The fast signal diffusion limit in Keller-Segel(-fluid) systems

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Abstract

This paper deals with convergence of solutions to a class of parabolic Keller-Segel systems, possibly coupled to the (Navier-)Stokes equations in the framework of the full model

$$\begin{cases} \partial_t n_\varepsilon + u_\varepsilon \cdot \nabla n_\varepsilon &= \Delta n_\varepsilon - \nabla \cdot \Big(n_\varepsilon S(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon \Big) + f(x, n_\varepsilon, c_\varepsilon), \\ \varepsilon \partial_t c_\varepsilon + u_\varepsilon \cdot \nabla c_\varepsilon &= \Delta c_\varepsilon - c_\varepsilon + n_\varepsilon, \\ \partial_t u_\varepsilon + \kappa (u_\varepsilon \cdot \nabla) u_\varepsilon &= \Delta u_\varepsilon + \nabla P_\varepsilon + n_\varepsilon \nabla \phi, \qquad \nabla \cdot u_\varepsilon = 0 \end{cases}$$

to solutions of the parabolic-elliptic counterpart formally obtained on taking $\varepsilon \searrow 0$. In smoothly bounded physical domains $\Omega \subset \mathbb{R}^N$ with $N \geq 1$, and under appropriate assumptions on the model ingredients, we shall first derive a general result which asserts certain strong and pointwise convergence properties whenever asserting that supposedly present bounds on ∇c_ε and u_ε are bounded in $L^\lambda((0,T); L^q(\Omega))$ and in $L^\infty((0,T); L^r(\Omega))$, respectively, for some $\lambda \in (2, \infty]$, $q > N$ and $r > \max\{2, N\}$ such that $\frac{1}{\lambda} + \frac{N}{2q} < \frac{1}{2}$. To our best knowledge, this seems to be the first rigorous mathematical result on a fast signal diffusion limit in a chemotaxis-fluid system.

This general result will thereafter be concretized in the context of two examples: Firstly, for an unforced Keller-Segel-Navier-Stokes system we shall establish a statement on global classical solutions under suitable smallness conditions on the initial data, and show that these solutions approach a global classical solution to the respective parabolic-elliptic simplification.

We shall secondly derive a corresponding convergence property for arbitrary solutions to fluid-free Keller-Segel systems with logistic source terms, which in spatially one-dimensional settings turn out to allow for a priori estimates compatible with our general theory. Building on the latter in conjunction with a known result on emergence of large densities in the associated parabolic-elliptic limit system, we will finally discover some quasi-blowup phenomenon for the fully parabolic Keller-Segel system with logistic source and suitably small parameter $\varepsilon > 0$.

Key words: chemotaxis; Keller-Segel; Navier-Stokes; fast signal diffusion limit
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1 Introduction

The Keller-Segel system and its parabolic-elliptic simplification. To describe chemotactic aggregation of cellular slime molds which move towards relatively high concentrations of a chemical secreted by the amoebae themselves, Keller and Segel [17] proposed cross-diffusive parabolic systems of the form

$$\begin{cases} n_t &= d_1 \Delta n - a_1 \nabla \cdot (n \nabla c), \\ c_t &= d_2 \Delta c - a_2 c + a_3 n, \end{cases}$$

where the unknown functions $n = n(x,t)$ and $c = c(x,t)$ denote the cell density and the concentration of the chemical substance at place x and time t, respectively, and where d_1, d_2, a_1, a_2, a_3 are positive numbers. By substituting

$$\frac{a_1}{d_1} = S, \quad \frac{d_1}{d_2} = \varepsilon, \quad \frac{a_2}{d_2} = \gamma \quad \text{and} \quad \frac{a_3}{d_2} = \alpha,$$

and replacing $d_1 t$ with t, from this we obtain the system

$$\begin{cases} n_t &= \Delta n - \nabla \cdot (nS\nabla c), \\ \varepsilon c_t &= \Delta c - \gamma c + \alpha n, \end{cases} \tag{1.1}$$

which in the limit $\varepsilon \searrow 0$ formally approaches the corresponding parabolic-elliptic system, with the second identity therein replaced with the inhomogeneous Helmholtz equation $-\Delta c + \gamma c = \alpha n$.

As is well-known from quite a large literature, with regard to technical purposes the latter simplification goes along with substantial advantages for mathematical analysis, in summary leading to much a deeper knowledge on parabolic-elliptic Keller-Segel systems than currently available for their fully parabolic relatives. Examples already include the mere detection of exploding solutions, typifying the probably most characteristic effect of the considered cross-diffusive interaction, which in fact could be accomplished for parabolic-elliptic systems already rather early ([14], [28], [29], [1], [3]), while for the full system (1.1) with positive ε, corresponding results on generic blow-up, thus going beyond particular examples ([13]), seem to require significantly stronger efforts and hence have been achieved only a few years ago ([52], [26]). Likewise, while considerable qualitative knowledge on the respective blow-up mechanisms has been collected for parabolic-elliptic systems (see e.g. [30], [32], [34], [2], [39], [38], [37]), only little information seems available for general blow-up solutions to (1.1) when $\varepsilon > 0$ ([30], [25], [55]).

More generally, by providing accessibility to numerous tools, especially from the analysis of scalar parabolic problems, resorting to parabolic-elliptic simplifications has made it possible to reveal further qualitative properties of Keller-Segel-type systems, inter alia also in the framework of global solutions ([16], [15], [27], [4], [56]), and partially even including couplings to additional quantities such as fluid flows or haptotactic attractants ([18], [19], [40]).

Problem setting and main objectives. In line with the above, it seems natural to seek for some appropriate control of the error made when approximating a fully parabolic system of Keller-Segel type by its parabolic-elliptic simplification, especially in cases when the considered biological situation is such that the respective signal diffuses much faster than individuals in the cell population, in the context of (1.1) thus meaning that $\varepsilon > 0$ is small. Indeed, even in the context of the classical system (1.1) already the question concerning mere convergence of solutions as $\varepsilon \searrow 0$, apart from partially being addressed by numerical considerations ([21]), seems to lack a rigorous answer up to now.

The goal of the present work consists in establishing a first result in this direction, with a main focus being on deriving an approach robust enough so as to be not necessarily restricted to the prototypical system (1.1), but rather capable of adequately treating more complex types of interaction, possibly also with further components. In order to include an example for the latter which appears to be of increasing interest in the recent literature, we shall address this problem in the context of the class of Keller-Segel systems possibly coupled to the (Navier-)Stokes equations from fluid mechanics, and for a fixed number $T > 0$ and arbitrary $\varepsilon > 0$, we will accordingly be concerned with solutions to the class of systems given by

$$\begin{cases} \partial_t n_\varepsilon + u_\varepsilon \cdot \nabla n_\varepsilon &= \Delta n_\varepsilon - \nabla \cdot \Big(n_\varepsilon S(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon \Big) + f(x, n_\varepsilon, c_\varepsilon), \qquad x \in \Omega,\ t \in (0,T), \\ \varepsilon \partial_t c_\varepsilon + u_\varepsilon \cdot \nabla c_\varepsilon &= \Delta c_\varepsilon - c_\varepsilon + n_\varepsilon, \qquad x \in \Omega,\ t \in (0,T), \\ \partial_t u_\varepsilon + \kappa (u_\varepsilon \cdot \nabla) u_\varepsilon &= \Delta u_\varepsilon + \nabla P_\varepsilon + n_\varepsilon \nabla \phi, \qquad \nabla \cdot u_\varepsilon = 0, \qquad x \in \Omega,\ t \in (0,T), \\ (\nabla n_\varepsilon - n_\varepsilon S(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon) \cdot \nu = \frac{\partial c_\varepsilon}{\partial \nu} = 0, \quad u_\varepsilon = 0, & \qquad x \in \partial\Omega,\ t \in (0,T), \\ n_\varepsilon(x,0) = n_0(x), \quad c_\varepsilon(x,0) = c_0(x), \quad u_\varepsilon(x,0) = u_0(x), & \qquad x \in \Omega. \end{cases} \tag{1.2}$$

In cases of nontrivial fluid velocity fields u_ε and potential functions ϕ, (1.2) thus accounts for possible influences of liquid environments on the evolution of bacterial populations, and vice versa, through transport and buoyancy; a considerable relevance of chemotaxis-fluid interaction mechanisms of this type has been suggested by experimental findings in several contexts, including the striking observations from [43] on convection-driven formation of plume-like aggregates in populations of *Bacillus subtilis* suspended to sessile water drops (cf. also the discussions and the references in [18] and in [19]). Apart from this, we shall include here the possibility that alternative to the choice $f \equiv 0$, the proliferation term f may e.g. represent a logistic-type source, possibly even reflecting competition with the quantity c such as typically present in taxis-type models from spatial ecology where c plays the role of a second species ([58]). Moreover, our approach will be general enough so as to allow for the chemotactic interaction in (1.2) to be described by the action of a matrix which may contain off-diagonal entries, and thus especially be able to account for rotational flux components such as proposed in the more recent modeling literature ([59]) but yet understood only rudimentarily from an analytical point of view ([5], [45], [46], [22], [54]). Correspondigly, we shall suppose that with some $K_f > 0$ and some nonincreasing $f_0 : [0,\infty) \to \mathbb{R}$ with $f_0(0) \geq 0$,

$$\begin{cases} f \in C^1(\overline{\Omega} \times [0,\infty)^2) \qquad \text{is such that } f(x,0,c) \geq 0 \quad \text{for all } (x,c) \in \overline{\Omega} \times [0,\infty), \qquad \text{and that} \\ f_0(n) \leq f(x,n,c) \leq K_f \cdot (n+1) \quad \text{for all } (x,n,c) \in \overline{\Omega} \times [0,\infty)^2, \end{cases} \tag{1.3}$$

that $S = (S_{ij})_{i,j \in \{1,...,N\}}$ is such that for all $(i,j) \in \{1,...,N\}^2$,

$$\begin{cases} S_{ij} \in C^2(\overline{\Omega} \times [0,\infty)^2), \qquad \text{and that} \\ |S_{ij}(x,n,c)| \leq K_S \quad \text{for all } (x,n,c) \in \overline{\Omega} \times [0,\infty)^2 \end{cases} \tag{1.4}$$

with a positive constant K_S, and that apart from that the parameter κ is any real number and the gravitational potential in (1.2) satisfies

$$\phi \in W^{2,\infty}(\Omega). \tag{1.5}$$

As for the initial data, our standing assumptions will be that

$$\begin{cases} n_0 \in W^{1,\infty}(\Omega) \quad \text{is nonnegative with } n_0 \not\equiv 0, \\ c_0 \in W^{1,\infty}(\Omega) \quad \text{is nonnegative,} \quad \text{and that} \\ u_0 \in W^{2,\infty}(\Omega;\mathbb{R}^N) \quad \text{satisfies } \nabla\cdot u_0 \equiv 0 \text{ and } u_0|_{\partial\Omega} = 0. \end{cases} \tag{1.6}$$

Our plan is to firstly derive a general result on convergence of solutions to (1.2) to solutions of the associated parabolic-elliptic counterpart, and to secondly concretize this in the framework of two particular examples. We shall thereby obtain corresponding approximation results both for certain small-data solutions to an unforced chemotaxis-Navier-Stokes system, and for arbitrary solutions to a one-dimensional fluid-free logistic Keller-Segel model, where as a by-product, the latter outcome will imply an apparently new result on spontaneous emergence of arbitrarily large densities in the fully parabolic case for suitably small $\varepsilon > 0$.

Main results I. A general statement on the limit $\varepsilon \searrow 0$ in (1.2). Accordingly, we shall first examine the relationship between solutions to (1.2) and those to

$$\begin{cases} n_t + u\cdot\nabla n &= \Delta n - \nabla\cdot(nS(x,n,c)\cdot\nabla c) + f(x,n,c), & x\in\Omega,\ t\in(0,T), \\ u\cdot\nabla c &= \Delta c - c + n, & x\in\Omega,\ t\in(0,T), \\ u_t + \kappa(u\cdot\nabla)u &= \Delta u + \nabla P + n\nabla\phi, \qquad \nabla\cdot u = 0, & x\in\Omega,\ t\in(0,T), \\ (\nabla n - nS(x,n,c)\cdot\nabla c)\cdot\nu = \frac{\partial c}{\partial\nu} = 0, \quad u = 0, & & x\in\partial\Omega,\ t\in(0,T), \\ n(x,0) = n_0(x), u(x,0) = u_0(x), & & x\in\Omega, \end{cases} \tag{1.7}$$

in a setting as general as possible. Our main result in this respect identifies a condition, yet on a given family of solutions to (1.2) itself, as sufficient for strong, and especially a.e. pointwise convergence, in the following sense.

Theorem 1.1 *Let $N \geq 1$ and $\Omega \subset \mathbb{R}^N$ be a bounded convex domain with smooth boundary, and assume that (1.6) holds, that $\kappa \in \mathbb{R}$, and that f, ϕ and S comply with (1.3), (1.5) and (1.4). Furthermore, suppose that $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,\infty)$ is such that $\varepsilon_j \searrow 0$ as $j\to\infty$, and that for some $T > 0$, $((n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon))_{\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}}$ is such that for each $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ solves (1.2) classically in $\Omega\times(0,T)$ with $n_\varepsilon \geq 0$ and $c_\varepsilon \geq 0$ in $\Omega\times(0,T)$, and such that*

$$\sup_{\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}} \|\nabla c_\varepsilon\|_{L^\lambda((0,T);L^q(\Omega))} < \infty \tag{1.8}$$

as well as

$$\sup_{\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}} \|u_\varepsilon\|_{L^\infty((0,T);L^r(\Omega))} < \infty \tag{1.9}$$

with some $\lambda \in (2,\infty]$, $q > N$ and $r > \max\{2, N\}$ satisfying

$$\frac{1}{\lambda} + \frac{N}{2q} < \frac{1}{2}. \tag{1.10}$$

Then there exist a subsequence $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ of $(\varepsilon_j)_{j\in\mathbb{N}}$ and a classical solution (n,c,u,P) of (1.7) in $\Omega\times(0,T)$ with the properties that

$$n_\varepsilon \to n \qquad \text{in } C^0(\overline{\Omega}\times[0,T]), \tag{1.11}$$

$$n_\varepsilon \rightharpoonup n \qquad \text{in } L^2((0,T);W^{1,2}(\Omega)), \tag{1.12}$$

$$c_\varepsilon \to c \qquad \text{in } L^\infty_{loc}((0,T];C^0(\overline{\Omega}))\cap L^2_{loc}((0,T];W^{1,2}(\Omega)), \tag{1.13}$$

$$\nabla c_\varepsilon \overset{\star}{\rightharpoonup} \nabla c \qquad \text{in } \bigcap_{\widehat{q}>N} L^\infty((0,T);L^{\widehat{q}}(\Omega))\cap L^\infty((\Omega\times(0,T)) \qquad \text{and} \tag{1.14}$$

$$u_\varepsilon \to u \qquad \text{in } C^0(\overline{\Omega}\times[0,T];\mathbb{R}^N)\cap C^{2,1}_{loc}(\overline{\Omega}\times(0,T];\mathbb{R}^N) \tag{1.15}$$

as $\varepsilon=\varepsilon_{j_k}\searrow 0$.

Remark. i) We underline that the above assumption on convexity of Ω is mainly of technical nature and could actually be removed by additional efforts based on a differential geometrical property due to [25] in quite a straightforward manner. In order to keep our reasoning as focused as possible, however, we refrain from giving details on this here. Similarly, since in essential places we will rely on convenient compactness features conveniently available in bounded domains, we shall not address possible extensions to unbounded domains.
ii) The restriction to subsequences in the statement of Theorem 1.1 is mainly due to the circumstance that in the full generality of the described setting we are not aware of an appropriate uniqueness result for the limit problem (1.7); however, for special cases in which e.g. $S\equiv id$ and fluid coupling is disregarded, the availability of corresponding uniqueness statements (see [42], for instance) in fact allows for natural extensions of the above, so as to assert convergence actually along the entire given sequence $(\varepsilon_j)_{j\in\mathbb{N}}$; an example for such a refined application of Theorem 1.1 can be found in the context of Theorem 1.3 below.
iii) We emphasize that Theorem 1.1 presupposes the existence of solutions to (1.2) throughout the considered time interval. In cases in which blow-up is expected, this especially restricts applicability of the above, in quite a natural manner, to local-in-time frameworks. Apart from that, possible challenges concerning existence theories for (1.2) are entirely disregarded here; while statements on local and also on global smooth solvability are available for numerous particular versions of (1.2) ([7], [49], [5], [51]), a comprehensive theory in this regard seems yet lacking, especially in cases of nondiagonal S.

Main results II. The fast signal diffusion limit for small-data solutions to a Keller-Segel-Navier-Stokes system. As a first application of the latter, let us consider the case when $f\equiv 0$ in the Keller-Segel-Navier-Stokes system (1.2) in arbitrary spatial dimensions $N\geq 2$. Then in light of well-known results on taxis-driven blow-up of some solutions to both the fully parabolic problem (1.2) as well as its parabolic-elliptic counterpart (1.7) already in the simple case $u\equiv 0$ ([29], [13], [52]), regular behavior throughout the arbitrary time interval $(0,T)$ can be expected only under appropriate additional assumptions on the initial data. In deriving the following consequence of Theorem 1.1 on this particular system, we shall accordingly restrict our considerations to solutions emanating from suitably small initial data. In this context we will see the following.

Theorem 1.2 *Let $N\geq 2$ and $\Omega\subset\mathbb{R}^N$ be a bounded convex domain with smooth boundary, let $\kappa\in\mathbb{R}$, $p>N$, $q>N$ and $r>N$, and assume that (1.5) and (1.4) are valid. Then there exists*

$\delta = \delta(p, q, r) > 0$ with the property that whenever n_0, c_0 and u_0 comply with (1.6) and satisfy

$$\|n_0\|_{L^p(\Omega)} \le \delta, \qquad \|\nabla c_0\|_{L^q(\Omega)} \le \delta \qquad \text{and} \qquad \|u_0\|_{L^r(\Omega)} \le \delta, \tag{1.16}$$

for all $\varepsilon > 0$ the problem (1.2) with $f \equiv 0$ possesses a global classical solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$. Moreover, given any $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0, \infty)$ satisfying $\varepsilon_j \searrow 0$ as $j \to \infty$, one can find a subsequence $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ and a global classical solution (n, c, u, P) of (1.7) with $f \equiv 0$ such that for each $T > 0$, (1.11)-(1.15) hold as $\varepsilon = \varepsilon_{j_k} \searrow 0$.

Remark. As a price to be paid in order to cover the general problem (1.2) in its full complexity, by inter alia requiring an essentially non-explicit smallness assumption on n_0 with respect to the norm in L^p with $p > N$ the above theorem is unable to cover e.g. the full mass-subcritical regime, as described by the mere condition $\int_\Omega n_0 < 4\pi$, of the simple two-component Keller-Segel system in planar domains, as obtained on letting $S \equiv id$, $f \equiv 0$ and $u_\varepsilon \equiv 0$ in (1.2) ([31]). Extensions capable of adequately coping with such critical situations apparently need to appropriately account for respectively present subtle structural features, such as e.g. expressed in particular energy inequalities. Such concentrations on particular versions of (1.2) form a topic of interest on its own, going beyond the focus of the present study.

Main results III. A growth phenomenon in a fully parabolic one-dimensional Keller-Segel system with logistic source. As a second application of our general theory, we shall consider the family of fluid-free one-dimensional Keller-Segel systems with logistic source, as given by

$$\begin{cases} n_{\varepsilon t} = Dn_{\varepsilon xx} - (n_\varepsilon c_{\varepsilon x})_x + an_\varepsilon - bn_\varepsilon^2, & x \in (0,1),\ t > 0, \\ \varepsilon c_{\varepsilon t} = c_{\varepsilon xx} - c_\varepsilon + n_\varepsilon, & x \in (0,1),\ t > 0, \\ n_{\varepsilon x}(0,t) = n_{\varepsilon x}(1,t) = c_{\varepsilon x}(0,t) = c_{\varepsilon x}(1,t) = 0, & t > 0, \\ n_\varepsilon(x,0) = n_0(x), \quad c_\varepsilon(x,0) = c_0(x), & x \in (0,1), \end{cases} \tag{1.17}$$

for $\varepsilon > 0$, with $a \in \mathbb{R}$ and $b \ge 0$, and with nonnegative functions $n_0 \in W^{1,\infty}((0,1))$ and $c_0 \in W^{1,\infty}((0,1))$. We note that upon replacing n_ε by $\tilde{n}_\varepsilon(x, \tilde{t}) := n_\varepsilon(x,t)$ with $\tilde{t} := Dt$ for $(x,t) \in [0,1] \times [0,\infty)$, this problem indeed takes the form (1.2), and that as a well-known fact, for each $\varepsilon > 0$ there exists a global classical solution $(n_\varepsilon, c_\varepsilon)$ for which both n_ε and c_ε are nonnegative and bounded throughout $(0,1) \times (0,\infty)$ ([49]). In fact, by relying on suitable embedding properties available in this one-dimensional context we shall firstly see that as $\varepsilon \searrow 0$, these solutions approach solutions to the corresponding parabolic-elliptic counterpart, namely, the problem

$$\begin{cases} n_t = Dn_{xx} - (nc_x)_x + an - bn^2, & x \in (0,1),\ t > 0, \\ 0 = c_{xx} - c + n, & x \in (0,1),\ t > 0, \\ n_x(0,t) = n_x(1,t) = c_x(0,t) = c_x(1,t) = 0, & t > 0, \\ n(x,0) = n_0(x), & x \in (0,1), \end{cases} \tag{1.18}$$

in the following sense:

Theorem 1.3 *Let $D > 0$, $a \in \mathbb{R}$ and $b \ge 0$, and suppose that n_0 and c_0 are such that (1.6) holds. Then for all $T > 0$, the solutions $(n_\varepsilon, c_\varepsilon)$ of (1.17) have the property that as $\varepsilon \searrow 0$, (1.11)-(1.14) hold with the unique classical solution $(n, c) \in (C^0([0,1] \times [0,T]) \cap C^{2,1}([0,1] \times (0,T))) \times C^{2,0}([0,1] \times (0,T))$ of (1.18).*

Building on this result, we shall secondly discover that solutions to the fully parabolic problem (1.17) can spontaneously generate arbitrarily large densities, possibly at intermediate time scales, provided that the parameters D and ε satisfy appropriate smallness conditions. A similar growth phenomenon had been detected in certain versions of the parabolic-elliptic problem before, and for further discussion, and also for numerical simulations indicating a temporally intermediat character of such large-density occurrences, we may refer to [53].

Theorem 1.4 *Let $a \in \mathbb{R}$ and $b \in [0,1)$. Then there exist $T > 0$ and a nonnegative function $n_0 \in W^{1,\infty}((0,1))$ with the following property: For all $M > 0$ one can find $D_0 > 0$ such that for each $D \in (0, D_0)$ and any nonnegative $c_0 \in W^{1,\infty}((0,1))$ there exist $x_0 \in (0,1), t_0 \in (0,T)$ and $\varepsilon_0 > 0$ such that for any choice of $\varepsilon \in (0, \varepsilon_0)$, the corresponding solution $(n_\varepsilon, c_\varepsilon)$ of (1.17) satisfies*

$$n_\varepsilon(x_0, t_0) \geq M. \tag{1.19}$$

Key steps in our analysis. The crucial role of the assumptions from Theorem 1.1, and especially of the inequality (1.10) therein, will already become clear in Section 2, in which we will derive some ε-independent estimates for general solutions to (1.2) under presupposed bounds on ∇c_ε and u_ε of the considered form. Complementing these estimates by further compactness properties will allow for passing to the limit along subsequences, with regard to the components n_ε and u_ε already in the flavor claimed in Theorem 1.1; as for c_ε, however, due to lacking uniform parabolicity (formally, $\varepsilon c_{\varepsilon t} \to 0$ as $\varepsilon \to 0$) in the equation describing its evolution we will at that stage only be able to conclude a weak convergence property in $L^2((0,T); W^{1,2}(\Omega))$.

A key step will thereafter consist in improving this knowledge, which will be achieved through several steps: After firstly showing that the limit c satisfies its respective subproblem of (1.7) in a weak sense, we can exploit the correspondingly satisfied integral identity to successively establish Hölder regularity of c, ∇c and $D^2 c$ in Section 5.1. The main step will then be accomplished by ensuring L^2 integrability of c_t, locally away from the temporal origin, in Section 5.2. Our derivation thereof will rely on suitably estimating the difference quotients

$$z_h(x,t) := \frac{c(x,t+h) - c(x,t)}{h} =: z_h^1(x,t) + z_h^2(x,t), \qquad x \in \Omega,\ t \in (\tau, T - h_0),$$

where z_h^1 and z_h^2 denote the classical solution of two linear elliptic equations, whose forcing terms involve the time derivatives of n and u. Thanks to the availability of appropriate regularity information on the latter, by utilizing standard elliptic regularity theory we will infer that indeed c_t belongs to $L^2_{loc}(\overline{\Omega} \times (0,T])$. This in turn will allow us to adequately control the difference $c_\varepsilon - c$ through analyzing a parabolic equation therefor, and hence verify Theorem 1.1 in Section 6.

Sections 7 and 8 will thereafter be devoted to the proofs of Theorem 1.2 and of Theorems 1.3 and 1.4, respectively.

2 Some general estimates

In this section we collect some estimates which are valid for general solutions to systems of the form (1.2), and which are independent of the particular choice of $\varepsilon > 0$, partially under presupposed bounds resembling those in (1.8) and (1.9). These estimates will firstly be used as a fundament for our proof of Theorem 1.1, and secondly some of them will afterwards serve as helpful ingredients for the derivation of Theorem 1.2 in Section 7.

Let us start with a fairly evident observation.

Lemma 2.1 *Suppose that (1.3), (1.5) and (1.6) hold, and let $\kappa \in \mathbb{R}$ and $T > 0$. Then there exists $C = C(T) > 0$ such that whenever $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ is a classical solution of (1.2) in $\Omega \times (0, T)$ for some $\varepsilon > 0$, we have $n_\varepsilon \geq 0$ and $c_\varepsilon \geq 0$ in $\Omega \times (0, T)$ as well as*

$$\|n_\varepsilon(\cdot, t)\|_{L^1(\Omega)} \leq C \qquad \textit{for all } t \in (0, T). \tag{2.1}$$

PROOF. According to the lower bound for f_0 and hence for f in (1.3), nonnegativity of n_ε results from an application of the maximum principle to the first equation in (1.2). In view of the second equation therein, by the same token this in turn entails nonnegativity also of c_ε.

Next, integrating the first equation in (1.2) shows that since $\nabla \cdot u_\varepsilon \equiv 0$, due to the upper bound for f from (1.3) we have

$$\frac{d}{dt} \int_\Omega n_\varepsilon = \int_\Omega f(x, n_\varepsilon, c_\varepsilon) \leq K_f \int_\Omega n_\varepsilon + K_f |\Omega| \qquad \text{for all } t \in (0, T),$$

from which (2.1) readily results upon a time integration. $\square$

The next lemma already makes full use of supposedly present bounds in the style of (1.8) and (1.9), and especially of the relation (1.10) involving the parameters therein.

Lemma 2.2 *Suppose that (1.3), (1.4), (1.5) and (1.6) hold, and let $\kappa \in \mathbb{R}$. Then for all $T > 0$, $L > 0$, $\lambda \in (2, \infty], q > N, r > N$ such that (1.10) holds, there exists $C = C(T, \lambda, q, r, K_S, L) > 0$ such that whenever $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ is a classical solution of (1.2) in $\Omega \times (0, T)$ for some $\varepsilon > 0$ fulfilling*

$$\|\nabla c_\varepsilon\|_{L^\lambda((0,T);L^q(\Omega))} \leq L \tag{2.2}$$

and

$$\|u_\varepsilon(\cdot, t)\|_{L^r(\Omega)} \leq L \qquad \textit{for all } t \in (0, T), \tag{2.3}$$

we have

$$\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C \qquad \textit{for all } t \in (0, T). \tag{2.4}$$

PROOF. Omitting the subscript ε for notational convenience, without loss of generality assuming that $\lambda < \infty$ and following an essentially well-established procedure (cf. e.g. [41]), we estimate

$$M(T') := \sup_{t \in (0, T')} \|n(\cdot, t)\|_{L^\infty(\Omega)}, \qquad T' \in (0, T)$$

by representing n via an associated Duhamel formula. Indeed, using the maximum principle and (1.3) as well as known smoothing properties of the Neumann heat semigroup $(e^{t\Delta})_{t \geq 0}$ in Ω (see Lemma 1.3

in [50]) we see that fixing any $\mu_1 = \mu_1(\lambda, q) \in (N, q)$ and $\mu_2 = \mu_2(r) \in (N, r)$ such that $\frac{1}{\lambda} + \frac{N}{2\mu_1} < \frac{1}{2}$, with some $C_1 = C_1(\lambda, q, r) > 0$ we have

$$\begin{aligned}
n(\cdot, t) &= e^{t\Delta} n_0 - \int_0^t e^{(t-s)\Delta} \nabla \cdot \Big(n(\cdot, s) S(\cdot, n(\cdot, s), c(\cdot, s)) \cdot \nabla c(\cdot, s) \Big) ds \\
&\quad - \int_0^t e^{(t-s)\Delta} \nabla \cdot (n(\cdot, s) u(\cdot, s)) ds + \int_0^t e^{(t-s)\Delta} f(\cdot, n(\cdot, s), c(\cdot, s)) ds \\
&\le \|n_0\|_{L^\infty(\Omega)} + C_1 \int_0^t (t-s)^{-\frac{1}{2} - \frac{N}{2\mu_1}} \|n(\cdot, s) S(\cdot, n(\cdot, s), c(\cdot, s)) \cdot \nabla c(\cdot, s)\|_{L^{\mu_1}(\Omega)} ds \\
&\quad + C_1 \int_0^t (t-s)^{-\frac{1}{2} - \frac{N}{2\mu_2}} \|n(\cdot, s) u(\cdot, s)\|_{L^{\mu_2}(\Omega)} ds + C_1 \int_0^t (t-s)^{-\frac{1}{2}} \|n(\cdot, s) + 1\|_{L^N(\Omega)} ds
\end{aligned} \tag{2.5}$$

for all $t \in (0, T)$. Here using (1.4) and the Hölder inequality along with (2.1) and our hypotheses (2.2) and (2.3), we find positive constants $C_2 = C_2(K_S), C_3 = C_3(K_S), C_4 = C_4(T, K_S), C_5 = C_5(T, r, L)$ and $C_6 = C_6(T)$ such that

$$\begin{aligned}
\|n(\cdot, s) S(\cdot, n(\cdot, s), c(\cdot, s)) \cdot \nabla c(\cdot, s)\|_{L^{\mu_1}(\Omega)} &\le C_2 \|n(\cdot, s)\|_{L^{\frac{q\mu_1}{q-\mu_1}}(\Omega)} \|\nabla c(\cdot, s)\|_{L^q(\Omega)} \\
&\le C_3 \|n(\cdot, s)\|_{L^\infty(\Omega)}^{a_1} \|n(\cdot, s)\|_{L^1(\Omega)}^{1-a_1} \|\nabla c(\cdot, s)\|_{L^q(\Omega)} \\
&\le C_4 M^{a_1}(T') \|\nabla c(\cdot, s)\|_{L^q(\Omega)} \qquad \text{for all } s \in (0, T')
\end{aligned}$$

and

$$\begin{aligned}
\|n(\cdot, s) u(\cdot, s)\|_{L^{\mu_2}(\Omega)} &\le \|n(\cdot, s)\|_{L^{\frac{r\mu_2}{r-\mu_2}}(\Omega)} \|u(\cdot, s)\|_{L^r(\Omega)} \\
&\le \|n(\cdot, s)\|_{L^\infty(\Omega)}^{a_2} \|n(\cdot, s)\|_{L^1(\Omega)}^{1-a_2} \|u(\cdot, s)\|_{L^r(\Omega)} \\
&\le C_5 M^{a_2}(T') \qquad \text{for all } s \in (0, T')
\end{aligned}$$

as well as

$$\begin{aligned}
\|n(\cdot, s) + 1\|_{L^N(\Omega)} &\le \|n(\cdot, s) + 1\|_{L^\infty(\Omega)}^{a_3} \|n(\cdot, s) + 1\|_{L^1(\Omega)}^{1-a_3} \\
&\le C_6 M^{a_3}(T') + C_6 \qquad \text{for all } s \in (0, T')
\end{aligned}$$

with $a_1 := \frac{q\mu_1 - q + \mu_1}{q\mu_1} \in (0,1)$, $a_2 := \frac{ru_2 - r + \mu_2}{r\mu_2} \in (0,1)$ and $a_3 := \frac{N-1}{N} \in (0,1)$. Since the inequalities $\frac{1}{\lambda} + \frac{N}{2\mu_1} < \frac{1}{2}$ and $\mu_2 > N$ moreover warrant that $(\frac{1}{2} + \frac{N}{2\mu_1}) \cdot \frac{\lambda}{\lambda - 1} < 1$ and $\frac{1}{2} + \frac{N}{2\mu_2} < 1$, by using the Hölder inequality two more times we thus infer from (2.5) and the nonnegativity of n that there exist $C_8 = C_8(T, \lambda, q, r, K_S, L) > 0$, $C_9 = C_9(T, \lambda, q) > 0$ and $C_{10} = C_{10}(r) > 0$ such that

$$\begin{aligned}
\|n(\cdot, t)\|_{L^\infty(\Omega)} &\le C_8 + C_8 M^{a_1}(T') \cdot \left\{ \int_0^t (t-s)^{-(\frac{1}{2} + \frac{N}{2\mu_1}) \cdot \frac{\lambda}{\lambda - 1}} ds \right\}^{\frac{\lambda - 1}{\lambda}} \cdot \left\{ \int_0^t \|\nabla c(\cdot, s)\|_{L^q(\Omega)}^\lambda ds \right\}^{\frac{1}{\lambda}} \\
&\quad + C_8 M^{a_2}(T') \cdot \int_0^t (t-s)^{-\frac{1}{2} - \frac{N}{2\mu_2}} ds + C_8 M^{a_3}(T') \\
&\le C_8 + C_8 C_9 M^{a_1}(T') + C_8 C_{10} M^{a_2}(T') + C_8 M^{a_3}(T') \qquad \text{for all } t \in (0, T').
\end{aligned}$$

Hence, by Young's inequality,

$$M(T') \le C_{11} + C_{11} M^a(T') \qquad \text{for all } T' \in (0,T),$$

where $a := \max\{a_1, a_2, a_3\}$ satisfies $a \in (0,1)$, and where $C_{11} := 2C_8 + C_8 C_9 + C_8 C_{10}$. As therefore

$$M(T') \le \max\Big\{1\,,\,(2C_{11})^{\frac{1}{1-a}}\Big\} \qquad \text{for all } T' \in (0,T),$$

we have thus established (2.4). $\square$

As a consequence of the latter estimate for n_ε, by means of quite a similar argument, essentially well-established in the theory of the Navier-Stokes system, we can again use the boundedness assumption (2.3) in order to appropriately control the fluid velocity field as follows.

Lemma 2.3 *Suppose that (1.3), (1.4), (1.5) and (1.6) hold, and let $\kappa \in \mathbb{R}$. Then for all $T > 0$, $L > 0$, $\lambda \in (2,\infty], q > N, r > \max\{2, N\}$ fulfilling (1.10), there exist $\alpha = \alpha(r) \in (\frac{1}{2}, 1)$, $\rho = \rho(r) > \max\{1, \frac{N}{2\alpha}\}$, $\theta = \theta(r) \in (0,1)$ and $C = C(T, \lambda, q, r, K_S, \kappa, L) > 0$ such that if for some $\varepsilon > 0$ $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ is a classical solution of (1.2) in $\Omega \times (0,T)$ for which (2.2) and (2.3) are valid, then*

$$\|A^\alpha u_\varepsilon(\cdot,t)\|_{L^\rho(\Omega)} \le C \qquad \textit{for all } t \in (0,T) \tag{2.6}$$

and

$$\|u_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[0,T])} \le C, \tag{2.7}$$

where $A := -\mathcal{P}\Delta$ denotes the realization of the Stokes operator in $L^2(\Omega;\mathbb{R}^2)$, defined on its domain $D(A) := W^{2,2}(\Omega;\mathbb{R}^2) \cap W_0^{1,2}(\Omega;\mathbb{R}^2) \cap L^2_\sigma(\Omega)$ with $L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega;\mathbb{R}^2) \mid \nabla \cdot u = 0\}$, and with $\mathcal{P}$ representing the Helmholtz projection of $L^2(\Omega;\mathbb{R}^2)$ onto $L^2_\sigma(\Omega)$.

Proof. Since $r > N$ and thus $\frac{1}{2} + \frac{N}{2r} < 1$, it is possible to fix $\alpha = \alpha(r) \in (\frac{1}{2}, 1)$ close to $\frac{1}{2}$ such that

$$\alpha + \frac{N}{2r} < 1, \tag{2.8}$$

and thereafter take $\beta \in (\frac{1}{2}, 1)$ such that $\beta < \alpha$. Then using that $\frac{N}{2\alpha} < N < r$ and that also $\frac{r}{r-1} < r$, we can pick $\rho = \rho(r) > \max\{1, \frac{N}{2\alpha}\}$ such that $\rho \le r$ and $\rho > \frac{r}{r-1}$, observing that the latter ensures that $\mu = \mu(r) := \frac{r\rho}{r+\rho}$ satisfies $\mu > 1$. Again dropping the index ε and as moreover $\mu < \rho$, relying on a variation-of-constants representation of u we may employ known smoothing properties of the Stokes semigroup $(e^{-tA})_{t\ge 0}$ ([10]) to find $C_1 = C_1(r,\kappa) > 0$ such that

$$\begin{aligned}
\|A^\alpha u(\cdot,t)\|_{L^\rho(\Omega)} &= \bigg\| A^\alpha e^{-tA} u_0 - \kappa \int_0^t A^\alpha e^{-(t-s)A} \mathcal{P}\Big[(u(\cdot,s)\cdot\nabla)u(\cdot,s)\Big] ds \\
&\qquad + \int_0^t A^\alpha e^{-(t-s)A} \mathcal{P}[n(\cdot,s)\nabla\phi] ds \bigg\|_{L^\rho(\Omega)} \\
&\le \|A^\alpha u_0\|_{L^\rho(\Omega)} + C_1 \int_0^t (t-s)^{-\alpha-\frac{N}{2}(\frac{1}{\mu}-\frac{1}{\rho})} \|(u(\cdot,s)\cdot\nabla)u(\cdot,s)\|_{L^\mu(\Omega)} ds \\
&\qquad + C_1 \int_0^t (t-s)^{-\alpha} \|n(\cdot,s)\|_{L^\rho(\Omega)} ds \qquad \text{for all } t \in (0,T).
\end{aligned} \tag{2.9}$$

Here since clearly $\mu < r$, we can employ the Hölder inequality to see that thanks to (2.3) and the inequalities $\alpha > \beta > \frac{1}{2}$ and $\rho \le r$, the continuity of the embedding $D(A_\rho^\beta) \hookrightarrow W^{1,\rho}(\Omega;\mathbb{R}^N)$ ([9] [12]) and a well-known interpolation property (see Theorem 14.1 in Part 2 of [9]) guarantee that with some $C_2 = C_2(r) > 0, C_3 = C_3(r) > 0$ and $C_4 = C_4(r) > 0$ we have

$$\begin{aligned}
\|(u(\cdot,s)\cdot\nabla)u(\cdot,s)\|_{L^\mu(\Omega)} &\le \|u(\cdot,s)\|_{L^r(\Omega)}\|\nabla u(\cdot,s)\|_{L^{\frac{r\mu}{r-\mu}}(\Omega)} \\
&\le L\|\nabla u(\cdot,s)\|_{L^\rho(\Omega)} \\
&\le C_2 L\|A^\beta u(\cdot,s)\|_{L^\rho(\Omega)} \\
&\le C_3 L\|A^\alpha u(\cdot,s)\|_{L^\rho(\Omega)}^a \|u(\cdot,s)\|_{L^\rho(\Omega)}^{1-a} \\
&\le C_4 L\|A^\alpha u(\cdot,s)\|_{L^\rho(\Omega)}^a \|u(\cdot,s)\|_{L^r(\Omega)}^{1-a} \\
&\le C_4 L^{2-a} M^a(T') \qquad \text{for all } s \in (0,T') \text{ and any } T' \in (0,T)
\end{aligned}$$

if we let $a := \frac{\beta}{\alpha} \in (0,1)$ and

$$M(T') := \sup_{t\in(0,T')} \|A^\alpha u(\cdot,t)\|_{L^\rho(\Omega)}, \qquad T' \in (0,T).$$

As Lemma 2.2 in particular implies the existence of $C_5 = C_5(T,\lambda,q,r,K_S,L) > 0$ such that

$$\|n(\cdot,t)\|_{L^\rho(\Omega)} \le C_5 \qquad \text{for all } t \in (0,T),$$

noting that $\alpha < 1$ and that

$$\alpha + \frac{N}{2}\Big(\frac{1}{\mu} - \frac{1}{\rho}\Big) = \alpha + \frac{N}{2}\Big(\frac{r+\rho}{r\rho} - \frac{1}{\rho}\Big) = \alpha + \frac{N}{2r} < 1$$

by (2.8), we thus conclude from (2.9) and (1.6) that there exists $C_6 = C_6(T,\lambda,q,r,K_S,\kappa,L) > 0$ such that

$$M(T') \le C_6 + C_6 M^a(T') \qquad \text{for all } T' \in (0,T),$$

which implies (2.6) due to the fact that $a < 1$.

Now by a straightforward adaptation of a well-known reasoning (see, for instance, the proof of (2.36) in Lemma 2.8 of [48]), in quite a similar manner it is furthermore possible to find $\theta_1 = \theta_1(r) \in (0,1)$ and $C_7 = C_7(T,\lambda,q,r,K_S,\kappa,L) > 0$ fulfilling

$$\|A^\alpha u(\cdot,t) - A^\alpha u(\cdot,t_0)\|_{L^\rho(\Omega)} \le C_7|t-t_0|^{\theta_1} \qquad \text{for all } t \in (0,T) \text{ and } t_0 \in (0,T),$$

which finally implies (2.7) due to the fact that $D(A_\rho^\alpha) \hookrightarrow C^{\theta_2}(\overline{\Omega};\mathbb{R}^N)$ for any θ_2 from the nonempty interval $(0, 2\alpha - \frac{N}{\rho})$ ([12]). □

Let us finally prepare an argument that will, before becoming substantial for the derivation of Theorem 1.2 in Section 7, inter alia reveal in Lemma 5.8 that the assumptions (1.8) and (1.9) actually imply boundedness of ∇c in $L^\infty((0,T);L^{\widehat{q}}(\Omega))$ for arbitrarily large $\widehat{q}$. The following lemma is the only place in this paper where convexity of Ω is explicitly needed.

Lemma 2.4 *For all $p > \max\{N, 2\}$, $q \geq 2$ and $r \in (2, \infty]$ there exists $C = C(p, q) > 0$ such that if $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ is a classical solution of (1.2) in $\Omega \times (0, T)$ for some $\varepsilon > 0$ and $T > 0$, then*

$$\begin{aligned}
&\frac{\varepsilon}{q}\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^q + \frac{1}{4}\int_\Omega |\nabla c_\varepsilon|^{q-2}|D^2 c_\varepsilon|^2 + \Big(1 - \frac{1}{q^2}\Big)\int_\Omega |\nabla c_\varepsilon|^q \\
&\quad \leq \; C\|n_\varepsilon\|^q_{L^p(\Omega)} + C\|u_\varepsilon\|^2_{L^r(\Omega)}\|\nabla c_\varepsilon\|^q_{L^{\frac{qr}{r-2}}(\Omega)} \qquad \text{for all } t \in (0, T),
\end{aligned} \tag{2.10}$$

where we interpret $\frac{qr}{r-2}$ as coinciding with q if $r = \infty$.

Proof. Once more omitting the subscript ε for convenience, by means of the second equation in (1.2), we see that for all $t \in (0, T)$,

$$\begin{aligned}
\frac{\varepsilon}{q}\frac{d}{dt}\int_\Omega |\nabla c|^q &= \int_\Omega |\nabla c|^{q-2}\nabla c \cdot \nabla\Big\{\Delta c - c + n - u \cdot \nabla c\Big\} \\
&= \frac{1}{2}\int_\Omega |\nabla c|^{q-2}\Delta|\nabla c|^2 - \int_\Omega |\nabla c|^{q-2}|D^2 c|^2 \\
&\quad - \int_\Omega |\nabla c|^q + \int_\Omega |\nabla c|^{q-2}\nabla c \cdot \nabla n - \int_\Omega |\nabla c|^{q-2}\nabla c \cdot \nabla(u \cdot \nabla c) \\
&= \frac{1}{2}\int_{\partial\Omega} |\nabla c|^{q-2}\frac{\partial|\nabla c|^2}{\partial\nu} - \frac{q-2}{2}\int_\Omega |\nabla c|^{q-4}|\nabla|\nabla c|^2|^2 - \int_\Omega |\nabla c|^{q-2}|D^2 c|^2 \\
&\quad - \int_\Omega |\nabla c|^q + \int_\Omega |\nabla c|^{q-2}\nabla c \cdot \nabla n - \int_\Omega |\nabla c|^{q-2}\nabla c \cdot \nabla(u \cdot \nabla c) \\
&\leq - \int_\Omega |\nabla c|^{q-2}|D^2 c|^2 - \int_\Omega |\nabla c|^q \\
&\quad - \int_\Omega n|\nabla c|^{q-2}\Delta c - (q-2)\int_\Omega n|\nabla c|^{q-4}\nabla c \cdot (D^2 c \cdot \nabla c) \\
&\quad + \int_\Omega (u \cdot \nabla c)|\nabla c|^{q-2}\Delta c + (q-2)\int_\Omega (u \cdot \nabla c)|\nabla c|^{q-4}\nabla c \cdot (D^2 c \cdot \nabla c),
\end{aligned} \tag{2.11}$$

because of $\frac{\partial|\nabla c|^2}{\partial\nu} \leq 0$ on $\partial\Omega \times (0, T)$ due to the convexity of Ω ([24]), and because of $q \geq 2$. Here two applications of Young's inequality and the Hölder inequality show that abbreviating $C_1(q) := \sqrt{2} + q - 2$ we have

$$\begin{aligned}
&- \int_\Omega n|\nabla c|^{q-2}\Delta c - (q-2)\int_\Omega n|\nabla c|^{q-4}\nabla c \cdot (D^2 c \cdot \nabla c) \\
&\qquad \leq \; C_1(q)\int_\Omega n|\nabla c|^{q-2}|D^2 c| \\
&\qquad \leq \; \frac{1}{4}\int_\Omega |\nabla c|^{q-2}|D^2 c|^2 + C_1^2(q)\int_\Omega n^2|\nabla c|^{q-2} \\
&\qquad \leq \; \frac{1}{4}\int_\Omega |\nabla c|^{q-2}|D^2 c|^2 + C_1^2(q)\|n\|^2_{L^p(\Omega)}\|\nabla c\|^{q-2}_{L^{\frac{p(q-2)}{p-2}}(\Omega)}
\end{aligned} \tag{2.12}$$

and

$$\int_\Omega (u \cdot \nabla c)|\nabla c|^{q-2}\Delta c + (q-2)\int_\Omega (u \cdot \nabla c)|\nabla c|^{q-4}\nabla c \cdot (D^2 c \cdot \nabla c)$$

$$
\begin{aligned}
&\le C_1(q)\int_\Omega |u||\nabla c|^{q-1}|D^2c| \\
&\le \frac{1}{4}\int_\Omega |\nabla c|^{q-2}|D^2c|^2 + C_1^2(q)\int_\Omega |u|^2|\nabla c|^q \\
&\le \frac{1}{4}\int_\Omega |\nabla c|^{q-2}|D^2c|^2 + C_1^2(q)\|u\|_{L^r(\Omega)}^2\|\nabla c\|^q_{L^{\frac{qr}{r-2}}(\Omega)}
\end{aligned}
\tag{2.13}
$$

for all $t \in (0,T)$. Now since $p > \max\{N,2\}$ and $q \ge 2$ ensure that $p(q-2)(N-2) < N(p-2)q$, it follows that $W^{1,2}(\Omega)$ is continuously embedded into $L^{\frac{2p(q-2)}{(p-2)q}}(\Omega)$. Again using Young's inequality, we can therefore find $C_2(p,q) > 0$ and $C_3(p,q) > 0$ such that

$$
\begin{aligned}
C_1^2(q)\|n\|^2_{L^p(\Omega)}\|\nabla c\|^{q-2}_{L^{\frac{p(q-2)}{p-2}}(\Omega)} &\le C_2(p,q)\|n\|^2_{L^p(\Omega)}\cdot\Big\{\Big\|\nabla|\nabla c|^{\frac{q}{2}}\Big\|^2_{L^2(\Omega)} + \Big\||\nabla c|^{\frac{q}{2}}\Big\|^2_{L^2(\Omega)}\Big\}^{\frac{q-2}{q}} \\
&\le \frac{1}{q^2}\cdot\Big\{\Big\|\nabla|\nabla c|^{\frac{q}{2}}\Big\|^2_{L^2(\Omega)} + \Big\||\nabla c|^{\frac{q}{2}}\Big\|^2_{L^2(\Omega)}\Big\} + C_3(p,q)\|n\|^q_{L^p(\Omega)} \\
&= \frac{1}{4}\int_\Omega |\nabla c|^{q-4}|D^2c\cdot\nabla c|^2 + \frac{1}{q^2}\int_\Omega|\nabla c|^q + C_3(p,q)\|n\|^q_{L^p(\Omega)} \\
&\le \frac{1}{4}\int_\Omega |\nabla c|^{q-2}|D^2c|^2 + \frac{1}{q^2}\int_\Omega|\nabla c|^q + C_3(p,q)\|n\|^q_{L^p(\Omega)}
\end{aligned}
$$

for all $t \in (0,T)$. Therefore, (2.12) and (2.13) when inserted into (2.11) show that

$$
\begin{aligned}
\frac{\varepsilon}{q}\frac{d}{dt}\int_\Omega |\nabla c|^q &+ \frac{1}{4}\int_\Omega |\nabla c|^{q-2}|D^2c|^2 + \Big(1-\frac{1}{q^2}\Big)\int_\Omega|\nabla c|^q \\
&\le C_3(p,q)\|n\|^q_{L^p(\Omega)} + C_1^2(q)\|u\|^2_{L^r(\Omega)}\|\nabla c\|^q_{L^{\frac{qr}{r-2}}(\Omega)} \qquad \text{for all } t\in(0,T),
\end{aligned}
$$

which directly results in (2.10). $\square$

3 Regularity and compactness properties implied by the hypotheses from Theorem 1.1

Next concentrating on the particular setup created by Theorem 1.1, in this part we will augment the estimates from the previous section by further compactness properties which will allow for passing to the limit, already partially in the flavor claimed in Theorem 1.1.

Firstly, the L^∞ bound from Lemma 2.2 can quite immediately be improved into an estimate in some Hölder space by means of standard parabolic theory.

Lemma 3.1 *Suppose that the assumptions of Theorem 1.1 are satisfied. Then there exist $\theta \in (0,1)$ and $C > 0$ such that*

$$
\|n_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[0,T])} \le C \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}. \tag{3.1}
$$

Proof. We rewrite the first equation in (1.2) in the form

$$
\partial_t n_\varepsilon = \nabla\cdot a_\varepsilon(x,t,\nabla n_\varepsilon) + b_\varepsilon(x,t), \qquad x\in\Omega,\ t\in(0,T),
$$

with

$$a_\varepsilon(x,t,\xi) := \xi - n_\varepsilon(x,t) S(x, n_\varepsilon(x,t), c_\varepsilon(x,t)) \cdot \nabla c_\varepsilon(x,t) - n_\varepsilon(x,t) u_\varepsilon(x,t), \ (x,t,\xi) \in \Omega \times (0,T) \times \mathbb{R}^N,$$

and

$$b_\varepsilon(x,t) := f(x, n_\varepsilon(x,t), c_\varepsilon(x,t)), \qquad (x,t) \in \Omega \times (0,T),$$

Then due to Young's inequality and (1.3), Lemma 2.2 and Lemma 2.3 yield positive constants C_1 and C_2 such that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,

$$a_\varepsilon(x,t,\xi) \cdot \xi \ge \frac{|\xi|^2}{2} - C_1 |\nabla c_\varepsilon(x,t)|^2 - C_1 \qquad \text{for all } (x,t,\xi) \in \Omega \times (0,T) \times \mathbb{R}^N$$

and

$$|a_\varepsilon(x,t,\xi)| \le |\xi| + C_2 |\nabla c_\varepsilon(x,t)| + C_2 \qquad \text{for all } (x,t,\xi) \in \Omega \times (0,T) \times \mathbb{R}^N$$

as well as

$$|b_\varepsilon(x,t)| \le C_3 \qquad \text{for all } (x,t) \in \Omega \times (0,T).$$

Since (2.2) provides a bound for $|\nabla c_\varepsilon|^2$ in $L^{\frac{\lambda}{2}}((0,T); L^{\frac{q}{2}}(\Omega))$, with the exponents therein satisfying $\frac{2}{\lambda} + \frac{N}{2\cdot\frac{q}{2}} = \frac{2}{\lambda} + \frac{N}{q} < 1$ by (1.10), the estimate (3.1) directly results on applying a standard result on Hölder regularity in scalar parabolic equations ([33, Theorem 1.3, Remark 1.4]). $\Box$

Thanks to standard Schauder estimates for the Stokes system, the latter directly entails bounds for u_ε even in higher-order Hölder spaces, at least locally away from the initial time.

Lemma 3.2 *Under the assumptions of Theorem 1.1, for each $\tau \in (0,T)$ one can find $\theta \in (0,1)$ and $C > 0$ such that*

$$\|u_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[\tau,T])} \le C \qquad \textit{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}. \tag{3.2}$$

Proof. Thanks to the estimates provided by Lemma 2.2 and Lemma 2.3, this follows upon a straightforward application of well-known Schauder theory for the linear inhomogeneous Stokes evolution equation (see, for instance, Proposition 1.1 in [36]). $\Box$

As a further implication of Lemma 2.2, by way of a standard testing procedure we can also obtain further bounds for the second solution component which, if the parameter s therein is chosen large, may partially go beyond the information invested through (1.8).

Lemma 3.3 *Suppose that the assumptions of Theorem 1.1 are satisfied with some $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,\infty)$. Then for all $s \ge 2$ there exists $C(s) > 0$ such that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,*

$$\int_\Omega c_\varepsilon^s(\cdot,t) \le C(s) \qquad \textit{for all } t \in (0,T), \tag{3.3}$$

and that

$$\int_0^T \int_\Omega c_\varepsilon^{s-2} |\nabla c_\varepsilon|^2 \le C(s). \tag{3.4}$$

PROOF. We multiply the second equation in (1.2) by c_ε^{s-1} and use Young's inequality to see that since $\nabla\cdot u_\varepsilon\equiv 0$,

$$\frac{\varepsilon}{s}\frac{d}{dt}\int_\Omega c_\varepsilon^s+(s-1)\int_\Omega c_\varepsilon^{s-2}|\nabla c_\varepsilon|^2+\int_\Omega c_\varepsilon^s=\int_\Omega n_\varepsilon c_\varepsilon^{s-1}\le\frac{s-1}{s}\int_\Omega c_\varepsilon^s+\frac{1}{s}\int_\Omega n_\varepsilon^s \qquad \text{for all } t\in(0,T),$$

so that Lemma 2.2 implies the existence of $C_1>0$ such that for all $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}$, $y_\varepsilon(t):=\int_\Omega c_\varepsilon^s(\cdot,t),\ t\in[0,T]$, satisfies

$$\varepsilon y_\varepsilon'(t)+y_\varepsilon(t)+s(s-1)\int_\Omega c_\varepsilon^{s-2}|\nabla c_\varepsilon|^2\le C_1 \qquad \text{for all } t\in(0,T).$$

This firstly entails by a comparison argument, or alternatively by a direct calculation, that

$$y_\varepsilon(t)\le\max\Big\{\int_\Omega c_0^s\,,\,C_1\Big\} \qquad \text{for all } t\in(0,T) \text{ and } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}},$$

and secondly ensures upon integration that

$$s(s-1)\int_0^T\int_\Omega c_\varepsilon^{s-2}|\nabla c_\varepsilon|^2\le\varepsilon\int_\Omega c_0^s+C_1T \qquad \text{for all } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}},$$

so that both (3.3) and (3.4) directly follow, because $(\varepsilon_j)_{j\in\mathbb{N}}$ is bounded. □

Similarly, Lemma 2.2 together with the latter entails an estimate for ∇n_ε:

Lemma 3.4 *Suppose that the assumptions of Theorem 1.1 are satisfied with some $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,\infty)$. Then there exists $C>0$ such that*

$$\int_0^T\int_\Omega|\nabla n_\varepsilon|^2\le C \qquad \textit{for all } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}. \tag{3.5}$$

PROOF. Using n_ε as a test function in the first equation in (1.2) and relying on (1.4), (1.3) as well as Young's inequality, we find that again since $\nabla\cdot u_\varepsilon\equiv 0$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega n_\varepsilon^2+\int_\Omega|\nabla n_\varepsilon|^2 &= \int_\Omega n_\varepsilon(S(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon)\cdot\nabla n_\varepsilon+\int_\Omega n_\varepsilon f(x,n_\varepsilon,c_\varepsilon)\\ &\le \frac{1}{2}\int_\Omega|\nabla n_\varepsilon|^2+\frac{K_S^2}{2}\|n_\varepsilon\|_{L^\infty(\Omega)}^2\int_\Omega|\nabla c_\varepsilon|^2+K_f\int_\Omega n_\varepsilon+K_f\int_\Omega n_\varepsilon^2\end{aligned}$$

for all $t\in(0,T)$. In view of the bounds provided by Lemma 2.2 and Lemma 3.3, upon a time integration this readily yields (3.5). □

To prepare a useful ingredient for our subsequent analysis concerning the time regularity of the limit c to be obtained, we note the following weak but eventually helpful regularity information on $\partial_t n_\varepsilon$. For its formulation and for later reference, let us agree on using the abbreviation $W_N^{2,2}(\Omega):=\{\psi\in W^{2,2}(\Omega)\mid\frac{\partial\psi}{\partial\nu}=0 \text{ on } \partial\Omega\}$.

Lemma 3.5 *Under the assumptions from Theorem 1.1, there exists $C>0$ such that*

$$\int_0^T\|\partial_t n_\varepsilon(\cdot,t)\|_{(W_N^{2,2}(\Omega))^\star}^2dt\le C \qquad \textit{for all } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}. \tag{3.6}$$

PROOF. For fixed $t \in (0,T)$ and $\psi \in W_N^{2,2}(\Omega)$, using (1.2), (1.4) and (1.3) together with the Hölder inequality, we see that since $\frac{\partial\psi}{\partial\nu} = 0$ on $\partial\Omega$,

$$\begin{aligned}
\left| \int_\Omega \partial_t n_\varepsilon(\cdot,t)\psi \right| &= \left| \int_\Omega n_\varepsilon \Delta\psi + \int_\Omega n_\varepsilon (S(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon)\cdot\nabla\psi + \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla\psi + \int_\Omega f(x,n_\varepsilon,c_\varepsilon)\psi \right| \\
&\le \|n_\varepsilon\|_{L^2(\Omega)} \|\Delta\psi\|_{L^2(\Omega)} + K_S \|n_\varepsilon\|_{L^\infty(\Omega)} \|\nabla c_\varepsilon\|_{L^q(\Omega)} \|\nabla\psi\|_{L^{\frac{q}{q-1}}(\Omega)} \\
&\quad + \|n_\varepsilon\|_{L^\infty(\Omega)} \|u_\varepsilon\|_{L^r(\Omega)} \|\nabla\psi\|_{L^{\frac{r}{r-1}}(\Omega)} + K_f \|n_\varepsilon + 1\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)}
\end{aligned}$$

As the inequalities $q > N$ and $r > N$ warrant that $W_N^{2,2}(\Omega)$ is continuously embedded into both $W^{1,\frac{q}{q-1}}(\Omega)$ and $W^{1,\frac{r}{r-1}}(\Omega)$, this implies the existence of $C_1 > 0$ such that for all $t \in (0,T)$ and $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,

$$\|\partial_t n_\varepsilon(\cdot,t)\|_{(W_N^{2,2}(\Omega))^\star} \le C_1 \cdot \Big\{ \|n_\varepsilon\|_{L^\infty(\Omega)} + 1 \Big\} \cdot \Big\{ \|\nabla c_\varepsilon\|_{L^q(\Omega)} + \|u_\varepsilon\|_{L^r(\Omega)} + 1 \Big\}$$

so that (3.6) becomes a consequence of Lemma 2.2 when combined with (1.8) and (1.9) due to the fact that the exponent therein satisfies $\lambda > 2$. □

Based on the estimates collected so far, we can now extract a subsequence and identify a limit triple (n,c,u) as follows.

Lemma 3.6 *Suppose that the assumptions of Theorem 1.1 hold. Then there exist a subsequence $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ of $(\varepsilon_j)_{j\in\mathbb{N}}$, a number $\theta \in (0,1)$ and functions*

$$\begin{cases} n \in C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[0,T]), \\ c \in L^2((0,T);W^{1,2}(\Omega)) \qquad \text{and} \\ u \in C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[0,T];\mathbb{R}^N) \cap C^{2,1}(\overline{\Omega}\times(0,T];\mathbb{R}^N) \end{cases} \tag{3.7}$$

such that as $\varepsilon = \varepsilon_{j_k} \searrow 0$,

$$n_\varepsilon \to n \qquad \text{in } C^0(\overline{\Omega}\times[0,T]), \tag{3.8}$$

$$n_\varepsilon \rightharpoonup n \qquad \text{in } L^2((0,T);W^{1,2}(\Omega)), \tag{3.9}$$

$$c_\varepsilon \rightharpoonup c \qquad \text{in } L^2((0,T);W^{1,2}(\Omega)) \qquad \text{and} \tag{3.10}$$

$$u_\varepsilon \to u \qquad \text{in } C^0(\overline{\Omega}\times[0,T];\mathbb{R}^N) \cap C^{2,1}_{loc}(\overline{\Omega}\times(0,T];\mathbb{R}^N), \tag{3.11}$$

and that moreover

$$\partial_t n_\varepsilon \rightharpoonup n_t \qquad \text{in } L^2((0,T);(W_N^{2,2}(\Omega))^\star). \tag{3.12}$$

PROOF. By means of a standard subsequence extraction procedure inter alia relying on the Arzelá-Ascoli theorem, this can readily be derived from Lemma 3.1, Lemma 3.4, Lemma 3.3, Lemma 2.3, Lemma 3.2 and Lemma 3.5. □

4 Solution properties of u

Thanks to the favorable convergence features of both u_ε itself and the quantity n_ε determining the forcing term in the fluid subsystem of (1.2), it is rather evident that the limit u obtained in Lemma 3.6 indeed satisfies its respective subproblem from (1.7):

Lemma 4.1 *If the assumptions of Theorem 1.1 hold, then the functions n and u gained in Lemma 3.6 have the property that with some $P \in C^{1,0}(\Omega \times (0,T))$ we have*

$$u_t + \kappa(u \cdot \nabla)u = \Delta u + \nabla P + n\nabla\phi, \quad \nabla \cdot u = 0 \qquad \textit{for all } x \in \Omega \textit{ and } t \in (0,T), \tag{4.1}$$

and that $u(x,t) = 0$ for all $x \in \partial\Omega$ and $t \in (0,T)$.

Proof. In view of (1.2) and the convergence properties in (3.8) and (3.11), this follows from arguments well-established in the theory of the Navier-Stokes equations (see, for instance, Chapter V in [35]). □

5 Regularity and solution properties of c. Strong convergence of c_ε

In view of the singular limit taken when passing from (1.2) to (1.7), it may not be surprising that corresponding questions concerning regularity in the limit process $c_\varepsilon \to c$, as well as solution properties of the obtained limit, are more delicate. Indeed, for appropriately taking $\varepsilon \searrow 0$ in nonlinear expressions involving the second solution component, and especially in the taxis term in (1.2), the yet weak convergence information in (3.10) seems insufficient.

5.1 Hölder regularity of $c, \nabla c$ and D^2c. Solution properties of c

Suitable improvement of our knowledge in this respect will form the goal of this key section, and our analysis in this direction will be launched by the following observation on validity of the Neumann problem for second equation in (1.7) at least in some weak sense.

Lemma 5.1 *Let the hypotheses from Theorem 1.1 be satisfied, and let n, c and u be as provided by Lemma 3.6. Then there exists a null set $N \subset (0,T)$ such that for all $t \in (0,T) \setminus N$, $c(\cdot,t) \in W^{1,2}(\Omega)$ with*

$$\int_\Omega \nabla c \cdot \nabla\psi + \int_\Omega c\psi = \int_\Omega n\psi - \int_\Omega (u \cdot \nabla c)\psi \qquad \textit{for all } \psi \in W^{1,2}(\Omega). \tag{5.1}$$

Proof. Let us first make sure that for all $\varphi \in C_0^\infty(\overline{\Omega} \times (0,T))$ we have

$$\int_0^T \int_\Omega \nabla c \cdot \nabla\varphi + \int_0^T \int_\Omega c\varphi = \int_0^T \int_\Omega n\varphi - \int_0^T \int_\Omega (u \cdot \nabla c)\varphi. \tag{5.2}$$

For the verification of this, given any such φ we use the second equation in (1.2) to see that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,

$$-\varepsilon \int_0^T \int_\Omega c_\varepsilon \varphi_t + \int_0^T \int_\Omega \nabla c_\varepsilon \cdot \nabla\varphi + \int_0^T \int_\Omega c_\varepsilon\varphi = \int_0^T \int_\Omega n_\varepsilon\varphi - \int_0^T \int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon)\varphi. \tag{5.3}$$

Since (3.10) and (3.8) warrant that with $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ as found in Lemma 3.6 we have

$$\int_0^T\int_\Omega \nabla c_\varepsilon\cdot\nabla\varphi \to \int_0^T\int_\Omega \nabla c\cdot\nabla\varphi, \quad \int_0^T\int_\Omega c_\varepsilon\varphi \to \int_0^T\int_\Omega c\varphi \quad \text{and} \quad \varepsilon\int_0^T\int_\Omega c_\varepsilon\varphi_t \to 0$$

as well as

$$\int_0^T\int_\Omega n_\varepsilon\varphi \to \int_0^T\int_\Omega n\varphi$$

as $\varepsilon=\varepsilon_{j_k}\searrow 0$, and since combining (3.10) with (3.11) yields

$$\int_0^T\int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)\varphi \to \int_0^T\int_\Omega (u\cdot\nabla c)\varphi \qquad \text{as } \varepsilon=\varepsilon_{j_k}\searrow 0,$$

the identity (5.2) indeed results from (5.3).

We next rely on the separability of $W^{1,2}(\Omega)$ and a mollification argument in fixing $(\psi_i)_{i\in\mathbb{N}}\subset C^\infty(\overline{\Omega})$ such that $X_0:=\{\psi_i \mid i\in\mathbb{N}\}$ is dense in $W^{1,2}(\Omega)$, and thereupon use that all the functionals $\zeta_i^{(\iota)}$, $i\in\mathbb{N}, \iota\in\{1,2,3\}$, defined by

$$\zeta_i^{(1)}(t) := \int_\Omega \nabla c(\cdot,t)\cdot\nabla\psi_i, \quad \zeta_i^{(2)}(t) := \int_\Omega c(\cdot,t)\psi_i \quad \text{and} \quad \zeta_i^{(3)}(t) := \int_\Omega (u(\cdot,t)\cdot\nabla c(\cdot,t))\psi_i$$

for $t\in(0,T)$ and $i\in\mathbb{N}$, belong to $L^1((0,T))$. Therefore, namely, for each $i\in\mathbb{N}$ we can fix a null set $\mathcal{N}_i\subset(0,T)$ such that any $t\in(0,T)\setminus\mathcal{N}_i$ is a Lebesgue point of $\zeta_i^{(\iota)}$ for $\iota\in\{1,2,3\}$, whence letting $\mathcal{N}:=\Big(\bigcup_{i\in\mathbb{N}}\mathcal{N}_i\Big)\cup\{t\in(0,T)\mid c(\cdot,t)\notin W^{1,2}(\Omega)\}$ we have found a null set $\mathcal{N}\subset(0,T)$ such that $(0,T)\setminus\mathcal{N}$ exclusively contains common Lebesgue points of all $\zeta_i^{(\iota)}$ for $i\in\mathbb{N}$ and $\iota\in\{1,2,3\}$, and such that moreover $c(\cdot,t)\in W^{1,2}(\Omega)$ for all $t\in(0,T)\setminus\mathcal{N}$.

Now for fixed $t_0\in(0,T)\setminus\mathcal{N}$ and $h\in(0,T-t_0)$ we choose $(\chi_l)_{l\in\mathbb{N}}\subset C_0^\infty((0,T))$ such that

$$\chi_l \overset{\star}{\rightharpoonup} \chi_{(t_0,t_0+h)} \quad \text{in } L^\infty((0,T)) \qquad \text{as } l\to\infty, \tag{5.4}$$

where as usual $\chi_{(t_0,t_0+h)}$ denotes the characteristic function of the set (t_0,t_0+h), and apply (5.2) for fixed $l\in\mathbb{N}$ and $\psi\in X_0$ to $\varphi(x,t):=\chi_l(t)\cdot\psi(x)$, $(x,t)\in\Omega\times(0,T)$, to see that

$$\int_0^T\int_\Omega \chi_l\nabla c\cdot\nabla\psi + \int_0^T\int_\Omega \chi_l c\psi = \int_0^T\int_\Omega \chi_l n\psi - \int_0^T\int_\Omega \chi_l(u\cdot\nabla c)\psi \qquad \text{for all } l\in\mathbb{N},$$

by (5.4) implying that

$$\frac{1}{h}\int_{t_0}^{t_0+h}\int_\Omega \nabla c\cdot\nabla\psi + \frac{1}{h}\int_{t_0}^{t_0+h}\int_\Omega c\psi = \frac{1}{h}\int_{t_0}^{t_0+h}\int_\Omega n\psi - \frac{1}{h}\int_{t_0}^{t_0+h}\int_\Omega (u\cdot\nabla c)\psi \quad \text{for all } h\in(0,T-t_0).$$

Thanks to the Lebesgue point property of t_0 as well as the continuity of n in $\overline{\Omega}\times(0,T)$ asserted by Lemma 3.6, we may let $h\searrow 0$ here to see that

$$\int_\Omega \nabla c(\cdot,t_0)\cdot\nabla\psi + \int_\Omega c(\cdot,t_0)\psi = \int_\Omega n(\cdot,t_0)\psi - \int_\Omega (u(\cdot,t_0)\cdot\nabla c(\cdot,t_0))\psi \qquad \text{for all } \psi\in X_0,$$

which, by density of X_0 in $W^{1,2}(\Omega)$, upon a further approximation argument readily entails (5.1). $\Box$

Due to our knowledge on Hölder continuity of n and u, the identity (5.1) can be seen to entail that c actually enjoys some further regularity properties.

Lemma 5.2 *Under the assumptions of Theorem 1.1 and with c and $\mathcal{N}$ taken from Lemma 3.6 and Lemma 5.1, one can find $\theta \in (0,1)$ and $C > 0$ fulfilling*

$$\|c(\cdot,t)\|_{W^{1,2}(\Omega)} \le C \qquad \textit{for all } t \in (0,T) \setminus \mathcal{N} \tag{5.5}$$

and

$$\|c(\cdot,t) - c(\cdot,s)\|_{W^{1,2}(\Omega)} \le C|t-s|^\theta \qquad \textit{for all } t \in (0,T) \setminus \mathcal{N} \textit{ and } s \in (0,T) \setminus \mathcal{N}. \tag{5.6}$$

In particular, on redefining $c(\cdot,t)$ for $t \in \mathcal{N} \cup \{0,T\}$ if necessary, we can achieve that

$$c \in C^\theta([0,T]; W^{1,2}(\Omega)). \tag{5.7}$$

Proof. We first observe that for $t \in (0,T) \setminus \mathcal{N}$ we may apply (5.1) to $\psi := c(\cdot,t) \in W^{1,2}(\Omega)$ to see that due to Young's inequality,

$$\begin{aligned}
\int_\Omega |\nabla c(\cdot,t)|^2 + \int_\Omega c^2(\cdot,t) &= \int_\Omega n(\cdot,t)c(\cdot,t) - \int_\Omega (u(\cdot,t)\cdot\nabla c(\cdot,t))c(\cdot,t) \\
&= \int_\Omega n(\cdot,t)c(\cdot,t) \\
&\le \frac{1}{2}\int_\Omega c^2(\cdot,t) + \frac{1}{2}\int_\Omega n^2(\cdot,t),
\end{aligned}$$

because $\nabla \cdot u(\cdot,t) \equiv 0$ in Ω. By boundedness of n in $\Omega \times (0,T)$, as implied by Lemma 3.6, this directly establishes (5.5).

Next, for fixed $t \in (0,T) \setminus \mathcal{N}$ and $s \in (0,T) \setminus \mathcal{N}$, we let $z(x) := c(x,t) - c(x,s)$, $x \in \Omega$. Then $z \in W^{1,2}(\Omega)$ by Lemma 5.1, whence z is an admissible test function in (5.1) evaluated both at t and at s. Subtracting the respectively obtained identities

$$\int_\Omega \nabla c(\cdot,t)\cdot\nabla z + \int_\Omega c(\cdot,t)z = \int_\Omega n(\cdot,t)z - \int_\Omega (u(\cdot,t)\cdot\nabla c(\cdot,t))z$$

and

$$\int_\Omega \nabla c(\cdot,s)\cdot\nabla z + \int_\Omega c(\cdot,s)z = \int_\Omega n(\cdot,s)z - \int_\Omega (u(\cdot,s)\cdot\nabla c(\cdot,s))z,$$

we thus obtain that

$$\int_\Omega |\nabla z|^2 + \int_\Omega z^2 = \int_\Omega (n(\cdot,t)-n(\cdot,s))z - \int_\Omega \Big\{(u(\cdot,t)-u(\cdot,s))\cdot\nabla c(\cdot,t)\Big\}\cdot z - \int_\Omega (u(\cdot,s)\cdot\nabla z)\cdot z,$$

whence if according to Lemma 3.6 and (5.5) we let $\theta_1 \in (0,1)$, $C_1 > 0$, $C_2 > 0$ and $C_3 > 0$ be such that

$$|n(x,\tilde t) - n(x,\tilde s)| \le C_1|\tilde t - \tilde s|^{\frac{\theta_1}{2}} \quad \text{and} \quad |u(x,\tilde t) - u(x,\tilde s)| \le C_2|\tilde t - \tilde s|^{\frac{\theta_1}{2}}$$
$$\text{for all } x \in \Omega,\ \tilde t \in (0,T) \text{ and } \tilde s \in (0,T)$$

as well as

$$\|\nabla c(\cdot,\tilde{t})\|_{L^2(\Omega)} \le C_3 \qquad \text{for all } \tilde{t} \in (0,T) \setminus \mathcal{N},$$

then it follows that

$$\int_\Omega |\nabla c(\cdot,t) - \nabla c(\cdot,s)|^2 + \frac{1}{2}\int_\Omega (c(\cdot,t) - c(\cdot,s))^2 \le C_1^2|\Omega| \cdot |t-s|^{\theta_1} + C_2^2 C_3^2 |t-s|^{\theta_1}$$

and that thus (5.6) holds. The conclusion (5.7) thereby becomes evident. $\square$

A second stage of our bootstrap-type argument now even yields some spatial $C^{2+\theta}$ regularity information, as well as validity of the sub-problem of (1.7) in question in the classical sense:

Lemma 5.3 *Suppose that the assumptions from Theorem 1.1 hold, and let n, c and u be as in Lemma 3.6. Then there exist $\theta \in (0,1)$ and $C > 0$ such that*

$$\|c(\cdot,t)\|_{C^{2+\theta}(\overline{\Omega})} \le C \qquad \textit{for all } t \in (0,T). \tag{5.8}$$

Moreover,

$$-\Delta c + c = n - u \cdot \nabla c \qquad \textit{for all } x \in \Omega \textit{ and } t \in (0,T) \tag{5.9}$$

as well as

$$\frac{\partial c}{\partial \nu} = 0 \qquad \textit{for all } x \in \partial\Omega \textit{ and } t \in (0,T) \tag{5.10}$$

in the classical sense.

Proof. Let us first make sure that there exist $q_\star > N$ and $C_1 > 0$ such that with $\mathcal{N} \subset (0,T)$ taken from Lemma 5.1 we have

$$\|c(\cdot,t)\|_{W^{2,q_\star}(\Omega)} \le C_1 \qquad \text{for all } t \in (0,T) \setminus \mathcal{N}. \tag{5.11}$$

To see this, we observe that since trivially $\frac{N}{(N-2)_+} > 1$, it is possible to fix a positive integer k_0 and numbers $q_1, ..., q_{k_0}$ such that $q_1 = 2, q_{k_0} > N$ and $q_k \le q_{k+1} < \frac{Nq_k}{(n-q_k)_+}$ whenever $1 \le k < k_0$. Then since Lemma 3.6 and Lemma 5.2 warrant the existence of $\theta_1 \in (0,1)$, $C_2 > 0$, $C_3 > 0$ and $C_4 > 0$ such that

$$\|n(\cdot,t)\|_{C^{\theta_1}(\overline{\Omega})} \le C_2, \quad \|u(\cdot,t)\|_{C^{\theta_1}(\overline{\Omega})} \le C_3 \quad \text{and} \quad \|\nabla c(\cdot,t)\|_{L^2(\Omega)} \le C_4 \qquad \text{for all } t \in (0,T), \tag{5.12}$$

it follows that $h(\cdot,t) := n(\cdot,t) - u(\cdot,t)\cdot\nabla c(\cdot,t)$, $t \in (0,T)$, has the property that by definition of q_1,

$$\|h(\cdot,t)\|_{L^{q_1}(\Omega)} \le C_5 := C_2 + C_3C_4 \qquad \text{for all } t \in (0,T).$$

As from Lemma 5.1 we know that for each $t \in (0,T) \setminus \mathcal{N}$, $c(\cdot,t) \in W^{1,2}(\Omega)$ is a weak solution, in the standard sense specified in (5.1), of the Neumann boundary value problem for $-\Delta c(\cdot,t) + c(\cdot,t) = h(\cdot,t)$ in Ω, elliptic estimates therefor (see, for instance, Chapter 9 in [11]) provide $C_6 > 0$ such that

$$\|c(\cdot,t)\|_{W^{2,q_1}(\Omega)} \le C_6\|h(\cdot,t)\|_{L^{q_1}(\Omega)} \le C_5C_6 \qquad \text{for all } t \in (0,T) \setminus \mathcal{N}.$$

In the case $N=1$ in which $q_1>N$, this already establishes (5.11), while if $N\ge 2$ and hence $k_0>1$, the inequality $q_2<\frac{Nq_1}{(N-q_1)_+}$ ensures continuity of the embedding $W^{2,q_1}(\Omega)\hookrightarrow W^{1,q_2}(\Omega)$, whence (5.12) actually implies boundedness of $(h(\cdot,t))_{t\in(0,T)\setminus\mathcal{N}}$ in $L^{q_2}(\Omega)$.

Repeating this procedure, after finitely many steps we conclude that indeed (5.11) holds with $q_\star:=q_{k_0}$ and some appropriately large $C_1>0$. Since $q_\star>N$, in view of the continuous embedding $W^{2,q_\star}(\Omega)\hookrightarrow C^{1+\theta_2}(\overline{\Omega})$ for any fixed $\theta_2\in(0,1-\frac{N}{q_\star})$ this entails boundedness of $(\nabla c(\cdot,t))_{t\in(0,T)\setminus\mathcal{N}}$ in $C^{\theta_2}(\overline{\Omega})$ and thus, again through (5.12), of $(h(\cdot,t))_{t\in(0,T)\setminus\mathcal{N}}$ in $C^{\theta_3}(\overline{\Omega})$ for some $\theta_3\in(0,1)$. Now elliptic Schauder theory (see, for instance, Chapter 6 in [11]) applies to the Neumann boundary value problem for $-\Delta c(\cdot,t)+c(\cdot,t)=h(\cdot,t)$ in Ω and provides $C_7>0$ such that

$$\|c(\cdot,t)\|_{C^{2+\theta_3}(\overline{\Omega})}\le C_7 \qquad \text{for all } t\in(0,T)\setminus\mathcal{N},$$

which due to the time continuity property expressed in (5.7) clearly extends so as to remain valid actually for all $t\in(0,T)$. This clearly entails (5.8) and, as a consequence of (5.1), also (5.9) and thus (5.10). □

By straightforward interpolation, combining the latter two lemmata finally provides Hölder continuity also in time of $c,\nabla c$ and D^2c.

Lemma 5.4 *Under the assumptions from Theorem 1.1, there exist $\theta\in(0,1)$ and $C>0$ such that*

$$\|c(\cdot,t)-c(\cdot,s)\|_{C^{2+\theta}(\overline{\Omega})}\le C|t-s|^\theta \qquad \textit{for all } t\in(0,T) \textit{ and } s\in(0,T), \tag{5.13}$$

where c is taken from Lemma 3.6.

Proof. In line with Lemma 5.2, let us fix $\theta_1\in(0,1)$ such that $c\in C^{\theta_1}([0,T];W^{1,2}(\Omega))$, and thereafter choose any $\theta_2\in(0,\theta_1)$. Then by straightforward interpolation, we can find $a\in(0,1)$ and $C_1>0$ such that

$$\|\psi\|_{C^{2+\theta_2}(\overline{\Omega})}\le C_1\|\psi\|^a_{C^{2+\theta_1}(\overline{\Omega})}\|\psi\|^{1-a}_{W^{1,2}(\Omega)} \qquad \text{for all } \psi\in C^{2+\theta_1}(\overline{\Omega}).$$

Therefore,

$$\|c(\cdot,t)-c(\cdot,s)\|_{C^{2+\theta_2}(\overline{\Omega})}\le C_1\cdot\Big\{\|c(\cdot,t)\|_{C^{2+\theta_1}(\overline{\Omega})}+\|c(\cdot,s)\|_{C^{2+\theta_1}(\overline{\Omega})}\Big\}^a\cdot\|c(\cdot,t)-c(\cdot,s)\|^{1-a}_{W^{1,2}(\Omega)}$$

for all $t\in(0,T)$ and $s\in(0,T)$, so that the claim readily results from Lemma 5.3 and Lemma 5.2 if we let $\theta:=(1-a)\theta_1$ and take $C>0$ appropriately large. □

5.2 Regularity of c_t. Strong convergence properties of c_ε and ∇c_ε

Now an observation crucial for our derivation of strong convergence properties of c_ε is contained in the following statement on local L^2 integrability of c_t away from $t=0$. Using the validity of the elliptic subproblem in (1.7) for c as a starting point, besides relying on the local boundedness of u_t in $\overline{\Omega}\times(0,T]$ our argument essentially utilizes the time regularity information on n_t provided by Lemma 3.6 through Lemma 3.5.

Lemma 5.5 *Under the assumptions from Theorem 1.1, the function c obtained in Lemma 3.6 satisfies*

$$c_t \in L^2_{loc}(\overline{\Omega} \times (0,T]). \tag{5.14}$$

PROOF. We pick $\tau \in (0,T)$ and $h_0 \in (0, T-\tau)$, and for $h \in (0,h_0)$ we let

$$z_h(x,t) := \frac{c(x,t+h) - c(x,t)}{h}, \qquad x \in \Omega, \ t \in (\tau, T-h_0).$$

Then using Lemma 5.3 we see that for each fixed $t \in (\tau, T-h_0)$, $z_h(\cdot,t) \in C^2(\overline{\Omega})$ is a classical solution of the Neumann boundary value problem for

$$\begin{aligned} -\Delta z_h(\cdot,t) + z_h(\cdot,t) &= \frac{1}{h} \cdot \Big\{ n(\cdot,t+h) - u(\cdot,t+h) \cdot \nabla c(\cdot,t+h) \Big\} - \frac{1}{h} \cdot \Big\{ n(\cdot,t) - u(\cdot,t) \cdot \nabla c(\cdot,t) \Big\} \\ &= g_h(\cdot,t) - u(\cdot,t) \cdot \nabla z_h(\cdot,t) \qquad \text{in } \Omega, \end{aligned} \tag{5.15}$$

where for $h \in (0,h_0)$,

$$g_h(x,t) := \frac{n(x,t+h) - n(x,t)}{h} - \frac{u(x,t+h) - u(x,t)}{h} \cdot \nabla c(x,t+h), \qquad x \in \Omega, \ t \in (\tau, T-h_0).$$

Next, for $h \in (0,h_0)$ and $t \in (0, T-h_0)$ we furthermore let $z_h^{(1)}(\cdot,t)$ and $z_h^{(2)}(\cdot,t)$ denote the classical solutions of

$$\begin{cases} -\Delta z_h^{(1)}(\cdot,t) + z_h^{(1)}(\cdot,t) = g_h(\cdot,t), & x \in \Omega, \\ \frac{\partial z_h^{(1)}}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{5.16}$$

and

$$\begin{cases} -\Delta z_h^{(2)}(\cdot,t) + z_h^{(2)}(\cdot,t) = -u(\cdot,t) \cdot \nabla z_h(\cdot,t), & x \in \Omega, \\ \frac{\partial z_h^{(2)}}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{5.17}$$

noting that their existence in the space $C^2(\overline{\Omega})$ is asserted by standard elliptic theory (see, for instance, Chapter 6 in [11]) due to the fact that both $g_h(\cdot,t)$ and $u(\cdot,t) \cdot \nabla z_h(\cdot,t)$ are Hölder continuous in $\overline{\Omega}$ by Lemma 3.6, Lemma 5.3 and the inclusion $z_h(\cdot,t) \in C^2(\overline{\Omega})$.

Then according to a uniqueness property of classical solutions to the Neumann problem associated with the inhomogeneous Helmholtz equation in (5.15), it follows that

$$z_h(\cdot,t) = z_h^{(1)}(\cdot,t) + z_h^{(2)}(\cdot,t) \qquad \text{for all } t \in (\tau, T-t_0), \tag{5.18}$$

and in order to successively derive L^2 bounds for $z_h^{(1)}$ and then for $z_h^{(2)}$, we first rewrite (5.16) in the form $z_h^{(1)}(\cdot,t) := B^{-1} g_h(\cdot,t)$, where B denotes the realization of $-\Delta + 1$ in $W_N^{2,2}(\Omega)$. Then since B^{-1} obviously is nonexpansive on $L^2(\Omega)$, we can estimate

$$\begin{aligned} \|z_h^{(1)}(\cdot,t)\|_{L^2(\Omega)} &\le \left\| B^{-1} \frac{n(\cdot,t+h) - n(\cdot,t)}{h} \right\|_{L^2(\Omega)} + \left\| B^{-1} \Big\{ \frac{u(\cdot,t+h) - u(\cdot,t)}{h} \cdot \nabla c(\cdot,t+h) \Big\} \right\|_{L^2(\Omega)} \\ &\le \left\| B^{-1} \frac{n(\cdot,t+h) - n(\cdot,t)}{h} \right\|_{L^2(\Omega)} + \left\| \frac{u(\cdot,t+h) - u(\cdot,t)}{h} \cdot \nabla c(\cdot,t+h) \right\|_{L^2(\Omega)} \end{aligned}$$

$$
\begin{aligned}
&\leq \left\| B^{-1}\frac{n(\cdot,t+h)-n(\cdot,t)}{h}\right\|_{L^2(\Omega)} + \left\|\frac{u(\cdot,t+h)-u(\cdot,t)}{h}\right\|_{L^\infty(\Omega)} \|\nabla c(\cdot,t+h)\|_{L^2(\Omega)} \\
&= \left\|\frac{1}{h}\int_t^{t+h} B^{-1}n_t(\cdot,s)ds\right\|_{L^2(\Omega)} + \left\|\frac{1}{h}\int_t^{t+h} u_t(\cdot,s)ds\right\|_{L^\infty(\Omega)} \cdot \|\nabla c(\cdot,t+h)\|_{L^2(\Omega)} \\
&\qquad\qquad\qquad \text{for all } t\in(\tau,T-h_0) \text{ and any } h\in(0,h_0).
\end{aligned}
$$

Since u_t is bounded in $\Omega\times(\tau,T)$ by Lemma 3.6, and since Lemma 5.2 implies boundedness of $(0,T)\ni t\mapsto \|\nabla c(\cdot,t)\|_{L^2(\Omega)}$, we thus obtain $C_1=C_1(\tau)>0$ such that for all $h\in(0,h_0)$,

$$
\|z_h^{(1)}(\cdot,t)\|_{L^2(\Omega)}^2 \leq 2\left\|\frac{1}{h}\int_t^{t+h} B^{-1}n_t(\cdot,s)ds\right\|_{L^2(\Omega)}^2 + C_1 \qquad \text{for all } t\in(\tau,T-h_0).
$$

Therefore, by integration using the Fubini theorem as well as the Cauchy-Schwarz inequality,

$$
\begin{aligned}
\int_\tau^{T-h_0} \|z_h^{(1)}(\cdot,t)\|_{L^2(\Omega)}^2 dt &\leq C_1T + \frac{2}{h^2}\int_\tau^{T-h_0}\left\|\int_t^{t+h} B^{-1}n_t(\cdot,s)ds\right\|_{L^2(\Omega)}^2 dt \\
&\leq C_1T + \frac{2}{h}\int_\tau^{T-h_0}\int_t^{t+h} \|B^{-1}n_t(\cdot,s)\|_{L^2(\Omega)}^2 dsdt \\
&= C_1T + \frac{2}{h}\int_\tau^{\tau+h}\int_\tau^{s} \|B^{-1}n_t(\cdot,s)\|_{L^2(\Omega)}^2 dtds \\
&\quad + \frac{2}{h}\int_{\tau+h}^{T-h_0}\int_{s-h}^{s} \|B^{-1}n_t(\cdot,s)\|_{L^2(\Omega)}^2 dtds \\
&\quad + \frac{2}{h}\int_{T-h_0}^{T-h_0+h}\int_{s-h}^{T-h_0} \|B^{-1}n_t(\cdot,s)\|_{L^2(\Omega)}^2 dtds \\
&\leq C_1T + 2\int_\tau^{T} \|B^{-1}n_t(\cdot,s)\|_{L^2(\Omega)}^2 ds \qquad \text{for all } h\in(0,T-h_0-\tau).
\end{aligned}
$$

Now since standard elliptic regularity theory (see, for instance, Theorem 21.1 in [57]) ensures that B^{-1} maps $L^2(\Omega)$ continuously into $W_N^{2,2}(\Omega)$, and that thus with some $C_2>0$ we have $\|B^{-1}\psi\|_{L^2(\Omega)} \leq C_2\|\psi\|_{(W_N^{2,2}(\Omega))^\star}$ for all $\psi\in(W_N^{2,2}(\Omega))^\star$, this entails that

$$
\int_\tau^{T-h_0}\int_\Omega (z_h^{(1)})^2 \leq C_3 = C_3(\tau) := C_1T + 2C_2^2\int_\tau^T \|n_t(\cdot,t)\|_{(W_N^{2,2}(\Omega))^\star}^2 dt \tag{5.19}
$$

for all $h\in 0,T-h_0-\tau)$ with C_3 being finite thanks to Lemma 3.5.

Next, in order to estimate $z_h^{(2)}$, we test (5.17) against $z_h^{(2)}(\cdot,t)$ and recall the decomposition (5.18) to infer that by solenoidality of u and by Young's inequality,

$$
\begin{aligned}
\int_\Omega |\nabla z_h^{(2)}(\cdot,t)|^2 + \int_\Omega (z_h^{(2)}(\cdot,t))^2 &= -\int_\Omega (u(\cdot,t)\cdot\nabla z_h^{(1)}(\cdot,t))z_h^{(2)}(\cdot,t) - \int_\Omega (u(\cdot,t)\cdot\nabla z_h^{(2)}(\cdot,t))z_h^{(2)}(\cdot,t) \\
&= \int_\Omega z_h^{(1)}(\cdot,t)(u(\cdot,t)\cdot\nabla z_h^{(2)}(\cdot,t))
\end{aligned}
$$

$$\begin{aligned}
&\le \int_\Omega |\nabla z_h^{(2)}(\cdot,t)|^2 + \frac{1}{4}\int_\Omega (z_h^{(1)}(\cdot,t))^2 |u(\cdot,t)|^2 \\
&\le \int_\Omega |\nabla z_h^{(2)}(\cdot,t)|^2 + \frac{1}{4}\|u(\cdot,t)\|_{L^\infty(\Omega)}^2 \int_\Omega (z_h^{(1)}(\cdot,t))^2 \\
&\qquad \text{for all } t \in (\tau, T-h_0) \text{ and each } h \in (0, T-h_0-\tau).
\end{aligned}$$

As once more relying on Lemma 3.6 we can find $C_4 > 0$ such that $\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C_4$ for all $t \in (0,T)$, in view of (5.19) this implies that

$$\int_\tau^{T-h_0} \int_\Omega (z_h^{(2)})^2 \le \frac{C_4^2}{4} \int_\tau^{T-h_0} \int_\Omega (z_h^{(1)})^2 \le \frac{C_3 C_4^2}{4} \qquad \text{for all } h \in (0, T-h_0-\tau)$$

and that thus, again by (5.19),

$$\int_\tau^{T-h_0} \int_\Omega z_h^2 \le 2C_3 + \frac{C_3 C_4^2}{2} \qquad \text{for all } h \in (0, T-h_0-\tau)$$

because of (5.18). Consequently, there exist $(h_i)_{i\in\mathbb{N}} \subset (0, T-h_0-\tau)$ and $z \in L^2(\Omega \times (\tau, T-h_0))$ such that $h_i \to 0$ and $z_{h_i} \rightharpoonup z$ in $L^2(\Omega \times (\tau, T-h_0))$ as $i \to \infty$, so that since by definition of distributional derivatives z must coincide with c_t a.e. in $\Omega \times (\tau, T-h_0)$, (5.14) results from the fact that $\tau \in (0,T)$ and $h_0 \in (0, T-\tau)$ were arbitrary. $\square$

On the basis of this, we can now in fact derive some strong convergence property of c_ε by analyzing the difference $c_\varepsilon - c$ through a parabolic equation satisfied by the latter, in which the crucial source term εc_t can appropriately be controlled using (5.14).

Lemma 5.6 *Suppose that the assumptions of Theorem 1.1 are satisfied with some $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,\infty)$, and let (n,c,u,P) and $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ be as provided by Lemma 3.6. Then*

$$c_\varepsilon \to c \qquad \text{in } L^\infty_{loc}((0,T]; L^2(\Omega)) \tag{5.20}$$

and

$$\nabla c_\varepsilon \to \nabla c \qquad \text{in } L^2_{loc}(\overline{\Omega} \times (0,T]) \tag{5.21}$$

as $\varepsilon = \varepsilon_{j_k} \searrow 0$.

Proof. For $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, we let

$$z_\varepsilon(x,t) := c_\varepsilon(x,t) - c(x,t), \qquad (x,t) \in \Omega \times (0,T),$$

and

$$y_\varepsilon(t) := \int_\Omega z_\varepsilon^2(x,t)\,dx, \qquad t \in (0,T).$$

Then since $z_\varepsilon \in L^\infty(\Omega \times (0,T))$ and $\partial_t z_\varepsilon = \partial_t c_\varepsilon - c_t \in L^2_{loc}((0,T]; L^2(\Omega))$ by Lemma 2.4 with $q=2$ and Lemma 5.5, it follows from a standard argument that y_ε belongs to $W^{1,2}_{loc}((0,T])$ and is therefore locally absolutely continuous in $(0,T]$ with

$$y_\varepsilon'(t) = 2\int_\Omega z_\varepsilon(\cdot,t)\partial_t z_\varepsilon(\cdot,t) \qquad \text{for a.e. } t \in (0,T).$$

As herein by (1.2) and Lemma 5.3,

$$\begin{aligned}
\varepsilon\partial_t z_\varepsilon &= \Delta c_\varepsilon - c_\varepsilon + n_\varepsilon - u_\varepsilon\cdot\nabla c_\varepsilon - \varepsilon c_t \\
&= \Delta z_\varepsilon + \Delta c - z_\varepsilon - c + n_\varepsilon - u_\varepsilon\cdot\nabla c_\varepsilon - \varepsilon c_t \\
&= \Delta z_\varepsilon - z_\varepsilon + (n_\varepsilon - n) - (u_\varepsilon - u)\cdot\nabla c - u_\varepsilon\cdot\nabla z_\varepsilon - \varepsilon c_t \qquad \text{a.e. in } \Omega \text{ for a.e. } t\in(0,T),
\end{aligned}$$

on integrating by parts and using Young's inequality we obtain that

$$\begin{aligned}
&\frac{\varepsilon}{2}y_\varepsilon'(t) + \int_\Omega |\nabla z_\varepsilon|^2 + \int_\Omega z_\varepsilon^2 \\
&\qquad = \int_\Omega (n_\varepsilon - n)z_\varepsilon - \int_\Omega \Big\{(u_\varepsilon - u)\cdot\nabla c\Big\} z_\varepsilon - \int_\Omega (u_\varepsilon\cdot\nabla z_\varepsilon)z_\varepsilon - \varepsilon\int_\Omega c_t z_\varepsilon \\
&\qquad \le \frac{3}{4}\int_\Omega z_\varepsilon^2 + \int_\Omega (n_\varepsilon - n)^2 + \int_\Omega |u_\varepsilon - u|^2|\nabla c|^2 + \varepsilon^2\int_\Omega c_t^2 \qquad \text{for a.e. } t\in(0,T),
\end{aligned}$$

because $\nabla\cdot u_\varepsilon \equiv 0$ a.e. in $\Omega\times(0,T)$. Hence

$$\frac{\varepsilon}{2}y_\varepsilon'(t) + \frac{1}{4}y_\varepsilon(t) + \int_\Omega |\nabla z_\varepsilon|^2 \le h_\varepsilon + \varepsilon^2\int_\Omega c_t^2(\cdot,t) \qquad \text{for a.e. } t\in(0,T), \tag{5.22}$$

where according to Lemma 3.6,

$$h_\varepsilon := |\Omega|\cdot\|n_\varepsilon - n\|^2_{L^\infty(\Omega\times(0,T))} + \|u_\varepsilon - u\|^2_{L^\infty(\Omega\times(0,T))}\cdot\|\nabla c\|^2_{L^\infty((0,T);L^2(\Omega))}$$

satisfies

$$h_\varepsilon \to 0 \qquad \text{as } \varepsilon = \varepsilon_{j_k} \searrow 0. \tag{5.23}$$

Given any $\tau\in(0,T)$ and an arbitrary $\eta>0$ we use (5.23) and Lemma 5.5 along with the boundedness of $(y_\varepsilon)_{\varepsilon\in(\varepsilon_{j_k})_{k\in\mathbb{N}}}$ in $L^\infty((0,T))$, as entailed by Lemma 3.3 and Lemma 5.3, to fix $\varepsilon_0>0$ small enough such that whenever $\varepsilon\in(\varepsilon_{j_k})_{k\in\mathbb{N}}$ is such that $\varepsilon<\varepsilon_0$, we have

$$4h_\varepsilon \le \frac{\eta}{3} \tag{5.24}$$

and

$$2\varepsilon\int_{\frac{\tau}{2}}^T\int_\Omega c_t^2 \le \frac{\eta}{3} \tag{5.25}$$

as well as

$$y_\varepsilon\Big(\frac{\tau}{2}\Big)\cdot e^{-\frac{\tau}{4\varepsilon}} \le \frac{\eta}{3}. \tag{5.26}$$

Then from (5.22) we infer on dropping the nonnegative last summand on the left that the absolutely continuous function $[\frac{\tau}{2},T]\ni t\mapsto e^{\frac{1}{2\varepsilon}(t-\frac{\tau}{2})}y_\varepsilon(t)$ satisfies

$$\frac{d}{dt}\Big\{e^{\frac{1}{2\varepsilon}(t-\frac{\tau}{2})}y_\varepsilon(t)\Big\} \le e^{\frac{1}{2\varepsilon}(t-\frac{\tau}{2})}\Big\{\frac{2h_\varepsilon}{\varepsilon} + 2\varepsilon\int_\Omega c_t^2(\cdot,t)\Big\} \qquad \text{for a.e. } t\in\Big(\frac{\tau}{2},T\Big),$$

and therefore we obtain using (5.26), (5.24) and (5.25) that

$$\begin{aligned}
y_\varepsilon(t) &\le y_\varepsilon\Big(\frac{\tau}{2}\Big)\cdot e^{-\frac{1}{2\varepsilon}(t-\frac{\tau}{2})} + \frac{2h_\varepsilon}{\varepsilon}\int_{\frac{\tau}{2}}^t e^{-\frac{1}{2\varepsilon}(t-s)}ds + 2\varepsilon\int_{\frac{\tau}{2}}^t e^{-\frac{1}{2\varepsilon}(t-s)}\cdot\int_\Omega c_t^2(\cdot,s)ds \\
&\le y_\varepsilon\Big(\frac{\tau}{2}\Big)\cdot e^{-\frac{1}{2\varepsilon}(t-\frac{\tau}{2})} + 4h_\varepsilon\cdot\Big\{1-e^{-\frac{1}{2\varepsilon}(t-\frac{\tau}{2})}\Big\} + 2\varepsilon\int_{\frac{\tau}{2}}^t\int_\Omega c_t^2 \\
&\le \frac{\eta}{3}+\frac{\eta}{3}+\frac{\eta}{3}=\eta \qquad \text{for all } t\in(\tau,T).
\end{aligned}$$

This in conjunction with Lemma 5.5 and Lemma 3.3 implies that for each $\tau\in(0,T)$ we have

$$z_\varepsilon\to 0 \quad \text{in } L^\infty((\tau,T);L^2(\Omega)) \qquad \text{as } \varepsilon=\varepsilon_{j_k}\searrow 0. \tag{5.27}$$

Going back to (5.22), we see upon direct integration therein that for each $\tau\in(0,T)$ we moreover have

$$\int_\tau^T\int_\Omega|\nabla z_\varepsilon|^2 \le \frac{\varepsilon}{2}y_\varepsilon(\tau)+h_\varepsilon\cdot(T-\tau)+\varepsilon^2\int_\tau^T\int_\Omega c_t^2 \qquad \text{for all } \varepsilon\in(\varepsilon_{j_k})_{k\in\mathbb{N}},$$

so that again by means of (5.23), Lemma 5.5 and the boundedness of $(y_\varepsilon)_{\varepsilon\in(\varepsilon_{j_k})_{k\in\mathbb{N}}}$ in $L^\infty((0,T))$ we infer that for any such τ,

$$z_\varepsilon\to 0 \quad \text{in } L^2((\tau,T);W^{1,2}(\Omega)) \qquad \text{as } \varepsilon=\varepsilon_{j_k}\searrow 0.$$

Together with (5.27), this shows that both (5.20) and (5.21) hold. $\square$

5.3 A bound for ∇c_ε in $L^\infty((0,T);L^{\widehat{q}}(\Omega))$ for arbitrarily large $\widehat{q}$

Let us conclude this section by providing some additional integrability information on the signal gradient ∇c_ε on the basis of the differential inequality from Lemma 2.4.

Our reasoning will involve the following elementary interpolation inequality.

Lemma 5.7 *Let $q\ge 2$. Then for all $\varphi\in C^2(\overline{\Omega})$ such that $\varphi\cdot\frac{\partial\varphi}{\partial\nu}=0$ on $\partial\Omega$, we have*

$$\int_\Omega|\nabla\varphi|^q \le (\sqrt{N}+q-2)^{\frac{2q}{q+2}}\cdot\Big\{\int_\Omega|\nabla\varphi|^{q-2}|D^2\varphi|^2\Big\}^{\frac{q}{q+2}}\cdot\Big\{\int_\Omega|\varphi|^q\Big\}^{\frac{2}{q+2}}. \tag{5.28}$$

Proof. We integrate by parts and use the Hölder inequality to see that

$$\begin{aligned}
\int_\Omega|\nabla\varphi|^q &= \int_\Omega|\nabla\varphi|^{q-2}\nabla\varphi\cdot\nabla\varphi \\
&= -\int_\Omega\varphi|\nabla\varphi|^{q-2}\Delta\varphi-(q-2)\int_\Omega\varphi|\nabla\varphi|^{q-4}\nabla\varphi\cdot(D^2\varphi\cdot\nabla\varphi) \\
&\le (\sqrt{N}+q-2)\int_\Omega|\varphi|\cdot|\nabla\varphi|^{q-2}\cdot|D^2\varphi| \\
&\le (\sqrt{N}+q-2)\Big\{\int_\Omega|\varphi|^q\Big\}^{\frac{1}{q}}\cdot\Big\{\int_\Omega|\nabla\varphi|^q\Big\}^{\frac{q-2}{2q}}\cdot\Big\{\int_\Omega|\nabla\varphi|^{q-2}|D^2\varphi|^2\Big\}^{\frac{1}{2}},
\end{aligned}$$

from which (5.28) can readily be derived. $\square$

Indeed, we can thereby achieve the following.

Lemma 5.8 *Assume the hypotheses from Theorem 1.1, and let $\widehat{q} > \max\{N, 2\}$. Then there exists $C = C(\widehat{q}) > 0$ such that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,*

$$\int_\Omega |\nabla c_\varepsilon(\cdot,t)|^{\widehat{q}} \le C \qquad \textit{for all } t \in (0,T). \tag{5.29}$$

In particular,

$$\sup_{\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}} \|c_\varepsilon\|_{L^\infty(\Omega\times(0,T))} < \infty. \tag{5.30}$$

Proof. Since $(n_\varepsilon)_{\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}}$ and $(u_\varepsilon)_{\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}}$ are bounded in $L^\infty(\Omega\times(0,T))$ and in $L^\infty(\Omega\times(0,T);\mathbb{R}^N)$ according to Lemma 2.2 and Lemma 2.3, respectively, from Lemma 2.4 we infer the existence of $C_1 = C_1(\widehat{q}) > 0$ such that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,

$$\frac{\varepsilon}{\widehat{q}}\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^{\widehat{q}} + \frac{1}{4}\int_\Omega |\nabla c_\varepsilon|^{\widehat{q}-2}|D^2 c_\varepsilon|^2 + \int_\Omega |\nabla c_\varepsilon|^{\widehat{q}} \le C_1 + C_1\int_\Omega |\nabla c_\varepsilon|^{\widehat{q}} \quad \text{for all } t\in(0,T). \tag{5.31}$$

Here we may use that Lemma 3.3 warrants boundedness of $(c_\varepsilon)_{\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}}$ in $L^\infty((0,T);L^{\widehat{q}}(\Omega))$ to see that as a consequence of Lemma 5.7 and Young's inequality, with some positive constants $C_i = C_i(\widehat{q})$, $i \in \{2,3,4\}$, we have

$$\begin{aligned}
C_1\int_\Omega |\nabla c_\varepsilon|^{\widehat{q}} &\le C_2\cdot\Big\{\int_\Omega |\nabla c_\varepsilon|^{\widehat{q}-2}|D^2 c_\varepsilon|^2\Big\}^{\frac{\widehat{q}}{\widehat{q}+2}} \cdot \Big\{\int_\Omega c_\varepsilon^{\widehat{q}}\Big\}^{\frac{2}{\widehat{q}+2}} \\
&\le C_3\cdot\Big\{\int_\Omega |\nabla c_\varepsilon|^{\widehat{q}-2}|D^2 c_\varepsilon|^2\Big\}^{\frac{\widehat{q}}{\widehat{q}+2}} \\
&\le \frac{1}{4}\int_\Omega |\nabla c_\varepsilon|^{\widehat{q}-2}|D^2 c_\varepsilon|^2 + C_4 \qquad \text{for all } t\in(0,T) \text{ and any } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}},
\end{aligned}$$

so that (5.31) entails the inequality

$$\frac{\varepsilon}{\widehat{q}}\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^{\widehat{q}} + \int_\Omega |\nabla c_\varepsilon|^{\widehat{q}} \le C_1 + C_4 \qquad \text{for all } t\in(0,T) \text{ and } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}},$$

from which (5.29) directly follows. As $\widehat{q} > N$ and hence $W^{1,\widehat{q}}(\Omega) \hookrightarrow L^\infty(\Omega)$, again due to the boundedness property from Lemma 3.3 this in turn implies (5.30). □

6 Solution properties of n. Proof of Theorem 1.1

Now knowing that $c_\varepsilon \to c$ also in the pointwise sense, we can readily pass to the limit also in the first equation in (1.2).

Lemma 6.1 *Suppose that the assumptions of Theorem 1.1 are satisfied with some $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,\infty)$, and let (n,c,u,P) be as provided by Lemma 3.6. Then in the classical pointwise sense we have*

$$n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(nS(x,n,c)\cdot\nabla c) + f(x,n,c) \qquad \textit{for all } x\in\Omega \textit{ and } t\in(0,T) \tag{6.1}$$

as well as

$$(\nabla n - nS(x,n,c)\cdot\nabla c)\cdot\nu = 0 \qquad \textit{for all } x\in\partial\Omega \textit{ and } t\in(0,T). \tag{6.2}$$

PROOF. Let us first make sure that for arbitrary $\varphi \in C_0^\infty(\overline{\Omega} \times [0,T))$,

$$-\int_0^T \int_\Omega n\varphi_t - \int_\Omega n_0\varphi(\cdot,0) = -\int_0^T \int_\Omega \nabla n \cdot \nabla\varphi + \int_0^T \int_\Omega n(S(x,n,c)\cdot\nabla c)\cdot\nabla\varphi + \int_0^T \int_\Omega nu\cdot\nabla\varphi + \int_0^T \int_\Omega f(x,n,c)\varphi. \qquad (6.3)$$

To see this, given any such φ and $\varepsilon \in (\varepsilon_j)j \in \mathbb{N}$ we use (1.2) to find that

$$-\int_0^T \int_\Omega n_\varepsilon\varphi_t - \int_\Omega n_0\varphi(\cdot,0) = -\int_0^T \int_\Omega \nabla n_\varepsilon \cdot \nabla\varphi + \int_0^T \int_\Omega n_\varepsilon(S(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon)\cdot\nabla\varphi + \int_0^T \int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\varphi + \int_0^T \int_\Omega f(x,n_\varepsilon,c_\varepsilon)\varphi, \qquad (6.4)$$

where by (3.8), (3.9) and (3.11), clearly

$$\int_0^T \int_\Omega n_\varepsilon\varphi_t \to \int_0^T \int_\Omega n\varphi_t, \qquad \int_0^T \int_\Omega \nabla n_\varepsilon\cdot\nabla\varphi \to \int_0^T \int_\Omega \nabla n\cdot\nabla\varphi \qquad \text{and}$$
$$\int_0^T \int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\varphi \to \int_0^T \int_\Omega nu\cdot\nabla\varphi$$

as $\varepsilon = \varepsilon_{j_k} \searrow 0$, where $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ is as provided by Lemma 3.6. Apart from this, thanks to Lemma 5.6 we know that $c_\varepsilon \to c$ a.e. in $\Omega\times(0,T)$ and hence, again by (3.8), that also $S(\cdot,n_\varepsilon,c_\varepsilon) \to S(\cdot,n,c)$ and $f(\cdot,n_\varepsilon,c_\varepsilon) \to f(\cdot,n,c)$ a.e. in $\Omega\times(0,T)$. Since combining (1.4) and (1.3) with (3.8) and (5.30) moreover ensures boundedness of $(S(\cdot,n_\varepsilon,c_\varepsilon))_{\varepsilon\in(\varepsilon_{j_k})_{k\in\mathbb{N}}}$ in $L^\infty(\Omega\times(0,T);\mathbb{R}^{N\times N})$ and of $(f(\cdot,n_\varepsilon,c_\varepsilon))_{\varepsilon\in(\varepsilon_{j_k})_{k\in\mathbb{N}}}$ in $L^\infty(\Omega\times(0,T))$, by means of the dominated convergence theorem and a well-known argument ([54, Lemma A.4]) we conclude that as $\varepsilon = \varepsilon_{j_k} \searrow 0$, not only $f(\cdot,n_\varepsilon,c_\varepsilon) \to f(\cdot,n,c)$ in $L^1(\Omega\times(0,T))$ but also

$$n_\varepsilon S(\cdot,n_\varepsilon,c_\varepsilon) \to nS(\cdot,n,c) \qquad \text{in } L^2(\Omega\times(0,T);\mathbb{R}^{N\times N}).$$

In conjunction with (3.10), these properties warrant that

$$\int_0^T \int_\Omega f(x,n_\varepsilon,c_\varepsilon)\varphi \to \int_0^T \int_\Omega f(x,n,c)\varphi$$

and

$$\int_0^T \int_\Omega n_\varepsilon(S(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon)\cdot\nabla\varphi \to \int_0^T \int_\Omega n(S(x,n,c)\cdot\nabla c)\cdot\nabla\varphi$$

as $\varepsilon = \varepsilon_{j_k} \searrow 0$, so that (6.3) becomes a consequence of (6.4).

Thus knowing that $n \in L^2((0,T);W^{1,2}(\Omega))$ forms a weak solution of the initial-boundary value problem (6.1)-(6.2) in the standard generalized sense from e.g. [20], we may invoke classical results from parabolic regularity theory to conclude from the Hölder continuity of $n, c, \nabla c, D^2c$ and u in $\overline{\Omega}\times[0,T]$, as stated by Lemma 3.6 and Lemma 5.4, that firstly $n \in C^{1+\theta_1,\frac{1+\theta_1}{2}}(\overline{\Omega}\times(0,T])$ for some $\theta_1 \in (0,1)$

(see, for instance, Theorem 1.1 in [23]), and that secondly, as a consequence thereof, due to this additional information on Hölder continuity of ∇n we can find $\theta_2 \in (0,1)$ such that we even have $n \in C^{2+\theta_2, 1+\frac{\theta_2}{2}}(\overline{\Omega} \times (0,T])$. Therefore, (6.3) warrants validity of (6.1) and (6.2) in the classical pointwise sense through a standard variational argument. $\square$

The proof of our main result on the parabolic-elliptic limit in (1.2) is now almost immediate:

PROOF of Theorem 1.1. With $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ and (n,c,u,P) taken as in Lemma 3.6 and Lemma 4.1, from Lemma 3.6 and Lemma 5.6 we immediately infer that (1.11), (1.12) and (1.15) hold, and that moreover

$$c_\varepsilon \to c \qquad \text{in } L^\infty_{loc}((0,T];L^2(\Omega)) \cap L^2_{loc}((0,T];W^{1,2}(\Omega)) \tag{6.5}$$

as $\varepsilon = \varepsilon_{j_k} \searrow 0$. Observing that for each fixed $\widehat{q} > N$, Lemma 5.8 provides $C_1 > 0$ such that

$$\|c_\varepsilon(\cdot,t)\|_{W^{1,\widehat{q}}(\Omega)} \le C_1 \qquad \text{for all } t \in (0,T) \text{ and each } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}, \tag{6.6}$$

we firstly obtain (1.14) as an immediate consequence thereof, and can furthermore secondly complete the derivation of (1.13): As $\widehat{q} > N$, namely, the Gagliardo-Nirenberg inequality yields $a \in (0,1)$ and $C_2 > 0$ such that for all $t \in (0,T)$ and $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,

$$\begin{aligned} \|c_\varepsilon(\cdot,t) - c(\cdot,t)\|_{C^0(\overline{\Omega})} &\le C_2 \|c_\varepsilon(\cdot,t) - c(\cdot,t)\|^a_{W^{1,\widehat{q}}(\Omega)} \|c_\varepsilon(\cdot,t) - c(\cdot,t)\|^{1-a}_{L^2(\Omega)} \\ &\le C_2 \Big\{ \|c_\varepsilon(\cdot,t)\|_{W^{1,\widehat{q}}(\Omega)} + \|c(\cdot,t)\|_{W^{1,\widehat{q}}(\Omega)} \Big\}^a \cdot \|c_\varepsilon(\cdot,t) - c(\cdot,t)\|^{1-a}_{L^2(\Omega)}, \end{aligned}$$

so that combining (6.5) with (6.6) shows that indeed $c_\varepsilon \to c$ also in $L^\infty_{loc}((0,T);C^0(\overline{\Omega}))$. Finally, Lemma 4.1 in conjunction with Lemma 5.3 and Lemma 6.1 guarantees that in fact (n,c,u,P) solves (1.7) classically in $\Omega \times (0,T)$. $\square$

7 Small-data solutions to an unforced Keller-Segel-Navier-Stokes system. Proof of Theorem 1.2

The purpose of this section consists in providing a first exemplary application of Theorem 1.1, namely in the framework of Theorem 1.2. To this end, as for general S the no-flux boundary conditions in (1.2) need not reduce to separate homogeneous Neumann boundary conditions for n_ε and c_ε, following [54] we introduce an appropriate regularization in which S vanishes near the lateral boundary. More precisely, let us fix $(\rho_\eta)_{\eta\in(0,1)} \subset C_0^\infty(\Omega)$ and $(\chi_\eta)_{\eta\in(0,1)} \subset C^\infty([0,\infty))$ such that

$$0 \le \rho_\eta \le 1 \text{ in } \Omega \quad \text{ with } \quad \rho_\eta \nearrow 1 \text{ in } \Omega \text{ as } \eta \searrow 0,$$

and that

$$0 \le \chi_\eta \le 1 \text{ in } [0,\infty) \quad \text{ with } \quad \chi_\eta \equiv 0 \text{ in } [\tfrac{1}{\eta},\infty) \text{ and } \chi_\eta \nearrow 1 \text{ in } [0,\infty) \text{ as } \eta \searrow 0. \tag{7.1}$$

For $\eta \in (0,1)$, we then define

$$S_\eta(x,n,c) := \rho_\eta(x) \cdot \chi_\eta(n) \cdot S(x,n,c), \qquad (x,n,c) \in \bar{\Omega} \times [0,\infty)^2, \tag{7.2}$$

and observe that $S_\eta \in C^2(\bar{\Omega} \times [0,\infty)^2;\mathbb{R}^{N\times N})$.

Given $\varepsilon > 0$, for $\eta \in (0,1)$ we consider the approximate versions of (1.2) given by

$$\begin{cases} \partial_t n_{\varepsilon\eta} + u_{\varepsilon\eta} \cdot \nabla n_{\varepsilon\eta} = \Delta n_{\varepsilon\eta} - \nabla \cdot \Big(n_{\varepsilon\eta} S_\eta(x, n_{\varepsilon\eta}, c_{\varepsilon\eta}) \cdot \nabla c_{\varepsilon\eta} \Big), & x \in \Omega,\ t > 0, \\ \varepsilon \partial_t c_{\varepsilon\eta} + u_{\varepsilon\eta} \cdot \nabla c_{\varepsilon\eta} = \Delta c_{\varepsilon\eta} - c_{\varepsilon\eta} + n_{\varepsilon\eta}, & x \in \Omega,\ t > 0, \\ \partial_t u_{\varepsilon\eta} + \kappa (u_{\varepsilon\eta} \cdot \nabla) u_{\varepsilon\eta} = \Delta u_{\varepsilon\eta} + \nabla P_{\varepsilon\eta} + n_{\varepsilon\eta} \nabla \phi, \qquad \nabla \cdot u_{\varepsilon\eta} = 0, & x \in \Omega,\ t > 0, \\ \frac{\partial n_{\varepsilon\eta}}{\partial \nu} = \frac{\partial c_{\varepsilon\eta}}{\partial \nu} = 0, \quad u_{\varepsilon\eta} = 0, & x \in \partial\Omega,\ t > 0, \\ n_{\varepsilon\eta}(x,0) = n_0(x), \quad c_{\varepsilon\eta}(x,0) = c_0(x), \quad u_{\varepsilon\eta}(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{7.3}$$

By means of a well-established construction involving the contraction mapping principle, one can firstly show that for each $\varepsilon > 0$ and $\eta \in (0,1)$ there exists a quadruple $(n_{\varepsilon\eta}, c_{\varepsilon\eta}, u_{\varepsilon\eta}, P_{\varepsilon\eta})$, nonnegative in its first two components, that solves (7.3) classically in $\Omega \times (0, T_{max,\varepsilon\eta})$ for some $T_{max,\varepsilon\eta} \le +\infty$. Using that $S_\eta(x,n,c) = 0$ whenever $n \ge \frac{1}{\eta}$, by following a series of standard arguments (see e.g. [47, 51]) one can thereupon readily verify on the basis of suitable a priori estimates that actually $T_{max,\varepsilon\eta} = +\infty$, and that hence this problem possesses a globally defined classical solution $(n_{\varepsilon\eta}, c_{\varepsilon\eta}, u_{\varepsilon\eta}, P_{\varepsilon\eta})$ for which $n_{\varepsilon\eta}$ and $c_{\varepsilon\eta}$ are nonnegative in $\Omega \times (0,\infty)$.

In order to derive appropriate bounds for these solutions, independently of ε and η, we start by again using Lemma 2.4 to refine the differential inequality appearing therein as follows.

Lemma 7.1 *Let $N \ge 2, p > N, q > 2$ and $r > N$. Then there exists $K_1(p,q,r) > 0$ such that for all $\varepsilon > 0$ and any $\eta \in (0,1)$,*

$$\begin{aligned} \frac{\varepsilon}{q} \frac{d}{dt} \int_\Omega |\nabla c_{\varepsilon\eta}|^q &+ \Big\{ \frac{1}{4} - K_1(p,q,r) \|u_{\varepsilon\eta}\|^2_{L^r(\Omega)} \Big\} \cdot \int_\Omega |\nabla c_{\varepsilon\eta}|^{q-2} |D^2 c_{\varepsilon\eta}|^2 \\ &+ \Big\{ 1 - \frac{1}{q^2} - K_1(p,q,r) \|u_{\varepsilon\eta}\|^2_{L^r(\Omega)} \Big\} \cdot \int_\Omega |\nabla c_{\varepsilon\eta}|^q \\ &\le K_1(p,q,r) \|n_{\varepsilon\eta}\|^q_{L^p(\Omega)} \qquad \textit{for all } t > 0. \end{aligned} \tag{7.4}$$

PROOF. Using Lemma 2.4 as a starting point, thanks to (2.10) we can fix $C_1 = C_1(p,q,r) > 0$ such that for all $\varepsilon > 0$ and $\eta \in (0,1)$,

$$\begin{aligned} \frac{\varepsilon}{q} \frac{d}{dt} \int_\Omega |\nabla c_{\varepsilon\eta}|^q + \frac{1}{4} \int_\Omega |\nabla c_{\varepsilon\eta}|^{q-2} |D^2 c_{\varepsilon\eta}|^2 + \Big(1 - \frac{1}{q^2}\Big) \int_\Omega |\nabla c_{\varepsilon\eta}|^q \\ \le C_1 \|n_{\varepsilon\eta}\|^q_{L^p(\Omega)} + C_1 \|u_{\varepsilon\eta}\|^2_{L^r(\Omega)} \|\nabla c_{\varepsilon\eta}\|^q_{L^{\frac{qr}{r-2}}(\Omega)} \qquad (7.5) \\ = C_1 \|n_{\varepsilon\eta}\|^q_{L^p(\Omega)} + C_1 \|u_{\varepsilon\eta}\|^2_{L^r(\Omega)} \Big\| |\nabla c_{\varepsilon\eta}|^{\frac{q}{2}} \Big\|^2_{L^{\frac{2r}{r-2}}(\Omega)} \qquad \text{for all } t > 0. \qquad (7.6) \end{aligned}$$

Here since $r > N$ and thus $\frac{2r}{r-2} < \frac{2N}{N-2}$, we may use the continuity of the embedding $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2r}{r-2}}(\Omega)$ to find $C_2 = C_2(p,q,r) > 0$ such that

$$\begin{aligned} C_1 \|u_{\varepsilon\eta}\|^2_{L^r(\Omega)} \Big\| |\nabla c_{\varepsilon\eta}|^{\frac{q}{2}} \Big\|^2_{L^{\frac{2r}{r-2}}(\Omega)} &\le C_2 \|u_{\varepsilon\eta}\|^2_{L^r(\Omega)} \cdot \Big\{ \Big\| \nabla |\nabla c_{\varepsilon\eta}|^{\frac{q}{2}} \Big\|^2_{L^2(\Omega)} + \Big\| |\nabla c_{\varepsilon\eta}|^{\frac{q}{2}} \Big\|^2_{L^2(\Omega)} \Big\} \\ &= C_2 \|u_{\varepsilon\eta}\|^2_{L^r(\Omega)} \cdot \Big\{ \frac{q^2}{4} \int_\Omega |\nabla c_{\varepsilon\eta}|^{q-4} |D^2 c_{\varepsilon\eta} \cdot \nabla c_{\varepsilon\eta}|^2 + \int_\Omega |\nabla c_{\varepsilon\eta}|^q \Big\} \\ &\le \frac{q^2}{4} C_2 \|u_{\varepsilon\eta}\|^2_{L^r(\Omega)} \int_\Omega |\nabla c_{\varepsilon\eta}|^{q-2} |D^2 c_{\varepsilon\eta}|^2 + C_2 \|u_{\varepsilon\eta}\|^2_{L^r(\Omega)} \int_\Omega |\nabla c_{\varepsilon\eta}|^q \end{aligned}$$

for all $t > 0$. When inserted into (7.5), this yields (7.4) on letting $K_1(p,q,r) := \max\{C_1, \frac{q^2}{4}C_2\}$, for instance. $\Box$

In consequence, if $u_{\varepsilon\eta}$ is suitably small, then also $\nabla c_{\varepsilon\eta}$ can be estimated in a favorable manner:

Lemma 7.2 *Let $N \geq 2, p > N$, $q > 2$ and $r > N$. Then there exist $\delta_1(p,q,r) > 0$ and $K_2(p,q,r) > 0$ such that if $\varepsilon > 0$, $\eta \in (0,1)$ and $T > 0$ are such that*

$$\|u_{\varepsilon\eta}(\cdot,t)\|_{L^r(\Omega)} \leq \delta_1(p,q,r) \qquad \textit{for all } t \in (0,T), \tag{7.7}$$

then

$$\|\nabla c_{\varepsilon\eta}(\cdot,t)\|_{L^q(\Omega)} \leq \max\left\{\|\nabla c_0\|_{L^q(\Omega)}\,,\, K_2 \cdot \sup_{s\in(0,t)} \|n_{\varepsilon\eta}(\cdot,s)\|_{L^p(\Omega)}\right\} \qquad \textit{for all } t \in (0,T). \tag{7.8}$$

Proof. With $K_1(p,q,r) > 0$ taken from Lemma 7.1, we let $\delta_1(p,q,r) := \frac{1}{\sqrt{4K_1(p,q,r)}}$ and then obtain from (7.4) that if (7.7) holds for some $\varepsilon > 0$, $\eta \in (0,1)$ and $T > 0$, then $y(t) := \int_\Omega |\nabla c_{\varepsilon\eta}(\cdot,t)|^q$, $t \geq 0$, satisfies

$$\frac{\varepsilon}{q} \cdot y'(t) + \frac{1}{2} y(t) \leq K_1(p,q,r)\|n_\varepsilon(\cdot,t)\|^q_{L^p(\Omega)} \qquad \text{for all } t \in (0,T).$$

Therefore, if given any $t \in (0,T)$ we let $M(t) := K_1(p,q,r) \cdot \sup_{s\in(0,t)} \|n_\varepsilon(\cdot,s)\|^q_{L^p(\Omega)}$, then

$$\frac{\varepsilon}{q} \cdot y'(s) + \frac{1}{2} y(s) \leq M(t) \qquad \text{for all } s \in (0,t),$$

so that a comparison argument yields the inequality

$$y(s) \leq \max\left\{y(0)\,,\, 2M(t)\right\} \qquad \text{for all } s \in [0,t].$$

When evaluated at $s = t$, this precisely leads to (7.8) upon defining $K_2(p,q,r) := (2K_1(p,q,r))^{\frac{1}{q}}$. $\Box$

Now the above hypothesis can be fulfilled if u_0 and $n_{\varepsilon\eta}$ are appropriately small:

Lemma 7.3 *Let $N \geq 2, p > 1$ and $r > N$ be such that*

$$p > \frac{Nr}{N+2r}. \tag{7.9}$$

Then for all $\delta > 0$ there exists $\delta_3(\delta,p,r) > 0$ such that if

$$\|u_0\|_{L^r(\Omega)} \leq \delta_3(\delta,p,r), \tag{7.10}$$

and if for some $\varepsilon > 0, \eta \in (0,1)$ and $T > 0$ we have

$$\|n_{\varepsilon\eta}(\cdot,t)\|_{L^p(\Omega)} \leq \delta_3(\delta,p,r) \qquad \textit{for all } t \in (0,T), \tag{7.11}$$

then

$$\|u_{\varepsilon\eta}(\cdot,t)\|_{L^r(\Omega)} \leq \delta \qquad \textit{for all } t \in (0,T). \tag{7.12}$$

PROOF. By relying on known regularization features of the Stokes semigroup $(e^{-tA})_{t\geq 0}$ ([10]), let us fix $C_1 = C_1(r) > 0$, $C_2 = C_2(r) > 0$, $C_3 = C_3(p,r) > 0$ and $\mu > 0$ such that for all $t > 0$,

$$\|e^{-tA}\varphi\|_{L^r(\Omega)} \leq C_1\|\varphi\|_{L^r(\Omega)} \qquad \text{for all } \varphi \in L^r_\sigma(\Omega) \tag{7.13}$$

and

$$\|e^{-tA}\mathcal{P}\nabla\cdot\varphi\|_{L^r(\Omega)} \leq C_2 t^{-\frac{1}{2}-\frac{N}{2r}} e^{-\mu t}\|\varphi\|_{L^{\frac{r}{2}}(\Omega)} \qquad \text{for all } \varphi \in C^1(\overline{\Omega};\mathbb{R}^{N\times N}) \text{ such that } \varphi = 0 \text{ on } \partial\Omega \tag{7.14}$$

as well as

$$\|e^{-tA}\mathcal{P}\varphi\|_{L^r(\Omega)} \leq C_3 t^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{r})_+} e^{-\mu t}\|\varphi\|_{L^p(\Omega)} \qquad \text{for all } \varphi \in C^0(\overline{\Omega};\mathbb{R}^N). \tag{7.15}$$

We then fix $\delta > 0$ and may without loss of generality assume that

$$C_2C_4|\kappa|\delta^2 \leq \frac{\delta}{6}, \tag{7.16}$$

where $C_4 = C_4(r) := \int_0^\infty \sigma^{-\frac{1}{2}-\frac{N}{2r}} e^{-\mu\sigma} d\sigma < \infty$ since $r > N$. Noting that thanks to (7.9) we moreover know that also $C_5 = C_5(p,r) := \int_0^\infty \sigma^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{r})_+} e^{-\mu\sigma} d\sigma$ is finite, we thereupon pick $\delta_3 = \delta_3(\delta,p,r) > 0$ small enough such that both

$$C_1\delta_3 \leq \frac{\delta}{6} \tag{7.17}$$

and

$$C_3C_5\|\nabla\phi\|_{L^\infty(\Omega)}\delta_3 \leq \frac{\delta}{6} \tag{7.18}$$

hold, and suppose that (7.10) and (7.11) are satisfied with some $\varepsilon > 0, \eta \in (0,1)$ and $T > 0$. Then since $u_{\varepsilon\eta}$ clearly is a mild solution of its respective subproblem in (7.3), we may use (7.13), (7.14) and (7.15) along with the Hölder inequality to estimate

$$\begin{aligned}
&\|u_{\varepsilon\eta}(\cdot,t)\|_{L^r(\Omega)} \\
&= \left\| e^{-tA}u_0 - \kappa\int_0^t e^{-(t-s)A}\mathcal{P}\nabla\cdot(u_{\varepsilon\eta}(\cdot,s)\otimes u_{\varepsilon\eta}(\cdot,s))ds + \int_0^t e^{-(t-s)A}\mathcal{P}[n(\cdot,s)\nabla\phi]ds \right\|_{L^r(\Omega)} \\
&\leq C_1\|u_0\|_{L^r(\Omega)} + C_2|\kappa|\int_0^t (t-s)^{-\frac{1}{2}-\frac{N}{2r}} e^{-\mu(t-s)}\|u_{\varepsilon\eta}(\cdot,s)\otimes u_{\varepsilon\eta}(\cdot,s)\|_{L^{\frac{r}{2}}(\Omega)}ds \\
&\quad + C_3\int_0^t (t-s)^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{r})_+} e^{-\mu(t-s)}\|n_{\varepsilon\eta}(\cdot,s)\nabla\phi\|_{L^p(\Omega)}ds \\
&\leq C_1\|u_0\|_{L^r(\Omega)} + C_2|\kappa|\int_0^t (t-s)^{-\frac{1}{2}-\frac{N}{2r}} e^{-\mu(t-s)}\|u_{\varepsilon\eta}(\cdot,s)\|^2_{L^r(\Omega)}ds \\
&\quad + C_3\|\nabla\phi\|_{L^\infty(\Omega)}\int_0^t (t-s)^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{p})_+} e^{-\mu(t-s)}\|n_{\varepsilon\eta}(\cdot,s)\|_{L^p(\Omega)}ds \qquad \text{for all } t \in [0,T].
\end{aligned} \tag{7.19}$$

In order to verify that this implies the inequality

$$M(T_0) < \delta \qquad \text{for all } T_0 \in [0,T] \tag{7.20}$$

for

$$M(T_0) := \sup_{t\in[0,T_0]} \|u_{\varepsilon\eta}(\cdot,t)\|_{L^r(\Omega)}, \qquad T_0 \in [0,T],$$

assuming (7.20) to be false we could make use of the continuity of M and e.g. combine (7.19) with (7.16) to find $T_\star \in (0,T]$ such that $M(T_0) < \delta$ for all $T_0 \in [0,T_\star)$ but $M(T_\star) = \delta$. According to (7.19) and our hypotheses (7.10) and (7.11) in conjunction with (7.16), (7.17) and (7.18), however, this would mean that

$$\begin{aligned}
\delta = M(T_\star) &\le C_1\delta_3 + C_2|\kappa|\delta^2 \int_0^t (t-s)^{-\frac12-\frac{N}{2r}} e^{-\mu(t-s)}ds \\
&\quad + C_3\|\nabla\phi\|_{L^\infty(\Omega)}\delta_3 \int_0^t (t-s)^{-\frac{N}{2}(\frac1p-\frac1r)_+} e^{-\mu(t-s)}ds \\
&\le C_1\delta_3 + C_2C_4|\kappa|\delta^2 + C_3C_5\|\nabla\phi\|_{L^\infty(\Omega)}\delta_3 \\
&\le \frac{\delta}{6} + \frac{\delta}{6} + \frac{\delta}{6} = \frac{\delta}{2},
\end{aligned}$$

which is absurd. Therefore, (7.20) and hence (7.12) must be valid. □

The hypotheses of the latter lemma, however, are satisfied if $\nabla c_{\varepsilon\eta}$ and $u_{\varepsilon\eta}$ are conveniently small:

Lemma 7.4 *Let $N \ge 2$. Then for all $p > 1, q > N$ and $r > N$ there exists $\delta_2(p,q,r) > 0$ such that if for some $\varepsilon > 0$, $\eta \in (0,1)$ and $T > 0$ we have*

$$\|\nabla c_{\varepsilon\eta}(\cdot,t)\|_{L^q(\Omega)} \le \delta_2(p,q,r) \qquad \text{for all } t \in (0,T) \tag{7.21}$$

and

$$\|u_{\varepsilon\eta}(\cdot,t)\|_{L^r(\Omega)} \le \delta_2(p,q,r) \qquad \text{for all } t \in (0,T), \tag{7.22}$$

then

$$\|n_{\varepsilon\eta}(\cdot,t)\|_{L^p(\Omega)} \le 2\|n_0\|_{L^p(\Omega)} \qquad \text{for all } t \in (0,T). \tag{7.23}$$

Proof. Given $p > 1, q > N$ and $r > N$, by using a well-known smoothing property of the Neumann heat semigroup $(e^{t\Delta})_{t\ge0}$ ([50]) we can fix $\mu > 0, C_1 = C_1(p,q) > 0$ and $C_2 = C_2(p,r) > 0$ such that whenever $t > 0$,

$$\begin{aligned}
\|e^{t\Delta}\nabla\cdot\varphi\|_{L^p(\Omega)} &\le \min\Big\{C_1 t^{-\frac12-\frac{N}{2q}}e^{-\mu t}\|\varphi\|_{L^{\frac{pq}{p+q}}(\Omega)},\ C_2 t^{-\frac12-\frac{N}{2r}}e^{-\mu t}\|\varphi\|_{L^{\frac{pr}{p+r}}(\Omega)}\Big\} \\
&\qquad \text{for all } \varphi \in C^1(\overline{\Omega};\mathbb{R}^N) \text{ such that } \varphi\cdot\nu = 0 \text{ on } \partial\Omega,
\end{aligned} \tag{7.24}$$

and thereupon let $\delta_2 = \delta_2(p,q,r) > 0$ be small enough such that

$$C_1C_3K_S\delta_2 \le \frac14 \tag{7.25}$$

and

$$C_2C_4\delta_2 \le \frac14, \tag{7.26}$$

where $C_2 = C_2(q) := \int_0^\infty \sigma^{-\frac{1}{2}-\frac{N}{2q}} e^{-\mu\sigma} d\sigma$ and where $C_3 = C_3(r) := \int_0^\infty \sigma^{-\frac{1}{2}-\frac{N}{2r}} e^{-\mu\sigma} d\sigma$ are finite due to our assumptions that $q > N$ and $r > N$.

Then assuming (7.21) and (7.22) to be valid for some $\varepsilon > 0, \eta \in (0,1)$ ad $T > 0$, we may employ a variation-of-constants representation associated with the first equation in (7.3) to see that thanks to the contractivity of $(e^{t\Delta})_{t\ge 0}$ on $L^p(\Omega)$,

$$\begin{aligned}
&\|n_{\varepsilon\eta}(\cdot,t)\|_{L^p(\Omega)} \\
= \ & \bigg\| e^{t\Delta} n_0 - \int_0^t e^{(t-s)\Delta} \nabla \cdot \Big(n_{\varepsilon\eta}(\cdot,s) S_\eta(\cdot, n_{\varepsilon\eta}(\cdot,s), c_{\varepsilon\eta}(\cdot,s)) \cdot \nabla c_{\varepsilon\eta}(\cdot,s) \Big) ds \\
&\quad - \int_0^t e^{(t-s)\Delta} \nabla \cdot \Big(n_{\varepsilon\eta}(\cdot,s) u_{\varepsilon\eta}(\cdot,s) \Big) ds \bigg\|_{L^p(\Omega)} \\
\le \ & \|n_0\|_{L^p(\Omega)} + C_1 \int_0^t (t-s)^{-\frac{1}{2}-\frac{N}{2q}} e^{-\mu(t-s)} \Big\| n_{\varepsilon\eta}(\cdot,s) S_\eta(\cdot, n_{\varepsilon\eta}(\cdot,s), c_{\varepsilon\eta}(\cdot,s)) \cdot \nabla c_{\varepsilon\eta}(\cdot,s) \Big\|_{L^{\frac{pq}{p+q}}(\Omega)} ds \\
&+ C_2 \int_0^t (t-s)^{-\frac{1}{2}-\frac{N}{2r}} e^{-\mu(t-s)} \|n_{\varepsilon\eta}(\cdot,s) u_{\varepsilon\eta}(\cdot,s)\|_{L^{\frac{pr}{p+r}}(\Omega)} ds
\end{aligned} \tag{7.27}$$

for all $t \in (0,T)$. Here by the Hölder inequality, (1.4), and (7.21), we know that abbreviating $M := \|n_{\varepsilon\eta}\|_{L^\infty((0,T);L^p(\Omega))}$ we have

$$\begin{aligned}
\Big\| n_{\varepsilon\eta}(\cdot,s) S_\eta(\cdot, n_{\varepsilon\eta}(\cdot,s), c_{\varepsilon\eta}(\cdot,s)) \cdot \nabla c_{\varepsilon\eta}(\cdot,s) \Big\|_{L^{\frac{pq}{p+q}}(\Omega)} &\le K_S \|n_{\varepsilon\eta}(\cdot,s)\|_{L^p(\Omega)} \|\nabla c_{\varepsilon\eta}(\cdot,s)\|_{L^q(\Omega)} \\
&\le K_S \delta_2 M \qquad \text{for all } s \in (0,T),
\end{aligned}$$

while similarly (7.22) guarantees that

$$\begin{aligned}
\|n_{\varepsilon\eta}(\cdot,s) u_{\varepsilon\eta}(\cdot,s)\|_{L^{\frac{pr}{p+r}}(\Omega)} &\le \|n_{\varepsilon\eta}(\cdot,s)\|_{L^p(\Omega)} \|u_{\varepsilon\eta}(\cdot,s)\|_{L^r(\Omega)} \\
&\le \delta_2 M \qquad \text{for all } s \in (0,T).
\end{aligned}$$

Therefore, we may use (7.25) and (7.26) to infer from (7.27) that

$$\begin{aligned}
\|n_{\varepsilon\eta}(\cdot,t)\|_{L^p(\Omega)} &\le \|n_0\|_{L^p(\Omega)} + C_1 K_S \delta_2 M \int_0^t (t-s)^{-\frac{1}{2}-\frac{N}{2q}} e^{-\mu(t-s)} ds \\
&\quad + C_2 \delta_2 M \int_0^t (t-s)^{-\frac{1}{2}-\frac{N}{2r}} e^{-\mu(t-s)} ds \\
&\le \|n_0\|_{L^p(\Omega)} + C_1 C_3 K_S \delta_2 M + C_2 C_4 \delta_2 M \\
&\le \|n_0\|_{L^p(\Omega)} + \frac{M}{4} + \frac{M}{4} \qquad \text{for all } t \in (0,T),
\end{aligned}$$

which implies that

$$M \le \|n_0\|_{L^p(\Omega)} + \frac{M}{2}$$

and hence completes the proof. □

Now a self-map type argument combines the latter lemmata so as to make sure that for suitably small initial data, all the above assumptions can be fulfilled simultaneously:

Lemma 7.5 *Let $N \geq 2, p > \max\{2, N\}, q > N$ and $r > N$. Then there exists $C = C(p, q, r) > 0$ such that if n_0, c_0 and u_0 satisfy (1.6) with*

$$\|n_0\|_{L^p(\Omega)} \leq \frac{1}{C}, \qquad \|\nabla c_0\|_{L^q(\Omega)} \leq \frac{1}{C} \qquad \text{and} \qquad \|u_0\|_{L^r(\Omega)} \leq \frac{1}{C}, \tag{7.28}$$

then for all $\varepsilon > 0$ and $\eta \in (0,1)$, the solution of (7.3) has the properties that

$$\|n_{\varepsilon\eta}(\cdot,t)\|_{L^p(\Omega)} \leq C, \quad \|\nabla c_{\varepsilon\eta}(\cdot,t)\|_{L^q(\Omega)} \leq C \quad \text{and} \quad \|u_{\varepsilon\eta}(\cdot,t)\|_{L^r(\Omega)} \leq C \quad \text{for all } t > 0. \tag{7.29}$$

Proof. Given $p > \max\{2, N\}, q > N$ and $r > N$, we take $\delta_1 = \delta_1(p,q,r) > 0$ and $K_2 = K_2(p,q,r) > 0$ from Lemma 7.2 and let $\delta_2 = \delta_2(p,q,r) > 0$ be as provided by Lemma 7.4. Then since $p > N > \frac{Nr}{N+2r}$, an application of Lemma 7.3 to $\delta := \min\{\delta_1, \delta_2\}$ yields $\delta_3 = \delta_3(p,q,r) > 0$ with the property that whenever (7.10) and (7.11) hold for some $\varepsilon > 0, \eta \in (0,1)$ and $T > 0$, we have

$$\|u_{\varepsilon\eta}(\cdot,t)\|_{L^r(\Omega)} \leq \delta_1 \qquad \text{for all } t \in (0,T) \tag{7.30}$$

and

$$\|u_{\varepsilon\eta}(\cdot,t)\|_{L^r(\Omega)} \leq \delta_2 \qquad \text{for all } t \in (0,T). \tag{7.31}$$

We now suppose that n_0, c_0 and u_0 comply with (1.6) and are such that

$$3K_2\|n_0\|_{L^p(\Omega)} \leq \delta_2 \tag{7.32}$$

and

$$\|\nabla c_0\|_{L^q(\Omega)} \leq \delta_2 \tag{7.33}$$

as well as

$$\|u_0\|_{L^r(\Omega)} \leq \delta_3 \tag{7.34}$$

and

$$3\|n_0\|_{L^p(\Omega)} \leq \delta_3, \tag{7.35}$$

and we claim that then for each $\varepsilon > 0$ and $\eta \in (0,1)$, the obviously well-defined element

$$T \equiv T_{\varepsilon\eta} := \sup\left\{\widehat{T} > 0 \;\middle|\; \|n_{\varepsilon\eta}(\cdot,t)\|_{L^p(\Omega)} < 3\|n_0\|_{L^p(\Omega)} \text{ for all } t \in (0,\widehat{T})\right\}$$

of $(0,\infty]$ actually satisfies $T_{\varepsilon\eta} = \infty$.

To see this, we note that by definition of T,

$$\|n_{\varepsilon\eta}(\cdot,t)\|_{L^p(\Omega)} < 3\|n_0\|_{L^p(\Omega)} \qquad \text{for all } t \in (0,T), \tag{7.36}$$

which in conjunction with (7.34) and (7.35) allows for an application of Lemma 7.3 to conclude that in fact both (7.30) and (7.31) hold. In particular, (7.30) enables us to employ Lemma 7.2 to see that thanks to (7.33), again (7.36), and (7.32),

$$\begin{aligned} \|\nabla c_{\varepsilon\eta}(\cdot,t)\|_{L^q(\Omega)} &\leq \max\left\{\|\nabla c_0\|_{L^q(\Omega)}\,,\, K_2 \cdot 3\|n_0\|_{L^p(\Omega)}\right\} \\ &\leq \delta_2 \qquad \text{for all } t \in (0,T), \end{aligned} \tag{7.37}$$

which in turn, when combined with (7.31), makes it possible to infer from Lemma 7.4 that

$$\|n_{\varepsilon\eta}(\cdot,t)\|_{L^p(\Omega)} \le 2\|n_0\|_{L^p(\Omega)} \qquad \text{for all } t \in (0,T).$$

As $n_0 \not\equiv 0$ by (1.6), by continuity of $n_{\varepsilon\eta}$ this shows that indeed $T_{\varepsilon\eta}$ cannot be finite for any $\varepsilon > 0$ and $\eta \in (0,1)$, and that thus (7.29) results as a consequence of (7.36), (7.37) and (7.30) if in accordance with (7.32)-(7.35), the constant C in (7.28) and (7.29) is chosen suitably large. □

In fact, we have thereby proved the essential body of Theorem 1.2 already:

PROOF of Theorem 1.2. According to Lemma 7.5, there exists $\delta = \delta(p,q,r) > 0$ such that (1.16) implies the boundedness properties in (7.29) uniformly with respect to $\varepsilon > 0$ and $\eta \in (0,1)$. Thanks to the estimates thereby implied through Lemma 2.2, Lemma 2.3 and Lemma 5.6, by means of a standard subsequence extraction procedure this can readily be seen to entail, for each $\varepsilon > 0$, the existence of a global classical solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ of (1.2) which in fact has the properties that $n_{\varepsilon\eta_l} \to n_\varepsilon$, $c_{\varepsilon\eta_l} \to c_\varepsilon$ and $u_{\varepsilon\eta_l} \to u_\varepsilon$ a.e. in $\Omega \times (0,\infty)$ with some $(\eta_l)_{l\in\mathbb{N}} \subset (0,1)$ such that $\eta_l \searrow 0$ as $l \to \infty$ ([5]).
The remaining part of the statement then directly results from Theorem 1.1 and the boundedness features of $(\nabla c_\varepsilon)_{\varepsilon>0}$ and $(u_\varepsilon)_{\varepsilon>0}$ implied by (7.29). □

8 A logistic Keller-Segel system. Proof of Theorems 1.3 and 1.4

As a second example for taking a parabolic-elliptic limit along the lines of Theorem 1.1, in this section we shall consider the one-dimensional logistic Keller-Segel system (1.17) for fixed $D > 0, a \in \mathbb{R}, b \ge 0$ and $\varepsilon > 0$.

Again we start by stating an almost immediate basic property.

Lemma 8.1 *Let $T > 0$. Then there exists $C(T) > 0$ such that for any $\varepsilon > 0$,*

$$\int_0^1 c_\varepsilon(\cdot,t) \le C(T) \qquad \textit{for all } t \in (0,T) \tag{8.1}$$

PROOF. As an immediate consequence of Lemma 2.1, we obtain $C_1(T) > 0$ such that

$$\int_0^1 n_\varepsilon(\cdot,t) \le C_1(T) \qquad \text{for all } t \in (0,T).$$

Thereupon, using (1.17) we can estimate

$$\varepsilon \frac{d}{dt} \int_0^1 c_\varepsilon(\cdot,t) + \int_0^1 c_\varepsilon(\cdot,t) = \int_0^1 n_\varepsilon(\cdot,t) \le C_1(T) \qquad \text{for all } t \in (0,T),$$

which by comparison implies that

$$\int_0^1 c_\varepsilon(\cdot,t) \le \max\left\{ \int_0^1 c_0\,,\, C_1(T)\right\} \qquad \text{for all } t \in (0,T),$$

as intended. □

Now in the spatially one-dimensional setting considered here, the availability of favorable embeddings allows us to conclude the following from an essentially well-established testing procedure.

Lemma 8.2 *Let $T > 0$. Then there exists $C(T) > 0$ such that*

$$\int_0^T \int_0^1 c_{\varepsilon xx}^2 \le C(T) \qquad \textit{for all } \varepsilon \in (0,1). \tag{8.2}$$

PROOF. By referring to both PDEs in (1.17) and employing Young's inequality, we see that whenever $\varepsilon > 0$,

$$\begin{aligned}
\frac{d}{dt}\Big\{ \int_0^1 n_\varepsilon \ln n_\varepsilon(\cdot,t) + \frac{\varepsilon}{2}\int_0^1 c_{\varepsilon x}^2(\cdot,t)\Big\} &+ D\int_0^1 \frac{n_{\varepsilon x}^2}{n_\varepsilon}(\cdot,t) + \int_0^1 c_{\varepsilon xx}^2(\cdot,t) + \int_0^1 c_{\varepsilon x}^2(\cdot,t) \\
&= -2\int_0^1 n_\varepsilon c_{\varepsilon xx}(\cdot,t) + a\int_0^1 n_\varepsilon \ln n_\varepsilon(\cdot,t) - b\int_0^1 n_\varepsilon^2 \ln n_\varepsilon(\cdot,t) + a\int_0^1 n_\varepsilon(\cdot,t) - b\int_0^1 n_\varepsilon^2(\cdot,t) \\
&\le \frac{1}{2}\int_0^1 c_{\varepsilon xx}^2(\cdot,t) + a\int_0^1 n_\varepsilon \ln n_\varepsilon(\cdot,t) - b\int_0^1 n_\varepsilon^2 \ln n_\varepsilon(\cdot,t) + a\int_0^1 n_\varepsilon(\cdot,t) + \int_0^1 n_\varepsilon^2(\cdot,t)
\end{aligned} \tag{8.3}$$

for all $t \in (0,T)$. Since it can readily be verified by elementary analysis that thanks to the nonnegativity of b there exists $C_1 > 0$ with the property that

$$a\xi\ln\xi - b\xi^2\ln\xi + a\xi + \xi^2 \le 2\xi^2 + C_1 \qquad \text{for all } \xi > 0,$$

and since the Gagliardo-Nirenberg inequality, Young's inequality and Lemma 2.1 provide $C_2 > 0$ and $C_3(T) > 0$ such that for all $\varepsilon > 0$ we have

$$\begin{aligned}
2\int_0^1 n_\varepsilon^2(\cdot,t) &= 2\|\sqrt{n_\varepsilon}\|_{L^4((0,1))}^4 \\
&\le C_2\|(\sqrt{n_\varepsilon})_x\|_{L^2((0,1))}\|\sqrt{n_\varepsilon}\|_{L^2((0,1))}^3 + C_2\|\sqrt{n_\varepsilon}\|_{L^2((0,1))}^4 \\
&\le D\int_0^1 \frac{n_{\varepsilon x}^2}{n_\varepsilon}(\cdot,t) + C_3(T) \qquad \text{for all } t \in (0,T),
\end{aligned}$$

from (8.3) it thus follows that for any such ε,

$$\frac{d}{dt}\Big\{ \int_0^1 n_\varepsilon \ln n_\varepsilon(\cdot,t) + \frac{\varepsilon}{2}\int_0^1 c_{\varepsilon x}^2(\cdot,t)\Big\} + \frac{1}{2}\int_0^1 c_{\varepsilon xx}^2(\cdot,t) \le C_4(T) := C_1 + C_3(T) \quad \text{for all } t \in (0,T).$$

Hence, when resorting to $\varepsilon \in (0,1)$ we infer that

$$\begin{aligned}
\int_0^1 n_\varepsilon(\cdot,T)\ln n_\varepsilon(\cdot,T) + \frac{\varepsilon}{2}\int_0^1 c_{\varepsilon x}^2(\cdot,T) + \frac{1}{2}\int_0^T\int_0^1 c_{\varepsilon xx}^2 &\le \int_0^1 n_0 \ln n_0 + \frac{\varepsilon}{2}\int_0^1 c_{0x}^2 + C_4(T)T \\
&\le \int_0^1 n_0 \ln n_0 + \frac{1}{2}\int_0^1 c_{0x}^2 + C_4(T)T,
\end{aligned}$$

which entails (8.2) due to the fact that $\int_0^1 n_\varepsilon(\cdot,T)\ln n_\varepsilon(\cdot,T) \ge -\frac{1}{e}$. $\square$

In conjunction with the L^1 information from Lemma 8.1, the latter entails an estimate for $c_{\varepsilon x}$ compatible with (1.10):

Lemma 8.3 *For any $T > 0$ one can find $C(T) > 0$ with the property that*

$$\int_0^T \|c_{\varepsilon x}(\cdot,t)\|_{L^\infty((0,1))}^{\frac{5}{2}} dt \le C(T) \qquad \text{for all } \varepsilon \in (0,1). \tag{8.4}$$

Proof. As the Gagliardo-Nirenberg inequality says that with some $C_1 > 0$ we have

$$\|c_{\varepsilon x}\|_{L^\infty((0,1))}^{\frac{5}{2}} \le C_1 \|c_{\varepsilon xx}\|_{L^2((0,1))}^2 \|c_\varepsilon\|_{L^1((0,1))}^{\frac{1}{2}} + C_1 \|c_\varepsilon\|_{L^1((0,1))}^{\frac{5}{2}} \qquad \text{for all } t > 0 \text{ and each } \varepsilon > 0,$$

the claim results upon integrating and combining Lemma 8.2 with Lemma 8.1. □

We can thereby directly pass to the limit $\varepsilon \searrow 0$ by means of Theorem 1.1:

Proof of Theorem 1.3. We pick any $q > 5$ and then obtain as a particular consequence of Lemma 8.3 that for each $T > 0$, $(c_{\varepsilon x})_{\varepsilon \in (0,1)}$ is bounded in $L^{\frac{5}{2}}((0,T); L^q((0,1)))$. Since this choice of q precisely ensures that $\frac{2}{5} + \frac{1}{2q} < \frac{1}{2}$, the conclusion follows by applying Theorem 1.1 with $u_0 \equiv 0$ and $\phi \equiv 0$, and recalling from standard literature ([42], [6]) a well-known uniqueness property of (1.18) within the indicated class. □

Thanks to a known result on spontaneous emergence of large densities in the limit problem (1.18) for suitably small $D > 0$, our statement from Theorem 1.3 enables us to finally draw a similar conclusion also for the fully parabolic problem when the parameter ε therein is appropriately small.

Proof of Theorem 1.4. According to a result from [53, Theorem 1.1] on the parabolic-elliptic problem (1.18), we can pick some nonnegative $n_0 \in W^{1,\infty}((0,1))$ which is such that there exists $T > 0$ having the property that to arbitrary $M > 0$ there corresponds some $D_0 > 0$ such that for each $D \in (0, D_0)$, the solution $(n,c) \equiv (n_D, c_D)$ of (1.18) satisfies

$$n_D(x_0(D), t_0(D)) \ge 2M \tag{8.5}$$

with some $x_0(D) \in (0,1)$ and $t_0(D) \in (0,T)$. Now keeping n_0, T and M fixed, given any such D and arbitrary nonnegative $c_0 \in W^{1,\infty}((0,1))$ we may employ Theorem 1.3 to see that the associated solutions $(n_{D\varepsilon}, c_{D\varepsilon})$ of (1.17) approximate (n_D, c_D) in the sense that, inter alia, $n_{D\varepsilon} \to n_D$ in $C^0([0,1] \times [0,T])$ as $\varepsilon \searrow 0$. In particular, we can therefore find $\varepsilon_0 > 0$ such that $n_{D\varepsilon} \ge n_D - M$ in $(0,1) \times (0,T)$ for all $\varepsilon \in (0, \varepsilon_0)$, which when evaluated at $(x_0(D), t_0(D))$ and combined with (8.5) directly yields (1.19). □

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