Large time behavior in a forager-exploiter model with different taxis strategies for two groups in search of food

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Abstract

This work deals with a taxis cascade model for food consumption in two populations, namely foragers directly orienting their movement upwards gradients of food concentration, and exploiters taking a parasitic strategy in search of food via tracking higher forager densities. As a consequence, the dynamics of both populations is adapted to the space distribution of food which is dynamically modified in time and space by the two populations. This model extends classical one-species chemotaxis-consumption systems by additionally accounting for a second taxis mechanism coupled to the first in a consecutive manner.

It is rigorously proved that for all suitably regular initial data, an associated Neumann-type initial-boundary value problem for the spatially one-dimensional version of this model possesses a globally defined bounded classical solution. Moreover, it is asserted that the considered two populations will approach spatially homogeneous distributions in the large timelimit, provided that either the total population number of foragers *or* that of exploiters is appropriately small.

Key words: chemotaxis; global existence; large time behavior
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1 Introduction

It is well-known that social interactions in mixed-species groups may lead to rich spatial patterns, already in some cases of populations consisting of merely two fractions, and the ambition to understand fundamental principles for such complex dynamics has been attracting considerable interest during the past decade in the literature near the borderline regions between, inter alia, biology, behavioral sciences and mathematics (cf. [10], [20], [12] and [4], for instance). Since living systems are complex and usually involve processes at quite different scales, the modeling of such systems is quite challenging and the corresponding mathematical descriptions vary from micro- and mesoscopic to macroscopic models depending on various assumptions and on different mathematical tools (see [1] for a recent review). Microscopic models are individual-based and developed by kinetic theory, such as the notable Boltzmann equation; macroscopic models commonly describe collective dynamics and are derived from hydrodynamic theory; and mesoscopic models bridge microscopic with macroscopic models. Remarkable collective behavior in self-propelled systems often appear in our natural biological systems, such as flocking of birds.

With the macroscopic formation of shearwater flocks through attraction to kittiwake foragers in Alaska forming a paradigmatic example, a rather clearly traceable type of social interplay, in the comparatively simple case of only two involved species, is constituted by so-called forager-exploiter interaction: The members of a first population, the "foragers", search for food by directly moving upward gradients of the nutrient concentration; contrary to this, as "exploiters" the individuals of a second population pursue a more indirect strategy by rather orienting their movement toward regions of higher forager population densities. The interaction between foragers and exploiters generates rich dynamics and in this regard several spatial models provide novel insights into the evolution of group foraging ([18], [20], [22]). The N-persons forager-exploiter game model constructed by Vickery et al. in [22] explores the frequency of occurrence of the different foraging strategies and predicts that as group size increases, the frequency of the scrounging strategy should increase. The model in [22] assumes that the number of discovered food patches in one time unit never exceeds one. By relaxing this assumption and assuming that the number of discovered food patches depends on the foragers' variable encounter rate with patches, the authors in [18] proposed a modified forager-exploiter game model which reveals that the proportion of scroungers will increase when resource is spatiotemporally clumped, but that scroungers will decrease if the resource is evenly distributed. Under the assumption that the whole group size is fixed, in [20] a taxis cascade model was developed, and it was found that taxis in producer-scrounger groups can give rise to pattern formation; more precisely, fluctuating spatiotemporal patterns can be sustained between such groups.

Besides presupposing that foragers and scroungers as well as the food resources undergo random diffusion, the model in [20] assumes the latter to be degraded by both foragers and scroungers upon contact, and to possibly be supplied by an external source and spontaneously decaying. According to the above, foragers move up food gradients, whereas scroungers follow gradients of the forager density. Based on these assumptions, as a macroscopic model for the spatio-temporal evolution of the population densities $u = u(x,t)$ and $v = v(x,t)$ of foragers and scroungers in such contexts, additionally accounting for the nutrient density $w = w(x,t)$ as a third unknown the authors in [20] propose the parabolic PDE system

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla w), \\ v_t = \Delta v - \chi_2 \nabla \cdot (v \nabla u), \\ w_t = d\Delta v - \lambda(u+v)w - \mu w + r, \end{cases} \tag{1.1}$$

with positive parameters χ_1, χ_2, d and λ, and with $\mu \geq 0$ and $r \geq 0$. In its most characteristic part, (1.1) accounts for the group-specific strategies of directed motion by postulating taxis-type cross-diffusion mechanisms to be responsible in both cases.

As a particular feature thereby generated, (1.1) contains a sequential coupling of two taxis processes, which may be expected to considerably increase the mathematical complexity of (1.1) when compared, for instance, to the corresponding one-species chemotaxis-consumption system, depending on the application context sometimes also referred to as prey-taxis system ([16], [17], [31]), that is obtained on letting $v \equiv 0$ in (1.1), and that is hence given by

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla w), \\ w_t = d\Delta v - \lambda uw - \mu w + r. \end{cases} \tag{1.2}$$

In the prototypical case when $\mu = r = 0$, namely, the structure of the latter is artless enough so as to allow for a meaningful energy structure that can be used as a technical basis for a comprehensive theory of global classical well-posedness in low-dimensional boundary value problems in which the spatial dimension n satisfies $n \leq 2$, of global weak solvability when $n = 3$, and even of asymptotic stabilization toward spatially homogeneous equilibria whenever $n \leq 3$ ([21]). Adaptations of such approaches have been utilized to address several variants of (1.2), partially even involving further components and interaction mechanisms ([15], [31], [28], [26]), but capturing the additional intricacy induced by the second taxis interaction in (1.1) seems beyond the abilities of such methods.

Due to the presence of the first taxis mechanism in (1.1), only little information on the spatial regularity of u seems available. In line with this, the sequential taxis process involving ∇u in the second equation in (1.1) seems to increase the mathematical delicacy of (1.1) quite substantially. In particular, to the best of our knowledge it is yet left completely open by the analytical literature how far the coupling of the nutrient taxis mechanism from (1.2) to a further cross-diffusion process sensitive to the gradient of the first population may lead to substantial destabilization of the tendency toward homogeneity, as known to occur in (1.2); Indeed, when $n \leq 2$, neither any blow-up phenomenon nor any nontrivial pattern formation can occur in (1.2), as asserted in [21]; but the same issue for (1.1) largely remains unknown due to its increased complexity. Only under some restrictive assumptions on r and the initial data $w|_{t=0}$, in essence ensuring that w remains below a suitably small threshold throughout evolution, it has recently been possible to achieve some results on global existence of certain generalized solutions to (1.1) in multi-dimensional cases, as well as on their large time stabilization toward constants when moreover $r = r(x,t)$ decays suitably in time ([30]).

On the other hand, numerical simulations as well as formal considerations in [20] indicate that the quantity of the total population $\int_\Omega u$ of foragers or of the total population $\int_\Omega v$ of scroungers plays a crucial role in determining the large time behavior or formation of patterns of two groups, clearly providing motivation to confirm such biologically important features by means of rigorous analysis.

Apart from that, in light of the strong singularity-supporting potential of chemotactic cross-diffusion, well-known as a striking feature e.g. of the classical Keller-Segel system ([13], [25]), already answers to questions from basic theory of global solvability in (1.1) thus seem far from obvious, even in the simplest case in which $n = 1$; in fact, the numerical experiments reported in [20] indicate quite a rich dynamical potential of (1.1) already in such one-dimensional frameworks.

Main results. The purpose of the present work now consists in making sure that the evident mathematical challenges notwithstanding, and especially despite the apparent lack of any favorable energy structure, at least the one-dimensional version of (1.1) does not only allow for a rather comprehensive theory of classical solvability, but beyond this is even accessible to an essentially exhaustive qualitative analysis in parameter constellations for which formal considerations predict asymptotic homogenization: In particular, we shall develop an analytical approach which firstly enables us to assert global existence of bounded classical solutions for widely arbitrary initial data, and which secondly is subtle enough so as to allow for a conclusion on large time stabilization toward constant steady states under an additional smallness assumption on the total population sizes of foragers and scroungers that quite precisely seems to match a corresponding condition formally identified as essentially necessary and sufficient therefor by means of a linear stability analysis in [20].

To take this more precise, in a bounded open interval $\Omega \subset \mathbb{R}$ let us consider the initial-boundary value problem for (1.1) given by

$$\begin{cases} u_t &= u_{xx} - \chi_1 (u w_x)_x, & x \in \Omega,\ t > 0, \\ v_t &= v_{xx} - \chi_2 (v u_x)_x, & x \in \Omega,\ t > 0, \\ w_t &= d w_{xx} - \lambda(u+v)w - \mu w + r, & x \in \Omega,\ t > 0, \\ u_x = v_x = w_x = 0, & & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), & & x \in \Omega, \end{cases} \tag{1.3}$$

where $\chi_1, \chi_2, d, \lambda$ and μ are positive constants and r is nonnegative, and where the initial data are such that

$$\begin{cases} u_0 \in W^{1,\infty}(\Omega) \text{ is nonnegative with } u_0 \not\equiv 0, \\ v_0 \in W^{1,\infty}(\Omega) \text{ is nonnegative with } v_0 \not\equiv 0, \qquad \text{and that} \\ w_0 \in W^{1,\infty}(\Omega) \text{ is positive in } \overline{\Omega}. \end{cases} \tag{1.4}$$

Then the first of our main results asserts global existence of bounded classical solutions in the following flavor.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}$ be a bounded open interval, and let $\chi_1, \chi_2, d, \lambda$ and μ be positive and r be nonnegative. Then for any choice of (u_0, v_0, w_0) fulfilling (1.4), the problem (1.3) possesses a global classical solution (u, v, w) which is uniquely determined by the properties that*

$$u, v \text{ and } w \text{ belong to } C^0([0,\infty); W^{1,2}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \tag{1.5}$$

and which is such that $u > 0$ and $v > 0$ in $\overline{\Omega} \times (0,\infty)$ as well as $w > 0$ in $\overline{\Omega} \times [0,\infty)$. Moreover, this solution is bounded in the sense that there exists $C > 0$ such that

$$\|u(\cdot,t)\|_{W^{1,2}(\Omega)} + \|v(\cdot,t)\|_{W^{1,2}(\Omega)} + \|w(\cdot,t)\|_{W^{1,2}(\Omega)} \le C \qquad \text{for all } t > 0. \tag{1.6}$$

Next intending to identify circumstances under which the diffusion processes in (1.3) are sufficiently strong so as to warrant relaxation into homogeneous states, we note that in light of corresponding results on asymptotically diffusive behavior in chemotaxis systems with essentially superlinear nonlinear ingredients ([5], [6], [24]) it seems far from audacious to conjecture that such dynamics can always be observed if solutions remain small *in all their components*, and *with respect to suitably fine topologies*, throughout evolution.

In order to approach a more subtle picture in this regard, we recall that a linear stability analysis detailed in [20] suggests to expect, in the normalized case when $\Omega = (0,1)$ and the spatial averages $\overline{u}_0 := \frac{1}{|\Omega|}\int_\Omega u_0$ and $\overline{v}_0 := \frac{1}{|\Omega|}v_0$ satisfying $\overline{u}_0 + \overline{v}_0 = 1$, the relation

$$\frac{8(\lambda+\mu)^2(d+1)}{\lambda r \chi_1 \overline{u}_0 \overline{v}_0} + \frac{2(d+1)}{\overline{v}_0} \gtrsim \chi_2 \tag{1.7}$$

as the decisive condition for prevalence of homogeneity. In particular, this condition is entirely independent of w_0, and moreover it involves u_0 and v_0 exclusively through their L^1 norms as their biologically best interpretable derivate. If, more generally, $\overline{u}_0 + \overline{v}_0$ is supposed to remain below a given number in an arbitrary bounded Ω, then assuming (1.7) evidently becomes equivalent to imposing a certain smallness hypothesis on $\min\{\overline{u}_0, \overline{v}_0\}$. Roughly speaking, this predicts that as long as either the average population of foragers or that of scroungers is suitably small, the two populations as well as food will eventually be homogenously distributed over the spatial habitat. The second of our main results now provides a rigorous mathematical counterpart of this formal consideration.

Theorem 1.2 *Suppose that $\Omega \subset \mathbb{R}$ is a bounded open interval, that $\chi_1, \chi_2, d, \lambda$ and μ are positive, and that $r \ge 0$. Then for all $M > 0$ one can find $\varepsilon(M) > 0$ with the property that whenever u_0, v_0 and w_0 are such that besides (1.4) we have*

$$\int_\Omega u_0 + \int_\Omega v_0 \le M \tag{1.8}$$

as well as

$$\int_\Omega u_0 \le \varepsilon(M) \qquad \textit{or} \qquad \int_\Omega v_0 \le \varepsilon(M), \tag{1.9}$$

there exist $C = C(u_0, v_0, w_0) > 0$ and $\alpha = \alpha(u_0, v_0, w_0) > 0$ such that the solution (u, v, w) of (1.3) satisfies

$$\|u(\cdot,t) - \overline{u}_0\|_{L^\infty(\Omega)} + \|v(\cdot,t) - \overline{v}_0\|_{L^\infty(\Omega)} + \|w(\cdot,t) - w_\star\|_{L^\infty(\Omega)} \le Ce^{-\alpha t} \qquad \textit{for all } t > 0, \tag{1.10}$$

where $w_\star$ is the nonnegative constant given by

$$w_\star := \frac{r}{\lambda(\overline{u}_0 + \overline{v}_0) + \mu}. \tag{1.11}$$

2 Local existence and an explicit L^∞ bound for w

The following basic statement on local existence and extensibility can be obtained from standard theory on evolution systems of parabolic type.

Lemma 2.1 *Suppose that $\Omega \subset \mathbb{R}$ is a bounded open interval, that $\chi_1, \chi_2, d, \lambda$ and μ are positive and $r \ge 0$, and that (1.4) holds. Then there exist $T_{max} \in (0, \infty]$ and nonnegative functions u, v, w, uniquely determined by the requirement that*

$$u, v \text{ and } w \text{ are elements of } C^0([0, T_{max}); W^{1,2}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})), \tag{2.1}$$

which solve (1.3) in the classical sense in $\Omega \times (0, T_{max})$, and which are such that

$$\text{if } T_{max} < \infty, \text{ then} \quad \limsup_{t \nearrow T_{max}} \Big\{ \|u(\cdot,t)\|_{W^{1,q}(\Omega)} + \|v(\cdot,t)\|_{W^{1,q}(\Omega)} + \|w(\cdot,t)\|_{W^{1,q}(\Omega)} \Big\} = \infty \qquad \text{for all } q > 1. \tag{2.2}$$

Moreover, $u > 0$ and $v > 0$ in $\overline{\Omega} \times (0, T_{max})$ and $w > 0$ in $\overline{\Omega} \times [0, T_{max})$, and we have

$$\int_\Omega u(\cdot,t) = \int_\Omega u_0 \quad \text{and} \quad \int_\Omega v(\cdot,t) = \int_\Omega v_0 \qquad \text{for all } t \in (0, T_{max}). \tag{2.3}$$

Proof. Since all eigenvalues of the diffusion matrix

$$\mathcal{A}(u, v, w) := \begin{pmatrix} 1 & 0 & -\chi_1 u \\ -\chi_2 v & 1 & 0 \\ 0 & 0 & d \end{pmatrix}$$

are positive, the general theory on local existence and maximal extension from [2] is applicable to (1.3); in particular, the existence of a uniquely determined maximal classical solution follows from [2, Theorems 14.4 and 14.6], and the extensibility criterion (2.2) is ensured by [2, Theorem 15.5]. Moreover, the strong maximum principle along with our assumptions $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$ in (1.4) yield positivity of u and v in $\overline{\Omega} \times (0, T_{max})$, whereas a simple comparison argument shows that $w(x,t) > 0$ in $\overline{\Omega} \times [0, T_{max})$ thanks to our assumption that $w_0 > 0$. Finally, the mass conservation properties in (2.3) immediately result from integration in the first equation and the second equation in (1.3), respectively. □

Constituting another basic but important feature of (1.3), the following pointwise bound on w is an immediate consequence of the maximum principle.

Lemma 2.2 *We have*

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{r}{\mu} + \|w_0\|_{L^\infty(\Omega)} e^{-\mu t} \qquad \text{for all } t \in (0, T_{max}). \tag{2.4}$$

Proof. We let $\overline{w}(x,t) := y(t)$ for $x \in \overline{\Omega}$ and $t \ge 0$, where

$$y(t) := y_0 e^{-\mu t} + \frac{r}{\mu}(1 - e^{-\mu t}), \qquad t \ge 0,$$

denotes the solution of $y'(t) + \mu y(t) = r$, $t > 0$, with $y(0) = y_0 := \|w_0\|_{L^\infty(\Omega)}$. Then $\overline{w}(\cdot, 0) = y_0 \ge w(\cdot, 0)$ in Ω as well as $\frac{\partial \overline{w}}{\partial \nu} = 0$ on $\partial\Omega \times (0, \infty)$. As moreover, by nonnegativity of λ, u and v,

$$\begin{aligned} \overline{w}_t - d\overline{w}_{xx} + \lambda(u+v)w + \mu\overline{w} - r &\ge \overline{w}_t - d\overline{w}_{xx} + \mu\overline{w} - r \\ &\ge y' + \mu y - r = 0 \qquad \text{in } \Omega \times (0, T_{max}), \end{aligned}$$

by means of a comparison argument we conclude that $w \le \overline{w}$ in $\Omega \times (0, T_{max})$, and that thus

$$w(x,t) \le y_0 e^{-\mu t} + \frac{r}{\mu} \qquad \text{for all } x \in \Omega \text{ and } t \in (0, T_{max}),$$

which is equivalent to (2.4). □

3 Further estimates. Linking regularity to the size of $\int_\Omega u_0 + \int_\Omega v_0$

The goal of this section is to reveal further regularity properties of the above local solution which on the one hand will allow for its global extension, but which on the other will also prepare our subsequent qualitative analysis. For this purpose, the dependence of the obtained estimates on the averages $\overline{u}_0$ and $\overline{v}_0$ as well as on their sum, as appearing in (1.8), will carefully be traced throughout this section.

3.1 A bound for w_x in L^q

In a first step we shall employ parabolic smoothing estimates to see that the mere mass conservation properties in (2.3) entail L^q bounds for the chemotactic gradient acting in the first equation from (1.3). In not requiring any restriction other than that q be finite, we here already make essential use of our assumption on the spatial framework to be one-dimensional.

Lemma 3.1 *There exists $\alpha > 0$ such that for all $M > 0$ and any $q > 1$ one can find $K(M,q) > 0$ with the property that whenever u_0, v_0 and w_0 satisfy (1.4) as well as (1.8), there exists $C = C(u_0, v_0, w_0) > 0$ such that*

$$\|w_x(\cdot,t)\|_{L^q(\Omega)} \le K(M,q) + Ce^{-\alpha t} \qquad \textit{for all } t \in (0, T_{max}). \tag{3.1}$$

Proof. Relying on known regularization features of the Neumann heat semigroup $(e^{\sigma\Delta})_{\sigma\ge 0}$ on Ω with $\Delta := (\cdot)_{xx}$ ([24]), let us fix $c_1(q) > 0$ and $c_2(q) > 0$ such that for all $t > 0$,

$$\|\partial_x e^{dt\Delta}\varphi\|_{L^q(\Omega)} \le c_1(q)\|\varphi\|_{W^{1,\infty}(\Omega)} \qquad \text{for all } \varphi \in W^{1,\infty}(\Omega) \tag{3.2}$$

and

$$\|\partial_x e^{dt\Delta}\varphi\|_{L^q(\Omega)} \le c_2(q)(1 + t^{-1+\frac{1}{2q}})\|\varphi\|_{L^1(\Omega)} \qquad \text{for all } \varphi \in C^0(\overline{\Omega}). \tag{3.3}$$

Then representing w according to

$$w(\cdot,t) = e^{t(d\Delta-\mu)}w_0 + \int_0^t e^{(t-s)(d\Delta-\mu)}\Big\{ -\lambda(u(\cdot,s)+v(\cdot,s))w(\cdot,s) + r\Big\}ds, \qquad t \in (0, T_{max}),$$

we can combine (3.2) with (3.3) to estimate

$$\begin{aligned}
&\|w_x(\cdot,t)\|_{L^q(\Omega)} \\
&\quad\le c_1(q)e^{-\mu t}\|w_0\|_{W^{1,\infty}(\Omega)} \\
&\qquad + c_2(q)\int_0^t e^{-\mu(t-s)}\Big(1+(t-s)^{-1+\frac{1}{2q}}\Big)\Big\| -\lambda(u(\cdot,s)+v(\cdot,s))w(\cdot,s) + r\Big\|_{L^1(\Omega)} ds \\
&\quad\le c_1(q)e^{-\mu t}\|w_0\|_{W^{1,\infty}(\Omega)} \\
&\qquad + c_2(q)\lambda\int_0^t e^{-\mu(t-s)}\Big(1+(t-s)^{-1+\frac{1}{2q}}\Big)\cdot\Big\{\|u(\cdot,s)\|_{L^1(\Omega)} + \|v(\cdot,s)\|_{L^1(\Omega)}\Big\}\|w(\cdot,s)\|_{L^\infty(\Omega)}ds \\
&\qquad + c_2(q)r|\Omega|\int_0^t e^{-\mu(t-s)}\Big(1+(t-s)^{-1+\frac{1}{2q}}\Big)ds \qquad \text{for all } t \in (0, T_{max}).
\end{aligned} \tag{3.4}$$

Here using (2.3) along with (1.8) and Lemma 2.2, we find that

$$
\begin{aligned}
c_2(q)\lambda \int_0^t & e^{-\mu(t-s)} \Big(1+(t-s)^{-1+\frac{1}{2q}}\Big) \cdot \Big\{ \|u(\cdot,s)\|_{L^1(\Omega)} + \|v(\cdot,s)\|_{L^1(\Omega)} \Big\} \|w(\cdot,s)\|_{L^\infty(\Omega)} ds \\
&\le c_2(q)\lambda \Big\{ \int_\Omega u_0 + \int_\Omega v_0 \Big\} \cdot \int_0^t e^{-\mu(t-s)} \Big(1+(t-s)^{-1+\frac{1}{2q}}\Big) \cdot \Big\{ \frac{r}{\mu} + \|w_0\|_{L^\infty(\Omega)} e^{-\mu s} \Big\} ds \\
&\le c_2(q)\lambda \Big\{ \int_\Omega u_0 + \int_\Omega v_0 \Big\} \cdot \Big\{ \frac{r}{\mu} \cdot \Big(\frac{2}{\mu} + 2q\Big) + \|w_0\|_{L^\infty(\Omega)} \cdot (te^{-\mu t} + 2qt^{\frac{1}{2q}} e^{-\mu t}) \Big\} \\
&\le \frac{2c_2(q)\lambda M r(1+q\mu)}{\mu^2} + c_2(q)\lambda M \|w_0\|_{L^\infty(\Omega)} \cdot (te^{-\mu t} + 2qt^{\frac{1}{2q}} e^{-\mu t}) \qquad \text{for all } t \in (0, T_{max})
\end{aligned}
$$

because

$$
\int_0^t e^{-\mu(t-s)} \Big(1+(t-s)^{-1+\frac{1}{2q}}\Big) ds = \int_0^t e^{-\mu s}(1+s^{-1+\frac{1}{2q}}) ds \le \int_0^\infty e^{-\mu s}(1+s^{-1+\frac{1}{2q}}) ds \le \frac{2}{\mu} + 2q
$$

and

$$
\int_0^t e^{-\mu t} \Big(1+(t-s)^{-1+\frac{1}{2q}}\Big) ds = te^{-\mu t} + 2qt^{\frac{1}{2q}} e^{-\mu t},
$$

whence moreover estimating

$$
te^{-\mu t} \le \frac{2}{\mu} e^{-\frac{\mu}{2}t} \qquad \text{as well as} \qquad 2qt^{\frac{1}{2q}} e^{-\mu t} \le 2q\Big(\frac{1}{\mu q}\Big)^{\frac{1}{2q}} e^{-\frac{\mu}{2}t}
$$

for $t > 0$, from (3.4) we infer that

$$
\begin{aligned}
\|w_x(\cdot,t)\|_{L^q(\Omega)} \le\ & \frac{2c_2(q)\lambda M r(1+q\mu)}{\mu^2} + \frac{2c_2(q) r(1+q\mu)|\Omega|}{\mu} \\
& + \Big\{ c_1(q)\|w_0\|_{W^{1,\infty}(\Omega)} + c_2(q)\lambda M \Big[\frac{2}{\mu} + 2q\Big(\frac{1}{\mu q}\Big)^{\frac{1}{2q}}\Big] \|w_0\|_{L^\infty(\Omega)} \Big\} \cdot e^{-\frac{\mu}{2}t}
\end{aligned}
$$

for all $t \in (0, T_{max})$, which is precisely of the claimed form with suitably chosen $K(M,q) > 0$ and $C = C(u_0, v_0, w_0) > 0$, and with $\alpha := \frac{\mu}{2}$. □

3.2 Estimating u in L^∞ and u_x in a spatio-temporal L^2 norm

Through a standard testing procedure performed using the first equation in (1.3), when applied to $q := 2$ the latter has a first consequence on regularity of u as well as its gradient.

Lemma 3.2 *There exists $\alpha > 0$ with the property that given any $M > 0$ one can choose $K(M) > 0$ such that if (1.4) and (1.8) hold, there exists $C = C(u_0, v_0, w_0) > 0$ such that*

$$
\int_\Omega u^2(\cdot,t) \le K(M)\overline{u}_0^2 + Ce^{-\alpha t} \qquad \textit{for all } t \in (0, T_{max}) \tag{3.5}
$$

and

$$
\int_t^{t+\tau} \int_\Omega u_x^2 \le K(M)\overline{u}_0^2 + Ce^{-\alpha t} \qquad \textit{for all } t \in (0, T_{max} - \tau), \tag{3.6}
$$

where $\tau := \min\{1, \frac{1}{2}T_{max}\}$.

PROOF. Let us first apply Lemma 3.1 to $q := 2$ to fix positive constants α and $k_1(M)$ such that whenever (1.4) and (1.8) hold, one can find $c_1 = c_1(u_0, v_0, w_0) > 0$ such that

$$\|w_x(\cdot,t)\|_{L^2(\Omega)} \le k_1(M) + c_1 e^{-\alpha t} \qquad \text{for all } t \in (0, T_{max}), \tag{3.7}$$

where without loss of generality we may assume that $k_1(M) \ge 1$ and $\alpha < \frac{1}{6}$. Apart from that, by means of the Gagliardo-Nirenberg inequality and Young's inequality we can choose $c_2 > 0$, $c_3 > 0$ and $c_4 > 0$ such that

$$\|\varphi\|_{L^\infty(\Omega)} \le c_2 \|\varphi_x\|_{L^2(\Omega)}^{\frac{2}{3}} \|\varphi\|_{L^1(\Omega)}^{\frac{1}{3}} + c_2 \|\varphi\|_{L^1(\Omega)} \qquad \text{for all } \varphi \in W^{1,2}(\Omega), \tag{3.8}$$

that

$$\frac{1}{2}\|\varphi\|_{L^2(\Omega)}^2 \le \frac{1}{4}\|\varphi_x\|_{L^2(\Omega)}^2 + c_3 \|\varphi\|_{L^1(\Omega)}^2 \qquad \text{for all } \varphi \in W^{1,2}(\Omega), \tag{3.9}$$

and that

$$c_2 \chi_1 |\Omega|^{\frac{1}{3}} ab \le \frac{1}{8} a^{\frac{6}{5}} + c_4 b^6 \qquad \text{for all } a \ge 0 \text{ and } b \ge 0. \tag{3.10}$$

Now assuming (1.4) and (1.8), we test the first equation in (1.3) by u to see using the Cauchy-Schwarz inequality as well as (3.7), (3.8), (3.9) and (2.3), that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 + \frac{1}{2}\int_\Omega u^2 + \int_\Omega u_x^2 \\
&= \chi_1 \int_\Omega u u_x w_x + \frac{1}{2}\int_\Omega u^2 \\
&\le \chi_1 \|u\|_{L^\infty(\Omega)} \|u_x\|_{L^2(\Omega)} \|w_x\|_{L^2(\Omega)} + \frac{1}{2}\int_\Omega u^2 \\
&\le \chi_1 \|u\|_{L^\infty(\Omega)} \|u_x\|_{L^2(\Omega)} \cdot \Big\{k_1(M) + c_1 e^{-\alpha t}\Big\} + \frac{1}{2}\int_\Omega u^2 \\
&\le c_2 \chi_1 \|u_x\|_{L^2(\Omega)}^{\frac{5}{3}} \|u\|_{L^1(\Omega)}^{\frac{1}{3}} \cdot \Big\{k_1(M) + c_1 e^{-\alpha t}\Big\} + c_2 \chi_1 \|u_x\|_{L^2(\Omega)} \|u\|_{L^1(\Omega)} \cdot \Big\{k_1(M) + c_1 e^{-\alpha t}\Big\} \\
&\quad + \frac{1}{4}\|u_x\|_{L^2(\Omega)}^2 + c_3 \|u\|_{L^1(\Omega)}^2 \\
&= c_2 \chi_1 |\Omega|^{\frac{1}{3}} \overline{u}_0^{\frac{1}{3}} \|u_x\|_{L^2(\Omega)}^{\frac{5}{3}} \cdot \Big\{k_1(M) + c_1 e^{-\alpha t}\Big\} + c_2 \chi_1 |\Omega| \overline{u}_0 \|u_x\|_{L^2(\Omega)} \cdot \Big\{k_1(M) + c_1 e^{-\alpha t}\Big\} \\
&\quad + \frac{1}{4}\|u_x\|_{L^2(\Omega)}^2 + c_3 |\Omega|^2 \overline{u}_0^2 \qquad \text{for all } t \in (0, T_{max}),
\end{aligned} \tag{3.11}$$

where by (3.10),

$$c_2 \chi_1 |\Omega|^{\frac{1}{3}} \overline{u}_0^{\frac{1}{3}} \|u_x\|_{L^2(\Omega)}^{\frac{5}{3}} \cdot \Big\{k_1(M) + c_1 e^{-\alpha t}\Big\} \le \frac{1}{8}\|u_x\|_{L^2(\Omega)}^2 + c_4 \overline{u}_0^2 \cdot \Big\{k_1(M) + c_1 e^{-\alpha t}\Big\}^6$$

and where once more by Young's inequality,

$$c_2 \chi_1 |\Omega| \overline{u}_0 \|u_x\|_{L^2(\Omega)} \cdot \Big\{k_1(M) + c_1 e^{-\alpha t}\Big\} \le \frac{1}{8}\|u_x\|_{L^2(\Omega)}^2 + 2 c_2^2 \chi_1^2 |\Omega|^2 \overline{u}_0^2 \cdot \Big\{k_1(M) + c_1 e^{-\alpha t}\Big\}^2$$

for all $t \in (0, T_{max})$. Since $k_1(M) \geq 1$ and thus

$$\Big\{k_1(M) + c_1 e^{-\alpha t}\Big\}^2 \leq \Big\{k_1(M) + c_1 e^{-\alpha t}\Big\}^6 \leq 32 \cdot \Big\{k_1^6(M) + c_1^6 e^{-6\alpha t}\Big\} \qquad \text{for all } t > 0,$$

from (3.11) we altogether obtain that

$$\frac{d}{dt}\int_\Omega u^2 + \int_\Omega u^2 + \int_\Omega u_x^2 \leq k_2(M)\overline{u}_0^2 + c_5 e^{-6\alpha t} \qquad \text{for all } t \in (0, T_{max}) \tag{3.12}$$

with

$$k_2(M) := 2c_3|\Omega|^2 + 64c_4 k_1^6(M) + 128c_2^2\chi_1^2|\Omega|^2 k_1^6(M)$$

and

$$c_5 \equiv c_5(u_0, v_0, w_0) := 64c_1^6 c_4 \overline{u}_0^2 + 128 c_1^6 c_2^2 \chi_1^2 |\Omega|^2 \overline{u}_0^2.$$

Using that $6\alpha < 1$, we may invoke Lemma 6.1 to firstly conclude from (3.12) that

$$\int_\Omega u^2(\cdot, t) \leq \left\{\int_\Omega u_0^2 + \frac{c_5}{1 - 6\alpha}\right\} \cdot e^{-6\alpha t} + k_2(M)\overline{u}_0^2 \qquad \text{for all } t \in (0, T_{max}), \tag{3.13}$$

and that thus (3.5) holds with evident choices of the constants therein. After that, by direct integration of (3.12) we see that since $\tau \leq 1$,

$$\begin{aligned}
\int_t^{t+\tau}\int_\Omega u_x^2 &\leq \int_\Omega u^2(\cdot, t) + k_2(M)\overline{u}_0^2 + c_5\int_t^{t+\tau} e^{-6\alpha s} ds \\
&\leq \int_\Omega u^2(\cdot, t) + k_2(M)\overline{u}_0^2 + c_5 e^{-6\alpha t} \qquad \text{for all } t \in (0, T_{max} - \tau),
\end{aligned}$$

which shows that (3.13) also entails (3.6). □

By again going back to Lemma 3.1, in light of the outcome of Lemma 3.2 we can now once more employ heat semigroup estimates to actually improve the topological setting in (3.5) so as to involve the respective L^∞ norm.

Lemma 3.3 *There exists $\alpha > 0$ such that if $M > 0$, then one can fix $K(M) > 0$ such that under the assumptions (1.4) and (1.8) it is possible to find $C = C(u_0, v_0, w_0) > 0$ such that*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq K(M)\overline{u}_0 + Ce^{-\alpha t} \qquad \textit{for all } t \in (0, T_{max}). \tag{3.14}$$

PROOF. We begin by employing Lemma 3.2 and Lemma 3.1 to take $\alpha_1 \in (0, 1)$ and $\alpha_2 \in (0, 1)$ such that given $M > 0$ we can find $k_1(M) > 0$ and $k_2(M) > 0$ with the property that if (1.4) and (1.8) hold, then with some $c_i = c_i(u_0, v_0, w_0) > 0$, $i \in \{1, 2\}$, we have

$$\|u(\cdot, t)\|_{L^2(\Omega)} \leq k_1(M)\overline{u}_0 + c_1 e^{-\alpha_1 t} \qquad \text{for all } t \in (0, T_{max})$$

and

$$\|w_x(\cdot, t)\|_{L^4(\Omega)} \leq k_2(M) + c_2 e^{-\alpha_2 t} \qquad \text{for all } t \in (0, T_{max})$$

and hence, by the Hölder inequality,

$$\begin{aligned}\|u(\cdot,t)w_x(\cdot,t)\|_{L^{\frac{4}{3}}(\Omega)} &\le \|u(\cdot,t)\|_{L^2(\Omega)}\|w_x(\cdot,t)\|_{L^4(\Omega)} \\ &\le k_1(M)k_2(M)\overline{u}_0 + c_1k_2(M)e^{-\alpha_1 t} + c_2k_1(M)\overline{u}_0e^{-\alpha_2 t} + c_1c_2e^{-(\alpha_1+\alpha_2)t} \\ &\le k_1(M)k_2(M)\overline{u}_0 + c_3e^{-\alpha t} \qquad \text{for all } t\in(0,T_{max}) \qquad (3.15)\end{aligned}$$

with $\alpha := \min\{\alpha_1,\alpha_2\}$ and $c_3 = c_3(u_0,v_0,w_0) := c_1k_2(M) + c_2k_1(M)\overline{u}_0 + c_1c_2$. Next, parabolic smoothing estimates ([24], [9]) provide $c_4 > 0$ and $c_5 > 0$ such that for all $t > 0$,

$$\|e^{t\Delta}\varphi_x\|_{L^\infty(\Omega)} \le c_4(1+t^{-\frac{7}{8}})\|\varphi\|_{L^{\frac{4}{3}}(\Omega)} \qquad \text{for all } \varphi\in C^1(\overline{\Omega}) \text{ such that } \varphi_x = 0 \text{ on } \partial\Omega$$

and

$$\|e^{t\Delta}\varphi\|_{L^\infty(\Omega)} \le c_5(1+t^{-\frac{1}{2}})\|\varphi\|_{L^1(\Omega)} \qquad \text{for all } \varphi\in C^0(\overline{\Omega}),$$

so that henceforth assuming (1.4) and (1.8) for some $M > 0$, and rewriting the first equation in (1.3) in the form

$$u_t - u_{xx} + u = -\chi_1(uw_x)_x + u \qquad \text{in } \Omega\times(0,T_{max}),$$

by means of an associated variation-of-constants representation we may estimate

$$\begin{aligned}\|u(\cdot,t)\|_{L^\infty(\Omega)} &= \left\| e^{t(\Delta-1)}u_0 - \chi_1\int_0^t e^{(t-s)(\Delta-1)}\partial_x\Big(u(\cdot,s)w_x(\cdot,s)\Big)ds + \int_0^t e^{(t-s)(\Delta-1)}u(\cdot,s)ds\right\|_{L^\infty(\Omega)} \\ &\le e^{-t}\|e^{t\Delta}u_0\|_{L^\infty(\Omega)} + c_4\chi_1\int_0^t e^{-(t-s)}\Big(1+(t-s)^{-\frac{7}{8}}\Big)\|u(\cdot,s)w_x(\cdot,s)\|_{L^{\frac{4}{3}}(\Omega)}ds \\ &\quad + c_5\int_0^t e^{-(t-s)}\Big(1+(t-s)^{-\frac{1}{2}}\Big)\|u(\cdot,s)\|_{L^1(\Omega)}ds \qquad \text{for all } t\in(0,T_{max}). \qquad (3.16)\end{aligned}$$

Here by the maximum principle and the fact that $\alpha \le 1$,

$$e^{-t}\|e^{t\Delta}u_0\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)}e^{-t} \le \|u_0\|_{L^\infty(\Omega)}e^{-\alpha t} \qquad \text{for all } t > 0, \qquad (3.17)$$

whereas (2.3) ensures that

$$\begin{aligned}c_5\int_0^t e^{-(t-s)}\Big(1+(t-s)^{-\frac{1}{2}}\Big)\|u(\cdot,s)\|_{L^1(\Omega)}ds &= c_5|\Omega|\overline{u}_0\int_0^t e^{-(t-s)}\Big(1+(t-s)^{-\frac{1}{2}}\Big)ds \\ &\le c_5c_6|\Omega|\overline{u}_0 \qquad \text{for all } t\in(0,T_{max}) \qquad (3.18)\end{aligned}$$

with $c_6 := \int_0^\infty e^{-\sigma}(1+\sigma^{-\frac{1}{2}})d\sigma < \infty$. Moreover, thanks to (3.15) we have

$$\begin{aligned}&c_4\chi_1\int_0^t e^{-(t-s)}\Big(1+(t-s)^{-\frac{7}{8}}\Big)\|u(\cdot,s)w_x(\cdot,s)\|_{L^{\frac{4}{3}}(\Omega)}ds \\ &\quad\le c_4\chi_1k_1(M)k_2(M)\overline{u}_0\int_0^t e^{-(t-s)}\Big(1+(t-s)^{-\frac{7}{8}}\Big)ds \\ &\qquad + c_3c_4\chi_1\int_0^t e^{-(t-s)}\Big(1+(t-s)^{-\frac{7}{8}}\Big)e^{-\alpha s}ds \qquad \text{for all } t\in(0,T_{max}), \qquad (3.19)\end{aligned}$$

where

$$\int_0^t e^{-(t-s)}\Big(1+(t-s)^{-\frac{7}{8}}\Big)ds \le c_7 := \int_0^\infty e^{-\sigma}(1+\sigma^{-\frac{7}{8}})d\sigma \qquad \text{for all } t>0,$$

and where

$$\int_0^t e^{-(t-s)}\Big(1+(t-s)^{-\frac{7}{8}}\Big)e^{-\alpha s}ds = e^{-\alpha t}\int_0^t e^{-(1-\alpha)\sigma}(1+\sigma^{-\frac{7}{8}})d\sigma \le c_8 e^{-\alpha t} \qquad \text{for all } t>0$$

with $c_8 := \int_0^\infty e^{-(1-\alpha)\sigma}(1+\sigma^{-\frac{7}{8}})d\sigma$ being finite thanks to our restriction that $\alpha<1$.
Inserting (3.17)-(3.19) into (3.16) thus shows that for all $t\in(0,T_{max})$,

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \Big\{c_5c_6|\Omega| + c_4c_7\chi_1k_1(M)k_2(M)\Big\}\cdot\overline{u}_0 + \Big\{\|u_0\|_{L^\infty(\Omega)} + c_3c_4c_8\chi_1\Big\}\cdot e^{-\alpha t},$$

and therefore establishes (3.14) upon the observation that c_4,c_5,c_6 and c_7 do not depend on our particular choice of u_0,v_0 and w_0. □

3.3 Space-time L^2 bounds for v and for w_{xx}

Now unlike in the analysis of (1.2), for globally extending our solution the bounds obtained in Lemma 3.3 and Lemma 3.1 seem yet insufficient: In view of the second equation in (1.3) it seems that for the detection of appropriate estimates for the second solution component, further information on the respectively relevant cross-diffusive gradient u_x seems in order. To prepare our derivation thereof in the next section, let us here provide some preliminary bounds on v, v_x and w_{xx} useful for that purpose.

We begin with a basic space-time integrability feature of $(\ln(v+1))_x$ which by another straightforward testing procedure can be seen to be quite a direct consequence of our present knowledge on u_x from Lemma 3.2.

Lemma 3.4 *One can find $\alpha>0$ with the property that to any $M>0$ there corresponds some $K(M)>0$ such that if (1.4) and (1.8) hold, then there exists $C=C(u_0,v_0,w_0)>0$ fulfilling*

$$\int_t^{t+\tau}\int_\Omega \frac{v_x^2}{(v+1)^2} \le K(M) + Ce^{-\alpha t} \qquad \textit{for all } t\in(0,T_{max}-\tau), \tag{3.20}$$

where $\tau := \min\{1,\frac{1}{2}T_{max}\}$.

Proof. We multiply the second equation in (1.3) by $\frac{1}{v+1}$ and integrate by parts to see that due to Young' inequality,

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \ln(v+1) &= \int_\Omega \frac{v_x^2}{(v+1)^2} - \chi_2\int_\Omega \frac{v}{(v+1)^2}u_xv_x \\
&\ge \frac{1}{2}\int_\Omega \frac{v_x^2}{(v+1)^2} - \frac{\chi_2^2}{2}\int_\Omega \frac{v^2}{(v+1)^2}u_x^2 \\
&\ge \frac{1}{2}\int_\Omega \frac{v_x^2}{(v+1)^2} - \frac{\chi_2^2}{2}\int_\Omega u_x^2 \qquad \text{for all } t\in(0,T_{max}).
\end{aligned}$$

As $0 \le \ln(\xi+1) \le \xi$ for all $\xi \ge 0$, further integration shows that thanks to (2.3),

$$\begin{aligned}\frac{1}{2}\int_t^{t+\tau}\int_\Omega \frac{v_x^2}{(v+1)^2} &\le \int_\Omega \ln\Big(v(\cdot,t+\tau)+1\Big) - \int_\Omega \ln\Big(v(\cdot,t)+1\Big) + \frac{\chi^2}{2}\int_t^{t+\tau}\int_\Omega u_x^2 \\ &\le \int_\Omega v_0 + \frac{\chi^2}{2}\int_t^{t+\tau}\int_\Omega u_x^2 \qquad \text{for all } t\in(0,T_{max},\tau),\end{aligned}$$

so that (3.20) becomes a consequence of Lemma 3.2. □

Again thanks to the fact that the considered setting is one-dimensional, a simple interpolation argument shows that the above entails a space time bound on v itself, rather than on the quantity $\ln(v+1)$ addressed in Lemma 3.4.

Lemma 3.5 *There exists $\alpha > 0$ such that whenever $M > 0$, one can pick $K(M) > 0$ such that if (1.4) and (1.8) hold, then with some $C = C(u_0, v_0, w_0) > 0$ we have*

$$\int_t^{t+\tau}\int_\Omega v^2 \le K(M) + Ce^{-\alpha t} \qquad \textit{for all } t \in (0, T_{max}-\tau), \tag{3.21}$$

where again $\tau := \min\{1, \frac{1}{2}T_{max}\}$.

Proof. According to the one-dimensional Gagliardo-Nirenberg inequality, we can fix $c_1 > 0$ such that

$$\|\varphi\|_{L^4(\Omega)}^4 \le c_1\|\varphi_x\|_{L^1(\Omega)}^2\|\varphi\|_{L^2(\Omega)}^2 + c_1\|\varphi\|_{L^2(\Omega)}^4 \qquad \text{for all } \varphi\in W^{1,1}(\Omega),$$

which when applied to $\sqrt{v(\cdot,t)+1}$, $t\in(0,T_{max})$, shows that since $\|\sqrt{v+1}\|_{L^2(\Omega)}^2 = \int_\Omega v_0 + |\Omega|$ for all $t \in (0,T_{max})$ by (2.3),

$$\begin{aligned}\int_\Omega v^2 \le \int_\Omega (v+1)^2 &= \|\sqrt{v+1}\|_{L^4(\Omega)}^4 \\ &\le c_1\|\partial_x\sqrt{v+1}\|_{L^1(\Omega)}^2\|\sqrt{v+1}\|_{L^2(\Omega)}^2 + c_1\|\sqrt{v+1}\|_{L^2(\Omega)}^4 \\ &= \frac{c_1}{4}\cdot\Big\{\int_\Omega v_0 + |\Omega|\Big\}\cdot\Big\{\int_\Omega \frac{|v_x|}{\sqrt{v+1}}\Big\}^2 + c_1\cdot\Big\{\int_\Omega v_0+|\Omega|\Big\}^2\end{aligned}$$

for all $t \in (0, T_{max})$. Once more in view of (2.3), using the Cauchy-Schwarz inequality we see that herein

$$\begin{aligned}\Big\{\int_\Omega \frac{|v_x|}{\sqrt{v+1}}\Big\}^2 &\le \Big\{\int_\Omega (v+1)\Big\}\cdot\int_\Omega \frac{v_x^2}{(v+1)^2} \\ &\le \Big\{\int_\Omega v_0 + |\Omega|\Big\}\cdot\int_\Omega\frac{v_x^2}{(v+1)^2} \qquad \text{for all } t\in(0,T_{max}),\end{aligned}$$

whence altogether, after a time integration,

$$\int_t^{t+\tau}\int_\Omega v^2 \le c_1\cdot\Big\{\int_\Omega v_0+|\Omega|\Big\}^2\cdot\Big\{\frac{1}{4}\int_\Omega \frac{v_x^2}{(v+1)^2}+1\Big\} \qquad \text{for all } t\in(0,T_{max}-\tau)$$

due to the fact that $\tau \le 1$. The claimed statement thus readily results from Lemma 3.4. $\Box$

Thus having at hand spatio-temporal integral estimates for v and, through e.g. Lemma 3.3, also for u, we have collected sufficient regularity information on all the source terms in the third equation from (1.3), when considered as a semilinear heat equation, so as to obtain the following as the outcome of a further standard testing process.

Lemma 3.6 *One can find $\alpha > 0$ in such a way that for each $M > 0$ there exists $K(M) > 0$ such that assuming (1.4) and (1.8) entails that with some $C = C(u_0, v_0, w_0) > 0$,*

$$\int_t^{t+\tau} \int_\Omega w_{xx}^2 \le \frac{K(M)}{\tau} + Ce^{-\alpha t} \qquad \textit{for all } t \in (0, T_{max} - \tau), \tag{3.22}$$

where once more $\tau := \min\{1, \frac{1}{2}T_{max}\}$.

Proof. By means of Lemma 3.2 and Lemma 3.5, we can find $\alpha_1 > 0$ and $\alpha_2 > 0$ such that given any $M > 0$ one can pick $k_1(M) > 0$ and $k_2(M) > 0$ such that whenever (1.4) and (1.8) hold, for the corresponding solution of (1.3) we have

$$\int_\Omega u^2 \le k_1(M) + c_1 e^{-\alpha_1 t} \qquad \text{for all } t \in (0, T_{max}) \tag{3.23}$$

and

$$\int_t^{t+\tau} \int_\Omega v^2 \le k_2(M) + c_2 e^{-\alpha_2 t} \qquad \text{for all } t \in (0, T_{max} - \tau) \tag{3.24}$$

with some $c_i = c_i(u_0, v_0, w_0) > 0$, $i \in \{1, 2\}$, and $\tau = \min\{1, \frac{1}{2}T_{max}\}$.

Now indeed assuming (1.4) and (1.8), we use w_{xx} as a test function for the third equation in (1.3) to see that by Young's inequality,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega w_x^2 + \mu \int_\Omega w_x^2 + d\int_\Omega w_{xx}^2 &= \lambda \int_\Omega uww_{xx} + \lambda \int_\Omega vww_{xx} \\ &\le \frac{d}{2}\int_\Omega w_{xx}^2 + \frac{\lambda^2}{d}\int_\Omega u^2w^2 + \frac{\lambda^2}{d}\int_\Omega v^2w^2 \\ &\le \frac{d}{2}\int_\Omega w_{xx}^2 + \frac{\lambda^2}{d}\|w\|_{L^\infty(\Omega)}^2 \cdot \left\{ \int_\Omega u^2 + \int_\Omega v^2 \right\} \end{aligned}$$

for all $t \in (0, T_{max})$ and hence

$$\frac{d}{dt}\int_\Omega w_x^2 + 2\mu\int_\Omega w_x^2 + d\int_\Omega w_{xx}^2 \le \frac{2\lambda^2}{d}\|w\|_{L^\infty(\Omega)}^2 \cdot \left\{ \int_\Omega u^2 + \int_\Omega v^2 \right\} \qquad \text{for all } t \in (0, T_{max}). \tag{3.25}$$

Here aiming at an application of Lemma 6.2, we pick any $\alpha > 0$ such that $\alpha < 2\mu$ and $\alpha \le \min\{\alpha_1, \alpha_2\}$, and combine (3.23) and (3.24) with the outcome of Lemma 2.2 and Young's inequality to estimate

$$\int_t^{t+\tau} \left\{ \frac{2\lambda^2}{d}\|w(\cdot, s)\|_{L^\infty(\Omega)}^2 \cdot \left\{ \int_\Omega u^2(\cdot, s) + \int_\Omega v^2(\cdot, s) \right\} \right\} ds$$

$$
\begin{aligned}
&\le \frac{2\lambda^2}{d}\int_t^{t+\tau}\Big\{\frac{r}{\mu}+\|w_0\|_{L^\infty(\Omega)}e^{-\mu t}\Big\}^2\cdot\Big\{\int_\Omega u^2(\cdot,s)+\int_\Omega v^2(\cdot,s)\Big\}ds\\
&\le \frac{4\lambda^2}{d}\cdot\Big\{\frac{r^2}{\mu^2}+\|w_0\|^2_{L^\infty(\Omega)}e^{-2\mu t}\Big\}\cdot\Big\{\int_t^{t+\tau}\int_\Omega u^2+\int_t^{t+\tau}\int_\Omega v^2\Big\}\\
&\le \frac{4\lambda^2}{d}\cdot\Big\{\frac{r^2}{\mu^2}+\|w_0\|^2_{L^\infty(\Omega)}e^{-2\mu t}\Big\}\cdot\Big\{k_1(M)+k_2(M)+c_1e^{-\alpha_1 t}+c_2e^{-\alpha_2 t}\Big\}
\end{aligned}
$$

for all $t\in(0,T_{max}-\tau)$, where we have used that $\tau\le 1$ and that hence $\int_t^{t+\tau}e^{-\beta s}ds\le e^{-\beta t}$ for all $t>0$ and any $\beta>0$. Since $\alpha\le\min\{\alpha_1,\alpha_2,2\mu\}$, this readily implies that

$$
\int_t^{t+\tau}\Big\{\frac{2\lambda^2}{d}\|w(\cdot,s)\|^2_{L^\infty(\Omega)}\cdot\Big\{\int_\Omega u^2(\cdot,s)+\int_\Omega v^2(\cdot,s)\Big\}\Big\}ds\le k_3(M)+c_4e^{-\alpha t}\qquad\text{for all } t\in(0,T_{max}-\tau) \tag{3.26}
$$

with $k_3(M):=\frac{4\lambda^2r^2}{d\mu^2}(k_1(M)+k_2(M))$ and

$$
c_4\equiv c_4(u_0,v_0,w_0):=\frac{4\lambda^2r^2}{d\mu^2}(c_1+c_2)+\frac{4\lambda^2}{d}\|w_0\|^2_{L^\infty(\Omega)}(k_1(M)+k_2(M)+c_1+c_2).
$$

Upon employing Lemma 6.2, we thus obtain that (3.25) firstly entails the inequality

$$
\int_\Omega w_x^2\le\frac{k_3(M)}{2\mu\tau}+k_3(M)+c_5e^{-\alpha t}\qquad\text{for all } t\in(0,T_{max})
$$

if we let

$$
c_5\equiv c_5(u_0,v_0,w_0):=\frac{e^\alpha}{\tau}\cdot\Big\{\int_\Omega w_{0x}^2+c_4+\frac{c_4}{2\mu-\alpha}+k_3(M)\Big\}+c_4e^\alpha.
$$

Thereupon, directly integrating (3.25) shows that again due to (3.26),

$$
\begin{aligned}
d\int_t^{t+\tau}\int_\Omega w_{xx}^2 &\le \int_\Omega w_x^2(\cdot,t)+k_3(M)+c_4e^{-\alpha t}\\
&\le \frac{k_3(M)}{2\mu\tau}+2k_3(M)+(c_4+c_5)e^{-\alpha t}\qquad\text{for all } t\in(0,T_{max}-\tau),
\end{aligned}
$$

which yields (3.22) upon again recalling that $\tau\le 1$. □

3.4 Estimating u_x in L^2

Thanks to Lemma 3.6, we now have appropriate information on the coefficient functions $a(x,t):=-\chi_1 w_x$ and $b(x,t):=-\chi_1 w_{xx}$ in the identity $u_t=u_{xx}+a(x,t)u_x+b(x,t)u$ to see that again due to a variational argument, u_x indeed enjoys the following integrability features which go substantially beyond those obtained in Lemma 3.2.

Lemma 3.7 *There exists $\alpha > 0$ such that for arbitrary $M > 0$ it is possible to choose $K(M) > 0$ in such a way that whenever (1.4) and (1.8) are satisfied, there exists $C = C(u_0, v_0, w_0) > 0$ such that writing $\tau := \min\{1, \frac{1}{2}T_{max}\}$ we have*

$$\int_\Omega u_x^2(\cdot,t) \le \frac{K(M)}{\tau^2} + Ce^{-\alpha t} \qquad \textit{for all } t \in (0, T_{max}) \tag{3.27}$$

and

$$\int_t^{t+\tau} \int_\Omega u_{xx}^2 \le \frac{K(M)}{\tau^2} + Ce^{-\alpha t} \qquad \textit{for all } t \in (0, T_{max} - \tau). \tag{3.28}$$

PROOF. As a consequence of Lemma 3.3 and Lemma 3.6, we may pick $\alpha_1 > 0$ and $\alpha_2 > 0$ such that for all $M > 0$ we can find $k_1(M) > 0$ and $k_2(M) > 0$ with the property that under the assumptions (1.4) and (1.8) one may fix $c_1 = c_1(u_0, v_0, w_0) > 0$ and $c_2 = c_2(u_0, v_0, w_0) > 0$ fulfilling

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}^2 \le k_1(M) + c_1 e^{-\alpha_1 t} \qquad \text{for all } t \in (0, T_{max}) \tag{3.29}$$

and

$$\int_t^{t+\tau} \int_\Omega w_{xx}^2 \le \frac{k_2(M)}{\tau} + c_2 e^{-\alpha_2 t} \qquad \text{for all } t \in (0, T_{max} - \tau). \tag{3.30}$$

Apart from that, we combine the Gagliardo-Nirenberg inequality with Young's inequality to obtain $c_3 > 0$ and $c_4 > 0$ such that

$$\|\varphi_x\|_{L^4(\Omega)}^2 \le c_3 \|\varphi_{xx}\|_{L^2(\Omega)} \|\varphi\|_{L^\infty(\Omega)} \qquad \text{for all } \varphi \in W^{2,2}(\Omega) \tag{3.31}$$

and that

$$\int_\Omega \varphi_x^2 \le \frac{1}{2} \int_\Omega \varphi_{xx}^2 + c_4 \Big\{ \int_\Omega |\varphi| \Big\}^2 \qquad \text{for all } \varphi \in W^{2,2}(\Omega) \tag{3.32}$$

Now assuming (1.4) and (1.8) to be valid for some $M > 0$, we integrate by parts in the first equation from (1.3) and use the Cauchy-Schwarz inequality along with (3.31), Young's inequality and (3.32) to see that for all $t \in (0, T_{max})$,

$$\begin{aligned} \frac{d}{dt}\int_\Omega u_x^2 + \int_\Omega u_x^2 + 2\int_\Omega u_{xx}^2 &= 2\chi_1 \int_\Omega u_x w_x u_{xx} + 2\chi_1 \int_\Omega u u_{xx} w_{xx} + \int_\Omega u_x^2 \\ &= -\chi_1 \int_\Omega u_x^2 w_{xx} + 2\chi_1 \int_\Omega u u_{xx} w_{xx} + \int_\Omega u_x^2 \\ &\le \chi_1 \|u_x\|_{L^4(\Omega)}^2 \|w_{xx}\|_{L^2(\Omega)} + 2\chi_1 \|u\|_{L^\infty(\Omega)} \|u_{xx}\|_{L^2(\Omega)} \|w_{xx}\|_{L^2(\Omega)} + \int_\Omega u_x^2 \\ &\le (c_3+2)\chi_1 \|u\|_{L^\infty(\Omega)} \|u_{xx}\|_{L^2(\Omega)} \|w_{xx}\|_{L^2(\Omega)} + \int_\Omega u_x^2 \\ &\le \int_\Omega u_{xx}^2 + \frac{(c_3+2)^2\chi_1^2}{2} \|u\|_{L^\infty(\Omega)}^2 \|w_{xx}\|_{L^2(\Omega)}^2 + c_4 \Big\{ \int_\Omega u_0 \Big\}^2 \end{aligned}$$

because of (2.3). In view of the hypothesis (1.8), this shows that abbreviating $c_5 := \frac{(c_3+2)^2\chi_1^2}{2}$ we have

$$\frac{d}{dt}\int_\Omega u_x^2 + \int_\Omega u_x^2 + \int_\Omega u_{xx}^2 \le c_5 \|u\|_{L^\infty(\Omega)}^2 \int_\Omega w_{xx}^2 + c_4 M^2 \qquad \text{for all } t \in (0, T_{max}), \tag{3.33}$$

where thanks to (3.29) and (3.30), fixing any $\alpha \in (0,1)$ such that $\alpha \le \min\{\alpha_1, \alpha_2\}$ we can estimate

$$\begin{aligned}
\int_t^{t+\tau} & \Big\{ c_5 \|u(\cdot,s)\|_{L^\infty(\Omega)}^2 \int_\Omega w_{xx}^2(\cdot,s) + c_4 M^2 \Big\} ds \\
& \le c_5 \cdot \Big\{ k_1(M) + c_1 e^{-\alpha_1 t} \Big\} \cdot \Big\{ \frac{k_2(M)}{\tau} + c_2 e^{-\alpha_2 t} \Big\} + c_4 M^2 \\
& \le \frac{k_3(M)}{\tau} + c_6 e^{-\alpha t} \qquad \text{for all } t \in (0, T_{max} - \tau)
\end{aligned} \tag{3.34}$$

with $k_3(M) := c_5 k_1(M) k_2(M) + c_4 M^2$ and $c_6 \equiv c_6(u_0, v_0, w_0) := c_2 c_5 k_1(M) + \frac{c_1 c_5 k_2(M)}{\tau} + c_1 c_2 c_5$. As a consequence of Lemma 6.2 and other restriction that $\alpha < 1$, from (3.33) we thus infer that writing $c_7 \equiv c_7(u_0, v_0, w_0) := \frac{e^\alpha}{\tau} \cdot \Big\{ \int_\Omega u_{0x}^2 + c_6 + \frac{c_6}{1-\alpha} + \frac{k_3(M)}{\tau} \Big\} + c_6 e^\alpha$ we have

$$\int_\Omega u_x^2 \le \frac{k_3(M)}{\tau^2} + \frac{k_3(M)}{\tau} + c_7 e^{-\alpha t} \qquad \text{for all } t \in (0, T_{max}), \tag{3.35}$$

and that hence, by integration of (3.33) and again using (3.34),

$$\begin{aligned}
\int_t^{t+\tau} \int_\Omega u_{xx}^2 & \le \int_\Omega u_x^2(\cdot,t) + \frac{k_3(M)}{\tau} + c_6 e^{-\alpha t} \\
& \le \frac{k_3(M)}{\tau^2} + \frac{2k_3(M)}{\tau} + (c_6 + c_7) e^{-\alpha t} \qquad \text{for all } t \in (0, T_{max} - \tau).
\end{aligned} \tag{3.36}$$

Since $\tau \le 1$, the claimed properties directly result from (3.35) and (3.36). □

3.5 An L^2 bound for v_x

We can thereby gradually improve our knowledge on the second solution component, firstly addressing v itself in the course of a further testing procedure:

Lemma 3.8 *There exists $\alpha > 0$ such that for all $M > 0$ one can fix $K(M) > 0$ having the property that whenever (1.4) and (1.8) hold, with some $C = C(u_0, v_0, w_0) > 0$ we have*

$$\int_\Omega v^4(\cdot,t) \le \frac{K(M)\overline{v}_0^4}{\tau^{10}} + Ce^{-\alpha t} \qquad \textit{for all } t \in (0, T_{max}). \tag{3.37}$$

Proof. On the basis of Lemma 3.7, it is possible to pick $\alpha_1 \in (0,1)$ in such a way that given $M > 0$ we can choose $k_1(M) > 0$ which is such that if (1.4) and (1.8) hold,

$$\Big\{ \int_\Omega u_x^2 \Big\}^5 \le \frac{k_1(M)}{\tau^{10}} + c_1 e^{-\alpha_1 t} \qquad \text{for all } t \in (0, T_{max}) \tag{3.38}$$

with some $c_1 = c_1(u_0, v_0, w_0) > 0$. Once more relying on the Gagliardo-Nirenberg inequality and Young's inequality, we furthermore fix $c_2 > 0$, $c_3 > 0$ and $c_4 > 0$ such that

$$\|\varphi\|_{L^\infty(\Omega)} \le c_2 \|\varphi_x\|_{L^2(\Omega)}^{\frac{4}{5}} \|\varphi\|_{L^{\frac{1}{2}}(\Omega)}^{\frac{1}{5}} + c_2 \|\varphi\|_{L^{\frac{1}{2}}(\Omega)} \qquad \text{for all } \varphi \in W^{1,2}(\Omega), \tag{3.39}$$

that

$$\int_\Omega \varphi^2 \le \|\varphi_x\|_{L^2(\Omega)}^2 + c_3\|\varphi\|_{L^{\frac{1}{2}}(\Omega)}^2 \qquad \text{for all } \varphi \in W^{1,2}(\Omega), \tag{3.40}$$

and that

$$6c_2\chi_2 ab \le a^{\frac{10}{9}} + c_4 b^{10} \qquad \text{for all } a \ge 0 \text{ and } b \ge 0. \tag{3.41}$$

Then supposing that (1.4) and (1.8) to be satisfied, we use the second equation in (1.3) to see that due to the Cauchy-Schwarz inequality and (2.3), applications of (3.39), (3.41), (3.40) and Young's inequality show that for all $t \in (0, T_{max})$,

$$\begin{aligned}
\frac{d}{dt}\int_\Omega v^4 + \int_\Omega v^4 + 3\int_\Omega (v^2)_x^2 &= 6\chi_2 \int_\Omega v^2 u_x (v^2)_x + \int_\Omega v^4 \\
&\le 6\chi_2 \|v^2\|_{L^\infty(\Omega)} \|u_x\|_{L^2(\Omega)} \|(v^2)_x\|_{L^2(\Omega)} + \int_\Omega v^4 \\
&\le 6c_2\chi_2 \|(v^2)_x\|_{L^2(\Omega)}^{\frac{9}{5}} \|v^2\|_{L^{\frac{1}{2}}(\Omega)}^{\frac{1}{5}} \|u_x\|_{L^2(\Omega)} \\
&\quad + 6c_2\chi_2 \|(v^2)_x\|_{L^2(\Omega)} \|v^2\|_{L^{\frac{1}{2}}(\Omega)} \|u_x\|_{L^2(\Omega)} \\
&\quad + \int_\Omega v^4 \\
&\le \|(v^2)_x\|_{L^2(\Omega)}^2 + c_4 \|v^2\|_{L^{\frac{1}{2}}(\Omega)}^2 \|u_x\|_{L^2(\Omega)}^{10} \\
&\quad + \|(v^2)_x\|_{L^2(\Omega)}^2 + 9c_2^2\chi_2^2 \|v^2\|_{L^{\frac{1}{2}}(\Omega)}^2 \|u_x\|_{L^2(\Omega)}^2 \\
&\quad + \|(v^2)_x\|_{L^2(\Omega)}^2 + c_3 \|v^2\|_{L^{\frac{1}{2}}(\Omega)}^2 \\
&= 3\int_\Omega (v^2)_x^2 + \Big\{ c_4 \|u_x\|_{L^2(\Omega)}^{10} + 9c_2^2\chi_2^2 \|u_x\|_{L^2(\Omega)}^2 + c_3 \Big\} \cdot \|v_0\|_{L^1(\Omega)}^4.
\end{aligned} \tag{3.42}$$

Since here, by using Young's inequality and relying on the fact that $\tau \le 1$, we can estimate

$$\begin{aligned}
&\Big\{ c_4 \|u_x\|_{L^2(\Omega)}^{10} + 9c_2^2\chi_2^2 \|u_x\|_{L^2(\Omega)}^2 + c_3 \Big\} \cdot \|v_0\|_{L^1(\Omega)}^4 \\
&\quad \le (c_4 + 9c_2^2\chi_2^2)\|v_0\|_{L^1(\Omega)}^4 \cdot \Big\{ \int_\Omega u_x^2 \Big\}^5 + (9c_2^2\chi_2^2 + c_3)\|v_0\|_{L^1(\Omega)}^4 \\
&\quad \le \frac{k_2(M)\overline{v}_0^4}{\tau^{10}} + c_5 e^{-\alpha_1 t} \qquad \text{for all } t \in (0, T_{max})
\end{aligned}$$

with $k_2(M) := \Big\{(c_4 + 9c_2^2\chi_2^2)k_1(M) + 9c_2^2\chi_2^2 + c_3\Big\} \cdot |\Omega|^4$ and $c_5 \equiv c_5(u_0, v_0, w_0) := c_1 \cdot (9c_2^2\chi_2^2 + c_4)\|v_0\|_{L^1(\Omega)}^4$, from (3.42) we thus infer that

$$\frac{d}{dt}\int_\Omega v^4 + \int_\Omega v^4 \le \frac{k_2(M)\overline{v}_0^4}{\tau^{10}} + c_5 e^{-\alpha_1 t} \qquad \text{for all } t \in (0, T_{max}).$$

Through Lemma 6.1, applicable here since $\alpha_1 < 1$, this entails that

$$\int_\Omega v^4 \le \frac{k_2(M)\overline{v}_0^4}{\tau^{10}} + \Big\{ \int_\Omega v_0^4 + \frac{c_5}{1-\alpha_1} \Big\} \cdot e^{-\alpha_1 t} \qquad \text{for all } t \in (0, T_{max})$$

and hence completes the proof. □

Yet concentrating on v itself, we next resort to a semigroup-based argument once more to turn the above into an esimate involving the norm in $L^\infty(\Omega)$.

Lemma 3.9 *One can find $\alpha > 0$ in such a manner that for each $M > 0$ there exists $K(M) > 0$ such that if (1.4) and (1.8) are satisfied, then*

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{K(M)\overline{v}_0}{\tau^{\frac{7}{2}}} + Ce^{-\alpha t} \qquad \text{for all } t \in (0, T_{max}) \tag{3.43}$$

with some $C = C(u_0, v_0, w_0) > 0$.

Proof. A verification of this can be achieved in a way quite similar to that in Lemma 3.3: By the Hölder inequality as well as Lemma 3.8 and Lemma 3.7, we see that with some $\alpha \in (0,1)$, given any $M > 0$ we can find $k_1(M) > 0$ such that if (1.4) and (1.8) hold, there exists $c_1 = c_1(u_0, v_0, w_0) > 0$ fulfilling

$$\|vu_x\|_{L^{\frac{4}{3}}(\Omega)} \le \|v\|_{L^4(\Omega)}\|u_x\|_{L^2(\Omega)} \le \frac{k_1(M)\overline{v}_0}{\tau^{\frac{7}{2}}} + c_1e^{-\alpha t} \qquad \text{for all } t \in (0, T_{max}).$$

Henceforth assuming (1.4) and (1.8), we combine this with known regularization features of the Neumann heat semigroup and (2.3) to see that with some positive constants c_2 and c_3 independent of u_0, v_0 and w_0 we have

$$\begin{aligned}
\|v(\cdot,t)\|_{L^\infty(\Omega)} &= \left\| e^{t(\Delta-1)}v_0 - \chi_2 \int_0^t e^{(t-s)(\Delta-1)}\partial_x\Big(v(\cdot,s)u_x(\cdot,s)\Big)ds + \int_0^t e^{(t-s)(\Delta-1)}v(\cdot,s)ds \right\|_{L^\infty(\Omega)} \\
&\le e^{-t}\|e^{t\Delta}v_0\|_{L^\infty(\Omega)} + c_2\int_0^t e^{-(t-s)}\Big(1+(t-s)^{-\frac{7}{8}}\Big)\|v(\cdot,s)u_x(\cdot,s)\|_{L^{\frac{4}{3}}(\Omega)}ds \\
&\quad + c_2\int_0^t e^{-(t-s)}\Big(1+(t-s)^{-\frac{1}{2}}\Big)\|v(\cdot,s)\|_{L^1(\Omega)}ds \\
&\le e^{-t}\|v_0\|_{L^\infty(\Omega)} + \frac{c_2k_1(M)\overline{v}_0}{\tau^{\frac{7}{2}}}\int_0^t e^{-(t-s)}\Big(1+(t-s)^{-\frac{7}{8}}\Big)ds \\
&\quad + c_1c_2\int_0^t e^{-(t-s)}\Big(1+(t-s)^{-\frac{7}{8}}\Big)e^{-\alpha s}ds \\
&\quad + c_2|\Omega|\overline{v}_0\int_0^t e^{-(t-s)}\Big(1+(t-s)^{-\frac{1}{2}}\Big)ds \\
&\le e^{-t}\|v_0\|_{L^\infty(\Omega)} + \frac{c_2k_1(M)\overline{v}_0}{\tau^{\frac{7}{2}}}\int_0^\infty e^{-\sigma}(1+\sigma^{-\frac{7}{8}})d\sigma \\
&\quad + c_1c_2e^{-\alpha t}\int_0^\infty e^{-(1-\alpha)\sigma}(1+\sigma^{-\frac{7}{8}})d\sigma \\
&\quad + c_2|\Omega|\overline{v}_0\int_0^\infty e^{-\sigma}(1+\sigma^{-\frac{1}{2}})d\sigma \qquad \text{for all } t \in (0, T_{max}),
\end{aligned}$$

which readily yields (3.43) due to the inequalities $\alpha < 1$ and $\tau \le 1$. □

As we are now in a position quite identical to that encountered immediately before Lemma 3.7, we can repeat the argument thereof to finally derive the following gradient estimate for the crucial second solution component.

Lemma 3.10 *There exists $\alpha > 0$ such that for all $M > 0$ one can choose $K(M) > 0$ with the property that if (1.4) and (1.8) hold, the with some $C = C(u_0, v_0, w_0) > 0$, we have*

$$\int_\Omega v_x^2(\cdot,t) \le \frac{K(M)}{\tau^{10}} + Ce^{-\alpha t} \qquad \text{for all } t \in (0, T_{max}). \tag{3.44}$$

Proof. The claimed inequality can be derived by means of an essentially verbatim copy of the argument from Lemma 3.7, instead of referring to Lemma 3.3 and Lemma 3.6 now relying on Lemma 3.9 and (3.28); we may therefore refrain from giving details here. □

4 Global existence and boundedness. Proof of Theorem 1.1

Now asserting global extensibility of our solution actually reduces to a mere collection of our previously obtained estimates, where at this stage neither any knowledge on the precise dependence thereof on M or on $\overline{u}_0$ and $\overline{v}_0$ is needed, nor do we rely on the exponentially decaying contributions to the above inequalities.

Lemma 4.1 *For all u_0, v_0 and w_0 fulfilling (1.4), we have $T_{max} = \infty$, and furthermore we can find $C = C(u_0, v_0, w_0) > 0$ such that (1.6) holds.*

Proof. In view of the extensibility criterion (2.2) from Lemma 2.1, for any fixed (u_0, v_0, w_0) satisfying (1.4) we may apply Lemma 3.7, Lemma 3.10 and Lemma 3.1 to $M := \int_\Omega u_0 + \int_\Omega v_0$ and $q := 2$ and thereby readily obtain that indeed T_{max} cannot be finite, and that hence moreover (1.6) is a consequence of (3.27), (3.44) and (3.1). □

In other words, we thereby already have derived our main result on global classical solvability in (1.3):

Proof of Theorem 1.1. We only need to combine Lemma 4.1 with Lemma 2.1. □

5 Convergence for small values of $\min\{\overline{u}_0, \overline{v}_0\}$. Proof of Theorem 1.2

Next, in contrast to our development of the above existence statement, our investigation of the large time behavior in (1.3), as forming the objective of this section, will considerably benefit from the more detailed information provided by our estimates from Section 3.

5.1 Identifying a conditional energy functional

The following lemma basically only collects the essence of what will be needed from Section 3 for our subsequent qualitative analysis.

Lemma 5.1 *Let $M > 0$. Then there exists $K(M) > 0$ with the property that if (1.4) and (1.8) hold, then one can find $t_0 = t_0(u_0, v_0, w_0) \ge 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le K(M)\overline{u}_0 \qquad \text{for all } t > t_0 \tag{5.1}$$

and

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le K(M)\overline{v}_0 \qquad \textit{for all } t > t_0 \tag{5.2}$$

as well as

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} \le K(M) \qquad \textit{for all } t > t_0. \tag{5.3}$$

Proof. From Lemma 3.3, Lemma 3.9 and Lemma 2.2 we infer the existence of $\alpha > 0$ such that whenever $M > 0$, one can find $k_1(M) > 0$ such that if (1.4) and (1.8) hold, we have

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le k_1(M)\overline{u}_0 + e^{-\alpha t} \qquad \text{for all } t > 0 \tag{5.4}$$

and

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le k_1(M)\overline{v}_0 + e^{-\alpha t} \qquad \text{for all } t > 0 \tag{5.5}$$

as well as

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} \le k_1(M) + e^{-\alpha t} \qquad \text{for all } t > 0, \tag{5.6}$$

where we note that in light of the fact that $T_{max} = \infty$ we now know that the number τ in Lemma 3.9 actually satisfies $\tau = 1$. For $t_0 := \frac{1}{\alpha} \cdot \ln_+ \frac{1}{k_1(M)\cdot\min\{\overline{u}_0,\overline{v}_0,1\}}$ and with $K(M) := 2k_1(M)$, the claimed inequalities now directly result from (5.4)-(5.6). □

Now a key toward our proof of stabilization can be found in the following observation on a genuine energy-type structure in (1.3) *when restricted to trajectories corresponding to initial data compatible with (1.8) and (1.9).* The presence of such conditional energy functionals, interpretable as a rigorous mathematical manifestation of superlinear dependence on the unknown in the crucial nonlinearities, has been used in several studies on asymptotic behavior in related chemotaxis problems in the recent few years (see e.g. [23], [7], [31], [26], [32] or also [29] for an incomplete collection); in comparison to most of these, the seemingly most unique feature of the present situation consists in that here it is possible to relax the smallness condition appearing therein in such a substantial manner that in its remaining part it merely reduces to a smallness assumption essentially equivalent to (1.8)-(1.9):

Lemma 5.2 *Let $M > 0$. Then there exists $\delta(M) > 0$ such that if u_0, v_0 and w_0 are such that if beyond (1.4) and (1.8) we have*

$$\Big\{\int_\Omega u_0\Big\} \cdot \Big\{\int_\Omega v_0\Big\}^2 \le \delta(M), \tag{5.7}$$

then it is possible to find $b = b(u_0,v_0,w_0) > 0$ and $t_0 = t_0(u_0,v_0,w_0) > 0$ with the property that

$$\mathcal{F}(t) := \int_\Omega u(\cdot,t)\ln\frac{u(\cdot,t)}{\overline{u}_0} + b\int_\Omega v(\cdot,t)\ln\frac{v(\cdot,t)}{\overline{v}_0} + \frac{\chi_1}{2\lambda}\int_\Omega \frac{w_x^2(\cdot,t)}{w(\cdot,t)}, \qquad t > 0, \tag{5.8}$$

and

$$\mathcal{D}(t) := \frac{1}{2}\int_\Omega \frac{u_x^2(\cdot,t)}{u(\cdot,t)} + \frac{b}{2}\int_\Omega \frac{v_x^2(\cdot,t)}{v(\cdot,t)} + \frac{\chi_1\mu}{4\lambda}\int_\Omega \frac{w_x^2(\cdot,t)}{w(\cdot,t)}, \qquad t > 0, \tag{5.9}$$

satisfy

$$\mathcal{F}'(t) \le -\mathcal{D}(t) \qquad \textit{for all } t > t_0. \tag{5.10}$$

PROOF. Given $M > 0$, we first apply Lemma 5.1 to fix $k_1(M) > 0$ with the property that for any choice of (u_0, v_0, w_0) complying with (1.4) and (1.8) we can find $t_0(u_0, v_0, w_0) \ge 0$ such that for all $t > t_0$,

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le k_1(M)\overline{u}_0, \quad \|v(\cdot,t)\|_{L^\infty(\Omega)} \le k_1(M)\overline{v}_0 \quad \text{and} \quad \|w(\cdot,t)\|_{L^\infty(\Omega)} \le k_1(M), \tag{5.11}$$

and we thereupon claim that the intended conclusion holds if we let

$$\delta(M) := \frac{\mu|\Omega|^3}{8\chi_1\chi_2^2\lambda k_1^4(M)}. \tag{5.12}$$

To see this, we fix any (u_0, v_0, w_0) fulfilling (1.4) and (1.8) as well as (5.7), and abbreviating $t_0 := t_0(u_0, v_0, w_0)$, $L_u := k_1(M)\overline{u}_0$, $L_v := k_1(M)\overline{v}_0$ and $L_w := k_1(M)$ we infer from (5.12) that it is possible to pick $b = b(u_0, v_0, w_0) > 0$ such that

$$\frac{4\chi_1\lambda L_v L_w}{\mu} \le b \le \frac{1}{2\chi_2^2 L_u L_v}. \tag{5.13}$$

We then let $\mathcal{F}$ and $\mathcal{D}$ be as accordingly defined through (5.8) and (5.9), and in order to verify (5.10) we integrate by parts in (1.3) and use (2.3) to compute

$$\frac{d}{dt}\int_\Omega u \ln \frac{u}{\overline{u}_0} = \frac{d}{dt}\int_\Omega u \ln u = -\int_\Omega \frac{u_x^2}{u} + \chi_1 \int_\Omega u_x w_x \tag{5.14}$$

and

$$\frac{d}{dt}\int_\Omega v \ln \frac{v}{\overline{v}_0} = \frac{d}{dt}\int_\Omega v \ln v = -\int_\Omega \frac{v_x^2}{v} + \chi_2 \int_\Omega u_x v_x \tag{5.15}$$

as well as

$$\begin{aligned} \frac{d}{dt}\int_\Omega \frac{w_x^2}{w} &= 2\int_\Omega \frac{w_x}{w}\cdot\Big\{ d w_{xxx} - \lambda u_x w - \lambda u w_x - \lambda v_x w - \lambda v w_x - \mu w_x \Big\} \\ &\quad - \int_\Omega \frac{w_x^2}{w^2}\cdot\Big\{ d w_{xx} - \lambda u w - \lambda v w - \mu w + r \Big\} \\ &= -2d\int_\Omega \frac{w_{xx}^2}{w} + d\int_\Omega \frac{w_x^2 w_{xx}}{w} \\ &\quad -2\lambda\int_\Omega u_x w_x - 2\lambda \int_\Omega v_x w_x - \lambda\int_\Omega \frac{u}{w} w_x^2 - \lambda \int_\Omega \frac{v}{w} w_x^2 - \mu\int_\Omega \frac{w_x^2}{w} \end{aligned} \tag{5.16}$$

for $t > 0$. Here once more integrating by parts we see that

$$\int_\Omega \frac{w_x^4}{w^3} = -\frac{1}{2}\int_\Omega \Big(\frac{1}{w^2}\Big)_x w_x^3 = \frac{3}{2}\int_\Omega \frac{w_x^2 w_{xx}}{w^2} \qquad \text{for all } t > 0,$$

which by the Cauchy-Schwarz inequality firstly entails that

$$\int_\Omega \frac{w_x^4}{w^3} \le \frac{3}{2}\Big\{\int_\Omega \frac{w_{xx}^2}{w}\Big\}^{\frac{1}{2}} \cdot \Big\{\int_\Omega \frac{w_x^4}{w^3}\Big\}^{\frac{1}{2}} \qquad \text{for all } t > 0$$

and hence

$$\int_\Omega \frac{w_x^4}{w^3} \le \frac{9}{4} \int_\Omega \frac{w_{xx}^2}{w} \qquad \text{for all } t>0,$$

and which, as a consequence, secondly shows that thus in (5.16) we can estimate

$$-2d\int_\Omega \frac{w_{xx}^2}{w} + d\int_\Omega \frac{w_x^2 w_{xx}}{w} = -2d\int_\Omega \frac{w_{xx}^2}{w} + \frac{2d}{3}\int_\Omega \frac{w_x^4}{w^3} \le -\frac{d}{2}\int_\Omega \frac{w_{xx}^2}{w} \le 0 \qquad \text{for all } t>0.$$

Upon combining (5.14)-(5.16) and neglecting two further well-signed summands, we therefore obtain that

$$\mathcal{F}'(t) + \int_\Omega \frac{u_x^2}{u} + b\int_\Omega \frac{v_x^2}{v} + \frac{\chi_1\mu}{2\lambda}\int_\Omega \frac{w_x^2}{w} \le b\chi_2 \int_\Omega u_x v_x - \chi_1 \int_\Omega v_x w_x \qquad \text{for all } t>0. \tag{5.17}$$

Here by Young's inequality and (5.11),

$$\begin{aligned} b\chi_2\int_\Omega u_x v_x &\le \frac{1}{2}\int_\Omega \frac{u_x^2}{u} + \frac{b^2\chi_2^2}{2}\int_\Omega u v_x^2 \\ &\le \frac{1}{2}\int_\Omega \frac{u_x^2}{u} + \frac{b^2\chi_2^2}{2}\|u\|_{L^\infty(\Omega)}\|v\|_{L^\infty(\Omega)}\int_\Omega \frac{v_x^2}{v} \\ &\le \frac{1}{2}\int_\Omega \frac{u_x^2}{u} + \frac{b^2\chi_2^2 L_u L_v}{2}\int_\Omega \frac{v_x^2}{v} \\ &\le \frac{1}{2}\int_\Omega \frac{u_x^2}{u} + \frac{b}{4}\int_\Omega \frac{v_x^2}{v} \qquad \text{for all } t>t_0, \end{aligned} \tag{5.18}$$

because thanks to the right inequality in (5.13) we know that

$$\frac{\frac{b^2\chi_2^2 L_u L_v}{2}}{\frac{b}{4}} = 2b\chi_2^2 L_u L_v \le 1.$$

Likewise, Young's inequality together with (5.11) moreover shows that

$$\begin{aligned} -\chi_1\int_\Omega v_x w_x &\le \frac{b}{4}\int_\Omega \frac{v_x^2}{v} + \frac{\chi_1^2}{b}\int_\Omega v w_x^2 \\ &\le \frac{b}{4}\int_\Omega \frac{v_x^2}{v} + \frac{\chi_1^2}{b}\|v\|_{L^\infty(\Omega)}\|w\|_{L^\infty(\Omega)}\int_\Omega \frac{w_x^2}{w} \\ &\le \frac{b}{4}\int_\Omega \frac{v_x^2}{v} + \frac{\chi_1^2 L_v L_w}{b}\int_\Omega \frac{w_x^2}{w} \\ &\le \frac{b}{4}\int_\Omega \frac{v_x^2}{v} + \frac{\chi_1\mu}{4\lambda}\int_\Omega \frac{w_x^2}{w} \qquad \text{for all } t>t_0, \end{aligned} \tag{5.19}$$

since by the left restriction in (5.13),

$$\frac{\frac{\chi_1^2 L_v L_w}{b}}{\frac{\chi_1\mu}{4\lambda}} = \frac{4\chi_1\lambda L_v L_w}{b\mu} \le 1.$$

It thus remains to insert (5.18) and (5.19) into (5.17) to end up with (5.10). □

5.2 Exponential convergence. Proof of Theorem 1.2

A first and rather immediate consequence of (5.10) when combined with well-known inequalities of logarithmic Sobolev and Csiszár-Kullback type yields convergence already at algebraic rates, but yet with respect to spatial L^1 norms only.

Lemma 5.3 *Given $M > 0$, let $\delta(M) > 0$ be as in Lemma 5.2, and suppose that (1.4), (1.8) and (5.7) hold. Then there exist $C = C(u_0, v_0, w_0) > 0$ and $\alpha = \alpha(u_0, v_0, w_0) > 0$ such that*

$$\|u(\cdot,t) - \overline{u}_0\|_{L^1(\Omega)} + \|v(\cdot,t) - \overline{v}_0\|_{L^1(\Omega)} \le Ce^{-\alpha t} \qquad \textit{for all } t > 0. \tag{5.20}$$

Proof. Given (u_0, v_0, w_0) such that (1.4), (1.8) and (5.7) hold, we take $b = b(u_0, v_0, w_0) > 0$ and $t_0 = t_0(u_0, v_0, w_0) > 0$ as provided by Lemma 5.2, and recall that according to a logarithmic Sobolev inequality ([11], [19]) and (2.3) there exists $c_1 > 0$ such that

$$\int_\Omega u \ln \frac{u}{\overline{u}_0} + b \int_\Omega v \ln \frac{v}{\overline{v}_0} \le c_1 \cdot \left\{ \frac{1}{2} \int_\Omega \frac{u_x^2}{u} + \frac{b}{2} \int_\Omega \frac{v_x^2}{v} \right\} \qquad \text{for all } t > 0.$$

Writing $c_2 := \max\{c_1, \frac{2}{\mu}\}$, for $\mathcal{F}$ and $\mathcal{D}$ as in (5.8) and (5.9) we thus obtain that

$$\mathcal{F}(t) \le c_2 \mathcal{D}(t) \qquad \text{for all } t > 0,$$

so that (5.10) implies the autonomous ODI

$$\mathcal{F}'(t) \le -\frac{1}{c_2}\mathcal{F}(t) \qquad \text{for all } t > t_0.$$

Upon integration, this entails that

$$\mathcal{F}(t) \le \mathcal{F}(t_0) e^{-\frac{t-t_0}{c_2}} \qquad \text{for all } t > t_0$$

and thereby establishes (5.20) with suitably large $C > 0$ and $\alpha := \frac{1}{2c_2}$, because according to a Csiszár-Kullback inequality ([8], [3]) and (2.3) we can find $c_3 > 0$ fulfilling

$$\|u(\cdot,t) - \overline{u}_0\|^2_{L^1(\Omega)} \le c_3 \int_\Omega u(\cdot,t) \ln \frac{u(\cdot,t)}{\overline{u}_0} \le c_3 \mathcal{F}(t)$$

and

$$\|v(\cdot,t) - \overline{v}_0\|^2_{L^1(\Omega)} \le c_3 \int_\Omega v(\cdot,t) \ln \frac{v(\cdot,t)}{\overline{v}_0} \le \frac{c_3}{b} \mathcal{F}(t)$$

for all $t > 0$. □

Thanks to the temporally uniform H^1 bounds for both u and v known from Theorem 1.1, a straightforward interpolation finally asserts exponential convergence also with respect to L^∞ norms.

Lemma 5.4 *Under the assumptions of Lemma 5.3, one can find $C = C(u_0, v_0, w_0) > 0$ and $\alpha = \alpha(u_0, v_0, w_0) > 0$ such that*

$$\|u(\cdot,t) - \overline{u}_0\|_{L^\infty(\Omega)} \le Ce^{-\alpha t} \qquad \textit{for all } t > 0 \tag{5.21}$$

and

$$\|v(\cdot,t) - \overline{v}_0\|_{L^\infty(\Omega)} \le Ce^{-\alpha t} \qquad \textit{for all } t > 0. \tag{5.22}$$

PROOF. By means of a Gagliardo-Nirenberg interpolation, we can find $c_1 > 0$ such that

$$\|\varphi - \overline{\varphi}\|_{L^\infty(\Omega)} \le c_1 \|\varphi_x\|_{L^2(\Omega)}^{\frac{2}{3}} \|\varphi - \overline{\varphi}\|_{L^1(\Omega)}^{\frac{1}{3}} \qquad \text{for all } \varphi \in W^{1,2}(\Omega),$$

whereas Theorem 1.1 warrants the existence of $c_2 > 0$ such that

$$\|u_x(\cdot,t)\|_{L^2(\Omega)} \le c_2 \quad \text{and} \quad \|v_x(\cdot,t)\|_{L^2(\Omega)} \le c_2 \qquad \text{for all } t > 0.$$

In view of (2.3), we therefore obtain that for all $t > 0$,

$$\|u(\cdot,t) - \overline{u}_0\|_{L^\infty(\Omega)} + \|v(\cdot,t) - \overline{u}_0\|_{L^\infty(\Omega)} \le c_1 c_2^{\frac{2}{3}} \|u(\cdot,t) - \overline{u}_0\|_{L^1(\Omega)}^{\frac{1}{3}} + c_1 c_2^{\frac{2}{3}} \|v(\cdot,t) - \overline{v}_0\|_{L^1(\Omega)}^{\frac{1}{3}},$$

which due to Lemma 5.3 implies both (5.21) and (5.22). □

On the basis of a final testing procedure independent from the above, again combined with an interpolation argument of the above flavor, we can lastly derive exponential and spatially uniform stabilization also in the third solution component.

Lemma 5.5 *Suppose that the hypotheses of Lemma 5.3 are satisfied. Then there exist $C = C(u_0, v_0, w_0) > 0$ and $\alpha = \alpha(u_0, v_0, w_0) > 0$ such that with $w_\star \ge 0$ given by (1.11) we have*

$$\|w(\cdot,t) - w_\star\|_{L^\infty(\Omega)} \le Ce^{-\alpha t} \qquad \text{for all } t > 0. \tag{5.23}$$

PROOF. Using that by definition of $w_\star$ we have $\{\lambda(\overline{u}_0 + \overline{v}_0) + \mu\} w_\star = r$, upon testing the third equation in (1.3) by $w - w_\star$ we obtain that for all $t > 0$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (w-w_\star)^2 + d\int_\Omega w_x^2 &= \int_\Omega \Big\{ -\lambda(u+v)w - \mu w + r \Big\} \cdot (w - w_\star) \\
&= \int_\Omega \Big\{ -\lambda(\overline{u}_0 + \overline{v}_0)w - \mu w + r \Big\} \cdot (w - w_\star) \\
&\quad -\lambda \int_\Omega (u - \overline{u}_0) w (w - w_\star) - \lambda \int_\Omega (v - \overline{v}_0) w (w - w_\star) \\
&= -c_1 \int_\Omega (w - w_\star)^2 \\
&\quad -\lambda \int_\Omega (u - \overline{u}_0) w (w - w_\star) - \lambda \int_\Omega (v - \overline{v}_0) w (w - w_\star) \qquad (5.24)
\end{aligned}$$

with $c_1 := \lambda(\overline{u}_0 + \overline{v}_0) + \mu > 0$. Here since $\|w(\cdot,t)\|_{L^\infty(\Omega)} \le c_2 := \frac{r}{\mu} + \|w_0\|_{L^\infty(\Omega)}$ for all $t > 0$ by Lemma 2.2, Young's inequality shows that

$$\begin{aligned}
-\lambda \int_\Omega (u - \overline{u}_0) w (w - w_\star) &\le \frac{c_1}{4} \int_\Omega (w - w_\star)^2 + \frac{\lambda^2}{c_1} \int_\Omega (u - \overline{u}_0)^2 w^2 \\
&\le \frac{c_1}{4} \int_\Omega (w - w_\star)^2 + \frac{c_2^2 \lambda^2}{c_1} \int_\Omega (u - \overline{u}_0)^2 \qquad \text{for all } t > 0,
\end{aligned}$$

and similarly estimating the rightmost summand in (5.24) we altogether infer that

$$\frac{d}{dt}\int_\Omega (w - w_\star)^2 + c_1 \int_\Omega (w - w_\star)^2 \le c_3 \int_\Omega (u - \overline{u}_0)^2 + c_3 \int_\Omega (v - \overline{v}_0)^2 \qquad \text{for all } t > 0 \tag{5.25}$$

if we let $c_3 := \frac{2c_2^2\lambda^2}{c_1}$. As Lemma 5.4 provides $c_4 > 0$ and $\alpha_1 \in (0, c_1)$ fulfilling

$$\int_\Omega \Big(u(\cdot,t) - \overline{u}_0\Big)^2 + \int_\Omega \Big(v(\cdot,t) - \overline{v}_0\Big)^2 \le c_4 e^{-\alpha_1 t} \qquad \text{for all } t > 0,$$

through e.g. Lemma 6.1 we readily conclude from (5.25) that

$$\int_\Omega \Big(w(\cdot,t) - w_\star\Big)^2 \le c_5 e^{-\alpha_1 t} \qquad \text{for all } t > 0$$

with $c_5 := \int_\Omega (w_0 + w_\star)^2 + \frac{c_4}{c_1 - \alpha_1}$.

Now since Lemma 4.1 in conjunction with Lemma 3.1 asserts the existence of $c_6 > 0$ such that $\|w(\cdot,t) - w_\star\|_{W^{1,2}(\Omega)} \le c_6$ for all $t > 0$, and hence by the Gagliardo-Nirenberg inequality we can find $c_7 > 0$ such that

$$\|w(\cdot,t) - w_\star\|_{L^\infty(\Omega)} \le c_7 \|w(\cdot,t) - w_\star\|_{W^{1,2}(\Omega)}^{\frac{1}{2}} \|w(\cdot,t) - w_\star\|_{L^2(\Omega)}^{\frac{1}{2}} \qquad \text{for all } t > 0,$$

this implied that

$$\|w(\cdot,t) - w_\star\|_{L^\infty(\Omega)} \le c_5^{\frac{1}{4}} c_6^{\frac{1}{2}} c_7 e^{-\frac{\alpha_1}{4} t} \qquad \text{for all } t > 0$$

and hence entails (5.23). □

Our main statements on temporal asymptotics thereby become almost evident:

PROOF of Theorem 1.2. For $M > 0$ we take $\delta(M) > 0$ as provided by Lemma 5.2 and let $\varepsilon(M) > 0$ be small enough such that $\varepsilon(M) \le \frac{\delta(M)}{M^2}$ and $\varepsilon(M) \le \sqrt{\frac{\delta}{M)}} M$. It can then readily be verified that assuming (1.8) together with (1.9) entails (5.7), so that for completing the proof it is sufficient to collect the statements from Lemma 5.4 and Lemma 5.5. □

6 Appendix: Two statements on ODE comparison

Let us finally state two elementary results of quite straightforward ODE comparison arguments, in view of our above applications with particular focus on the respective dependence on the parameters appearing therein. We begin with a simple observation that has been used in the proofs of Lemma 3.2, Lemma 3.8 and Lemma 5.5.

Lemma 6.1 *Let $\kappa > 0, a \ge 0, b \ge 0$ and $\alpha > 0$ be such that $\alpha < \kappa$, and suppose that $y \in C^0([0,T)) \cap C^1((0,T))$ is a nonnegative function satisfying*

$$y'(t) + \kappa y(t) \le a e^{-\alpha t} + b \qquad \text{for all } t \in (0,T) \tag{6.1}$$

with some $T \in (0,\infty]$. Then

$$y(t) \le \Big(y(0) + \frac{a}{\kappa - \alpha}\Big) e^{-\alpha t} + \frac{b}{\kappa} \qquad \text{for all } t \in (0,T). \tag{6.2}$$

Proof. We let $c_1 := y(0) + \frac{a}{\kappa-\alpha}$ and $\overline{y}(t) := c_1 e^{-\alpha t} + \frac{b}{\kappa}$ for $t \ge 0$. Then clearly $\overline{y}(0) > y(0)$, and since moreover $\overline{y}'(t) + \kappa\overline{y}(t) - ae^{-\alpha t} - b = \{(\kappa-\alpha)c_1 - a\}e^{-\alpha t} = (\kappa-\alpha)y(0)e^{-\alpha t} \ge 0$ for all $t > 0$ thanks to the nonnegativity of y and our assumption that $\alpha < \kappa$, the inequality in (6.2) results from a comparison argument. □

Our second statement in this direction, as used in crucial places in Lemma 3.6 and Lemma 3.7, merely imposes some hypothesis on temporal averages of the respective force term, and therefore requires a slightly more subtle argument:

Lemma 6.2 *Let $\kappa > 0, a \ge 0, b \ge 0$ and $\alpha \in (0,\kappa)$, and assume that with some $T \in (0,\infty]$ and $\tau \in (0,T)$ such that $\tau \le 1$, the nonnegative functions $y \in C^0([0,T)) \cap C^1((0,T))$ and $f \in L^1_{loc}([0,T))$ are such that*

$$y'(t) + \kappa y(t) \le f(t) \qquad \text{for all } t \in (0,T) \tag{6.3}$$

and

$$\int_t^{t+\tau} f(s)ds \le ae^{-\alpha t} + b \qquad \text{for all } t \in [0,T-\tau). \tag{6.4}$$

Then

$$y(t) \le \left\{ \Big(y(0) + a + \frac{a}{\kappa-\alpha} + b\Big) \cdot \frac{e^\alpha}{\tau} + ae^\alpha \right\} \cdot e^{-\alpha t} + \frac{b}{\kappa\tau} + b \qquad \text{for all } t \in (0,T). \tag{6.5}$$

Proof. Letting $z(t) := \int_t^{t+\tau} y(s)ds$ for $t \in [0,T-\tau)$, by two integrations of (6.3), from (6.4) we obtain that firstly

$$z'(t) + \kappa z(t) \le ae^{-\alpha t} + b \qquad \text{for all } t \in (0,T-\tau), \tag{6.6}$$

and that secondly,

$$y(t) \le y(t_0) + \int_{t_0}^t f(s)ds \le y(t_0) + ae^{-\alpha t_0} + b \qquad \text{for all } t_0 \in [0,T) \text{ and } t \in (t_0,T) \text{ such that } t \le t_0 + \tau, \tag{6.7}$$

whence in particular

$$y(t) \le y(0) + a + b \qquad \text{for all } t \in (0,\tau] \tag{6.8}$$

As thus

$$z(0) = \int_0^\tau y(s)ds \le y(0) + a + b$$

due to our assumption that $\tau \le 1$, by applying Lemma 6.1 to (6.6) we see that

$$\begin{aligned} z(t) &\le \Big(z(0) + \frac{a}{\kappa-\alpha}\Big)e^{-\alpha t} + \frac{b}{\kappa} \\ &\le \Big(y(0) + a + \frac{a}{\kappa-\alpha} + b\Big)e^{-\alpha t} + \frac{b}{\kappa} \qquad \text{for all } t \in (0,T-\tau), \end{aligned}$$

which implies that

$$z(t-\tau) \le \Big(y(0) + a + \frac{a}{\kappa-\alpha} + b\Big)e^{-\alpha(t-\tau)} + \frac{b}{\kappa} \qquad \text{for all } t \in (\tau,T). \tag{6.9}$$

In view of the definition of $z(t-\tau)$, this especially means that whenever $t \in (\tau, T)$, we can find $t_0(t) \in (t-\tau, t)$ fulfilling

$$y(t_0(t)) \le \frac{1}{\tau} \cdot \Big(y(0) + a + \frac{a}{\kappa - \alpha} + b \Big) \cdot e^{-\alpha(t-\tau)} + \frac{b}{\kappa\tau}.$$

Again employing (6.7), and moreover using the that $e^{-\alpha t_0} < e^{-\alpha(t-\tau)}$ due to the inclusion $t_0 \in (t-\tau, t)$, we conclude that for any such t,

$$\begin{aligned} y(t) &\le y(t_0(t)) + a e^{-\alpha t_0} + b \\ &\le \Big\{ \frac{1}{\tau} \Big(y(0) + a + \frac{a}{\kappa-\alpha} + b \Big) e^{\alpha\tau} + a e^{\alpha\tau} \Big\} \cdot e^{-\alpha t} + \frac{b}{\kappa\tau} + b, \end{aligned}$$

from which (6.5) immediately follows once more due to the inequality $\tau \le 1$.

If $t \in (0, \tau]$, however, we infer (6.5) directly from (6.8), because clearly

$$y(0) + a + b \le \frac{1}{\tau} \Big(y(0) + a + b \Big) e^{\alpha\tau} e^{-\alpha t}$$

for all $t \in (0, \tau]$. □

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