A chemotaxis-haptotaxis system with haptoattractant remodeling: Boundedness enforced by mild saturation of signal production

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Abstract

We consider the chemotaxis-haptotaxis system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1-u-w), \\ v_t = \Delta v - v + f(u), \\ w_t = -vw + \eta w(1-u-w), \end{cases}$$

in a bounded convex domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, where χ, ξ, μ and η are positive constants, and where $f \in C^1([0,\infty))$ is a given function fulfilling $f(0) \ge 0$ and

$$f(s) \le K_f (s+1)^\alpha \qquad \text{for all } s \ge 0$$

with some $K_f > 0$ and $\alpha > 0$.

It is asserted that whenever

$$\alpha < \begin{cases} \frac{3}{2} & \text{if } n = 1, \\ \frac{n+6}{2(n+2)} & \text{if } n \ge 2, \end{cases}$$

the Neumann boundary problem with suitably regular initial data possesses a unique global and bounded classical solution.

Key words: chemotaxis-haptotaxis system; nonlinear signal production; global existence; boundedness; maximal Sobolev regularity
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1 Introduction

Taxis mechanism are known to play an important role in the process of cancer cell invasion of neighboring tissue. In addition to random motion, cancer cells are able to direct their migration toward the higher concentration of enzymes secreted by themselves, while they are moreover attracted by matrix molecules adhering to the tissue and consequently their movement is biased toward the higher density of tissue as well. Cancer cells usually undergo proliferation and death and compete surviving space with adjacent tissue. The diffusible enzymes can degrade the tissue, which is assumed to have a certain ability to recover to a normal level, and which competes for space with cancer cells.

A renowned macroscopic model accounting for these mechanisms, as proposed by Chaplain and Lolas ([4], [5]), gives rise to studying the parabolic-parabolic-ODE initial-boundary value problem given by

$$\begin{cases} u_t &= \Delta u - \chi\nabla\cdot(u\nabla v) - \xi\nabla\cdot(u\nabla w) + \mu u(1-u-w), & x\in\Omega,\ t>0, \\ v_t &= \Delta v - v + f(u), & x\in\Omega,\ t>0, \\ w_t &= -vw + \eta w(1-u-w), & x\in\Omega,\ t>0, \\ (\nabla u - \chi u\nabla v - \xi u\nabla w)\cdot\nu = \frac{\partial v}{\partial\nu} = 0, & & x\in\partial\Omega,\ t>0, \\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), \quad w(x,0)=w_0(x), & & x\in\Omega, \end{cases} \tag{1.1}$$

in a bounded domain $\Omega\subset\mathbb{R}^n$ with smooth boundary, and in fact a considerable literature has addressed questions from existence and qualitative theories therefor under various circumstances. In particular, quite a comprehensive knowledge could be generated in the case when $\eta=0$, hence reflecting situations in which spontaneous tissue remodeling can be neglected. For such systems, various results on global existence of classical solutions are available under mild assumptions inter alia allowing for the most prototypical choice $f(u)=u$ ([18], [19], [24], cf. also [1] for a survey), and furthermore some results on boundedness and even on large time stabilization could be achieved, partially upon some additional assumptions on the initial data w_0 and on the model parameters μ and χ ([10], [24], [27]), or upon some further simplification replacing the second identity in (1.1) by an elliptic equation ([21], [22]), or upon alternative model modifications by e.g. including nonlinear cell diffusion ([13], [31], [26]).

As for full versions of the system (1.1) which do contain tissue remodeling terms by admitting η to be positive, only little seems known, which from a mathematical perspective might be viewed as reflecting some additional destabilizing potential induced by the reaction term $-\eta uw$, then entering (1.1) in a nontrivial manner, through possible effects on formation of sharp haptoattractant gradients ∇w which due to the lack of diffusion cannot expected to be regularized during evolution. Actually, in the case when $f(u)=u$ the only results we are aware of in this field are either restricted to spatially two-dimensional settings, in which in fact a result on global existence and boundedness could recently be derived ([16], cf. also the precedents [15] and [23]), or on the construction of weak solutions, which were indeed found to exist also in three-dimensional domains ([17]), but the regularity information on which is yet rather poor and especially does not rule out the possibility of finite-time emergence of singularities.

Main results. The purpose of the present work consists in examining to which extent the regularity properties of the full system (1.1), thus including tissue remodeling, benefit from certain saturation effects in the production of enzymes at large densities of the tumor population. In particular, we shall subsequently consider (1.1), with positive parameters χ,ξ,μ and η, under that standing assumptions that

$$f \in C^1([0,\infty)) \quad \text{is such that } f(0) \geq 0, \tag{1.2}$$

and that the growth of f at large values of u is controlled from above according to

$$f(s) \leq K_f (s+1)^\alpha \qquad \text{for all } s \geq 0 \tag{1.3}$$

with some $K_f > 0$ and $\alpha > 0$. Our hypotheses concerning the initial data will be that

$$\begin{cases} u_0, v_0 \text{ and } w_0 \text{ are nonnegative functions from } C^{2+\vartheta}(\overline{\Omega}) \text{ for some } \vartheta > 0, \\ \text{with } \frac{\partial u_0}{\partial \nu} = \frac{\partial v_0}{\partial \nu} = \frac{\partial w_0}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.4}$$

Addressing precisely this setup, for the physically relevant three-dimensional case a recent work ([3]) has provided a result on global existence of bounded classical solutions under the condition that f grows at most in a considerably sublinear manner in the sense that in (1.3) we have $\alpha < \frac{5}{6}$ and hence

$$1 - \alpha > \frac{1}{6}. \tag{1.5}$$

The goal of this work is to develop an apparently novel type of mathematical approach, at its core based on an argument relying on well-known maximal Sobolev regularity estimates for inhomogeneous heat equations (see Section 4), which allows for a significant reduction of the gap between saturated and linear signal production expressed in (1.5), and our main results will reveal that (1.5) can actually be replaced with the assumption that merely

$$1 - \alpha > \frac{1}{10} \tag{1.6}$$

when $n = 3$. More precisely, and more generally, we shall derive the following.

Theorem 1.1 *Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with smooth boundary, let u_0, v_0 and w_0 satisfy (1.4), and let $\chi > 0, \xi > 0, \mu > 0$ and $\eta > 0$ be such that $\mu > \xi\eta \max\{1, \|w_0\|_{L^\infty(\Omega)}\}$. Then if f satisfies (1.2) and (1.3) with some $\alpha > 0$ fulfilling*

$$\alpha < \begin{cases} \frac{3}{2} & \text{if } n = 1, \\ \frac{n+6}{2(n+2)} & \text{if } n \geq 2, \end{cases} \tag{1.7}$$

the problem (1.1) admits a global classical solution (u, v, w) which is bounded in the sense that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \qquad \text{for all } t > 0 \tag{1.8}$$

with some $C > 0$.

2 Local existence and a convenient extensibility criterion

Following a rather standard reduction step (see e.g. [6], [7], [25] and [20] for some precedents), we conveniently transform (1.1) by the substitution

$$a := e^{-\xi w} u, \tag{2.1}$$

through which, namely, in the framework of classical solutions the problem (1.1) becomes equivalent to

$$\begin{cases} a_t = e^{-\xi w}\nabla\cdot(e^{\xi w}\nabla a) - \chi e^{-\xi w}\nabla\cdot(e^{\xi w}a\nabla v) + \xi avw + (\mu-\xi\eta w)a(1-e^{\xi w}a - w), & x\in\Omega,\ t>0,\\ v_t = \Delta v - v + f(e^{\xi w}a), & x\in\Omega,\ t>0,\\ w_t = -vw + \eta w(1-w-e^{\xi w}a), & x\in\Omega,\ t>0,\\ \frac{\partial a}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ a(x,0)=e^{-\xi w_0(x)}u_0(x), \quad v(x,0)=v_0(x), \quad w(x,0)=w_0(x), & x\in\Omega. \end{cases} \tag{2.2}$$

In this setting, the following statement on local existence and extensibility can be derived by means of quite well-established arguments.

Lemma 2.1 *Suppose that $n\ge 1$ and that $\Omega\subset\mathbb{R}^n$ is a bounded domain with smooth boundary, that χ,ξ,μ and η are positive, and that (1.4) holds. Then there exist T_{max} and a uniquely determined triple (a,w,v) of nonnegative functions a,v and w belonging to $C^{2,1}(\overline{\Omega}\times[0,T_{max}))$ which form a classical solution of (2.2) in $\Omega\times(0,T_{max})$, and which are such that*

$$\textit{if } T_{max}<\infty, \textit{ then } \quad \limsup_{t\nearrow T_{max}} \Big\{ \|a(\cdot,t)\|_{L^\infty(\Omega)} + \|\nabla w(\cdot,t)\|_{L^q(\Omega)} \Big\} = \infty \qquad \textit{for all } q\ge n+2. \tag{2.3}$$

Furthermore, with $u:=e^{\xi w}a$ and $\tau:=\min\{1,\frac{1}{2}T_{max}\}$ we have

$$\int_\Omega u(\cdot,t) \le m := \max\Big\{|\Omega|\,,\int_\Omega u_0\Big\} \qquad \textit{for all } t\in(0,T_{max}) \tag{2.4}$$

and

$$\int_t^{t+\tau}\int_\Omega u^2 \le \frac{(1+\mu)m}{\mu} \qquad \textit{for all } t\in(0,T_{max}-\tau) \tag{2.5}$$

as well as

$$w(x,t)\le M:=\max\Big\{1\,,\|w_0\|_{L^\infty(\Omega)}\Big\} \qquad \textit{for all } x\in\Omega \textit{ and } t\in(0,T_{max}). \tag{2.6}$$

Proof. On the basis of a standard fixed point argument, the claims concerning local existence and extensibility can be verified by minor adaptation of well-documented reasonings, e.g. the proof detailed for a related two-dimensional analogue in [23, Lemmata 2.1 and 2.2], to the present multi-dimensional setting. The nonnegativity of a,v and w as well as (2.6) thereafter follow from the maximum principle, and since an integration in the first equation from (1.1) shows that

$$\frac{d}{dt}\int_\Omega u = \mu\int_\Omega u - \mu\int_\Omega u^2 - \mu\int_\Omega uw \le \mu\int_\Omega u - \mu\int_\Omega u^2 \qquad \text{for all } t\in(0,T_{max}), \tag{2.7}$$

due to the fact that $\int_\Omega u^2 \ge \frac{1}{|\Omega|}\big(\int_\Omega u\big)^2$ by the Cauchy-Schwarz inequality we readily infer both (2.4) and (2.5). $\Box$

Throughout the sequel, without further explicit mentioning we shall suppose that $n\ge 1$ and $\Omega\subset\mathbb{R}^n$ is a bounded convex domain with smooth boundary, that (1.4) holds, that χ,ξ,μ and η are positive and $\mu > \xi\eta M$ with $M > 0$ taken from (2.6), and let a, v, w and u as well as T_{max} be as correspondingly provided by Lemma 2.1 and (2.1).

In order to conveniently relax the extensibility criterion (2.3), let us make sure that L^∞ bounds for a already entail spatial L^q bounds for ∇w with arbitrarily large finite q, at least locally in time:

Lemma 2.2 *Suppose that*

$$\sup_{t\in(0,T_{max})} \|a(\cdot,t)\|_{L^\infty(\Omega)} < \infty. \tag{2.8}$$

Then for all $p \geq 2$ and any $T > 0$ one can find $C(p,T) > 0$ such that

$$\int_\Omega |\nabla w(\cdot,t)|^{p+2} \leq C(p,T) \qquad \textit{for all } t \in (0, \min\{T_{max}, T\}). \tag{2.9}$$

PROOF. Rewriting the first equation in (2.2) in the form

$$a_t = \Delta a + g(x,t)\cdot\nabla a + h(x,t), \qquad x \in \Omega,\ t \in (0,T_{max}), \tag{2.10}$$

we see that thanks to (2.8), the coefficient functions

$$g(x,t) := \xi\nabla w - \chi\nabla v, \qquad x \in \Omega,\ t \in (0,T_{max}),$$

and

$$h(x,t) := -\chi\xi a\nabla v\cdot\nabla w - \chi a\Delta v + \xi avw + (\mu - \xi\eta w)a(1 - e^{\xi w}a - w), \qquad x \in \Omega,\ t \in (0,T_{max}),$$

satisfy

$$|g(x,t)| \leq c_1|\nabla w| + c_1 \qquad \text{for all } x \in \Omega \text{ and } t \in (0,T_{max})$$

as well as

$$|h(x,t)| \leq c_2|\nabla w| + c_2|\Delta v| + c_2 \qquad \text{for all } x \in \Omega \text{ and } t \in (0,T_{max})$$

with some $c_1 > 0$ and $c_2 > 0$, because (2.8) clearly entails boundedness of v and ∇v in $\Omega \times (0,T_{max})$ through standard parabolic estimates (see e.g. [11, Lemma 4.1]). Therefore, by means of Young's inequality we see that with some $c_3 = c_3(p) > 0$ and $c_4 = c_4(p) > 0$ we have

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega |\nabla a|^p &= \int_\Omega |\nabla a|^{p-2}\nabla a\cdot\nabla(\Delta a + g\cdot\nabla a + h) \\
&= \frac{1}{2}\int_\Omega |\nabla a|^{p-2}\Delta|\nabla a|^2 - \int_\Omega |\nabla a|^{p-2}|D^2a|^2 + \int_\Omega |\nabla a|^{p-2}\nabla a\cdot\nabla(g\cdot\nabla a + h) \\
&\leq -\int_\Omega |\nabla a|^{p-2}|D^2a|^2 - \int_\Omega (g\cdot\nabla a + h)\cdot\Big\{|\nabla a|^{p-2}\Delta a + (p-2)|\nabla a|^{p-4}\nabla a\cdot(D^2a\cdot\nabla a)\Big\} \\
&\leq -\int_\Omega |\nabla a|^{p-2}|D^2a|^2 + (\sqrt{n} + p - 2)\int_\Omega |g\cdot\nabla a + h|\cdot|\nabla a|^{p-2}|D^2a| \\
&\leq -\frac{1}{2}\int_\Omega |\nabla a|^{p-2}|D^2a|^2 + c_3\int_\Omega |g\cdot\nabla a + h|^2\cdot|\nabla a|^{p-2} \\
&\leq -\frac{1}{2}\int_\Omega |\nabla a|^{p-2}|D^2a|^2 + c_4\int_\Omega |\nabla a|^p|\nabla w|^2 + c_4\int_\Omega |\nabla a|^p|\Delta v|^2 + c_4\int_\Omega |\nabla a|^p \\
&\quad + c_4\int_\Omega |\nabla w|^2 + c_4\int_\Omega |\Delta v|^2 + c_4
\end{aligned} \tag{2.11}$$

for all $t \in (0,T_{max})$, because $\nabla a \cdot \nabla\Delta a = \frac{1}{2}\Delta|\nabla a|^2 - |D^2 a|^2$, because $\frac{\partial|\nabla a|^2}{\partial\nu} \le 0$ on $\partial\Omega \times (0,T_{max})$ by convexity of Ω ([14]), and because $|\Delta a| \le \sqrt{n}|D^2 a|$. We now note that thanks to the latter we may also infer from (2.8) that there exists $c_5 = c_5(p) > 0$ such that

$$\begin{aligned}
\int_\Omega |\nabla a|^{p+2} &= -\int_\Omega a|\nabla a|^p\Delta a - p\int_\Omega a|\nabla a|^{p-2}\nabla a\cdot(D^2 a\cdot\nabla a) \\
&\le (\sqrt{n}+p)\|a\|_{L^\infty(\Omega)}\int_\Omega |\nabla a|^p|D^2 a| \\
&\le c_5\Big\{\int_\Omega |\nabla a|^{p+2}\Big\}^{\frac{1}{2}}\cdot\Big\{\int_\Omega |\nabla a|^{p-2}|D^2 a|^2\Big\}^{\frac{1}{2}} \qquad \text{for all } t\in(0,T_{max})
\end{aligned}$$

and hence

$$\int_\Omega |\nabla a|^{p+2} \le c_5^2\int_\Omega |\nabla a|^{p-2}|D^2 a|^2 \qquad \text{for all } t\in(0,T_{max}). \tag{2.12}$$

Again using Young's inequality, we thus readily see that (2.11) implies that

$$\int_\Omega |\nabla a|^p + \frac{p}{4}\int_0^t\int_\Omega |\nabla a|^{p-2}|D^2 a|^2 \le c_6 t\cdot\sup_{s\in(0,t)}\int_\Omega |\nabla w(\cdot,s)|^{p+2} + c_6\int_0^t\int_\Omega |\Delta v|^{p+2} + c_7\cdot(t+1) \tag{2.13}$$

for all $t \in (0,T_{max})$, with some $c_6 = c_6(p) > 0$ and $c_7 = c_7(p,\int_\Omega |\nabla a(\cdot,0)|^p) > 0$, where maximal Sobolev regularity estimates ([8]) along with (2.8) and (2.6) yield $c_8 = c_8(p,\int_\Omega |\Delta v_0(\cdot,0)|^{p+2},T) > 0$ satisfying

$$\int_0^t\int_\Omega |\Delta v|^{p+2} \le c_8 \qquad \text{for all } t\in(0,\min\{T_{max},T\}). \tag{2.14}$$

As, independently, on testing the third equation in (2.2) by $|\nabla w|^p\nabla w$ we easily find $c_9 = c_9(p) > 0$ and $c_{10} = c_{10}(p,\int_\Omega |\nabla w(\cdot,0)|^{p+2}) > 0$ such that

$$\int_\Omega |\nabla w|^{p+2} \le c_9\int_0^t\int_\Omega |\nabla a|^{p+2} + c_{10} \qquad \text{for all } t\in(0,T_{max}), \tag{2.15}$$

in view of (2.12) this entails that if we restrict t so as to satisfy $t \in [0,t_0]$ with $t_0 = t_0(p) \in (0,\min\{T_{max},T\})$ fulfilling $c_5^2 c_6 c_9 t_0 \le \frac{p}{8}$, then as a consequence of (2.13) we obtain that

$$\begin{aligned}
\frac{p}{8}\int_0^t\int_\Omega |\nabla a|^{p-2}|D^2 a|^2 &\le c_7\cdot(t+1) + c_6\cdot c_8 + c_6 c_{10} t \\
&\le c_7\cdot(T+1) + c_6\cdot c_8 + c_6 c_{10} T \qquad \text{for all } t\in[0,t_0].
\end{aligned}$$

When combined with (2.12) and (2.15), this establishes (2.9) for any such t, and since t_0 could be chosen so as to depend only on p but not on, e.g., $(a,v,w)(\cdot,0)$, we may repeat this procedure finitely many times if necessary to conclude that (2.9) actually holds for all $t \in (0,\min\{T_{max},T\})$. □

Thereby, (2.3) actually reduces to the following.

Lemma 2.3 *The solution of (2.2) actually has the property that*

$$\text{if } T_{max} < \infty, \text{ then } \quad \limsup_{t\nearrow T_{max}} \|a(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{2.16}$$

Proof. This directly follows by combining Lemma 2.1 with Lemma 2.2. □

3 Implications of gradient estimates for v on integrability of a

In order to concretize our goal to be subsequently pursued, let us state the following observation which forms the starting point therefor, and which can be obtained by means of a standard testing procedure. The following lemma is the only place in this paper in which our standing assumption $\mu > \xi\eta M$ is explicitly utilized.

Lemma 3.1 *Let $p > 1$. Then there exists $C(p) > 0$ such that*

$$\frac{d}{dt}\int_\Omega e^{\xi w}a^p + \frac{1}{C(p)}\int_\Omega |\nabla a^{\frac{p}{2}}|^2 + \frac{1}{C(p)}\int_\Omega a^{p+1} \le C(p)\int_\Omega a^p|\nabla v|^2 + C(p)\int_\Omega v^{p+1} + C(p) \tag{3.1}$$

for all $t \in (0, T_{max})$.

Proof. This can be seen in a straightforward manner on testing the first equation in (2.2) against a^{p-1}: Thanks to Young's inequality and (2.6), namely, from (2.2) we see that with $c_1 := \mu - \xi\eta M > 0$ and some $c_2 = c_2(p) > 0$,

$$\begin{aligned}
\frac{d}{dt}\int_\Omega e^{\xi w}a^p &= p\int_\Omega a^{p-1}\cdot\Big\{\nabla\cdot(e^{\xi w}\nabla a) - \chi\nabla\cdot(e^{\xi w}a\nabla v) + \xi e^{\xi w}avw + (\mu-\xi\eta w)ae^{\xi w}(1-e^{\xi w}a-w)\Big\} \\
&\quad +\xi\int_\Omega e^{\xi w}a^p\cdot\Big\{-vw+\eta w(1-w-e^{\xi w}a)\Big\} \\
&= -p(p-1)\int_\Omega e^{\xi w}a^{p-2}|\nabla a|^2 + p(p-1)\chi\int_\Omega e^{\xi w}a^{p-1}\nabla a\cdot\nabla v \\
&\quad +p\xi\int_\Omega e^{\xi w}a^p vw \\
&\quad +p\int_\Omega(\mu-\xi\eta w)e^{\xi w}a^p - p\int_\Omega(\mu-\xi\eta w)e^{2\xi w}a^{p+1} - p\int_\Omega(\mu-\xi\eta w)e^{\xi w}a^p w \\
&\quad -\xi\int_\Omega e^{\xi w}a^p vw + \xi\eta\int_\Omega e^{\xi w}a^p w - \xi\eta\int_\Omega e^{\xi w}a^p w^2 - \xi\eta\int_\Omega e^{2\xi w}a^{p+1}w \\
&\le -\frac{p(p-1)}{2}\int_\Omega e^{\xi w}a^{p-2}|\nabla a|^2 + \frac{p(p-1)\chi^2}{2}\int_\Omega e^{\xi w}a^p|\nabla v|^2 \\
&\quad +p\xi\int_\Omega e^{\xi w}a^p vw + p\mu\int_\Omega e^{\xi w}a^p - pc_1\int_\Omega e^{2\xi w}a^{p+1} + \xi\eta\int_\Omega e^{\xi w}a^p w \\
&\le -\frac{p(p-1)}{2}\int_\Omega e^{\xi w}a^{p-2}|\nabla a|^2 + \frac{p(p-1)\chi^2}{2}\int_\Omega e^{\xi w}a^p|\nabla v|^2 \\
&\quad -\frac{pc_1}{2}\int_\Omega e^{2\xi w}a^{p+1} + c_2\int_\Omega e^{-(p-1)\xi w}\cdot(v^{p+1}w^{p+1} + w^{p+1} + 1)
\end{aligned}$$

for all $t \in (0, T_{max})$. Once more recalling (2.6), from this we readily infer (3.1). □

In order to draw appropriate consequences of this, but also for later reference, let us recall from [30, Lemma 3.4] the following statement on control of the inhomogeneous part in a Duhamel formula associated with the ODI $y'(t) + \lambda y(t) \le h(t)$ with given nonnegative h and $\lambda > 0$.

Lemma 3.2 *Let $T \in (0,\infty]$ and $\tau \in (0,T)$, and let $h \in L^1_{loc}((0,T))$ be nonnegative and such that with some $b > 0$,*

$$\int_t^{t+\tau} h(s)ds \le b \qquad \text{for all } t \in (0, T-\tau).$$

Then for any choice of $\lambda > 0$,

$$\int_0^t e^{-\lambda(t-s)} h(s)ds \le \frac{b\tau}{1-e^{-\lambda\tau}} \qquad \text{for all } t \in (0,T).$$

Now estimating the crucial first integral on the right of (3.1) by means of two different strategies depending on whether $n = 1$ or $n \ge 2$, using Lemma 3.2 we obtain from Lemma 3.1 the following criterion for the derivation of estimates for a which are of a similar flavor of those implied by (2.4) and (2.5), but which go beyond these when the parameter p addressed below satisfies $p > 1$.

Lemma 3.3 *Suppose that $p > 1$ is such that with $\tau := \min\{1, \frac{1}{2}T_{max}\}$ we have*

$$\begin{cases} \sup_{t\in(0,T_{max}-\tau)} \int_t^{t+\tau} \|v(\cdot,s)\|^{2p+2}_{W^{1,2}(\Omega)} ds < \infty & \text{if } n = 1, \\ \sup_{t\in(0,T_{max}-\tau)} \int_t^{t+\tau} \|v(\cdot,s)\|^{2p+2}_{W^{1,2p+2}(\Omega)} ds < \infty & \text{if } n \ge 2. \end{cases} \tag{3.2}$$

Then there exists $C = C(p) > 0$ such that

$$\int_\Omega a^p(\cdot,t) \le C(p) \qquad \text{for all } t \in (0, T_{max}) \tag{3.3}$$

and

$$\int_t^{t+\tau}\int_\Omega a^{p+1} \le C(p) \qquad \text{for all } t \in (0, T_{max}-\tau). \tag{3.4}$$

Proof. According to Lemma 3.1, we can fix $c_1 = c_1(p) > 0$ and $c_2 = c_2(p) > 0$ such that

$$\frac{d}{dt}\int_\Omega e^{\xi w}a^p + c_1\int_\Omega |\nabla a^{\frac{p}{2}}|^2 + c_1\int_\Omega a^{p+1} \le c_2\int_\Omega a^p|\nabla v|^2 + c_2\int_\Omega v^{p+1} + c_2 \qquad \text{for all } t \in (0,T_{max}), \tag{3.5}$$

and in the case $n \ge 2$ we use Young's inequality to see that with some $c_3 = c_3(p) > 0$ we herein have

$$c_2\int_\Omega a^p|\nabla v|^2 \le \frac{c_1}{3}\int_\Omega a^{p+1} + c_3\int_\Omega |\nabla v|^{2p+2} \qquad \text{for all } t \in (0,T_{max}).$$

Since clearly, by the same token and (2.6),

$$c_2\int_\Omega v^{p+1} \le c_2\int_\Omega v^{2p+2} + \frac{c_2|\Omega|}{4} \qquad \text{for all } t \in (0,T_{max})$$

as well as

$$\frac{c_1}{3}\int_\Omega a^{p+1} \ge c_4\int_\Omega e^{\xi w}a^p - c_5 \qquad \text{for all } t \in (0,T_{max}) \tag{3.6}$$

with some $c_4 = c_4(p) > 0$ and $c_5 = c_5(p) > 0$, from (3.5) we thus infer that in this case,

$$\frac{d}{dt}\int_\Omega e^{\xi w}a^p + c_4\int_\Omega e^{\xi w}a^p + \frac{c_1}{3}\int_\Omega a^{p+1} \le c_3\int_\Omega |\nabla v|^{2p+2} + c_2\int_\Omega v^{2p+2} + \frac{c_2|\Omega|}{4} + c_5$$

for all $t \in (0, T_{max})$. Thanks to (3.2), on integration using Lemma 3.2 from this we readily obtain both (3.3) and (3.4).

In the case $n = 1$, in (3.5) we rather make use of the second summand on the left: By means of the Gagliardo-Nirenberg inequality, (2.4) and Young's inequality, namely, we see that with some positive constants $c_6 = c_6(p), c_7 = c_7(p)$ and $c_8 = c_8(p)$ we have

$$\begin{aligned} c_2\int_\Omega a^p v_x^2 &\le c_2\|a^{\frac{p}{2}}\|^2_{L^\infty(\Omega)}\|v_x\|^2_{L^2(\Omega)} \\ &\le c_6\cdot\Big\{\|(a^{\frac{p}{2}})_x\|^{\frac{2p}{p+1}}_{L^2(\Omega)}\|a^{\frac{p}{2}}\|^{\frac{2}{p+1}}_{L^{\frac{2}{p}}(\Omega)} + \|a^{\frac{p}{2}}\|^2_{L^{\frac{2}{p}}(\Omega)}\Big\}\cdot\|v_x\|^2_{L^2(\Omega)} \\ &\le c_7\cdot\Big\{\|(a^{\frac{p}{2}})_x\|^{\frac{2p}{p+1}}_{L^2(\Omega)} + 1\Big\}\cdot\|v_x\|^2_{L^2(\Omega)} \\ &\le c_1\|(a^{\frac{p}{2}})_x\|^2_{L^2(\Omega)} + c_8\|v_x\|^{2p+2}_{L^2(\Omega)} + c_8 \qquad \text{for all } t \in (0, T_{max}). \end{aligned}$$

As furthermore combining the Gagliardo-Nirenberg inequality with Young's inequality we easily find $c_9 = c_9(p) > 0$ fulfilling

$$c_2\int_\Omega v^{p+1} \le c_9\|v\|^{2p+2}_{W^{1,2}(\Omega)} + c_9 \qquad \text{for all } t \in (0, T_{max}),$$

again relying on an inequality of the form in (3.6) we thereby obtain $c_{10} = c_{10}(p) > 0$ and $c_{11} = c_{11}(p) > 0$ such that

$$\frac{d}{dt}\int_\Omega e^{\xi w}a^p + c_{10}\int_\Omega e^{\xi w}a^p + c_{10}\int_\Omega a^{p+1} \le c_{11}\|v\|^{2p+2}_{W^{1,2}(\Omega)} + c_{11} \qquad \text{for all } t \in (0, T_{max}),$$

and can hence derive (3.3) and (3.4) from (3.2) through Lemma 3.2 as before. □

4 The core: gradually improving bounds for $\int_\Omega a^p$ and $\int_t^{t+\tau}\int_\Omega a^{p+1}$

Inspired by the outcome of Lemma 3.3, our strategy now will consist in successively improving our knowledge on integrability of a, as measured through its norm in $L^\infty((0, T_{max}); L^p(\Omega))$ and the expression $\sup_{t\in(0,T_{max}-\tau)}\int_t^{t+\tau}\int_\Omega a^{p+1}$ for $p \ge 1$, using the basic information from (2.4) and (2.5) as a starting point. More precisely, our goal will be to show unboundedness of the set S given by

$$S := \Big\{p \ge 1 \ \Big| \ \sup_{t\in(0,T_{max})}\int_\Omega a^p(\cdot,t) < \infty \quad \text{and} \quad \sup_{t\in(0,T_{max}-\tau)}\int_t^{t+\tau}\int_\Omega a^{p+1} < \infty\Big\}, \tag{4.1}$$

where again $\tau := \min\{1, \frac{1}{2}T_{max}\}$, by means of a contradictory argument based on the fact that, as we shall see, the assumptions from Theorem 1.1 essentially warrant the existence of $\delta > 0$ such that whenever $p \in S$, we also have $p + \delta \in S$. In light of Lemma 3.3, the latter step amounts to developing the bounds for a implied by a supposedly valid inclusion $p \in S$ into appropriate estimates for v and its gradient.

Though rather elementary, a first and quite fundamental observation relates such bounds for a to certain integrability properties of the inhomogeneity $f(u)$ in the second eqation from (1.1).

Lemma 4.1 *Let $p \geq 1$ belong to S. Then for all $\sigma \in (\frac{p}{\alpha}, \frac{p+1}{\alpha}]$ one can find $C(p,\sigma) > 0$ such that*

$$\int_t^{t+\tau} \|f(u(\cdot,s))\|_{L^\sigma(\Omega)}^{\frac{\sigma}{\alpha\sigma-p}} ds \leq C(p,\sigma) \qquad \textit{for all } t \in (0, T_{max}-\tau) \tag{4.2}$$

with $\tau = \min\{1, \frac{1}{2}T_{max}\}$.

Proof. According to the fact that $p \in S$, by boundedness of Ω we can find $c_1 = c_1(p) > 0$ and $c_2 = c_2(p) > 0$ such that

$$\|u(\cdot,t)+1\|_{L^p(\Omega)} \leq c_1 \qquad \text{for all } t \in (0, T_{max})$$

and

$$\int_t^{t+\tau} \|u(\cdot,s)+1\|_{L^{p+1}(\Omega)}^{p+1} ds \leq c_2 \qquad \text{for all } t \in (0, T_{max}-\tau).$$

Using (1.3), by means of the Hölder inequality we can therefore estimate

$$\begin{aligned}
\int_t^{t+\tau} \|f(u(\cdot,s))\|_{L^\sigma(\Omega)}^{\frac{\sigma}{\alpha\sigma-p}} ds &\leq K_f^{\frac{\sigma}{\alpha\sigma-p}} \int_t^{t+\tau} \|u(\cdot,s)+1\|_{L^{\alpha\sigma}(\Omega)}^{\frac{\alpha\sigma}{\alpha\sigma-p}} ds \\
&\leq K_f^{\frac{\sigma}{\alpha\sigma-p}} \int_t^{t+\tau} \|u(\cdot,s)+1\|_{L^{p+1}(\Omega)}^{p+1} \|u(\cdot,s)+1)\|_{L^p(\Omega)}^{\frac{p(p+1-\alpha\sigma)}{\alpha\sigma-p}} ds \\
&\leq c_1^{\frac{p(p+1-\alpha\sigma)}{\alpha\sigma-p}} c_2 K_f^{\frac{\sigma}{\alpha\sigma-p}}
\end{aligned}$$

for all $t \in (0, T_{max}-\tau)$. □

A first application thereof, here yet exclusively used for the choice $\sigma = \frac{p+1}{\alpha}$ in (4.2), entails a temporally uniform bound for v in a conveniently small L^q space.

Lemma 4.2 *Assume that $\alpha \in (0,2)$, and suppose that $p \geq 1$ is such that $p \in S$. Then there exists $C(p) > 0$ with the property that*

$$\int_\Omega v^{\frac{p+1}{\alpha}}(\cdot,t) \leq C(p) \qquad \textit{for all } t \in (0, T_{max}). \tag{4.3}$$

Proof. Integrating by parts in the second equation from (1.1), for $t \in (0, T_{max})$ we compute

$$\begin{aligned}
\frac{\alpha}{p+1}\frac{d}{dt}\int_\Omega v^{\frac{p+1}{\alpha}} &= \int_\Omega v^{\frac{p+1-\alpha}{\alpha}} \cdot \Big\{\Delta v - v + f(u)\Big\} \\
&= -\frac{p+1-\alpha}{\alpha}\int_\Omega v^{\frac{p+1-2\alpha}{\alpha}}|\nabla v|^2 - \int_\Omega v^{\frac{p+1}{\alpha}} + \int_\Omega f(u) v^{\frac{p+1-\alpha}{\alpha}},
\end{aligned} \tag{4.4}$$

where the first summand on the right is nonpositive due to the fact that $\alpha < 2$ implies that $p+1-\alpha \ge 2-\alpha > 0$. Since for the same reason we also have $\frac{p+1}{\alpha} > 1$, we may furthermore invoke Young's inequality, which allows us to estimate the rightmost integral in (4.4) according to

$$\int_\Omega f(u) v^{\frac{p+1-\alpha}{\alpha}} \le \frac{p+1-\alpha}{p+1} \int_\Omega v^{\frac{p+1}{\alpha}} + \frac{\alpha}{p+1} \int_\Omega f^{\frac{p+1}{\alpha}}(u) \qquad \text{for all } t \in (0, T_{max}).$$

From (4.4) we therefore obtain that

$$\frac{d}{dt} \int_\Omega v^{\frac{p+1}{\alpha}} + \int_\Omega v^{\frac{p+1}{\alpha}} \le h(t) := \int_\Omega f^{\frac{p+1}{\alpha}}(u) \qquad \text{for all } t \in (0, T_{max}),$$

where the inclusion $p \in S$ along with (2.6) ensures the existence of $c_1 = c_1(p) > 0$ such that $\int_t^{t+\tau} h(s)ds \le c_1$ for all $t \in (0, T_{max} - \tau)$, again with $\tau := \min\{1, \frac{1}{2} T_{max}\}$. An application of Lemma 3.2 thus shows that

$$\int_\Omega v^{\frac{p+1}{\alpha}} \le \int_\Omega v_0^{\frac{p+1}{\alpha}} + \frac{c_1}{1 - e^{-\tau}} \qquad \text{for all } t \in (0, T_{max})$$

and hence establishes (4.3). □

4.1 Space-time estimates for v, ∇v and $D^2 v$ via maximal Sobolev regularity

Secondly, through maximal Sobolev regularity features of the inhomogeneous heat equation satisfied by v, Lemma 4.1 implies space-time bounds for v even involving second-order spatial derivatives:

Lemma 4.3 *Suppose that $p \ge 1$ is such that $p > \alpha - 1$ and $p \in S$. Then for all $\sigma \in (\frac{p}{\alpha}, \frac{p+1}{\alpha}]$ with $\sigma > 1$, there exists $C(p, \sigma) > 0$ satisfying*

$$\int_t^{t+\tau} \|v(\cdot, s)\|_{W^{2,\sigma}(\Omega)}^{\frac{\sigma}{\alpha\sigma - p}} ds \le C(p, \sigma) \qquad \text{for all } t \in (0, T_{max} - \tau), \tag{4.5}$$

where again $\tau = \min\{1, \frac{1}{2} T_{max}\}$.

Proof. Following a temporal localization procedure in the style of that e.g. from [29], we fix $\zeta_0 \in C^\infty(\mathbb{R})$ such that $0 \le \zeta_0 \le 1$ on $\mathbb{R}$, $\zeta_0 \equiv 0$ in $(-\infty, -\tau]$ and $\zeta_0 \equiv 1$ in $[0, \infty)$, and for $t_0 \in [0, T_{max})$ we let $\zeta(t) \equiv \zeta(t_0)(t) := \zeta_0(t - t_0)$, $t \ge 0$, as well as

$$z(\cdot, t) \equiv z^{(t_0)}(\cdot, t) := \zeta^{(t_0)}(t) \cdot \Big\{ v(\cdot, t) - e^{t(\Delta - 1)} v_0 \Big\}, \qquad t \in [(t_0 - \tau)_+, T_{max}).$$

Then

$$\begin{cases} z_t = \Delta z - z + g(x,t), & x \in \Omega,\ t \in \Big((t_0 - \tau)_+, T_{max} \Big), \\ \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega,\ t \in \Big((t_0 - \tau)_+, T_{max} \Big), \\ z(x, (t_0 - \tau)_+) = 0, & x \in \Omega, \end{cases} \tag{4.6}$$

with

$$g(\cdot, t) \equiv g^{(t_0)}(\cdot, t) := \zeta(t) f(u(\cdot, t)) + \zeta'(t) v(\cdot, t) - \zeta'(t) \cdot e^{t(\Delta - 1)} v_0, \qquad t \in \Big((t_0 - \tau)_+, T_{max} \Big),$$

where according to our hypotheses on v_0 in (1.4), we see that with some $c_1 > 0$ independent of $t_0 \in [0, T_{max})$ we have

$$|g(x,t)| \le c_1 f(u) + c_1 v + c_1 \qquad \text{all } x \in \Omega \text{ and } t \in (0, T_{max}). \tag{4.7}$$

Now since our assumption $p > \alpha - 1$ ensures that $\frac{p+1}{\alpha} > 1$, and that furthermore in both cases $\alpha \le 1$ and $\alpha > 1$ we also have $\frac{\sigma}{\alpha\sigma - p} > 1$, we may invoke a well-known result from maximal Sobolev regularity theory ([8], [9]) to find $c_2 = c_2(p,\sigma) > 0$ such that

$$\int_{(t_0-\tau)_+}^{t_0+\tau} \|z(\cdot,t)\|_{W^{2,\sigma}(\Omega)}^{\frac{\sigma}{\alpha\sigma-p}} dt \le c_2 \int_{(t_0-\tau)_+}^{t_0+\tau} \|g(\cdot,t)\|_{L^\sigma(\Omega)}^{\frac{\sigma}{\alpha\sigma-p}} dt \qquad \text{for all } t_0 \in [0, T_{max}). \tag{4.8}$$

As Lemma 4.1 and Lemma 4.2 provide $c_3 = c_3(p) > 0$ and $c_4 = c_4(p) > 0$ such that

$$\int_{(t_0-\tau)_+}^{t_0+\tau} \|f(u(\cdot,t))\|_{L^\sigma(\Omega)}^{\frac{\sigma}{\alpha\sigma-p}} dt \le c_3 \qquad \text{for all } t_0 \in [0, T_{max})$$

and that

$$\|v(\cdot,t)\|_{L^{\frac{p+1}{\alpha}}(\Omega)} \le c_4 \qquad \text{for all } t \in (0, T_{max}),$$

and that hence also

$$\begin{aligned}\int_{(t_0-\tau)_+}^{t_0+\tau} \|v(\cdot,t)\|_{L^\sigma(\Omega)}^{\frac{\sigma}{\alpha\sigma-p}} dt &\le |\Omega|^{\frac{p+1-\alpha\sigma}{(p+1)(\alpha\sigma-p)}} \int_{(t_0-\tau)_+}^{t_0+\tau} \|v(\cdot,t)\|_{L^{\frac{p+1}{\alpha}}(\Omega)}^{\frac{\sigma}{\alpha\sigma-p}} dt \\ &\le 2\tau |\Omega|^{\frac{p+1-\alpha\sigma}{(p+1)(\alpha\sigma-p)}} c_4^{\frac{\sigma}{\alpha\sigma-p}} \qquad \text{for all } t_0 \in [0, T_{max}),\end{aligned}$$

from (4.7) and (4.8) we obtain $c_5 = c_5(p,\sigma) > 0$ such that

$$\int_{(t_0-\tau)_+}^{t_0+\tau} \|z(\cdot,t)\|_{W^{2,\sigma}(\Omega)}^{\frac{\sigma}{\alpha\sigma-p}} dt \le c_5 \qquad \text{for all } t_0 \in [0, T_{max}).$$

Since $z^{(t_0)} \equiv v - e^{t(\Delta-1)} v_0$ in $\Omega \times (t_0, t_0+\tau)$ by construction, together with our assumptions on v_0 in (1.4) this particularly establishes (4.5). $\Box$

4.2 Temporally uniform $W^{1,q}$ bounds for v via L^p-L^q estimates

An argument independent from that from Lemma 4.3, rather relying on direct application of smoothing estimates for the Neumann heat semigroup to a Duhamel representation of v, shows that Lemma 4.1 furthermore implies temporally uniform bounds for v in certain $W^{1,q}$ spaces.

Lemma 4.4 *Let $p \ge 1$ belong to S, and suppose that $\sigma \in (\frac{p}{\alpha}, \frac{p+1}{\alpha}]$ and $q \ge 1$ are such that $\sigma \ge 1$ and*

$$2p > (2\alpha - 1)\sigma \tag{4.9}$$

as well as

$$\frac{1}{q} > \frac{2\alpha\sigma - \sigma - 2p + n}{n\sigma}. \tag{4.10}$$

Then there exists $C(p,q,\sigma) > 0$ such that

$$\|v(\cdot,t)\|_{W^{1,q}(\Omega)} \le C(p,q,\sigma) \qquad \textit{for all } t \in (0, T_{max}). \tag{4.11}$$

PROOF. As a consequence of (4.9), we know that

$$\frac{2\alpha\sigma-\sigma-2p+n}{n\sigma}<\frac{1}{\sigma},$$

whence we may assume without loss of generality that besides (4.10), q also satisfies $q>\sigma$. Therefore, we can employ a well-known smoothing property of the Neumann heat semigroup ([28]) to fix $c_1=c_1(p,q,\sigma)>0$ such that for all $\varphi\in C^0(\overline{\Omega})$,

$$\|e^{t\Delta}\varphi\|_{W^{1,q}(\Omega)}\le c_1\cdot\Big(1+t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{q})}\Big)\|\varphi\|_{L^\sigma(\Omega)}\qquad\text{for all } t>0.$$

Furthermore picking $c_2=c_2(q)>0$ such that for all $\varphi\in W^{1,\infty}(\Omega)$ we have

$$\|e^{t\Delta}\varphi\|_{W^{1,q}(\Omega)}\le c_2\|\varphi\|_{W^{1,\infty}(\Omega)}\qquad\text{for all } t>0,$$

on the basis of a Duhamel formula associated with the second equation in (1.1), by using the Hölder inequality we can estimate

$$\begin{aligned}
\|v(\cdot,t)\|_{W^{1,q}(\Omega)} &= \left\|e^{t(\Delta-1)}v_0+\int_0^t e^{(t-s)(\Delta-1)}f(u(\cdot,s))ds\right\|_{W^{1,q}(\Omega)}\\
&\le c_2e^{-t}\|v_0\|_{W^{1,\infty}(\Omega)}+c_1\int_0^t\Big\{1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{q})}\Big\}e^{-(t-s)}\|f(u(\cdot,s))\|_{L^\sigma(\Omega)}ds\\
&\le c_2\|v_0\|_{W^{1,\infty}(\Omega)}\\
&\quad +c_1\cdot\bigg\{\int_0^t\Big\{1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{q})}\Big\}^{\frac{\sigma}{\sigma-\alpha\sigma+p}}e^{-(t-s)}ds\bigg\}^{\frac{\sigma-\alpha\sigma+p}{\sigma}}\\
&\qquad\qquad\times\bigg\{\int_0^t e^{-(t-s)}\|f(u(\cdot,s))\|_{L^\sigma(\Omega)}^{\frac{\sigma}{\alpha\sigma-p}}ds\bigg\}^{\frac{\alpha\sigma-p}{\sigma}}
\end{aligned}\tag{4.12}$$

for all $t\in(0,T_{max})$. Here we use that according to our restrictions on σ we may invoke Lemma 4.1 to find $c_3=c_3(p,\sigma)>0$ such that $h(t):=\|f(u(\cdot,t))\|_{L^\sigma(\Omega)}^{\frac{\sigma}{\alpha\sigma-p}}$, $t\in(0,T_{max})$, satisfies $\int_t^{t+\tau}h(s)ds\le c_3$ for all $t\in(0,T_{max}-\tau)$, where $\tau=\min\{1,\frac{1}{2}T_{max}\}$, and that hence, by Lemma 3.2,

$$\int_0^t e^{-(t-s)}h(s)ds\le\frac{c_3\tau}{1-e^{-\tau}}\qquad\text{for all } t\in(0,T_{max}).$$

To derive (4.11) from (4.12), it is therefore sufficient to observe that thanks to our hypothesis (4.10),

$$\beta:=\Big\{\frac{1}{2}+\frac{n}{2}\Big(\frac{1}{\sigma}-\frac{1}{q}\Big)\Big\}\cdot\frac{\sigma}{\sigma-\alpha\sigma+p}$$

has the property that

$$\beta<\Big\{\frac{1}{2}+\frac{n}{2}\Big(\frac{1}{\sigma}-\frac{2\alpha\sigma-\sigma-2p+n}{n\sigma}\Big)\Big\}\cdot\frac{\sigma}{\sigma-\alpha\sigma+p}=1,$$

so that

$$\begin{aligned}
\int_0^t \Big\{1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{q})}\Big\}^{\frac{\sigma}{\sigma-\alpha\sigma+p}} e^{-(t-s)}ds &\le 2^{\frac{\sigma}{\sigma-\alpha\sigma+p}}\cdot\Big\{\int_0^t e^{-(t-s)}ds+\int_0^t (t-s)^{-\beta}e^{-(t-s)}ds\Big\} \\
&\le 2^{\frac{\sigma}{\sigma-\alpha\sigma+p}}\cdot\Big\{1+\Gamma(1-\beta)\Big\}
\end{aligned}$$

for all $t>0$. □

Upon optimizing the choice of σ here, we obtain a corresponding statement that will allow for somewhat more convenient handling in the sequel.

Corollary 4.5 *Let $p\ge 1$ belong to S, and let $q\ge 1$ be such that*

$$q<\begin{cases} \frac{n}{2\alpha-2p+n-1} & \textit{if } p>\frac{n}{2}, p>\alpha-\frac{1}{2} \textit{ and } p<\alpha, \\ \frac{np}{(n\alpha-p)_+} & \textit{if } p>\frac{n}{2} \textit{ and } p\ge\alpha, \\ \frac{n(p+1)}{[(n+2)\alpha-p-1]_+} & \textit{if } p\le\frac{n}{2} \textit{ and } p>2\alpha-1. \end{cases} \tag{4.13}$$

Then there exists $C(p,q)>0$ fulfilling

$$\|v(\cdot,t)\|_{W^{1,q}(\Omega)}\le C(p,q) \qquad \textit{for all } t\in(0,T_{max}). \tag{4.14}$$

PROOF. In view of Lemma 4.4, we only need to make sure that for any $q\ge 1$ fulfilling (4.13) we can find $\sigma\ge 1$ such that $\sigma\in(\frac{p}{\alpha},\frac{p+1}{\alpha}]$ and that both (4.9) and (4.10) hold.

To motivate our selection of σ, guided by (4.10) let us observe that

$$\psi(\sigma):=\frac{2\alpha\sigma-\sigma-2p+n}{n\sigma}, \qquad \sigma>0,$$

satisfies $\psi'(\sigma)=\frac{2p-n}{n\sigma^2}$ for all $\sigma>0$, so that ψ is increasing if $p>\frac{n}{2}$ and nonincreasing when $p\le\frac{n}{2}$. Consequently, if we let $J:=(\frac{p}{\alpha},\frac{p+1}{\alpha}]\cap[1,\infty)$, then

$$\inf_{\sigma\in J}\psi(\sigma)=\begin{cases} \psi(1) & \text{if } \frac{n}{2}<p<\alpha, \\ \psi(\frac{p}{\alpha}) & \text{if } p>\frac{n}{2} \text{ and } p\ge\alpha, \\ \psi(\frac{p+1}{\alpha}) & \text{if } p\le\frac{n}{2}. \end{cases} \tag{4.15}$$

In particular, since $\psi(1)=\frac{2\alpha-2p+n-1}{n}>0$, in the case when $p>\frac{n}{2}$ is such that $p>\alpha-\frac{1}{2}$ and $p<\alpha$, we take $\sigma=1$ in Lemma 4.4 to infer that any $q\ge 1$ fulfilling $q<\frac{n}{2\alpha-2p+n-1}$ satisfies (4.10). Noting that in this case the restriction $p>\alpha-\frac{1}{2}$ guarantees that our choice of σ is moreover consistent with (4.9), we see that indeed (4.14) results from Lemma 4.4 in this range of p.

Next, if $p>\frac{n}{2}$ and $p\ge\alpha$, computing

$$\psi\Big(\frac{p}{\alpha}\Big)=\frac{n\alpha-p}{np}$$

and observing that

$$2p-(2\alpha-1)\cdot\frac{p}{\alpha}=\frac{p}{\alpha}>0,$$

by means of (4.15) and a continuity argument we conclude that given any $q \ge 1$ such that $q < \frac{np}{(n\alpha-p)_+}$ we can pick $\sigma \in J$ such that besides (4.10) also (4.9) holds, and that hence (4.14) again becomes a consequence of Lemma 4.4.

Likewise, if $p \le \frac{n}{2}$ is such that $p > 2\alpha - 1$, then in view of (4.15) we choose $\sigma = \frac{p+1}{\alpha}$ to see that then any $q < \frac{n(p+1)}{[(n+2)\alpha-p-1]_+}$ satisfies

$$\frac{1}{q} > \frac{(n+2)\alpha - p - 1}{n(p+1)} = \psi\Big(\frac{p+1}{\alpha}\Big),$$

and that furthermore

$$2p - (2\alpha-1)\sigma = \frac{p-(2\alpha-1)}{\alpha} > 0,$$

so that the claim can be derived from Lemma 4.4 also in this case. $\square$

4.3 Estimates suitable for Lemma 3.3 via interpolation

We next interpolate between the inequalities provided by Lemma 4.3 and Corollary 4.5 to derive bounds for v which can directly be used in Lemma 3.3. First concentrating on the case $n \ge 2$, by exclusively relying on the choice $\sigma := \frac{p+1}{\alpha}$ in Lemma 4.3 we will thereby obtain the following.

Lemma 4.6 *Let $n \ge 2$ and $\alpha \in (\frac{1}{2}, 1)$, and suppose that $p \ge 1$ belongs to S. Then for all $r > 1$ fulfilling*

$$r < \begin{cases} \frac{(n+2)(p+1)}{(n+2)\alpha-p-1} & \text{if } p \le \frac{n}{2}, \\ \frac{n(p+1)}{(n\alpha-p)_+} & \text{if } p > \frac{n}{2}, \end{cases} \tag{4.16}$$

one can find $C(p,r) > 0$ such that

$$\int_t^{t+\tau} \|v(\cdot,s)\|_{W^{1,r}(\Omega)}^r ds \le C(p,r) \qquad \text{for all } t \in (0, T_{max}-\tau), \tag{4.17}$$

where again $\tau = \min\{1, \frac{1}{2}T_{max}\}$.

Proof. We first note that our assumption that $\alpha < 1$ ensures that both $p \ge \alpha > \alpha - 1$ and $p > 2\alpha - 1$, which allows us to employ Corollary 4.5 as well as Lemma 4.3, regardless of the sign of $p - \frac{n}{2}$. Indeed, when applied to $\sigma := \frac{p+1}{\alpha}$ the latter provides $c_1 = c_1(p) > 0$ such that

$$\int_t^{t+\tau} \|v(\cdot,s)\|_{W^{2,\frac{p+1}{\alpha}}(\Omega)}^{\frac{p+1}{\alpha}} ds \le c_1 \qquad \text{for all } t \in (0, T_{max}-\tau), \tag{4.18}$$

which we combine with the outcome of Corollary 4.5 as follows:

In the case $p \le \frac{n}{2}$, observing that then necessarily $(n+2)\alpha - p - 1 > \frac{n+2}{2} - \frac{n}{2} - 1 = 0$ due to our restriction that $\alpha > \frac{1}{2}$, given any $r > 1$ such that $r < \frac{(n+2)(p+1)}{(n+2)\alpha-p-1}$ we see that the latter inequality ensures that

$$\frac{n\alpha r}{p+1} - n < \frac{n\alpha}{p+1} \cdot \frac{(n+2)(p+1)}{(n+2)\alpha-p-1} - n = \frac{n(p+1)}{(n+2)\alpha-p-1}$$

and moreover, in particular, warrants that

$$\frac{(n\alpha-p-1)r}{p+1}-n<\max\Big\{0\,,\,\frac{(n\alpha-p-1)(n+2)}{(n+2)\alpha-p-1}-n\Big\}=\max\Big\{0\,,\,\frac{-2(p+1)}{(n+2)\alpha-p-1}\Big\}=0 \tag{4.19}$$

and that hence

$$\Big(\frac{n\alpha r}{p+1}-n\Big)-r<0. \tag{4.20}$$

As furthermore clearly

$$\frac{n(p+1)}{(n+2)\alpha-p-1}-1=\frac{(n+1)(p+1)-(n+2)\alpha}{(n+2)\alpha-p-1}>\frac{2(n+1)-(n+2)}{(n+2)\alpha-p-1}=\frac{n}{(n+2)\alpha-p-1}>0,$$

we thus see that it is possible to pick $q=q(p,r)\ge 1$ such that

$$q<\frac{n(p+1)}{(n+2)\alpha-p-1} \tag{4.21}$$

and

$$q<r, \tag{4.22}$$

but that on the other hand

$$q>\frac{n\alpha r}{p+1}-n. \tag{4.23}$$

Now according to (4.21), Corollary 4.5 applies so as to yield $c_2=c_2(p,r)>0$ such that

$$\|v(\cdot,t)\|_{W^{1,q}(\Omega)}\le c_2 \qquad \text{for all } t\in(0,T_{max}), \tag{4.24}$$

whereas (4.19) along with (4.22) ensures continuity of the embeddings $W^{2,\frac{p+1}{\alpha}}(\Omega)\hookrightarrow W^{1,r}(\Omega)\hookrightarrow W^{1,q}(\Omega)$, whence based on the Gagliardo-Nirenberg inequality we can find $c_3=c_3(p,r)>0$ such that

$$\|v\|_{W^{1,r}(\Omega)}^r\le c_3\|v\|_{W^{2,\frac{p+1}{\alpha}}(\Omega)}^{rb}\|v\|_{W^{1,q}(\Omega)}^{r(1-b)} \qquad \text{for all } t\in(0,T_{max}) \tag{4.25}$$

with

$$b\equiv b(p,r):=\frac{\frac{n}{q}-\frac{n}{r}}{1+\frac{n}{q}-\frac{n\alpha}{p+1}}\in(0,1) \tag{4.26}$$

thanks to (4.22) and (4.20). As (4.23) guarantees that $nr<\frac{(p+1)(n+q)}{\alpha}$ and thus

$$rb=\frac{\frac{nr}{q}-n}{1+\frac{n}{q}-\frac{n\alpha}{p+1}}<\frac{\frac{(p+1)(n+q)}{q\alpha}-n}{1+\frac{n}{q}-\frac{n\alpha}{p+1}}=\frac{p+1}{\alpha}, \tag{4.27}$$

the estimate (4.24) together with (4.18) implies that

$$\begin{aligned}
\int_t^{t+\tau}\|v(\cdot,s)\|_{W^{1,r}(\Omega)}^r ds &\le c_3\int_t^{t+\tau}\|v(\cdot,s)\|_{W^{2,\frac{p+1}{\alpha}}(\Omega)}^{rb}\|v(\cdot,s)\|_{W^{1,q}(\Omega)}^{r(1-b)}ds\\
&\le c_2^{r(1-b)}c_3\int_t^{t+\tau}\|v(\cdot,s)\|_{W^{2,\frac{p+1}{\alpha}}(\Omega)}^{rb}ds\\
&\le c_2^{r(1-b)}c_3\cdot\Big\{\tau+\int_t^{t+\tau}\|v(\cdot,s)\|_{W^{2,\frac{p+1}{\alpha}}(\Omega)}^{\frac{p+1}{\alpha}}ds\Big\}\\
&\le c_2^{r(1-b)}c_3\cdot(\tau+c_1) \qquad \text{for all } t\in(0,T_{max}-\tau),
\end{aligned} \tag{4.28}$$

as claimed.

Next, in the case when $\frac{n}{2} < p < n\alpha$ we proceed quite similarly, first verifying that then (4.16) implies that $\frac{n\alpha r}{p+1} - n < \frac{np}{n\alpha-p}$, and noting that (4.19) and (4.20) remain unchanged, whereas evidently $\frac{np}{n\alpha-p} > 1$ due to the fact that $\frac{n\alpha}{n+1} < \frac{n}{n+1} < 1 \le p$. We can therefore fix a number $q = q(p,r)$ which again has the properties that $1 \le q < r$ and that (4.23) holds, and which moreover satisfies $q < \frac{np}{n\alpha-p}$ as well as (4.27) with b as accordingly defined through (4.26), the latter statement being valid due to (4.23). Consequently, we may argue precisely as in our derivation of (4.24) and (4.28) to infer that (4.17) is valid also in this case.

Finally, if $p \ge n\alpha$ and $r \in (1,\infty)$ is arbitrary, the claim directly results on applying Corollary 4.5 to $q := r$. $\square$

Our one-dimensional counterpart thereof, now choosing σ in Lemma 4.3 significantly smaller than before, reads as follows.

Lemma 4.7 *Let $n = 1$ and $\alpha \in (1,2)$, and assume that $p \ge 1$ is such that $p \in S$, and that $r > 1$ is such that*

$$r < \frac{4}{(4\alpha - 4p - 1)_+}. \tag{4.29}$$

Then there exists $C(p,r) > 0$ with the property that

$$\int_t^{t+\tau} \|v(\cdot,s)\|_{W^{1,2}(\Omega)}^r ds \le C(p,r) \qquad \text{for all } t \in (0, T_{max} - \tau), \tag{4.30}$$

where still $\tau = \min\{1, \frac{1}{2}T_{max}\}$.

Proof. First, if $p > \alpha - \frac{1}{4}$, then we observe that $\frac{1}{2(\alpha-p)_+} > 2$ and hence conclude from Corollary 4.5 that in both cases $p \ge \alpha$ and $p \in (\alpha - \frac{1}{4}, \alpha)$ we can find $c_1 = c_1(p) > 0$ such that

$$\|v(\cdot,t)\|_{W^{1,2}(\Omega)} \le c_1 \qquad \text{for all } t \in (0, T_{max}),$$

from which (4.30) immediately follows.

If $p \le \alpha - \frac{1}{4}$, then given any $r > 1$ fulfilling (4.29) we see that

$$\Big(2 - \frac{2}{(\alpha-p)r}\Big) - \frac{1}{2(\alpha-p)} = \frac{(4\alpha - 4p - 1)r - 4}{2(\alpha-p)r} < 0,$$

so that it is possible to fix $q \in (0,2)$ such that

$$2 - \frac{2}{(\alpha-p)r} < q < \frac{1}{2(\alpha-p)}. \tag{4.31}$$

Since the left inequality herein ensures that $r < \frac{2}{(\alpha-p)(2-q)}$, and since on the other hand

$$\rho(\sigma) := \frac{2(q\sigma + \sigma - q)}{(\alpha\sigma - p)(2-q)}, \qquad \sigma > 1,$$

satisfies $\rho(\sigma) \to \frac{2}{(\alpha-p)(2-q)}$ as $\sigma \searrow 1$, noting that $\frac{p+1}{\alpha} \ge \frac{2}{\alpha} > 1$ we can finally pick $\sigma \in (1, \frac{p+1}{\alpha}]$ such that still

$$r < \rho(\sigma). \tag{4.32}$$

Next, thanks to the second inequality in (4.31) we may apply Corollary 4.5 to find $c_2 = c_2(p,r) > 0$ such that

$$\|v(\cdot,t)\|_{W^{1,q}(\Omega)} \le c_2 \qquad \text{for all } t \in (0, T_{max}), \tag{4.33}$$

whereas the inclusions $\sigma \in (1, \frac{p+1}{\alpha}] \subset (\frac{p}{\alpha}, \frac{p+1}{\alpha}]$ enable us to invoke Lemma 4.3 to see that there exists $c_3 = c_3(p,r) > 0$ such that

$$\int_t^{t+\tau} \|v(\cdot,s)\|_{W^{2,\sigma}(\Omega)}^{\frac{\sigma}{\alpha\sigma-p}} ds \le c_3 \qquad \text{for all } t \in (0, T_{max}-\tau). \tag{4.34}$$

Now combining the Gagliardo-Nirenberg inequality with (4.33) shows that with some $c_4 = c_4(p,r) > 0$ and with $b = b(p,r) := (\frac{1}{q} - \frac{1}{2})/(1 + \frac{1}{q} - \frac{1}{\sigma})$ we have

$$\begin{aligned} \int_t^{t+\tau} \|v(\cdot,s)\|_{W^{1,2}(\Omega)}^r ds &\le c_4 \int_t^{t+\tau} \|v(\cdot,s)\|_{W^{2,\sigma}(\Omega)}^{rb} \|v(\cdot,s)\|_{W^{1,q}(\Omega)}^{r(1-b)} ds \\ &\le c_2^{r(1-b)} c_4 \int_t^{t+\tau} \|v(\cdot,s)\|_{W^{2,\sigma}(\Omega)}^{rb} ds \qquad \text{for all } t \in (0, T_{max}-\tau). \end{aligned}$$

As (4.32) guarantees that herein

$$rb < \frac{\rho(\sigma)\cdot\left(\frac{1}{q}-\frac{1}{2}\right)}{1+\frac{1}{q}-\frac{1}{\sigma}} = \frac{\sigma}{\alpha\sigma-p},$$

by means of Young's inequality we may rely on (4.34) in estimating

$$\begin{aligned} \int_t^{t+\tau} \|v(\cdot,s)\|_{W^{1,2}(\Omega)}^r ds &\le c_2^{r(1-b)} c_4 \cdot \left\{ \tau + \int_t^{t+\tau} \|v(\cdot,s)\|_{W^{2,\sigma}(\Omega)}^{\frac{\sigma}{\alpha\sigma-p}} ds \right\} \\ &\le c_2^{r(1-b)} c_4 \cdot (\tau + c_3) \qquad \text{for all } t \in (0, T_{max}-\tau), \end{aligned}$$

as intended. □

4.4 Closing the circle. Proof of Theorem 1.1

We are now prepared to make sure that the upper bounds on α from Theorem 1.1 are sufficient to ensure that, through Lemma 3.3 and our results from Section 4.3, the set S from (4.1) indeed cannot be bounded.

Specifically, in the multi-dimensional case Lemma 4.6 and Lemma 3.3 entail the following.

Lemma 4.8 *Let $n \ge 2$, and assume that $\alpha > \frac{1}{2}$ is such that*

$$\alpha < \frac{n+6}{2(n+2)}. \tag{4.35}$$

Then there exists $\delta > 0$ with the property that whenever $p \ge 1$ is such that $p \in S$, we also have $p + \delta \in S$.

Proof. Since (4.35) ensures that both $n+6-2(n+2)\alpha$ and $1-\alpha$ are positive, and since $\alpha > \frac{1}{2}$ entails that also $n\alpha - 1 \ge 2\alpha - 1 > 0$, we can fix $\delta > 0$ such that

$$\delta < \frac{2[n+6-2(n+2)\alpha]}{n+2} \tag{4.36}$$

and

$$\delta < \frac{2n(1-\alpha)}{n\alpha - 1}. \tag{4.37}$$

Then assuming that $p \ge 1$ is such that p belongs to S, in the case when $p \le \frac{n}{2}$ we use that by (4.35), $(n+2)\alpha - 2 \le (n+2)\cdot\frac{n+6}{2(n+2)} - 2 = \frac{n+2}{2}$ to see that as a consequence of (4.36),

$$\begin{aligned}
\Big\{2(p+\delta)+2\Big\} - \frac{(n+2)(p+1)}{(n+2)\alpha - p - 1} &= 2\delta - \frac{[n+2p+4-2(n+2)\alpha](p+1)}{(n+2)\alpha - p - 1} \\
&\le 2\delta - \frac{[n+6-2(n+2)\alpha]\cdot 2}{(n+2)\alpha - 2} \\
&\le 2\delta - \frac{4[n+6-2(n+2)\alpha]}{n+2} \\
&< 0 \qquad \text{if } p \le \frac{n}{2}.
\end{aligned} \tag{4.38}$$

In presence of larger p, we similarly obtain

$$\Big\{2(p+\delta)+2\Big\} - \frac{n(p+1)}{(n\alpha - p)_+} < 0 \qquad \text{if } p > \frac{n}{2}, \tag{4.39}$$

which in fact is trivial when $p \ge n\alpha$, while in the case $p < n\alpha$ we can derive (4.39) by using (4.37) according to

$$\begin{aligned}
\Big\{2(p+\delta)+2\Big\} - \frac{n(p+1)}{n\alpha - p} &= 2\delta - \frac{(n+2p-2n\alpha)(p+1)}{n\alpha - p} \\
&\le 2\delta - \frac{(2n-2n\alpha)\cdot 2}{n\alpha - p} \\
&< 0.
\end{aligned}$$

Now on the basis of (4.38) and (4.39), we can employ Lemma 4.6 to infer that in both cases $1 \le p \le \frac{n}{2}$ and $p > \frac{n}{2}$ we can find $c_1 = c_1(p) > 0$ such that

$$\int_t^{t+\tau} \|v(\cdot,s)\|_{W^{1,2(p+\delta)+2}(\Omega)}^{2(p+\delta)+2} ds \le c_1 \qquad \text{for all } t \in (0, T_{max} - \tau)$$

with $\tau = \min\{1, \frac{1}{2}T_{max}\}$. An application of Lemma 3.3 therefore shows that also $p+\delta \in S$. □

Similarly, combining Lemma 4.7 with Lemma 3.3 yields a one-dimensional analogue of the latter:

Lemma 4.9 *Suppose that $n = 1$ and that $\alpha > 1$ is such that*

$$\alpha < \frac{3}{2}. \tag{4.40}$$

Then one can find $\delta > 0$ such that if $p \ge 1$ belongs to S, then also $p + \delta \in S$.

PROOF. Since $\alpha < \frac{3}{2}$, the function

$$\phi(z) := 4z^2 + (5-4\alpha)z + 3 - 4\alpha, \qquad z \ge 1,$$

has the properties that $c_1 := \phi(1) = 12 - 8\alpha > 0$ and

$$\phi'(z) = 8z + 5 - 4\alpha \ge 13 - 4\alpha > 0 \qquad \text{for all } z > 1.$$

Therefore, $\phi(z) \ge c_1$ for all $z \ge 1$, so that we may pick $\delta > 0$ small enough fulfilling

$$\delta < \frac{c_1}{(4\alpha - 5)_+}. \tag{4.41}$$

By definition of ϕ, this ensures that for any choice of $p \ge 1$ we have

$$2(p+\delta) + 2 < \frac{4}{(4\alpha - 4p - 1)_+}, \tag{4.42}$$

because in the nontrivial case when $1 \le p < \alpha - \frac{1}{4}$, using (4.41) we see that

$$\begin{aligned} \Big\{2(p+\delta)+2\Big\} - \frac{4}{4\alpha - 4p - 1} &= 2\delta + \frac{2(p+1)(4\alpha - 4p - 1) - 4}{4\alpha - 4p - 1} \\ &= 2\delta - \frac{2\phi(p)}{4\alpha - 4p - 1} \\ &\le 2\delta - \frac{2c_1}{(4\alpha - 5)_+} \\ &< 0. \end{aligned}$$

Now according to (4.42), Lemma 4.7 applies so as to yield $c_2 = c_2(p) > 0$ such that with $\tau = \min\{1, \frac{1}{2}T_{max}\}$ we have

$$\int_t^{t+\tau} \|v(\cdot,s)\|_{W^{1,2}(\Omega)}^{2(p+\delta)+2} ds \le c_2 \qquad \text{for all } t \in (0, T_{max} - \tau),$$

whence the claim again results from Lemma 3.3. □

By straightforward application of the previous two results, and by once more recalling (2.4) and (2.5), we can finally verify our main result on global existence and boundedness in (1.1):

PROOF of Theorem 1.1. Using (2.4) and (2.5) together with (2.1), we see that the set S from (4.1) is not empty, for $1 \in S$. Furthermore, since without loss of generality we may assume that $\alpha > 1$ if $n = 1$, and that $\alpha > \frac{1}{2}$ whenever $n \ge 2$, through Lemma 4.9 and Lemma 4.8 the hypothesis (1.7) warrants that in both cases $n = 1$ and $n \ge 2$ we can find $\delta > 0$ such that $S + \delta \subset S$, which clearly entails that S is unbounded. In particular, this implies that for all $p \ge 1$ there exists $c_1 = c_1(p) > 0$ such that

$$\int_\Omega a^p(\cdot,t) \le c_1(p) \qquad \text{for all } t \in (0, T_{max}).$$

An adaptation of the Moser-Alikakos iteration procedure, e.g. in the style of that used in [23, Lemma 3.12], thereafter yields $c_2 > 0$ such that actually

$$\|a(\cdot,t)\|_{L^\infty(\Omega)} \le c_2 \qquad \text{for all } t \in (0,T_{max}),$$

whereupon both the statement on global existence and the estimate in (1.8) result by applying Lemma 2.3 and once more recalling that $e^{\xi w} \le e^{\xi M}$ in $\Omega \times (0,T_{max})$ according to (2.6). □

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