Boundedness and stabilization in a population model with cross-diffusion for one species

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Abstract

This work studies the two-species Shigesada-Kawasaki-Teramoto model with cross-diffusion for one species, as given by

$$\begin{cases} u_t = \Delta[(d_1 + a_{11}u + a_{12}v)u] + \mu_1 u(1 - u - a_1 v), \\ v_t = \Delta[(d_2 + a_{22}v)v] + \mu_2 v(1 - v - a_2 u), \end{cases} \qquad (\star)$$

with positive parameters d_1, d_2 and a_{11}, and nonnegative constants $a_{12}, a_{22}, \mu_1, \mu_2, a_1$ and a_2. Beyond some statements on global existence, the literature apparently provides only few results on qualitative behavior of solutions; in particular, questions related to boundedness as well as to large time asymptotics in $(\star)$ seem unsolved so far.

In the present paper it is inter alia shown that if $n \le 9$ and $\Omega \subset \mathbb{R}^n$ is a bounded convex domain with smooth boundary, then whenever $u_0 \in W^{1,\infty}(\Omega)$ and $v_0 \in W^{1,\infty}(\Omega)$ are nonnegative, the associated Neumann initial-boundary value problem for $(\star)$ possesses a global classical solution which in fact is bounded in the sense that

$$u \in L^\infty(\Omega \times (0,\infty)) \qquad \text{and} \qquad v \in L^\infty((0,\infty); W^{1,p}(\Omega)) \quad \text{for all } p > n. \qquad (\star\star)$$

Moreover, the asymptotic behavior of arbitrary nonnegative solutions enjoying the boundedness property is studied in the general situation when $n \ge 1$ is arbitrary and Ω no longer necessarily convex. If $a_1 \in (0,1)$, then in both of the cases $a_2 > 1$ and $a_2 \in (0,1)$, an explicit smallness condition on a_{12} is identified as sufficient for stabilization of any nontrivial solutions toward a corresponding unique nontrivial spatially homogeneous steady state. If $a_1 \ge 1$ and $a_2 \in (0,1)$, then without any further assumption all nonzero solutions are seen to approach the equilibrium $(0,1)$. As a by product, this particularly improves previous knowledge on nonexistence of nonconstant equilibria of $(\star)$.

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1 Introduction

We consider the initial-boundary value problem

$$\begin{cases} u_t = \Delta[(d_1 + a_{11}u + a_{12}v)u] + \mu_1 u(1-u-a_1 v), & x\in\Omega,\ t>0,\\ v_t = \Delta[(d_2 + a_{22}v)v] + \mu_2 v(1-v-a_2 u), & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

in a bounded domain $\Omega\subset\mathbb{R}^n$ with smooth boundary, as initially proposed by Shigesada, Kawasaki and Teramoto ([26]) to model processes of spatial segregation in populations of interacting species. In this context, $u=u(x,t)$ and $v=v(x,t)$ stand for the densities of two competing species, d_1, d_2, a_{11} and a_{22} represent their respective random diffusion and self-diffusion rates, and a_{12} measures the ability of the first species to repulsively cross-diffuse; the parameters μ_1, μ_2, a_1 and a_2 indicate the rates of logistic proliferation, as well as of interspecific competition, in the style of usual Lotka-Volterra-type kinetics.

Since its introduction in 1979, along with several simplifications and also a slightly more general variant accounting for cross-diffusive movement also of the second species, (1.1) has received considerable attention. In the case when a_{11} is merely assumed nonnegative but a_{12} and a_{22} are allowed to be positive, the only available result addressing smooth solutions seems to go back to [18], where global existence of classical solutions to (1.1) was proved in two-dimensional domains; for arbitrary $n\geq 1$, at least global weak solutions can be constructed ([4], [5]; cf. also [30] for some results on a variant of (1.1) involving more general than logistic-type degradation terms, and [3] as well as [15] for examples of subtle approaches even capable of covering some multi-species cases in the presence of a certain so-called detailed balance of system ingredients). In the case when a_{11} is positive, exploiting a correspondingly present strongly dampening effect of self-diffusion lead to results on global classical solvability firstly for $n\leq 5$ ([6], [14]), and by a refined analysis thereafter for any $n\leq 9$ ([31]), and recently for arbitrary $n\geq 1$ ([10]).

Focusing on issues of existence theory, the above works exclusively concentrate on questions of global solvability, and in particular they do not provide any information on the large time behavior of solutions, with the case $\mu_1=\mu_2=0$ of trivial kinetics forming an exception ([5]). To the best of our knowledge, even boundedness of solutions could be established only for very specific, and hence quite non-generic, parameter constellations up to now ([11]).

It is the purpose of the present work to undertake a natural next step in the analsyis by providing some information on the large time behavior of solutions to (1.1) under mild assumptions on the system parameters. Our first particular objective consists in developing an approach which, relying on the additional assumptions that Ω be convex and that, more essentially, the space dimension satisfies $n\leq 9$, asserts boundedness in suitable spaces; in fact, our analysis for its derivation will give an independent proof of the global existence result already known from [31] and [10].

Theorem 1.1 *Let $n\leq 9$ and $\Omega\subset\mathbb{R}^n$ be a bounded convex domain with smooth boundary, and suppose that the parameters d_1, d_2 and a_{11} are positive and a_{12}, a_{22}, a_1, a_2, μ_1 and μ_2 nonnegative. Then for all nonnegative functions u_0 and v_0 from $W^{1,\infty}(\Omega)$, the problem (1.1) possesses a global classical solution* *which for any $\vartheta > n$ is uniquely determined by the inclusion*

$$(u,v) \in \Big(C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)) \cap L^\infty_{loc}([0,\infty); W^{1,\vartheta}(\Omega)) \Big)^2,$$

such that both u and v are nonnegative in $\bar{\Omega} \times (0,\infty)$, and such that (u,v) is bounded in the sense that given any $p > 1$ one can find $K > 0$ fulfilling

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,p}(\Omega)} \le K \qquad \text{for all } t > 0. \tag{1.2}$$

The boundedness property (1.2) discovered above will thereafter enable us to address the problem of describing in more detail the large time behavior of the solutions constructed in Theorem 1.1, with a particular focus on the question how far the presence of cross-diffusion may influence properties of stabilization toward spatially homogeneous steady states which are well-known in the corresponding semilinear case obtained on letting $a_{11} = a_{12} = a_{22} = 0$ in (1.1). Here the crucial role of (1.2) will already become clear during the formulation of our results in this respect, which namely will apply to any spatial dimension and widely arbitrary solutions of (1.1) that will merely be assumed to satisfy (1.2) for some $p > n$.

Indeed, we firstly recall that in the latter situation, the additional assumption that

$$a_1 \in (0,1) \qquad \text{and} \qquad a_2 > 1 \tag{1.3}$$

ensures that for any choice of reasonably regular initial data $0 \not\equiv u_0 \ge 0$ and $v_0 \ge 0$, the corresponding version of (1.1) possesses a uniquely determined global classical solution satisfying $u(\cdot,t) \to 1$ and $v(\cdot,t) \to 0$ in $L^\infty(\Omega)$ as $t \to \infty$, hence reflecting complete invasion of the entire spatial habitat by the species which more efficiently degrades the other (see [34] for an analysis of a generalized multi-species case in absence of diffusion, and [28] for the diffusible two-species case as well as a generalization to small chemotactic interaction).

Our main result in this direction asserts that with regard to the parameters a_1 and a_2, the sufficiency of (1.3) for u to outcompete v remains unaffected also when cross-diffusion is introduced, provided that the diffusivities in (1.1) are suitably large in comparison to cross-diffusion; more precisely:

Theorem 1.2 *Suppose that $n \ge 1$ is arbitrary, that a_1 and a_2 satisfy (1.3), and that $d_1 > 0$, $d_2 > 0$, $\mu_1 > 0, \mu_2 > 0$ and $a_{12} \ge 0$ are such that*

$$\frac{a_{12}^2}{d_1 d_2} \le \frac{\mu_1}{\mu_2} \cdot \frac{4(2-a_1)(a_2-1)}{a_2}. \tag{1.4}$$

Let $(u,v) \in (C^{2,1}(\bar{\Omega} \times (0,\infty)))^2$ be any classical solution of the boundary value problem in (1.1) for which u and v are nonnegative with $u \not\equiv 0$, and for which (1.2) holds with some $p > n$ and some $K > 0$. Then

$$u(\cdot,t) \to 1 \quad \text{in } C^2(\bar{\Omega}) \qquad \text{as } t \to \infty \tag{1.5}$$

and

$$v(\cdot,t) \to 0 \quad \text{in } C^2(\bar{\Omega}) \qquad \text{as } t \to \infty. \tag{1.6}$$

We will next be concerned with the corresponding question in the case when yet $a_1 \in (0,1)$ but in addition also a_2 is suitably small, in which a flat coexistence state is known to determine the asymptotics in the above semilinear situation (cf. e.g. [2] for a proof actually covering a more general cross-diffusive generalization thereof). More precisely assuming that

$$a_1 \in (0,1) \qquad \text{and} \qquad a_2 \in [0,1), \tag{1.7}$$

we shall reveal a global asymptotic stability property of the homogeneous equilibrium $(u_\star, v_\star)$ given by

$$u_\star := \frac{1-a_1}{1-a_1a_2} \qquad \text{and} \qquad v_\star := \frac{1-a_2}{1-a_1a_2}, \tag{1.8}$$

again under an appropriate assumption on relative smallness of cross-diffusion:

Theorem 1.3 *Let $n \ge 1$ and a_1 and a_2 be such that (1.7) holds, and let d_1, d_2, μ_1 and μ_2 be positive and $a_{12} \ge 0$ fulfill*

$$\frac{a_{12}^2}{d_1d_2} < \frac{\mu_1}{\mu_2} \cdot \frac{4a_1(1-a_2)}{a_2(1-a_1)}. \tag{1.9}$$

Then whenever $(u,v) \in (C^{2,1}(\bar\Omega \times (0,\infty)))^2$ is a classical solution of the boundary value problem in (1.1) such that $0 \not\equiv u \ge 0$ and $0 \not\equiv v \ge 0$ and that (1.2) is valid with some $p > n$ and $K > 0$, we have

$$u(\cdot,t) \to u_\star \quad \text{in } C^2(\bar\Omega) \qquad \text{as } t \to \infty \tag{1.10}$$

and

$$v(\cdot,t) \to v_\star \quad \text{in } C^2(\bar\Omega) \qquad \text{as } t \to \infty, \tag{1.11}$$

where $u_\star$ and $v_\star$ are given by (1.8).

The smallness conditions (1.4) and (1.9) on the ratio $q := \frac{a_{12}^2}{d_1d_2}$ seem in very good accordance with a well-known result from a corresponding steady-state analysis asserting nonexistence of nonconstant equilibria for (1.1) whenever precisely this expression q is small ([17], [23]; cf. also some further corresponding nonexistence results under alternative smallness assumptions on a_{12}, as achieved in [21]). On the other hand, under the hypothesis that the cross-diffusion coefficient a_{12} is sufficiently large, a recent study shows that (1.1) does possesses nonconstant positive steady states at least when $a_2 > 1$, provided that the remaining parameters lie in a suitable range ([19]). This indicates that appropriate smallness requirements on a_{12} may indeed be necessary for stabilization results of the above form.

Surprisingly, no such further assumption on a_{12} will be necessary for the second species to outcompete the first in the situation dual to that from Theorem 1.2. Namely, in the case when

$$a_1 \ge 1 \qquad \text{and} \qquad a_2 \in (0,1), \tag{1.12}$$

we shall see that without any restriction on the size of a_{12} all nontrivial solutions will approach the equilibrium $(0,1)$ in the large time limit:

Theorem 1.4 *Assume that $n \geq 1$ and that a_1 and a_2 satisfy (1.12). Then for each solution $(u,v) \in (C^{2,1}(\bar\Omega \times (0,\infty)))^2$ of the boundary value problem in (1.1) which is such that $u \geq 0$ and $0 \not\equiv v \geq 0$ and that (1.2) is satisfied with some $p > n$ and $K > 0$, we have*

$$u(\cdot,t) \to 0 \quad \text{in } C^2(\bar\Omega) \qquad \text{as } t \to \infty \tag{1.13}$$

and

$$v(\cdot,t) \to 1 \quad \text{in } C^2(\bar\Omega) \qquad \text{as } t \to \infty. \tag{1.14}$$

In summary, Theorem 1.2, Theorem 1.3 and Theorem 1.4 confirm asymptotic preference of the spatially homogeneous equilibria dictated by the Lotka-Volterra kinetics in (1.1), in both the strongly asymmetric and the weakly competitive situations determined by (1.3), (1.12) and (1.7), provided that cross-diffusion is suitably small in the cases when u is not eventually outcompeted. With regard to the creation of a yet more complete picture, e.g. for applications of (1.1) in spatial ecology, an interesting open problem consists in deciding to which extent our above conditions on a_{12} can be viewed as approaching optimality.

Without any further comment, as an immediate by-product of Theorem 1.2, Theorem 1.3 and Theorem 1.4 let us furthermore state the following result on nonexistence of nonconstant steady-state solutions to (1.1) which complements the recent findings gained in [21]; in particular, in the case when $a_1 \geq 1$ and $a_2 \in (0,1)$ this significantly extends the latter by not relying on any further assumption such as $d_1 \geq d_2$ or $\frac{a_{12}}{d_2} < \frac{1}{a_2}$ required therein.

Corollary 1.5 *Let $n \geq 1$, and let $d_1 > 0, d_2 > 0, a_{11} \geq 0, a_{12} \geq 0, a_1 > 0$ and $a_2 > 0$, and suppose that $(u,v) \in (C^1(\bar\Omega) \cap C^2(\Omega))^2$ is a steady-state solution of (1.1) such that u and v are nonnegative.*

i) If

$$a_1 \in (0,1) \qquad \text{and} \qquad a_2 > 1, \tag{1.15}$$

and if (1.4) holds, then unless $u \equiv 0$,

$$(u,v) \equiv (1,0). \tag{1.16}$$

ii) In the case when

$$a_1 \in (0,1) \qquad \text{and} \qquad a_2 \in [0,1), \tag{1.17}$$

and when moreover (1.9) is valid and $u \not\equiv 0 \not\equiv v$, we have

$$(u,v) \equiv (u_\star, v_\star). \tag{1.18}$$

iii) If

$$a_1 \geq 1 \qquad \text{and} \qquad a_2 \in (0,1), \tag{1.19}$$

then unless $v \equiv 0$,

$$(u,v) \equiv (0,1). \tag{1.20}$$

Since our analysis will strongly rely on the absence of cross-diffusion in the second equation in (1.1), neither of our results covers the case of doubly cross-diffusive systems. In fact, except for one-dimensional settings where some results on global existence and boundedness are available ([12], [27]), even with regard to a basic existence theory the understanding of such systems seems far from complete; beyond global existence within suitably generalized solution frameworks ([4], [5], [7], [3]), [15]), or for systems with suitably small cross-diffusion coefficients ([8]), fairly little seems known about the solution behavior especially in multi-dimensional situations. Only in the special case when both diffusivities coincide and self-diffusion is entirely neglected, the existence of global smooth solutions as well as their uniform boundedness have recently been established ([22]).

Challenges and main ideas. The main intention in our strategy toward deriving the boundedness result in Theorem 1.1 is to suitably control any possibly destabilizing effect of cross-diffusion by making appropriate use of the strongly dampening properties of the nonlinear diffusion mechanism in the first equation in (1.1) at large densities of u. A similar intuition has been pursued in numerous studies on various types of related cross-diffusive systems of chemotaxis type involving diffusivities $D(u)$ which become suitably strong at large values of u. In fact, some parallels to such taxis systems can be drawn by observing that despite evident structural differences becoming transparent when expanding $\Delta(uv) = \nabla \cdot (u\nabla v) + \nabla \cdot (v\nabla u)$, the cross-diffusive interaction in (1.1) can, heuristically, be guessed to share essential features with chemotactic cross-diffusion, in classical Keller-Segel systems becoming manifest in terms of the form $\nabla \cdot (u\nabla v)$, with regard to their strength as destabilizing nonlinearities counteracting diffusion. Indeed, for several types of such systems an appropriately organized analysis leads to the conclusion that linear growth of $D(u)$ with respect to u warrants boundedness, within large classes of the respective evolution equation for v, in particular containing the choice made in (1.1) when $a_{22} = 0$ ([25], [20], [32]).

Unlike in the latter situation, however, in the case $a_{22} > 0$ when also v diffuses in a nonlinear manner, a substantial technical obstacle seems to consist in a lack of appropriate knowledge on regularity properties of v in dependence on supposedly present regularity properties of u, especially in cases when the latter are rather weak. In particular, this apparently obstructs any detection of energy-like properties in (1.1) which involve coupled functionals of the form

$$\int_\Omega u^p + B\int_\Omega |\nabla v|^q \tag{1.21}$$

for some $B > 0$ and $p > 1$ whenever $q > 2$, as playing a corresponding role when $a_{22} = 0$. This is due to the circumstance that even with regard to the scalar equation $v_t = \Delta\phi(v) + f(x,t)$ in $\Omega \times (0,T)$ for given f and $T > 0$, in the case $\phi' \not\equiv const.$, the functional $\int_\Omega |\nabla v|^q$ for $q > 2$ seems to enjoy favorable quasi-Lyapunov properties only along solutions for which $(v(\cdot,t))_{t\in(0,T)}$ is a priori known to be equicontinuous in $\bar\Omega$; in consequence, tracking the time evolution thereof appears to be inadequate if in accordance with basic properties of (1.1) (cf. Lemma 2.2) the function f herein is merely assumed to belong to $L^\infty((0,T);L^1(\Omega))$ and hence insufficient to enforce estimates on the modulus of continuity of v.

Nevertheless attempting to adequately respect the coupling in (1.1), in a first step of our a priori estimation procedure we shall therefore restrict ourselves to the study of a functional related to the choice $q = 2$, namely,

$$\int_\Omega u^p + B\int_\Omega |\nabla\phi(v)|^2, \tag{1.22}$$

with suitably chosen $B > 0$ and $\phi(s) := d_2 s + a_{22}s^2$, $s \geq 0$ (Lemma 3.4). In view of the increasing strength of cross-diffusive contributions at large values of p, this approach will in a natural manner be restricted to a bounded range for p (Lemma 3.5), but in the case $n \le 9$ this range will contain suitably large values of p so as to allow for the conclusion that the regularity properties of u thereby gained are sufficient to warrant bounds for v in some spaces of Hölder continuous functions (Lemma 4.1). Along with an Ehrling-type interpolation lemma for such uniformly continuous functions (Lemma 10.1), this will enable us to trace functionals of the form in (1.21) for some arbitrarily large p and q, resulting in estimates for u and v with respect to norms in spatial L^p spaces for any finite $p > 1$ (Lemma 5.1). Straightforward bootstrap arguments relying on standard parabolic regularity theory (Lemma 5.2, Lemma 6.1, Lemma 6.2 and Lemma 6.3) will thereafter establish the claim of Theorem 1.1.

The proof of the stabilization result in Theorem 1.2 will be based on investigating functionals of the form

$$\int_\Omega (u - 1 - \ln u) + \frac{\alpha}{2}\int_\Omega v^2 + \beta \int_\Omega v,$$

which can be seen to constitute a *genuine* energy functional for (1.1) if the system coefficients satisfy the hypotheses of Theorem 1.2 and the free parameters $\alpha > 0$ and $\beta > 0$ are adjusted properly (Lemma 7.3, Lemma 7.4, Lemma 7.5 and Lemma 7.6).
Likewise, our derivation of the results on coexistence and outcompetition stated in Theorem 1.3 and Theorem 1.4 will rely on corresponding Lyapunov properties of

$$\int_\Omega \Big(u - u_\star - u_\star \ln \frac{u}{u_\star}\Big) + \alpha \int_\Omega \Big(v - v_\star - v_\star \ln \frac{v}{v_\star}\Big)$$

and of

$$\int_\Omega u + \beta \int_\Omega (v - 1 - \ln v)$$

for suitably chosen $\alpha > 0$ and $\beta > 0$, respectively (Section 8 and Section 9).

It is well conceivable that this basic strategy toward our asymptotic analysis of (1.1) can be adapted so as to appropriately cover also variants of (1.1) involving several species, such as those addressed in [3] and in [15], or also more general power law nonlinearities in the style of those considered in [30]. As a prerequitite forming an apparently necessary fundament of our approach, however, developing a theory of global solvability by bounded functions, in the flavor of e.g. the results from Theorem 1.1, seems to require substantial further efforts and thus goes beyond the scope of the present work.

2 Preliminaries

Let us recall the following local existence and extensibility result from the standard literature ([1]).

Lemma 2.1 *Let $n \ge 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, let d_1, d_2, a_{11} be positive and a_{12}, a_{22}, a_1, a_2, μ_1 and μ_2 be nonnegative, and assume that u_0 and v_0 are nonnegative functions from $W^{1,\infty}(\Omega)$. Then there exist $T_{max} \in (0,\infty]$ and a classical solution (u,v) of (1.1) in $\Omega \times (0,T_{max})$, for each $\vartheta > n$ uniquely determined by the requirement that*

$$(u,v) \in \Big(C^0(\bar\Omega \times [0,T_{max})) \cap C^{2,1}(\bar\Omega \times (0,T_{max})) \cap L^\infty_{loc}([0,T_{max}); W^{1,\vartheta}(\Omega))\Big)^2,$$

such that u and v are nonnegative in $\Omega \times (0, T_{max})$, and such that

$$\text{either } T_{max} = \infty, \text{ or}$$
$$\lim_{t \nearrow T_{max}} \sup \Big(\|u(\cdot,t)\|_{W^{1,\vartheta}(\Omega)} + \|v(\cdot,t)\|_{W^{1,\vartheta}(\Omega)} \Big) = \infty \quad \text{for all } \vartheta > n. \tag{2.1}$$

Throughout the sequel, unless otherwise stated we shall assume that Ω, u_0 and v_0 are as specified in Lemma 2.1, and let (u, v) denote the corresponding solution, defined up to its maximal existence time $T_{max} \le \infty$.

Some basic properties of any such solution are immediate.

Lemma 2.2 *The solution of (1.1) satisfies*

$$\frac{d}{dt} \int_\Omega u = \mu_1 \int_\Omega u - \mu_1 \int_\Omega u^2 - \mu_1 a_1 \int_\Omega uv \qquad \text{for all } t \in (0, T_{max}), \tag{2.2}$$

and in particular we have

$$\int_\Omega u(\cdot,t) \le m := \max\Big\{ \int_\Omega u_0, \, |\Omega| \Big\} \qquad \text{for all } t \in (0, T_{max}). \tag{2.3}$$

Moreover,

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le M := \max\{\|v_0\|_{L^\infty(\Omega)}, \, 1\} \qquad \text{for all } t \in (0, T_{max}). \tag{2.4}$$

Proof. From an integration of the first equation in (1.1) we immediately obtain (2.2). Since both u and v are nonnegative, and since $\int_\Omega u^2 \ge \frac{1}{|\Omega|}(\int_\Omega u)^2$ for all $t \in (0, T_{max})$ by the Cauchy-Schwarz inequality, this implies that

$$\begin{aligned} \frac{d}{dt} \int_\Omega u &\le \mu_1 \int_\Omega u - \mu_1 \int_\Omega u^2 \\ &\le \mu_1 \int_\Omega u - \frac{\mu_1}{|\Omega|} \Big(\int_\Omega u \Big)^2 \qquad \text{for all } t \in (0, T_{max}). \end{aligned}$$

On an ODE comparison, this entails (2.3), whereas (2.4) is a direct consequence of the maximum principle. $\square$

In order to prepare our further analysis, and especially a suitable testing technique respecting the structure of nonlinear diffusion in the second equation in (1.1), let us introduce the function $\phi : [0, \infty) \to \mathbb{R}$ defined by setting

$$\phi(s) := d_2 s + a_{22} s^2 \qquad \text{for } s \in \mathbb{R}, \tag{2.5}$$

and fix the following elementary features thereof for frequent reference in the remaining part of the paper.

Lemma 2.3 *The function ϕ from (2.5) satisfies*

$$\phi(v(x,t)) \le d_2 M + a_{22} M^2 \qquad \text{for all } x \in \Omega \text{ and } t \in (0, T_{max}) \tag{2.6}$$

and

$$d_2 \le \phi'(v(x,t)) \le d_2 + 2a_{22} M \qquad \text{for all } x \in \Omega \text{ and } t \in (0, T_{max}), \tag{2.7}$$

with $M > 0$ given by (2.4).

Proof. All three inequalities are evident consequences of (2.4) and the nonnegativity of v. $\square$

3 A spatio-temporal L^2 bound on $\nabla u^{\frac{p+1}{2}}$ for $p < \frac{3n}{(n-2)_+}$

The purpose of this section consists in deriving a space-time L^2 for ∇u^λ for suitable $\lambda > 2$, which will appear as part of the dissipation rate in a quasi-energy inequality related to a coupled functional that involves spatial L^p norms of u and a Dirichlet integral for $\phi(v)$ with ϕ as in (2.5).

As a first step toward this, let us perform two standard testing procedures to the first and second equations in (1.1) separately. The result of the former is straighforward:

Lemma 3.1 *Let $p \ge 2$. Then for all $t \in (0, T_{max})$, we have*

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + (p-1)a_{11}\int_\Omega u^{p-1}|\nabla u|^2 \le \frac{(p-1)a_{12}^2}{4a_{11}}\int_\Omega u^{p-1}|\nabla v|^2 + \mu_1\int_\Omega u^p. \tag{3.1}$$

Proof. We multiply the first equation in (1.1) by u^{p-1} and integrate by parts to find on dropping four nonpositive summands on the resulting right-hand side that

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p &= -(p-1)d_1\int_\Omega u^{p-2}|\nabla u|^2 - 2(p-1)a_{11}\int_\Omega u^{p-1}|\nabla u|^2 \\
&\quad -(p-1)a_{12}\int_\Omega u^{p-2}v|\nabla u|^2 - (p-1)a_{12}\int_\Omega u^{p-1}\nabla u\cdot\nabla v \\
&\quad +\mu_1\int_\Omega u^p - \mu_1\int_\Omega u^{p+1} - \mu_1 a_1\int_\Omega u^p v \\
&\le -2(p-1)a_{11}\int_\Omega u^{p-1}|\nabla u|^2 - (p-1)a_{12}\int_\Omega u^{p-1}\nabla u\cdot\nabla v + \mu_1\int_\Omega u^p
\end{aligned}$$

for all $t \in (0, T_{max})$. Since by Young's inequality we can estimate

$$-(p-1)a_{12}\int_\Omega u^{p-1}\nabla u\cdot\nabla v \le (p-1)a_{11}\int_\Omega u^{p-1}|\nabla u|^2 + \frac{(p-1)a_{12}^2}{4a_{11}}\int_\Omega u^{p-1}|\nabla v|^2$$

for all $t \in (0, T_{max})$, this implies (3.1). □

In adaptation to the nonlinear diffusion mechanism in the second equation in (1.1), our second testing procedure will involve, rather than v itself, the transformed variable $\phi(v)$ with ϕ taken from (2.5).

Lemma 3.2 *Let ϕ be as in (2.5). Then for all $t \in (0, T_{max})$, we have*

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |\nabla\phi(v)|^2 + \frac{d_2}{2}\int_\Omega |\Delta\phi(v)|^2 &\le 6\mu_2^2 a_2^2 M^2(d_2 + 2a_{22}M)\int_\Omega u^2 \\
&\quad +6\mu_2^2 M^2(1+M^2)(d_2+2a_{22}M)|\Omega|,
\end{aligned} \tag{3.2}$$

with $M > 0$ as determined by (2.4).

Proof. Letting $g(x,t) := \mu_2 v(x,t)(1 - v(x,t) - a_2 u(x,t))$ for $(x,t) \in \Omega \times (0, T_{max})$, we test the accordingly rewritten second equation in (1.1), that is, the identity

$$v_t = \Delta\phi(v) + g(x,t), \qquad x \in \Omega,\ t \in (0, T_{max}), \tag{3.3}$$

by $\partial_t \phi(v) \equiv \phi'(v) v_t$ to obtain

$$\int_\Omega \phi'(v) v_t^2 = -\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\phi(v)|^2 + \int_\Omega \phi'(v) v_t \cdot g \qquad \text{for all } t \in (0, T_{max}). \tag{3.4}$$

Here invoking Young's inequality we see that

$$\int_\Omega \phi'(v) v_t \cdot g \le \frac{1}{2}\int_\Omega \phi'(v) v_t^2 + \frac{1}{2}\int_\Omega \phi'(v) g^2 \qquad \text{for all } t \in (0, T_{max}),$$

and using the same token along with (3.3) we can estimate

$$\begin{aligned} \frac{1}{2}\int_\Omega \phi'(v) v_t^2 &= \frac{1}{2}\int_\Omega \phi'(v)|\Delta\phi(v)|^2 + \int_\Omega \phi'(v)\Delta\phi(v) g + \frac{1}{2}\int_\Omega \phi'(v) g^2 \\ &\ge \frac{1}{4}\int_\Omega \phi'(v)|\Delta\phi(v)|^2 - \frac{1}{2}\int_\Omega \phi'(v) g^2 \qquad \text{for all } t \in (0, T_{max}). \end{aligned}$$

Therefore, (3.4) implies that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\phi(v)|^2 + \frac{1}{4}\int_\Omega \phi'(v)|\Delta\phi(v)|^2 &= \frac{1}{4}\int_\Omega \phi'(v)|\Delta\phi(v)|^2 - \int_\Omega \phi'(v) v_t^2 + \int_\Omega \phi'(v) v_t \cdot g \\ &\le \left\{\frac{1}{2}\int_\Omega \phi'(v) v_t^2 + \frac{1}{2}\int_\Omega \phi'(v) g^2\right\} - \int_\Omega \phi'(v) v_t^2 \\ &\quad + \left\{\frac{1}{2}\int_\Omega \phi'(v) v_t^2 + \frac{1}{2}\int_\Omega \phi'(v) g^2\right\} \\ &= \int_\Omega \phi'(v) g^2 \qquad \text{for all } t \in (0, T_{max}). \end{aligned} \tag{3.5}$$

In view of (2.7), and since Young's inequality and (2.4) imply that

$$g^2 = \mu_2^2 v^2 (1 - v - a_2 u)^2 \le 3\mu_2^2 v^2 (1 + v^2 + a_2^2 u^2) \le 3\mu_2^2 a_2^2 M^2 u^2 + 3\mu_2^2 M^2 (1 + M^2)$$

in $\Omega \times (0, T_{max})$, from (3.5) we readily infer that (3.2) holds. $\square$

In order to suitably estimate the first term on the right of (3.1), we shall make use of the following basic interpolation property. As the elementary proof will show, the inequality (3.6) continues to hold also in the case $n \ge 4$ when the requirements that $W^{2,2}(\Omega) \hookrightarrow W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$, as usually made in the framework of standard Gagliardo-Nirenberg inequalities, are not met.

Lemma 3.3 *Suppose that $n \ge 1$, and that $\Omega \subset \mathbb{R}^n$ is s bounded domain with smooth boundary. Then there exists $C > 0$ such that for all $\varphi \in C^2(\bar\Omega)$ fulfilling $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$ we have*

$$\|\nabla\varphi\|_{L^4(\Omega)}^4 \le C\|\Delta\varphi\|_{L^2(\Omega)}^2 \|\varphi\|_{L^\infty(\Omega)}^2. \tag{3.6}$$

Proof. We integrate by parts to see that

$$\int_\Omega |\nabla\varphi|^4 = \int_\Omega |\nabla\varphi|^2 \nabla\varphi \cdot \nabla\varphi = -\int_\Omega \varphi|\nabla\varphi|^2\Delta\varphi - \int_\Omega \varphi\nabla\varphi \cdot \nabla|\nabla\varphi|^2, \tag{3.7}$$

where by the Cauchy-Schwarz inequality,

$$-\int_\Omega \varphi|\nabla\varphi|^2\Delta\varphi \le \|\varphi\|_{L^\infty(\Omega)}\|\nabla\varphi\|^2_{L^4(\Omega)}\|\Delta\varphi\|_{L^2(\Omega)}, \tag{3.8}$$

and where since $\nabla|\nabla\varphi|^2 = 2D^2\varphi\cdot\nabla\varphi$ we can similarly estimate

$$-\int_\Omega \varphi\nabla\varphi\cdot\nabla|\nabla\varphi|^2 \le 2\|\varphi\|_{L^\infty(\Omega)}\|\nabla\varphi\|^2_{L^4(\Omega)}\|D^2\varphi\|_{L^2(\Omega)}. \tag{3.9}$$

As herein standard elliptic regularity ([9]) provides $c_1 > 0$ fulfilling

$$\|D^2\varphi\|_{L^2(\Omega)} \le c_1\|\Delta\varphi\|_{L^2(\Omega)},$$

from (3.7)-(3.9) it follows that

$$\|\nabla\varphi\|^2_{L^4(\Omega)} \le (1+2c_1)\|\varphi\|_{L^\infty(\Omega)}\|\Delta\varphi\|_{L^2(\Omega)},$$

which implies (3.6) with $C := (1+2c_1)^2$. □

Now the key to our first regularity statement on u beyond (2.3) will consist in the following implication, to be achieved through the analysis of an energy-like functional on the basis of Lemma 3.1 and Lemma 3.2, which will form the core of an recursive argument in Lemma 3.5.

Lemma 3.4 *Suppose that $p_0 \ge 1$ is such that $(n-4)p_0 \le 4n$ and*

$$\sup_{t\in(0,T_{max})} \|u(\cdot,t)\|_{L^{p_0}(\Omega)} < \infty. \tag{3.10}$$

Then there exists $C > 0$ such that with $p := \frac{2p_0}{n} + 3$ and $\tau := \min\{1, \frac{1}{2}T_{max}\}$ we have

$$\int_t^{t+\tau}\int_\Omega u^{p-1}|\nabla u|^2 \le C \qquad \textit{for all } t\in(0,T_{max}-\tau). \tag{3.11}$$

Proof. In order to prepare the construction of an entropy-like functional, according to (3.10) let us fix $c_1 > 0$ such that

$$\int_\Omega u^{p_0}(\cdot,t) \le c_1 \qquad \text{for all } t\in(0,T_{max}) \tag{3.12}$$

and apply the Gagliardo-Nirenberg inequality to find $c_2 > 0$ fulfilling

$$\|\varphi\|^{\frac{4p_0+4n}{p_0+2n}}_{L^{\frac{4p_0+4n}{p_0+2n}}(\Omega)} \le c_2\|\nabla\varphi\|^2_{L^2(\Omega)}\|\varphi\|^{\frac{2p_0}{p_0+2n}}_{L^{\frac{np_0}{p_0+2n}}(\Omega)} + c_2\|\varphi\|^{\frac{4p_0+4n}{p_0+2n}}_{L^{\frac{np_0}{p_0+2n}}(\Omega)} \qquad \text{for all } \varphi\in W^{1,2}(\Omega), \tag{3.13}$$

where we note that our assumption $(n-4)p_0 \le 4n$ warrants that

$$\frac{np_0}{p_0+2n} \le \frac{4p_0+4n}{p_0+2n} \le \frac{2n}{(n-2)_+}$$

and that hence $W^{1,2}(\Omega) \hookrightarrow L^{\frac{4p_0+4n}{p_0+2n}}(\Omega) \subset L^{\frac{np_0}{p_0+2n}}(\Omega)$.
Moreover, we abbreviate $c_3 := \frac{(p-1)a_{12}^2}{a_{11}}$ as well as

$$c_4 := d_2 M + a_{22} M^2 \tag{3.14}$$

with M as in (2.4), and invoke Lemma 3.3 to pick $c_5 > 0$ such that

$$\|\nabla\varphi\|_{L^4(\Omega)}^4 \le c_5 \|\Delta\varphi\|_{L^2(\Omega)}^2 \|\varphi\|_{L^\infty(\Omega)}^2 \qquad \text{for all } \varphi \in C^2(\bar\Omega) \text{ such that } \tfrac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega. \tag{3.15}$$

We then choose $\eta > 0$ small enough fulfilling

$$\frac{(p+1)^2}{2} c_1^{\frac{2}{n}} c_2 \eta \le \frac{(p-1)a_{11}}{2} \tag{3.16}$$

and thereafter fix $b > 0$ suitably large such that

$$\frac{c_3^2 c_4^2 c_5}{4 d_2^4 \eta} \le \frac{b d_2}{4}. \tag{3.17}$$

Upon these specifications, with ϕ as in (2.5) we introduce

$$y(t) := \frac{1}{p} \int_\Omega u^p(\cdot,t) + b \int_\Omega |\nabla\phi(v(\cdot,t))|^2, \qquad t \in [0, T_{max}), \tag{3.18}$$

and to arrange the derivation of an appropriate autonomous ordinary differential inequality for y, we first go back to Lemma 3.1 to see that by definition of c_3 we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p + (p-1)a_{11}\int_\Omega u^{p-1}|\nabla u|^2 \le c_3 \int_\Omega u^{p-1}|\nabla v|^2 + (\mu_1+1)\int_\Omega u^p \qquad \text{for all } t \in (0, T_{max}). \tag{3.19}$$

Here with η as fixed above, by Young's inequality we can estimate

$$c_3 \int_\Omega u^{p-1}|\nabla v|^2 \le \eta \int_\Omega u^{2p-2} + \frac{c_3^2}{4\eta}\int_\Omega |\nabla v|^4 \qquad \text{for all } t \in (0, T_{max}), \tag{3.20}$$

where recalling (2.7) and (2.6), by means of (3.15), (3.14) and (3.17) we see that

$$\begin{aligned}
\frac{c_3^2}{4\eta}\int_\Omega |\nabla v|^4 &= \frac{c_3^2}{4\eta}\int_\Omega \frac{1}{\phi'^4(v)}|\nabla\phi(v)|^4 \\
&\le \frac{c_3^2}{4d_2^4\eta}\int_\Omega |\nabla\phi(v)|^4 \\
&\le \frac{c_3^2 c_5}{4d_2^4\eta}\|\Delta\phi(v)\|_{L^2(\Omega)}^2 \|\phi(v)\|_{L^\infty(\Omega)}^2 \\
&\le \frac{c_3^2 c_4^2 c_5}{4d_2^4\eta}\int_\Omega |\Delta\phi(v)|^2 \\
&\le \frac{b d_2}{4}\int_\Omega |\Delta\phi(v)|^2 \qquad \text{for all } t \in (0, T_{max}).
\end{aligned} \tag{3.21}$$

As Lemma 3.2 yields $c_6 > 0$ such that

$$\frac{d}{dt}\int_\Omega |\nabla\phi(v)|^2 + \frac{d_2}{2}\int_\Omega |\Delta\phi(v)|^2 \le c_6\int_\Omega u^2 + c_6 \qquad \text{for all } t\in(0,T_{max}),$$

from (3.19)-(3.21) we thus infer that

$$\begin{aligned} \frac{d}{dt}\left\{\frac{1}{p}\int_\Omega u^p + b\int_\Omega |\nabla\phi(v)|^2\right\} + \int_\Omega u^p + (p-1)a_{11}\int_\Omega u^{p-1}|\nabla u|^2 + \frac{bd_2}{4}\int_\Omega |\Delta\phi(v)|^2 \\ \le \ (\mu_1+1)\int_\Omega u^p + \eta\int_\Omega u^{2p-2} + bc_6\int_\Omega u^2 + bc_6 \qquad \text{for all } t\in(0,T_{max}). \end{aligned} \tag{3.22}$$

Here we may use that $p > 2$ and hence $2p-2 > p$ as well as $2p-2 > 2$ in employing Young's inequality again to find $c_7 > 0$ such that

$$(\mu_1+1)\int_\Omega u^p + \eta\int_\Omega u^{2p-2} + bc_6\int_\Omega u^2 \le 2\eta\int_\Omega u^{2p-2} + c_7 \qquad \text{for all } t\in(0,T_{max}), \tag{3.23}$$

where an application of (3.13) to $\varphi := u^{\frac{p+1}{2}}$ shows that according to our definition of p we have

$$\begin{aligned} 2\eta\int_\Omega u^{2p-2} &= 2\eta\|u^{\frac{p+1}{2}}\|_{L^{\frac{4p-4}{p+1}}(\Omega)}^{\frac{4p-4}{p+1}} \\ &= 2\eta\|u^{\frac{p+1}{2}}\|_{L^{\frac{4p_0+4n}{p_0+2n}}(\Omega)}^{\frac{4p_0+4n}{p_0+2n}} \\ &\le 2c_2\eta\|\nabla u^{\frac{p+1}{2}}\|_{L^2(\Omega)}^2\|u^{\frac{p+1}{2}}\|_{L^{\frac{np_0}{p_0+2n}}(\Omega)}^{\frac{2p_0}{p_0+2n}} + 2c_2\eta\|u^{\frac{p+1}{2}}\|_{L^{\frac{np_0}{p_0+2n}}(\Omega)}^{\frac{4p_0+4n}{p_0+2n}} \end{aligned} \tag{3.24}$$

for all $t\in(0,T_{max})$. Since $\frac{p+1}{2} = \frac{p_0+2n}{n}$ and hence

$$\|u^{\frac{p+1}{2}}\|_{L^{\frac{np_0}{p_0+2n}}(\Omega)}^{\frac{np_0}{p_0+2n}} = \int_\Omega u^{p_0} \le c_1 \qquad \text{for all } t\in(0,T_{max}),$$

by (3.16) this entails that

$$\begin{aligned} 2\eta\int_\Omega u^{2p-2} &\le 2c_1^{\frac{2}{n}}c_2\eta\|\nabla u^{\frac{p+1}{2}}\|_{L^2(\Omega)}^2 + 2c_1^{\frac{4p_0+4n}{np_0}}c_2\eta \\ &= \frac{(p+1)^2}{2}c_1^{\frac{2}{n}}c_2\eta\int_\Omega u^{p-1}|\nabla u|^2 + 2c_1^{\frac{4p_0+4n}{np_0}}c_2\eta \\ &\le \frac{(p-1)a_{11}}{2}\int_\Omega u^{p-1}|\nabla u|^2 + c_8 \qquad \text{for all } t\in(0,T_{max}) \end{aligned} \tag{3.25}$$

with $c_8 := 2c_1^{\frac{4p_0+4n}{np_0}}c_2\eta$.

Finally, invoking the Poincaré inequality to fix $c_9 > 0$ such that

$$\int_\Omega |\nabla\varphi|^2 \le c_9\int_\Omega |\Delta\varphi|^2 \qquad \text{for all } \varphi\in W^{2,2}(\Omega) \text{ fulfilling } \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega,$$

combining (3.23), (3.24) and (3.25) we conclude from (3.22) that

$$\frac{d}{dt}\left\{\frac{1}{p}\int_\Omega u^p + b\int_\Omega |\nabla\phi(v)|^2\right\} + \int_\Omega u^p + \frac{bd_2}{4c_9}\int_\Omega |\nabla\phi(v)|^2 + \frac{(p-1)a_{11}}{2}\int_\Omega u^{p-1}|\nabla u|^2$$
$$\le c_{10} := bc_6 + c_7 + c_8 \qquad \text{for all } t\in(0,T_{max}),$$

and that hence with y as in (3.18), $h(t) := \frac{(p-1)a_{11}}{2}\int_\Omega u^{p-1}(\cdot,t)|\nabla u(\cdot,t)|^2$, $t\in(0,T_{max})$, and $c_{11} := \min\{p, \frac{d_2}{4c_9}\}$ we have

$$y'(t) + c_{11}y(t) + h(t) \le c_{10} \qquad \text{for all } t\in(0,T_{max}). \tag{3.26}$$

By nonnegativity of h, on an ODE comparison this firstly implies that

$$y(t) \le c_{12} := \max\left\{\frac{1}{p}\int_\Omega u_0^p + b\int_\Omega |\nabla\phi(v_0))|^2\,,\,\frac{c_{10}}{c_{11}}\right\} \qquad \text{for all } t\in(0,T_{max}),$$

whereafter a direct integration of (3.26) in time shows that

$$y(t+\tau) + \int_t^{t+\tau} h(s)ds \le y(t) + c_{10}\tau \le c_{12} + c_{10} \qquad \text{for all } t\in(0,T_{max}-\tau),$$

because $\tau\le 1$. As also y is nonnegative, this entails (3.11). □

A straightforward induction thereupon implies validity of (3.11) actually for any $p < \frac{3n}{(n-2)_+}$.

Lemma 3.5 *For any choice of $p>2$ satisfying $p<\frac{3n}{(n-2)_+}$, there exists $C=C(p)>0$ such that*

$$\int_t^{t+\tau}\int_\Omega u^{p-1}|\nabla u|^2 \le C \qquad \textit{for all } t\in(0,T_{max}-\tau) \tag{3.27}$$

with $\tau=\min\{1,\frac{1}{2}T_{max}\}$.

Proof. We recursively define $(p_k)_{k\in\{0,1,2,...\}}\subset[1,\infty)$ by letting $p_0 := 1$ and $p_k := \frac{2p_{k-1}}{n}+3$ for $k\ge 1$. Then it can easily be verified that $p_k \nearrow \frac{3n}{(n-2)_+}$ as $k\to\infty$, so that since successively applying Lemma 3.4 yields

$$\sup_{t\in(0,T_{max}-\tau)}\int_t^{t+\tau}\int_\Omega u^{p_k-1}|\nabla u|^2 < \infty \qquad \text{for all } k\in\{1,2,3,...\},$$

it follows that for each p in the indicated range the inequality (3.27) holds with suitably large $C=C(p)>0$. □

4 A Hölder estimate for v

Now in the case $n\le 9$, Lemma 3.5 together with the Sobolev embedding theorem and (2.3) implies spatio-temporal integrability propeties of u itself which by means of standard regularity theory for quasilinear parabolic equations can be seen to ensure bounds for v in appropriate Hölder spaces.

Lemma 4.1 *Let $n \le 9$. Then there exist $\theta \in (0,1)$ and $C > 0$ such that*

$$\|v\|_{C^{\theta,\frac{\theta}{2}}(\bar\Omega\times[t,t+\tau])} \le C \qquad \textit{for all } t \in (0, T_{max} - \tau), \tag{4.1}$$

where again $\tau = \min\{1, \frac{1}{2}T_{max}\}$.

Proof. We divide the argument into two steps.

Step 1. Let us first make sure that there exist $q > 1, r > 1$ and $c_1 > 0$ such that

$$\frac{1}{r} + \frac{n}{2q} < 1, \tag{4.2}$$

and that

$$\int_t^{t+\tau} \|u(\cdot,s)\|^r_{L^q(\Omega)} ds \le c_1 \qquad \text{for all } t \in (0, T_{max} - \tau). \tag{4.3}$$

To see this, we note that our hypothesis $n \le 9$ warrants that $(n-2)^2 < 6n$ and hence $\frac{(n-2)_+}{3} < \frac{2n}{(n-2)_+}$, whence it is possible to fix $\kappa \ge 1$ such that

$$\frac{(n-2)_+}{3} < \kappa < \frac{2n}{(n-2)_+}. \tag{4.4}$$

Here the left inequality guarantees that $\frac{n}{\kappa} < \frac{3n}{(n-2)_+}$, so that we can find $p > 2$ fulfilling

$$\frac{n}{\kappa} < p < \frac{3n}{(n-2)_+}, \tag{4.5}$$

and thereupon define

$$q := \frac{(p+1)\kappa}{2} \qquad \text{and} \qquad r := p + 1, \tag{4.6}$$

noting that then clearly $q > 1$ and $r > 1$, and that thanks to the left inequality in (4.5) we have

$$\frac{1}{r} + \frac{n}{2q} = \frac{1}{p+1}\Big(1 + \frac{n}{\kappa}\Big) < 1,$$

as required in (4.2).
Now according to the second inequality in (4.5), Lemma 3.5 applies so as to yield $c_2 > 0$ such that

$$\int_t^{t+\tau}\int_\Omega |\nabla u^{\frac{p+1}{2}}|^2 \le c_2 \qquad \text{for all } t \in (0, T_{max} - \tau). \tag{4.7}$$

As the second restriction expressed in (4.4) ensures that $W^{1,2}(\Omega) \hookrightarrow L^\kappa(\Omega)$, in view of (2.3) we infer from (4.7) te existence of $c_3 > 0$ such that

$$\int_t^{t+\tau} \|u^{\frac{p+1}{2}}(\cdot,s)\|^2_{L^\kappa(\Omega)} ds \le c_3 \qquad \text{for all } t \in (0, T_{max} - \tau),$$

that is,

$$\int_t^{t+\tau} \|u(\cdot,s)\|^{p+1}_{L^{\frac{(p+1)\kappa}{2}}(\Omega)} ds \le c_3 \qquad \text{for all } t \in (0, T_{max} - \tau),$$

which due to (4.6) is equivalent to (4.3) with $c_1 = c_3$.

Step 2. We proceed to verify that the conclusion of the lemma holds.
For this purpose, we write the second equation in (1.1) in the form

$$v_t = \nabla \cdot A(x,t,\nabla v) + B(x,t), \qquad x \in \Omega,\ t \in (0, T_{max}),$$

where

$$A(x,t,\xi) := \Big(d_2 + 2a_{22} v(x,t)\Big)\xi, \qquad (x,t,\xi) \in \Omega \times (0,T_{max}) \times \mathbb{R}^n,$$

and

$$B(x,t) := \mu_2 v(x,t)\Big(1 - v(x,t) - a_2 u(x,t)\Big), \qquad (x,t) \in \Omega \times (0, T_{max}),$$

define measurable functions satisfying

$$A(x,t,\xi)\cdot\xi = \Big(d_2 + 2a_{22}v(x,t)\Big)|\xi|^2 \ge d_2|\xi|^2 \qquad \text{for all } (x,t,\xi) \in \Omega \times (0,T_{max}) \times \mathbb{R}^n$$

and

$$|A(x,t,\xi)| \le (d_2 + 2a_{22}M)|\xi| \qquad \text{for all } (x,t,\xi) \in \Omega \times (0,T_{max}) \times \mathbb{R}^n$$

as well as

$$|B(x,t)| \le \mu_2 M(1+M) + \mu_2 a_2 M u(x,t) \qquad \text{for all } (x,t) \in \Omega \times (0,T_{max})$$

according to (2.4). Since (4.3) therefore guarantees that

$$\sup_{t\in(0,T_{max}-\tau)} \int_t^{t+\tau} \|B(\cdot,s)\|^r_{L^q(\Omega)} ds < \infty,$$

and since v is bounded in $\Omega \times (0, T_{max})$ and v_0 is Hölder continuous in $\bar{\Omega}$ by assumption, in view of (4.2) a standard result on Hölder regularity in quasilinear parabolic equations ([24, Theorem 1.3, Remark 1.4]) becomes applicable so as to yield $\theta \in (0,1)$ and $C > 0$ such that (4.1) holds. □

5 Boundedness of u

The crucial progress provided by Lemma 4.1 consists in the circumstance that it facilitates access of our analysis to an interpolation inequality addressing integrals for derivatives of functions with bounded modulus of continuity, as observed in [22, Lemma 5.1] for a special case and generalized in Lemma 10.1 in the appendix. In the following lemma, this will enable us to suitably estimate certain lower-order integrals obtained when tracking the time evolution of suitably weighted L^{2p} norms of ∇v also in the case when unlike in Lemma 3.2 and Lemma 3.4, the parameter p herein is chosen to be larger than 1. We remark that through the analysis of the functional from (5.3), at this stage we make essential use of our overall assumption that d_2 be positive (cf. e.g. the definition of c_1 in the proof below).

Lemma 5.1 *Suppose that $n \le 9$, and assume that Ω is convex. Then for all $p \ge 2$ there exists $C = C(p) > 0$ such that*

$$\int_\Omega u^p(\cdot,t) \le C \qquad \textit{for all } t \in (0,T_{max}) \tag{5.1}$$

and

$$\int_\Omega |\nabla v(\cdot,t)|^{2p} \le C \qquad \textit{for all } t \in (0,T_{max}). \tag{5.2}$$

PROOF. Without loss of generality we may assume that $p \ge \frac{n}{2} - 1$. We then abbreviate $z := \phi(v)$ with ϕ as given by (2.5), and prepare an analysis of

$$y(t) := \frac{1}{p}\int_\Omega u^p(\cdot,t) + \frac{1}{2p}\int_\Omega |\nabla z(\cdot,t)|^{2p}, \qquad t \in (0,T_{max}), \tag{5.3}$$

by writing the second equation in (1.1) in the form

$$z_t = \phi'(v)v_t = \phi'(v)\Delta z + g, \qquad x \in \Omega,\ t \in (0,T_{max}),$$

with

$$g(x,t) := \mu_2\phi'(v(x,t))v(x,t)\Big(1 - v(x,t) - a_2u(x,t)\Big), \qquad (x,t) \in \Omega \times (0,T_{max}). \tag{5.4}$$

By straightforward computation using the identity $\nabla z \cdot \nabla\Delta z = \frac{1}{2}\Delta|\nabla z|^2 - |D^2z|^2$, from this we obtain that

$$\begin{aligned}
\frac{1}{2p}\frac{d}{dt}\int_\Omega |\nabla z|^{2p} &= \int_\Omega |\nabla z|^{2p-2}\nabla z \cdot \nabla\Big\{\phi'(v)\Delta z + g\Big\} \\
&= \int_\Omega \phi'(v)|\nabla z|^{2p-2}\nabla z \cdot \nabla\Delta z + \int_\Omega \phi''(v)|\nabla z|^{2p-2}(\nabla z \cdot \nabla v)\Delta z \\
&\quad + \int_\Omega |\nabla z|^{2p-2}\nabla z \cdot \nabla g \\
&= \frac{1}{2}\int_\Omega \phi'(v)|\nabla z|^{2p-2}\Delta|\nabla z|^2 - \int_\Omega \phi'(v)|\nabla z|^{2p-2}|D^2z|^2 \\
&\quad + \int_\Omega \frac{\phi''(v)}{\phi'(v)}|\nabla z|^{2p}\Delta z + \int_\Omega |\nabla z|^{2p-2}\nabla z \cdot \nabla g \qquad \text{for all } t \in (0,T_{max}), \quad (5.5)
\end{aligned}$$

where since $\frac{\partial|\nabla z|^2}{\partial\nu} \le 0$ on $\partial\Omega$ by convexity of Ω and the fact that $\frac{\partial z}{\partial\nu} = 0$ on $\partial\Omega$ ([16]), integrating by parts and invoking Young's inequality shows that for all $t \in (0,T_{max})$ we have

$$\begin{aligned}
\frac{1}{2}\int_\Omega \phi'(v)|\nabla z|^{2p-2}\Delta|\nabla z|^2 &= -\frac{1}{2}\int_\Omega \nabla\Big\{\phi'(v)|\nabla z|^{2p-2}\Big\} \cdot \nabla|\nabla z|^2 + \frac{1}{2}\int_{\partial\Omega} \phi'(v)|\nabla z|^{2p-2}\frac{\partial|\nabla z|^2}{\partial\nu} \\
&\le -\frac{p-1}{2}\int_\Omega \phi'(v)|\nabla z|^{2p-4}\Big|\nabla|\nabla z|^2\Big|^2 - \frac{1}{2}\int_\Omega \frac{\phi''(v)}{\phi'(v)}|\nabla z|^{2p-2}\nabla z \cdot \nabla|\nabla z|^2 \\
&\le -\frac{p-1}{4}\int_\Omega \phi'(v)|\nabla z|^{2p-4}\Big|\nabla|\nabla z|^2\Big|^2 + \frac{1}{4(p-1)}\int_\Omega \frac{\phi''^2(v)}{\phi'^3(v)}|\nabla z|^{2p+2} \\
&\le -\frac{p-1}{4}\int_\Omega \phi'(v)|\nabla z|^{2p-4}\Big|\nabla|\nabla z|^2\Big|^2 + c_1\int_\Omega |\nabla z|^{2p+2} \qquad (5.6)
\end{aligned}$$

with $c_1 := \frac{a_{22}^2}{(p-1)d_2^3}$, where we have used (2.7) and the fact that $\phi'' \equiv 2a_{22}$.
Similarly, the second last term in (5.5) can be estimated against the third last by using the pointwise inequality $|\Delta z| \le \sqrt{n}|D^2 z|$ according to

$$\begin{aligned}\left|\int_\Omega \frac{\phi''(v)}{\phi'(v)}|\nabla z|^{2p}\Delta z\right| &\le \sqrt{n}\int_\Omega \frac{\phi''(v)}{\phi'(v)}|\nabla z|^{2p}|D^2 z| \\ &\le \frac{1}{2}\int_\Omega \phi'(v)|\nabla z|^{2p-2}|D^2 z|^2 + \frac{n}{2}\int_\Omega \frac{\phi''^2(v)}{\phi'^3(v)}|\nabla z|^{2p+2} \\ &\le \frac{1}{2}\int_\Omega \phi'(v)|\nabla z|^{2p-2}|D^2 z|^2 + c_2\int_\Omega |\nabla z|^{2p+2} \qquad \text{for all } t\in(0,T_{max}) \end{aligned} \tag{5.7}$$

with $c_2 := \frac{2na_{22}^2}{d_2^3}$.
Finally, observing that by (5.4) and (2.4) we have

$$|g(x,t)| \le c_3 \cdot \Big(u(x,t)+1\Big) \qquad \text{for all } x\in\Omega \text{ and } t\in(0,T_{max})$$

with $c_3 := \max\{\mu_2(d_2+2a_{22}M)M(1+M)\,,\, \mu_2 a_2(d_2+2a_{22}M)M\}$, upon one more integration by parts we see that due to (2.7), in the rightmost integral in (5.5) we have

$$\begin{aligned}\int_\Omega |\nabla z|^{2p-2}\nabla z\cdot\nabla g &= -\int_\Omega g|\nabla z|^{2p-2}\Delta z - (p-1)\int_\Omega g|\nabla z|^{2p-4}\nabla z\cdot\nabla|\nabla z|^2 \\ &\le \frac{1}{4}\int_\Omega \phi'(v)|\nabla z|^{2p-2}|D^2z|^2 + \frac{nc_3^2}{d_2}\int_\Omega (u+1)^2|\nabla z|^{2p-2} \\ &\quad +\frac{p-1}{4}\int_\Omega \phi'(v)|\nabla z|^{2p-4}\Big|\nabla|\nabla z|^2\Big|^2 + \frac{(p-1)c_3^2}{d_2}\int_\Omega (u+1)^2|\nabla z|^{2p-2} \\ &\le \frac{1}{4}\int_\Omega \phi'(v)|\nabla z|^{2p-2}|D^2z|^2 + \frac{p-1}{4}\int_\Omega \phi'(v)|\nabla z|^{2p-4}\Big|\nabla|\nabla z|^2\Big|^2 \\ &\quad +\frac{(n+p-1)c_3^2}{d_2}\int_\Omega (u+1)^{p+1} + \frac{(n+p-1)c_3^2}{d_2}\int_\Omega |\nabla z|^{2p+2} \\ &\le \frac{1}{4}\int_\Omega \phi'(v)|\nabla z|^{2p-2}|D^2z|^2 + \frac{p-1}{4}\int_\Omega \phi'(v)|\nabla z|^{2p-4}\Big|\nabla|\nabla z|^2\Big|^2 \\ &\quad +c_4\int_\Omega u^{p+1} + c_4\int_\Omega|\nabla z|^{2p+2} + c_4 \qquad \text{for all } t\in(0,T_{max}), \end{aligned} \tag{5.8}$$

where $c_4 := 2^{p+1}\cdot\frac{(n+p-1)c_3^2}{d_2}\cdot\max\{1,|\Omega|\}$. As clearly

$$\frac{1}{2}\int_\Omega |\nabla z|^{2p} \le \frac{1}{2}\int_\Omega |\nabla z|^{2p+2} + \frac{|\Omega|}{2} \qquad \text{for all } t\in(0,T_{max}),$$

combining (5.5)-(5.8) shows that writing $c_5 := \max\{c_4\,,\, c_1+c_2+c_4+\frac{1}{2}\,,\, c_4+\frac{|\Omega|}{2}\}$ we have

$$\frac{1}{2p}\frac{d}{dt}\int_\Omega |\nabla z|^{2p} + \frac{1}{2}\int_\Omega |\nabla z|^{2p} + \frac{d_2}{4}\int_\Omega |\nabla z|^{2p-2}|D^2z|^2 \le c_5\int_\Omega u^{p+1} + c_5\int_\Omega |\nabla z|^{2p+2} + c_5 \tag{5.9}$$

for all $t \in (0, T_{max})$, again because $\phi'(v) \geq d_2$ by (2.7).
We now recall Lemma 3.1 which implies that if we let $c_6 := \frac{4(p-1)a_{11}}{(p+1)^2}$ and $c_7 := \max\{\frac{(p-1)a_{12}^2}{4a_{11}}, \mu_1 + 1\}$, then

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p + c_6 \int_\Omega |\nabla u^{\frac{p+1}{2}}|^2 \leq c_7 \int_\Omega u^{p-1}|\nabla v|^2 + c_7 \int_\Omega u^p \qquad \text{for all } t \in (0, T_{max}), \tag{5.10}$$

where once more using Young's inequality and (2.7) we find that

$$c_7 \int_\Omega u^p \leq c_7 \int_\Omega u^{p+1} + c_7|\Omega| \qquad \text{for all } t \in (0, T_{max}) \tag{5.11}$$

and

$$\begin{aligned} c_7 \int_\Omega u^{p-1}|\nabla v|^2 &\leq c_7 \int_\Omega u^{p+1} + c_7 \int_\Omega |\nabla v|^{p+1} \\ &\leq c_7 \int_\Omega u^{p+1} + c_7 \int_\Omega |\nabla v|^{2p+2} + c_7|\Omega| \\ &\leq c_7 \int_\Omega u^{p+1} + \frac{c_7}{d_2^{2p+2}} \int_\Omega |\nabla z|^{2p+2} + c_7|\Omega| \qquad \text{for all } t \in (0, T_{max}). \end{aligned} \tag{5.12}$$

In summary, from (5.9)-(5.12) we infer that for all $t \in (0, T_{max})$, the function y defined in (5.3) satisfies

$$y'(t) + py(t) + \frac{d_2}{4}\int_\Omega |\nabla z|^{2p-2}|D^2 z|^2 + c_6 \int_\Omega |\nabla u^{\frac{p+1}{2}}|^2 \leq c_8 \int_\Omega u^{p+1} + c_8 \int_\Omega |\nabla z|^{2p+2} + c_8 \tag{5.13}$$

with $c_8 := \max\{c_5 + 2c_7\,, c_5 + \frac{c_7}{d_2^{2p+2}}\,, c_5 + 2c_7|\Omega|\}$. Here we use an Ehrling-type lemma along with (2.3) to find $c_9 > 0$ such that

$$\begin{aligned} c_8 \int_\Omega u^{p+1} = c_8 \|u^{\frac{p+1}{2}}\|^2_{L^2(\Omega)} &\leq c_6 \|\nabla u^{\frac{p+1}{2}}\|^2_{L^2(\Omega)} + c_9 \|u^{\frac{p+1}{2}}\|^2_{L^{\frac{2}{p+1}}(\Omega)} \\ &\leq c_6 \int_\Omega |\nabla u^{\frac{p+1}{2}}|^2 + c_9 m^{p+1} \qquad \text{for all } t \in (0, T_{max}), \end{aligned} \tag{5.14}$$

whereas the Hölder bound for v derived in Lemma 4.1, clearly implying equicontinuity of $(z(\cdot,t))_{t\in(0,T_{max})}$ according to the definition of ϕ and the boundedness of v, allows for an application of the interpolation inequality stated in Lemma 10.1 below to conclude that with some $c_{10} > 0$ we have

$$\begin{aligned} c_8 \int_\Omega |\nabla z|^{2p+2} &\leq \frac{d_2}{4}\int_\Omega |\nabla z|^{2p-2}|D^2 z|^2 + c_{10}\|z\|^{2p+2}_{L^\infty(\Omega)} \\ &\leq \frac{d_2}{4}\int_\Omega |\nabla z|^{2p-2}|D^2 z|^2 + c_{10}(d_2 M + a_{22}M^2)^{2p+2} \qquad \text{for all } t \in (0, T_{max}), \end{aligned} \tag{5.15}$$

because $p \geq \frac{n}{2} - 1$, and again because of (2.4) and (2.6).
In light of (5.14) and (5.15), we thus see that (5.13) implies the inequality

$$y'(t) + py(t) \leq c_{11} := c_8 + c_9 m^{p+1} + c_{10}(d_2 M + a_{22}M^2)^{2p+2} \qquad \text{for all } t \in (0, T_{max}),$$

from which on an ODE comparison we obtain that

$$y(t) \le \max\left\{\frac{1}{p}\int_\Omega u_0^p + \frac{1}{2p}\int_\Omega |\nabla\phi(v_0)|^{2p}\,,\ \frac{c_{11}}{p}\right\} \qquad \text{for all } t\in(0,T_{max})$$

and hence infer that (5.1) and (5.2) hold. □

By means of a standard argument, the latter implies a bound for u in L^∞.

Lemma 5.2 *Suppose that $n \le 9$ and that Ω is convex. Then there exists $C>0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t\in(0,T_{max}).$$

Proof. Since the exponent p in Lemma 5.1 can be chosen arbitrarily large, this results from a straightforward application of a Moser-type iteration procedure (cf. [29, Lemma A.1] for a version precisely covering the present situation). □

6 Estimates for solutions and derivatives in Hölder spaces. Proof of Theorem 1.1

In this section we shall derive further estimates in spaces of Hölder continuous functions. Since we plan to apply these both to complete our proof of global existence as well as to prepare a subsequent compactness argument for the verification of the convergence statements in Theorem 1.2-Theorem 1.4, we formulate our results here in a way slightly more general for each of the latter purposes.

We begin with a second application of parabolic Hölder estimates to see that bounds in the flavor of (1.2) imply Hölder estimates for u and v in the following sense.

Lemma 6.1 *Let $n\ge 1$, $T\in(0,\infty]$ and $(u,v)\in(C^{2,1}(\bar\Omega\times(0,T)))^2$ be a classical solution of the boundary value problem in (1.1) in $\Omega\times(0,T)$ for which $u\ge 0$ and $v\ge 0$, and for which there exist $p>n$ and $K>0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,p}(\Omega)} \le K \qquad \textit{for all } t\in(0,T). \tag{6.1}$$

Then for each $\delta\in(0,\tau)$ there exist $\theta=\theta(\delta)\in(0,1)$ and $C=C(\delta)>0$ with the property that

$$\|u\|_{C^{\theta,\frac{\theta}{2}}(\bar\Omega\times[t,t+\tau])} \le C \qquad \textit{for all } t\in(\delta,T-\tau) \tag{6.2}$$

and

$$\|v\|_{C^{\theta,\frac{\theta}{2}}(\bar\Omega\times[t,t+\tau])} \le C \qquad \textit{for all } t\in(\delta,T-\tau), \tag{6.3}$$

where $\tau:=\min\{1,\frac{1}{2}T\}$.

Proof. We rewrite the first equation in (1.1) according to

$$u_t = \nabla\cdot A(x,t,\nabla u) + B(x,t), \qquad x\in\Omega,\ t\in(0,T),$$

with

$$A(x,t,\xi) := d_1\xi + 2a_{11}u(x,t)\xi + a_{12}v(x,t)\xi + a_{12}u(x,t)\nabla v(x,t), \qquad (x,t,\xi)\in\Omega\times(0,T)\times\mathbb{R}^n,$$

and

$$B(x,t) := \mu_1 u(x,t)\Big(1 - u(x,t) - a_1 v(x,t)\Big), \qquad (x,t) \in \Omega \times (0,T).$$

Here by Young's inequality we can estimate

$$\begin{aligned} A(x,t,\xi)\cdot\xi &\geq d_1|\xi|^2 + a_{12}u(x,t)\nabla v(x,t)\cdot\xi \\ &\geq \frac{d_1}{2}|\xi|^2 - \frac{a_{12}^2}{2d_1}u^2(x,t)|\nabla v(x,t)|^2 \qquad \text{for all } (x,t,\xi) \in \Omega\times(0,T)\times\mathbb{R}^n, \end{aligned}$$

and moreover we have

$$\begin{aligned} |A(x,t,\xi)| &\leq \Big(d_1 + 2a_{11}u(x,t) + a_{12}v(x,t)\Big)\cdot|\xi| + a_{12}u(x,t)|\nabla v(x,t)| \\ &\qquad\qquad\qquad \text{for all } (x,t,\xi) \in \Omega\times(0,T)\times\mathbb{R}^n \end{aligned}$$

and

$$|B(x,t)| \leq \mu_1 u(x,t)\cdot\Big(1 + u(x,t) + a_1 v(x,t)\Big) \qquad \text{for all } (x,t) \in \Omega\times(0,T).$$

Using that (6.1) warrants that $\nabla v \in L^\infty((0,T);L^p(\Omega;\mathbb{R}^n))$ and that u and also v is bounded due to the hypothesis $p > n$, we readily obtain (6.2) from [24, Theorem 1.3]. Likewise, relying only on the latter boundedness properies of u and v, the property (6.3) can be obtained from [24, Theorem 1.3] immediately. $\square$

By means of standard parabolic Schauder theory, we proceed to derive higher order estimates, firstly for the second solution component.

Lemma 6.2 *Let $n \geq 1$ and $T \in (0,\infty]$, and assume that $(u,v) \in (C^{2,1}(\bar\Omega\times(0,T)))^2$ solves the boundary value problem in (1.1) in $\Omega\times(0,T)$ and is such that u and v are nonnegative and that (6.1) holds with some $p > n$ and $K > 0$. Then for each $\delta \in (0,\tau)$ there exist $\theta = \theta(\delta) \in (0,1)$ and $C = C(\delta) > 0$ with the property that*

$$\|v\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar\Omega\times[t,t+\tau])} \leq C \qquad \textit{for all } t \in (\delta, T-\tau), \tag{6.4}$$

where again we have set $\tau := \min\{1, \frac{1}{2}T\}$.

Proof. Given $\delta > 0$, we fix $\chi \in C^\infty([0,\infty))$ such that $\chi \equiv 0$ in $[0,\frac{\delta}{2}]$ and $\chi \equiv 1$ in $[\delta,\infty)$, and observe that then with ϕ as in (2.5), the function z defined by

$$z(x,t) := \chi(t)\cdot\phi(v(x,t)), \qquad (x,t) \in \bar\Omega\times[0,T),$$

satisfies $\frac{\partial z}{\partial\nu} = 0$ on $\partial\Omega\times(0,\infty)$ and

$$z_t = A(x,t)\Delta z + B(x,t), \qquad x \in \Omega,\ t \in (0,T), \tag{6.5}$$

where

$$A(x,t) := \phi'(v(x,t)), \qquad (x,t) \in \Omega\times(0,T),$$

and

$$B(x,t) := \chi'(t)\phi(v(x,t)) + \mu_2\chi(t)\phi'(v(x,t)) \cdot v(x,t)\Big(1 - v(x,t) - a_2 u(x,t)\Big), \qquad (x,t) \in \Omega \times (0,T).$$

Since Lemma 6.1 entails the existence of $\theta_1 = \theta_1(\delta) \in (0,1)$ and $c_1 = c_1(\delta) > 0$ such that

$$\|A\|_{C^{\theta_1,\frac{\theta_1}{2}}(\bar\Omega\times[t,t+\tau])} + \|B\|_{C^{\theta_1,\frac{\theta_1}{2}}(\bar\Omega\times[t,t+\tau])} \le c_1 \qquad \text{for all } t \in (0, T-\tau),$$

and since $z(\cdot,0) \equiv 0$ is trivially smooth in $\bar\Omega$ and satisfies the first-order compatibility condition for the Neumann initial-boundary value problem associated with (6.5), the property (6.4) is a straightforward consequence of classical Schauder theory for inhomogeneous linear parabolic equations with Hölder continuous coefficients ([13, p. 320, Theorem 5.3]) and evident mapping properties of the C^∞ diffeomorphism ϕ on $[0,\infty)$. $\Box$

An extension of the above argument to the first equation in (1.1) requires to adequately respect the cross-diffusive interaction present therein. Thanks to the information on Hölder continuity of v_t entailed by the previous lemma, however, this can be accomplished with the following outcome.

Lemma 6.3 *Let $n \ge 1$, $T \in (0,\infty]$ and $(u,v) \in (C^{2,1}(\bar\Omega \times (0,T)))^2$ be a classical solution of the boundary value problem in (1.1) in $\Omega \times (0,T)$ with $u \ge 0$ and $v \ge 0$, which satisfies (6.1) with some $p > n$ and $K > 0$. Then writing $\tau := \min\{1, \frac{1}{2}T\}$, for each $\delta \in (0,\tau)$ one can find $\theta = \theta(\delta) \in (0,1)$ and $C = C(\delta) > 0$ fulfilling*

$$\|u\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar\Omega\times[t,t+\tau])} \le C \qquad \textit{for all } t \in (\delta, T-\tau). \tag{6.6}$$

Proof. We once more fix $\chi \in C^\infty([0,\infty))$ such that $\chi \equiv 0$ on $[0,\frac{\delta}{2}]$ and $\chi \equiv 1$ on $[\delta,\infty)$, and thereupon let

$$w(x,t) := \chi(t) \cdot \Big\{d_1 u(x,t) + a_{11}u^2(x,t) + a_{12}u(x,t)v(x,t)\Big\}, \qquad (x,t) \in \Omega \times (0,T).$$

Then using (1.1) we compute

$$\begin{aligned}
w_t &= \chi(t) \cdot \Big\{d_1 u_t + 2a_{11}uu_t + a_{12}vu_t + a_{12}uv_t\Big\} + \chi'(t) \cdot \Big\{d_1 u + a_{11}u^2 + a_{12}uv\Big\} \\
&= \Big\{d_1 + 2a_{11}u + a_{12}v\Big\} \cdot \chi(t)\Delta\Big\{d_1 u + a_{11}u^2 + a_{12}uv\Big\} \\
&\quad + \chi(t) \cdot \Big\{d_1 + 2a_{11}u + a_{12}v\Big\} \cdot \Big\{\mu_1 u - \mu_1 u^2 - \mu_1 a_1 uv\Big\} \\
&\quad + a_{12}\chi(t)uv_t + \chi'(t) \cdot \Big\{d_1 u + a_{11}u^2 + a_{12}uv\Big\} \\
&= A(x,t)\Delta w + B(x,t), \qquad x \in \Omega,\ t \in (0,T),
\end{aligned} \tag{6.7}$$

where

$$A(x,t) := d_1 + 2a_{11}u(x,t) + a_{12}v(x,t), \qquad (x,t) \in \Omega \times (0,T),$$

and

$$B(x,t) \;:=\; \chi(t)\cdot\Big\{d_1+2a_{11}u(x,t)+a_{12}v(x,t)\Big\}\cdot\Big\{\mu_1 u(x,t)-\mu_1 u^2(x,t)-\mu_1 a_1 u(x,t)v(x,t)\Big\}$$
$$+a_{12}\chi(t)u(x,t)v_t(x,t)+\chi'(t)\cdot\Big\{d_1 u(x,t)+a_{11}u^2(x,t)+a_{12}u(x,t)v(x,t)\Big\},$$
$$(x,t)\in\Omega\times(0,T).$$

Here due to Lemma 6.1, and moreover thanks to Lemma 6.2 and the fact that $\chi\equiv 0$ in $[0,\frac{\delta}{2}]$, it follows that there exist $\theta_1=\theta_1(\delta)\in(0,1)$ and $c_1=c_1(\delta)>0$ such that

$$\|A\|_{C^{\theta_1,\frac{\theta_1}{2}}(\bar\Omega\times[t,t+\tau])}+\|B\|_{C^{\theta_1,\frac{\theta_1}{2}}(\bar\Omega\times[t,t+\tau])}\le c_1 \qquad \text{for all } t\in(0,T-\tau),$$

that $w(\cdot,0)\equiv 0$, and that w satisfies the first-order compatibility condition corresponding to the Neumann initial-boundary value problem for (6.7), so that by parabolic Schauder theory ([13]) we can find $c_2=c_2(\delta)>0$ fulfilling

$$\|w\|_{C^{2+\theta_1,1+\frac{\theta_1}{2}}(\bar\Omega\times[t,t+\tau])}\le c_2 \qquad \text{for all } t\in(0,T-\tau). \tag{6.8}$$

Now since by definition of χ and w we can represent ∇u in $\Omega\times(\delta,T)$ according to

$$\nabla u=\frac{\nabla w-a_{12}u\nabla v}{d_1+2a_{11}u+a_{12}v}, \qquad (x,t)\in\Omega\times(\delta,T),$$

again in view of Lemma 6.1 and Lemma 6.2 we firstly conclude from (6.8) that with some $\theta_2=\theta_2(\delta)\in(0,1)$ and $c_3=c_3(\delta)>0$ we have

$$\|\nabla u\|_{C^{\theta_2,\frac{\theta_2}{2}}(\bar\Omega\times[t,t+\tau])}\le c_3 \qquad \text{for all } t\in(\delta,T-\tau). \tag{6.9}$$

Thereupon, for $i\in\{1,...n\}$ and $j\in\{1,...,n\}$ computing

$$\partial_{x_ix_j}u=\frac{\partial_{x_ix_j}w-a_{12}\partial_{x_j}u\partial_{x_i}v-a_{12}u\partial_{x_ix_j}v}{d_1+2a_{11}u+a_{12}v}-\frac{(2a_{11}u_{x_j}+a_{12}v_{x_j})(w_{x_i}-a_{11}uv_{x_i})}{(d_1+2a_{11}u+a_{12}v)^2},$$

for all $(x,t)\in\Omega\times(\delta,T)$, we similarly see, using (6.8), Lemma 6.1, Lemma 6.2 and now also (6.9), that there exist $\theta_3=\theta_3(\delta)\in(0,1)$ and $c_4=c_4(\delta)>0$ such that

$$\|D^2u\|_{C^{\theta_3,\frac{\theta_3}{2}}(\bar\Omega\times[t,t+\tau])}\le c_4 \qquad \text{for all } t\in(\delta,T-\tau), \tag{6.10}$$

and finally the identity

$$u_t=\frac{w_t-a_{12}uv_t}{d_1+2a_{11}u+a_{12}v}, \qquad (x,t)\in\Omega\times(\delta,T),$$

in view of (6.8), Lemma 6.1 and Lemma 6.2 warrants the existence of $\theta_4=\theta_4(\delta)\in(0,1)$ and $c_5=c_5(\delta)>0$ such that

$$\|u_t\|_{C^{\theta_4,\frac{\theta_4}{2}}(\bar\Omega\times[t,t+\tau])}\le c_5 \qquad \text{for all } t\in(\delta,T-\tau). \tag{6.11}$$

Combining (6.10) and (6.11) with Lemma 6.1 finally proves (6.6). □

Our main result on global existence and boundedness of solutions to (1.1) is now obvious.

PROOF of Theorem 1.1. We only need to combine the statements from Lemma 5.1 and Lemma 6.3, the latter being applied to the local solution from Lemma 2.1 and to $T:=T_{max}$, with the extensibility criterion (2.1). □

7 Dominance of the first species. Proof of Theorem 1.2

In order to simplify presentation, throughout this section we shall assume without further mentioning that the hypotheses of Theorem 1.2 are met, hence in particular supposing (u,v) to be a global classical solution of the boundary value problem in (1.1) to which Lemma 6.2 and Lemma 6.3 apply.

In order to construct a genuine energy functional under the parameter assumptions from Theorem 1.2, beyond the mass evolution law for the first component observed in Lemma 2.2 we shall need information on the time evolution of three further functionals. The first of these is addressed in the following.

Lemma 7.1 *We have*

$$\frac{d}{dt}\int_\Omega \ln u \ge -\frac{a_{12}^2}{4d_1}\int_\Omega |\nabla v|^2 + \mu_1|\Omega| - \mu_1\int_\Omega u - \mu_1 a_1\int_\Omega v \qquad \textit{for all } t>0. \tag{7.1}$$

PROOF. Since u is positive in $\bar\Omega\times(0,\infty)$ according to the strong maximum principle, we may multiply the first equation in (1.1) by $\frac{1}{u}$ to see using integration by parts that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \ln u &= \int_\Omega (d_1+2a_{11}u+a_{12}v)\frac{|\nabla u|^2}{u^2} + a_{12}\int_\Omega \frac{1}{u}\nabla u\cdot\nabla v + \mu_1|\Omega| - \mu_1\int_\Omega u - \mu_1 a_1\int_\Omega v \\
&\ge d_1\int_\Omega \frac{|\nabla u|^2}{u^2} + a_{12}\int_\Omega \frac{1}{u}\nabla u\cdot\nabla v + \mu_1|\Omega| - \mu_1\int_\Omega u - \mu_1 a_1\int_\Omega v \qquad \text{for all } t>0.
\end{aligned}$$

As by Young's inequality we can estimate

$$\left| a_{12}\int_\Omega \frac{1}{u}\nabla u\cdot\nabla v\right| \le d_1\int_\Omega \frac{|\nabla u|^2}{u^2} + \frac{a_{12}^2}{4d_1}\int_\Omega |\nabla v|^2 \qquad \text{for all } t>0,$$

this yields (7.1). □

As for the second solution component, we shall need the following two observations.

Lemma 7.2 *The solution of (1.1) satisfies*

$$\frac{d}{dt}\int_\Omega v = \mu_2\int_\Omega v - \mu_2\int_\Omega v^2 - \mu_2 a_2\int_\Omega uv \qquad \textit{for all } t>0 \tag{7.2}$$

as well as

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 \le -d_2\int_\Omega |\nabla v|^2 + \mu_2\int_\Omega v^2 - \mu_2\int_\Omega v^3 \qquad \textit{for all } t>0. \tag{7.3}$$

PROOF. The identity (7.2) directly results on integrating the second equation in (1.1) over Ω. Furthermore, on using v as a test function in (1.1) we see that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 = -\int_\Omega (d_2+2a_{22}v)|\nabla v|^2 + \mu_2\int_\Omega v^2 - \mu_2\int_\Omega v^3 - \mu_2 a_2\int_\Omega uv^2 \qquad \text{for all } t>0,$$

and hence conclude from the nonnegativity of both u and v that also (7.3) holds. □

Now combining the latter two lemmata we obtain a two-parameter family of candidates for an energy functional.

Lemma 7.3 *For $\beta > 0$ and*

$$\alpha \geq \frac{a_{12}^2}{4d_1d_2}, \tag{7.4}$$

we let

$$\mathcal{F}_{\alpha\beta}(t) := \int_\Omega \Big(u(\cdot,t) - 1 - \ln u(\cdot,t)\Big) + \frac{\alpha}{2}\int_\Omega v^2(\cdot,t) + \beta\int_\Omega v(\cdot,t), \qquad t > 0. \tag{7.5}$$

Then for any choice of $\varepsilon \in (0,1)$ we have

$$\mathcal{F}'_{\alpha\beta}(t) \leq -\varepsilon\mu_1\int_\Omega (u-1)^2 - \varepsilon\alpha\mu_2\int_\Omega v^3 + L_\varepsilon(\alpha,\beta)\int_\Omega v \qquad \textit{for all } t > 0, \tag{7.6}$$

where we have set

$$L_\varepsilon(\alpha,\beta) := \frac{1}{4(1-\varepsilon)\alpha\mu_2}\cdot\Big\{\frac{(\mu_1a_1+\beta\mu_2a_2)^2}{4(1-\varepsilon)\mu_1} + (\alpha-\beta)\mu_2\Big\}_+^2 - \beta\mu_2(a_2-1) \tag{7.7}$$

for $\alpha > 0, \beta > 0$ and $\varepsilon \in (0,1)$.

Proof. We combine the differential inequalities established in Lemma 7.1 and Lemma 7.2 and recall that $\frac{d}{dt}\int_\Omega u = \mu_1\int_\Omega u - \mu_1\int_\Omega u^2 - \mu_1a_1\int_\Omega uv$ for $t > 0$ to see on straightforward rearrangements into quadratic expressions that

$$\begin{aligned}
\mathcal{F}'_{\alpha\beta}(t) \;\leq\;& \mu_1\int_\Omega u - \mu_1\int_\Omega u^2 - \mu_1a_1\int_\Omega uv \\
& -\Big\{-\frac{a_{12}^2}{4d_1}\int_\Omega|\nabla v|^2 + \mu_1|\Omega| - \mu_1\int_\Omega u - \mu_1a_1\int_\Omega v\Big\} \\
& +\alpha\cdot\Big\{-d_2\int_\Omega|\nabla v|^2 + \mu_2\int_\Omega v^2 - \mu_2\int_\Omega v^3\Big\} \\
& +\beta\cdot\Big\{\mu_2\int_\Omega v - \mu_2\int_\Omega v^2 - \mu_2a_2\int_\Omega uv\Big\} \\
=\;& \Big\{\frac{a_{12}^2}{4d_1} - \alpha d_2\Big\}\cdot\int_\Omega|\nabla v|^2 \\
& -\mu_1\int_\Omega (u-1)^2 \\
& +(\mu_1a_1+\beta\mu_2)\int_\Omega v + (\alpha-\beta)\mu_2\int_\Omega v^2 - \alpha\mu_2\int_\Omega v^3 \\
& -(\mu_1a_1+\beta\mu_2a_2)\int_\Omega uv \qquad \text{for all } t > 0.
\end{aligned} \tag{7.8}$$

Here the first summand on the right is nonpositive thanks to (7.4), and in order to make appropriate use of the dissipative properties of the last we rewrite $uv = (u-1)v + v$ therein and use Young's inequality to estimate

$$\begin{aligned}
-(\mu_1a_1+\beta\mu_2a_2)\int_\Omega uv \;=\;& -(\mu_1a_1+\beta\mu_2a_2)\int_\Omega (u-1)v - (\mu_1a_1+\beta\mu_2a_2)\int_\Omega v \\
\leq\;& (1-\varepsilon)\mu_1\int_\Omega (u-1)^2 + \frac{(\mu_1a_1+\beta\mu_2a_2)^2}{4(1-\varepsilon)\mu_1}\int_\Omega v^2 - (\mu_1a_1+\beta\mu_2a_2)\int_\Omega v
\end{aligned}$$

for $t > 0$. From (7.8) we therefore obtain that for all $t > 0$,

$$\mathcal{F}'_{\alpha\beta}(t) \le -\varepsilon\mu_1 \int_\Omega (u-1)^2 \\ -\beta\mu_2(a_2-1)\int_\Omega v + l_\varepsilon(\alpha,\beta)\int_\Omega v^2 - \alpha\mu_2 \int_\Omega v^3$$

with

$$l_\varepsilon(\alpha,\beta) := \left\{ \frac{(\mu_1 a_1 + \beta\mu_2 a_2)^2}{4(1-\varepsilon)\mu_1} + (\alpha-\beta)\mu_2 \right\}.$$

As another application of Young's inequality shows that

$$l_\varepsilon(\alpha,\beta)\int_\Omega v^2 \le (1-\varepsilon)\alpha\mu_2 \int_\Omega v^3 + \frac{(l_\varepsilon)_+^2(\alpha,\beta)}{4(1-\varepsilon)\alpha\mu_2}\int_\Omega v \qquad \text{for all } t > 0,$$

this readily leads to (7.6) with $L_\varepsilon(\alpha,\beta)$ as given by (7.7). $\Box$

In order to suitably select the numbers α and β in $\mathcal{F}_{\alpha\beta}$, in view of an argument based on continuous dependence it seems adequate to study the corresponding problem in the limit case $\varepsilon \searrow 0$ in (7.6). We therefore define $L_0(\alpha,\beta) := \lim_{\varepsilon\searrow 0} L_\varepsilon(\alpha,\beta)$, that is, we let

$$L_0(\alpha,\beta) := \frac{1}{4\alpha\mu_2}\cdot\left\{ \frac{(\mu_1 a_1 + \beta\mu_2 a_2)^2}{4\mu_1} + (\alpha-\beta)\mu_2 \right\}_+^2 - \beta\mu_2(a_2-1), \qquad \alpha > 0,\ \beta > 0, \tag{7.9}$$

and are thus led to determining circumstances under which L_0 becomes negative. The following lemma reduces this to an associated negativity property of a polynomial with coefficients only involving the parameters a_1 and a_2.

Lemma 7.4 *Suppose that $a_2 > 1$, and let $\alpha > 0$ and $\beta > 0$. Then*

$$L_0(\alpha,\beta) < 0 \qquad \textit{if and only if} \qquad \Lambda\left(\sqrt{\frac{\mu_2}{\mu_1}}\alpha,\ \sqrt{\frac{\mu_2}{\mu_1}}\beta\right) < 0, \tag{7.10}$$

where

$$\Lambda(\xi,\eta) := 4\xi^2 - 8\sqrt{a_2-1}\,\xi\eta + a_2^2\eta^4 + 2(a_1a_2-2)\eta^2 + a_1^2, \qquad \xi > 0,\ \eta > 0. \tag{7.11}$$

Proof. If $L_0(\alpha,\beta) < 0$, then by definition of L_0 we have

$$\frac{(\mu_1 a_1 + \beta\mu_2 a_2)^2}{4\mu_1} + (\alpha-\beta)\mu_2 \le \left\{ \frac{(\mu_1 a_1 + \beta\mu_2 a_2)^2}{4\mu_1} + (\alpha-\beta)\mu_2 \right\}_+ < 2\sqrt{\alpha\beta(a_2-1)}\,\mu_2,$$

and hence

$$(\mu_1 a_1 + \mu_2 a_2\beta)^2 + 4(\alpha-\beta)\mu_1\mu_2 < 8\sqrt{\alpha\beta(a_2-1)}\,\mu_1\mu_2.$$

On dividing by $\mu_1\mu_2$ and substituting $\alpha = \frac{\mu_1}{\mu_2}\xi^2$ as well as $\beta = \frac{\mu_1}{\mu_2}\eta^2$, this can readily be seen to entail (7.10). The converse implication can be verified similarly. $\Box$

The arguments leading to the following statements are elementary, but since they contain the key observations underlying our conditions on a_1 and a_2 in Theorem 1.2, we include a proof here for completeness.

Lemma 7.5 *Suppose that $a_2 > 1$, and let the function Λ be as defined in (7.11).*
i) Λ possesses a critical point in $(0,\infty)^2$ if and only if $a_1 < 2$, and in that case this critical point is uniquely determined by the identity $(\xi,\eta) = (\xi_0,\eta_0)$, where

$$\xi_0 := \sqrt{\frac{(2-a_1)(a_2-1)}{a_2}} \qquad \textit{and} \qquad \eta_0 := \sqrt{\frac{2-a_1}{a_2}}, \tag{7.12}$$

and Λ attains a local minimum at (ξ_0,η_0).
ii) Let $a_1 < 2$. Then

$$\Lambda(\xi_0,\eta_0) < 0 \qquad \textit{if and only if} \qquad a_1 < 1. \tag{7.13}$$

Proof. We compute

$$\frac{\partial\Lambda(\xi,\eta)}{\partial\xi} = 8\xi - 8\sqrt{a_2-1}\eta, \qquad (\xi,\eta) \in (0,\infty)^2,$$

and

$$\frac{\partial\Lambda(\xi,\eta)}{\partial\eta} = -8\sqrt{a_2-1}\xi + 4a_2^2\eta^3 + 4(a_1a_2-2)\eta, \qquad (\xi,\eta) \in (0,\infty)^2,$$

to find that $(\xi,\eta) \in (0,\infty)^2$ is a critical point of Λ if and only if

$$\xi = \sqrt{a_2-1}\eta \qquad \text{and} \qquad 2\sqrt{a_2-1}\xi = a_2^2\eta^3 + (a_1a_2-2)\eta,$$

that is, if and only if

$$2\sqrt{a_2-1}^2\eta = a_2^2\eta^3 + (a_1a_2-2)\eta,$$

or, equivalently,

$$(2-a_1)a_2\eta = a_2^2\eta^3.$$

Since solving this with respect to $\eta > 0$ is possible precisely when $a_1 < 2$, this readily implies the claimed necessity and sufficiency of this condition for the existence of a critical point, as well as the formulae in (7.12). As moreover in this case we have

$$D^2\Lambda(\xi,\eta) = \begin{pmatrix} 8 & -8\sqrt{a_2-1} \\ -8\sqrt{a_2-1} & 12a_2^2\eta^2 + 4(a_1a_2-2) \end{pmatrix} \qquad \text{for all } (\xi,\eta) \in (0,\infty)^2$$

and hence

$$\begin{aligned} \det D^2\Lambda(\xi_0,\eta_0) &= 8 \cdot \Big\{12a_2^2 \cdot \frac{2-a_1}{a_2} + 4(a_1a_2-2)\Big\} - 64(a_2-1) \\ &= 96a_2(2-a_1) + 32(a_1a_2-2) - 64(a_2-1) \\ &= 64(2-a_1)a_2 > 0, \end{aligned}$$

it follows that in fact Λ attains a local minimum at (ξ_0,η_0).

ii) By (7.11) and (7.12), we see that indeed

$$\begin{aligned}
\Lambda(\xi_0,\eta_0) &= 4\cdot\frac{(2-a_1)(a_2-1)}{a_2} - 8\sqrt{a_2-1}\cdot\sqrt{\frac{(2-a_1)(a_2-1)}{a_2}}\cdot\sqrt{\frac{2-a_1}{a_2}} \\
&\quad +a_2^2\cdot\Big(\frac{2-a_1}{a_2}\Big)^2 + 2(a_1a_2-2)\cdot\frac{2-a_1}{a_2} + a_1^2 \\
&= \frac{8a_2-8-4a_1a_2+4a_1}{a_2} - \frac{16a_2-16-8a_1a_2+8a_1}{a_2} \\
&\quad +4-4a_1+a_1^2+\frac{4a_1a_2-2a_1^2a_2-8+4a_1}{a_2}+a_1^2 \\
&= 4a_1-4
\end{aligned}$$

is negative if and only if $a_1 < 1$. □

We are now ready to identify the parameter conditions from Theorem 1.2 as sufficient for $\mathcal{F}_{\alpha\beta}$ to become an energy functional for (1.1) when α and β are chosen suitably.

Lemma 7.6 *Suppose that*

$$a_2 > 1, \qquad \textit{and} \qquad a_1 \in [0,1) \tag{7.14}$$

as well as

$$\frac{a_{12}^2}{d_1d_2} \le \frac{\mu_1}{\mu_2}\cdot\frac{4(2-a_1)(a_2-1)}{a_2}. \tag{7.15}$$

Then there exist $\alpha > 0$, $\beta > 0$ and $\delta > 0$ such that the function $\mathcal{F}_{\alpha\beta}$ defined in (7.5) satisfies

$$\mathcal{F}'_{\alpha\beta}(t) \le -\delta\int_\Omega (u-1)^2 - \delta\int_\Omega v^3 \qquad \textit{for all } t>0. \tag{7.16}$$

Proof. According to the outcome of Lemma 7.5, with ξ_0 and η_0 taken from (7.12) we define

$$\alpha := \frac{\mu_1}{\mu_2}\xi_0^2 \qquad \text{and} \qquad \beta := \frac{\mu_1}{\mu_2}\eta_0^2,$$

and then obtain on combining Lemma 7.5 with Lemma 7.4 that as a consequence of (7.14) we have

$$L_0(\alpha,\beta) < 0.$$

In light of the definition (7.9) of $L_0(\alpha,\beta)$, we can therefore fix $\varepsilon \in (0,1)$ suitably small such that still

$$L_\varepsilon(\alpha,\beta) \le 0.$$

Now since (7.15) warrants that

$$\alpha = \frac{\mu_1}{\mu_2}\cdot\frac{(2-a_1)(a_2-1)}{a_2} \ge \frac{a_{12}^2}{4d_1d_2}$$

and that hence (7.4) is satisfied, from (7.6) we directly infer that indeed (7.16) holds if we let $\delta := \min\{\varepsilon\mu_1, \varepsilon\alpha\mu_2\}$, for instance. □

Along with the uniform continuity properties implied by Lemma 6.2 and Lemma 6.3, within the above parameter regime the global dissipative feature of (1.1) expressed in (7.16) finally implies stabilization in the claimed sense.

Proof of Theorem 1.2. As $(u(\cdot,t))_{t>1}$ and $(v(\cdot,t))_{t>1}$ are relatively compact in $C^2(\bar\Omega)$ by the Arzelá-Ascoli theorem, for the derivation of (1.5) and (1.6) it is sufficient to make sure that

$$u(\cdot,t) \to 1 \quad \text{in } L^\infty(\Omega) \qquad \text{as } t\to\infty \tag{7.17}$$

and

$$v(\cdot,t) \to 0 \quad \text{in } L^\infty(\Omega) \qquad \text{as } t\to\infty. \tag{7.18}$$

To verify this, we firstly invoke Lemma 7.6 to obtain $\alpha>0,\beta>0$ and $\delta>0$ such that

$$\delta\int_1^t\int_\Omega (u-1)^2 + \delta\int_1^t\int_\Omega v^3 \le \mathcal{F}_{\alpha\beta}(1) \qquad \text{for all } t>1$$

and that hence

$$\int_1^\infty\int_\Omega (u-1)^2 < \infty \qquad \text{and} \qquad \int_1^\infty\int_\Omega v^3 < \infty. \tag{7.19}$$

Now if (7.17) was false, then we could pick $(x_k)_{k\in\mathbb{N}}\subset\Omega$, $(t_k)_{k\in\mathbb{N}}\subset(1,\infty)$ and $\varepsilon>0$ such that $t_k\to\infty$ as $k\to\infty$ but

$$\Big|u(x_k,t_k)-1\Big| \ge \varepsilon \qquad \text{for all } k\in\mathbb{N}.$$

In view of the spatio-temporal equicontinuity property implied by Lemma 6.3, this would entail that on passing to a subsequence if necessary we could assume that

$$\Big|u(x,t)-1\Big| \ge \frac{\varepsilon}{2} \qquad \text{for all } x\in B_r(x_0), \text{ each } t\in(t_k,t_k+\eta) \text{ and any } k\in\mathbb{N}$$

with some $x_0\in\Omega, r>0$ and $\eta>0$. This, however, would imply that

$$\int_{t_k}^{t_k+\eta}\int_\Omega \Big(u(x,t)-1\Big)^2 dxdt \ge \eta|B_r(x_0)|\cdot\frac{\varepsilon^2}{4} > 0 \qquad \text{for all } k\in\mathbb{N}$$

and thereby contradict the fact that

$$\int_t^{t+\eta}\int_\Omega \Big(u(x,s)-1\Big)^2 dxds \to 0 \qquad \text{as } t\to\infty,$$

as asserted by (7.19). The corresponding convergence statement (7.18) can be verified in quite a similar manner. $\square$

8 Coexistence. Proof of Theorem 1.3

In this section we consider the case of weak competition, as determined by the assumptions that both $a_1\in(0,1)$ and $a_2\in(0,1)$. Tacitly assuming that (u,v) is a global solution with the properties and under the circumstances listed in Theorem 1.3, we begin with an elementary lemma asserting an asymptotic pointwise upper bound of v.

Lemma 8.1 *The second solution component satisfies*

$$\limsup_{t\to\infty} \|v(\cdot,t)\|_{L^\infty(\Omega)} \le 1. \tag{8.1}$$

PROOF. We let $\overline{v}(x,t) := y(t)$, $(x,t) \in \bar{\Omega} \times [0,\infty)$, where $y \in C^1([0,\infty))$ denotes the solution of

$$\begin{cases} y'(t) = \mu_2 y(t) \cdot (1 - y(t)), & t > 0, \\ y(0) = \|v\|_{L^\infty(\Omega\times(0,\infty))}. \end{cases}$$

Then since

$$\overline{v}_t - \Delta[(d_2 + a_{22}\overline{v})\overline{v}] - \mu_2\overline{v}(1 - \overline{v} - a_2 u) = y' - \mu_2 y(1 - y - a_2 u) \ge 0 \qquad \text{in } \Omega \times (0,\infty),$$

an application of the comparison principle readily shows that $v \le \overline{v}$ in $\Omega \times (0,\infty)$, so that (8.1) results from the fact that unless $v \equiv 0$ we have $y(t) \to 1$ as $t \to \infty$. $\Box$

Using this, we can construct a functional which in its basic flavor resembles that from Lemma 7.3, and which along each individual trajectory of (1.1) plays the role of an energy after some appropriate waiting time. Actually, the verification of this will turn out to be much simpler than that in Section 7.

Lemma 8.2 *Assume that $a_1 \in (0,1)$ and $a_2 \in (0,1)$, and that (1.9) holds. Then with $u_\star$ and $v_\star$ as given by (1.8), there exist $t_0 > 0$ and $\varepsilon > 0$ such that for*

$$\mathcal{F}^{(2)}(t) := \int_\Omega \Big(u(\cdot,t) - u_\star - u_\star \ln \frac{u(\cdot,t)}{u_\star} \Big) + \frac{\mu_1 a_1}{\mu_2 a_2} \int_\Omega \Big(v(\cdot,t) - v_\star - v_\star \ln \frac{v(\cdot,t)}{v_\star} \Big), \qquad t > 0, \tag{8.2}$$

we have

$$\mathcal{F}^{(2)}(t) \ge 0 \qquad \textit{for all } t > 0 \tag{8.3}$$

as well as

$$\frac{d}{dt}\mathcal{F}^{(2)}(t) + \varepsilon \int_\Omega \Big(u(\cdot,t) - u_\star \Big)^2 + \varepsilon \int_\Omega \Big(v(\cdot,t) - v_\star \Big)^2 \le 0 \qquad \textit{for all } t > t_0. \tag{8.4}$$

PROOF. Once more due to the strong maximum principle, our overall assumption that $u \not\equiv 0 \not\equiv v$ warrants that both u and v are positive in $\bar{\Omega} \times (0,\infty)$ and that hence $\mathcal{F}^{(2)}$ is well-defined. To prove its nonnegativity, we let $f(s) := s - 1 - \ln s$ for $s > 0$ and observe that $f(1) = 0 = f'(1)$ as well as $f''(s) > 0$ for all $s > 0$, so that $0 = f(1) = \min_{s>0} f(s)$. For all $t > 0$ we thus have

$$u(\cdot,t) - u_\star - u_\star \ln \frac{u(\cdot,t)}{u_\star} = u_\star f\Big(\frac{u(\cdot,t)}{u_\star}\Big) \ge 0$$

and

$$v(\cdot,t) - v_\star - v_\star \ln \frac{v(\cdot,t)}{v_\star} = v_\star f\Big(\frac{v(\cdot,t)}{v_\star}\Big) \ge 0,$$

which entail that indeed $\mathcal{F}^{(2)}(t) \ge 0$.

Next, in order to prepare our verification of (8.4) we note that our assumptions $a_1 \in (0,1)$ and $a_2 \in (0,1)$ imply that $\frac{1}{a_2} - a_1 > 0$, which enables us to fix some $\delta \in (0,1)$ that satisfies

$$\frac{1}{a_2} - \frac{a_1}{1-\delta} > 0. \tag{8.5}$$

Moreover, since a straightforward computation using (1.8) and (1.9) shows that

$$\begin{aligned}
\frac{\mu_1 a_1}{\mu_2 a_2} v_\star d_2 - \frac{a_{12}^2 u_\star}{4d_1} &= \frac{u_\star d_2}{4}\Big(\frac{\mu_1}{\mu_2} \cdot \frac{a_1}{a_2} \cdot \frac{4v_\star}{u_\star} - \frac{a_{12}^2}{d_1 d_2}\Big) \\
&= \frac{u_\star d_2}{4}\Big(\frac{\mu_1}{\mu_2} \cdot \frac{4a_1(1-a_2)}{a_2(1-a_1)} - \frac{a_{12}^2}{d_1 d_2}\Big) \\
&> 0,
\end{aligned}$$

it is possible to find some suitably small $\eta > 0$ such that still

$$\frac{\mu_1 a_1}{\mu_2 a_2} \cdot \frac{v_\star d_2}{(1+\eta)^2} - \frac{a_{12}^2 u_\star}{4d_1} > 0, \tag{8.6}$$

whereupon Lemma 8.1 applies so as to yield $t_0 > 0$ such that

$$v(x,t) \le 1 + \eta \qquad \text{for all } x \in \Omega \text{ and } t > t_0. \tag{8.7}$$

To derive (8.4) for this choice of t_0 and the number

$$\varepsilon := \min\Big\{\delta \ , \ \mu_1 a_1\Big(\frac{1}{a_2} - \frac{a_1}{1-\delta}\Big)\Big\} \tag{8.8}$$

which is positive due to (8.5), we first use (1.1) to compute

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big(u - u_\star - u_\star \ln\frac{u}{u_\star}\Big) \\
&\quad= \int_\Omega \Big(u_t - \frac{u_\star}{u} u_t\Big) \\
&\quad= \int_\Omega \mu_1 (u-u_\star)(1-u-a_1 v) - u_\star \int_\Omega (d_1 + 2a_{11}u + a_{12}v)\frac{|\nabla u|^2}{u^2} - a_{12}u_\star \int_\Omega \frac{\nabla u}{u}\cdot\nabla v \\
&\quad= -\mu_1 \int_\Omega (u-u_\star)^2 - \mu_1 a_1 \int_\Omega (u-u_\star)(v-v_\star) - u_\star \int_\Omega (d_1 + 2a_{11}u + a_{12}v)\frac{|\nabla u|^2}{u^2} \\
&\qquad\quad - a_{12}u_\star \int_\Omega \frac{\nabla u}{u}\cdot\nabla v \\
&\quad\le -\mu_1 \int_\Omega (u-u_\star)^2 - \mu_1 a_1 \int_\Omega (u-u_\star)(v-v_\star) - u_\star d_1 \int_\Omega \frac{|\nabla u|^2}{u^2} - a_{12}u_\star \int_\Omega \frac{\nabla u}{u}\cdot\nabla v
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big(v - v_\star - v_\star \ln \frac{v}{v_\star}\Big) \\
&\qquad = \; -\mu_2 \int_\Omega (v-v_\star)^2 - \mu_2 a_2 \int_\Omega (u-u_\star)(v-v_\star) - v_\star \int_\Omega (d_2 + 2a_{22}v)\frac{|\nabla v|^2}{v^2} \\
&\qquad \le \; -\mu_2 \int_\Omega (v-v_\star)^2 - \mu_2 a_2 \int_\Omega (u-u_\star)(v-v_\star) - v_\star d_2 \int_\Omega \frac{|\nabla v|^2}{v^2}
\end{aligned}$$

for $t > 0$. Here for large t we may use (8.7) to see that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big(v - v_\star - v_\star \ln \frac{v}{v_\star}\Big) \\
&\qquad \le \; -\mu_2 \int_\Omega (v-v_\star)^2 - \mu_2 a_2 \int_\Omega (u-u_\star)(v-v_\star) - \frac{v_\star d_2}{(1+\eta)^2}\int_\Omega |\nabla v|^2 \qquad \text{for all } t > t_0
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}^{(2)}(t) \; \le \; & -\mu_1 \int_\Omega (u-u_\star)^2 - 2\mu_1 a_1 \int_\Omega (u-u_\star)(v-v_\star) - \frac{\mu_1 a_1}{a_2}\int_\Omega (v-v_\star)^2 \\
& -u_\star d_1 \int_\Omega \frac{|\nabla u|^2}{u^2} - a_{12}u_\star \int_\Omega \frac{\nabla u}{u}\cdot\nabla v - \frac{\mu_1 a_1}{\mu_2 a_2}\cdot\frac{v_\star d_2}{(1+\eta)^2}\int_\Omega |\nabla v|^2 \qquad \text{for all } t > t_0. \qquad (8.9)
\end{aligned}$$

As thanks to Young's inequality we know that

$$-2\mu_1 a_1 \int_\Omega (u-u_\star)(v-v_\star) \le \mu_1(1-\delta)\int_\Omega (u-u_\star)^2 + \frac{\mu_1 a_1^2}{1-\delta}\int_\Omega (v-v_\star)^2$$

and

$$-a_{12}u_\star \int_\Omega \frac{\nabla u}{u}\cdot\nabla v \le u_\star d_1 \int_\Omega \frac{|\nabla u|^2}{u^2} + \frac{a_{12}^2 u_\star}{4d_1}\int_\Omega |\nabla v|^2$$

for all $t > 0$, from (8.9) and (8.6) we infer that indeed

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}^{(2)}(t) \; \le \; & -\delta \int_\Omega (u-u_\star)^2 - \mu_1 a_1\Big(\frac{1}{a_2} - \frac{a_1}{1-\delta}\Big)\int_\Omega (v-v_\star)^2 - \Big(\frac{\mu_1 a_1}{\mu_2 a_2}\cdot\frac{v_\star d_2}{(1+\eta)^2} - \frac{a_{12}^2 u_\star}{4d_1}\Big)\int_\Omega |\nabla v|^2 \\
\le \; & -\delta \int_\Omega (u-u_\star)^2 - \mu_1 a_1\Big(\frac{1}{a_2} - \frac{a_1}{1-\delta}\Big)\int_\Omega (v-v_\star)^2 \\
\le \; & -\varepsilon\cdot\Big\{\int_\Omega (u-u_\star)^2 + \int_\Omega (v-v_\star)^2\Big\} \qquad \text{for all } t > t_0 \qquad (8.10)
\end{aligned}$$

according to our definition (8.8) of ε. □

Relying on the energy inequality (8.4) together with the equicontinuity properties implied by Lemma 6.3 and Lemma 6.2, we can complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Taking $t_0 > 0$ and $\varepsilon > 0$ from Lemma 8.2, on integrating (8.4) we infer that with $\mathcal{F}^{(2)}$ as introduced in (8.2) we have

$$\varepsilon \int_{t_0}^{t} \int_\Omega (u-u_\star)^2 + \varepsilon \int_{t_0}^{t} \int_\Omega (v-v_\star)^2 \le \mathcal{F}^{(2)}(t_0) \qquad \text{for all } t > t_0$$

and thus

$$\int_{t_0}^{\infty} \int_\Omega (u-u_\star)^2 < \infty \qquad \text{and} \qquad \int_{t_0}^{\infty} \int_\Omega (v-v_\star)^2 < \infty. \tag{8.11}$$

As u and v are uniformly continuous in $\Omega \times (1,\infty)$ by Lemma 6.3 and Lemma 6.2, by means of an argument in the style of that presented in Theorem 1.2, this entails both (1.10) and (1.11). $\square$

9 Dominance of the second species. Proof of Theorem 1.4

Let us finally consider the case when $a_1 \ge 1 > a_2$, in which the independence of our main result from a_{12} rests on the following observation.

Lemma 9.1 *Let $a_1 \ge 1$ and $a_2 \in (0,1)$, and assume that $(u,v) \in (C^{2,1}(\bar\Omega \times (0,\infty)))^2$ is a classical solution of the boundary value problem in (1.1) such that $v \not\equiv 0$. Then there exists $\varepsilon > 0$ such that*

$$\mathcal{F}^{(3)}(t) := \int_\Omega u(\cdot,t) + \frac{\mu_1(2-a_2)}{\mu_2 a_2^2} \int_\Omega \Big(v(\cdot,t) - 1 - \ln v(\cdot,t) \Big), \qquad t > 0, \tag{9.1}$$

satisfies

$$\mathcal{F}^{(3)}(t) \ge 0 \qquad \textit{for all } t > 0 \tag{9.2}$$

as well as

$$\frac{d}{dt}\mathcal{F}^{(3)}(t) + \varepsilon \int_\Omega u^2(\cdot,t) + \varepsilon \int_\Omega \Big(v(\cdot,t) - 1 \Big)^2 \le 0 \qquad \textit{for all } t > 0. \tag{9.3}$$

Proof. Again noting that $v > 0$ in $\bar\Omega \times (0,\infty)$ by the strong maximum principle, we obtain that $\mathcal{F}^{(3)}$ is well-defined and satisfies (9.2) due to the nonnegativity of $0 < s \mapsto s - 1 - \ln s$. In order to derive (9.3), we note that since $a_2 < 1$ it is possible to fix $\eta \in (0,1)$ suitably small such that $\frac{1}{1-\eta} < 2 - a_2$, so that

$$\varepsilon_1 := \frac{\mu_1(2-a_2)}{a_2^2} - \frac{\mu_1}{(1-\eta)a_2^2}$$

is positive and hence also $\varepsilon := \min\{\varepsilon_1, \eta\mu_1\} > 0$. To verify (9.3), we first use (1.1) to see that

$$\begin{aligned} \frac{d}{dt}\int_\Omega u &= \mu_1 \int_\Omega u - \mu_1 \int_\Omega u^2 - \mu_1 a_1 \int_\Omega uv \\ &= -\mu_1(a_1-1)\int_\Omega u - \mu_1 \int_\Omega u^2 - \mu_1 a_1 \int_\Omega u(v-1) \qquad \text{for all } t > 0 \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}\int_\Omega \Big(v - 1 - \ln v \Big) &= -\mu_2 \int_\Omega (v-1)^2 - \mu_2 a_2 \int_\Omega u(v-1) - \int_\Omega (d_2 + 2a_{22}v)\frac{|\nabla v|^2}{v^2} \\ &\le -\mu_2 \int_\Omega (v-1)^2 - \mu_2 a_2 \int_\Omega u(v-1) \qquad \text{for all } t > 0. \end{aligned}$$

Therefore,

$$\frac{d}{dt}\mathcal{F}^{(3)}(t) \le -\mu_1\int_\Omega u^2 - \frac{\mu_1(2-a_2)}{a_2^2}\int_\Omega (v-1)^2 - \mu_1(a_1-1)\int_\Omega u - \Big(\mu_1 a_1 + \frac{\mu_1(2-a_2)}{a_2}\Big)\int_\Omega u(v-1) \quad (9.4)$$

for all $t>0$, where rearranging the latter two summands and using our assumption $a_1 \ge 1$ as well as Young's inequality shows that

$$\begin{aligned} -\mu_1(a_1-1)\int_\Omega u - \Big(\mu_1 a_1 + \frac{\mu_1(2-a_2)}{a_2}\Big)\int_\Omega u(v-1) \\ = \ -\mu_1(a_1-1)\int_\Omega uv - \frac{2\mu_1}{a_2}\int_\Omega u(v-1) \\ \le \ (1-\eta)\mu_1\int_\Omega u^2 + \frac{\mu_1}{(1-\eta)a_2^2}\int_\Omega (v-1)^2 \qquad \text{for all } t>0. \end{aligned}$$

Consequently, (9.4) implies that

$$\begin{aligned} \frac{d}{dt}\int_\Omega \mathcal{F}^{(3)}(t) \ &\le \ -\eta\mu_1\int_\Omega u^2 - \Big(\frac{\mu_1(2-a_2)}{a_2^2} - \frac{\mu_1}{(1-\eta)a_2^2}\Big)\int_\Omega (v-1)^2 \\ &\le \ -\varepsilon\cdot\Big\{\int_\Omega u^2 + \int_\Omega (v-1)^2\Big\} \qquad \text{for all } t>0, \end{aligned}$$

as claimed. □

Indeed, this enables us to derive Theorem 1.4 from Lemma 6.3 and Lemma 6.2.

Proof of Theorem 1.4. As Lemma 9.1 entails that with $\varepsilon>0$ and $\mathcal{F}^{(3)}$ as defined there we have

$$\int_1^t\int_\Omega u^2 + \int_1^t\int_\Omega (v-1)^2 \le \frac{\mathcal{F}^{(3)}(1)}{\varepsilon} \qquad \text{for all } t>1,$$

on the basis of Lemma 6.3 and Lemma 6.2 we readily obtain (1.13) and (1.14). □

10 Appendix: An interpolation lemma for equicontinuous families of functions

We finally include the Ehrling-type interpolation inequality used in Lemma 5.1, which actually generalizes a previously obtained result addressing the special case when $p=2$ in the following ([22, Lemma 5.1]). For completeness, let us include a proof here.

Lemma 10.1 *Let $n\ge 1$ and $\Omega\subset\mathbb{R}^n$ be a bounded domain with smooth boundary, and let $\omega:(0,\infty)\to(0,\infty)$ be nondecreasing. Then for all $p\ge 1$ and each $\varepsilon>0$ there exists $C(p,\varepsilon)>0$ such that*

$$\int_\Omega |\nabla\varphi|^{2p+2} \le \varepsilon\int_\Omega |\nabla\varphi|^{2p-2}|D^2\varphi|^2 + C(p,\varepsilon)\|\varphi\|_{L^\infty(\Omega)}^{2p+2} \qquad (10.1)$$

holds for all

$$\varphi \in \mathcal{S}_\omega := \Big\{ \tilde\varphi \in C^2(\bar\Omega) \ \Big| \ \frac{\partial\tilde\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega, \text{ and for all } \varepsilon' > 0, \text{ we have } |\tilde\varphi(x) - \tilde\varphi(y)| < \varepsilon' \text{ whenever } x, y \in \bar\Omega \text{ are such that } |x-y| < \omega(\varepsilon'). \Big\} \qquad (10.2)$$

Proof. Given $\varepsilon > 0$, we write

$$\varepsilon' := \sqrt{\frac{\varepsilon}{4n + 32p^2}} \quad \text{and} \quad \delta := \omega(\varepsilon') \qquad (10.3)$$

and then may use the compactness of $\bar\Omega$ to pick $N \in \mathbb{N}$ and $\{x_1, ..., x_N\} \subset \bar\Omega$ such that $\bar\Omega \subset \bigcup_{j=1}^N B_\delta(x_j)$. Moreover, we let $(\zeta_j)_{j\in\{1,...,N\}} \subset C^1(\bar\Omega)$ be an associated partition of unity such that $\zeta_j \ge 0$ in $\bar\Omega$, $\operatorname{supp}\zeta_j \subset B_\delta(x_j)$ for all $j \in \{1, ..., N\}$ and $\sum_{j=1}^N \zeta_j \equiv 1$ in $\bar\Omega$. Finally employing Young's inequality in choosing $c_1 > 0$ such that

$$AB \le \frac{1}{8N} A^{\frac{2p+2}{2p+1}} + c_1 B^{2p+2} \qquad \text{for all } A \ge 0 \text{ and } B \ge 0, \qquad (10.4)$$

we claim that (10.1) holds for all $\varphi \in \mathcal{S}_\omega$ if we set

$$C(p,\varepsilon) := 2^{2p+5} N c_1 c_2^{2p+2} \qquad (10.5)$$

with

$$c_2 := \max_{j\in\{1,...,N\}} \|\nabla\zeta_j\|_{L^{2p+2}(\Omega)}. \qquad (10.6)$$

To verify this, we introduce

$$I := \int_\Omega |\nabla\varphi|^{2p+2} \qquad \text{and} \qquad J := \int_\Omega |\nabla\varphi|^{2p-2}|D^2\varphi|^2$$

as well as

$$\overline{\varphi}_j := \varphi(x_j), \quad I_j := \int_\Omega |\nabla\varphi|^{2p+2}\zeta_j \quad \text{and} \quad J_j := \int_\Omega |\nabla\varphi|^{2p-2}|D^2\varphi|^2\zeta_j \qquad \text{for } j \in \{1, ..., N\}.$$

Then an integration by parts shows that for each $j \in \{1, ..., N\}$,

$$\begin{aligned} I_j &= \int_\Omega |\nabla\varphi|^{2p}\nabla\varphi\cdot\nabla(\varphi - \overline{\varphi}_j)\zeta_j \\ &= -\int_\Omega (\varphi - \overline{\varphi}_j)|\nabla\varphi|^{2p}\Delta\varphi\zeta_j - \int_\Omega (\varphi - \overline{\varphi}_j)\nabla\varphi\cdot\nabla|\nabla\varphi|^{2p}\zeta_j \\ &\quad -\int_\Omega (\varphi - \overline{\varphi}_j)|\nabla\varphi|^{2p}\nabla\varphi\cdot\nabla\zeta_j \\ &=: I_{j1} + I_{j2} + I_{j3}, \end{aligned} \qquad (10.7)$$

where boundary integrals vanish due to the fact that $\frac{\partial \varphi}{\partial \nu}|_{\partial\Omega} = 0$. Here the inclusion $\varphi \in \mathcal{S}_\omega$ along with our choice of δ implies that

$$|\varphi - \overline{\varphi}_j| \le \varepsilon' \qquad \text{in } \operatorname{supp} \zeta_j.$$

Since $|\Delta\varphi| \le \sqrt{n}|D^2\varphi|$ in Ω, in view of the Cauchy-Schwarz inequality and Young's inequality we can therefore estimate

$$\begin{aligned} I_{j1} &\le \varepsilon' \int_\Omega |\nabla\varphi|^{2p} |\Delta\varphi| \zeta_j \\ &\le \varepsilon' \cdot \Big(\int_\Omega |\nabla\varphi|^{2p+2} \zeta_j \Big)^{\frac{1}{2}} \Big(\int_\Omega |\nabla\varphi|^{2p-2} |\Delta\varphi|^2 \zeta_j \Big)^{\frac{1}{2}} \\ &\le \sqrt{n}\varepsilon' \sqrt{I_j}\sqrt{J_j} \\ &\le \frac{1}{2} I_j + \frac{n\varepsilon'^2}{2} J_j. \end{aligned} \tag{10.8}$$

In the same manner, by using the identity $\nabla|\nabla\varphi|^{2p} = 2p|\nabla\varphi|^{2p-2} D^2\varphi \cdot \nabla\varphi$ we find that

$$\begin{aligned} I_{j2} &\le 2p\varepsilon' \int_\Omega |\nabla\varphi|^{2p-2} \Big| \nabla\varphi \cdot (D^2\varphi \cdot \nabla\varphi) \Big| \zeta_j \\ &\le 2p\varepsilon' \int_\Omega |\nabla\varphi|^{2p} |D^2\varphi| \zeta_j \\ &\le 2p\varepsilon' \sqrt{I_j}\sqrt{J_j} \\ &\le \frac{1}{4} I_j + 4p^2 \varepsilon'^2 J_j. \end{aligned} \tag{10.9}$$

As for the last summand I_{j3} in (10.7), we trivially estimate $|\varphi - \overline{\varphi}_j| \le 2\|\varphi\|_{L^\infty(\Omega)}$ to derive from the Hölder inequality, (10.6) and (10.4) that

$$\begin{aligned} I_{j3} &\le 2\|\varphi\|_{L^\infty(\Omega)} \Big(\int_\Omega |\nabla\varphi|^{2p+2} \Big)^{\frac{2p+1}{2p+2}} \Big(\int_\Omega |\nabla\zeta_j|^{2p+2} \Big)^{\frac{1}{2p+2}} \\ &\le 2c_2 \|\varphi\|_{L^\infty(\Omega)} I^{\frac{2p+1}{2p+2}} \\ &\le \frac{1}{8N} I + c_1 \cdot \Big(2c_2 \|\varphi\|_{L^\infty(\Omega)} \Big)^{2p+2}. \end{aligned} \tag{10.10}$$

Collecting (10.7)-(10.10) shows that

$$\frac{1}{4} I_j \le \Big(\frac{n}{2} + 4p^2 \Big) \varepsilon'^2 J_j + \frac{1}{8N} I + 2^{2p+2} c_1 c_2^{2p+2} \|\varphi\|_{L^\infty(\Omega)}^{2p+2},$$

which on summation over $j \in \{1, ..., N\}$ entails that

$$\begin{aligned} I &= \sum_{j=1}^N I_j \\ &\le (2n + 16p^2)\varepsilon'^2 \cdot \sum_{j=1}^N J_j + 4N \cdot \Big(\frac{1}{8N} I + 2^{2p+2} c_1 c_2^{2p+2} \|\varphi\|_{L^\infty(\Omega)}^{2p+2} \Big) \\ &= (2n + 16p^2)\varepsilon'^2 J + \frac{1}{2} I + 2^{2p+4} N c_1 c_2^{2p+2} \|\varphi\|_{L^\infty(\Omega)}^{2p+2}. \end{aligned}$$

Consequently,

$$I \le (4n + 32p^2)\varepsilon'^2 J + 2^{2p+5} N c_1 c_2^{2p+2} \|\varphi\|_{L^\infty(\Omega)}^{2p+2},$$

which according to the definitions (10.3) and (10.5) of ε' and $C(p, \varepsilon)$ precisely proves (10.1). □

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