A three-dimensional Keller-Segel-Navier-Stokes system with logistic source: Global weak solutions and asymptotic stabilization

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Abstract

The Keller-Segel-Navier-Stokes system

$$\left\{ \begin{array}{rcl} n_t + u\cdot\nabla n &=& \Delta n - \chi\nabla\cdot(n\nabla c) + \rho n - \mu n^2, \\ c_t + u\cdot\nabla c &=& \Delta c - c + n, \\ u_t + (u\cdot\nabla)u &=& \Delta u + \nabla P + n\nabla\phi + f(x,t), \\ \nabla\cdot u &=& 0, \end{array} \right. \qquad (\star)$$

is considered in a bounded convex domain $\Omega \subset \mathbb{R}^3$ with smooth boundary, where $\phi \in W^{1,\infty}(\Omega)$ and $f \in C^1(\bar\Omega \times [0,\infty))$, and where $\chi > 0, \rho \in \mathbb{R}$ and $\mu > 0$ are given parameters.

It is proved that under the assumption that $\sup_{t>0}\int_t^{t+1} \|f(\cdot,s)\|_{L^{\frac{6}{5}}(\Omega)} ds$ be finite, for any sufficiently regular initial data (n_0, c_0, u_0) satisfying $n_0 \ge 0$ and $c_0 \ge 0$, the initial-value problem for $(\star)$ under no-flux boundary conditions for n and c and homogeneous Dirichlet boundary conditions for u possesses at least one globally defined solution in an appropriate generalized sense, and that this solution is uniformly bounded in with respect to the norm in $L^1(\Omega) \times L^6(\Omega) \times L^2(\Omega;\mathbb{R}^3)$. Moreover, under the explicit hypothesis that $\mu > \frac{\chi\sqrt{\rho_+}}{4}$, these solutions are shown to stabilize toward a spatially homogeneous state in their first two components by satisfying

$$\Big(n(\cdot,t), c(\cdot,t)\Big) \to \Big(\frac{\rho_+}{\mu}, \frac{\rho_+}{\mu}\Big) \quad \text{in } L^1(\Omega)\times L^p(\Omega) \quad \text{for all } p\in[1,6) \qquad \text{as } t\to\infty.$$

Finally, under an additional condition on temporal decay of f it is shown that also the third solution component equilibrates in that $u(\cdot,t) \to 0$ in $L^2(\Omega;\mathbb{R}^3)$ as $t \to \infty$.

Key words: chemotaxis, Navier-Stokes, large time behavior, generalized solution
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1 Introduction

Mechanisms of chemotaxis, that is, of the partially oriented movement of cells in response to a chemical signal, are known to play an important role in various biological processes, and thus have attracted great interest also at the level of a theoretical understanding ([20], [22]). Here the most thoroughly studied class of models, containing the classical Keller-Segel system ([27]) as the most celebrated representative, concentrates on the mere interaction between the cells and the signal substance, neglecting any interplay with further components. In particular, liquid environments of cell populations are considered to be essentially quiescent and without any significant effect on the cell motion.

Experiments indicate, however, that cells may well influence the motion of a surrounding fluid through bouyant forces due to differences in densities, and that vice versa the fluid-driven transport of cells and signal may substantially affect the overall behavior; for instance, populations of aerobic bacteria suspended in sessile drops of water may exhibit quite a complex but structured collective dynamics, inter alia involving the spontaneous formation of plume-like aggregates ([12], [59], cf. also [8]).

In this work we consider a model for such bioconvection processes in the case when the signal substance is produced by the cells themselves, thus covering numerous biologically relevant situations when cells actively use chemotaxis as a means of communication such as e.g. during the paradigmatic cluster formation in populations of *Escherichia coli* ([45]). Accordingly, we shall subsequently be concerned with the Keller-Segel system coupled to the incompressible Navier-Stokes equations,

$$\begin{cases} n_t + u\cdot\nabla n &= \Delta n - \chi\nabla\cdot(n\nabla c) + \rho n - \mu n^2, & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c &= \Delta c - c + n, & x\in\Omega,\ t>0,\\ u_t + (u\cdot\nabla)u &= \Delta u + \nabla P + n\nabla\phi + f(x,t), & x\in\Omega,\ t>0,\\ \nabla\cdot u &= 0, & x\in\Omega,\ t>0, \end{cases} \tag{1.1}$$

for the unknown population density n, the signal concentration c, the fluid velocity field u and the associated pressure P, in the physical domain $\Omega\subset\mathbb{R}^3$. Here the constant χ denotes the chemotactic sensitivity, and supposing henceforth that $\chi>0$ presumes that cells move toward increasing concentrations of the signal substance which is produced by themselves. The fluid motion is driven, besides by a possibly nonzero external force f, by the presence of cells, as mediated through the gravitational potential ϕ. Furthermore, in requiring throughout that $\mu>0$, we shall presuppose in (1.1) that on the considered time scales of tactic migration and convection, also degradation effects are relevant at large cell population densities, where in apparently good accordance with well-established modeling approaches ([20], [49]; cf. also the particular chemotaxis-fluid model introduced in [28]) we assume the underlying mechanisms to be of essentially (self-)competition type, thus leading to standard quadratic absorption. Apart from that, allowing for arbitrary $\rho\in\mathbb{R}$ and hence explicitly including the case $\rho=0$ enables us to cover also situations when individuals undergo quadratic decay but do not reproduce, such as typically occurring in bioconvection models from the context of broadcast spawning ([28], [29], [11]). For a recent rigorous derivation of (1.1) on the basis of fundamental principles from the kinetic theory of active particles, we refer to [1].

Mathematical challenges. The system (1.1) couples the delicate structures of three-dimensional fluid dynamics on the one hand, and of chemotactic cross-diffusion reinforced by signal production on the other. Indeed, even when posed without any external influence, the corresponding Navier-Stokes

system does not admit a satisfactory solution theory up to now; beyond classical results on global weak solvability ([36], [53]), on local existence of smooth solutions ([15], [63]), and on absence of self-similar blow-up ([47]), the question of global solvability in classes of suitably regular functions yet remains open except in cases when the initial data are appropriately small ([63]).
Besides this, also the corresponding chemotaxis-only subsystem of (1.1) obtained on letting $u \equiv 0$, that is, the system

$$\begin{cases} n_t = \Delta n - \chi\nabla \cdot (n\nabla c) + \rho n - \mu n^2, & x \in \Omega,\ t > 0, \\ c_t = \Delta c - c + n, & x \in \Omega,\ t > 0, \end{cases} \tag{1.2}$$

already possesses quite a delicate mathematical structure. In fact, if posed along with homogeneous Neumann boundary conditions in three-dimensional balls, in the borderline case $\mu = 0$, even when $\rho = 0$ this system is known to possess solutions which undergo a finite-time explosion in the sense that for appropriate initial data, up to some $T > 0$ a classical solution (n, c) exists but satisfies $\limsup_{t\nearrow T} \|n(\cdot, t)\|_{L^\infty(\Omega)} = \infty$ ([67]). This striking property of the classical Keller-Segel system has moreover been detected in numerous related frameworks such as in two-dimensional radially symmetric ([19], [25], [2], [50]) and also some nonradial settings ([46]), and is frequently interpreted as reflecting its ability to properly describe aggregation phenomena.

On the other hand, the presence of a quadratic degradation term in (1.2), as implied by requiring $\mu > 0$, may have a substantial regularizing effect in this respect: In the two-dimensional case, for instance, it is known that whenever $\mu > 0$, (1.2) possesses global bounded smooth solutions for widely arbitrary initial data ([48]); in the three-dimensional counterpart, a similar statement holds under the additional assumption that $\mu > \mu_0$ with some $\mu_0 = \mu_0(\chi, \rho, \Omega) > 0$ ([64]). Apparently, however, the literature so far leaves open the question whether blow-up may occur in the three-dimensional version of (1.2) in cases when μ is positive but small, in which only weak solutions are known to exist globally ([35]). Anyhow, the strongly destabilizing action of chemotactic cross-diffusion has recently been shown to enforce the occurrence of solutions which attain possibly finite but arbitrarily large values when $\mu > 0$ is suitably small ([75]; cf. also [34] for a precedent concerning a a related parabolic-elliptic variant). A further mathematical caveat reports finite-time blow-up of some solutions to a high-dimensional parabolic-elliptic modification of (1.2) with the death term $-\mu n^2$ replaced by certain superlinear but subquadratic terms ([65]).

Analysis of chemotaxis-fluid systems. As recent analytical findings show, already cell transport through a given fluid may substantially influence the solution behavior in certain Keller-Segel-type chemotaxis systems ([28], [29]); in fact, even complete suppression of blow-up may occur ([30]). To the best of our knowledge, however, the full mutual coupling of equations from fluid dynamics to chemotaxis systems, including buoyancy-driven feedback on the fluid motion of the form in (1.1), has been considered in the analytical literature mainly in cases when the signal substance is consumed by the cells, rather than produced as in (1.2). For such models, numerous results have been achieved in the past few years, addressing topics ranging from the mere existence of global solutions ([13], [73], [3], [6]), also in the context of nonlinear cell diffusion ([14], [55], [60], [9], [77]), to global boundedness ([66], [5], [24], [4]) and even asymptotic issues such as eventual regularity and stabilization ([68], [7], [74], [71], [76]).

In contrast to this, the knowledge on chemotaxis-fluid systems involving signal production such as in

(1.2) seems rather thin, although numerical simulations for a parabolic-elliptic variant of (1.1) posed in $\Omega = \mathbb{R}^2$ indicate interesting effects such as fluid-induced blow-up prevention in the sense that global regular solutions exist despite the fact that the same initial data enforce finite-time blow-up in the corresponding fluid-free Keller-Segel system ([43]).

Beyond a result on global existence of certain small-data solutions ([31]), the apparently first analytical result in this context which addresses large initial data asserts global existence of certain weak solutions in a two-dimensional simplification of (1.1) obtained on neglecting the nonlinear convection term $(u\cdot\nabla)u$ in the case $\rho = 0$ and $f \equiv 0$ ([11]). Global bounded smooth solutions of a boundary-value problem for the three-dimensional counterpart of this chemotaxis-Stokes variant of (1.1) have recently been shown to exist for any $\rho \ge 0$, suitably good-natured f and appropriately large $\mu > 0$ ([56]). In smoothly bounded two-dimensional domains, the full system (1.1) has been addressed in [57], where global bounded classical solutions have been constructed for arbitrary $\rho \ge 0$ and $\mu > 0$ and any f satisfying appropriate regularity and boundedness conditions. Moreover, the latter two works also provide some results on the asymptotic decay of solutions when $\rho = 0$. For certain chemotaxis-(Navier-)Stokes modifications of (1.1) involving some inhibition of chemotactic cross-diffusion at large cell or signal densities, results on global existence and partially also on boundedness in two- and three-dimensional settings have recently been established in [61] and [62] as well as in [39], [40] and [41].

Main results. The present work focuses on an initial-boundary value problem for the full Keller-Segel-Navier-Stokes system (1.1) in the delicate spatially three-dimensional framework in which any theory needs to adequately cope with the circumstance that apparently only quite weak regularity information is available in general; accordingly, already the construction of global solutions for arbitrarily large initial data, constituting our first objective, will require substantial efforts and, in particular, the design of a seemingly non-standard concept of generalized solvability. The second goal will thereafter consist in describing the large time behavior of these solutions under appropriate assumptions. Here since a considerably complex dynamical behavior must be expected to occur already in the Keller-Segel-growth system (1.2) according to numerical simulations ([21]) and analytical results on the associated set of equilibria ([32]), and since we do not expect the additional presence of a fluid interaction to have any regularizing effect in this regard, our exploration in this direction will concentrate on the case when $\mu > 0$ is adequately large, having in mind previous studies asserting uniqueness and global attractivity of nontrivial equilibria for the fluid-free system (1.2) if μ is large ([69]).

In order to prepare a precise statement of our main results in these respects, let us fix the mathematical framework by considering (1.1) in a bounded convex domain $\Omega \subset \mathbb{R}^3$ with smooth boundary, where $\chi > 0, \rho \in \mathbb{R}$ and $\mu > 0$, and where for simplicity we assume that

$$\phi \in W^{1,\infty}(\Omega) \qquad \text{and} \qquad f \in C^1(\bar\Omega \times [0,\infty);\mathbb{R}^3), \tag{1.3}$$

and that moreover f has the boundedness property

$$\sup_{t>0} \int_t^{t+1} \|f(\cdot,s)\|^2_{L^{\frac{6}{5}}(\Omega)} ds < \infty. \tag{1.4}$$

We shall consider (1.1) along with the initial conditions

$$n(x,0) = n_0(x), \quad c(x,0) = c_0(x) \quad \text{and} \quad u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.5}$$

and under the boundary conditions

$$\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0 \quad \text{and} \quad u = 0 \qquad \text{on } \partial\Omega. \tag{1.6}$$

Here our standing assumptions are that

$$\begin{cases} n_0 \in C^0(\bar{\Omega}) \quad \text{is nonnegative with } n_0 \not\equiv 0, \quad \text{that} \\ c_0 \in W^{1,\infty}(\Omega) \quad \text{is nonnegative,} \quad \text{and that} \\ u_0 \in W^{2,2}(\Omega) \cap W^{1,2}_{0,\sigma}(\Omega), \end{cases} \tag{1.7}$$

where $W^{1,2}_{0,\sigma}(\Omega)$ denotes the closure of $C^\infty_{0,\sigma}(\Omega) := C^\infty_0(\Omega) \cap L^2_\sigma(\Omega)$ with respect to the norm in $W^{1,2}(\Omega) \equiv W^{1,2}(\Omega;\mathbb{R}^3)$, with $L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega;\mathbb{R}^3) \mid \nabla\cdot\varphi = 0\}$ representing the space of all solenoidal vector fields in $L^2(\Omega)$.

In this context, we can in fact assert the following result on global existence in which, as substantiated more thoroughly near the end of this introduction and in Section 2, we refer to a generalized solution concept, apparently novel in this context, that turns out to be mild enough so as to favorably cooperate with the sparse regularity information to be gathered in the course of our subsequent analysis.

Theorem 1.1 *Let $\chi > 0$, $\rho \in \mathbb{R}$ and $\mu > 0$, and suppose that ϕ, f and (n_0, c_0, u_0) comply with (1.3), (1.4) and (1.7). Then the problem (1.1), (1.5), (1.6) possesses at least one global generalized solution (n, c, u) in terms of Definition 2.2 below. Furthermore, this solution satisfies*

$$\begin{cases} n \in L^2_{loc}(\bar{\Omega} \times [0,\infty)) \cap L^{\frac{16}{13}}_{loc}([0,\infty); W^{1,\frac{16}{13}}(\Omega)), \\ c \in L^\infty((0,\infty); L^6(\Omega)) \cap L^{\frac{8}{5}}_{loc}([0,\infty); W^{2,\frac{8}{5}}(\Omega)) \qquad \textit{and} \\ u \in L^\infty((0,\infty); L^2_\sigma(\Omega)) \cap L^{\frac{10}{3}}_{loc}(\bar{\Omega} \times [0,\infty)) \cap L^2_{loc}([0,\infty); W^{1,2}_{0,\sigma}(\Omega)), \end{cases} \tag{1.8}$$

and there exists $C > 0$ such that

$$\|n(\cdot,t)\|_{L^1(\Omega)} \le C, \quad \|c(\cdot,t)\|_{L^6(\Omega)} \le C \quad \textit{and} \quad \|u(\cdot,t)\|_{L^2(\Omega)} \le C \qquad \textit{for a.e. } t > 0. \tag{1.9}$$

Moreover, (n, c, u) can be obtained as the limit of solutions $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ to the regularized problems (3.1) below in the sense that there exists $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_j \to 0$ as $j \to \infty$ and

$$n_\varepsilon \to n, \quad c_\varepsilon \to c \quad \textit{and} \quad u_\varepsilon \to u \qquad \textit{a.e. in } \Omega \times (0,\infty)$$

as $\varepsilon = \varepsilon_j \to 0$.

Remark. By straightforward adaptation of our reasoning demonstrated below, it is possible to assert global existence of a generalized solution actually under much weaker assumptions on f than those stated in (1.3) and (1.4). In fact, it turns out that for the pure existence result no temporally uniform boundedness condition like (1.4) is required; in such cases, however, boundedness properties of u such as formulated in (1.9) can evidently no longer be expected.

Next, under an explicit condition on the size of μ relative to χ and ρ, we can make sure that all our solutions approach a spatially homogeneous equilibrium in their first two components. We note that even in the fluid-free case when $u \equiv 0$, this apparently provides some progress with regard to stabilization in the resulting logistic Keller-Segel system: For the latter, namely, though it has already been known that large values of $\frac{\mu}{\chi}$ enforce asymptotic stability of corresponding constant equilibria ([69]), findings on explicit lower bounds on $\frac{\mu}{\chi}$ ensuring this seem yet lacking.

Theorem 1.2 *Let $\chi > 0$ and $\rho \in \mathbb{R}$, and suppose that*

$$\mu > \frac{\chi\sqrt{\rho_+}}{4}, \tag{1.10}$$

where $\rho_+ := \max\{\rho, 0\}$. If ϕ, f and (n_0, c_0, u_0) satisfy (1.3), (1.4) and (1.7), and if (n, c, u) denotes the corresponding global generalized solution of (1.1), (1.5), (1.6) provided by Theorem 1.1, then there exists a null set $N \subset (0, \infty)$ such that

$$\Big\| n(\cdot, t) - \frac{\rho_+}{\mu} \Big\|_{L^1(\Omega)} \to 0 \quad \textit{and} \quad \Big\| c(\cdot, t) - \frac{\rho_+}{\mu} \Big\|_{L^p(\Omega)} \to 0 \quad \textit{for all } p \in [1, 6) \qquad \textit{as } (0, \infty) \setminus N \ni t \to \infty. \tag{1.11}$$

Finally, under an additional requirement on asymptotic decrease of f in a suitable sense it is also possible to assert decay of the fluid velocity with respect to the norm in $L^2(\Omega)$, as known to be the case in the unforced Navier-Stokes system.

Theorem 1.3 *Let $\chi > 0$, $\rho \in \mathbb{R}$ and $\mu > 0$ be such that (1.10) holds, and suppose that ϕ, f and (n_0, c_0, u_0) satisfy (1.3) and (1.7) as well as*

$$\int_t^{t+1} \|f(\cdot, s)\|^2_{L^{\frac{6}{5}}(\Omega)} ds \to 0 \qquad \textit{as } t \to \infty. \tag{1.12}$$

Then for any choice of (n_0, c_0, u_0) fulfilling (1.7), the global generalized solution of (1.1), (1.5), (1.6) given by Theorem 1.1 has the property that for some null set $N \subset (0, \infty)$ we have

$$\|u(\cdot, t)\|_{L^2(\Omega)} \to 0 \qquad \textit{as } (0, \infty) \setminus N \ni t \to \infty. \tag{1.13}$$

Plan of the paper. Unlike substantial bodies of precedent studies on chemotaxis-fluid systems involving signal consumption mechanisms ([13], [66], [73]), an existence theory for (1.1) can apparently not be based on a proper exploitation of energy-type inequalities yielding appropriate a priori information on regularity of solutions. In fact, already the fluid-free chemotaxis system (1.2) seems to lack any gradient-like structure whenever $\mu \neq 0$. Accordingly, our approach toward Theorem 1.1 will be based on alternative methods to verify the intuitive idea that the quadratic degradation term in the first equation of (1.1) should enforce suitable regularity properties, uniformly with respect to a small parameter ε involved in certain approximations of (1.1) which are known to admit global smooth solutions (see (3.1)).

Indeed, as first consequences of an easily obtained estimate on $\int_t^{t+1} \int_\Omega n^2$ for such approximate solutions (Lemma 3.2), we will achieve bounds for $\int_\Omega c^6$ (Lemma 3.6) as well as for $\int_\Omega |u|^2$ and $\int_t^{t+1} \int_\Omega |\nabla u|^2$ (Lemma 3.8); unlike in the analysis of (1.2), however, in the presence of coupling to fluid flows this initial information on n seems insufficient to directly entail bounds for ∇c, e.g. with respect to the norm in $L^2(\Omega)$ and uniformly in time, as can be gained in a standard manner for solutions of (1.2) upon testing the second equation therein by Δc (cf. e.g. [35]). In our alternative attempt to nevertheless derive some basic regularity properties of the signal gradient, a key role will be played by an analysis of the functional $\int_\Omega (|\nabla c|^2+1)^{\frac{2}{3}}$ in which we will rely on an interpolation inequality (Lemma 4.1) which allows to control $\int_\Omega (|\nabla c|^2+1)^{\frac{4}{3}}$ in terms of $\int_\Omega c^6$ and an integral involving certain second-order expressions which appear as a dissipated quantity in this context. Based on the previously gained information, this will finally yield estimates for $\int_t^{t+1}\int_\Omega c^8$, $\int_t^{t+1}\int_\Omega |\nabla c|^{\frac{8}{3}}$ and $\int_t^{t+1}\int_\Omega |D^2 c|^{\frac{8}{5}}$ (Lemma 4.3 and Lemma 4.4) and thereupon provide sufficient regularity of the chemotactic flux term in (1.1) so as to allow for successfully tracking the time evolution of the concave functional $\int_\Omega n^{\frac{3}{4}}$ with the conclusion that also $\int_t^{t+1}\int_\Omega n^{-\frac{5}{4}}|\nabla n|^2$ and $\int_t^{t+1}\int_\Omega |\nabla n|^{\frac{16}{13}}$ enjoy appropriate bounds (Lemma 5.1 and Lemma 5.2). Section 7 will then be devoted to a straightforward limit procedure and the verification that the regularity properties gathered above are sufficient to justify that the achieved limit indeed satisfies (1.1) in the generalized sense described in Section 2.

Finally, the verification of the stabilization results in Theorem 1.2 and Theorem 1.3 is based on the observation that in the nontrivial case $\rho > 0$, the largeness assumption (1.10) on μ warrants that at least formally,

$$\mathcal{F}_{n_\star,B}(n,c) := \int_\Omega \Big(n - n_\star - n_\star \ln \frac{n}{n_\star} \Big) + \frac{B}{2}\int_\Omega (c-n_\star)^2$$

acts as a Lyapunov functional for (1.1) when $n_\star := \frac{\rho}{\mu}$ and $B > 0$ lies in an appropriate range. Here mathematical obstacles in a rigorous justification of this property, mainly stemming from the lack of sufficient regularity of the limit gained before, can be overcome by firstly remaining at the level of approximate solutions and proving that the corresponding energy inequality for these solutions provides some quantitative convergence properties which are uniform with respect to the regularization parameter ε (Section 8.3). A key step in this context will consist in controlling $\mathcal{F}_{n_\star,B}(n,c)$ from above in terms of the corresponding dissipation rate (Section 8.2).

2 A generalized solution concept

To begin with, let us specify the solution concept which will be pursued in the sequel. With regard to the second and third equations in (1.1), our concept will basically coincide with the natural notions of respective weak solutions of the corresponding subproblems. According to a lack of knowledge on suitable regularity and compactness properties encountered below, however, concerning the crucial first solution component n we shall need to further weaken our concept so as to require that n, instead of fulfilling an integral *identity*, rather satisfies two integral *inequalities* which will, in a weak sense, identify n as a subsolution and supersolution of the respective evolution subsystem of (1.1). Here a genuine relaxation will stem from the fact that these inequalities need not necessarily involve n itself but may rather refer to transformed variants thereof; however, in contrast to related but apparently different notions of *renormalized solutions* ([10], [51]), all these transformations will be required to be globally injective in that the involved mappings Φ will be assumed strictly increasing throughout $[0,\infty)$. Let us first make this more precise, thus yet concentrating on the first equation in (1.1).

Definition 2.1 *Let $\Phi \in C^2([0,\infty))$ be nonnegative with $\Phi' > 0$ on $[0,\infty)$, and let $n_0 : \Omega \to [0,\infty)$, $c_0 : \Omega \to \mathbb{R}$ and $u_0 : \Omega \to \mathbb{R}^3$ are such that $\Phi(n_0) \in L^1(\Omega)$. Suppose that $n : \Omega \times (0,\infty) \to [0,\infty)$,*

$c : \Omega \times (0,\infty) \to \mathbb{R}$ and $u : \Omega \times (0,\infty) \to \mathbb{R}^3$ are such that $\nabla n, \nabla c$ and Δc are measurable, that

$$\Phi(n), \Phi''(n)|\nabla n|^2, \Phi(n)\Delta c, n\Phi'(n)\Delta c, n\Phi'(n) \text{ and } n^2\Phi'(n) \quad \text{belong to } L^1_{loc}(\bar\Omega \times [0,\infty)), \tag{2.1}$$

that moreover

$$\Phi'(n)\nabla n, \Phi(n)\nabla c \text{ and } \Phi(n)u \quad \text{lie in } L^1_{loc}(\bar\Omega \times [0,\infty); \mathbb{R}^3) \tag{2.2}$$

and that $\nabla \cdot u \equiv 0$ in $\mathcal{D}'(\Omega \times (0,\infty))$.
Then (n,c,u) will be called a weak Φ-subsolution *(resp., a* weak Φ-supersolution*) of the first equations in (1.1), (1.5), (1.6) if*

$$\begin{aligned}
-\int_0^\infty \int_\Omega \Phi(n)\varphi_t - \int_\Omega \Phi(n_0)\varphi(\cdot,0) \overset{(\geq)}{\leq} &-\int_0^\infty \int_\Omega \Phi''(n)|\nabla n|^2\varphi - \int_0^\infty \int_\Omega \Phi'(n)\nabla n \cdot \nabla\varphi \\
&+\chi \int_0^\infty \int_\Omega \Phi(n)\nabla c \cdot \nabla\varphi + \chi \int_0^\infty \int_\Omega \Phi(n)\Delta c\varphi - \chi \int_0^\infty \int_\Omega n\Phi'(n)\Delta c\varphi \\
&+\rho \int_0^\infty \int_\Omega n\Phi'(n)\varphi - \mu \int_0^\infty \int_\Omega n^2\Phi'(n)\varphi + \int_0^\infty \int_\Omega \Phi(n)u \cdot \nabla\varphi
\end{aligned} \tag{2.3}$$

holds for all nonnegative $\varphi \in C_0^\infty(\bar\Omega \times [0,\infty))$.

Now our concept of solutions for the whole system (1.1), (1.5), (1.6) reads as follows. Here any below, for given vectors $v \in \mathbb{R}^3$ and $w \in \mathbb{R}^3$ we define the matrix $v \otimes w$ by letting $(v \otimes w)_{ij} := v_i w_j$ for $i,j \in \{1,2,3\}$.

Definition 2.2 *A triple of functions*

$$\begin{cases} n \in L^1_{loc}(\bar\Omega \times [0,\infty)), \\ c \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)), \\ u \in L^1_{loc}([0,\infty); W_0^{1,1}(\Omega;\mathbb{R}^3)) \end{cases}$$

fulfilling

$$cu \in L^1_{loc}(\bar\Omega \times [0,\infty); \mathbb{R}^3) \qquad \text{and} \qquad u \otimes u \in L^1_{loc}(\Omega \times [0,\infty); \mathbb{R}^{3\times 3}),$$

as well as $n \geq 0$ a.e. in $\Omega \times (0,\infty)$, is said to be a generalized solution *of (1.1), (1.5), (1.6) if $\nabla \cdot u = 0$ in $\mathcal{D}'(\Omega \times (0,\infty))$, if*

$$\begin{aligned}
-\int_0^\infty \int_\Omega c\varphi_t - \int_\Omega c_0\varphi(\cdot,0) &= -\int_0^\infty \int_\Omega \nabla c \cdot \nabla\varphi - \int_0^\infty \int_\Omega c\varphi \\
&\quad + \int_0^\infty \int_\Omega n\varphi + \int_0^\infty \int_\Omega cu \cdot \nabla\varphi
\end{aligned} \tag{2.4}$$

for all $\varphi \in C_0^\infty(\bar\Omega \times [0,\infty))$ and

$$\begin{aligned}
-\int_0^\infty \int_\Omega u \cdot \varphi_t - \int_\Omega u_0 \cdot \varphi(\cdot,0) &= -\int_0^\infty \int_\Omega \nabla u \cdot \nabla\varphi + \int_0^\infty \int_\Omega (u \otimes u) \cdot \nabla\varphi \\
&\quad + \int_0^\infty \int_\Omega n\nabla\phi \cdot \varphi + \int_0^\infty \int_\Omega f \cdot \varphi
\end{aligned} \tag{2.5}$$

for all $\varphi \in C_0^\infty(\Omega \times [0,\infty); \mathbb{R}^3)$ fulfilling $\nabla \cdot \varphi \equiv 0$, and if there exist $\Phi_1, \Phi_2 \in C^2([0,\infty))$ such that $\Phi_1' > 0$ and $\Phi_2' > 0$ on $[0,\infty)$, and such that (n,c,u) is a weak Φ_1-subsolution and a weak Φ_2-supersolution of the first equations in (1.1), (1.5), (1.6) in the sense of Definition 2.1.

Remark. It can easily be checked by standard arguments, following those detailed in [72] in a relatd but simpler context, that each generalized solution (n,c,u) which is sufficiently smooth, e.g. in the sense that n and c belong to $C^0(\bar\Omega\times[0,\infty))\cap C^{2,1}(\bar\Omega\times(0,\infty))$ and that u lies in $C^0(\bar\Omega\times[0,\infty);\mathbb{R}^3)\cap C^{2,1}(\bar\Omega\times(0,\infty);\mathbb{R}^3)$, is actually classical; that is, in that case there exists $P\in C^{1,0}(\bar\Omega\times(0,\infty))$ such that (n,c,u,P) is a global classical solution of (1.1), (1.5), (1.6).

3 Approximate solutions and basic properties

In order to gain a solution of (1.1), (1.5), (1.6) through a suitable approximation procedure, we follow well-established approaches to regularize both the chemotactic sensitivity in the first equation in (1.1) as well as the nonlinear convective term in the Navier-Stokes subsystem of (1.1) ([44], [53], [63]). Accordingly, let us introduce the family of approximate problems given by

$$\begin{cases} n_{\varepsilon t}+u_\varepsilon\cdot\nabla n_\varepsilon = \Delta n_\varepsilon-\chi\nabla\cdot\big(\frac{n_\varepsilon}{1+\varepsilon n_\varepsilon}\nabla c_\varepsilon\big)+\rho n_\varepsilon-\mu n_\varepsilon^2, & x\in\Omega,\ t>0,\\ c_{\varepsilon t}+u_\varepsilon\cdot\nabla c_\varepsilon = \Delta c_\varepsilon-c_\varepsilon+n_\varepsilon, & x\in\Omega,\ t>0,\\ u_{\varepsilon t}+(Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon = \Delta u_\varepsilon+\nabla P_\varepsilon+n_\varepsilon\nabla\phi+f(x,t), & x\in\Omega,\ t>0,\\ \nabla\cdot u_\varepsilon = 0, & x\in\Omega,\ t>0,\\ \frac{\partial n_\varepsilon}{\partial\nu}=0,\quad \frac{\partial c_\varepsilon}{\partial\nu}=0,\quad u_\varepsilon=0, & x\in\partial\Omega,\ t>0,\\ n_\varepsilon(x,0)=n_0(x),\quad c_\varepsilon(x,0)=c_0(x),\quad u_\varepsilon(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{3.1}$$

for $\varepsilon\in(0,1)$, where Y_ε denotes the Yosida approximation determined by

$$Y_\varepsilon\varphi := (1+\varepsilon A)^{-1}\varphi,\qquad \varphi\in L^2_\sigma(\Omega). \tag{3.2}$$

Here and in the next lemma, by $A:=-\mathcal{P}\Delta$ we mean the realization of the Stokes operator in $L^2_\sigma(\Omega)$ with domain given by $D(A)=W^{2,2}(\Omega)\cap W^{1,2}_{0,\sigma}(\Omega)$, where $\mathcal{P}:L^2(\Omega)\to L^2_\sigma(\Omega)$ denotes the Helmholtz projection. It is well-known that A is self-adjoint and sectorial in $L^2_\sigma(\Omega)$ and hence possesses densely defined self-adjoint fractional powers A^α for any $\alpha\in\mathbb{R}$.

All these problems (3.1) are indeed globally solvable in the classical sense:

Lemma 3.1 *For each $\varepsilon\in(0,1)$, there exist functions*

$$\begin{cases} n_\varepsilon\in C^0(\bar\Omega\times[0,\infty))\cap C^{2,1}(\bar\Omega\times(0,\infty)),\\ c_\varepsilon\in C^0(\bar\Omega\times[0,\infty))\cap C^{2,1}(\bar\Omega\times(0,\infty)),\\ u_\varepsilon\in C^0(\bar\Omega\times[0,\infty))\cap C^{2,1}(\bar\Omega\times(0,\infty)) \qquad \textit{and}\\ P_\varepsilon\in C^{1,0}(\bar\Omega\times(0,\infty)) \end{cases}$$

with the property that $(n_\varepsilon,c_\varepsilon,u_\varepsilon,P_\varepsilon)$ solves (3.1) classically in $\bar\Omega\times(0,\infty)$.

Proof. Local existence, up to a maximal existence time $T_{max,\varepsilon}\in(0,\infty]$, can be seen e.g. by means of a contraction mapping argument involving standard regularity theories for the heat equation and the Stokes system ([33], [52], [54], [17], [16], [18]; cf. also [42] for an early approach), where

$$\text{if } T_{max,\varepsilon}<\infty,\quad \text{then for all } q>3 \text{ and } \alpha\in(\tfrac34,1),$$
$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}+\|c_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)}+\|A^\alpha u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\to\infty \qquad \text{as } t\nearrow T_{max,\varepsilon}; \tag{3.3}$$

details in a corresponding reasoning in a closely related framework can be found in [66, Lemma 2.1]. Thereafter, assuming $T_{max,\varepsilon}$ to be finite for contradiction and fixing any $q > 3$ and $\alpha \in (\frac{3}{4}, 1)$ for definiteness, due to the regularizing actions of the Yosida approximation in the third and the boundedness of the saturated chemotactic sensitivity $\frac{n_\varepsilon}{1+\varepsilon n_\varepsilon}$ in the first equation in (3.1), by following well-established a priori estimation procedures it is possible to derive bounds, possibly depending on ε, q and α but not on $t \in (\frac{1}{2}T_{max,\varepsilon}, T_{max,\varepsilon})$, for $n_\varepsilon(\cdot,t), c_\varepsilon(\cdot,t)$ and $A^\alpha u_\varepsilon(\cdot,t)$ in $L^\infty(\Omega), W^{1,q}(\Omega)$ and $L^2(\Omega)$, respectively (cf. the reasonings in [66, Lemma 5.4] and [73, Lemma 3.9], for instance). In view of (3.3), this means that in fact $T_{max,\varepsilon} = \infty$. $\square$

The following basic and essentially well-known properties of these solutions are due to the presence of the quadratic degradation term in the first equation in (3.1).

Lemma 3.2 *For each $\varepsilon \in (0,1)$, the solution of (3.1) satisfies*

$$\int_\Omega n_\varepsilon(x,t)dx \le m := \max\left\{ \int_\Omega n_0 \, , \, \frac{\rho_+|\Omega|}{\mu} \right\} \qquad \textit{for all } t > 0 \tag{3.4}$$

and

$$\int_t^{t+1} \int_\Omega n_\varepsilon^2(x,s)dxds \le \frac{(\rho_+ + 1)m}{\mu} \qquad \textit{for all } t > 0. \tag{3.5}$$

Proof. By the Hölder inequality, integrating the first equation in (3.1) implies that

$$\frac{d}{dt}\int_\Omega n_\varepsilon = \rho \int_\Omega n_\varepsilon - \mu \int_\Omega n_\varepsilon^2 \le \rho_+ \int_\Omega n_\varepsilon - \frac{\mu}{|\Omega|}\left(\int_\Omega n_\varepsilon\right)^2 \qquad \text{for all } t > 0, \tag{3.6}$$

which entails (3.4) on an ODE comparison. A time integration of the first identity in (3.6) thereafter yields (3.5). $\square$

The estimate (3.4) has an immediate consequence on the spatial L^1 norm of c_ε.

Lemma 3.3 *We have*

$$\int_\Omega c_\varepsilon(x,t)dx \le \max\left\{ \int_\Omega c_0 \, , \, m \right\} \qquad \textit{for all } t > 0, \tag{3.7}$$

where m is as given by (3.4).

Proof. We integrate the second equation in (3.1) and use Lemma 3.2 to find that

$$\frac{d}{dt}\int_\Omega c_\varepsilon + \int_\Omega c_\varepsilon = \int_\Omega n_\varepsilon \le m \qquad \text{for all } t > 0.$$

By comparison, this proves (3.7). $\square$

3.1 A bound for c_ε in $L^6(\Omega)$

On the basis of (3.5), we next plan to improve our knowledge on the spatial regularity of c_ε, having in mind that any approach toward this which proceeds by testing the second equation in (3.1) against powers of c_ε will not rely on any a priori information on regularity of the fluid profile, because $\nabla \cdot u_\varepsilon \equiv 0$. In order to make use of the spatio-temporal information provided by (3.5) in the course of our corresponding testing procedure in Lemma 3.6, we shall utilize the following elementary lemma which will also be referred to in several places below.

Lemma 3.4 *Let $t_0 \in \mathbb{R}, T \in (t_0, \infty]$, and suppose that the nonnegative function $h \in L^1_{loc}(\mathbb{R})$ has the property that there exist $\tau > 0$ and $b > 0$ such that*

$$\frac{1}{\tau}\int_t^{t+\tau} h(s)ds \le b \qquad \textit{for all } t \in (t_0, T). \tag{3.8}$$

Then for any choice of $a > 0$ we have

$$\int_{t_0}^t e^{-a(t-s)}h(s)ds \le \frac{b\tau}{1-e^{-a\tau}} \qquad \textit{for all } t \in [t_0, T). \tag{3.9}$$

Consequently, if $y \in C^0([t_0,T)) \cap C^1((t_0,T))$ has the property that

$$y'(t) + ay(t) \le h(t) \qquad \textit{for all } t \in (t_0, T),$$

then

$$y(t) \le e^{-a(t-t_0)}y(t_0) + \frac{b\tau}{1-e^{-a\tau}} \qquad \textit{for all } t \in [t_0, T), \tag{3.10}$$

and in particular

$$y(t) \le y(t_0) + \frac{b\tau}{1-e^{-a\tau}} \qquad \textit{for all } t \in [t_0, T). \tag{3.11}$$

Proof. Without loss of generality assuming that $t_0 = 0$ and $T \in (0, \infty)$, for fixed $t \in (0, T)$ we pick a nonnegative integer N such that $N\tau \le t < (N+1)\tau$, whence by nonnegativity of h we can estimate

$$\begin{aligned}\int_0^t e^{-a(t-s)}h(s)ds &\le \int_{t-(N+1)\tau}^t e^{-a(t-s)}h(s)ds = \sum_{j=0}^N \int_{t-(j+1)\tau}^{t-j\tau} e^{-a(t-s)}h(s)ds \\ &\le \sum_{j=0}^N e^{-ja\tau}\int_{t-(j+1)\tau}^{t-j\tau} h(s)ds.\end{aligned}$$

Since clearly (3.8) entails that

$$\int_{t-(j+1)\tau}^{t-j\tau} h(s)ds \le b\tau \qquad \text{for all } j \in \{0, ..., N\},$$

this implies that

$$\int_0^t e^{-a(t-s)}h(s)ds \le b\tau\sum_{j=0}^N e^{-ja\tau} \le b\tau\sum_{j=0}^\infty e^{-ja\tau} = \frac{b\tau}{1-e^{-a\tau}}.$$

This precisely yields (3.9) and hence shows that any function y with the indicated properties satisfies (3.10) and (3.11), because then

$$y(t) \le e^{-at}y(0) + \int_0^t e^{-a(t-s)}h(s)ds \qquad \text{for all } t \in (0, T)$$

according to a comparison argument. $\Box$

Let us furthermore note the following simple inequality.

Lemma 3.5 *Let $a \geq 0, b \geq 0$ and $\alpha \geq 1$. Then*

$$(a-b)_+^\alpha \geq 2^{1-\alpha} a^\alpha - b^\alpha. \tag{3.12}$$

PROOF. If $a \leq b$, then since $\alpha \geq 1$ we have $2^{1-\alpha}a^\alpha \leq a^\alpha \leq b^\alpha$, whence (3.12) is valid due to the fact that then $(a-b)_+ = 0$. On the other hand, if $a > b$ then (3.12) follows from the observation that by the Minkowski inequality, $a^\alpha = [(a-b)+b]^\alpha \leq 2^{\alpha-1}[(a-b)^\alpha + b^\alpha]$. □

We can now prove the following estimate, which is the first place in which the spatial dimension three enters our argument in a quantitative manner.

Lemma 3.6 *There exists $C > 0$ such that for each $\varepsilon \in (0,1)$ we have*

$$\int_\Omega c_\varepsilon^6(x,t)dx \leq C \qquad \textit{for all } t > 0. \tag{3.13}$$

PROOF. We multiply the second equation in (3.1) by c_ε^5 and use that $\nabla \cdot u_\varepsilon \equiv 0$ to see upon integration by parts that

$$\frac{1}{6}\frac{d}{dt}\int_\Omega c_\varepsilon^6 + 5\int_\Omega c_\varepsilon^4|\nabla c_\varepsilon|^2 + \int_\Omega c_\varepsilon^6 = \int_\Omega n_\varepsilon c_\varepsilon^5 \qquad \text{for all } t > 0, \tag{3.14}$$

where by the Cauchy-Schwarz inequality,

$$\int_\Omega n_\varepsilon c_\varepsilon^5 \leq \left(\int_\Omega n_\varepsilon^2\right)^{\frac{1}{2}} \left(\int_\Omega c_\varepsilon^{10}\right)^{\frac{1}{2}} \qquad \text{for all } t > 0. \tag{3.15}$$

Here we can use the Gagliardo-Nirenberg inequality to find $C_1 > 0$ fulfilling

$$\begin{aligned}
\left(\int_\Omega c_\varepsilon^{10}\right)^{\frac{1}{2}} &= \|c_\varepsilon^3\|_{L^{\frac{10}{3}}(\Omega)}^{\frac{5}{3}} \\
&\leq C_1\|\nabla c_\varepsilon^3\|_{L^2(\Omega)}\|c_\varepsilon^3\|_{L^2(\Omega)}^{\frac{2}{3}} + C_1\|c_\varepsilon^3\|_{L^{\frac{1}{3}}(\Omega)}^{\frac{5}{3}} \\
&= 3C_1\left(\int_\Omega c_\varepsilon^4|\nabla c_\varepsilon|^2\right)^{\frac{1}{2}}\left(\int_\Omega c_\varepsilon^6\right)^{\frac{1}{3}} + C_1\left(\int_\Omega c_\varepsilon\right)^5 \qquad \text{for all } t > 0.
\end{aligned} \tag{3.16}$$

Now since Lemma 3.3 asserts the existence of $C_2 > 0$ such that

$$\int_\Omega c_\varepsilon \leq C_2 \qquad \text{for all } t > 0, \tag{3.17}$$

combining (3.15) with (3.16) and once more employing Young's inequality we obtain

$$\begin{aligned}
\int_\Omega n_\varepsilon c_\varepsilon^5 &\leq 3C_1\left(\int_\Omega n_\varepsilon^2\right)^{\frac{1}{2}}\left(\int_\Omega c_\varepsilon^4|\nabla c_\varepsilon|^2\right)^{\frac{1}{2}}\left(\int_\Omega c_\varepsilon^6\right)^{\frac{1}{3}} + C_1C_2^5\left(\int_\Omega n_\varepsilon^2\right)^{\frac{1}{2}} \\
&\leq \int_\Omega c_\varepsilon^4|\nabla c_\varepsilon|^2 + \frac{9C_1^2}{4}\left(\int_\Omega n_\varepsilon^2\right)\left(\int_\Omega c_\varepsilon^6\right)^{\frac{2}{3}} + \frac{C_1C_2^5}{2}\left\{\int_\Omega n_\varepsilon^2 + 1\right\} \\
&\leq \int_\Omega c_\varepsilon^4|\nabla c_\varepsilon|^2 + \frac{9C_1^2}{4}\left(\int_\Omega n_\varepsilon^2\right)\left\{\frac{2}{3}\int_\Omega c_\varepsilon^6 + \frac{1}{3}\right\} + \frac{C_1C_2^5}{2}\left\{\int_\Omega n_\varepsilon^2 + 1\right\} \qquad \text{for all } t > 0.
\end{aligned}$$

Therefore, (3.14) implies that there exists $C_3 > 0$ such that

$$\frac{d}{dt}\int_\Omega c_\varepsilon^6 + 24\int_\Omega c_\varepsilon^3|\nabla c_\varepsilon|^2 \le C_3\Big\{\int_\Omega n_\varepsilon^2 + 1\Big\}\Big\{\int_\Omega c_\varepsilon^6 + 1\Big\} \qquad \text{for all } t>0, \tag{3.18}$$

where we use that once more employing the Gagliardo-Nirenberg inequality, the Hölder inequality and Young's inequality we can find $C_4 > 0$ satisfying

$$\begin{aligned}
\int_\Omega c_\varepsilon^6 &= \|c_\varepsilon^3\|_{L^2(\Omega)}^2 \\
&\le C_4\|\nabla c_\varepsilon^3\|_{L^2(\Omega)}^{\frac{30}{17}}\|c_\varepsilon^3\|_{L^{\frac{1}{3}}(\Omega)}^{\frac{4}{17}} + C_4\|c_\varepsilon^3\|_{L^{\frac{1}{3}}(\Omega)}^2 \\
&= 3^{\frac{30}{17}}C_4\Big(\int_\Omega c_\varepsilon^4|\nabla c_\varepsilon|^2\Big)^{\frac{15}{17}}\Big(\int_\Omega c_\varepsilon\Big)^{\frac{12}{17}} + C_4\Big(\int_\Omega c_\varepsilon\Big)^6 \qquad \text{for all } t>0.
\end{aligned}$$

According to (3.17), this means that with some $C_5 > 0$ we have

$$\int_\Omega c_\varepsilon^6 + 1 \le C_5\Big(\int_\Omega c_\varepsilon^4|\nabla c_\varepsilon|^2\Big)^{\frac{15}{17}} + C_5 \qquad \text{for all } t>0,$$

so that applying Lemma 3.5 to $\alpha := \frac{17}{15}$, $a := \frac{1}{C_5}\Big\{\int_\Omega c_\varepsilon^6 + 1\Big\}$ and $b := 1$ for fixed $t > 0$, we obtain that

$$\int_\Omega c_\varepsilon^4|\nabla c_\varepsilon|^2 \ge 2^{-\frac{2}{15}}C_5^{-\frac{17}{15}}\Big\{\int_\Omega c_\varepsilon^6 + 1\Big\}^{\frac{17}{15}} - 1 \qquad \text{for all } t>0$$

and that hence

$$\frac{\int_\Omega c_\varepsilon^4|\nabla c_\varepsilon|^2}{\int_\Omega c_\varepsilon^6 + 1} \ge 2^{-\frac{2}{15}}C_5^{-\frac{17}{15}}\Big\{\int_\Omega c_\varepsilon^6 + 1\Big\}^{\frac{2}{15}} - 1 \qquad \text{for all } t>0.$$

Now making use of the elementary inequality $z^{\frac{2}{15}} \ge \frac{2e}{15}\ln z$, valid for any $z > 0$, from this we infer that there exist $C_6 > 0$ and $C_7 > 0$ such that

$$\frac{24\int_\Omega c_\varepsilon^4|\nabla c_\varepsilon|^2}{\int_\Omega c_\varepsilon^6 + 1} \ge C_6\ln\Big\{\int_\Omega c_\varepsilon^6 + 1\Big\} - C_7 \qquad \text{for all } t>0,$$

whence returning to (3.18) we conclude that

$$\begin{aligned}
\frac{d}{dt}\ln\Big\{\int_\Omega c_\varepsilon^6 + 1\Big\} &= \frac{\frac{d}{dt}\int_\Omega c_\varepsilon^6}{\int_\Omega c_\varepsilon^6 + 1} \\
&\le -C_6\ln\Big\{\int_\Omega c_\varepsilon^6 + 1\Big\} + C_7 + C_3\Big\{\int_\Omega n_\varepsilon^2 + 1\Big\} \qquad \text{for all } t>0.
\end{aligned} \tag{3.19}$$

As Lemma 3.2 warrants that with m as in (3.4) we have

$$\int_t^{t+1}\Big\{C_7 + C_3\Big\{\int_\Omega n_\varepsilon^2(\cdot,s) + 1\Big\}\Big\}ds \le C_8 := C_7 + C_3\cdot\Big\{\frac{(\rho_+ + 1)m}{\mu} + 1\Big\} \qquad \text{for all } t>0,$$

in light of Lemma 3.4 the inequality (3.19) guarantees that

$$\ln\left\{\int_\Omega c_\varepsilon^6+1\right\} \le \ln\left\{\int_\Omega c_0^6+1\right\} + \frac{C_8}{1-e^{-C_6}} \qquad \text{for all } t>0,$$

and that hence (3.13) is valid. □

3.2 Basic estimates for u_ε

We next rely on the standard energy inequality associated with the fluid evolution system in (3.1) to derive some basic estimates for u_ε. For later reference, let us separately state the starting point therefor.

Lemma 3.7 *For all $\varepsilon\in(0,1)$, we have*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\phi + \int_\Omega f\cdot u_\varepsilon \qquad \textit{for all } t>0. \tag{3.20}$$

Proof. This immediately results on testing the third equation in (3.1) against u_ε. □

Now along with our overall boundedness assumption (1.4), the inequality (3.5) asserts the following.

Lemma 3.8 *Assume that (1.4) holds. Then there exists $C>0$ such that for all $\varepsilon\in(0,1)$,*

$$\int_\Omega |u_\varepsilon(x,t)|^2dx \le C \qquad \textit{for all } t>0 \tag{3.21}$$

and

$$\int_t^{t+1}\int_\Omega |\nabla u_\varepsilon(x,s)|^2dxds \le C \qquad \textit{for all } t>0 \tag{3.22}$$

as well as

$$\int_t^{t+1}\int_\Omega |u(x,s)|^{\frac{10}{3}}dxds \le C \qquad \textit{for all } t>0. \tag{3.23}$$

Proof. According to the Poincaré inequality and the fact that $W^{1,2}_{0,\sigma}(\Omega)\hookrightarrow L^6(\Omega)$, we can find $C_1>0$ and $C_2>0$ such that

$$\int_\Omega |\nabla u_\varepsilon|^2 \ge C_1\int_\Omega |u_\varepsilon|^2 \qquad \text{for all } t>0$$

and

$$\int_\Omega |\nabla u_\varepsilon|^2 \ge C_2\left(\int_\Omega |u|^6\right)^{\frac{1}{3}} \qquad \text{for all } t>0.$$

Writing $C_3 := \|\nabla\phi\|_{L^\infty(\Omega)}$, from (3.20) we therefore obtain on using the Hölder inequality, Young's inequality and the Minkowski inequality that

$$\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 + C_1\int_\Omega |u_\varepsilon|^2$$

$$
\begin{aligned}
&\leq 2\int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla\phi + 2\int_\Omega f\cdot u_\varepsilon \\
&\leq 2C_3\Big(\int_\Omega n_\varepsilon^2\Big)^{\frac12}\Big(\int_\Omega |u_\varepsilon|^2\Big)^{\frac12} + 2\Big(\int_\Omega |f|^{\frac65}\Big)^{\frac56}\Big(\int_\Omega |u_\varepsilon|^6\Big)^{\frac16} \\
&\leq \frac{2C_3}{\sqrt{C_1}}\Big(\int_\Omega n_\varepsilon^2\Big)^{\frac12}\Big(\int_\Omega |\nabla u_\varepsilon|^2\Big)^{\frac12} + \frac{2}{\sqrt{C_2}}\Big(\int_\Omega |f|^{\frac65}\Big)^{\frac56}\Big(\int_\Omega |\nabla u_\varepsilon|^2\Big)^{\frac12} \\
&= \left\{\frac{2C_3}{\sqrt{C_1}}\Big(\int_\Omega n_\varepsilon^2\Big)^{\frac12} + \frac{2}{\sqrt{C_2}}\Big(\int_\Omega |f|^{\frac65}\Big)^{\frac56}\right\}\cdot\Big(\int_\Omega |\nabla u_\varepsilon|^2\Big)^{\frac12} \\
&\leq \frac12\int_\Omega |\nabla u_\varepsilon|^2 + \frac12\left\{\frac{2C_3}{\sqrt{C_1}}\Big(\int_\Omega n_\varepsilon^2\Big)^{\frac12} + \frac{2}{\sqrt{C_2}}\Big(\int_\Omega |f|^{\frac65}\Big)^{\frac56}\right\}^2 \\
&\leq \frac12\int_\Omega |\nabla u_\varepsilon|^2 + C_4\int_\Omega n_\varepsilon^2 + C_4\Big(\int_\Omega |f|^{\frac65}\Big)^{\frac53} \qquad \text{for all } t>0
\end{aligned}
$$

with $C_4 := \max\{\frac{4C_3^2}{C_1}, \frac{4}{C_2}\}$. Thus,

$$
\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \frac12\int_\Omega |\nabla u_\varepsilon|^2 + C_1\int_\Omega |u_\varepsilon|^2 \leq C_4\int_\Omega n_\varepsilon^2 + C_4\Big(\int_\Omega |f|^{\frac65}\Big)^{\frac53} \qquad \text{for all } t>0, \tag{3.24}
$$

where

$$
C_4\int_t^{t+1}\int_\Omega n_\varepsilon^2(x,s)\,dxds \leq C_5 := \frac{(1+\rho_+)m}{\mu}C_4
$$

with m as in (3.4), and where by (1.4) we can find $C_6>0$ such that

$$
C_4\int_t^{t+1}\Big(\int_\Omega |f(x,s)|^{\frac65}\Big)^{\frac53} ds \leq C_6 \qquad \text{for all } t>0.
$$

An application of Lemma 3.4 thus shows that

$$
\int_\Omega |u_\varepsilon(\cdot,t)|^2 \leq C_7 := \int_\Omega |u_0|^2 + \frac{C_5+C_6}{1-e^{-C_1}} \qquad \text{for all } t>0,
$$

wereafter integrating (3.24) in time yields

$$
\frac12\int_t^{t+1}\int_\Omega |\nabla u_\varepsilon(\cdot,s)|^2 ds \leq \int_\Omega |u_\varepsilon(\cdot,t)|^2 + C_5 + C_6 \leq C_8 := C_7 + C_5 + C_6 \qquad \text{for all } t>0
$$

and thereby verifies (3.22).
Finally, by means of the Gagliardo-Nirenberg inequality we infer the existence of $C_9>0$ fulfilling

$$
\begin{aligned}
\int_t^{t+1}\int_\Omega |u_\varepsilon|^{\frac{10}{3}} &\leq C_9\int_t^{t+1} \|\nabla u_\varepsilon(\cdot,s)\|_{L^2(\Omega)}^2 \|u_\varepsilon(\cdot,s)\|_{L^2(\Omega)}^{\frac43} ds \\
&\leq C_9\cdot 2C_8\cdot C_7^{\frac23} \qquad \text{for all } t>0,
\end{aligned}
$$

and conclude. □

4 Higher regularity properties of c_ε

A crucial step in our analysis will consist in detecting further regularity properties of the second solution component. In fact, Lemma 4.3 and Lemma 4.4 shall inter alia reveal spatio-temporal estimates for ∇c_ε and $D^2 c_\varepsilon$ which will become important in passing to the limit $\varepsilon \searrow 0$ in those integrals in (2.3) that originate from the cross-diffusive term in (3.1). After a technical preparation, the key step toward this will be achieved in Lemma 4.2 through an examination of the evolution of the functional $\int_\Omega (|\nabla c_\varepsilon|^2+1)^{\frac{2}{3}}$ which is non-convex with respect to $|\nabla c_\varepsilon|^2$, with the restriction to the sublinear power $\frac{2}{3}$ herein being mainly determined by the integrability exponent appearing in Lemma 3.6.

4.1 An interpolation lemma

As a main ingredient for the proof of Lemma 4.2, let us separately state the following Gagliardo-Nirenberg-type interpolation inequality which may be viewed as a variant of a corresponding precedent established in [71, Lemma 3.8] for situations when the spatial L^∞ norm is involved, rather than that in $L^6(\Omega)$.

Lemma 4.1 *There exists $C>0$ with the property that whenever $\varphi \in C^2(\bar\Omega)$ is such that $\varphi \cdot \frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$, the inequality*

$$\begin{aligned}\int_\Omega \Big(|\nabla\varphi|^2+1\Big)^{\frac{4}{3}} \quad &\le \quad C\cdot \bigg\{\int_\Omega \Big(|\nabla\varphi|^2+1\Big)^{-\frac{1}{3}}|D^2\varphi|^2\bigg\} \cdot \bigg\{\Big(\int_\Omega \varphi^6\Big)^{\frac{2}{9}}+1\bigg\} \\ &\quad +C\cdot \bigg\{\Big(\int_\Omega \varphi^6\Big)^{\frac{4}{9}}+1\bigg\} \end{aligned} \tag{4.1}$$

holds.

PROOF. By Young's inequality, we find $C_1>0$ such that

$$\begin{aligned}\int_\Omega \Big(|\nabla\varphi|^2+1\Big)^{\frac{4}{3}} &= \int_\Omega \Big(|\nabla\varphi|^2+1\Big)^{\frac{1}{3}}|\nabla\varphi|^2 + \int_\Omega \Big(|\nabla\varphi|^2+1\Big)^{\frac{1}{3}} \\ &\le \int_\Omega \Big(|\nabla\varphi|^2+1\Big)^{\frac{1}{3}}|\nabla\varphi|^2 + \frac{1}{2}\int_\Omega \Big(|\nabla\varphi|^2+1\Big)^{\frac{4}{3}} + C_1, \end{aligned} \tag{4.2}$$

and using our assumption that $\varphi\cdot\frac{\partial\varphi}{\partial\nu}|_{\partial\Omega}=0$, integrating by parts we see that

$$\begin{aligned}\int_\Omega \Big(|\nabla\varphi|^2+1\Big)^{\frac{1}{3}}|\nabla\varphi|^2 &= -\int_\Omega \varphi\nabla\cdot\bigg\{\Big(|\nabla\varphi|^2+1\Big)^{\frac{1}{3}}\nabla\varphi\bigg\} \\ &= -\int_\Omega \varphi\Big(|\nabla\varphi|^2+1\Big)^{\frac{1}{3}}\Delta\varphi \\ &\quad -\frac{2}{3}\int_\Omega \varphi\Big(|\nabla\varphi|^2+1\Big)^{-\frac{2}{3}}(D^2\varphi\cdot\nabla\varphi)\cdot\nabla\varphi. \end{aligned}$$

Thanks to the pointwise inequality $|\Delta\varphi| \le \sqrt{3}|D^2\varphi|$, along with (4.2) this implies that

$$\int_\Omega \Big(|\nabla\varphi|^2+1\Big)^{\frac{4}{3}} \le C_2\int_\Omega |\varphi|\cdot\Big(|\nabla\varphi|^2+1\Big)^{\frac{1}{3}}|D^2\varphi| + 2C_1$$

with $C_2 := 2\sqrt{3} + \frac{4}{3}$, where we invoke the Hölder inequality to obtain

$$\begin{aligned}
\int_\Omega \Big(|\nabla\varphi|^2+1\Big)^{\frac{4}{3}} &\le C_2\Big\{\int_\Omega \Big(|\nabla\varphi|^2+1\Big)^{-\frac{1}{3}}|D^2\varphi|^2\Big\}^{\frac{1}{2}} \cdot \Big\{\int_\Omega \varphi^2\Big(|\nabla\varphi|^2+1\Big)\Big\}^{\frac{1}{2}} + 2C_1 \\
&\le C_2\Big\{\int_\Omega \Big(|\nabla\varphi|^2+1\Big)^{-\frac{1}{3}}|D^2\varphi|^2\Big\}^{\frac{1}{2}} \times \\
&\qquad \times\Big\{\int_\Omega \Big(|\nabla\varphi|^2+1\Big)^{\frac{4}{3}}\Big\}^{\frac{3}{8}} \cdot \Big\{\int_\Omega \varphi^8\Big\}^{\frac{1}{8}} \cdot +2C_1.
\end{aligned} \tag{4.3}$$

Now due to the Gagliardo-Nirenberg inequality we can find $C_3 > 0$ such that

$$\Big\{\int_\Omega \varphi^8\Big\}^{\frac{1}{8}} \le C_3\|\nabla\varphi\|_{L^{\frac{8}{3}}(\Omega)}^{\frac{1}{3}}\|\varphi\|_{L^6(\Omega)}^{\frac{2}{3}} + C_3\|\varphi\|_{L^6(\Omega)},$$

where clearly

$$\|\nabla\varphi\|_{L^{\frac{8}{3}}(\Omega)}^{\frac{1}{3}} = \Big\{\int_\Omega |\nabla\varphi|^{\frac{8}{3}}\Big\}^{\frac{1}{8}} \le \Big\{\int_\Omega \Big(|\nabla\varphi|^2+1\Big)^{\frac{4}{3}}\Big\}^{\frac{1}{8}}.$$

Therefore, (4.3) entails that

$$\begin{aligned}
\int_\Omega \Big(|\nabla\varphi|^2+1\Big)^{\frac{4}{3}} &\le C_2C_3\Big\{\int_\Omega \Big(|\nabla\varphi|^2+1\Big)^{-\frac{1}{3}}|D^2\varphi|^2\Big\}^{\frac{1}{2}} \cdot \Big\{\int_\Omega \Big(|\nabla\varphi|^2+1\Big)^{\frac{4}{3}}\Big\}^{\frac{3}{8}} \times \\
&\qquad \times\Big\{\Big(\int_\Omega \Big(|\nabla\varphi|^2+1\Big)^{\frac{4}{3}}\Big)^{\frac{1}{8}} \cdot \Big(\int_\Omega \varphi^6\Big)^{\frac{1}{9}} + \Big(\int_\Omega \varphi^6\Big)^{\frac{1}{6}}\Big\} + 2C_1,
\end{aligned}$$

meaning that

$$I := \int_\Omega \Big(|\nabla\varphi|^2+1\Big)^{\frac{4}{3}}, \quad J := \int_\Omega \Big(|\nabla\varphi|^2+1\Big)^{-\frac{1}{3}}|D^2\varphi|^2 \quad \text{and} \quad K := \int_\Omega \varphi^6$$

satisfy

$$I \le C_2C_3J^{\frac{1}{2}}I^{\frac{1}{2}}K^{\frac{1}{9}} + C_2C_3J^{\frac{1}{2}}I^{\frac{3}{8}}K^{\frac{1}{6}} + 2C_1. \tag{4.4}$$

Here three applications of Young's inequality provide positive constants C_4 and C_5 such that

$$C_2C_3J^{\frac{1}{2}}I^{\frac{1}{2}}K^{\frac{1}{9}} \le \frac{1}{2}I + C_4JK^{\frac{2}{9}}$$

and

$$\begin{aligned}
C_2C_3J^{\frac{1}{2}}I^{\frac{3}{8}}K^{\frac{1}{6}} &\le \frac{1}{4}I + C_5J^{\frac{4}{5}}K^{\frac{4}{15}} \\
&= \frac{1}{4}I + C_5(JK^{\frac{2}{9}})^{\frac{4}{5}} \cdot K^{\frac{4}{45}} \\
&\le \frac{1}{4}I + C_5JK^{\frac{2}{9}} + C_5K^{\frac{4}{9}}.
\end{aligned}$$

In conclusion, (4.4) shows that

$$\frac{1}{4} I \le (C_4 + C_5) J K^{\frac{2}{9}} + C_5 K^{\frac{4}{9}} + 2C_1$$

and thereby proves (4.1). $\square$

4.2 Bounds for ∇c_ε and $D^2 c_\varepsilon$

We can now apply an appropriate testing procedure to the second equation in (3.1) to gain the following two estimates, only the second of which will actually be used in the sequel.

Lemma 4.2 *There exists $C > 0$ such that for any $\varepsilon \in (0,1)$ we have*

$$\int_\Omega |\nabla c_\varepsilon(x,t)|^{\frac{4}{3}} dx \le C \qquad \text{for all } t > 0 \tag{4.5}$$

and

$$\int_t^{t+1} \int_\Omega \Big(|\nabla c_\varepsilon|^2 + 1 \Big)^{-\frac{1}{3}} |D^2 c_\varepsilon|^2 \le C \qquad \text{for all } t > 0. \tag{4.6}$$

Proof. By straightforward differentiation, using the second equation in (3.1) we obtain

$$\begin{aligned} \frac{3}{4} \frac{d}{dt} \int_\Omega \Big(|\nabla c_\varepsilon|^2 + 1 \Big)^{\frac{2}{3}} &= \int_\Omega \Big(|\nabla c_\varepsilon|^2 + 1 \Big)^{-\frac{1}{3}} \nabla c_\varepsilon \cdot \nabla c_{\varepsilon t} \\ &= \int_\Omega \Big(|\nabla c_\varepsilon|^2 + 1 \Big)^{-\frac{1}{3}} \nabla c_\varepsilon \cdot \nabla \Delta c_\varepsilon - \int_\Omega \Big(|\nabla c_\varepsilon|^2 + 1 \Big)^{-\frac{1}{3}} |\nabla c_\varepsilon|^2 \\ &\quad + \int_\Omega \Big(|\nabla c_\varepsilon|^2 + 1 \Big)^{-\frac{1}{3}} \nabla c_\varepsilon \cdot \nabla n_\varepsilon \\ &\quad - \int_\Omega \Big(|\nabla c_\varepsilon|^2 + 1 \Big)^{-\frac{1}{3}} \nabla c_\varepsilon \cdot \nabla (u_\varepsilon \cdot \nabla c_\varepsilon) \qquad \text{for all } t > 0, \end{aligned} \tag{4.7}$$

where since $\nabla c_\varepsilon \cdot \nabla \Delta c_\varepsilon = \frac{1}{2} \Delta |\nabla c_\varepsilon|^2 - |D^2 c_\varepsilon|^2$, an integration by parts yields

$$\begin{aligned} \int_\Omega \Big(|\nabla c_\varepsilon|^2 + 1 \Big)^{-\frac{1}{3}} \nabla c_\varepsilon \cdot \nabla \Delta c_\varepsilon &= \frac{1}{2} \int_\Omega \Big(|\nabla c_\varepsilon|^2 + 1 \Big)^{-\frac{1}{3}} \Delta |\nabla c_\varepsilon|^2 - \int_\Omega \Big(|\nabla c_\varepsilon|^2 + 1 \Big)^{-\frac{1}{3}} |D^2 c_\varepsilon|^2 \\ &\le \frac{1}{6} \int_\Omega \Big(|\nabla c_\varepsilon|^2 + 1 \Big)^{-\frac{4}{3}} \Big| \nabla |\nabla c_\varepsilon|^2 \Big|^2 - \int_\Omega \Big(|\nabla c_\varepsilon|^2 + 1 \Big)^{-\frac{1}{3}} |D^2 c_\varepsilon|^2 \end{aligned}$$

for all $t > 0$, because $\frac{\partial |\nabla c_\varepsilon|^2}{\partial \nu} \le 0$ on $\partial\Omega$ by convexity of Ω and the boundary condition $\frac{\partial c_\varepsilon}{\partial \nu}|_{\partial\Omega} = 0$ ([37, Lemme 2.I.1]). Estimating

$$\Big| \nabla |\nabla c_\varepsilon|^2 \Big|^2 = 4 |D^2 c_\varepsilon \cdot \nabla c_\varepsilon|^2 \le 4 |D^2 c_\varepsilon|^2 |\nabla c_\varepsilon|^2,$$

we thereby infer from (4.7) that

$$\begin{aligned} \frac{3}{4} \frac{d}{dt} \int_\Omega \Big(|\nabla c_\varepsilon|^2 + 1 \Big)^{\frac{2}{3}} &+ \frac{1}{3} \int_\Omega \Big(|\nabla c_\varepsilon|^2 + 1 \Big)^{-\frac{1}{3}} |D^2 c_\varepsilon|^2 + \int_\Omega \Big(|\nabla c_\varepsilon|^2 + 1 \Big)^{-\frac{1}{3}} |\nabla c_\varepsilon|^2 \\ &\le \int_\Omega \Big(|\nabla c_\varepsilon|^2 + 1 \Big)^{-\frac{1}{3}} \nabla c_\varepsilon \cdot \nabla n_\varepsilon - \int_\Omega \Big(|\nabla c_\varepsilon|^2 + 1 \Big)^{-\frac{1}{3}} \nabla c_\varepsilon \cdot \nabla (u_\varepsilon \cdot \nabla c_\varepsilon) \end{aligned} \tag{4.8}$$

for all $t > 0$. Here in the first integral on the right we also integrate by parts and use the Cauchy-Schwarz inequality to find that

$$
\begin{aligned}
\int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{1}{3}} \nabla c_\varepsilon \cdot \nabla n_\varepsilon &= -\int_\Omega n_\varepsilon \nabla \cdot \Big\{ \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{1}{3}} \nabla c_\varepsilon \Big\} \\
&= -\int_\Omega n_\varepsilon \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{1}{3}} \Delta c_\varepsilon \\
&\quad + \frac{2}{3} \int_\Omega n_\varepsilon \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{4}{3}} (D^2 c_\varepsilon \cdot \nabla c_\varepsilon) \cdot \nabla c_\varepsilon \\
&\le \Big\{ \int_\Omega n_\varepsilon^2 \Big\}^{\frac{1}{2}} \cdot \Big\{ \int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{2}{3}} |\Delta c_\varepsilon|^2 \Big\}^{\frac{1}{2}} \\
&\quad + \frac{2}{3} \Big\{ \int_\Omega n_\varepsilon^2 \Big\}^{\frac{1}{2}} \cdot \Big\{ \int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{8}{3}} |\nabla c_\varepsilon|^4 |D^2 c_\varepsilon|^2 \Big\}^{\frac{1}{2}}
\end{aligned}
$$

for all $t > 0$. As $|\Delta c_\varepsilon|^2 \le 3|D^2 c_\varepsilon|^2$ and

$$
(z^2+1)^{-\frac{8}{3}} z^4 \le (z^2+1)^{-\frac{2}{3}} \le (z^2+1)^{-\frac{1}{3}} \qquad \text{for all } z \ge 0,
$$

in view of Young's inequality this implies that

$$
\begin{aligned}
\int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{1}{3}} \nabla c_\varepsilon \cdot \nabla n_\varepsilon &\le \Big(\sqrt{3}+\frac{2}{3}\Big) \cdot \Big\{ \int_\Omega n_\varepsilon^2 \Big\}^{\frac{1}{2}} \cdot \Big\{ \int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{1}{3}} |D^2 c_\varepsilon|^2 \Big\}^{\frac{1}{2}} \\
&\le \frac{1}{6} \int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{1}{3}} |D^2 c_\varepsilon|^2 + C_1 \int_\Omega n_\varepsilon^2 \qquad \text{for all } t > 0 \quad (4.9)
\end{aligned}
$$

with $C_1 := \frac{3}{2}(\sqrt{3}+\frac{2}{3})^2$.
Next, integrating by parts in the last integral in (4.8) we see that since $\nabla \cdot u_\varepsilon \equiv 0$, for all $t > 0$ we have

$$
\begin{aligned}
-\int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{1}{3}} \nabla c_\varepsilon \cdot \nabla (u_\varepsilon \cdot \nabla c_\varepsilon) &= -\int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{1}{3}} \nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon) \\
&\quad -\int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{1}{3}} u_\varepsilon \cdot (D^2 c_\varepsilon \cdot \nabla c_\varepsilon) \\
&= -\int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{1}{3}} \nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon) \\
&\quad -\frac{3}{4} \int_\Omega u_\varepsilon \cdot \nabla \Big(|\nabla c_\varepsilon|^2+1\Big)^{\frac{2}{3}} \\
&= -\int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{1}{3}} \nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon), \qquad (4.10)
\end{aligned}
$$

where by the Cauchy-Schwarz inequality,

$$
-\int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{1}{3}} \nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon) \le \Big\{ \int_\Omega |\nabla u_\varepsilon|^2 \Big\}^{\frac{1}{2}} \cdot \Big\{ \int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{2}{3}} |\nabla c_\varepsilon|^4 \Big\}^{\frac{1}{2}}. \qquad (4.11)
$$

Now using that by Lemma 3.6 we have

$$\int_\Omega c_\varepsilon^6 \le C_2 \qquad \text{for all } t>0$$

with some $C_2>0$, by employing Lemma 4.1 we gain positive constants C_3 and C_4 such that

$$\begin{aligned}
\int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{2}{3}} |\nabla c_\varepsilon|^4 &\le \int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{\frac{4}{3}} \\
&\le C_3 \Big\{ \int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{1}{3}} |D^2 c_\varepsilon|^2 \Big\} \cdot \Big\{ \Big(\int_\Omega c_\varepsilon^6 \Big)^{\frac{2}{9}} + 1 \Big\} \\
&\quad + C_3 \Big\{ \Big(\int_\Omega c_\varepsilon^6 \Big)^{\frac{4}{9}} + 1 \Big\} \\
&\le C_4 \Big\{ \int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{1}{3}} |D^2 c_\varepsilon|^2 \Big\} + C_4 \qquad \text{for all } t>0,
\end{aligned}$$

so that employing Young's inequality we infer from (4.10) and (4.11) that

$$\begin{aligned}
-\int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{1}{3}} \nabla c_\varepsilon \cdot \nabla (u_\varepsilon \cdot \nabla c_\varepsilon) &\le \sqrt{C_4} \Big\{ \int_\Omega |\nabla u_\varepsilon|^2 \Big\}^{\frac{1}{2}} \cdot \Big\{ \int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{1}{3}} |D^2 c_\varepsilon|^2 + 1 \Big\}^{\frac{1}{2}} \\
&\le \frac{1}{12} \int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{1}{3}} |D^2 c_\varepsilon|^2 + \frac{1}{12} + 3C_4 \int_\Omega |\nabla u_\varepsilon|^2 \qquad (4.12)
\end{aligned}$$

for all $t>0$. Finally, the third term on the left of (4.8) can be estimated from below by observing that

$$\begin{aligned}
\int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{\frac{2}{3}} &= \int_{\{|\nabla c_\varepsilon|\ge 1\}} \Big(|\nabla c_\varepsilon|^2+1\Big)^{\frac{2}{3}} + \int_{\{|\nabla c_\varepsilon|<1\}} \Big(|\nabla c_\varepsilon|^2+1\Big)^{\frac{2}{3}} \\
&\le \int_{\{|\nabla c_\varepsilon|\ge 1\}} \Big(|\nabla c_\varepsilon|^2+1\Big)^{\frac{2}{3}} + 2^{\frac{2}{3}}|\Omega| \qquad \text{for all } t>0,
\end{aligned}$$

and that hence the validity of

$$\frac{(z^2+1)^{\frac{2}{3}}}{(z^2+1)^{-\frac{1}{3}} z^2} = 1 + \frac{1}{z^2} \le 2 \qquad \text{for all } z \ge 1$$

implies that

$$\int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{\frac{2}{3}} \le 2 \int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{1}{3}} |\nabla c_\varepsilon|^2 + 2^{\frac{2}{3}}|\Omega| \qquad \text{for all } t>0,$$

that is,

$$\int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{1}{3}} |\nabla c_\varepsilon|^2 \ge \frac{1}{2} \int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{\frac{2}{3}} - 2^{-\frac{1}{3}}|\Omega| \qquad \text{for all } t>0.$$

Together with (4.9) and (4.12) inserted into (4.8), this shows that

$$\frac{3}{4}\frac{d}{dt}\int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{\frac{2}{3}} + \frac{1}{12}\int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{1}{3}}|D^2c_\varepsilon|^2 + \frac{1}{2}\int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{\frac{2}{3}} \le C_1\int_\Omega n_\varepsilon^2 + 3C_4\int_\Omega |\nabla u_\varepsilon|^2 + C_5 \qquad \text{for all } t>0 \tag{4.13}$$

with $C_5 := \frac{1}{12}+2^{-\frac{1}{3}}|\Omega|$. Since from Lemma 3.2 and Lemma 3.8 we know that there exists $C_6>0$ such that

$$\int_t^{t+1}\Big\{C_1\int_\Omega n_\varepsilon^2(\cdot,s)+3C_4\int_\Omega |\nabla u_\varepsilon(\cdot,s)|^2+C_5\Big\}ds \le C_6 \qquad \text{for all } t>0,$$

as a consequence of Lemma 3.4 we see that (4.13) implies the inequality

$$\int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{\frac{2}{3}} \le C_7 := \int_\Omega \Big(|\nabla c_0|^2+1\Big)^{\frac{2}{3}} + \frac{4C_6}{3(1-e^{-\frac{2}{3}})} \qquad \text{for all } t>0,$$

which proves (4.5). Thereupon an integration of (4.13) yields

$$\frac{1}{12}\int_t^{t+1}\int_\Omega \Big(|\nabla c_\varepsilon(\cdot,s)|^2+1\Big)^{-\frac{1}{3}}|D^2c_\varepsilon(\cdot,s)|^2ds \le \frac{3}{4}\int_\Omega \Big(|\nabla c_\varepsilon(\cdot,t)|^2+1\Big)^{\frac{2}{3}} + C_6 \le \frac{3}{4}C_7+C_6$$

for all $t>0$, which establishes (4.6). □

When combined with Lemma 3.6 and Lemma 4.1, (4.6) readily yields space-time estimates for ∇c_ε, and also for c_ε itself, involving conveniently high integrability powers.

Lemma 4.3 *There exists $C>0$ such that whenever $\varepsilon\in(0,1)$, we have*

$$\int_t^{t+1}\int_\Omega |\nabla c_\varepsilon(x,s)|^{\frac{8}{3}}dxds \le C \qquad \textit{for all } t>0 \tag{4.14}$$

and

$$\int_t^{t+1}\int_\Omega c_\varepsilon^8(x,s)dxds \le C \qquad \textit{for all } t>0. \tag{4.15}$$

Proof. In view of the interpolation inequality provided by Lemma 4.1, the bounds on $\int_\Omega c_\varepsilon^6$ and $\int_t^{t+1}\int_\Omega \Big(|\nabla c_\varepsilon|^2+1\Big)^{-\frac{1}{3}}|D^2c_\varepsilon|^2$ asserted by Lemma 3.6 and Lemma 4.2 directly yield (4.14). To verify (4.15), we invoke the Gagliardo-Nirenberg inequality to pick $C_1>0$ such that

$$\int_t^{t+1}\int_\Omega c_\varepsilon^8 \le C_1\int_t^{t+1}\Big\{\|\nabla c_\varepsilon(\cdot,s)\|_{L^{\frac{8}{3}}(\Omega)}^{\frac{8}{3}}\|c_\varepsilon(\cdot,s)\|_{L^6(\Omega)}^{\frac{16}{3}} + \|c_\varepsilon(\cdot,s)\|_{L^6(\Omega)}^8\Big\}ds \qquad \text{for all } t>0,$$

whence combining the result of Lemma 3.6 with (4.14) completes the proof. □

By another simple interpolation, we can moreover turn (4.6) into an estimate for D^2c_ε which does no longer involve $|\nabla c_\varepsilon|^2$ as a weight.

Lemma 4.4 *There exists $C > 0$ with the property that for each $\varepsilon \in (0,1)$, the solution of (3.1) satisfies*

$$\int_t^{t+1} \int_\Omega |D^2 c_\varepsilon(x,s)|^{\frac{8}{5}} dxds \le C \qquad \text{for all } t > 0. \tag{4.16}$$

PROOF. Observing that by Young's inequality we have the pointwise inequality

$$\begin{aligned} |D^2 c_\varepsilon|^{\frac{8}{5}} &= \Big\{ \Big(|\nabla c_\varepsilon|^2 + 1 \Big)^{-\frac{1}{3}} |D^2 c_\varepsilon|^2 \Big\}^{\frac{4}{5}} \cdot \Big(|\nabla c_\varepsilon|^2 + 1 \Big)^{\frac{4}{15}} \\ &\le \Big(|\nabla c_\varepsilon|^2 + 1 \Big)^{-\frac{1}{3}} |D^2 c_\varepsilon|^2 + \Big(|\nabla c_\varepsilon|^2 + 1 \Big)^{\frac{4}{3}} \qquad \text{for all } x \in \Omega \text{ and } t > 0, \end{aligned}$$

the claim immediately results from Lemma 4.2 and Lemma 4.3. $\Box$

5 A bound for ∇n_ε

A next set of requirements made in Definition 2.1 consists in the regularity properties of ∇n expressed in (2.1) and (2.2); in particular, since we are finally planning to use a strictly concave function Φ in (2.3) when verifying a supersolution property for our limit object n in Lemma 7.5 below, we apparently need to derive an adequate bound for $|\nabla n_\varepsilon|^2$. When multiplied by a weight function which decays sufficiently fast at large values of n_ε, this indeed becomes possible on analyzing another non-convex functional, namely $\int_\Omega n_\varepsilon^{\frac{3}{4}}$:

Lemma 5.1 *There exists $C > 0$ such that*

$$\int_t^{t+1} \int_\Omega n_\varepsilon^{-\frac{5}{4}}(x,s) |\nabla n_\varepsilon(x,s)|^2 dxds \le C \qquad \text{for all } t > 0 \tag{5.1}$$

and each $\varepsilon \in (0,1)$.

PROOF. As n_ε is positive in $\bar\Omega \times (0,\infty)$ by the strong maximum principle, we may test the first equation in (3.1) against $n_\varepsilon^{-\frac{1}{4}}$ to see that

$$\frac{4}{3} \frac{d}{dt} \int_\Omega n_\varepsilon^{\frac{3}{4}} = \frac{1}{4} \int_\Omega n_\varepsilon^{-\frac{5}{4}} |\nabla n_\varepsilon|^2 - \frac{\chi}{4} \int_\Omega \frac{n_\varepsilon^{-\frac{1}{4}}}{1 + \varepsilon n_\varepsilon} \nabla n_\varepsilon \cdot \nabla c_\varepsilon + \rho \int_\Omega n_\varepsilon^{\frac{3}{4}} - \mu \int_\Omega n_\varepsilon^{\frac{7}{4}} \qquad \text{for all } t > 0,$$

so that

$$\begin{aligned} \int_t^{t+1} \int_\Omega n_\varepsilon^{-\frac{5}{4}} |\nabla n_\varepsilon|^2 &= \frac{16}{3} \int_\Omega n_\varepsilon^{\frac{3}{4}}(\cdot, t+1) - \frac{16}{3} \int_\Omega n_\varepsilon^{\frac{3}{4}}(\cdot, t) \\ &\quad + \chi \int_t^{t+1} \int_\Omega \frac{n_\varepsilon^{-\frac{1}{4}}}{1 + \varepsilon n_\varepsilon} \nabla n_\varepsilon \cdot \nabla c_\varepsilon \\ &\quad - 4\rho \int_t^{t+1} \int_\Omega n_\varepsilon^{\frac{3}{4}} + 4\mu \int_t^{t+1} \int_\Omega n_\varepsilon^{\frac{7}{4}} \qquad \text{for all } t > 0. \end{aligned} \tag{5.2}$$

Here we once more integrate by parts to see writing

$$\psi_\varepsilon(s) := \int_0^s \frac{d\sigma}{\sigma^{\frac{1}{4}}(1+\varepsilon\sigma)}, \qquad s \ge 0,$$

that due to Young's inequality,

$$\begin{aligned}
\int_t^{t+1}\int_\Omega \frac{n_\varepsilon^{-\frac{1}{4}}}{1+\varepsilon n_\varepsilon}\nabla n_\varepsilon \cdot \nabla c_\varepsilon &= \int_t^{t+1}\int_\Omega \nabla\psi_\varepsilon(n_\varepsilon)\cdot\nabla c_\varepsilon \\
&= -\int_t^{t+1}\int_\Omega \psi_\varepsilon(n_\varepsilon)\Delta c_\varepsilon \\
&\le \int_t^{t+1}\int_\Omega \psi_\varepsilon^{\frac{8}{3}}(n_\varepsilon) + \int_t^{t+1}\int_\Omega |\Delta c_\varepsilon|^{\frac{8}{5}} \qquad \text{for all } t>0.
\end{aligned}$$

Since

$$\psi_\varepsilon(s) \le \int_0^s \frac{d\sigma}{\sigma^{\frac{1}{4}}} = \frac{4}{3}s^{\frac{3}{4}} \qquad \text{for all } s \ge 0,$$

and since with m as in (3.4) we have

$$\int_t^{t+1}\int_\Omega n_\varepsilon^2 \le C_1 := \frac{(\rho_+ + 1)m}{\mu} \qquad \text{for all } t>0$$

by Lemma 3.2, in view of Lemma 4.4 we thus infer that there exists $C_2>0$ such that

$$\int_t^{t+1}\int_\Omega \frac{n_\varepsilon^{-\frac{1}{4}}}{1+\varepsilon n_\varepsilon}\nabla n_\varepsilon\cdot\nabla c_\varepsilon \le \Big(\frac{4}{3}\Big)^{\frac{8}{3}}\int_t^{t+1}\int_\Omega n_\varepsilon^2 + \int_t^{t+1}\int_\Omega |\Delta c_\varepsilon|^{\frac{8}{5}} \le C_2 \qquad \text{for all } t>0.$$

As Young's inequality and Lemma 3.2 moreover show that

$$\frac{16}{3}\int_\Omega n_\varepsilon^{\frac{3}{4}}(\cdot,t+1) \le \frac{16}{3}\Big\{\int_\Omega n_\varepsilon(\cdot,t+1) + |\Omega|\Big\} \le \frac{16}{3}(m+|\Omega|) \qquad \text{for all } t>0$$

and

$$-4\rho\int_t^{t+1}\int_\Omega n_\varepsilon^{\frac{3}{4}} \le 4\rho_-\Big\{\int_t^{t+1}\int_\Omega n_\varepsilon + |\Omega|\Big\} \le 4\rho_-(m+|\Omega|) \qquad \text{for all } t>0$$

as well as

$$4\mu\int_t^{t+1}\int_\Omega n_\varepsilon^{\frac{7}{4}} \le 4\mu\Big\{\int_t^{t+1}\int_\Omega n_\varepsilon^2 + |\Omega|\Big\} \le 4\mu(C_1+|\Omega|) \qquad \text{for all } t>0,$$

from (5.2) we readily derive (5.1). $\square$

Once more by interpolation, this also implies an unweighted integral bound for ∇n_ε in an L^p space with some $p>1$.

Lemma 5.2 *There exists $C > 0$ with the property that for all $\varepsilon \in (0,1)$ we have*

$$\int_t^{t+1} \int_\Omega |\nabla n_\varepsilon(x,s)|^{\frac{16}{13}} dxds \le C \qquad \textit{for all } t > 0. \tag{5.3}$$

PROOF. Since by Young's inequality we can estimate

$$|\nabla n_\varepsilon|^{\frac{16}{13}} = \Big\{ n_\varepsilon^{-\frac{5}{4}} |\nabla n_\varepsilon|^2 \Big\}^{\frac{8}{13}} \cdot n_\varepsilon^{\frac{10}{13}} \le n_\varepsilon^{-\frac{5}{4}} |\nabla n_\varepsilon|^2 + n_\varepsilon^2 \qquad \text{for } x \in \Omega \text{ and } t > 0,$$

this is an immediate consequence of Lemma 5.1 and Lemma 3.2. □

6 Estimates for time derivatives

In a straightforward manner, the estimates gained above can be seen to imply certain properties of the respective time derivatives.

Lemma 6.1 *There exists $C > 0$ such that for all $\varepsilon \in (0,1)$,*

$$\int_0^T \|n_{\varepsilon t}(\cdot,t)\|_{(W^{3,2}(\Omega))^\star} dt \le C \cdot (T+1) \qquad \textit{for all } T > 0 \tag{6.1}$$

and

$$\int_0^T \int_\Omega |c_{\varepsilon t}(x,t)|^{\frac{40}{27}} dxdt \le C \cdot (T+1) \qquad \textit{for all } T > 0 \tag{6.2}$$

as well as

$$\int_0^T \|u_{\varepsilon t}(\cdot,t)\|_{(W_{0,\sigma}^{1,2}(\Omega))^\star}^{\frac{4}{3}} dt \le C \cdot (T+1) \qquad \textit{for all } T > 0. \tag{6.3}$$

PROOF. For fixed $t > 0$, we test the first equation in (3.1) against an arbitrary $\psi \in C^\infty(\bar\Omega)$ to see by several applications of the Hölder inequality that

$$\begin{aligned}
\Big| \int_\Omega n_{\varepsilon t} \psi \Big| &= \Big| -\int_\Omega \nabla n_\varepsilon \cdot \nabla\psi + \chi \int_\Omega \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} \nabla c_\varepsilon \cdot \nabla\psi + \rho \int_\Omega n_\varepsilon \psi - \mu \int_\Omega n_\varepsilon^2 \psi + \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla\psi \Big| \\
&\le \|\nabla n_\varepsilon\|_{L^{\frac{16}{13}}(\Omega)} \|\nabla\psi\|_{L^{\frac{16}{3}}(\Omega)} + \chi \|n_\varepsilon\|_{L^2(\Omega)} \|\nabla c_\varepsilon\|_{L^{\frac{8}{3}}(\Omega)} \|\nabla\psi\|_{L^8(\Omega)} \\
&\quad + |\rho| \|n_\varepsilon\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)} + \mu \|n_\varepsilon\|_{L^2(\Omega)}^2 \|\psi\|_{L^\infty(\Omega)} + \|n_\varepsilon\|_{L^2(\Omega)} \|u_\varepsilon\|_{L^{\frac{10}{3}}(\Omega)} \|\nabla\psi\|_{L^5(\Omega)}.
\end{aligned}$$

Since $W^{3,2}(\Omega) \hookrightarrow W^{1,8}(\Omega) \hookrightarrow W^{1,\frac{16}{3}}(\Omega) \hookrightarrow W^{1,5}(\Omega) \hookrightarrow L^\infty(\Omega) \hookrightarrow L^2(\Omega)$, in view of Young's inequality this implies the existence of positive constants C_1 and C_2 such that

$$\begin{aligned}
\|n_{\varepsilon t}(\cdot,t)\|_{(W^{3,2}(\Omega))^\star} &\le C_1 \cdot \Big\{ \|\nabla n_\varepsilon\|_{L^{\frac{16}{13}}(\Omega)} + \|n_\varepsilon\|_{L^2(\Omega)} \|\nabla c_\varepsilon\|_{L^{\frac{8}{3}}(\Omega)} \\
&\qquad + \|n_\varepsilon\|_{L^2(\Omega)} + \|n_\varepsilon\|_{L^2(\Omega)}^2 + \|n_\varepsilon\|_{L^2(\Omega)} \|u_\varepsilon\|_{L^{\frac{10}{3}}(\Omega)} \Big\} \\
&\le C_2 \cdot \Big\{ \|\nabla n_\varepsilon\|_{L^{\frac{16}{13}}(\Omega)}^{\frac{16}{13}} + \|n_\varepsilon\|_{L^2(\Omega)}^2 + \|\nabla c_\varepsilon\|_{L^{\frac{8}{3}}(\Omega)}^{\frac{8}{3}} + \|u_\varepsilon\|_{L^{\frac{10}{3}}(\Omega)}^{\frac{10}{3}} + 1 \Big\}.
\end{aligned}$$

According to the bounds provided by Lemma 5.2, Lemma 3.2, Lemma 4.3 and Lemma 3.8, this readily yields (6.1).
Next, to derive (6.2) we only need to observe that by the Minkowski inequality and Young's inequality, with some $C_3 > 0$ and $C_4 > 0$ we have the pointwise inequality

$$\begin{aligned}
|c_{\varepsilon t}|^{\frac{40}{27}} &= \Big|\Delta c_\varepsilon - c_\varepsilon + n_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon\Big|^{\frac{40}{27}} \\
&\le C_3 \cdot \Big\{ |\Delta c_\varepsilon|^{\frac{40}{27}} + c_\varepsilon^{\frac{40}{27}} + n_\varepsilon^{\frac{40}{27}} + |u_\varepsilon|^{\frac{40}{27}} |\nabla c_\varepsilon|^{\frac{40}{27}} \Big\} \\
&\le C_4 \cdot \Big\{ |\Delta c_\varepsilon|^{\frac{8}{5}} + c_\varepsilon^{8} + n_\varepsilon^{2} + |u_\varepsilon|^{\frac{10}{3}} + |\nabla c_\varepsilon|^{\frac{8}{3}} + 1 \Big\} \qquad \text{for all } x \in \Omega \text{ and } t > 0,
\end{aligned}$$

because $\frac{40}{27} < \frac{8}{5}$, and because $\frac{3}{10} + \frac{3}{8} = \frac{27}{40}$. Therefore, (6.2) results from Lemma 4.4, Lemma 4.3, Lemma 3.2 and Lemma 3.8.
Finally, for the proof of (6.3) we pick $t > 0$ and multiply the third equation in (3.1) by an arbitrary solenoidal $\psi \in C_0^\infty(\Omega)$ to see on using the Hölder inequality that

$$\begin{aligned}
\Big| \int_\Omega u_{\varepsilon t} \cdot \psi \Big| &= \Big| - \int_\Omega \nabla u_\varepsilon \cdot \nabla \psi + \int_\Omega (Y_\varepsilon u_\varepsilon \otimes u_\varepsilon) \cdot \nabla \psi + \int_\Omega n_\varepsilon \nabla \phi \cdot \psi + \int_\Omega f \cdot \psi \Big| \\
&\le \|\nabla u_\varepsilon\|_{L^2(\Omega)} \|\nabla \psi\|_{L^2(\Omega)} + \|Y_\varepsilon u_\varepsilon\|_{L^6(\Omega)} \|u_\varepsilon\|_{L^3(\Omega)} \|\nabla \psi\|_{L^2(\Omega)} \\
&\quad + C_5 \|n_\varepsilon\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)} + \|f\|_{L^{\frac{6}{5}}(\Omega)} \|\psi\|_{L^6(\Omega)}
\end{aligned}$$

with $C_5 := \|\nabla \phi\|_{L^\infty(\Omega)}$. Since $W^{1,2}_{0,\sigma}(\Omega) \hookrightarrow L^6(\Omega)$, and since

$$\|\nabla Y_\varepsilon u_\varepsilon\|_{L^2(\Omega)} = \|A^{\frac{1}{2}} Y_\varepsilon u_\varepsilon\|_{L^2(\Omega)} = \|Y_\varepsilon A^{\frac{1}{2}} u_\varepsilon\|_{L^2(\Omega)} \le \|A^{\frac{1}{2}} u_\varepsilon\|_{L^2(\Omega)} = \|\nabla u_\varepsilon\|_{L^2(\Omega)} \qquad \text{for all } t > 0$$

due to the fact that Y_ε is nonexpansive on $L^2_\sigma(\Omega)$, we conclude that with some $C_6 > 0$ and $C_7 > 0$ we have

$$\begin{aligned}
\|u_{\varepsilon t}(\cdot, t)\|^{\frac{4}{3}}_{(W^{1,2}_{0,\sigma}(\Omega))^\star} &\le C_6 \cdot \Big\{ \|\nabla u_\varepsilon\|^{\frac{4}{3}}_{L^2(\Omega)} + \|\nabla u_\varepsilon\|^{\frac{4}{3}}_{L^2(\Omega)} \|u_\varepsilon\|^{\frac{4}{3}}_{L^3(\Omega)} + \|n_\varepsilon\|^{\frac{4}{3}}_{L^2(\Omega)} + \|f\|^{\frac{4}{3}}_{L^{\frac{6}{5}}(\Omega)} \Big\} \\
&\le C_6 \cdot \Big\{ \|\nabla u_\varepsilon\|^{2}_{L^2(\Omega)} + \|\nabla u_\varepsilon\|^{\frac{4}{3}}_{L^2(\Omega)} \|u_\varepsilon\|^{\frac{4}{3}}_{L^3(\Omega)} + \|n_\varepsilon\|^{2}_{L^2(\Omega)} + \|f\|^{2}_{L^{\frac{6}{5}}(\Omega)} + 1 \Big\} \qquad (6.4)
\end{aligned}$$

for all $t > 0$. As the Gagliardo-Nirenberg inequality provides $C_8 > 0$ such that

$$\begin{aligned}
\|\nabla u_\varepsilon\|^{\frac{4}{3}}_{L^2(\Omega)} \|u_\varepsilon\|^{\frac{4}{3}}_{L^3(\Omega)} &\le C_8 \|\nabla u_\varepsilon\|^{\frac{4}{3}}_{L^2(\Omega)} \cdot \Big\{ \|\nabla u_\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega)} \|u_\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega)} \Big\}^{\frac{4}{3}} \\
&= C_8 \|\nabla u_\varepsilon\|^{2}_{L^2(\Omega)} \|u_\varepsilon\|^{\frac{2}{3}}_{L^2(\Omega)} \qquad \text{for all } t > 0,
\end{aligned}$$

combining (6.4) with the outcome of Lemma 3.8 and Lemma 3.2 as well as our assumption (1.4) we immediately obtain (6.3). $\square$

7 Passing to the limit. Proof of Theorem 1.1

We are now in the position to extract a suitable sequence of numbers ε along which the respective solutions approach a limit in convenient topologies.

Lemma 7.1 *There exist $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_j \searrow 0$ as $j\to\infty$, and such that as $\varepsilon = \varepsilon_j \searrow 0$ we have*

$$
\begin{aligned}
& n_\varepsilon \to n \qquad \text{in } L^p_{loc}(\bar\Omega\times[0,\infty)) \quad \text{for all } p\in[1,2) \quad \text{and a.e. in } \Omega\times(0,\infty), && (7.1)\\
& n_\varepsilon \rightharpoonup n \qquad \text{in } L^2_{loc}(\bar\Omega\times[0,\infty)), && (7.2)\\
& \nabla n_\varepsilon \rightharpoonup \nabla n_\varepsilon \qquad \text{in } L^{\frac{16}{13}}_{loc}(\bar\Omega\times[0,\infty)), && (7.3)\\
& \nabla(n_\varepsilon+1)^\beta \rightharpoonup \nabla(n+1)^\beta \qquad \text{in } L^2_{loc}(\bar\Omega\times[0,\infty)) \quad \text{for all } \beta\in\Big(0,\frac{3}{8}\Big] && (7.4)
\end{aligned}
$$

and

$$
\begin{aligned}
& c_\varepsilon \to c \qquad \text{in } L^p_{loc}(\bar\Omega\times[0,\infty)) \quad \text{for all } p\in[1,8) \quad \text{and a.e. in } \Omega\times(0,\infty), && (7.5)\\
& \nabla c_\varepsilon \rightharpoonup \nabla c \qquad \text{in } L^{\frac{8}{3}}_{loc}(\bar\Omega\times[0,\infty)), && (7.6)\\
& D^2 c_\varepsilon \rightharpoonup D^2 c \qquad \text{in } L^{\frac{8}{5}}_{loc}(\bar\Omega\times[0,\infty)) && (7.7)
\end{aligned}
$$

as well as

$$
\begin{aligned}
& u_\varepsilon \to u \qquad \text{in } L^2_{loc}(\bar\Omega\times[0,\infty)) \quad \text{and a.e. in } \Omega\times(0,\infty), && (7.8)\\
& u_\varepsilon(\cdot,t) \to u(\cdot,t) \quad \text{in } L^2(\Omega) \qquad \text{for a.e. } t>0, && (7.9)\\
& u_\varepsilon \rightharpoonup u \qquad \text{in } L^{\frac{10}{3}}_{loc}(\bar\Omega\times[0,\infty)) \qquad \text{and} && (7.10)\\
& \nabla u_\varepsilon \rightharpoonup \nabla u \qquad \text{in } L^2_{loc}(\bar\Omega\times[0,\infty)) && (7.11)
\end{aligned}
$$

with some limit functions n, c and u which are such that $n\ge 0$ and $c\ge 0$ a.e. in $\Omega\times(0,\infty)$ and satisfy (1.8) and (1.9).

Proof. First, combining Lemma 5.2 with Lemma 3.2 we see that $(n_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^{\frac{16}{13}}_{loc}([0,\infty);W^{1,\frac{16}{13}}(\Omega))$, whereas $(n_{\varepsilon t})_{\varepsilon\in(0,1)}$ is bounded in $L^1_{loc}([0,\infty);(W^{3,2}(\Omega))^\star)$ by Lemma 6.1. Accordingly, a variant of the Aubin-Lions lemma ([58]) asserts that $(n_\varepsilon)_{\varepsilon\in(0,1)}$ is relatively compact in e.g. $L^{\frac{16}{13}}_{loc}(\bar\Omega\times[0,\infty))$ with respect to the strong topology therein. We can thus pick $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ such that $\varepsilon_j\searrow 0$ as $j\to\infty$ and such that

$$
n_\varepsilon \to n \qquad \text{a.e. in } \Omega\times(0,\infty) \tag{7.12}
$$

as well as (7.3) hold as $\varepsilon=\varepsilon_j\searrow 0$, where in view of (3.5) we may assume that moreover (7.2) is valid. Since Lemma 5.1 in particular warrants the existence of $C_1>0$ such that for any $\beta\in(0,\frac{3}{8}]$ we have

$$
\int_t^{t+1}\int_\Omega |\nabla(n_\varepsilon+1)^\beta|^2 \;=\; \beta^2\int_t^{t+1}\int_\Omega (n_\varepsilon+1)^{2\beta-2}|\nabla n_\varepsilon|^2
$$

$$
\begin{aligned}
&\leq \beta^2 \int_t^{t+1} \int_\Omega (n_\varepsilon+1)^{-\frac{5}{4}} |\nabla n_\varepsilon|^2 \\
&\leq \beta^2 \int_t^{t+1} \int_\Omega n_\varepsilon^{-\frac{5}{4}} |\nabla n_\varepsilon|^2 \\
&\leq C_1 \qquad \text{for all } t>0 \text{ and } \varepsilon \in (0,1),
\end{aligned}
$$

using that $(n_\varepsilon+1)^\beta \to (n+1)^\beta$ a.e. in $\Omega \times (0,\infty)$ by (7.12) we clearly can simultaneously also achieve (7.4) for any such β.
To complete the proof of (7.1), it is sufficient to consider the case $p \in (1,2)$, in which we again invoke Lemma 3.2 to see that $(n_\varepsilon^p)_{\varepsilon\in(0,1)}$ is bounded in $L^{\frac{2}{p}}_{loc}(\bar\Omega \times [0,\infty))$, whence on extracting a further subsequence if necessary we may assume that for any such p we also have

$$
n_\varepsilon^p \rightharpoonup n^p \qquad \text{in } L^{\frac{2}{p}}_{loc}(\bar\Omega \times [0,\infty)) \quad \text{as } \varepsilon=\varepsilon_j \searrow 0.
$$

For fixed $T>0$, this in particular implies that $\int_0^T \int_\Omega n_\varepsilon^p \to \int_0^T \int_\Omega n^p$ and hence, by uniform convexity of $L^p(\Omega \times (0,T))$ for $p>1$, that $n_\varepsilon \to n$ in $L^p(\Omega \times (0,T))$ as $\varepsilon=\varepsilon_j \searrow 0$, as desired.

Likewise, using boundedness of $(c_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^{\frac{8}{3}}_{loc}([0,\infty); W^{1,\frac{8}{3}}(\Omega))$ and of $(c_{\varepsilon t})_{\varepsilon\in(0,1)}$ in the space $L^{\frac{40}{27}}_{loc}([0,\infty); L^{\frac{40}{27}}(\Omega))$, as asserted by Lemma 4.3 and Lemma 6.1, from the Aubin-Lions lemma we infer the existence of a further subsequence, suitably relabeled for notational convenience, such that as $\varepsilon=\varepsilon_j \searrow 0$ we have

$$
c_\varepsilon \to c \qquad \text{a.e. in } \Omega \times (0,\infty) \tag{7.13}
$$

as well as (7.6) and, in view of Lemma 4.4, also (7.7). As moreover $(c_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^8_{loc}(\bar\Omega \times [0,\infty))$ by Lemma 4.3, arguing as above we see that we can also achieve that $c_\varepsilon \to c$ in $L^p_{loc}(\bar\Omega \times [0,\infty))$ for all $p \in [1,8)$ as $\varepsilon=\varepsilon_j \searrow 0$.

The properties (7.8)-(7.11) can similarly be obtained on combining the bounds on $(u_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^2_{loc}([0,\infty); W^{1,2}_{0,\sigma}(\Omega)) \cap L^{\frac{10}{3}}_{loc}(\bar\Omega \times [0,\infty))$ from Lemma 3.8 with the boundedness of $(u_{\varepsilon t})_{\varepsilon\in(0,1)}$ in $L^{\frac{4}{3}}_{loc}([0,\infty); (W^{1,2}_{0,\sigma}(\Omega))^\star)$, as provided by Lemma 6.1 (cf. also [53, p. 329] for a comment on (7.9)).

Clearly, both n and c inherit nonnegativity of n_ε and c_ε, whereas the additional boundedness properties $n \in L^\infty((0,\infty); L^1(\Omega))$, $c \in L^\infty((0,\infty); L^6(\Omega))$ and $u \in L^\infty((0,\infty); L^2_\sigma(\Omega))$ in (1.9) directly result on using Lemma 3.2, Lemma 3.6 and Lemma 3.8 in conjunction with the pointwise convergence statements contained in (7.1), (7.5) and (7.8). $\Box$

In investigating the solution properties of the limit (n,c,u) gained above, we first concentrate on the second and the third equations in (1.1) which indeed are satisfied in the natural weak sense, as postulated in Definition 2.2.

Lemma 7.2 *Let n, c and u be as in Lemma 7.1. Then (2.4) and (2.5) hold.*

Proof. Multiplying the second equation in (3.1) by $\varphi \in C_0^\infty(\bar\Omega \times [0,\infty))$, we see that

$$
\begin{aligned}
-\int_0^\infty \int_\Omega c_\varepsilon \varphi_t - \int_\Omega c_0 \varphi(\cdot,0) &= -\int_0^\infty \int_\Omega \nabla c_\varepsilon \cdot \nabla\varphi - \int_0^\infty \int_\Omega c_\varepsilon \varphi \\
&\quad + \int_0^\infty \int_\Omega n_\varepsilon \varphi + \int_0^\infty \int_\Omega c_\varepsilon u_\varepsilon \cdot \nabla\varphi
\end{aligned} \tag{7.14}
$$

for all $\varepsilon \in (0,1)$, where thanks to (7.5), (7.6) and (7.1) we have

$$-\int_0^\infty\int_\Omega c_\varepsilon\varphi_t \to -\int_0^\infty\int_\Omega c\varphi_t, \qquad -\int_0^\infty\int_\Omega \nabla c_\varepsilon\cdot\nabla\varphi \to -\int_0^\infty\int_\Omega \nabla c\cdot\nabla\varphi,$$
$$-\int_0^\infty\int_\Omega c_\varepsilon\varphi \to -\int_0^\infty\int_\Omega c\varphi \quad\text{and}\quad \int_0^\infty\int_\Omega n_\varepsilon\varphi \to \int_0^\infty\int_\Omega n\varphi$$

as $\varepsilon = \varepsilon_j \searrow 0$. Since (7.10) combined with the outcome of (7.5) for $p := \frac{10}{7} < 8$ implies that $c_\varepsilon u_\varepsilon \rightharpoonup cu$ in $L^1_{loc}(\bar\Omega\times[0,\infty))$ and hence

$$\int_0^\infty\int_\Omega c_\varepsilon u_\varepsilon\cdot\nabla\varphi \to \int_0^\infty\int_\Omega cu\cdot\nabla\varphi$$

as $\varepsilon = \varepsilon_j \searrow 0$, (7.14) implies (2.4).
We next test the third equation in (3.1) by an arbitrary $\varphi \in C_0^\infty(\Omega\times[0,\infty))$ with $\nabla\cdot\varphi \equiv 0$ to obtain the identity

$$\begin{aligned} -\int_0^\infty\int_\Omega u_\varepsilon\cdot\varphi_t - \int_\Omega u_0\cdot\varphi(\cdot,0) &= -\int_0^\infty\int_\Omega \nabla u_\varepsilon\cdot\nabla\varphi + \int_0^\infty\int_\Omega (Y_\varepsilon u_\varepsilon\otimes u_\varepsilon)\cdot\nabla\varphi \\ &\quad + \int_0^\infty\int_\Omega n_\varepsilon\nabla\phi\cdot\varphi + \int_0^\infty\int_\Omega f\cdot\varphi \end{aligned} \tag{7.15}$$

for all $\varepsilon \in (0,1)$, in which by (7.8), (7.11) and (7.1),

$$-\int_0^\infty\int_\Omega u_\varepsilon\cdot\varphi_t \to -\int_0^\infty\int_\Omega u\cdot\varphi_t, \qquad -\int_0^\infty\int_\Omega \nabla u_\varepsilon\cdot\nabla\varphi \to -\int_0^\infty\int_\Omega \nabla u\cdot\nabla\varphi \qquad\text{and}$$
$$\int_0^\infty n_\varepsilon\nabla\phi\cdot\varphi \to \int_0^\infty\int_\Omega n\nabla\phi\cdot\varphi$$

as $\varepsilon = \varepsilon_j \searrow 0$. Since according to a well-known argument (cf. [53, p. 331] or [73, Proof of Lemma 4.1], for instance), (7.8) implies that also

$$Y_\varepsilon u_\varepsilon \to u \qquad \text{in } L^2_{loc}(\bar\Omega\times[0,\infty)),$$

it furthermore follows that $Y_\varepsilon u_\varepsilon \otimes u_\varepsilon \to u\otimes u$ in $L^1_{loc}(\bar\Omega\times[0,\infty))$ and therefore

$$\int_0^\infty\int_\Omega (Y_\varepsilon u_\varepsilon\otimes u_\varepsilon)\cdot\nabla\varphi \to \int_0^\infty\int_\Omega (u\otimes u)\cdot\nabla\varphi$$

as $\varepsilon = \varepsilon_j \searrow 0$, whereby (2.5) becomes a consequence of (7.15). □

Next concerned with the solution properties of n, let us proceed to make sure that this first component enjoys a subsolution property in the sense of Definition 2.1.

Lemma 7.3 *Let (n, c, u) be as given by Lemma 7.1. Then n is a weak Φ-subsolution of the first equation in (1.1) for*

$$\Phi(s) := s, \qquad s \geq 0. \tag{7.16}$$

Proof. Multiplying the first equation in (3.1) by an arbitrary nonnegative $\varphi \in C_0^\infty(\bar\Omega \times [0,\infty))$ results in

$$\begin{aligned}
\mu \int_0^\infty \int_\Omega n_\varepsilon^2 \varphi &= \int_0^\infty \int_\Omega n_\varepsilon \varphi_t + \int_\Omega n_0 \varphi(\cdot,0) - \int_0^\infty \int_\Omega \nabla n_\varepsilon \cdot \nabla\varphi + \chi \int_0^\infty \int_\Omega \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} \nabla c_\varepsilon \cdot \nabla \varphi \\
&\quad + \rho \int_0^\infty \int_\Omega n_\varepsilon \varphi + \int_0^\infty \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla\varphi
\end{aligned} \tag{7.17}$$

for all $\varepsilon \in (0,1)$, where making use of (7.1) and (7.3) we directly see that

$$\int_0^\infty \int_\Omega n_\varepsilon \varphi_t \to \int_0^\infty \int_\Omega n\varphi_t, \qquad -\int_0^\infty \nabla n_\varepsilon \cdot \nabla\varphi \to -\int_0^\infty \int_\Omega \nabla n \cdot \nabla \varphi \qquad \text{and}$$
$$\rho \int_0^\infty \int_\Omega n_\varepsilon \varphi \to \rho \int_0^\infty \int_\Omega n\varphi$$

as $\varepsilon = \varepsilon_j \searrow 0$. Furthermore, applying (7.1) to $p := \frac{8}{5} < 2$ shows that

$$\frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} \to n \qquad \text{in } L^{\frac{8}{5}}_{loc}(\bar\Omega \times [0,\infty)) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0,$$

so that due to (7.6) we have

$$\chi \int_0^\infty \int_\Omega \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} \nabla c_\varepsilon \cdot \nabla\varphi \to \chi \int_0^\infty \int_\Omega n \nabla c \cdot \nabla \varphi$$

as $\varepsilon = \varepsilon_j \searrow 0$, whereas invoking (7.1) with $p := \frac{10}{7} < 2$ we see that

$$\int_0^\infty \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla\varphi \to \int_0^\infty n u \cdot \nabla\varphi$$

as $\varepsilon = \varepsilon_j \searrow 0$ thanks to (7.10). According to the pointwise convergence property in (7.1) and the nonnegativity of φ, from (7.17) we thus infer on using Fatou's lemma that

$$\begin{aligned}
\mu \int_0^\infty \int_\Omega n^2 \varphi &\le \liminf_{\varepsilon=\varepsilon_j \searrow 0} \left\{ \mu \int_0^\infty \int_\Omega n_\varepsilon^2 \varphi \right\} \\
&= \int_0^\infty \int_\Omega n\varphi_t + \int_\Omega n_0 \varphi(\cdot,0) - \int_0^\infty \int_\Omega \nabla n \cdot \nabla \varphi + \chi \int_0^\infty \int_\Omega n \nabla c \cdot \nabla\varphi \\
&\quad + \rho \int_0^\infty \int_\Omega n\varphi + \int_0^\infty \int_\Omega n u \cdot \nabla \varphi,
\end{aligned}$$

which for Φ as in (7.16) is equivalent to the desired inequality in (2.3). $\Box$

For the proof of Theorem 1.1 it remains to assert a corresponding supersolution property. To accomplish this in Lemma 7.5, let us first derive from Lemma 7.1 some further approximation features which will be used therefor.

Lemma 7.4 *Let $(\varepsilon_j)_{j\in\mathbb{N}}$ and n be as provided by Lemma 7.1, and suppose that $\alpha \in (0, \frac{3}{4})$. Then*

$$(n_\varepsilon+1)^\alpha \to (n+1)^\alpha \qquad \textit{in } L^p_{loc}(\bar\Omega\times[0,\infty)) \textit{ for all } p\in\Big[1,\frac{2}{\alpha}\Big), \tag{7.18}$$

$$(n_\varepsilon+1)^{\alpha-\frac{3}{8}} \to (n+1)^{\alpha-\frac{3}{8}} \qquad \textit{in } L^p_{loc}(\bar\Omega\times[0,\infty)) \textit{ for all } p\in\Big[1,\frac{2}{(\alpha-\frac{3}{8})_+}\Big), \tag{7.19}$$

$$n_\varepsilon(n_\varepsilon+1)^{\alpha-1} \to n(n+1)^{\alpha-1} \qquad \textit{in } L^p_{loc}(\bar\Omega\times[0,\infty)) \textit{ for all } p\in\Big[1,\frac{2}{\alpha}\Big), \tag{7.20}$$

$$n_\varepsilon^2(n_\varepsilon+1)^{\alpha-1} \to n^2(n+1)^{\alpha-1} \qquad \textit{in } L^p_{loc}(\bar\Omega\times[0,\infty)) \textit{ for all } p\in\Big[1,\frac{2}{\alpha+1}\Big) \textit{ and} \tag{7.21}$$

$$\frac{n_\varepsilon}{(n_\varepsilon+1)^{1-\alpha}(1+\varepsilon n_\varepsilon)} \to n(n+1)^{\alpha-1} \qquad \textit{in } L^p_{loc}(\bar\Omega\times[0,\infty)) \textit{ for all } p\in\Big[1,\frac{2}{\alpha}\Big), \tag{7.22}$$

and writing

$$\psi_\varepsilon(s) := \alpha\int_0^s \frac{d\sigma}{(\sigma+1)^{1-\alpha}(1+\varepsilon\sigma)^2}, \qquad s\ge 0,\ \varepsilon\in(0,1), \tag{7.23}$$

we have

$$\psi_\varepsilon(n_\varepsilon) \to (n+1)^\alpha \qquad \textit{in } L^p_{loc}(\bar\Omega\times[0,\infty)) \textit{ for all } p\in\Big[1,\frac{2}{\alpha}\Big). \tag{7.24}$$

Proof. We first claim that for all $\beta\in(0,2)$ and any $q\in(1,\frac{2}{\beta})$ we have

$$(n_\varepsilon+1)^\beta \to (n+1)^\beta \qquad \text{in } L^q_{loc}(\bar\Omega\times[0,\infty)) \tag{7.25}$$

as $\varepsilon=\varepsilon_j\searrow 0$. Indeed, this follows by an argument quite similar to that used in Lemma 7.1: Since $((n_\varepsilon+1)^\beta))_{\varepsilon\in(0,1)}$ is bounded in $L^{\frac{2}{\beta}}_{loc}(\bar\Omega\times[0,\infty))$ according to Lemma 3.2, using that $n_\varepsilon\to n$ a.e. in $\Omega\times(0,\infty)$ we first obtain that for all $T>0$ we have $(n_\varepsilon+1)^\beta \rightharpoonup (n+1)^\beta$ in $L^{\frac{2}{\beta}}(\Omega\times(0,T)) \subset L^q(\Omega\times(0,T))$ as $\varepsilon=\varepsilon_j\searrow 0$. As furthermore from the same source we know that $((n_\varepsilon+1)^{q\beta})_{\varepsilon\in(0,1)}$ is bounded in $L^{\frac{2}{q\beta}}_{loc}(\bar\Omega\times[0,\infty))$, for all $T>0$ we also obtain that $(n_\varepsilon+1)^{q\beta}\rightharpoonup(n+1)^{q\beta}$ in $L^{\frac{2}{q\beta}}(\Omega\times(0,T))$ and hence $\int_0^T\int_\Omega(n_\varepsilon+1)^{q\beta}\to\int_0^T\int_\Omega(n+1)^{q\beta}$ as $\varepsilon=\varepsilon_j\searrow 0$, implying (7.25).
When applied to $\beta:=\alpha$ and to $\beta:=\alpha-\frac{3}{8}$, respectively, this immediately proves (7.18) and also (7.19) when $\alpha>\frac{3}{8}$, whereas to derive the latter in the case $\alpha\le\frac{3}{8}$ we only need to note that as $\varepsilon=\varepsilon_j\searrow 0$, for each $\gamma\ge 0$ we have

$$(n_\varepsilon+1)^{-\gamma}\to(n+1)^{-\gamma} \qquad \text{in } L^p_{loc}(\bar\Omega\times[0,\infty)) \qquad \text{for all } p\in[1,\infty) \tag{7.26}$$

by (7.1) and the dominated convergence theorem. On decomposing

$$n_\varepsilon(n_\varepsilon+1)^{\alpha-1} = (n_\varepsilon+1)^\alpha - (n_\varepsilon+1)^{\alpha-1}$$

and again using (7.26), from (7.18) we easily obtain (7.20), and, similarly, rewriting

$$n_\varepsilon^2(n_\varepsilon+1)^{\alpha-1} = (n_\varepsilon+1)^{\alpha+1} - 2(n_\varepsilon+1)^\alpha + (n_\varepsilon+1)^{\alpha-1},$$

by an application of (7.25) to $\beta:=\alpha+1$, and again of (7.18) and (7.26), we readily deduce (7.21).

To verify (7.22), we split

$$\begin{aligned}
\frac{n_\varepsilon}{(n_\varepsilon+1)^{1-\alpha}(1+\varepsilon n_\varepsilon)} - n(n+1)^{\alpha-1} &= \Big\{\frac{n_\varepsilon}{(n_\varepsilon+1)^{1-\alpha}(1+\varepsilon n_\varepsilon)} - \frac{n}{(n+1)^{1-\alpha}(1+\varepsilon n_\varepsilon)}\Big\} \\
&\quad + \Big\{\frac{n}{(n+1)^{1-\alpha}(1+\varepsilon n_\varepsilon)} - \frac{n}{(n+1)^{1-\alpha}}\Big\} \\
&=: w_{1\varepsilon} + w_{2\varepsilon}, \qquad (7.27)
\end{aligned}$$

and use (7.20) to see that for all $T>0$ and $p \in [1, \frac{2}{\alpha})$,

$$\int_0^T \int_\Omega |w_{1\varepsilon}|^p \le \int_0^T \int_\Omega \Big| n_\varepsilon (n_\varepsilon+1)^{\alpha-1} - n(n+1)^{\alpha-1}\Big|^p \to 0 \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0. \qquad (7.28)$$

Since $\xi_\varepsilon := \frac{1}{1+\varepsilon n_\varepsilon} - 1$ satisfies

$$|\xi_\varepsilon| = \frac{\varepsilon n_\varepsilon}{1+\varepsilon n_\varepsilon} \le 1 \qquad \text{in } \Omega \times (0,\infty)$$

as well as $\xi_\varepsilon \to 0$ a.e. in $\Omega \times (0,\infty)$ as $\varepsilon = \varepsilon_j \searrow 0$, by the dominated convergence theorem we moreover find that for any such T and p,

$$\int_0^T \int_\Omega |w_{2\varepsilon}|^p \le \int_0^T \int_\Omega |\xi_\varepsilon|^p \cdot \Big(n(n+1)^{\alpha-1}\Big)^p \to 0 \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0,$$

because clearly $(n(n+1)^{\alpha-1})^p$ belongs to $L^1(\Omega \times (0,T))$ by e.g. (7.20). Combined with (7.28) and (7.27), this establishes (7.22).
Finally, for the derivation of (7.24) we also use a decomposition according to

$$\psi_\varepsilon(n_\varepsilon) - (n+1)^\alpha = \Big\{\psi_\varepsilon(n_\varepsilon) - \psi_\varepsilon(n)\Big\} + \Big\{\psi_\varepsilon(n) - (n+1)^\alpha\Big\} =: z_{1\varepsilon} + z_{2\varepsilon}, \qquad (7.29)$$

where using that

$$\begin{aligned}
\Big|\psi_\varepsilon(s_2) - \psi_\varepsilon(s_1)\Big| &= \alpha \int_{s_1}^{s_2} \frac{d\sigma}{(\sigma+1)^{1-\alpha}(1+\varepsilon\sigma)^2} \\
&\le \alpha \int_{s_1}^{s_2} \frac{d\sigma}{(\sigma+1)^{1-\alpha}} \\
&= (s_2+1)^\alpha - (s_1+1)^\alpha \qquad \text{whenever } 0 \le s_1 \le s_2,
\end{aligned}$$

we readily infer from (7.18) that for all $T>0$ and $p \in [1, \frac{2}{\alpha})$,

$$\int_0^T \int_\Omega |z_{1\varepsilon}|^p \le \int_0^T \int_\Omega \Big|(n_\varepsilon+1)^\alpha - (n+1)^\alpha\Big|^p \to 0 \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0. \qquad (7.30)$$

Since furthermore

$$\begin{aligned}
0 \le (n+1)^\alpha - \psi_\varepsilon(n) &= \alpha \int_0^n \frac{1}{(\sigma+1)^{1-\alpha}} \cdot \Big\{1 - \frac{1}{(1+\varepsilon\sigma)^2}\Big\} d\sigma \\
&=: \eta_\varepsilon = \eta_\varepsilon(x,t) \qquad \text{in } \Omega \times (0,\infty),
\end{aligned}$$

where $\eta_\varepsilon \to 0$ a.e. in $\Omega \times (0,\infty)$ as $\varepsilon = \varepsilon_j \searrow 0$ and

$$\eta_\varepsilon^p \le \left\{ \alpha \int_0^n \frac{d\sigma}{(\sigma+1)^{1-\alpha}} \right\}^p = (n+1)^{p\alpha} \qquad \text{in } \Omega \times (0,\infty),$$

and since $(n+1)^{p\alpha} \in L^1_{loc}(\bar\Omega \times [0,\infty))$ for all $p \in [1,\frac{2}{\alpha})$ by (7.18), we may once again invoke the dominated convergence theorem to see that for all $T>0$ we have

$$\int_0^T \int_\Omega |z_{2\varepsilon}|^p \le \int_0^T \int_\Omega \eta_\varepsilon^p \to 0 \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0,$$

which together with (7.30) and (7.29) proves (7.24). □

We can thereupon verify the following.

Lemma 7.5 *Let (n,c,u) be as given by Lemma 7.1. Then for any choice of $\alpha \in (0,\frac{3}{4})$, n is a weak Φ-supersolution of the first equation in (1.1) for*

$$\Phi(s) := (s+1)^\alpha, \qquad s \ge 0. \tag{7.31}$$

Proof. We first compute

$$\begin{aligned} \partial_t(n_\varepsilon+1)^\alpha \;=\;& \alpha(n_\varepsilon+1)^{\alpha-1}\Delta n_\varepsilon \\ & -\chi \cdot \frac{\alpha}{(n_\varepsilon+1)^{1-\alpha}(1+\varepsilon n_\varepsilon)^2}\nabla n_\varepsilon \cdot \nabla c_\varepsilon - \alpha\chi \cdot \frac{n_\varepsilon}{(n_\varepsilon+1)^{1-\alpha}(1+\varepsilon n_\varepsilon)}\Delta c_\varepsilon \\ & +\alpha\rho n_\varepsilon(n_\varepsilon+1)^{\alpha-1} - \alpha\mu n_\varepsilon^2(n_\varepsilon+1)^{\alpha-1} \\ & -u_\varepsilon \cdot \nabla(n_\varepsilon+1)^\alpha, \qquad x \in \Omega,\ t>0, \end{aligned} \tag{7.32}$$

where we note that

$$\frac{\alpha}{(n_\varepsilon+1)^{1-\alpha}(1+\varepsilon n_\varepsilon)^2}\nabla n_\varepsilon = \nabla\psi_\varepsilon(n_\varepsilon)$$

with ψ_ε as defined in (7.23). Thus, testing (7.32) by an arbitrary nonnegative $\varphi \in C_0^\infty(\bar\Omega \times [0,\infty))$ we obtain that

$$\begin{aligned} -\int_0^\infty \int_\Omega (n_\varepsilon+1)^\alpha \varphi_t - \int_\Omega (n_0+1)^\alpha \varphi(\cdot,0) \\ = \alpha(1-\alpha)\int_0^\infty \int_\Omega (n_\varepsilon+1)^{\alpha-2}|\nabla n_\varepsilon|^2\varphi - \alpha \int_0^\infty \int_\Omega (n_\varepsilon+1)^{\alpha-1}\nabla n_\varepsilon \cdot \nabla\varphi \\ +\chi \int_0^\infty \int_\Omega \psi_\varepsilon(n_\varepsilon)\nabla c_\varepsilon \cdot \nabla\varphi + \chi \int_0^\infty \int_\Omega \psi_\varepsilon(n_\varepsilon)\Delta c_\varepsilon \varphi \\ -\alpha\chi \int_0^\infty \int_\Omega \frac{n_\varepsilon}{(n_\varepsilon+1)^{1-\alpha}(1+\varepsilon n_\varepsilon)}\Delta c_\varepsilon \varphi \\ +\alpha\rho \int_0^\infty \int_\Omega n_\varepsilon(n_\varepsilon+1)^{\alpha-1}\varphi - \alpha\mu \int_0^\infty \int_\Omega n_\varepsilon^2(n_\varepsilon+1)^{\alpha-1}\varphi \\ + \int_0^\infty \int_\Omega (n_\varepsilon+1)^\alpha u_\varepsilon \cdot \nabla\varphi \end{aligned} \tag{7.33}$$

for all $\varepsilon \in (0,1)$. Here we apply Lemma 7.4 to see that since $(n_\varepsilon+1)^\alpha \to (n+1)^\alpha$ and $n_\varepsilon(n_\varepsilon+1)^{\alpha-1} \to n(n+1)^{\alpha-1}$ as well as $n_\varepsilon^2(n_\varepsilon+1)^{\alpha-1} \to n^2(n+1)^{\alpha-1}$ in $L^1_{loc}(\bar\Omega \times [0,\infty))$ by (7.18), (7.20) and (7.21), we have

$$-\int_0^\infty \int_\Omega (n_\varepsilon+1)^\alpha \varphi_t \to -\int_0^\infty \int_\Omega (n+1)^\alpha \varphi_t \tag{7.34}$$

and

$$\alpha\rho \int_0^\infty \int_\Omega n_\varepsilon(n_\varepsilon+1)^{\alpha-1}\varphi \to \alpha\rho \int_0^\infty \int_\Omega n(n+1)^{\alpha-1}\varphi \tag{7.35}$$

as well as

$$-\alpha\mu \int_0^\infty \int_\Omega n_\varepsilon^2(n_\varepsilon+1)^{\alpha-1}\varphi \to -\alpha\mu \int_0^\infty \int_\Omega n^2(n+1)^{\alpha-1}\varphi \tag{7.36}$$

as $\varepsilon = \varepsilon_j \searrow 0$. Moreover, combining the fact that $(\Delta c_\varepsilon)_{\varepsilon=\varepsilon_j\searrow 0}$ is weakly convergent in $L^{\frac{8}{5}}_{loc}(\bar\Omega \times [0,\infty))$, as asserted by (7.7), with an application of (7.24) and (7.22) to $p := \frac{8}{3}$, which is possible since according to our assumption $\alpha < \frac{3}{4}$ we have $\frac{8}{3} < \frac{2}{\alpha}$, we see that

$$\chi \int_0^\infty \int_\Omega \psi_\varepsilon(n_\varepsilon)\Delta c_\varepsilon \varphi \to \chi \int_0^\infty \int_\Omega (n+1)^\alpha \Delta c \varphi \tag{7.37}$$

and

$$-\alpha\chi \int_0^\infty \int_\Omega \frac{n_\varepsilon}{(n_\varepsilon+1)^{1-\alpha}(1+\varepsilon n_\varepsilon)}\Delta c_\varepsilon \varphi \to -\alpha\chi \int_0^\infty \int_\Omega n(n+1)^{\alpha-1}\Delta c \varphi \tag{7.38}$$

as $\varepsilon = \varepsilon_j \searrow 0$. Similarly, the convergence property of $(\nabla c_\varepsilon)_{\varepsilon=\varepsilon_j\searrow 0}$ in $L^{\frac{8}{3}}_{loc}(\bar\Omega \times [0,\infty))$ in (7.6) can be complemented by invoking (7.24) for $p := \frac{8}{5} < \frac{2}{\alpha}$ to obtain

$$\chi \int_0^\infty \int_\Omega \psi_\varepsilon(n_\varepsilon)\nabla c_\varepsilon \cdot \nabla\varphi \to \chi \int_0^\infty \int_\Omega (n+1)^\alpha \nabla c \cdot \nabla\varphi \tag{7.39}$$

as $\varepsilon = \varepsilon_j \searrow 0$, and since $(u_\varepsilon)_{\varepsilon=\varepsilon_j\searrow 0}$ is weakly convergent in $L^{\frac{10}{3}}_{loc}(\bar\Omega \times [0,\infty))$ by (7.10), taking $p := \frac{10}{7} < \frac{2}{\alpha}$ in (7.18) yields

$$\int_0^\infty \int_\Omega (n_\varepsilon+1)^\alpha u_\varepsilon \cdot \nabla\varphi \to \int_0^\infty \int_\Omega (n+1)^\alpha u \cdot \nabla\varphi \tag{7.40}$$

as $\varepsilon = \varepsilon_j \searrow 0$.

Next, in the second integral on the right of (7.33) we use that $\nabla(n_\varepsilon+1)^{\frac{3}{8}} = \frac{3}{8}(n_\varepsilon+1)^{-\frac{5}{8}}\nabla n_\varepsilon$ to decompose

$$-\alpha \int_0^\infty \int_\Omega (n_\varepsilon+1)^{\alpha-1}\nabla n_\varepsilon \cdot \nabla\varphi = -\frac{8\alpha}{3}\int_0^\infty \int_\Omega (n_\varepsilon+1)^{\alpha-\frac{3}{8}}\nabla(n_\varepsilon+1)^{\frac{3}{8}} \cdot \nabla\varphi,$$

so that invoking (7.4) with $\beta := \frac{3}{8}$ and applying (7.19) to $p := 2 < \frac{2}{(\alpha-\frac{3}{8})_+}$ shows that

$$\begin{aligned} -\alpha \int_0^\infty \int_\Omega (n_\varepsilon+1)^{\alpha-1}\nabla n_\varepsilon \cdot \nabla\varphi &\to -\frac{8\alpha}{3}\int_0^\infty \int_\Omega (n+1)^{\alpha-\frac{3}{8}}\nabla(n+1)^{\frac{3}{8}} \cdot \nabla\varphi \\ &= -\alpha \int_0^\infty \int_\Omega (n+1)^{\alpha-1}\nabla n \cdot \nabla\varphi \end{aligned} \tag{7.41}$$

as $\varepsilon = \varepsilon_j \searrow 0$.

Again relying on (7.4), now applied to $\beta := \frac{\alpha}{2}$, in view of a standard argument based on lower semicontinuity of the norm in $L^2(\Omega \times (0,\infty))$ with respect to weak convergence, we thus infer from (7.33)-(7.41) that

$$\begin{aligned}
\alpha(1-\alpha)\int_0^\infty\int_\Omega (n+1)^{\alpha-2}|\nabla n|^2\varphi &= \frac{4(1-\alpha)}{\alpha}\int_0^\infty\int_\Omega \Big|\nabla(n+1)^{\frac{\alpha}{2}}\Big|^2\varphi \\
&\le \liminf_{\varepsilon=\varepsilon_j\searrow 0}\left\{\frac{4(1-\alpha)}{\alpha}\int_0^\infty\int_\Omega \Big|\nabla(n_\varepsilon+1)^{\frac{\alpha}{2}}\Big|^2\varphi\right\} \\
&= \liminf_{\varepsilon=\varepsilon_j\searrow 0}\left\{\alpha(1-\alpha)\int_0^\infty\int_\Omega (n_\varepsilon+1)^{\alpha-2}|\nabla n_\varepsilon|^2\varphi\right\} \\
&= -\int_0^\infty\int_\Omega (n+1)^\alpha\varphi_t - \int_\Omega (n_0+1)^\alpha\varphi(\cdot,0) \\
&\quad +\alpha\int_0^\infty\int_\Omega (n+1)^{\alpha-1}\nabla n\cdot\nabla\varphi \\
&\quad -\chi\int_0^\infty\int_\Omega (n+1)^\alpha\nabla c\cdot\nabla\varphi - \chi\int_0^\infty\int_\Omega (n+1)^\alpha\Delta c\varphi \\
&\quad +\alpha\chi\int_0^\infty\int_\Omega n(n+1)^{\alpha-1}\Delta c\varphi \\
&\quad -\alpha\rho\int_0^\infty\int_\Omega n(n+1)^{\alpha-1}\varphi + \alpha\mu\int_0^\infty\int_\Omega n^2(n+1)^{\alpha-1}\varphi \\
&\quad -\int_0^\infty\int_\Omega (n+1)^\alpha u\cdot\nabla\varphi
\end{aligned}$$

holds for any such φ. By definition (7.31) of Φ, this precisely yields the claimed supersolution property in (2.3). $\Box$

In summary, (n,c,u) indeed is a weak solution in the desired flavor:

Proof of Theorem 1.1. We only need to collect the results of Lemma 7.1, Lemma 7.2, Lemma 7.3 and Lemma 7.5. $\Box$

8 Large time behavior: The case $\rho > 0$

We next address the large time behavior of the solutions gained above under the largeness assumption (1.10) on μ. Here in the case $\rho \le 0$ when the first equation in (1.1) does actually not contain a production term, it is not surprising that solutions decay in their first two components in the large time limit (cf. Lemma 9.1 below for the main argument justifying this intuition). We therefore first consider the case when ρ is positive, which will be significantly more involved. Fortunately, the assumption (1.10) warrants that at least formally, (1.1) possesses a Lyapunov functional, containing the first two solution components, which is such that the dissipation rate in the corresponding energy inequality is adequately large as long as solutions are far from the spatially homogeneous equilibrium $(\frac{\rho}{\mu},\frac{\rho}{\mu})$. Due to the lack of knowledge on appropriate regularity of solutions, however, the precise verification of this latter circumstance will require some arguments which seem not straightforward, and which will be the objective of Section 8.2. Furthermore, in order to remain at the level of suitably smooth functions as long as possible, our approach will require the derivation of decay properties which are essentially independent of $\varepsilon \in (0,1)$.

8.1 An energy functional for (3.1)

Given a positive number $n_\star$, we let $\zeta_{n_\star} : (0,\infty) \to \mathbb{R}$ be defined by

$$\zeta_{n_\star}(s) := s - n_\star - n_\star \ln \frac{s}{n_\star}, \qquad s > 0. \tag{8.1}$$

Then $\zeta_{n_\star}$ is convex with $\zeta_{n_\star}(n_\star) = \zeta'_{n_\star}(n_\star) = 0$, so that $\zeta_{n_\star}(s) \ge 0$ for all $s > 0$.
In particular, for each $B > 0$ and any nonnegative continuous $n : \bar\Omega \to (0,\infty)$ and $c : \bar\Omega \to \mathbb{R}$,

$$\mathcal{F}_{n_\star,B}(n,c) := \int_\Omega \zeta_{n_\star}(n) + \frac{B}{2} \int_\Omega (c - n_\star)^2 \tag{8.2}$$

is well-defined and nonnegative with $\mathcal{F}_{n_\star,B}(n_\star, n_\star) = 0$.

In fact, $\mathcal{F}_{n_\star,B}$ plays the role of an energy functional for (3.1) in the following sense.

Lemma 8.1 *Let $\chi > 0$ and $\rho > 0$, and suppose that*

$$\mu > \frac{\chi\sqrt{\rho}}{4}. \tag{8.3}$$

Then there exist $B > 0$ and $C > 0$ such that writing $n_\star := \frac{\rho}{\mu}$, with $\mathcal{F}_{n_\star,B}$ as in (8.2) we have

$$\begin{aligned} \frac{d}{dt}\mathcal{F}_{n_\star,B}\Big(n_\varepsilon(\cdot,t), c_\varepsilon(\cdot,t)\Big) \quad &+ \quad C\bigg\{ \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon^2} + \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega (n_\varepsilon - n_\star)^2 + \int_\Omega (c_\varepsilon - n_\star)^2 \bigg\} \le 0 \\ &\le \quad 0 \qquad \textit{for all } t > 0 \end{aligned} \tag{8.4}$$

and any $\varepsilon \in (0,1)$.

Proof. According to (8.3), we have

$$\frac{\chi^2 n_\star}{4} < 4\mu,$$

whence it is possible to fix $B > 0$ such that

$$\frac{\chi^2 n_\star}{4} < B < 4\mu. \tag{8.5}$$

Here the former inequality ensures the existence of some suitably small $\theta \in (0,1)$ such that still

$$\frac{\chi^2 n_\star}{4(1-\theta)} < B, \tag{8.6}$$

whereas due to the latter inequality in (8.5) we can find $\eta \in (0,1)$ fulfilling

$$\frac{B}{4(1-\eta)} < \mu. \tag{8.7}$$

With this value of B fixed, for $\varepsilon \in (0,1)$ we use the first two equations in (3.1) and the fact that $\nabla \cdot u_\varepsilon = 0$ to compute

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}_{n_\star,B}(n_\varepsilon,c_\varepsilon) &= \int_\Omega n_{\varepsilon t} - n_\star \int_\Omega \frac{n_{\varepsilon t}}{n_\varepsilon} + B\int_\Omega (c_\varepsilon - n_\star)c_{\varepsilon t} \\
&= \rho\int_\Omega n_\varepsilon - \mu\int_\Omega n_\varepsilon^2 - n_\star \int_\Omega \frac{1}{n_\varepsilon}\cdot\Big\{\Delta n_\varepsilon - \chi\nabla\cdot\Big(\frac{n_\varepsilon}{1+\varepsilon n_\varepsilon}\nabla c_\varepsilon\Big) + \rho n_\varepsilon - \mu n_\varepsilon^2 - u_\varepsilon\cdot\nabla n_\varepsilon\Big\} \\
&\quad + B\int_\Omega (c_\varepsilon - n_\star)\cdot\Big\{\Delta c_\varepsilon - c_\varepsilon + n_\varepsilon - u_\varepsilon\cdot\nabla c_\varepsilon\Big\} \\
&= \rho\int_\Omega n_\varepsilon - \mu\int_\Omega n_\varepsilon^2 \\
&\quad - n_\star\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon^2} + n_\star\chi\int_\Omega \frac{\nabla n_\varepsilon}{n_\varepsilon(1+\varepsilon n_\varepsilon)}\cdot\nabla c_\varepsilon - n_\star\rho|\Omega| + n_\star\mu\int_\Omega n_\varepsilon \\
&\quad - B\int_\Omega |\nabla c_\varepsilon|^2 - B\int_\Omega (c_\varepsilon - n_\star)^2 + B\int_\Omega (c_\varepsilon - n_\star)(n_\varepsilon - n_\star) \qquad \text{for all } t>0.
\end{aligned} \tag{8.8}$$

Here it can easily be checked that thanks to the definition of $n_\star$ we have

$$\rho\int_\Omega n_\varepsilon - \mu\int_\Omega n_\varepsilon^2 - n_\star\rho|\Omega| + n_\star\mu\int_\Omega n_\varepsilon = -\mu\int_\Omega (n_\varepsilon - n_\star)^2 \qquad \text{for all } t>0, \tag{8.9}$$

and two applications of Young's inequality show that

$$n_\star\chi\int_\Omega \frac{\nabla n_\varepsilon}{n_\varepsilon(1+\varepsilon n_\varepsilon)}\cdot\nabla c_\varepsilon \le (1-\theta)n_\star\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon^2} + \frac{n_\star\chi^2}{4(1-\theta)}\int_\Omega |\nabla c_\varepsilon|^2 \qquad \text{for all } t>0 \tag{8.10}$$

and

$$B\int_\Omega (c_\varepsilon - n_\star)(n_\varepsilon - n_\star) \le (1-\eta)B\int_\Omega (c_\varepsilon - n_\star)^2 + \frac{B}{4(1-\eta)}\int_\Omega (n_\varepsilon - n_\star)^2 \qquad \text{for all } t>0. \tag{8.11}$$

Collecting (8.8)-(8.11), we thus infer that

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}_{n_\star,B}(n_\varepsilon,c_\varepsilon) &\le -\theta n_\star\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon^2} - \Big(B - \frac{n_\star\chi^2}{4(1-\theta)}\Big)\int_\Omega |\nabla c_\varepsilon|^2 \\
&\quad - \Big(\mu - \frac{B}{4(1-\eta)}\Big)\int_\Omega (n_\varepsilon - n_\star)^2 - \eta B\int_\Omega (c_\varepsilon - n_\star)^2 \qquad \text{for all } t>0.
\end{aligned}$$

As (8.6) and (8.7) assert that both $B - \frac{n_\star\chi^2}{4(1-\theta)}$ and $\mu - \frac{B}{4(1-\eta)}$ are positive, this establishes (8.4) upon an evident choice of C. $\square$

As an immediate consequence, we obtain the following which will firstly serve as a fundament for our proof of stabilization in the first two solution components, and which moreover, through the second inequality implicitly contained in (8.13), will later on be useful for achieving decay of the fluid velocity field in the proof of Theorem 1.3.

Corollary 8.2 *Let $\chi > 0$ and $\rho > 0$, and suppose that (8.3) holds. Then with $n_\star := \frac{\rho}{\mu}$ and $B > 0$ as given by Lemma 8.1, and with $\mathcal{F}_{n_\star,B}$ as defined in (8.2), for all $\varepsilon \in (0,1)$ we have*

$$\mathcal{F}_{n_\star,B}\Big(n_\varepsilon(\cdot,t), c_\varepsilon(\cdot,t)\Big) \le \mathcal{F}_{n_\star,B}\Big(n_\varepsilon(\cdot,t_0), c_\varepsilon(\cdot,t_0)\Big) \qquad \textit{whenever } 0 \le t_0 < t, \tag{8.12}$$

and there exists $C > 0$ such that

$$\int_0^\infty \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon^2} + \int_0^\infty \int_\Omega (n_\varepsilon - n_\star)^2 + \int_0^\infty \int_\Omega (c_\varepsilon - n_\star)^2 \le C \qquad \textit{for all } \varepsilon \in (0,1). \tag{8.13}$$

Proof. This directly results on integrating (8.4) in time. □

8.2 Estimating the energy in terms of the dissipation rate

In order to take full advantage of the dissipation mechanism expressed by (8.4), it seems desirable to relate the size of the energy by the corresponding dissipation rate. Here we underline that a substantial obstacle toward this stems from the fact that in the context of arbitrary positive functions, for each fixed $n_\star > 0$ and $B > 0$ the functional $\mathcal{F}_{n_\star,B}$ apparently may attain arbitrarily large values even under the constraint that the dissipation rate remains bounded; this is due to the singularity of $\zeta_{n_\star}$ at the origin, and can accordingly be seen on e.g. choosing $c \equiv n_\star$ and $n \equiv \delta$ for suitably small $\delta > 0$. When resorting to functions for which additionally a certain smallness condition on $n - n_\star$ is satisfied, however, one can make proper use of the term $\int_\Omega \frac{|\nabla n|^2}{n^2}$ to control $\mathcal{F}_{n_\star,B}(n,c)$. This can be achieved by means of the following variant of the Poincaré inequality.

Lemma 8.3 *There exists $C_P > 0$ such that*

$$\int_\Omega h^2 \le C_P \int_\Omega |\nabla h|^2 \qquad \textit{for all } h \in W^{1,2}(\Omega) \textit{ satisfying } \Big|\{h=0\}\Big| \ge \frac{|\Omega|}{2}.$$

Proof. This is part of the statement proved in [26, Corollary 8.1.4]. □

In fact, we can thereby derive the following control of the possibly singular contribution to $\mathcal{F}_{n_\star,B}$ in terms of integrals appearing in the dissipation rate in (8.4).

Lemma 8.4 *For $n_\star > 0$, let $\zeta_{n_\star}$ be as defined in (8.1). Then taking $C_P > 0$ from Lemma 8.3, we have*

$$\int_\Omega \zeta_{n_\star}(\varphi) \le n_\star \sqrt{C_P|\Omega|} \bigg(\int_\Omega \frac{|\nabla\varphi|^2}{\varphi^2} \bigg)^{\frac{1}{2}} + \frac{\sqrt{8|\Omega|}}{n_\star} \bigg(\int_\Omega (\varphi - n_\star)^2 \bigg)^{\frac{1}{2}} + \frac{1}{n_\star} \int_\Omega (\varphi - n_\star)^2 \tag{8.14}$$

for any positive $\varphi \in C^1(\bar\Omega)$ satisfying

$$\int_\Omega (\varphi - n_\star)^2 \le \frac{n_\star^2 |\Omega|}{8}. \tag{8.15}$$

Proof. Assuming without loss of generality that

$$I := \int_\Omega \frac{|\nabla\varphi|^2}{\varphi^2} \qquad \text{and} \qquad J := \int_\Omega (\varphi - n_\star)^2 \tag{8.16}$$

are both positive, using the Chebyshev inequality we see that

$$\Big|\Big\{(\varphi-n_\star)^2>\frac{2J}{|\Omega|}\Big\}\Big| \le \frac{\int_\Omega(\varphi-n_\star)^2}{\frac{2J}{|\Omega|}} = \frac{|\Omega|}{2}$$

and hence

$$\Big|\Big\{\varphi\ge n_\star-\sqrt{\frac{2J}{|\Omega|}}\Big\}\Big| \ge \Big|\Big\{|\varphi-n_\star|\le\sqrt{\frac{2J}{|\Omega|}}\Big\}\Big| \ge \frac{|\Omega|}{2}.$$

Therefore,

$$h(x):=\Big(-\ln\frac{\varphi(x)}{n_\star-\sqrt{\frac{2J}{|\Omega|}}}\Big)_+$$

belongs to $W^{1,2}(\Omega)$ and satisfies

$$\Big|\{h=0\}\Big| = \Big|\Big\{\varphi\ge n_\star-\sqrt{\frac{2J}{|\Omega|}}\Big\}\Big| \ge \frac{|\Omega|}{2},$$

whence Lemma 8.3 shows that

$$\int_\Omega\Big(-\ln\frac{\varphi(x)}{n_\star-\sqrt{\frac{2J}{|\Omega|}}}\Big)_+^2 = \int_\Omega h^2 \le C_P\int_\Omega|\nabla h|^2 = C_P\int_{\{\varphi<n_\star-\sqrt{\frac{2J}{|\Omega|}}\}}\frac{|\nabla\varphi|^2}{\varphi^2} \le C_P I. \tag{8.17}$$

In order to estimate the leftmost integral from below, we make use of the fact that

$$\frac{1}{n_\star}\sqrt{\frac{2J}{|\Omega|}} \le \frac12 \tag{8.18}$$

according to our assumption (8.15), so that since $\ln(1-z)\ge -2z$ for all $z\in[0,\frac12]$ we firstly infer that

$$\begin{aligned}
\int_{\{\varphi<n_\star-\sqrt{\frac{2J}{|\Omega|}}\}}\Big(-\ln\frac{\varphi}{n_\star}\Big) &= \int_{\{\varphi<n_\star-\sqrt{\frac{2J}{|\Omega|}}\}}\Big(-\ln\frac{\varphi}{n_\star-\sqrt{\frac{2J}{|\Omega|}}}\Big) - \int_{\{\varphi<n_\star-\sqrt{\frac{2J}{|\Omega|}}\}}\ln\Big(1-\frac{1}{n_\star}\sqrt{\frac{2J}{|\Omega|}}\Big) \\
&\le \int_{\{\varphi<n_\star-\sqrt{\frac{2J}{|\Omega|}}\}}\Big(-\ln\frac{\varphi}{n_\star-\sqrt{\frac{2J}{|\Omega|}}}\Big) - |\Omega|\ln\Big(1-\frac{1}{n_\star}\sqrt{\frac{2J}{|\Omega|}}\Big) \\
&\le \int_{\{\varphi<n_\star-\sqrt{\frac{2J}{|\Omega|}}\}}\Big(-\ln\frac{\varphi}{n_\star-\sqrt{\frac{2J}{|\Omega|}}}\Big) + 2|\Omega|\cdot\frac{1}{n_\star}\sqrt{\frac{2J}{|\Omega|}} \\
&= \int_\Omega\Big(-\ln\frac{\varphi}{n_\star-\sqrt{\frac{2J}{|\Omega|}}}\Big)_+ + \frac{\sqrt{8|\Omega|}}{n_\star}\sqrt{J}.
\end{aligned}$$

Thanks to the Cauchy-Schwarz inequality, from (8.17) we thus obtain that

$$\begin{aligned}
\int_{\{\varphi<n_\star-\sqrt{\frac{2J}{|\Omega|}}\}} \zeta_{n_\star}(\varphi) &= \int_{\{\varphi<n_\star-\sqrt{\frac{2J}{|\Omega|}}\}} (\varphi-n_\star) + n_\star \int_{\{\varphi<n_\star-\sqrt{\frac{2J}{|\Omega|}}\}} \Big(-\ln\frac{\varphi}{n_\star}\Big) \\
&\le n_\star \int_{\{\varphi<n_\star-\sqrt{\frac{2J}{|\Omega|}}\}} \Big(-\ln\frac{\varphi}{n_\star}\Big) \\
&\le n_\star \int_\Omega \Bigg(-\ln\frac{\varphi}{n_\star-\sqrt{\frac{2J}{|\Omega|}}}\Bigg)_+ + \frac{\sqrt{8|\Omega|}}{n_\star}\sqrt{J} \\
&\le n_\star\sqrt{|\Omega|}\Bigg(\int_\Omega \Bigg(-\ln\frac{\varphi}{n_\star-\sqrt{\frac{2J}{|\Omega|}}}\Bigg)_+^2\Bigg)^{\frac{1}{2}} + \frac{\sqrt{8|\Omega|}}{n_\star}\sqrt{J} \\
&\le n_\star\sqrt{|\Omega|}\sqrt{C_P}\sqrt{I} + \frac{\sqrt{8|\Omega|}}{n_\star}\sqrt{J}.
\end{aligned} \tag{8.19}$$

As for the corresponding integral covering the region where $\varphi \ge n_\star - \sqrt{\frac{2J}{|\Omega|}}$, in view of the fact that

$$n_\star - \sqrt{\frac{2J}{|\Omega|}} \ge \frac{n_\star}{2} \tag{8.20}$$

by (8.18), we introduce

$$\psi(s) := \zeta_{n_\star}(s) - \frac{1}{n_\star}(s-n_\star)^2, \qquad s \ge \frac{n_\star}{2},$$

and then obtain that $\psi(n_\star)=0$ and

$$\begin{aligned}
(s-n_\star)\cdot\psi'(s) &= (s-n_\star)\cdot\Big\{1-\frac{n_\star}{s}-\frac{2}{n_\star}(s-n_\star)\Big\} \\
&= (s-n_\star)^2\cdot\Big\{\frac{1}{s}-\frac{2}{n_\star}\Big\} \\
&\le 0 \qquad \text{for all } s \ge \frac{n_\star}{2}.
\end{aligned}$$

As a consequence, for all $s \ge \frac{n_\star}{2}$ we have $\psi(s) \le 0$ and hence

$$\zeta_{n_\star}(s) \le \frac{1}{n_\star}(s-n_\star)^2,$$

by (8.20) implying that

$$\int_{\{\varphi\ge n_\star-\sqrt{\frac{2J}{|\Omega|}}\}} \zeta_{n_\star}(\varphi) \le \int_{\{\varphi\ge\frac{n_\star}{2}\}} \zeta_{n_\star}(\varphi) \le \frac{1}{n_\star}\int_{\{\varphi\ge\frac{n_\star}{2}\}} (\varphi-n_\star)^2 \le \frac{1}{n_\star}\int_\Omega (\varphi-n_\star)^2.$$

On adding this to (8.19), recalling (8.16) we readily arrive at (8.14). □

8.3 ε-independent stabilization

With Lemma 8.4 at hand, we can now make sure that (8.4) implies the following decay property of $\mathcal{F}_{n_\star,B}(n_\varepsilon, c_\varepsilon)$ which is uniform with respect to $\varepsilon \in (0,1)$.

Lemma 8.5 *Let $\rho > 0$, $\chi > 0$ and $\mu > \frac{\chi\sqrt{\rho}}{4}$, and let $n_\star := \frac{\rho}{\mu}$ and B be as in Lemma 8.1. Then for all $\delta > 0$ there exists $T(\delta) > 0$ such that for all $\varepsilon \in (0,1)$, with $\mathcal{F}_{n_\star,B}$ as in (8.2) we have*

$$\mathcal{F}_{n_\star,B}\Big(n_\varepsilon(\cdot,t), c_\varepsilon(\cdot,t)\Big) \le \delta \qquad \textit{for all } t \ge T(\delta). \tag{8.21}$$

PROOF. According to Corollary 8.2, we can fix $C_1 > 0$ such that

$$\int_0^\infty \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon^2} + \int_0^\infty \int_\Omega (n_\varepsilon - n_\star)^2 + \int_0^\infty \int_\Omega (c_\varepsilon - n_\star)^2 \le C_1 \qquad \text{for all } \varepsilon \in (0,1), \tag{8.22}$$

whereas since n_ε is strictly positive in $\bar{\Omega}\times(0,\infty)$ by the strong maximum principle, Lemma 8.4 provides $C_2 = C_2(n_\star) > 0$ such that whenever

$$\int_\Omega \Big(n_\varepsilon(\cdot,t) - n_\star\Big)^2 \le \frac{n_\star^2|\Omega|}{8}, \tag{8.23}$$

with $\zeta_{n_\star}$ as in (8.1) we have

$$\int_\Omega \zeta_{n_\star}(n_\varepsilon(\cdot,t)) \le C_2\bigg(\int_\Omega \frac{|\nabla n_\varepsilon(\cdot,t)|^2}{n_\varepsilon^2(\cdot,t)}\bigg)^{\frac{1}{2}} + C_2\bigg(\int_\Omega \Big(n_\varepsilon(\cdot,t) - n_\star\Big)^2\bigg)^{\frac{1}{2}} + C_2\int_\Omega \Big(n_\varepsilon(\cdot,t) - n_\star\Big)^2. \tag{8.24}$$

Now given $\delta > 0$, we pick positive numbers η_1 and η_2 such that

$$C_2\sqrt{\eta_1} \le \frac{\delta}{4} \tag{8.25}$$

and

$$C_2\sqrt{\eta_2} + C_2\eta_2 \le \frac{\delta}{4} \tag{8.26}$$

as well as

$$\eta_2 \le \frac{n_\star^2|\Omega|}{8} \tag{8.27}$$

and thereafter choose $T = T(\delta)$ suitably large satisfying

$$T \ge \max\Big\{\frac{C_1}{\eta_1}, \frac{C_1}{\eta_2}, \frac{C_1 B}{\delta}\Big\}. \tag{8.28}$$

Then (8.22) implies that for each $\varepsilon \in (0,1)$ we have

$$\frac{1}{T}\int_0^T \bigg\{ \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon^2} + \int_\Omega (n_\varepsilon - n_\star)^2 + \int_\Omega (c_\varepsilon - n_\star)^2 \bigg\} \le \frac{C_1}{T},$$

whence for any such ε we can find $t_\varepsilon \in (0,T)$ such that

$$\int_\Omega \frac{|\nabla n_\varepsilon(\cdot,t_\varepsilon)|^2}{n_\varepsilon^2(\cdot,t_\varepsilon)} + \int_\Omega \Big(n_\varepsilon(\cdot,t_\varepsilon) - n_\star\Big)^2 + \int_\Omega \Big(c_\varepsilon(\cdot,t_\varepsilon) - n_\star\Big)^2 \le \frac{C_1}{T}.$$

By (8.28), this in particular implies that

$$\int_\Omega \frac{|\nabla n_\varepsilon(\cdot,t_\varepsilon)|^2}{n_\varepsilon^2(\cdot,t_\varepsilon)} \le \frac{C_1}{T} \le \eta_1 \tag{8.29}$$

and

$$\int_\Omega \Big(n_\varepsilon(\cdot,t_\varepsilon)-n_\star\Big)^2 \le \frac{C_1}{T} \le \eta_2 \tag{8.30}$$

as well as

$$\frac{B}{2}\int_\Omega \Big(c_\varepsilon(\cdot,t_\varepsilon)-n_\star\Big)^2 \le \frac{C_1 B}{2T} \le \frac{\delta}{2}, \tag{8.31}$$

where combining (8.30) with (8.27) shows that (8.23) is valid for $t := t_\varepsilon$, meaning that (8.24) becomes applicable so as to warrant that, by (8.29) and (8.30),

$$\int_\Omega \zeta_{n_\star}(n_\varepsilon(\cdot,t_\varepsilon)) \le C_2\sqrt{\eta_1} + C_2\sqrt{\eta_2} + C_2\eta_2.$$

Thanks to (8.25) and (8.26), this entails that

$$\int_\Omega \zeta_{n_\star}(n_\varepsilon(\cdot,t_\varepsilon)) \le \frac{\delta}{4}+\frac{\delta}{4}=\frac{\delta}{2}$$

and that due to (8.31), by definition (8.2) of $\mathcal{F}_{n_\star,B}$ we thus have

$$\begin{aligned}
\mathcal{F}_{n_\star,B}(n_\varepsilon(\cdot,t_\varepsilon),c_\varepsilon(\cdot,t_\varepsilon)) &= \int_\Omega \zeta_{n_\star}(n_\varepsilon(\cdot,t_\varepsilon)) + \frac{B}{2}\int_\Omega \Big(c_\varepsilon(\cdot,t_\varepsilon)-n_\star\Big)^2 \\
&\le \frac{\delta}{2}+\frac{\delta}{2}=\delta.
\end{aligned}$$

Since $t_\varepsilon < T$, in view of the monotonicity property (8.12) asserted by Corollary 8.2 this directly proves (8.21). □

In order to translate the latter into a corresponding decay property referring to usual spatial L^p norms, we shall need to bound $\int_\Omega \zeta_{n_\star}(n)$ from below appropriately.

Lemma 8.6 *For $n_\star > 0$, let $\zeta_{n_\star}$ be as defined in (8.1). Then*

$$\int_\Omega |\varphi-n_\star| \le \sqrt{3n_\star|\Omega|}\Big(\int_\Omega \zeta_{n_\star}(\varphi)\Big)^{\frac{1}{2}} + \Big(\frac{1}{\ln 2-\frac{1}{2}}+\frac{1}{1-2\ln\frac{3}{2}}\Big)\int_\Omega \zeta_{n_\star}(\varphi) \tag{8.32}$$

for all $\varphi \in C^0(\bar\Omega)$ fulfilling $\varphi > 0$ in $\bar\Omega$.

Proof. We split

$$\int_\Omega |\varphi-n_\star| = \int_{\{\varphi<\frac{n_\star}{2}\}}(n_\star-\varphi) + \int_{\{|\varphi-n_\star|\le\frac{n_\star}{2}\}}|\varphi-n_\star| + \int_{\{\varphi>\frac{3n_\star}{2}\}}(\varphi-n_\star) \tag{8.33}$$

and first observe that by downward monotonicity of $\zeta_{n_\star}$ in $(0,n_\star)$,

$$\zeta_{n_\star}(s) \ge \zeta_{n_\star}\Big(\frac{n_\star}{2}\Big) = \frac{n_\star}{2} - n_\star - n_\star\ln\frac{1}{2} = n_\star\Big(\ln 2-\frac{1}{2}\Big) \qquad \text{for all } s\in\Big(0,\frac{n_\star}{2}\Big),$$

so that

$$\int_{\{\varphi<\frac{n_\star}{2}\}}(n_\star-\varphi) \le \int_{\{\varphi<\frac{n_\star}{2}\}} n_\star \le \frac{1}{\ln 2-\frac{1}{2}}\int_{\{\varphi<\frac{n_\star}{2}\}}\zeta_{n_\star}(\varphi) \le \frac{1}{\ln 2-\frac{1}{2}}\int_\Omega \zeta_{n_\star}(\varphi), \tag{8.34}$$

because $\zeta_{n_\star}$ is nonnegative.
Next, in view of the second integral on the right of (8.33) we let

$$\psi_1(s) := \zeta_{n_\star}(s) - \frac{1}{3n_\star}(s-n_\star)^2, \qquad s\in\Big[\frac{n_\star}{2},\frac{3n_\star}{2}\Big],$$

to see that

$$\begin{aligned}(s-n_\star)\psi_1'(s) &= (s-n_\star)\cdot\Big\{1-\frac{n_\star}{s}-\frac{1}{3n_\star}\cdot 2(s-n_\star)\Big\}\\ &= (s-n_\star)^2\cdot\Big\{\frac{1}{s}-\frac{2}{3n_\star}\Big\}\\ &\ge 0 \qquad \text{for all } s\in\Big[\frac{n_\star}{2},\frac{3n_\star}{2}\Big].\end{aligned}$$

Therefore,

$$\psi_1(s)\ge\psi_1(n_\star)=0 \qquad \text{for all } s\in\Big[\frac{n_\star}{2},\frac{3n_\star}{2}\Big]$$

and hence

$$(s-n_\star)^2\le 3n_\star\zeta_{n_\star}(s) \qquad \text{for all } s\in\Big[\frac{n_\star}{2},\frac{3n_\star}{2}\Big],$$

so that invoking the Cauchy-Schwarz inequality we can estimate

$$\begin{aligned}\int_{\{|\varphi-n_\star|\le\frac{n_\star}{2}\}}|\varphi-n_\star| &\le \sqrt{|\Omega|}\Big(\int_{\{|\varphi-n_\star|\le\frac{n_\star}{2}\}}(\varphi-n_\star)^2\Big)^{\frac{1}{2}}\\ &\le \sqrt{3n_\star|\Omega|}\Big(\int_{\{|\varphi-n_\star|\le\frac{n_\star}{2}\}}\zeta_{n_\star}(\varphi)\Big)^{\frac{1}{2}}\\ &\le \sqrt{3n_\star|\Omega|}\Big(\int_\Omega\zeta(\varphi)\Big)^{\frac{1}{2}}.\end{aligned}\tag{8.35}$$

Finally, writing

$$\psi_2(s) := n_\star\ln\frac{s}{n_\star} - 2\ln\frac{3}{2}\cdot(s-n_\star), \qquad s\ge\frac{3n_\star}{2},$$

we see that since $3\ln\frac{3}{2}=\ln\frac{27}{8}>1$, we have

$$\psi_2'(s)=\frac{n_\star}{s}-2\ln\frac{3}{2}\le\frac{2}{3}-2\ln\frac{3}{2}<0 \qquad \text{for all } s\ge\frac{3n_\star}{2},$$

so that

$$\psi_2(s) \le \psi_2\Big(\frac{3n_\star}{2}\Big) = n_\star \ln\frac{3}{2} - 2\ln\frac{3}{2}\cdot\frac{n_\star}{2} = 0 \qquad \text{for all } s \ge \frac{3n_\star}{2}$$

and hence

$$\begin{aligned}
\zeta_{n_\star}(s) &= s - n_\star - \Big\{\psi_2(s) + 2\ln\frac{3}{2}\cdot(s-n_\star)\Big\} \\
&\ge \Big(1 - 2\ln\frac{3}{2}\Big)\cdot(s-n_\star) \qquad \text{for all } s \ge \frac{3n_\star}{2}.
\end{aligned}$$

As $2\ln\frac{3}{2} = \ln\frac{9}{4} < 1$, this entails that

$$\int_{\{\varphi>\frac{3n_\star}{2}\}} (\varphi - n_\star) \le \frac{1}{1-2\ln\frac{3}{2}} \int_{\{\varphi>\frac{3n_\star}{2}\}} \zeta_{n_\star}(\varphi) \le \frac{1}{1-2\ln\frac{3}{2}} \int_\Omega \zeta_{n_\star}(\varphi)$$

and thus, when combined with (8.34), (8.35) and (8.33), proves (8.32). $\Box$

We can thereby draw the following consequence of Lemma 8.5.

Lemma 8.7 *Suppose that $\rho > 0$, $\chi > 0$ and $\mu > \frac{\chi\sqrt{\rho}}{4}$. Then for all $\delta > 0$ there exists $T(\delta) > 0$ such that for all $\varepsilon \in (0,1)$ we have*

$$\Big\|n_\varepsilon(\cdot,t) - \frac{\rho}{\mu}\Big\|_{L^1(\Omega)} + \Big\|c_\varepsilon(\cdot,t) - \frac{\rho}{\mu}\Big\|_{L^2(\Omega)} \le \delta \qquad \textit{for all } t \ge T(\delta). \tag{8.36}$$

PROOF. Writing $n_\star := \frac{\rho}{\mu}$ and

$$C_1 := \max\Big\{\sqrt{3n_\star|\Omega|}\,,\ \frac{1}{\ln 2 - \frac{1}{2}} + \frac{1}{1-2\ln\frac{3}{2}}\Big\}, \tag{8.37}$$

given $\delta > 0$ we fix $\delta_1 > 0$ such that

$$C_1\sqrt{\delta_1} + C_1\delta_1 \le \frac{\delta}{2} \tag{8.38}$$

and

$$\sqrt{\frac{2\delta_1}{B}} \le \frac{\delta}{2}. \tag{8.39}$$

Then Lemma 8.5 provides $T = T(\delta) > 0$ with the property that for each $\varepsilon \in (0,1)$, the functional $\mathcal{F}_{n_\star,B}$ introduced in (8.2) satisfies

$$\mathcal{F}_{n_\star,B}(n_\varepsilon(\cdot,t), c_\varepsilon(\cdot,t)) \le \delta_1 \qquad \text{for all } t \ge T. \tag{8.40}$$

By definition of $\mathcal{F}_{n_\star,B}$, this in particular means that with $\zeta_{n_\star}$ as in (8.1), for all $\varepsilon \in (0,1)$ we have

$$\int_\Omega \zeta_{n_\star}(n_\varepsilon(\cdot,t)) \le \delta_1 \qquad \text{for all } t \ge T \tag{8.41}$$

and

$$\frac{B}{2}\int_\Omega \Big(c_\varepsilon(\cdot,t) - n_\star\Big)^2 \le \delta_1 \qquad \text{for all } t \ge T,$$

where the latter ensures that

$$\|c_\varepsilon(\cdot,t) - n_\star\|_{L^2(\Omega)} \le \sqrt{\frac{2\delta_1}{B}} \le \frac{\delta}{2} \qquad \text{for all } t \ge T \tag{8.42}$$

due to (8.39). In light of Lemma 8.6 and (8.37), however, (8.41) guarantees that

$$\begin{aligned} \|n_\varepsilon(\cdot,t) - n_\star\|_{L^1(\Omega)} &\le C_1\Big(\int_\Omega \zeta_{n_\star}(n_\varepsilon(\cdot,t))\Big)^{\frac{1}{2}} + C_1\int_\Omega \zeta_{n_\star}(n_\varepsilon(\cdot,t)) \\ &\le C_1\sqrt{\delta_1} + C_1\delta_1 \\ &\le \frac{\delta}{2} \qquad \text{for all } t \ge T \end{aligned}$$

by (8.38), which along with (8.42) establishes (8.36). □

9 Stabilization. Proof of Theorem 1.2

9.1 ε-independent decay estimates in the case $\rho \le 0$

As a last preparation for the proof of Theorem 1.2, let us refine the argument from Lemma 3.2 and Lemma 3.3 to derive the following quantitative decay estimates for n_ε and c_ε with respect to the norms in $L^1(\Omega)$ when ρ is nonpositive. As a by-product, we thereby moreover obtain the inequality (9.3) which will be useful for the proof of Theorem 1.3.

Lemma 9.1 *Let $\chi > 0, \mu > 0$ and $\rho \le 0$. Then there exists $C > 0$ such that for all $\varepsilon \in (0,1)$ we have*

$$\int_\Omega n_\varepsilon(x,t)dx \le \frac{C}{t+1} \qquad \textit{for all } t > 0 \tag{9.1}$$

and

$$\int_\Omega c_\varepsilon(x,t)dx \le \frac{C}{t+1} \qquad \textit{for all } t > 0 \tag{9.2}$$

as well as

$$\int_0^\infty \int_\Omega n_\varepsilon^2(x,t)dxdt \le C. \tag{9.3}$$

Proof. We repeat the integration procedure from Lemma 3.2 to see that since $\rho \le 0$,

$$\frac{d}{dt}\int_\Omega n_\varepsilon = \rho\int_\Omega n_\varepsilon - \mu\int_\Omega n_\varepsilon^2 \le -\mu\int_\Omega n_\varepsilon^2 \qquad \text{for all } t > 0, \tag{9.4}$$

which immediately yields (9.3). Moreover, employing the Cauchy-Schwarz inequality we infer from (9.4) that $y(t) := \int_\Omega n_\varepsilon(\cdot,t)$, $t \ge 0$, satisfies $y'(t) \le -\frac{\mu}{|\Omega|}y^2(t)$ for all $t \ge 0$ and hence

$$y(t) \le \frac{y(0)}{1 + \frac{\mu}{|\Omega|}y(0)t} \le \frac{C_1}{t+1} \qquad \text{for all } t > 0 \tag{9.5}$$

with $C_1 := \max\{\int_\Omega n_0\,,\,\frac{|\Omega|}{\mu}\}$.
Having thus proved (9.1), to derive (9.2) we recall that by the second equation in (3.1), $z(t) := \int_\Omega c_\varepsilon(\cdot,t)$, $t \ge 0$, fulfils $z'(t) = -z(t) + y(t)$ and therefore

$$z'(t) + z(t) - \frac{C_1}{t+1} \le 0 \qquad \text{for all } t > 0$$

by (9.5). Thus, if we fix $C_2 := \max\{4C_1\,,\,2\int_\Omega c_0\}$ and let $\overline{z}(t) := \frac{C_2}{t+2}$, then $\overline{z}(0) \ge \frac{C_2}{2} \ge \int_\Omega c_0 = z(0)$ and

$$\begin{aligned}
\overline{z}'(t) + \overline{z}(t) - \frac{C_1}{t+1} &= -\frac{C_2}{(t+2)^2} + \frac{C_2}{t+2} - \frac{C_1}{t+1} \\
&= \frac{C_2}{t+2}\cdot\Big\{1 - \frac{1}{t+2} - \frac{C_1}{C_2}\cdot\frac{t+2}{t+1}\Big\} \\
&\ge \frac{C_2}{t+2}\cdot\Big\{\frac{1}{2} - \frac{C_1}{C_2}\cdot 2\Big\} \\
&\ge 0 \qquad \text{for all } t > 0,
\end{aligned}$$

so that by comparison we conclude that $\overline{z}(t) \ge z(t)$ for all $t \ge 0$, which clearly implies (9.2). $\square$

9.2 Proof of Theorem 1.2

Proof of Theorem 1.2. Along with the Fubini-Tonelli theorem, Lemma 7.1 provides $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ and a null set $N \subset (0,\infty)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$ and

$$n_\varepsilon(\cdot,t) \to n(\cdot,t) \quad \text{and} \quad c_\varepsilon(\cdot,t) \to c(\cdot,t) \qquad \text{a.e. in } \Omega \text{ for all } t \in (0,\infty)\setminus N$$

as $\varepsilon = \varepsilon_j \searrow 0$. In view of Fatou's lemma and the fact that $L^2(\Omega) \subset L^1(\Omega)$, in both cases $\rho > 0$ and $\rho \le 0$ we then obtain from Lemma 8.7 and Lemma 9.1 that

$$\Big\|n(\cdot,t) - \frac{\rho_+}{\mu}\Big\|_{L^1(\Omega)} \to 0 \quad \text{and} \quad \Big\|c(\cdot,t) - \frac{\rho_+}{\mu}\Big\|_{L^1(\Omega)} \to 0 \qquad \text{as } (0,\infty)\setminus N \ni t \to \infty.$$

Here the former statement precisely yields decay of n as claimed in (1.11), whereas the second one can be combined with the fact that $C_1 := \sup_{t\in(0,\infty)\setminus N}\int_\Omega c^6(\cdot,t)$ is finite, as asserted by Lemma 3.6 and Fatou's lemma, to see upon interpolating by means of the Hölder inequality that for any $p \in [1,6)$,

$$\|c(\cdot,t)\|_{L^p(\Omega)} \le C_1^{\frac{6(p-1)}{5p}}\,\|c(\cdot,t)\|_{L^1(\Omega)}^{\frac{6-p}{5p}} \to 0 \qquad \text{as } (0,\infty)\setminus N \ni t \to \infty,$$

which completes the proof of (1.11). $\square$

10 Decay of u. Proof of Theorem 1.3

Let us finally make sure that in the case $\mu > \frac{\chi\sqrt{\rho_+}}{4}$, the stabilization properties obtained above are sufficient to ensure asymptotic decay of u, provided that the external force f satisfies (1.12). More precisely, our proof for this will make use of the following information yielded by Corollary 8.2 and Lemma 9.1.

Lemma 10.1 *Let $\chi > 0$, $\rho \in \mathbb{R}$ and $\mu > \frac{\chi\sqrt{\rho_+}}{4}$. Then*

$$\int_0^\infty \int_\Omega \Big(n(x,t) - \frac{\rho_+}{\mu}\Big)^2 dxdt < \infty. \tag{10.1}$$

Proof. If $\rho \le 0$, then this results from Lemma 9.1, Lemma 7.1 and Fatou's lemma, while when $\rho > 0$ we can conclude similarly, relying on Corollary 8.2 in this case. $\square$

Now once more exploiting the basic identity (3.20), this time in a more elaborate manner, yields the claimed statement on decay of u.

Proof of Theorem 1.3. Let us first verify that the approximation process documented in Lemma 7.1 occurs at a sufficiently regular level so as to allow for an implication of the natual energy inequality associated with the Navier-Stokes subsystem of (1.1) in the sense that one can find positive constants C_1 and C_2 and a null set $N \subset (0,\infty)$ such that

$$\int_\Omega |u(\cdot,t)|^2 \le e^{-C_1(t-t_0)} \cdot \int_\Omega |u(\cdot,t_0)|^2 + \int_{t_0}^t e^{-C_1(t-s)} h(s)ds \qquad \text{for all } t_0 \in (0,\infty)\setminus N \text{ and each } t \in (t_0,\infty)\setminus N, \tag{10.2}$$

where

$$h(t) := C_2\|n(\cdot,t) - n_\star\|_{L^2(\Omega)}^2 + C_2\|f(\cdot,t)\|_{L^{\frac{6}{5}}(\Omega)}^2 \qquad \text{for } t > 0 \tag{10.3}$$

with $n_\star := \frac{\rho_+}{\mu}$. To see this, we first go back to Lemma 3.7 and make use of the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ and Young's inequality to find $C_3 > 0$ such that for any $\varepsilon \in (0,1)$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 &= \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla\phi + \int_\Omega f\cdot u_\varepsilon \\
&= \int_\Omega (n_\varepsilon - n_\star) u_\varepsilon \cdot \nabla\phi + \int_\Omega f \cdot u_\varepsilon \\
&\le \int_\Omega (n_\varepsilon - n_\star) u_\varepsilon \cdot \nabla\phi + \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 + C_3\|f\|_{L^{\frac{6}{5}}(\Omega)}^2 \qquad \text{for all } t > 0.
\end{aligned}$$

Thanks to the Poincaré inequality, this shows that with some $C_1 > 0$ we have

$$\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + 2C_1\int_\Omega |u_\varepsilon|^2 \le 2\int_\Omega (n_\varepsilon - n_\star)u_\varepsilon\cdot\nabla\phi + 2C_3\|f\|_{L^{\frac{6}{5}}(\Omega)}^2 \qquad \text{for all } t > 0. \tag{10.4}$$

Here we note that so far we have avoided estimating the first integral on the right in view of the circumstance that we do not know whether $(n_\varepsilon)_{\varepsilon\in(0,1)}$ is strongly precompact in $L^2_{loc}(\bar\Omega \times [0,\infty))$. To further circumvent any difficulty possibly stemming from this, we now multiply (10.4) by $e^{C_1 t}$ and integrate in time to see that for all $\varepsilon \in (0,1)$,

$$\begin{aligned}
&e^{C_1 t}\int_\Omega |u_\varepsilon(x,t)|^2 dx - e^{C_1 t_0}\int_\Omega |u_\varepsilon(x,t_0)|^2 dx - C_1\int_{t_0}^t\int_\Omega e^{C_1 s}|u_\varepsilon(x,s)|^2 dxds \\
&\qquad +2C_1\int_{t_0}^t\int_\Omega e^{C_1 s}|u_\varepsilon(x,s)|^2 dxds
\end{aligned}$$

$$\leq \ 2\int_{t_0}^t \int_\Omega e^{C_1 s}\Big(n_\varepsilon(x,s)-n_\star\Big)u_\varepsilon(x,s)\cdot\nabla\phi(x)dxds$$
$$+2C_3\int_{t_0}^t e^{C_1 s}\|f(\cdot,s)\|^2_{L^{\frac{6}{5}}(\Omega)}ds \qquad \text{whenever } 0\le t_0<t. \tag{10.5}$$

We next rely on the fact that according to Lemma 7.1 we can pick $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ and a null set $N\subset(0,\infty)$ such that $\varepsilon_j\searrow 0$ as $j\to\infty$ and

$$u_\varepsilon(\cdot,t)\to u(\cdot,t)\quad \text{in } L^2(\Omega) \qquad \text{for all } t\in(0,\infty)\setminus N \tag{10.6}$$

as $\varepsilon=\varepsilon_j\searrow 0$. In fact, this implies that if we take $t_0\in(0,\infty)\setminus N$ and $t\in(t_0,\infty)\setminus N$, then in (10.5) we have

$$\int_\Omega |u_\varepsilon(x,t_0)|^2dx\to\int_\Omega|u(x,t_0)|^2dx \quad \text{and} \int_\Omega |u_\varepsilon(x,t)|^2dx\to\int_\Omega|u(x,t)|^2dx \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0.$$

Since Lemma 7.1 moreover warrants that we can also achieve that $u_\varepsilon\to u$ a.e. in $\Omega\times(0,\infty)$ and in $L^2_{loc}(\bar\Omega\times[0,\infty))$ as well as $n_\varepsilon\rightharpoonup n$ in $L^2_{loc}(\bar\Omega\times[0,\infty))$ as $\varepsilon=\varepsilon_j\searrow 0$, from (10.5) and e.g. Fatou's lemma we infer that

$$\begin{aligned}
e^{C_1 t}\int_\Omega|u(x,t)|^2dx-e^{C_1 t_0}\int_\Omega|u(x,t_0)|^2dx \ &+ \ C_1\int_{t_0}^t\int_\Omega e^{C_1 s}|u(x,s)|^2dxds\\
&\le \ 2\int_{t_0}^t\int_\Omega e^{C_1 s}\Big(n(x,s)-n_\star\Big)u(x,s)\cdot\nabla\phi(x)dxds\\
&\quad +2C_3\int_{t_0}^t e^{C_1 s}\|f(\cdot,s)\|^2_{L^{\frac{6}{5}}(\Omega)}ds\\
&\quad \text{for all } t_0\in(0,\infty)\setminus N \text{ and } t\in(t_0,\infty)\setminus N. \qquad (10.7)
\end{aligned}$$

Now we proceed to estimate the first integral on the right-hand side herein by using Young's inequality to find that for any such t_0 and t,

$$\begin{aligned}
2\int_{t_0}^t\int_\Omega e^{C_1 s}\Big(n(x,s)-n_\star\Big)u(x,s)\cdot\nabla\phi(x)dxds \ &\le \ C_1\int_{t_0}^t\int_\Omega e^{C_1 s}|u(x,s)|^2dxds\\
&\quad +C_4\int_{t_0}^t e^{C_1 s}\|n(\cdot,s)-n_\star\|^2_{L^2(\Omega)}ds
\end{aligned}$$

holds with $C_4:=\frac{\|\nabla\phi\|^2_{L^\infty(\Omega)}}{C_1}$, so that (10.7) readily leads to (10.2) if we let $C_2:=\max\{C_4\,,\,2C_3\}$. Thereupon, the derivation of (1.13) is straightforward: According to (1.12) and the fact that

$$\int_t^{t+1}\|n(\cdot,s)-n_\star\|^2_{L^2(\Omega)}ds\to 0 \qquad \text{as } t\to\infty$$

by Lemma 10.1, given $\delta>0$ we can fix $t_0\in(0,\infty)\setminus N$ such that the function h in (10.3) satisfies

$$\int_t^{t+1}h(s)ds\le\frac{1-e^{-C_1}}{2}\cdot\delta \qquad \text{for all } t>t_0, \tag{10.8}$$

and thereafter fix $t_1 > t_0$ large enough fulfilling

$$C_5^2 e^{-C_1(t-t_0)} \le \frac{\delta}{2} \qquad \text{for all } t > t_1, \tag{10.9}$$

where

$$C_5 := \sup_{t\in(0,\infty)\setminus N} \|u(\cdot,t)\|_{L^2(\Omega)} \tag{10.10}$$

is finite thanks to Lemma 3.8 and (10.6). Then as a consequence of Lemma 3.4, (10.8) guarantees that in (10.2) we have

$$\int_{t_0}^t e^{-C_1(t-s)} h(s)ds \le \frac{\frac{1-e^{-C_1}}{2}\cdot\delta}{1-e^{-C_1}} = \frac{\delta}{2} \qquad \text{for all } t > t_0,$$

while due to our choice of t_0 we know from (10.9) and (10.10) that

$$e^{-C_1(t-t_0)} \cdot \int_\Omega |u(\cdot,t_0)|^2 \le \frac{\delta}{2} \qquad \text{for all } t > t_1.$$

Therefore, (10.2) implies that

$$\int_\Omega |u(\cdot,t)|^2 \le \delta \qquad \text{for all } t \in (t_1,\infty)\setminus N,$$

as desired. □

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