xi_+ = \max(\xi, 0)$ for $\xi \in \mathbb{R}$.

On the other hand, taking ε sufficiently small, it follows from (4.5), (4.6) and (3.14) that

$$J = w_r + \varepsilon rw^2 \leq 0 \quad \text{on the parabolic boundary of } Q := [T_0, T) \times (0, R), \quad (4.10)$$

where $T_0 = T/2$. Note that this remains true with $R = 1$ in case $\Omega = \mathbb{R}^n$ by the end of Step 2 and (4.3). Set

$$E = \{\tau \in [T_0, T); J \leq 0 \text{ on } [0, R] \times [T_0, \tau]\} \neq \emptyset$$

and assume for contradiction that $T_1 := \sup E < T$. Set $T_2 = (T_1 + T)/2$. Then there exists $\eta > 0$ such that $\frac{1}{w(0, t)} \geq \eta$ on $[T_0, T_2]$. Since $J \leq 0$ on $[0, R] \times [T_0, T_1]$, by continuity, there exists $T_3 \in (T_1, T_2)$ such that $J < \eta/(C_9R)$ on $[0, R] \times [T_0, T_3]$. It follows from (4.9) that

$$\tilde{\mathcal{P}}J \leq 0 \quad \text{in } (0, R) \times (T_0, T_3]. \quad (4.11)$$

Note that, by the definition of $\tilde{\mathcal{P}}$ in Step 3, we may write

$$\tilde{\mathcal{P}}J = J_t - J_{rr} + a_1(r, t)J_r + \frac{n+1}{r^2}J + a_2(r, t)J \quad \text{in } (0, R) \times [T_0, T),$$

for some functions $a_1, a_2 : (0, R) \times [T_0, T) \rightarrow \mathbb{R}$ with $a_2(r, t)$ bounded on $(0, R) \times (T_0, T_3]$. Setting $\hat{J} := e^{-\lambda t}J$ with $\lambda > \|a_2\|_{L^\infty((0, R) \times (T_0, T_3])}$, we see that \hat{J} cannot attain a positive local maximum in $(0, R) \times (T_0, T_3]$. Indeed at such a point, we would have

$$0 \leq \hat{J}_t - \hat{J}_{rr} + a_1(r, t)\hat{J}_r \leq e^{-\lambda t}[\tilde{\mathcal{P}}J - (\lambda + a_2)J] < 0,$$

which is impossible. Consequently, owing to (4.10), we have $J \leq 0$ in $[0, R] \times [T_0, T_3]$. But this contradicts the definition of T_1 . It follows that $T_1 = T$ and we conclude from (4.8) that $\varepsilon r^2 w - 2 \leq -2w/w(0, t)$, hence

$$\frac{1}{w(r, t)} \geq \frac{\varepsilon r^2}{2} + \frac{1}{w(0, t)} \quad \text{on } [0, R] \times [T_0, T). \quad (4.12)$$

Since $u = rw_r + nw \leq nw$ and $u(0, t) = nw(0, t)$, we get

$$\frac{1}{u(r, t)} \geq \frac{\varepsilon r^2}{2n} + \frac{1}{u(0, t)} \quad \text{on } [0, R] \times [T_0, T),$$

and estimate (1.12) follows, hence in particular (1.8).

Since $B(u_0) = \{0\}$, the existence of $U(x) := \lim_{t \rightarrow T} u(x, t)$ for all $x \in \bar{\Omega} \setminus \{0\}$ is an immediate consequence of interior parabolic estimates, and we get (1.9). By dominated convergence, owing to (1.8), we also have convergence in $L^1(B_R)$. The proof of Theorems 1.1 and 1.3 is complete. \square

5 Lower estimate: proof of Theorem 1.2

PROOF OF THEOREM 1.2.

Step 1. *Lower estimate of $w(\cdot, T)$.* First, similarly as in [14, Lemma 4.5], we find $C_1 > 0$ such that

$$|w_r| \leq C_1 w^{3/2}(0, t) \quad \text{in } Q := (0, R) \times (0, T). \quad (5.1)$$

Indeed, for all $(r, t) \in Q$, using $w_t \geq 0$, $w_r \leq 0$, $w \geq \tilde{\mu} \geq 0$, we obtain

$$\frac{\partial}{\partial r} \left(\frac{1}{2} w_r^2 + \frac{n}{3} w^3 \right) = (w_{rr} + n w^2) w_r = \left(w_t - \frac{n+1}{r} w_r - n b r w_r (w - \tilde{\mu}) + \mu w \right) w_r \leq 0.$$

This guarantees

$$\left(\frac{1}{2} w_r^2 + \frac{n}{3} w^3 \right) (r, t) \leq \frac{n}{3} w^3(0, t),$$

hence (5.1).

We next use a modification of an argument from [36] (see also [31, p.192]). First, since $B(u_0) = \{0\}$ by Theorem 1.1 and

$$w(0, t) = \|w(t)\|_\infty \rightarrow \infty \quad \text{as } t \rightarrow T \quad (5.2)$$

by (3.2) and (3.10), we may assume that

$$w(0, t) > 2w(R, t), \quad T - \delta < t < T,$$

by taking δ small enough (with $R := 1$ in the case $\Omega = \mathbb{R}^n$). Therefore, for all $t \in (T - \delta, T)$, there exists $r_0(t) \in (0, R)$ such that $w(r_0(t), t) = \frac{1}{2}w(0, t)$. Note that, since $w_r < 0$ in $(0, R] \times (0, T)$, the implicit function theorem guarantees that $r_0(t)$ is unique and is a continuous function of t . By (5.2), this implies $r_0(t) \rightarrow 0$ as $t \rightarrow T$. In view of (5.1), it follows that with some $C_2 > 0$,

$$-w_r \leq C_2 w^{3/2}, \quad 0 \leq r \leq r_0(t).$$

Integrating, we get

$$w^{-1/2}(r_0(t), t) \leq w^{-1/2}(0, t) + C_3 r_0(t) = (2w(r_0(t), t))^{-1/2} + C_3 r_0(t),$$

hence $w(r_0(t), t) \geq C_4 (r_0(t))^{-2}$ with appropriate constants $C_3 > 0$ and $C_4 > 0$. Using $w_t \geq 0$, it follows that

$$w(r_0(t), T) \geq C_4 (r_0(t))^{-2}, \quad 0 < t < T.$$

Since r_0 is continuous and $r_0(t) \rightarrow 0$ as $t \rightarrow T$, we deduce that the range $r_0((T - \delta, T))$ contains an interval of the form $(0, \eta)$, hence

$$w(r, T) \geq C_4 r^{-2}, \quad 0 < r < \eta.$$

Step 2. *Lower estimate of U .* Going back to U , we have thus proved that

$$\int_0^r s^{n-1} U(s) ds = r^n w(r, T) \geq C_4 r^{n-2}, \quad 0 < r < \eta. \quad (5.3)$$

On the other hand, since $n \geq 3$ and Theorem 1.1 provides $C_5 > 0$ such that $U(r) \leq C_5 r^{-2}$ for all $r \in (0, R)$, it follows that

$$\int_0^r s^{n-1} U(s) ds \leq C_5 \int_0^r s^{n-3} ds = \frac{C_5 r^{n-2}}{n-2}, \quad 0 < r < R. \quad (5.4)$$

Since U is nonincreasing, combining (5.3) and (5.4), we deduce that, for each $K > 0$,

$$\begin{aligned} \frac{(Kr)^n U(r)}{n} &\geq \int_r^{Kr} s^{n-1} U ds = \int_0^{Kr} s^{n-1} U ds - \int_0^r s^{n-1} U ds \\ &\geq \left(C_4 K^{n-2} - \frac{C_5}{n-2} \right) r^{n-2}, \quad 0 < r < \eta/K. \end{aligned}$$

Choosing $K := (2C_5 / ((n-2)C_4))^{1/(n-2)}$ and setting $C_6 := nC_5 K^{-n} / (n-2)$, we conclude that

$$U(r) \geq \frac{n}{(Kr)^n} \cdot \frac{C_5}{n-2} r^{n-2} = C_6 r^{-2}, \quad 0 < r < \eta/K.$$

□

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