Large time behavior in a predator-prey system with indirect pursuit-evasion interaction

Genglin Li*
College of Information and Technology,
Donghua University, Shanghai 201620, P.R. China

Youshan Tao#
School of Mathematical Sciences, Shanghai Jiao Tong University,
Shanghai 200240, P.R. China

Michael Winkler‡
Institut für Mathematik, Universität Paderborn,
33098 Paderborn, Germany

Abstract

In a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, this work considers the indirect pursuit-evasion model

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla w) + u(\lambda - u + av), \\ v_t = \Delta v + \xi \nabla \cdot (v \nabla z) + v(\mu - v - bu), \\ 0 = \Delta w - w + v, \\ 0 = \Delta z - z + u, \end{cases}$$

with positive parameters χ, ξ, λ, μ, a and b.

It is firstly asserted that when $n \le 3$, for any given suitably regular initial data the corresponding homogeneous Neumann initial-boundary problem admits a global and bounded smooth solution. Moreover, it is shown that if $b\lambda < \mu$ and under some explicit smallness conditions on χ and ξ, any nontrival bounded classical solution converges to the spatially homogeneous coexistence state in the large time limit; if $b\lambda > \mu$, however, then under an explicit smallness assumption on χ but without any restriction on ξ, any bounded classical solution (u, v) with $u \not\equiv 0$ stabilizes to $(\lambda, 0)$ as $t \to \infty$.

Key words: chemotaxis, attraction-repulsion, predator-prey, stabilization
MSC 2010: 35B40, 35B65, 35K57, 35Q92, 92C17

*1182028@mail.dhu.edu.cn
#taoys@sjtu.edu.cn
‡michael.winkler@math.uni-paderborn.de

1 Introduction

Lotka-Volterra type interplay, forming a class of paradigmatic interaction mechanisms in population dynamics, occurs in various biological and ecological processes, and the investigation of possible effects has been stimulated both theoretical biology and applied mathematics during the past century. A significant potential with regard to structure formation, preferably in contexts involving non-negligible migration of individuals, is indicated by numerous findings on colorful solution behavior already in simple reaction-diffusion systems combining Lotka-Volterra kinetics with undirected random diffusion ([15]).

To capture even more complex dynamics, and especially the occurrence of wave-like behavior, in yet simple two-component models of predator-prey type, the authors in [24] propose to additionally account for partially directed migration mechanisms reflecting, on the one hand, the ambition of predators to move toward prey-rich regions, and, on the other hand, a certain predisposition of prey individuals to move away from predator-populated areas. In the resulting model,

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + f(u,v), \\ v_t = \Delta v + \xi\nabla\cdot(v\nabla u) + g(u,v), \end{cases} \tag{1.1}$$

the quantities $u = u(x,t)$ and $v = v(x,t)$ denote the densities of the predator and the prey population, respectively, f and g represent the local kinetics, and the positive parameters χ and ξ measure the strength of attractive and repulsive directed migration, respectively. Indeed, the numerical simulations presented in [24] indicate that even in spatially one-dimensional frameworks and for f and g reflecting functional response of so-called Holling type III, despite their seemingly artless structure such systems can well describe the emergence of soliton-like taxis waves, as observed in experiments involving bacterial populations of E. coli on semi-solid nutrient media ([23]).

In fact, already the literature on related systems indicates that in fact the introduction of the two cross-diffusion mechanisms in (1.1) may go along with a substantial change of mathematical properties in comparison to those known for the corresponding taxis-free variants in which $\chi = \xi = 0$. For instance, choosing $\xi = 0, f \equiv 0$ and $g(u,v) = u - v$ shows that the celebrated Keller-Segel chemotaxis system ([14]), that is,

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v), \\ v_t = \Delta v - v + u, \end{cases} \tag{1.2}$$

can be viewed as a special case of (1.1), and it is well-known that even finite-time blow-up of solutions to corresponding Neumann problems in bounded n-dimensional domains will occur for suitably large initial data in the cases $n = 2$ ([9]) and $n \geq 3$ ([26]). The particularly delicate role of the attractive taxis mechanism therein is underlined by the observation that no such drastic aggregation phenomenon occurs when a single cross-diffusion term of the considered form is purely repulsive, such as in the variant of (1.2) given by

$$\begin{cases} u_t = \Delta u - u + v, \\ v_t = \Delta v + \xi\nabla\cdot(v\nabla u). \end{cases} \tag{1.3}$$

Indeed, an associated Neumann problem is known to admit global smooth solutions when $n = 2$, and at least some global weak solutions when $n \in \{3,4\}$, and each of these solutions approaches a spatially homogeneous equilibrium in the large time limit ([3]).

Now in contexts of production and degradation processes which are closer to the prototypical choices in Lotka-Volterra systems than those underlying (1.2) and (1.3), the explosion-supporting potential of attractive taxis can partially be compensated by suitably regularizing influences of the respective zero-order expressions involved. Setting $\xi = 0$ and $g(u,v) = -uv + g_0(v)$ in (1.1), for instance, reduces (1.1) to the so-called prey-taxis system ([13])

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u,v), \\ v_t = \Delta v - uv + g_0(v), \end{cases} \tag{1.4}$$

in which analytical findings indicate that at least in two-dimensional settings, the additional dissipation generated by the absorptive term $-uv$ therein may rule out the occurrence of blow-up under various assumptions on χ or on local kinetics functions f and g_0 ([19], [27]; cf. also [28], [12], [8], [16], [19], [29] and [21] for some closely related variants).

In contrast to its subsystems (1.2), (1.3) and (1.4), by simultaneously accounting for two taxis mechanisms the full model (1.1) can no longer be viewed as a triangular cross-diffusion system, which substantially reduces its accessibility to well-established analytical techniques, and which is reflected in an apparently complete absence of rigorous results concerned with global solutions to any version of (1.1) involving nontrivial choices of both χ and μ. In order to nevertheless achieve some insight into possible dynamical properties of pursuit-evasion processes, in this work we shall focus on a variant of (1.1) in which the respective tactic movements are oriented along gradients of some *indirectly* produced stimuli, rather than following individuals directly. In fact, assuming predators and preys to exert species-characteristic substances such as pheromones or scent marks, the authors in [25] propose the variant of the pursuit-evasion model (1.1) given by

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla w) + f(u,v), \\ v_t = \Delta v + \xi \nabla \cdot (v \nabla z) + g(u,v), \\ w_t = D_w \Delta w - \delta_w w + v, \\ z_t = D_z \Delta z - \delta_z z + u, \end{cases} \tag{1.5}$$

additionally containing the concentrations $w = w(x,t)$ and $z = z(x,t)$ of the respectively emitted chemicals, as well as the positive parameters D_w, D_z, δ_w and δ_z. Relying on the circumstance that chemicals diffuse substantially faster than individuals, we shall follow a corresponding and standard quasi-stationary approximation procedure, quite well-established in the context of chemotaxis systems ([11], [10]), and hence subsequently concentrate on the parabolic-elliptic simplification of (1.5) given by

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla w) + f(u,v), \\ v_t = \Delta v + \xi \nabla \cdot (v \nabla z) + g(u,v), \\ 0 = D_w \Delta w - \delta_w w + v, \\ 0 = D_z \Delta z - \delta_z z + u. \end{cases} \tag{1.6}$$

Beyond some analytical results addressing questions of global weak solvability and boundedness in two-dimensional boundary value problems ([7], [1]), numerical evidence indicates that even upon trivial choices of f, g, δ_w and δ_z, (1.6) may indeed generate various types of patterns ([6]).

Main results. The purpose of the present work is to investigate possible effects resulting from the interplay of the doubly cross-diffusive and indirectly mediated migration in (1.6) with zero-order kinetics genuinely related to Lotka-Volterra type predator-prey interaction. Accordingly, we shall henceforth consider the indirect pursuit-evasion system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla w) + u(\lambda - u + av), & x \in \Omega,\ t > 0, \\ v_t = \Delta v + \xi \nabla \cdot (v \nabla z) + v(\mu - v - bu), & x \in \Omega,\ t > 0, \\ 0 = \Delta w - w + v, & x \in \Omega,\ t > 0, \\ 0 = \Delta z - z + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.7}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, where $n \geq 1$, where χ, ξ, λ, μ, a and b are positive parameters, and where u_0 and v_0 are given suitably regular functions.

In this framework, the first of our results asserts global existence of smooth solutions to (1.7) for widely arbitrary initial data in any physically meaningful dimension:

Proposition 1.1 *Let $n \leq 3$, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and suppose that the parameters $\chi, \xi, \lambda, \mu, a$ and b are positive. Then for all nonnegative functions $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in C^0(\bar{\Omega})$, the problem (1.7) possesses a unique global classical solution (u, v, w, z) with*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ v \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ w \in C^{2,0}(\overline{\Omega} \times (0,\infty)), \\ z \in C^{2,0}(\overline{\Omega} \times (0,\infty)), \end{cases}$$

which is bounded in the sense that there exists $C > 0$ satisfying

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} + \|z(\cdot,t)\|_{L^\infty(\Omega)} \leq C \qquad \text{for all } t > 0. \tag{1.8}$$

Next concerned with the qualitative behavior of these solutions, we recall from well-known facts about the ODE system associated with (1.7) that merely the sign of the number $\mu - b\lambda$ decides about the existence of a spatially homogeneous equilibrium which is positive in both population components and hence reflects coexistence ([15]). In fact, our second result will reveal that the assumption $b\lambda < \mu$ therefor will retain its sufficiency with regard to asymptotic stability of this steady state, provided that an explicit smallness condition on the taxis coefficients χ and ξ is satisfied. We note that the following statement in this direction actually applies to any global bounded solution to (1.7), regardless of the space dimension $n \geq 1$, with unconditional applicability to widely arbitrary solutions when $n \leq 3$ due to Proposition 1.1.

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, let $\lambda > 0, \mu > 0, a > 0$ and $b > 0$ satisfy*

$$b\lambda < \mu, \tag{1.9}$$

assume that the positive parameter χ is such that

$$\chi < 4\sqrt{\frac{a(1+ab)}{b(\lambda+a\mu)}} \tag{1.10}$$

and suppose that the positive parameter ξ fulfills

$$\xi < 4\sqrt{\frac{b(1+ab)}{a(\mu-b\lambda)}}. \tag{1.11}$$

Then any nonnegative global bounded classical solution (u,v,w,z) of the boundary value problem in (1.7) with $u \not\equiv 0$ and $v \not\equiv 0$ satisfies

$$u(\cdot,t) \to u_\star := \frac{\lambda+a\mu}{1+ab} \quad \textit{and} \quad z(\cdot,t) \to u_\star \quad \textit{in } C^2(\bar\Omega) \qquad \textit{as } t \to \infty \tag{1.12}$$

as well as

$$v(\cdot,t) \to v_\star := \frac{\mu-b\lambda}{1+ab} \quad \textit{and} \quad w(\cdot,t) \to v_\star \quad \textit{in } C^2(\bar\Omega) \qquad \textit{as } t \to \infty. \tag{1.13}$$

If, conversely, $b\lambda > \mu$, then under an again explicit restriction on smallness of χ, but now without any constraint on the repulsive taxis coefficient ξ, the predators will asymptotically outcompete the prey population.

Theorem 1.3 *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, that $\lambda > 0, \mu > 0, a > 0$ and $b > 0$ are such that*

$$b\lambda > \mu, \tag{1.14}$$

that

$$\chi < \sqrt{\frac{16a}{b\lambda}}, \tag{1.15}$$

and that

$$\xi > 0 \quad \textit{is arbitrary.} \tag{1.16}$$

Then whenever (u,v,w,z) is a nonnegative bounded global classical solution of the boundary value problem in (1.7) satisfying $u \not\equiv 0$ and $v \not\equiv 0$, we have

$$u(\cdot,t) \to \lambda \quad \textit{and} \quad z(\cdot,t) \to \lambda \quad \textit{in } C^2(\bar\Omega) \qquad \textit{as } t \to \infty \tag{1.17}$$

as well as

$$v(\cdot,t) \to 0 \quad \textit{and} \quad w(\cdot,t) \to 0 \quad \textit{in } C^2(\bar\Omega) \qquad \textit{as } t \to \infty. \tag{1.18}$$

2 Global existence and boundedness. Proof of Proposition 1.1

The following basic result on local existence can be proved by adapting well-established approaches for parabolic-elliptic chemotaxis models (cf. [4] and [17], for instance).

Lemma 2.1 *Let $n \geq 1$, and suppose that the parameters $\chi, \xi, \lambda, \mu, a$ and b are positive. Then for all nonnegative functions $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in C^0(\bar{\Omega})$, there exist $T_{max} \in (0, \infty]$ and a unique quadruple (u, v, w, z) of nonnegative functions from $C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}))$ solving (1.7) classically in $\Omega \times (0, T_{max})$. Moreover,*

$$\begin{aligned}
&\textit{either } T_{max} = \infty, \textit{ or} \\
&\lim_{t \nearrow T_{max}} \sup \Big(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \Big) = \infty.
\end{aligned} \tag{2.1}$$

Some elemetary bounds on the respective total mass functionals will be of substantial importance in the sequel.

Lemma 2.2 *The solution of (1.7) satisfies*

$$\|u(\cdot,t)\|_{L^1(\Omega)} = \|z(\cdot,t)\|_{L^1(\Omega)} \leq m \qquad \textit{for all } t \in (0, T_{max}) \tag{2.2}$$

and

$$\|v(\cdot,t)\|_{L^1(\Omega)} = \|w(\cdot,t)\|_{L^1(\Omega)} \leq \frac{b}{a} m \qquad \textit{for all } t \in (0, T_{max}), \tag{2.3}$$

where $m := \max \Big\{ \int_\Omega u_0 + \frac{a}{b} \int_\Omega v_0,\ \frac{1}{4} \Big[(\lambda+1)^2 + \frac{a}{b}(\mu+1)^2 \Big] \cdot |\Omega| \Big\} > 0$.

Proof. Integrating the first and the second equation of (1.7) with respect to $x \in \Omega$, we see that

$$\begin{aligned}
\frac{d}{dt} \int_\Omega \Big(u + \frac{a}{b} v \Big) + \int_\Omega \Big(u + \frac{a}{b} v \Big) &= (\lambda+1) \int_\Omega u - \int_\Omega u^2 + \frac{a}{b} \Big\{ (\mu+1) \int_\Omega v - \int_\Omega v^2 \Big\} \\
&\leq \frac{1}{4} \Big[(\lambda+1)^2 + \frac{a}{b} (\mu+1)^2 \Big] \cdot |\Omega| \qquad \text{for all } t \in (0, T_{max}).
\end{aligned}$$

Thus, $y(t) := \int_\Omega \Big(u(\cdot,t) + \frac{a}{b} v(\cdot,t) \Big)$, $t \in [0, T_{max})$, satisfies

$$\frac{d}{dt} y(t) + y(t) \leq \frac{1}{4} \Big[(\lambda+1)^2 + \frac{a}{b} (\mu+1)^2 \Big] \cdot |\Omega| \qquad \text{for all } t \in (0, T_{max}).$$

which upon a simple ODE comparison shows that

$$y(t) \leq \max \Big\{ \int_\Omega u_0 + \frac{a}{b} \int_\Omega v_0,\ \frac{1}{4} \Big[(\lambda+1)^2 + \frac{a}{b} (\mu+1)^2 \Big] \cdot |\Omega| \Big\} \qquad \text{for all } t \in (0, T_{max}). \tag{2.4}$$

Since integrating the third and fourth equations in (1.7) we find that

$$\int_\Omega w = \int_\Omega v \quad \text{and} \quad \int_\Omega z = \int_\Omega u \qquad \text{for all } t \in (0, T_{max}),$$

from (2.4) we readily obtain (2.2) and (2.3) thanks to the nonnegativity of u and v. $\square$

Next making essential use of the assumption $n \leq 3$ underlying Proposition 1.1, we shall turn the above L^1 bounds for w and z into L^p estimates for u and v. Concerning the latter second solution component v, this can be viewed as a fairly obvious extension of a related result already observed in [20, Theorem 1.1] for a proliferation-free variant of (1.7); the corresponding argument for u, however, will require an additional consideration here, basically reducing to the observation that within the course of a standard L^p testing procedure, the nonlinear source term $+auv$ in (1.7) can be compared in strength with the chemotactic term $-\chi \nabla \cdot (u \nabla w)$ (see (2.12) below).

Lemma 2.3 *Let $n \leq 3$, $\chi > 0, \xi > 0, \lambda > 0, \mu > 0, a > 0$ and $b > 0$. Then for any finite $p > 1$ one can find $C(p) > 0$ such that the solution of (1.7) satisfies*

$$\|v(\cdot,t)\|_{L^p(\Omega)} \leq C(p) \qquad \textit{for all } t \in (0, T_{max}) \tag{2.5}$$

as well as

$$\|u(\cdot,t)\|_{L^p(\Omega)} \leq C(p) \qquad \textit{for all } t \in (0, T_{max}). \tag{2.6}$$

PROOF. In essence, the proof follows the idea from [20]. So, we only outline the main steps and point out that the two terms of local kinetics will not induce any new technical difficulty in the proof.
Step 1. In light of known results on elliptic boundary problem with inhomogeneities in $L^1(\Omega)$ ([2]) together with the L^1 bound for u asserted in (2.2), we find that for any $s \in [1, \frac{n}{n-1})$, there exists $c_1 = c_1(s) > 0$ such that

$$\|z(\cdot,t)\|_{W^{1,s}(\Omega)} \leq c_1 \qquad \text{for all } t \in (0, T_{max}).$$

This in conjunction with the Sobolev embedding, $W^{1,s}(\Omega) \hookrightarrow L^r(\Omega)$ for any $r \in [1, \frac{ns}{(n-s)_+})$, yields that for all $r > 1$ fulfilling $r < \frac{n}{(n-2)_+}$, there exists $c_2 = c_2(r) > 0$ such that

$$\|z(\cdot,t)\|_{L^r(\Omega)} \leq c_2 \qquad \text{for all } t \in (0, T_{max}) \tag{2.7}$$

(cf. [20, lemma 3.1] for more details).
Step 2. We multiply the second equation in (1.7) by v^{p-1} and integrate by parts using the identity $\Delta z = z - u$ to obtain that

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int_\Omega v^p + \frac{4(p-1)}{p^2}\int_\Omega |\nabla v^{\frac{p}{2}}|^2 &= -(p-1)\xi\int_\Omega v^{p-1}\nabla v\cdot\nabla z + \mu\int_\Omega v^p - \int_\Omega v^{p+1} - b\int_\Omega uv^p \\ &\leq \frac{p-1}{p}\xi\int_\Omega v^p\Delta z + \mu\int_\Omega v^p \\ &= \frac{p-1}{p}\xi\int_\Omega v^p(z-u) + \mu\int_\Omega v^p \\ &\leq \frac{p-1}{p}\xi\int_\Omega v^p z + \mu\int_\Omega v^p \qquad \text{for all } t \in (0, T_{max}) \end{aligned} \tag{2.8}$$

thanks to the nonnegativity of u and v. Here since $n \leq 3$, we can fix $r > 1$ such that

$$\frac{n}{2} < r < \frac{n}{(n-2)_+}, \tag{2.9}$$

and thus relying on the Hölder inequality, using (2.7) and invoking the Gagliardo-Nirenberg inequality along with (2.3) we can find $c_3 = c_3(p) > 0$ and $c_4 = c_4(p) > 0$ such that

$$\begin{aligned} \frac{p-1}{p}\xi\int_\Omega v^p z &\leq \frac{p-1}{p}\xi\Big(\int_\Omega v^{pr'}\Big)^{\frac{1}{r'}}\cdot\Big(\int_\Omega z^r\Big)^{\frac{1}{r}} \\ &\leq c_3\Big(\int_\Omega v^{pr'}\Big)^{\frac{1}{r'}} \\ &= c_3\|v^{\frac{p}{2}}\|^2_{L^{2r'}(\Omega)} \\ &\leq \frac{2(p-1)}{p^2}\int_\Omega |\nabla v^{\frac{p}{2}}|^2 + c_4 \qquad \text{for all } t \in (0, T_{max}) \end{aligned}$$

where $r' := \frac{r}{r-1}$ satisfies $2r' < \frac{2n}{(n-2)_+}$ due to the left inequality in (2.9). Inserting this into (2.8) yields

$$\frac{d}{dt}\int_\Omega v^p + \int_\Omega v^p + \frac{2(p-1)}{p}\int_\Omega |\nabla v^{\frac{p}{2}}|^2 \le (1+\mu p)\int_\Omega v^p + pc_4 \qquad \text{for all } t \in (0, T_{max}). \tag{2.10}$$

Now using the Poincaré inequality and noting that $\|v^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{p}} = \int_\Omega v \le \frac{b}{a}m$ for all $t \in (0, T_{max})$ due to (2.3), we further obtain $c_5 = c_5(p) > 0$ such that

$$(1+\mu p)\int_\Omega v^p \le \frac{2(p-1)}{p}\int_\Omega |\nabla v^{\frac{p}{2}}|^2 + c_5 \qquad \text{for all } t \in (0, T_{max}),$$

which combined with (2.10) entails that

$$\frac{d}{dt}\int_\Omega v^p + \int_\Omega v^p \le pc_4 + c_5 \qquad \text{for all } t \in (0, T_{max}).$$

Upon an ODE comparison, this results in

$$\int_\Omega v^p(\cdot, t) \le c_6 = c_6(p) := \max\left\{\int_\Omega v_0^p, \quad pc_4 + c_5\right\} \qquad \text{for all } t \in (0, T_{max}) \tag{2.11}$$

and thereby proves (2.5).

Step 3. By straightforward computation using three integrations by parts, similar to the derivation of (2.8) we have

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 &= (p-1)\chi\int_\Omega u^{p-1}\nabla u\cdot\nabla w + \lambda\int_\Omega u^p - \int_\Omega u^{p+1} + a\int_\Omega u^p v \\
&\le -\frac{p-1}{p}\chi\int_\Omega u^p\Delta w + \lambda\int_\Omega u^p + a\int_\Omega u^p v \\
&= -\frac{p-1}{p}\chi\int_\Omega u^p(w-v) + \lambda\int_\Omega u^p + a\int_\Omega u^p v \\
&\le \Big(\frac{p-1}{p}\chi + a\Big)\int_\Omega u^p v + \lambda\int_\Omega u^p \qquad \text{for all } t \in (0, T_{max}),
\end{aligned} \tag{2.12}$$

where relying on the estimate (2.11) and proceeding as in Step 2 to deal with the first summand on the right hand side of (2.12), we obtain $c_7 = c_7(p) > 0$ such that

$$\Big(\frac{p-1}{p}\chi + a\Big)\int_\Omega u^p v \le \frac{2(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_7 \qquad \text{for all } t \in (0, T_{max}).$$

Combining this with (2.12) entails

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{2(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \le \lambda\int_\Omega u^p + c_7 \qquad \text{for all } t \in (0, T_{max}),$$

from which (2.6) can be derived, similarly to to reasoning in Step 2, through an application of the Poincaré inequality and (2.2). □

We now in a position to complete the proof of global existence and boundedness of solutions to (1.7).

PROOF of Proposition 1.1. In view of Lemma 2.3 and standard elliptic regularity theory ([5]), fixing any $p > n$ we obtain bounds for both w and z in $L^\infty((0,T_{max}); W^{2,p}(\Omega))$, which along with a Sobolev embedding theorem implies

$$\|\nabla w(\cdot,t)\|_{L^\infty(\Omega)} + \|\nabla z(\cdot,t)\|_{L^\infty(\Omega)} \le c_1 \qquad \text{for all } t \in (0,T_{max})$$

with some $c_1 > 0$. Using this information together with Lemma 2.3 and performing a Moser-type iteration (cf. [18, Lemma A.1]), we obtain $c_2 > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \le c_2 \qquad \text{for all } t \in (0,T_{max}).$$

In conjunction with the extensibility criterion (2.1) in Lemma 2.1, this immediately leads to the statements in Proposition 1.1. □

3 Large time behavior of bounded solutions

3.1 A general observation on evolution of functionals involving logarithms

Our qualitative analysis of bounded solutions to (1.7) will rely on the construction of Lyapunov functionals on the basis of the following observations.

Lemma 3.1 *Any global classical solution of (1.7) with $u \not\equiv 0$ and $v \not\equiv 0$ satisfies*

$$\frac{d}{dt}\int_\Omega \ln u \ge -\frac{\chi^2}{4}\int_\Omega |\nabla w|^2 + \lambda|\Omega| - \int_\Omega u + a\int_\Omega v \qquad \textit{for all } t>0 \tag{3.1}$$

and

$$\frac{d}{dt}\int_\Omega \ln v \ge -\frac{\xi^2}{4}\int_\Omega |\nabla z|^2 + \mu|\Omega| - \int_\Omega v - b\int_\Omega u \qquad \textit{for all } t>0 \tag{3.2}$$

as well as

$$\frac{d}{dt}\int_\Omega u = \lambda\int_\Omega u - \int_\Omega u^2 + a\int_\Omega uv \qquad \textit{for all } t>0 \tag{3.3}$$

and

$$\frac{d}{dt}\int_\Omega v = \mu\int_\Omega v - \int_\Omega v^2 - b\int_\Omega uv \qquad \textit{for all } t>0. \tag{3.4}$$

Moreover,

$$\int_\Omega |\nabla w|^2 = -\int_\Omega (w-v_\star)^2 + \int_\Omega (v-v_\star)(w-v_\star) \qquad \textit{for all } t>0 \tag{3.5}$$

and

$$\int_\Omega |\nabla z|^2 = -\int_\Omega (z-u_\star)^2 + \int_\Omega (u-u_\star)(z-u_\star) \qquad \textit{for all } t>0. \tag{3.6}$$

PROOF. Since u is positive in $\overline{\Omega} \times (0,\infty)$ by the strong maximum principle, we may multiply the first equation in (1.7) by $\frac{1}{u}$ to obtain using integration by parts that

$$\frac{d}{dt}\int_\Omega \ln u = \int_\Omega \frac{|\nabla u|^2}{u^2} - \int_\Omega \frac{\chi}{u}\nabla u \cdot \nabla w + \lambda|\Omega| - \int_\Omega u + a\int_\Omega v \qquad \text{for all } t>0.$$

As by Young's inequality we can estimate

$$\left| - \int_\Omega \frac{\chi}{u}\nabla u \cdot \nabla w \right| \le \int_\Omega \frac{|\nabla u|^2}{u^2} + \frac{\chi^2}{4}\int_\Omega |\nabla w|^2 \qquad \text{for all } t>0,$$

this yields (3.1). Similarly, testing the second equation in (1.7) against $\frac{1}{v}$ leads to (3.2), whereas the identities (3.3) and (3.4) directly result on integrating the first two equations in (1.7) over Ω. Furthermore, we use $(w - v_\star)$ as a testing function for the third equation in (1.7) to derive (3.5), whereas (3.6) is a consequence of testing the fourth equation in (1.7) by $(z - u_\star)$. $\square$

3.2 Coexistence. Proof of Theorem 1.2

Now when the taxis parameters χ and ξ are suitably small, assuming (1.9) enables us to discover a gradient-like structure in (1.7) in the following sense:

Lemma 3.2 *Let (1.9), (1.10) and (1.11) hold. Then there exist positive numbers α, β, η and γ such that given any global classical solution of (1.7) with $u \not\equiv 0$ and $v \not\equiv 0$, letting*

$$\mathcal{F}^{(1)} := \int_\Omega \Big(u(\cdot,t) - u_\star - u_\star \ln \frac{u(\cdot,t)}{u_\star}\Big) + \frac{a}{b}\int_\Omega \Big(v(\cdot,t) - v_\star - v_\star \ln \frac{v(\cdot,t)}{u_\star}\Big), \qquad t>0, \tag{3.7}$$

we have

$$\frac{d}{dt}\mathcal{F}^{(1)}(t) \le -\alpha\int_\Omega (u-u_\star)^2 - \beta\int_\Omega (v-v_\star)^2 - \eta\int_\Omega (w-v_\star)^2 - \gamma\int_\Omega (z-u_\star)^2 \qquad \textit{for all } t>0. \tag{3.8}$$

PROOF. As our assumptions (1.9), (1.10) and (1.11) imply that $\frac{\chi^2}{16}u_\star < \frac{a}{b}$ and $\frac{a\xi^2}{16b}v_\star < 1$, we can fix some $\varepsilon \in (0,1)$ such that

$$\alpha := 1 - \frac{a\xi^2}{16b(1-\varepsilon)}v_\star > 0 \tag{3.9}$$

and

$$\beta := \frac{a}{b} - \frac{\chi^2}{16(1-\varepsilon)}u_\star > 0. \tag{3.10}$$

We then combine the differential inequalities (3.1) and (3.2) with the identities (3.3) and (3.4) to see on straightforward rearrangements that

$$\begin{aligned}\frac{d}{dt}\mathcal{F}^{(1)} \le\ & \lambda\int_\Omega u - \int_\Omega u^2 + a\int_\Omega uv \\ & -u_\star \cdot \Big\{ -\frac{\chi^2}{4}\int_\Omega |\nabla w|^2 + \lambda|\Omega| - \int_\Omega u + a\int_\Omega v \Big\}\end{aligned}$$

$$
\begin{aligned}
& +\frac{a}{b}\cdot\Big\{\mu\int_\Omega v-\int_\Omega v^2-b\int_\Omega uv\Big\} \\
& -\frac{a}{b}v_\star\cdot\Big\{-\frac{\xi^2}{4}\int_\Omega|\nabla z|^2+\mu|\Omega|-\int_\Omega v-b\int_\Omega u\Big\} \\
= \;& \frac{\chi^2}{4}u_\star\cdot\int_\Omega|\nabla w|^2+\frac{a\xi^2}{4b}v_\star\cdot\int_\Omega|\nabla z|^2 \\
& -\int_\Omega u^2+(\lambda+u_\star+av_\star)\int_\Omega u \\
& -\frac{a}{b}\int_\Omega v^2+\Big(\frac{a}{b}\mu-au_\star+\frac{a}{b}v_\star\Big)\int_\Omega v \\
& -\lambda u_\star|\Omega|-\frac{a}{b}\mu v_\star|\Omega| \qquad \text{for all } t>0.
\end{aligned}
$$

Since direct computation shows that

$$
\begin{aligned}
-\int_\Omega u^2+(\lambda+u_\star+av_\star)\int_\Omega u &= -\int_\Omega u^2+2u_\star\int_\Omega u \\
&= -\int_\Omega(u-u_\star)^2+u_\star^2|\Omega| \qquad \text{for all } t>0
\end{aligned}
$$

and

$$
\begin{aligned}
-\frac{a}{b}\int_\Omega v^2+\Big(\frac{a}{b}\mu-au_\star+\frac{a}{b}v_\star\Big)\int_\Omega v &= \frac{a}{b}\cdot\Big\{-\int_\Omega v^2+(\mu-bu_\star+v_\star)\int_\Omega v\Big\} \\
&= \frac{a}{b}\cdot\Big\{-\int_\Omega v^2+2v_\star\int_\Omega v\Big\} \\
&= \frac{a}{b}\cdot\Big\{-\int_\Omega(v-v_\star)^2+v_\star^2|\Omega|\Big\} \qquad \text{for all } t>0
\end{aligned}
$$

as well as

$$
\Big\{-\lambda u_\star-\frac{a}{b}\mu v_\star+u_\star^2+\frac{a}{b}v_\star^2\Big\}\cdot|\Omega|=0,
$$

this entails that

$$
\frac{d}{dt}\mathcal{F}^{(1)}\le\frac{\chi^2}{4}u_\star\cdot\int_\Omega|\nabla w|^2+\frac{a\xi^2}{4b}v_\star\cdot\int_\Omega|\nabla z|^2-\int_\Omega(u-u_\star)^2-\frac{a}{b}\int_\Omega(v-v_\star)^2
$$

for all $t>0$. From this and the identities (3.5) and (3.6) we infer that

$$
\begin{aligned}
\frac{d}{dt}\mathcal{F}^{(1)} \le\;& -\frac{\chi^2}{4}u_\star\int_\Omega(w-v_\star)^2+\frac{\chi^2}{4}u_\star\int_\Omega(v-v_\star)(w-v_\star) \\
& -\frac{a\xi^2}{4b}v_\star\int_\Omega(z-u_\star)^2+\frac{a\xi^2}{4b}v_\star\int_\Omega(u-u_\star)(z-u_\star) \\
& -\int_\Omega(u-u_\star)^2-\frac{a}{b}\int_\Omega(v-v_\star)^2 \qquad \text{for all } t>0, \qquad (3.11)
\end{aligned}
$$

where Young's inequality implies that

$$\frac{\chi^2}{4}u_\star\int_\Omega (v-v_\star)(w-v_\star) \le \frac{\chi^2}{16(1-\varepsilon)}u_\star\int_\Omega (v-v_\star)^2 + \frac{\chi^2}{4}(1-\varepsilon)u_\star\int_\Omega (w-v_\star)^2$$

for all $t>0$ and

$$\frac{a\xi^2}{4b}v_\star\int_\Omega (u-u_\star)(z-u_\star) \le \frac{a\xi^2}{16b(1-\varepsilon)}v_\star\int_\Omega (u-u_\star)^2 + \frac{a\xi^2}{4b}(1-\varepsilon)v_\star\int_\Omega (z-u_\star)^2$$

for all $t>0$. This together with (3.11) yields (3.8) with α and β defined as in (3.9) and (3.10), and with $\eta := \varepsilon\cdot\frac{\chi^2}{4}u_\star > 0$ and $\gamma := \varepsilon\cdot\frac{a\xi^2}{4b}v_\star > 0$. □

We now assert asymptotic coexistence in the flavor of and under the assumptions of Theorem 1.2.
PROOF of Theorem 1.2. By means of standard parabolic and elliptic Schauder theory in conjunction with Proposition 1.1, we first obtain $\theta\in(0,1)$ and $c_1>0$ with the property that for all $t>1$,

$$\|u\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar\Omega\times[t,t+1])} + \|v\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar\Omega\times[t,t+1])} + \|w\|_{C^{2+\theta,\frac{\theta}{2}}(\bar\Omega\times[t,t+1])} + \|z\|_{C^{2+\theta,\frac{\theta}{2}}(\bar\Omega\times[t,t+1])} \le c_1. \quad (3.12)$$

As $(u(\cdot,t))_{t>1}$ and $(v(\cdot,t))_{t>1}$ are relatively compact in $C^2(\bar\Omega)$ by the Arzelá-Ascoli theorem, for the proof of (1.12) and (1.13) it is evidently sufficient to derive that

$$u(\cdot,t)\to u_\star \quad\text{and}\quad z(\cdot,t)\to u_\star \quad \text{in } L^\infty(\Omega) \qquad \text{as } t\to\infty \quad (3.13)$$

and

$$v(\cdot,t)\to v_\star \quad\text{and}\quad w(\cdot,t)\to v_\star \quad \text{in } L^\infty(\Omega) \qquad \text{as } t\to\infty. \quad (3.14)$$

But since Lemma 3.2 asserts that if we take $\mathcal{F}^{(1)}$, α,β,η and γ as introduced there, then since $\mathcal{F}^{(1)}$ can readily seen to be nonnegative, for all $t>1$ we have

$$\alpha\int_1^t\int_\Omega (u-u_\star)^2 + \beta\int_1^t\int_\Omega (v-v_\star)^2 + \eta\int_1^t\int_\Omega (w-v_\star)^2 + \gamma\int_1^t\int_\Omega (z-u_\star)^2 \le \mathcal{F}^{(1)}.$$

In view of the spatio-temporal equicontinuity property implied by (3.12), the basic decay information contained herein actually entails (3.13) and (3.14) (cf. [22] for a detailed reasoning on this by a contradictory argument). □

4 Extinction of preys. Proof of Theorem 1.3

Our construction of a Lyapunov functional in the framework of supercritical values of $b\lambda$ will, in addition to (3.1), (3.3) and (3.4), rely on the following observation that can directly be obtained upon testing the third equation in (1.7) by w.

Lemma 4.1 *If (u,v,w,z) is any global classical solution of (1.7), then*

$$\int_\Omega |\nabla w|^2 = -\int_\Omega w^2 + \int_\Omega vw \qquad \textit{for all } t>0. \quad (4.1)$$

Now under the assumption (1.14), we can find an energy-type structure in (1.7) for any $\xi > 0$ if the pursuit parameter χ is appropriately small.

Lemma 4.2 *If (1.14), (1.15) and (1.16) are valid, one can find $\delta > 0$ and $\kappa > 0$ such that bwhenever (u, v, w, z) is a global classical solution of (1.7) with $u \not\equiv 0$ and $v \not\equiv 0$, then*

$$\mathcal{F}^{(2)}(t) := \int_\Omega \Big(u(\cdot,t) - \lambda - \lambda \ln \frac{u(\cdot,t)}{\lambda}\Big) + \frac{a}{b}\int_\Omega v(\cdot,t), \qquad t > 0, \tag{4.2}$$

satisfies

$$\frac{d}{dt}\mathcal{F}^{(2)}(t) \le -\int_\Omega (u-\lambda)^2 - \delta \int_\Omega v^2 - \kappa \int_\Omega w^2 \qquad \textit{for all } t > 0. \tag{4.3}$$

Proof. Observing that our assumption (1.15) implies that $\frac{a}{b} - \frac{\lambda\chi^2}{16} > 0$, let us pick some $\varepsilon \in (0,1)$ such that

$$\delta := \frac{a}{b} - \frac{\lambda\chi^2}{16(1-\varepsilon)} > 0. \tag{4.4}$$

Thus, given a global solution fulfilling $u \not\equiv 0$, we may recall (3.1), (3.3) and (3.4) to obtain that

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}^{(2)}(t) &\le \lambda\int_\Omega u - \int_\Omega u^2 + a\int_\Omega uv \\
&\quad -\lambda\cdot\Big\{-\frac{\chi^2}{4}\int_\Omega |\nabla w|^2 + \lambda|\Omega| - \int_\Omega u + a\int_\Omega v\Big\} \\
&\quad +\frac{a}{b}\cdot\Big\{\mu\int_\Omega v - \int_\Omega v^2 - b\int_\Omega uv\Big\} \\
&= -\int_\Omega (u-\lambda)^2 \\
&\quad -\frac{a}{b}\int_\Omega v^2 + \frac{\lambda\chi^2}{4}\int_\Omega |\nabla w|^2 \\
&\quad -\frac{a}{b}\cdot(\lambda b - \mu)\int_\Omega v \qquad \text{for all } t > 0,
\end{aligned}$$

where the last summand on the right is nonpositive by our assumption that $\lambda b - \mu > 0$, and where (4.1) along with Young's inequality implies that

$$\begin{aligned}
-\frac{a}{b}\int_\Omega v^2 + \frac{\lambda\chi^2}{4}\int_\Omega |\nabla w|^2 &= -\frac{a}{b}\int_\Omega v^2 + \frac{\lambda\chi^2}{4}\int_\Omega vw - \frac{\lambda\chi^2}{4}\int_\Omega w^2 \\
&\le -\frac{a}{b}\int_\Omega v^2 + \frac{\lambda\chi^2}{16(1-\varepsilon)}\int_\Omega v^2 + \frac{\lambda\chi^2}{4}(1-\varepsilon)\int_\Omega w^2 - \frac{\lambda\chi^2}{4}\int_\Omega w^2 \\
&= -\Big(\frac{a}{b} - \frac{\lambda\chi^2}{16(1-\varepsilon)}\Big)\int_\Omega v^2 - \frac{\lambda\chi^2}{4}\varepsilon\int_\Omega w^2 \qquad \text{for all } t > 0,
\end{aligned}$$

so that (4.3) results from (4.4) if we let $\kappa := \frac{\lambda\chi^2}{4}\varepsilon$. $\square$

Using the decay information of the functional $\mathcal{F}^{(2)}(t)$ provided by Lemma 4.2, we can finally verify the statement made in Theorem 1.3.

Proof of Theorem 1.3. From Lemma 4.2 we infer that

$$\int_1^\infty \int_\Omega (u-\lambda)^2 < \infty, \qquad \int_1^\infty \int_\Omega v^2 < \infty \quad \text{and} \quad \int_1^\infty \int_\Omega w^2 < \infty, \tag{4.5}$$

because the function $\mathcal{F}^{(2)}$ from (4.2) is nonneative. Since multiplying the last equation in (1.7) by $(z-\lambda)$ and using Young's inequality we obtain that

$$\begin{aligned}
\int_\Omega |\nabla(z-\lambda)|^2 + \int_\Omega (z-\lambda)^2 &= \int_\Omega (u-\lambda)(z-\lambda) \\
&\le \frac{1}{2}\int_\Omega (z-\lambda)^2 + \frac{1}{2}\int_\Omega (u-\lambda)^2 \qquad \text{for all } t>0,
\end{aligned}$$

and since thus, by nonnegativity of the first summand on the left-hand side herein,

$$\int_\Omega (z-\lambda)^2 \le \int_\Omega (u-\lambda)^2 \qquad \text{for all } t>0,$$

the first inequality in (4.5) guarantees that furthermore

$$\int_1^\infty \int_\Omega (z-\lambda)^2 \le \int_1^\infty \int_\Omega (u-\lambda)^2 < \infty. \tag{4.6}$$

On the basis of (4.5) and (4.6), proceeding as in the proof of Theorem 1.2 we readily obtain the intended convergence features in (1.17) and (1.18). $\Box$

Acknowledgment. Y. Tao acknowledges support of the *National Natural Science Foundation of China (No. 11861131003).* M. Winkler was supported by the *Deutsche Forschungsgemeinschaft* in the context of the project *Emergence of structures and advantages in cross-diffusion systems* (No. 411007140, GZ: WI 3707/5-1).

References

[1] Amorim, P., Telch, B., Villada, L.M.: *A reaction-diffusion predator-prey model with pursuit, evasion, and nonlocal sensing.* Math. Biosci. Eng. **16**, 5114-5145 (2019)

[2] Bréezis, H., Strauss, W.A.: *Semilinear second-order elliptic equations in L^1.* J. Math. Soc. Japan **25**, 565-590 (1973)

[3] Cieślak, T., P. Laurençot, P., Morales-Rodrigo, C.: *Global existence and convergence to steady-states in a chemorepulsion system.* Banach Center Publ. **81**, Polish Acad. Sci., Warsaw, 105-117 (2008)

[4] Cieślak, T., Winkler, M.: *Finite-time blow-up in a quasilinear system of chemotaxis.* Nonlinearity **21**, 1057-1076 (2008)

[5] Friedman, A.: *Partial Differential Equations.* Holt, Rinehart & Winston, New York (1969)

[6] Goudon, T., Nkonga, B., Rascle, M., et. al.: *Self-organized populations interacting under pursuitevasion dynamics.* Physica D: Nonlinear Phenomena **304-305**, 1-22 (2015)

[7] Goudon, T., Urrutia, L.: *Analysis of kinetic and macroscopic models of pursuit-evasion dynamics.* Commun. Math. Sci. **14**, 2253-2286 (2016)

[8] He, X., Zheng, S.: *Global boundedness of solutions in a reaction-diffusion system of predator-prey model with prey-taxis.* Appl. Math. Lett. **49**, 73-77 (2015)

[9] Herrero, M.A., Velázquez, J.J.L.: *A blow-up mechanism for a chemotaxis model.* Ann. Scuola Normale Superiore Pisa **24**, 633-683 (1997)

[10] Horstmann, D.: *From 1970 until present: the Keller-Segel model in chemotaxis and its consequences I.* Jahresber. DMV **105**, 103-165 (2003)

[11] Jäger, W., Luckhaus, S.: *On explosions of solutions to a system of partial differential equations modelling chemotaxis.* Trans. Amer. Math. Soc. **329**, 819-824 (1992)

[12] Jin, H.-Y., Wang, Z.-A.: *Global stability of prey-taxis systems.* J. Differential Eq. **262**, 1257-1290 (2017)

[13] Kareiva, P., Odell, G.: *Swarms of predators exhibit 'preytaxis' if individual predators use arearestricted search.* Am. Nat. **130**, 233-270 (1987)

[14] Keller, E.F., Segel, L.A.: *Initiation of slime mold aggregation viewed as an instability.* J. Theor. Biol. **26**, 399-415 (1970)

[15] Lou, Y., Ni, W.M.: *Diffusion, self-diffusion and cross-diffusion.* J. Differential Equations **131**, 79-131 (1996)

[16] Tao, Y.: *Global existence of classical solutions to a predator-prey model with nonlinear prey-taxis.* Nonlin. Anal. Real World Appl. **11**, 2056-2064 (2010)

[17] Tao, Y., Wang, Z.A.: *Competing effects of attraction vs. repulsion in chemotaxis.* Math. Models Methods Appl. Sci. **23**, 1-36 (2013)

[18] Tao, Y., Winkler, M.: *Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity.* J. Differential Equations **252**, 692-715 (2012)

[19] Tao, Y., Winkler, M.: *Eventual smoothness and stabilization of large-data solutions in a three-dimensional chemotaxis system with consumption of chemoattractant.* J. Differential Eq. **252**, 2520-2543 (2012)

[20] Tao, Y., Winkler, M.: *Boundedness vs. blow-up in a two-species chemotaxis system with two chemicals.* Discr. Cont. Dyn. Syst. B **20**, 3165-3183 (2015)

[21] Tao, Y., Winkler, M.: *Global smooth solvability of a parabolic-elliptic nutrient taxis system in domains of arbitrary dimension.* J. Differential Equations **267**, 388-406 (2019)

[22] Tao, Y., Winkler, M.: *Boundedness and stabilization in a population model with cross-diffusion for one species.* Proc. London Math. Soc. (3) **119**, 1598-1632 (2019)

[23] Tsyganov, M.A., Kresteva, I.B., Medvinsky, A.B., Ivanitsky, G.R.: *A novel mode bacterial population wave interaction.* Dokl. Akad. Nauk **333**, 532-536 (1993)

[24] Tsyganov, M.A., Brindley, J., Holden, A.V., Biktashev, V.N.: *Quasi-soliton interaction of pursuit-evasion waves in a predator-prey system.* Physical Review Letters **91**, 218102 (2003)

[25] Tyutyunov, Y., Titova, L., Arditi, R.: *A Minimal Model of Pursuit-Evasion in a Predator-Prey System.* Math. Model. Nat. Pheno. **2**, 122-134 (2007)

[26] Winkler, M.: *Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system.* J. Math. Pures Appl. **100**, 748-767 (2013), `arXiv:1112.4156v1`

[27] Winkler, M.: *Asymptotic homogenization in a three-dimensional nutrient taxis system involving food-supported proliferation.* J. Differential Equations **263**, 4826-4869 (2017)

[28] Wu, S., Shi, J., Wu, B.: *Global existence of solutions and uniform persistence of a diffusive predator-prey model with prey-taxis.* J. Differential Eq. **260**, 5847-5874 (2016)

[29] Xiang, T.: *Global dynamics for a diffusive predator-prey model with prey-taxis and classical Lotka-Volterra kinetics.* Nonlinear Anal. Real World Appl. **39**, 278-299 (2018)