Mathematical Models and Methods in Applied Sciences

Relaxation by nonlinear diffusion enhancement in a two-dimensional cross-diffusion model for urban crime propagation

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We consider a class of macroscopic models for the spatio-temporal evolution of urban crime, as originally going back to [28]. The focus here is on the question of how far a certain porous medium enhancement in the random diffusion of criminal agents may exert visible relaxation effects. It is shown that sufficient regularity of the nonnegative source terms in the system and a sufficiently strong nonlinear enhancement ensure that a corresponding Neumann-type initial-boundary value problem, posed in a smoothly bounded planar convex domain, admits locally bounded solutions for a wide class of arbitrary initial data. Furthermore, this solution is globally bounded under mild additional conditions on the source terms. These results are supplemented by numerical evidence which illustrates smoothing effects in solutions with sharply structured initial data in the presence of such porous medium type diffusion and support the existence of singular structures in the linear diffusion case, which is the type of diffusion proposed in [28].

Keywords: urban crime; porous medium diffusion; global existence; a priori estimates

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1. Introduction

This manuscript is concerned with an adaptation of a macroscopic model for the dynamics of urban crime, such as residential burglaries. In its original version, as proposed in [28], this model takes the form

$$\begin{cases} u_t = \nabla\cdot(D\nabla u) - 2\nabla\cdot\left(\frac{u}{v}\nabla v\right) - uv + B_1(x,t), & x\in\Omega,\ t>0,\\ v_t = \Delta v + uv - v + B_2(x,t), & x\in\Omega,\ t>0, \end{cases} \tag{1.1}$$

System (1.1) is derived from an agent-based lattice model that incorporates the movement of *criminals*, u, and a scalar field representing the *attractiveness of crime*, v, measuring the appeal of a location from a criminal agents perspective. The system is grounded on the assumptions of routine activity theory and the so-called repeat and near-victimization effect ([9], [27]). Routine activity theory asserts that crime revolves around three factors: a potential offender, a suitable target, and the absence of guardianship ([9]).

The repeat and near-repeat victimization effect states that criminal activity in a certain location increases the probability of another crime occurring at the same, or nearby, locations within a short period of time. This effect has been measured in real-life data for crimes like residential burglaries ([15], [27]). In (1.1), this self-exciting nature of crime is incorporated in the assumption that each crime increases the attractiveness field, giving rise to the summand $+uv$ in the second equation (following from the fact that uv is the expected number of crimes). The near-repeat victimization effect is incorporated in the diffusivity of the attractiveness value, leading to the term Δv in the second equation.

It is also assumed that when criminal agents commit a crime they exit the system, which gives rise to the term $-uv$ in the first equation. To counteract the exit of criminal agents, the criminal population is subject to growth that is determined by the known function B_1. Moreover, an assumed base attractiveness value gives rise to the the growth function, B_2, observed in the second equation. We remark that for the analysis done in [28], it is assumed that B_1 and B_2 are constant functions; however, the authors mention that spatially and temporally dependent functions B_1 and B_2 are more realistic. On a final note, in system (1.1), criminals are assumed to move with a combination of unconditional dispersal, $D\Delta u$, and conditional dispersal, $-2\nabla \cdot \left(\frac{u}{v}\nabla v\right)$, biased by high values of attractiveness. In fact, in system (1.1), D is a constant and thus the criminals move with a combination of linear dispersal and a chemotactic-like dispersal.

Before we discuss the model of focus, let us mention some related previous works. The fundamental issue of the well-posedness of (1.1) has been addressed by various authors. In [26], the existence of global solutions to (1.1) in a one-dimensional interval has been established; in two-dimensional balls and in the presence of radially symmetric initial data, at least some globally defined generalized solutions have been constructed ([36]). The well-posedness of certain variants of (1.1) have been addressed in [20] and [25]. Beyond the well-posedness theory, the existence of spatially heterogeneous equilibrium solutions and their qualitative properties have also been addressed. In [29], a weakly nonlinear analysis around the bifurcation point between the linear stability and instability of the constant solutions is performed. Global bifurcation of spatially heterogeneous steady states emanating from the unique constant equilibrium solution is investigated in [6]. The existence and stability of spike-type equilibrium solutions to some related problems has been studied in [4], [12],

[17], [18], [21] and [33], for instance. Finally, we note that generalizations to system (1.1) that include police dynamics have been proposed and analyzed in [16], [23], and [24]; or where criminals disperse through a Lévy process over Brownian motion [8]. For other related models and their analysis we refer the reader to [2], [8], [5], [7], [19] and [37].

The adaptation of (1.1) that we consider in this work is based on the premise that criminals might have a tendency to avoid regions with a high density of other criminals. For example, this is reasonable when criminals want to avoid competition, or even suspect that hotspot policing, a strategy where the police force is deployed to areas with high crime, is being employed ([29]). In such cases, criminals might want to avoid areas with a high police density (or equivalently areas with a high criminal density). The assumption that criminal agents tend to avoid police officers is a natural consequence of routine activity theory. Indeed, one of the factors needed for crime to occur, based on this theory, is absence of guardianship. Thus, criminal agents will tend not commit a crime in locations where there are police agents, but will instead choose to move away from areas with a high density of police.

A natural approach to incorporate such a change in the movement strategy of criminal agents consists in allowing the diffusion rate D to depend on u – recall that in (1.1), D is a constant. In particular, we assume that the diffusivity of criminals increases with u, thus modeling an overcrowding effect. Here we concentrate on the prototypical algebraic choice of $D = D(u)$, hence leading to porous medium type diffusion operators. In the framework of a full no-flux initial-boundary value problem we subsequently consider the variant of (1.1) given by

$$\begin{cases} u_t = \nabla\cdot(u^{m-1}\nabla u) - \chi\nabla\cdot\left(\frac{u}{v}\nabla v\right) - uv + B_1(x,t), & x\in\Omega,\ t>0,\\ v_t = \Delta v + uv - v + B_2(x,t), & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{1.2}$$

in a bounded domain $\Omega\subset\mathbb{R}^2$ with smooth boundary. Here, B_1 and B_2 are suitably regular nonnegative functions on $\Omega\times(0,\infty)$, $m>1$ is a given parameter and χ is allowed to attain any positive value, thus including the choice $\chi=2$ in (1.1) as a special case.

We note that in order to keep the modeling framework as simple as possible, in this work we do not independently model the dynamics of the police force by, e.g., describing their population density through an additional variable, but rather we make the simplifying assumption that the police force will match those of the criminal agents.

Main results: Blow-up suppression by strong diffusion enhancement. Due to the potentially substantial destabilizing character of the self-enhanced cross-diffusive interaction therein, systems of the form (1.1) seem to bring about significant challenges already at the level of basic solution theories. Accordingly, the few analytical findings available for (1.1) and related systems are either restricted to spatially one-dimensional settings ([26]), or address ranges of suitably small χ which do not contain the relevant choice $\chi = 2$ ([10]), or concentrate on certain small-data solutions in cases of sufficiently small B_1 and B_2 ([31]), or resort to strongly generalized concepts of solvability which do not a priori preclude the emergence of singularities within finite time ([13], [36]). Although apparently no analytical study has rigorously detected the occurrence of such phenomena yet, the outcome of numerical experiments supports the conjecture that indeed the linear diffusion mechanism in (1.1) is insufficient to rule out the possibility of explosions (cf. also Section 9).

In contrast to this, we shall see that the presence of suitably strong nonlinear diffusion enhancement entirely suppresses any such singular behavior in (1.2) within finite time intervals, as expressed in the following statement on global existence of locally bounded solutions:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with smooth boundary, and suppose that $\chi > 0$, that*

$$B_1 \text{ and } B_2 \text{ are nonnegative functions from } C^1(\overline{\Omega} \times [0,\infty)), \tag{B}$$

and that

$$m > \frac{3}{2}. \tag{1.3}$$

Then for any choice of functions u_0 and v_0 which are such that

$$\begin{cases} u_0 \in W^{1,\infty}(\Omega) & \text{is nonnegative, and that} \\ v_0 \in W^{1,\infty}(\Omega) & \text{is positive in } \overline{\Omega}, \end{cases} \tag{1.4}$$

the problem (1.2) possesses at least one global weak solution (u,v) in the sense of Definition 8.1 below. This solution is locally bounded in that

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty \qquad \text{for all } T > 0$$

and

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|v(\cdot,t)\|_{W^{1,q}(\Omega)} < \infty \qquad \text{for all } T > 0 \text{ and } q > 2.$$

Under quite mild additional assumptions on B_1 and B_2, particularly fulfilled by any nonnegative $B_1 = B_1(x) \in C^1(\overline{\Omega})$ and $0 \not\equiv B_2 = B_2(x) \in C^1(\overline{\Omega})$, solutions can be found which are in fact globally bounded, meaning that in such cases moreover even any infinite-time singularity formation is ruled out:

Theorem 1.2. *Assume that $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with smooth boundary, that $m > \frac{3}{2}$ and $\chi > 0$, and that (u_0, v_0) satisfies (1.4), and suppose furthermore that B_1 and B_2 are such that beyond (B) we have*

$$\sup_{(x,t)\in\Omega\times(0,\infty)} \Big\{ B_1(x,t) + B_2(x,t) \Big\} < \infty \tag{B1}$$

and

$$\liminf_{t\to\infty} \int_\Omega B_2(x,t)dx > 0. \tag{B2}$$

Then (1.2) admits a global weak solution according to Definition 8.1 which is globally bounded in the sense that

$$\operatorname*{ess\,sup}_{t>0} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty \tag{1.5}$$

and

$$\operatorname*{ess\,sup}_{t>0} \|v(\cdot,t)\|_{W^{1,q}(\Omega)} < \infty \qquad \text{for all } q > 2. \tag{1.6}$$

Accompanied and illustrated by outcomes of corresponding numerical simulations, to be presented in Section 9, these results quantitatively identify an effect of the considered diffusion strengthening on overcrowding prevention. This seems to indicate that nonlinear migration mechanisms of the said flavor may stabilize systems of the considered form by precluding a model breakdown due to the emergence of singularities. Viewed in the contexts of the addressed application seems to be of relevance, especially due to the nontrivial size of criminal agents. As partially seen in Section 9, the description of crime hotspot formation, as known to occur in associated typical real-life situations, is thereby transported to mathematical sceneries involving structured but bounded spatial profiles, rather than exploding solutions such as naturally going along with Keller-Segel type modeling of aggregation in populations of microbial individuals ([14], [35]; see also [3]).

2. Regularization and basic properties

In order to conveniently regularize (1.2), we combine the essence of the corresponding procedure in [36] with a standard non-degenerate approximation of porous medium type diffusion operators, and hence we shall subsequently consider the problems

$$\begin{cases} u_{\varepsilon t} = \nabla\cdot\big((u_\varepsilon+\varepsilon)^{m-1}\nabla u_\varepsilon\big) - \chi\nabla\cdot\Big(\frac{u_\varepsilon}{v_\varepsilon}\nabla v_\varepsilon\Big) - u_\varepsilon v_\varepsilon + B_1(x,t), & x\in\Omega,\ t>0,\\ v_{\varepsilon t} = \Delta v_\varepsilon + \frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} - v_\varepsilon + B_2(x,t), & x\in\Omega,\ t>0,\\ \frac{\partial u_\varepsilon}{\partial\nu} = \frac{\partial v_\varepsilon}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u_\varepsilon(x,0)=u_0(x), \quad v_\varepsilon(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{2.1}$$

for $\varepsilon\in(0,1)$, which indeed are all globally solvable in the classical sense:

Lemma 2.1. *Assume (B) and (1.4), and let $m>1$ and $\varepsilon\in(0,1)$. Then there exist functions*

$$\begin{cases} u_\varepsilon \in C^0(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)),\\ v_\varepsilon \in \bigcap_{p>2} C^0([0,\infty);W^{1,p}(\Omega)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)), \end{cases}$$

which solve (2.1) classically in $\overline{\Omega} \times [0,\infty)$, *and which are such that* $u_\varepsilon > 0$ *in* $\overline{\Omega} \times (0,\infty)$ *and* $v_\varepsilon > 0$ *in* $\overline{\Omega} \times [0,\infty)$.

Proof. This can be seen by a straightforward adaptation of the reasoning in [36] on the basis of standard results on local existence and extensibility, as provided e.g. by the general theory in [1]. □

Throughout the sequel, without further explicit mentioning we shall assume that (B) and (1.4) are satisfied, and for $m > 1$ and $\varepsilon \in (0,1)$ we let $(u_\varepsilon, v_\varepsilon)$ denote the solutions of (2.1) gained above.

In our respective formulation of statements on regularity of these solutions, we find it convenient to make use of the following notational convention concerning a certain time independence of constants under the hypotheses (B1) and (B2).

Definition 2.1. Let $K : (0,\infty) \to (0,\infty)$. We then say that K satisfies (K) if K has the property that

$$\sup_{T>0} K(T) < \infty \qquad \text{whenever (B1) and (B2) hold.}$$

With reference to this property, our first basic statement on a pointwise lower bound for the second solution component, resembling similar information found in [26] and [36] already, reads as follows.

Lemma 2.2. *Let* $m > 1$. *Then there exists* $K : (0,\infty) \to (0,\infty)$ *fulfilling (K) such that whenever* $T > 0$,

$$\frac{1}{v_\varepsilon(x,t)} \le K(T) \qquad \textit{for all } x \in \Omega, t \in (0,T) \textit{ and } \varepsilon \in (0,1). \tag{2.2}$$

Proof. Firstly, in view of the nonnegativity of $u_\varepsilon, v_\varepsilon$ and B_2 it follows by a comparison argument that

$$v_\varepsilon(x,t) \ge \left\{ \inf_{y\in\Omega} v_0(y) \right\} \cdot e^{-t} \qquad \text{for all } x \in \Omega, t > 0 \text{ and } \varepsilon \in (0,1). \tag{2.3}$$

Moreover, the convexity of Ω allows us to import from [11] a result on a pointwise positivity feature of the Neumann heat semigroup $(e^{t\Delta})_{t\ge 0}$ on Ω to fix $c_1 > 0$ fulfilling

$$e^{t\Delta}\psi \ge c_1 \int_\Omega \psi \quad \text{in } \Omega \qquad \text{for all } t > 1 \text{ and any nonnegative } \psi \in C^0(\overline{\Omega}),$$

whence again by the comparison principle, for arbitrary $t_0 \geq 0$ we can estimate

$$\begin{aligned}
v_\varepsilon(\cdot,t) &= e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)}\Big\{\frac{u_\varepsilon(\cdot,s)v_\varepsilon(\cdot,s)}{1+\varepsilon u_\varepsilon(\cdot,s)v_\varepsilon(\cdot,s)} + B_2(\cdot,s)\Big\}ds \\
&\geq c_1 \int_{t_0}^t e^{-(t-s)} \cdot \Big\{\int_\Omega B_2(\cdot,s)\Big\}ds \\
&\geq c_1 \cdot \Big\{\inf_{s>t_0}\int_\Omega B_2(\cdot,s)\Big\} \cdot \int_{t_0}^t e^{-(t-s)}ds \\
&= c_1 \cdot \Big\{\inf_{s>t_0}\int_\Omega B_2(\cdot,s)\Big\} \cdot (1-e^{-(t-t_0)}) \\
&\geq (1-e^{-1})c_1 \cdot \Big\{\inf_{s>t_0}\int_\Omega B_2(\cdot,s)\Big\} \quad \text{in } \Omega \qquad \text{for all } t > t_0+1. \qquad (2.4)
\end{aligned}$$

Combining (2.3) with (2.4) readily yields (2.2) with some K satisfying (K). □

Likewise, our second basic observation has quite closely related precedents in [26] and [36].

Lemma 2.3. *Let $m > 1$. Then there exists $K : (0,\infty) \to (0,\infty)$ such that (K) holds, and such that for all $T > 0$,*

$$\int_\Omega u_\varepsilon(\cdot,t) \leq K(T) \qquad \textit{for all } t \in (0,T) \textit{ and } \varepsilon \in (0,1), \qquad (2.5)$$

and that

$$\int_\Omega v_\varepsilon(\cdot,t) \leq K(T) \qquad \textit{for all } t \in (0,T) \textit{ and } \varepsilon \in (0,1). \qquad (2.6)$$

Proof. According to Lemma 2.2, we can find $k_1 : (0,\infty) \to (0,\infty)$ with the corresponding property (K) such that for all $T > 0$,

$$\frac{1}{v_\varepsilon} \leq k_1(T) \quad \text{in } \Omega \times (0,T) \qquad \text{for all } \varepsilon \in (0,1). \qquad (2.7)$$

Then letting

$$k_2(T) := \min\Big\{1\,,\,\frac{1}{2k_1(T)}\Big\}, \qquad T > 0, \qquad (2.8)$$

we use (2.1) to see that given any $T > 0$, for all $t > 0$ and each $\varepsilon \in (0,1)$ we have

$$\begin{aligned}
\frac{d}{dt}\Big\{2\int_\Omega u_\varepsilon + \int_\Omega v_\varepsilon\Big\} &+ k_2(T) \cdot \Big\{2\int_\Omega u_\varepsilon + \int_\Omega v_\varepsilon\Big\} \\
&= -2\int_\Omega u_\varepsilon v_\varepsilon + 2\int_\Omega B_1 \\
&\quad -\int_\Omega v_\varepsilon + \int_\Omega \frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} + \int_\Omega B_2 \\
&\quad +2k_2(T)\int_\Omega u_\varepsilon + k_2(T)\int_\Omega v_\varepsilon \\
&\leq -\int_\Omega u_\varepsilon v_\varepsilon + 2k_2(T)\int_\Omega u_\varepsilon + 2\int_\Omega B_1 + \int_\Omega B_2, \qquad (2.9)
\end{aligned}$$

because $k_2(T) \le 1$. Using that moreover $2k_2(T) \le \frac{1}{k_1(T)}$ and hence

$$-\int_\Omega u_\varepsilon v_\varepsilon + 2k_2(T)\int_\Omega u_\varepsilon \le -\frac{1}{k_1(T)}\int_\Omega u_\varepsilon + 2k_2(T)\int_\Omega u_\varepsilon \le 0$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$, by (2.7), from (2.9) we infer that

$$\frac{d}{dt}\Big\{2\int_\Omega u_\varepsilon + \int_\Omega v_\varepsilon\Big\} + k_2(T)\cdot\Big\{2\int_\Omega u_\varepsilon + \int_\Omega v_\varepsilon\Big\}$$
$$\le k_3(T) := \sup_{s\in(0,T)}\Big\{2\int_\Omega B_1 + \int_\Omega B_2\Big\}$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$. Therefore, an ODE comparison shows that

$$2\int_\Omega u_\varepsilon(\cdot,t) + \int_\Omega v_\varepsilon(\cdot,t) \le \max\Big\{2\int_\Omega u_0 + \int_\Omega v_0\,,\,\frac{k_3(T)}{k_2(T)}\Big\}$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$, from which both (2.5) and (2.6) result upon the observation that in view of (2.7), (K) holds for the function $\frac{k_3}{k_2}$. □

3. Estimates for v_ε in $W^{1,q}(\Omega)$ with $q \le 2$

The following estimate essentially reproduces a similar finding from [36] to the present framework involving slightly different hypotheses on B_1 and B_2.

Lemma 3.1. *Assume that $m > 1$, and let $p \in (0,1)$. Then there exists a function $K \equiv K^{(p)} : (0,\infty) \to (0,\infty)$ which satisfies (K) and is such that whenever $T > 0$,*

$$\int_t^{t+1}\int_\Omega v_\varepsilon^{p-2}|\nabla v_\varepsilon|^2 \le K(T) \qquad \textit{for all } t \in (0,T) \textit{ and } \varepsilon \in (0,1). \tag{3.1}$$

Proof. Relying on Lemma 2.3, we can fix a mapping $k_1 : (0,\infty) \to (0,\infty)$ which enjoys the boundedness feature in (K) and is such that for all $T > 0$,

$$\int_\Omega v_\varepsilon(\cdot,t) \le k_1(T) \qquad \text{for all } t \in (0,T+1) \text{ and } \varepsilon \in (0,1),$$

whence given $p \in (0,1)$ we can use Young's inequality to see that

$$\int_\Omega v_\varepsilon^p(\cdot,t) \le \int_\Omega \Big(v_\varepsilon(\cdot,t)+1\Big) \le k_1(T) + |\Omega| \qquad \text{for all } t \in (0,T+1) \text{ and } \varepsilon \in (0,1). \tag{3.2}$$

Since according to (2.1) we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega v_\varepsilon^p = (1-p)\int_\Omega v_\varepsilon^{p-2}|\nabla v_\varepsilon|^2 - \int_\Omega v_\varepsilon^p + \int_\Omega \frac{u_\varepsilon v_\varepsilon^p}{1+\varepsilon u_\varepsilon} + \int_\Omega v_\varepsilon^{p-1}B_2$$
$$\ge (1-p)\int_\Omega v_\varepsilon^{p-2}|\nabla v_\varepsilon|^2 - \int_\Omega v_\varepsilon^p \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1)$$

and thus

$$(1-p)\int_t^{t+1}\int_\Omega v_\varepsilon^{p-2}|\nabla v_\varepsilon|^2 \le \frac{1}{p}\int_\Omega v_\varepsilon^p(\cdot,t+1) - \frac{1}{p}\int_\Omega v_\varepsilon^p(\cdot,t) + \int_t^{t+1}\int_\Omega v_\varepsilon^p$$
$$\le \frac{1}{p}\int_\Omega v_\varepsilon^p(\cdot,t+1) + \int_t^{t+1}\int_\Omega v_\varepsilon^p$$

for all $t>0$ and $\varepsilon\in(0,1)$, utilizing (3.2) to estimate

$$\frac{1}{p}\int_\Omega v_\varepsilon^p(\cdot,t+1) + \int_t^{t+1}\int_\Omega v_\varepsilon^p \le \frac{1}{p}\cdot\big(k_1(T)+|\Omega|\big) + k_1(T) + |\Omega|$$

for all $t\in(0,T)$ and $\varepsilon\in(0,1)$. We arrive at (3.1) upon an evident choice of K. □

Besides being of independent use in some of our subsequent estimates (see Lemma 4.1 and Lemma 7.2), Lemma 3.1, through suitable interpolation involving Lemma 2.3, also entails the following boundedness property of v_ε with respect to the norm in $W^{1,q}(\Omega)$ for $q\in[1,2)$ arbitrarily close to 2.

Lemma 3.2. *Suppose that $m>1$ and let $q\in[1,2)$. Then there exists $K\equiv K^{(q)}:(0,\infty)\to(0,\infty)$ fulfilling (K) with the property that for all $T>0$, any $\varepsilon\in(0,1)$ and each $t\in(0,T)$ fulfilling $t\ge 2$ one can find $t_0=t_0(t,\varepsilon)\in(t-2,t-1)$ such that*

$$\|v_\varepsilon(\cdot,t_0)\|_{W^{1,q}(\Omega)} \le K(T). \tag{3.3}$$

Proof. We evidently need to define $K(T)$ for $T\ge 2$ only, and to achieve this we first employ Lemma 3.1 and Lemma 2.3 to find $k_i:(0,\infty)\to(0,\infty)$, $i\in\{1,2\}$, which comply with (K) and are such that whenever $T\ge 2$,

$$\int_{t-2}^{t-1}\int_\Omega v_\varepsilon^{-\frac{3}{2}}|\nabla v_\varepsilon|^2 \le k_1(T) \qquad \text{for all } t\in[2,T] \text{ and } \varepsilon\in(0,1) \tag{3.4}$$

and

$$\int_\Omega v_\varepsilon \le k_2(T) \qquad \text{for all } t\in(0,T) \text{ and } \varepsilon\in(0,1). \tag{3.5}$$

Moreover, given $q\in[1,2)$ we define $p=p(q):=\frac{3q}{2(2-q)}>1$ and make use of the continuity of the embedding $W^{1,2}(\Omega)\hookrightarrow L^{4p}(\Omega)$ to fix $c_1=c_1(q)>0$ such that

$$\|\varphi\|_{L^{4p}(\Omega)}^{4p} \le c_1\|\nabla\varphi\|_{L^2(\Omega)}^{4p} + c_1\|\varphi\|_{L^4(\Omega)}^{4p} \qquad \text{for all } \varphi\in W^{1,2}(\Omega). \tag{3.6}$$

Now letting $T\ge 2$ and $t\in[2,T]$ be arbitrary, from (3.4) we infer the existence of $t_0=t_0(t,\varepsilon)\in(t-2,t-1)$ such that

$$\int_\Omega v_\varepsilon^{-\frac{3}{2}}(\cdot,t_0)|\nabla v_\varepsilon(\cdot,t_0)|^2 \le k_1(T), \tag{3.7}$$

which in conjunction with (3.6) and (3.5) entails that

$$\begin{aligned}
\int_\Omega v_\varepsilon^p(\cdot,t_0) &= \|v_\varepsilon^{\frac{1}{4}}(\cdot,t_0)\|_{L^{4p}(\Omega)}^{4p} \\
&\le c_1\|\nabla v_\varepsilon^{\frac{1}{4}}(\cdot,t_0)\|_{L^2(\Omega)}^{4p} + c_1\|v_\varepsilon^{\frac{1}{4}}(\cdot,t_0)\|_{L^4(\Omega)}^{4p} \\
&= \frac{c_1}{4^{4p}} \cdot \Big\{ \int_\Omega v_\varepsilon^{-\frac{3}{2}}(\cdot,t_0)|\nabla v_\varepsilon(\cdot,t_0)|^2 \Big\}^{2p} + c_1 \cdot \Big\{ \int_\Omega v_\varepsilon(\cdot,t_0) \Big\}^p \\
&\le k_3(T) \equiv k_3^{(q)}(T) := \frac{c_1}{4^{4p}} \cdot k_1^{2p}(T) + c_1 k_p^2(T).
\end{aligned}$$

Once more combined with (3.7), due to Young's inequality and thanks to our definition of p this shows that

$$\begin{aligned}
\int_\Omega |\nabla v_\varepsilon(\cdot,t_0)|^q &= \int_\Omega \Big\{ v_\varepsilon^{-\frac{3}{2}}(\cdot,t_0)|\nabla v_\varepsilon(\cdot,t_0)|^2 \Big\}^{\frac{q}{2}} \cdot v_\varepsilon^{\frac{3q}{4}}(\cdot,t_0) \\
&\le \int_\Omega v_\varepsilon^{-\frac{3}{2}}(\cdot,t_0)|\nabla v_\varepsilon(\cdot,t_0)|^2 + \int_\Omega v_\varepsilon^{\frac{3q}{2(2-q)}}(\cdot,t_0) \\
&\le k_1(T) + k_3(T).
\end{aligned}$$

In view of (3.5), this implies the claimed boundedness property in $W^{1,q}(\Omega)$. $\square$

4. Superlinear integrability properties of u_ε

Our derivation of further regularity properties of v_ε will crucially rely on the following a priori information on the first solution component, obtained by means of a standard testing procedure on the basis of Lemma 3.1.

Lemma 4.1. *Let $m > 1$. Then there exists $K : (0,\infty) \to (0,\infty)$ satisfying (K) such that if $T > 0$ then*

$$\begin{cases}
\int_t^{t+1}\int_\Omega (u_\varepsilon+\varepsilon)^{m-1}(u_\varepsilon+1)^{m-3}|\nabla u_\varepsilon|^2 \le K(T) \quad \textit{for all } t\in(0,T) \textit{ and } \varepsilon\in(0,1), \\
\hfill \textit{if } m\in(1,2], \\
\int_t^{t+1}\int_\Omega (u_\varepsilon+\varepsilon)^{2m-4}|\nabla u_\varepsilon|^2 \le K(T) \quad \textit{for all } t\in(0,T) \textit{ and } \varepsilon\in(0,1) \\
\hfill \textit{if } m>2.
\end{cases} \tag{4.1}$$

Proof. We again return to Lemma 2.2 and additionally employ Lemma 3.1 to pick $k_i : (0,\infty) \to (0,\infty)$, $i \in \{1,2\}$, which comply with (K) and are such that if $T > 0$ then

$$v_\varepsilon \ge \frac{1}{k_1(T)} \quad \text{in } \Omega\times(0,T) \qquad \text{for all } \varepsilon\in(0,1) \tag{4.2}$$

and

$$\int_t^{t+1}\int_\Omega v_\varepsilon^{-\frac{3}{2}}|\nabla v_\varepsilon|^2 \le k_2(T) \qquad \text{for all } t\in(0,T) \text{ and } \varepsilon\in(0,1), \tag{4.3}$$

and to further prepare our argument for large m we utilize Young's inequality together with (B1) to find $k_3 : (0,\infty) \to (0,\infty)$ which is such that (K) holds and that if $m > 2$, then whenever $T > 0$,

$$\left\{ \frac{1}{k_1(T)} + \|B_1(\cdot,t)\|_{L^\infty(\Omega)} \right\} \cdot \xi^{m-2} \le \frac{1}{2k_1(T)} \xi^{m-1} + k_3(T) \qquad \text{for all } t \in (0,T). \tag{4.4}$$

Now for such m, we use $(u_\varepsilon+\varepsilon)^{m-2}$ as a test function in (2.1) to see that

$$\begin{aligned} &\frac{1}{m-1}\frac{d}{dt}\int_\Omega (u_\varepsilon+\varepsilon)^{m-1} + (m-2)\int_\Omega (u_\varepsilon+\varepsilon)^{2m-4}|\nabla u_\varepsilon|^2 \\ &\quad = (m-2)\chi \int_\Omega u_\varepsilon (u_\varepsilon+\varepsilon)^{m-3}\nabla u_\varepsilon \cdot \frac{\nabla v_\varepsilon}{v_\varepsilon} \\ &\qquad - \int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{m-2} v_\varepsilon + \int_\Omega (u_\varepsilon+\varepsilon)^{m-2} B_1 \qquad \text{for all } t>0 \text{ and } \varepsilon \in (0,1), \end{aligned} \tag{4.5}$$

where once more by Young's inequality, and by (4.2), given $T > 0$ we can estimate

$$\begin{aligned} &(m-2)\chi \int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{m-3}\nabla u_\varepsilon \cdot \frac{\nabla v_\varepsilon}{v_\varepsilon} \\ &\le \frac{m-2}{2}\int_\Omega (u_\varepsilon+\varepsilon)^{2m-4}|\nabla u_\varepsilon|^2 + \frac{(m-2)\chi^2}{2}\int_\Omega \Big(\frac{u_\varepsilon}{u_\varepsilon+\varepsilon}\Big)^2 \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} \\ &\le \frac{m-2}{2}\int_\Omega (u_\varepsilon+\varepsilon)^{2m-4}|\nabla u_\varepsilon|^2 + \frac{(m-2)\chi^2}{2} k_1^{\frac12}(T) \int_\Omega v_\varepsilon^{-\frac32}|\nabla v_\varepsilon|^2 \end{aligned}$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$, and where (4.2) together with (4.4) ensures that for all $T > 0$,

$$\begin{aligned} &-\int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{m-2} v_\varepsilon + \int_\Omega (u_\varepsilon+\varepsilon)^{m-2}B_1 \\ &\quad \le -\frac{1}{k_1(T)}\int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{m-2} + \int_\Omega (u_\varepsilon+\varepsilon)^{m-2}B_1 \\ &\quad = -\frac{1}{k_1(T)}\int_\Omega (u_\varepsilon+\varepsilon)^{m-1} + \frac{\varepsilon}{k_1(T)}\int_\Omega (u_\varepsilon+\varepsilon)^{m-2} + \int_\Omega (u_\varepsilon+\varepsilon)^{m-2}B_1 \\ &\quad \le -\frac{1}{k_1(T)}\int_\Omega (u_\varepsilon+\varepsilon)^{m-1} + \left\{\frac{1}{k_1(T)} + \|B_1(\cdot,t)\|_{L^\infty(\Omega)}\right\} \cdot \int_\Omega (u_\varepsilon+\varepsilon)^{m-2} \\ &\quad \le -\frac{1}{2k_1(T)}\int_\Omega (u_\varepsilon+\varepsilon)^{m-1} + k_3(T)|\Omega| \qquad \text{for all } t\in(0,T) \text{ and } \varepsilon\in(0,1). \end{aligned}$$

Therefore, (4.5) entails that whenever $T > 0$,

$$\begin{aligned} &\frac{1}{m-1}\frac{d}{dt}\int_\Omega (u_\varepsilon+\varepsilon)^{m-1} + \frac{m-2}{2}\int_\Omega (u_\varepsilon+\varepsilon)^{2m-4}|\nabla u_\varepsilon|^2 + \frac{1}{2k_1(T)}\int_\Omega (u_\varepsilon+\varepsilon)^{m-1} \\ &\quad \le \frac{(m-2)\chi^2}{2} k_1^{\frac12}(T)\int_\Omega v_\varepsilon^{-\frac32}|\nabla v_\varepsilon|^2 + k_3(T)|\Omega| \end{aligned} \tag{4.6}$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$, from which in light of (4.3) the respective inequality in (4.1) readily results upon an integration in time.

We are thus left with the case when $m \in (1,2]$, in which we let $\Phi(\xi) := \int_0^\xi \int_0^\sigma (\tau + 1)^{m-3} d\tau d\sigma$, $\xi \ge 0$, and noting that $\Phi''(\xi) = (\xi+1)^{m-3}$ for all $\xi \ge 0$ we once more resort to (2.1) to see that similarly to the above, given an arbitrary $T > 0$ we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \Phi(u_\varepsilon) &= \int_\Omega \Phi'(u_\varepsilon) \cdot \Big\{ \nabla \cdot \big((u_\varepsilon+\varepsilon)^{m-1}\nabla u_\varepsilon\big) - \chi \nabla \cdot \Big(\frac{u_\varepsilon}{v_\varepsilon}\nabla v_\varepsilon\Big) - u_\varepsilon v_\varepsilon + B_1 \Big\} \\
&= -\int_\Omega (u_\varepsilon+\varepsilon)^{m-1}\Phi''(u_\varepsilon)|\nabla u_\varepsilon|^2 + \chi \int_\Omega u_\varepsilon \Phi''(u_\varepsilon)\nabla u_\varepsilon \cdot \frac{\nabla v_\varepsilon}{v_\varepsilon} \\
&\quad - \int_\Omega u_\varepsilon \Phi'(u_\varepsilon) v_\varepsilon + \int_\Omega \Phi'(u_\varepsilon) B_1 \\
&= -\int_\Omega (u_\varepsilon+\varepsilon)^{m-1}(u_\varepsilon+1)^{m-3}|\nabla u_\varepsilon|^2 + \chi \int_\Omega u_\varepsilon (u_\varepsilon+1)^{m-3}\nabla u_\varepsilon \cdot \frac{\nabla v_\varepsilon}{v_\varepsilon} \\
&\quad - \int_\Omega u_\varepsilon \Phi'(u_\varepsilon) v_\varepsilon + \int_\Omega \Phi'(u_\varepsilon) B_1 \\
&\le -\frac{1}{2}\int_\Omega (u_\varepsilon+\varepsilon)^{m-1}(u_\varepsilon+1)^{m-3}|\nabla u_\varepsilon|^2 - \int_\Omega u_\varepsilon \Phi'(u_\varepsilon) v_\varepsilon \\
&\quad + \frac{\chi^2}{2}\int_\Omega u_\varepsilon^2 (u_\varepsilon+\varepsilon)^{1-m}(u_\varepsilon+1)^{m-3}\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} + \int_\Omega \Phi'(u_\varepsilon) B_1 \\
&\le -\frac{1}{2}\int_\Omega (u_\varepsilon+\varepsilon)^{m-1}(u_\varepsilon+1)^{m-3}|\nabla u_\varepsilon|^2 + \frac{\chi^2}{2} k_1^{\frac{1}{2}}(T) \int_\Omega v_\varepsilon^{-\frac{3}{2}} |\nabla v_\varepsilon|^2 \\
&\quad - \int_\Omega u_\varepsilon \Phi'(u_\varepsilon) v_\varepsilon + \int_\Omega \Phi'(u_\varepsilon) B_1
\end{aligned} \tag{4.7}$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$ because of the pointwise inequality

$$u_\varepsilon^2 (u_\varepsilon+\varepsilon)^{1-m}(u_\varepsilon+1)^{m-3} \le (u_\varepsilon+\varepsilon)^{3-m}(u_\varepsilon+1)^{m-3} \le 1,$$

valid throughout $\Omega \times (0,\infty)$ for each $\varepsilon \in (0,1)$ and any such m. Now from the definition of Φ we furthermore see that for all $\xi \ge 0$,

$$\Phi'(\xi) = \begin{cases} -\frac{1}{2-m}(\xi+1)^{m-2} + \frac{1}{2-m}, & \text{if } , m \in (1,2), \\ \ln(\xi+1), & \text{if } m = 2, \end{cases} \tag{4.8}$$

and

$$\Phi(\xi) = \begin{cases} -\frac{1}{(2-m)(m-1)}(\xi+1)^{m-1} + \frac{1}{2-m}\xi + \frac{1}{(2-m)(m-1)}, & \text{if } m \in (1,2), \\ (\xi+1)\ln(\xi+1) - \xi, & \text{if } m = 2, \end{cases}$$

from which it follows that for each $\xi \ge 0$,

$$\xi\Phi'(\xi) - \Phi(\xi) = \begin{cases} \frac{1}{m-1}(\xi+1)^{m-1} - \frac{1}{2-m}(\xi+1)^{m-2} - \frac{1}{(2-m)(m-1)}, & \text{if } m \in (1,2), \\ \xi - (\xi+1)\ln(\xi+1), & \text{if } m = 2. \end{cases}$$

Using that $(\xi+1)^{m-2} \le 1$ for all $\xi \ge 0$ when $m < 2$, and that $\ln(\xi+1) \le \xi$ for any $\xi \ge 0$, we thus obtain $c_1 > 0$ and $c_2 > 0$ such that in both cases,

$$\xi\Phi'(\xi) - \Phi(\xi) \ge -c_1 \qquad \text{for all } \xi \ge 0$$

and

$$\Phi'(\xi) \le \xi + c_2 \qquad \text{for all } \xi \ge 0.$$

As (4.8) moreover implies that $\Phi' \ge 0$ on $[0,\infty)$, once again going back to (4.2) and recalling Lemma 2.3, we can thus find $k_4 : (0,\infty) \to (0,\infty)$ fulfilling (K) such that for fixed $T > 0$ we can estimate the two rightmost summands in (4.7) according to

$$\begin{aligned} -\int_\Omega u_\varepsilon \Phi'(u_\varepsilon) v_\varepsilon + \int_\Omega \Phi'(u_\varepsilon) B_1 &\le -\frac{1}{k_1(T)} \int_\Omega u_\varepsilon \Phi'(u_\varepsilon) + \|B_1(\cdot,t)\|_{L^\infty(\Omega)} \int_\Omega \Phi'(u_\varepsilon) \\ &\le -\frac{1}{k_1(T)} \int_\Omega \big\{ \Phi(u_\varepsilon) + c_1 \big\} \\ &\quad + \|B_1(\cdot,t)\|_{L^\infty(\Omega)} \int_\Omega (u_\varepsilon + c_2) \\ &\le -\frac{1}{k_1(T)} \int_\Omega \Phi(u_\varepsilon) + k_4(T) \end{aligned}$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$. For any such T, from (4.7) we consequently derive the analogue of (4.6) given by

$$\begin{aligned} &\frac{d}{dt} \int_\Omega \Phi(u_\varepsilon) + \frac{1}{2} \int_\Omega (u_\varepsilon + \varepsilon)^{m-1} (u_\varepsilon + 1)^{m-3} |\nabla u_\varepsilon|^2 + \frac{1}{k_1(T)} \int_\Omega \Phi(u_\varepsilon) \\ &\quad \le \frac{\chi^2}{2} k_1^{\frac{1}{2}}(T) \int_\Omega v_\varepsilon^{-\frac{3}{2}} |\nabla v_\varepsilon|^2 + k_4(T) \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1), \end{aligned}$$

which due to (4.3) and the evident nonnegativity of Φ entails the claimed inequality in (4.1) also for such values of m. □

An interpolation of the latter with the L^1 bound provided by Lemma 2.3, namely, yields a spatio-temporal integral estimate for u_ε itself which involves superlinear summability powers conveniently increasing with m.

Lemma 4.2. *Let $m > 1$. Then there exists $K : (0,\infty) \to (0,\infty)$ satisfying (K) such that whenever $T > 0$,*

$$\int_t^{t+1} \int_\Omega u_\varepsilon^{2m-1} \le K(T) \qquad \textit{for all } t \in (0,T) \textit{ and } \varepsilon \in (0,1). \tag{4.9}$$

Proof. We fix $\rho \in C^0([0,\infty))$ such that $\rho \equiv 0$ on $[0,1]$, $\rho(\xi) = \xi^{m-2}$ for all $\xi \ge 2$ and $0 \le \rho(\xi) \le \xi^{m-2}$ for all $\xi \ge 0$, and let $P(\xi) := \int_0^\xi \rho(\sigma) d\sigma$ for $\xi \ge 0$. Then P belongs to $C^1([0,\infty))$ and satisfies $P(\xi) \le \frac{\xi^{m-1}}{m-1}$ as well as

$$P(\xi) \ge \int_2^\xi \rho(\sigma) d\sigma = \frac{\xi^{m-1} - 2^{m-1}}{m-1} \ge c_1 \xi^{m-1} \qquad \text{for all } \xi \ge 3 \tag{4.10}$$

with $c_1 := \frac{1-(\frac{2}{3})^{m-1}}{m-1} > 0$. Since thus

$$\|P(u_\varepsilon)\|_{L^{\frac{1}{m-1}}(\Omega)}^{\frac{1}{m-1}} \le (m-1)^{-\frac{1}{m-1}} \int_\Omega u_\varepsilon \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1)$$

and

$$\|\nabla P(u_\varepsilon)\|^2_{L^2(\Omega)} = \int_\Omega \rho^2(u_\varepsilon)|\nabla u_\varepsilon|^2 \le \int_{\{u_\varepsilon\ge 1\}} u_\varepsilon^{2m-4}|\nabla u_\varepsilon|^2$$

for all $t > 0$ and $\varepsilon \in (0,1)$, and since herein

$$\begin{aligned}\int_{\{u_\varepsilon\ge 1\}} u_\varepsilon^{2m-4}|\nabla u_\varepsilon|^2 &= \int_{\{u_\varepsilon\ge 1\}} u_\varepsilon^{m-1}\cdot u_\varepsilon^{m-3}|\nabla u_\varepsilon|^2 \\ &\le 2^{3-m}\int_\Omega (u_\varepsilon+\varepsilon)^{m-1}(u_\varepsilon+1)^{m-3}|\nabla u_\varepsilon|^2\end{aligned}$$

for all $t > 0$ and $\varepsilon \in (0,1)$ if $m \le 2$ and, clearly,

$$\int_{\{u_\varepsilon\ge 1\}} u_\varepsilon^{2m-4}|\nabla u_\varepsilon|^2 \le (u_\varepsilon+\varepsilon)^{2m-4}|\nabla u_\varepsilon|^2 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1)$$

if $m > 2$, by combining Lemma 2.3 with Lemma 4.1 we obtain functions $k_i : (0,\infty) \to (0,\infty)$, $i \in \{1,2\}$, for which (K) holda and which are such that when $T > 0$,

$$\|P(u_\varepsilon)\|_{L^{\frac{1}{m-1}}(\Omega)} \le k_1(T) \qquad \text{for all } t\in(0,T+1) \text{ and } \varepsilon\in(0,1) \tag{4.11}$$

and

$$\int_t^{t+1} \|\nabla P(u_\varepsilon(\cdot,s))\|^2_{L^2(\Omega)} ds \le k_2(T) \qquad \text{for all } t\in(0,T) \text{ and } \varepsilon\in(0,1). \tag{4.12}$$

As the Gagliardo-Nirenberg inequality provides $c_1 > 0$ fulfilling

$$\int_\Omega |\varphi|^{\frac{2m-1}{m-1}} \le c_1\|\nabla\varphi\|^2_{L^2(\Omega)}\|\varphi\|^{\frac{1}{m-1}}_{L^{\frac{1}{m-1}}(\Omega)} + c_1\|\varphi\|^{\frac{2m-1}{m-1}}_{L^{\frac{1}{m-1}}(\Omega)} \qquad \text{for all } \varphi\in W^{1,2}(\Omega),$$

we thus infer that for any $T > 0$,

$$\begin{aligned}\int_t^{t+1}\int_\Omega P^{\frac{2m-1}{m-1}}(u_\varepsilon) &\le c_1\int_t^{t+1}\|\nabla P(u_\varepsilon(\cdot,s))\|^2_{L^2(\Omega)}\|P(u_\varepsilon(\cdot,s))\|^{\frac{1}{m-1}}_{L^{\frac{1}{m-1}}(\Omega)} ds \\ &\quad + c_1\int_t^{t+1}\|P(u_\varepsilon(\cdot,s))\|^{\frac{2m-1}{m-1}}_{L^{\frac{1}{m-1}}(\Omega)} ds \\ &\le c_1 k_1^{\frac{1}{m-1}}(T)k_2(T) + c_1 k_1^{\frac{2m-1}{m-1}}(T)\end{aligned}$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$, from which (4.9) immediately follows thanks to (4.10) and the trivial fact that $u_\varepsilon^{2m-1} \le 3^{2m-1}$ in $\{u_\varepsilon \le 3\}$. $\square$

5. Estimating $\|v_\varepsilon\|_{W^{1,q}(\Omega)}$ for some $q > 2$ when $m > \frac{3}{2}$

Now an observation of crucial importance to our approach asserts a bound for v_ε with respect to the norm in $W^{1,q}(\Omega)$ with some $q > 2$, provided that the integrability exponent in Lemma 4.2 can be chosen to be superquadratic. This circumstance can be viewed as the core of our requirement on m in Theorem 1.1 and Theorem 1.2.

Lemma 5.1. *Let $m > \frac{3}{2}$. Then one can find a function $K : (0,\infty) \to (0,\infty)$ such that (K) holds, and that if $T > 0$ then*

$$\|v_\varepsilon(\cdot,t)\|_{W^{1,2m-1}(\Omega)} \le K(T) \qquad \textit{for all } t \in (0,T) \textit{ and } \varepsilon \in (0,1). \tag{5.1}$$

Proof. As $p := 2m-1$ satisfies $p > 2$, a Gagliardo-Nirenberg interpolation corresponding to the continuous embeddings $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega) \hookrightarrow L^1(\Omega)$ warrants the existence of $c_1 > 0$ such that

$$\|\varphi\|_{L^\infty(\Omega)} \le c_1 \|\varphi\|_{W^{1,p}(\Omega)}^a \|\varphi\|_{L^1(\Omega)}^{1-a} \qquad \text{for all } \varphi \in W^{1,p}(\Omega), \tag{5.2}$$

with the number $a := \frac{2p}{3p-2} \in (0,1)$ satisfying

$$\frac{1}{p} + \frac{p-1}{pa} = \frac{1}{p} + \frac{(p-1)(3p-2)}{2p^2} = \frac{3p^2-3p+2}{2p^2} > \frac{1}{2},$$

because $2p^2 - 3p + 2 = 2(p-1)^2 + p > 0$. We can therefore pick $q \in (1,2)$ sufficiently close to 2 such that

$$\frac{1}{q} < \frac{1}{p} + \frac{p-1}{pa}, \tag{5.3}$$

and thereupon invoke known smoothing properties of the Neumann heat semigroup $(e^{\sigma\Delta})_{\sigma\ge0}$ on Ω ([34]) to fix positive constants c_2, c_3 and c_4 fulfilling

$$\|e^{\sigma\Delta}\varphi\|_{W^{1,p}(\Omega)} \le c_2\sigma^{-\alpha}\|\varphi\|_{W^{1,q}(\Omega)} \qquad \text{for all } \sigma \in (0,2) \text{ and } \varphi \in C^1(\overline{\Omega}) \tag{5.4}$$

and

$$\|e^{\sigma\Delta}\varphi\|_{W^{1,p}(\Omega)} \le c_3\|\varphi\|_{W^{1,\infty}(\Omega)} \qquad \text{for all } \sigma \in (0,2) \text{ and } \varphi \in C^1(\overline{\Omega}) \tag{5.5}$$

as well as

$$\|e^{\sigma\Delta}\varphi\|_{W^{1,p}(\Omega)} \le c_4\sigma^{-\frac{1}{2}}\|\varphi\|_{L^p(\Omega)} \qquad \text{for all } \sigma \in (0,2) \text{ and } \varphi \in C^0(\overline{\Omega}), \tag{5.6}$$

where $\alpha := \frac{1}{q} - \frac{1}{p} > 0$. Apart from that, Lemma 2.3 together with Lemma 4.2, (B), Lemma 3.2 and (1.4) provides functions $k_i : (0,\infty) \to (0,\infty)$, $i \in \{1,2,3,4\}$, which satisfy (K) and are such that for all $T > 0$,

$$\|v_\varepsilon(\cdot,t)\|_{L^1(\Omega)} \le k_1(T) \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1) \tag{5.7}$$

and

$$\int_{(t-2)_+}^t \int_\Omega u_\varepsilon^p \le k_2(T) \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1) \tag{5.8}$$

as well as

$$\|B_2(\cdot,t)\|_{L^p(\Omega)} \le k_3(T) \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1), \tag{5.9}$$

and that for any such T, each $t_0 \in (0,T)$ and arbitrary $\varepsilon \in (0,1)$ we can find $t_\star \ge 0$ $t_\star = t_\star(t_0,\varepsilon) \ge 0$ with the properties that

$$t_\star \in \big((t_0-2)_+, (t_0-1)_+\big) \quad \text{and} \quad \|v_\varepsilon(\cdot,t_\star)\|_{W^{1,q}(\Omega)} \le k_4(T) \qquad \text{if } t_0 \ge 2, \tag{5.10}$$

and that

$$t_\star = 0 \quad \text{and} \quad \|v_\varepsilon(\cdot,t_\star)\|_{W^{1,\infty}(\Omega)} \le c_5 := \|v_0\|_{W^{1,\infty}(\Omega)} \qquad \text{if } t_0 \in (0,2). \tag{5.11}$$

Now given $T > 0, t_0 \in (0,T)$ and $\varepsilon \in (0,1)$, taking $t_\star = t_\star(t_0,\varepsilon)$ as thus specified we estimate the number

$$M := \begin{cases} \sup\limits_{t\in(t_\star,t_0]} \Big\{(t-t_\star)^\alpha \|v_\varepsilon(\cdot,t)\|_{W^{1,p}(\Omega)}\Big\} & \text{if } t_0 \ge 2, \\ \sup\limits_{t\in(t_\star,t_0]} \|v_\varepsilon(\cdot,t)\|_{W^{1,p}(\Omega)} & \end{cases} \tag{5.12}$$

by relying on a Duhamel representation associated with the second sub-problem of (2.1) to see that due to (5.6),

$$\begin{aligned} \|v_\varepsilon(\cdot,t)\|_{W^{1,p}(\Omega)} &= \bigg\| e^{(t-t_\star)(\Delta-1)} v_\varepsilon(\cdot,t_\star) + \int_{t_\star}^t e^{(t-s)(\Delta-1)} \frac{u_\varepsilon(\cdot,s)v_\varepsilon(\cdot,s)}{1+\varepsilon u_\varepsilon(\cdot,s)v_\varepsilon(\cdot,s)} ds \\ &\qquad\qquad + \int_{t_\star}^t e^{(t-s)(\Delta-1)} B_2(\cdot,s) ds \bigg\|_{W^{1,p}(\Omega)} \\ &\le e^{-(t-t_\star)} \|e^{(t-t_\star)\Delta} v_\varepsilon(\cdot,t_\star)\|_{W^{1,p}(\Omega)} \\ &\quad + c_4 \int_{t_\star}^t (t-s)^{-\frac12} e^{-(t-s)} \bigg\| \frac{u_\varepsilon(\cdot,s)v_\varepsilon(\cdot,s)}{1+\varepsilon u_\varepsilon(\cdot,s)v_\varepsilon(\cdot,s)} \bigg\|_{L^p(\Omega)} ds \\ &\quad + c_4 \int_{t_\star}^t (t-s)^{-\frac12} e^{-(t-s)} \|B_2(\cdot,s)\|_{L^p(\Omega)} ds \end{aligned} \tag{5.13}$$

for all $t \in (t_\star, t_0]$, because $t_0 - t_\star \in (0,2)$. Here if $t_0 \ge 2$, then by (5.4) and (5.10),

$$e^{-(t-t_\star)} \|e^{(t-t_\star)\Delta} v_\varepsilon(\cdot,t_\star)\|_{W^{1,p}(\Omega)} \le c_2 (t-t_\star)^{-\alpha} \|v_\varepsilon(\cdot,t_\star)\|_{W^{1,q}(\Omega)} \le c_2 k_4(T)(t-t_\star)^{-\alpha} \tag{5.14}$$

for all $t \in (t_\star, t_0]$, and if $t_0 < 2$, then by (5.5) and (5.11),

$$e^{-(t-t_\star)} \|e^{(t-t_\star)\Delta} v_\varepsilon(\cdot,t_\star)\|_{W^{1,p}(\Omega)} \le c_3 \|v_\varepsilon(\cdot,t_\star)\|_{W^{1,\infty}(\Omega)} \le c_3 c_5, \tag{5.15}$$

while (5.9) asserts that

$$\begin{aligned} c_4 \int_{t_\star}^t (t-s)^{-\frac12} e^{-(t-s)} \|B_2(\cdot,s)\|_{L^p(\Omega)} ds &\le c_4 k_3(T) \int_{t_\star}^t (t-s)^{-\frac12} ds \\ &= 2c_4 k_3(T)(t-t_\star)^{\frac12} \\ &\le 2^{\frac32} c_4 k_3(T) \end{aligned} \tag{5.16}$$

for all $t \in (t_\star, t_0]$. In order to appropriately cope with the crucial second last summand on the right of (5.13), we first concentrate on the case when $t_0 \ge 2$, in which we apply (5.2) together with (5.7) and recall our respective definition of M from (5.12) to find that

$$\begin{aligned} \bigg\| \frac{u_\varepsilon(\cdot,s)v_\varepsilon(\cdot,s)}{1+\varepsilon u_\varepsilon(\cdot,s)v_\varepsilon(\cdot,s)} \bigg\|_{L^p(\Omega)} &\le \|u_\varepsilon(\cdot,s)\|_{L^p(\Omega)} \|v_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)} \\ &\le c_1 \|u_\varepsilon(\cdot,s)\|_{L^p(\Omega)} \|v_\varepsilon(\cdot,s)\|_{W^{1,p}(\Omega)}^a \|v_\varepsilon(\cdot,s)\|_{L^1(\Omega)}^{1-a} \\ &\le c_1 k_1^{1-a}(T) M^a \|u_\varepsilon(\cdot,s)\|_{L^p(\Omega)} (s-t_\star)^{-\alpha a} \end{aligned}$$

for all $s \in (t_\star, t_0)$, so that by the Hölder inequality,

$$\begin{aligned} & c_4 \int_{t_\star}^t (t-s)^{-\frac{1}{2}} e^{-(t-s)} \Big\| \frac{u_\varepsilon(\cdot,s)v_\varepsilon(\cdot,s)}{1+\varepsilon u_\varepsilon(\cdot,s)v_\varepsilon(\cdot,s)} \Big\|_{L^p(\Omega)} ds \\ \le\ & c_1 c_4 k_1^{1-a}(T) M^a \int_{t_\star}^t (t-s)^{-\frac{1}{2}} (s-t_\star)^{-\alpha a} \|u_\varepsilon(\cdot,s)\|_{L^p(\Omega)} ds \\ \le\ & c_1 c_4 k_1^{1-a}(T) M^a \cdot \Big\{ \int_{t_\star}^t \int_\Omega u_\varepsilon^p \Big\}^{\frac{1}{p}} \cdot \Big\{ \int_{t_\star}^t (t-s)^{-\frac{p}{2(p-1)}} (s-t_\star)^{-\frac{p\alpha a}{p-1}} ds \Big\}^{\frac{p-1}{p}} \end{aligned} \tag{5.17}$$

for all $t \in (t_\star, t_0]$. Here,

$$\int_{t_\star}^t (t-s)^{-\frac{p}{2(p-1)}} (s-t_\star)^{-\frac{p\alpha a}{p-1}} ds = c_6 (t-t_\star)^{1-\frac{p}{2(p-1)}-\frac{p\alpha a}{p-1}} \qquad \text{for all } t > t_\star,$$

with $c_6 := \int_0^1 (1-\sigma)^{-\frac{p}{2(p-1)}} \sigma^{-\frac{p\alpha a}{p-1}} d\sigma$ being finite, because the inequalities $p > 2, q > 1$ and $a < 1$ imply that $\frac{p}{2(p-1)} < 1$ and $\frac{p\alpha a}{p-1} = \frac{(p-q)a}{(p-1)q} < a < 1$. In view of (5.8), from (5.17) we therefore obtain that

$$\begin{aligned} & c_4 \int_{t_\star}^t (t-s)^{-\frac{1}{2}} e^{-(t-s)} \Big\| \frac{u_\varepsilon(\cdot,s)v_\varepsilon(\cdot,s)}{1+\varepsilon u_\varepsilon(\cdot,s)v_\varepsilon(\cdot,s)} \Big\|_{L^p(\Omega)} ds \\ & \quad \le c_1 c_4 c_6^{\frac{p-1}{p}} k_1^{1-a}(T) k_2^{\frac{1}{p}}(T) M^a (t-t_\star)^{\frac{p-1}{p}-\frac{1}{2}-\alpha a} \qquad \text{for all } t \in (t_\star, t_0], \end{aligned} \tag{5.18}$$

which combined with (5.14) and (5.16) shows that (5.13) implies the inequality

$$\begin{aligned} (t-t_\star)^\alpha \|v_\varepsilon(\cdot,t)\|_{W^{1,p}(\Omega)} &\le c_2 k_4(T) + c_1 c_4 c_6^{\frac{p-1}{p}} k_1^{1-a}(T) k_2^{\frac{1}{p}}(T) M^a (t-t_\star)^{\frac{p-1}{p}-\frac{1}{2}-\alpha a+\alpha} \\ &\quad + 2^{\frac{3}{2}} c_4 k_3(T) (t-t_\star)^\alpha \\ &\le k_5(T) + k_5(T) M^a \qquad \text{for all } t \in (t_\star, t_0] \end{aligned} \tag{5.19}$$

with

$$k_5(T) := \max\Big\{ c_2 k_4(T) + 2^{\frac{3}{2}} c_4 k_3(T) \cdot 2^\alpha\,,\ c_1 c_4 c_6^{\frac{p-1}{p}} k_1^{1-a}(T) k_2^{\frac{1}{p}}(T) \cdot 2^{\frac{p-1}{p}-\frac{1}{2}-\alpha a+\alpha} \Big\},$$

because again since $p > 2$ and $a < 1$,

$$\frac{p-1}{p} - \frac{1}{2} - \alpha a + \alpha = \frac{p-2}{2p} + (1-a)\alpha > 0.$$

As a further consequence of the fact that $a < 1$, (5.19) finally entails that

$$M \le k_6(T) := \max\Big\{ 1\,,\ (2k_5(T))^{\frac{1}{1-a}} \Big\},$$

from which by the definition of M in (5.12) it particularly follows that whenever $t_0 \ge 2$,

$$\|v_\varepsilon(\cdot,t_0)\|_{W^{1,p}(\Omega)} \le (t_0-t_\star)^{-\alpha} M \le k_6(T), \tag{5.20}$$

because $t_0 - t_\star \ge 1$ and $\alpha \ge 0$.

If $t_0 \in (0,2)$, however, then referring to the respective part in (5.12) enables us to actually simplify the above reasoning so as to infer, in a way similar to that in (5.17) and (5.18), that

$$
\begin{aligned}
&c_4\int_{t_\star}^t (t-s)^{-\frac{1}{2}}e^{-(t-s)}\Big\|\frac{u_\varepsilon(\cdot,s)v_\varepsilon(\cdot,s)}{1+\varepsilon u_\varepsilon(\cdot,s)v_\varepsilon(\cdot,s)}\Big\|_{L^p(\Omega)}ds\\
&\qquad\le c_1c_4k_1^{1-a}(T)M^a\cdot\Big\{\int_0^t\int_\Omega u_\varepsilon^p\Big\}^{\frac{1}{p}}\cdot\Big\{\int_0^t (t-s)^{-\frac{p}{2(p-1)}}ds\Big\}^{\frac{p-1}{p}}\\
&\qquad\le c_1c_4c_7^{\frac{p-1}{p}}k_1^{1-a}(T)k_2^{\frac{1}{p}}(T)M^a \qquad \text{for all } t\in(t_\star,t_0]\equiv(0,t_0],
\end{aligned}
$$

with $c_7:=\int_0^2\sigma^{-\frac{p}{2(p-1)}}d\sigma\equiv\frac{2(p-1)}{p-2}\cdot 2^{\frac{p-2}{2(p-1)}}$. In this case now relying on (5.15) instead of (5.14), from (5.13) and (5.16) we thus infer that

$$
\|v_\varepsilon(\cdot,t)\|_{W^{1,p}(\Omega)}\le c_3c_5+c_1c_4c_7^{\frac{p-1}{p}}k_1^{1-a}(T)k_2^{\frac{1}{p}}(T)M^a+2^{\frac{3}{2}}c_4k_3(T) \text{ for all } t\in(0,t_0]
$$

and that hence

$$
M\le k_7(T)+k_7(T)M^a,
$$

where $k_7(T):=\max\{c_3c_5+2^{\frac{3}{2}}c_4k_3(T)\,,\,c_1c_4c_7^{\frac{p-1}{p}}k_1^{1-a}(T)k_2^{\frac{1}{p}}(T)\}$. Again since $a<1$, this especially shows that for any such t_0,

$$
\|v_\varepsilon(\cdot,t_0)\|_{W^{1,p}(\Omega)}\le M\le\max\Big\{1\,,\,(2k_7(T))^{\frac{1}{1-a}}\Big\},
$$

which together with (5.20) yields the claimed conclusion. □

6. Boundedness properties in $L^\infty(\Omega)\times W^{1,q}(\Omega)$ for arbitrary $q>2$

With the knowledge from Lemma 5.1 at hand, we can successively improve our information about regularity in the course of a three-step bootstrap procedure, the first part of which is concerned with bounds on u_ε in $L^p(\Omega)$ for arbitrarily large finite p.

Lemma 6.1. *Let $m>\frac{3}{2}$ and $p>\max\{1\,,m-1+\frac{2m-3}{2m-1}\}$. Then there exists $K\equiv K^{(p)}:(0,\infty)\to(0,\infty)$ such that (K) is valid and that whenever $T>0$,*

$$
\int_\Omega u_\varepsilon^p(\cdot,t)\le K(T) \qquad \textit{for all } t\in(0,T) \textit{ and } \varepsilon\in(0,1). \tag{6.1}
$$

Proof. On testing the first equation in (2.1) by u_ε^{p-1} and using Young's inequality, we see that

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u_\varepsilon^p+\frac{2(p-1)}{(m+p-1)^2}\int_\Omega|\nabla u_\varepsilon^{\frac{m+p-1}{2}}|^2\\
&\quad=\frac{p-1}{2}\int_\Omega u_\varepsilon^{m+p-3}|\nabla u_\varepsilon|^2-(p-1)\int_\Omega u_\varepsilon^{p-2}(u_\varepsilon+\varepsilon)^{m-1}|\nabla u_\varepsilon|^2\\
&\qquad+(p-1)\chi\int_\Omega u_\varepsilon^{p-1}\nabla u_\varepsilon\cdot\frac{\nabla v_\varepsilon}{v_\varepsilon}-\int_\Omega u_\varepsilon^p v_\varepsilon+\int_\Omega u_\varepsilon^{p-1}B_1\\
&\quad\le\frac{(p-1)\chi^2}{2}\int_\Omega u_\varepsilon^{-m+p-1}\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}-\int_\Omega u_\varepsilon^p v_\varepsilon+\int_\Omega u_\varepsilon^{p-1}B_1 \qquad \text{for all } t>0,
\end{aligned}
\tag{6.2}
$$

where using Lemma 2.2 along with (B) and again Young's inequality we can find $k_i : (0,\infty) \to (0,\infty)$, $i \in \{1,2,3\}$, fulfilling (K) and such that for all $T > 0$,

$$\begin{aligned} -\int_\Omega u_\varepsilon^p v_\varepsilon + \int_\Omega u_\varepsilon^{p-1} B_1 &\le -k_1(T) \int_\Omega u_\varepsilon^p + k_2(T) \int_\Omega u_\varepsilon^{p-1} \\ &\le -\frac{k_1(T)}{2} \int_\Omega u_\varepsilon^p + k_3(T) \end{aligned} \tag{6.3}$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$. Apart from that, Lemma 5.1 in conjunction with Lemma 2.2 entails the existence of $k_4 : (0,\infty) \to (0,\infty)$ such that (K) holds, and that if $T > 0$ then

$$\int_\Omega \frac{|\nabla v_\varepsilon|^{2m-1}}{v_\varepsilon^{2m-1}} \le k_4(T) \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1),$$

whence utilizing the Hölder inequality we find that for any such T,

$$\begin{aligned} &\frac{(p-1)\chi^2}{2} \int_\Omega u_\varepsilon^{-m+p-1} \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} \\ &\le \frac{(p-1)\chi^2}{2} \cdot \left\{ \int_\Omega u_\varepsilon^{\frac{(2m-1)(-m+p-1)}{2m-3}} \right\}^{\frac{2m-3}{2m-1}} \cdot \left\{ \int_\Omega \frac{|\nabla v_\varepsilon|^{2m-1}}{v_\varepsilon^{2m-1}} \right\}^{\frac{2m-3}{2m-1}} \\ &\le k_5(T) \cdot \left\{ \int_\Omega u_\varepsilon^{\frac{(2m-1)(-m+p-1)}{2m-3}} \right\}^{\frac{2m-3}{2m-1}} \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1) \end{aligned} \tag{6.4}$$

with $k_5(T) := \frac{(p-1)\chi^2}{2} k_4^{\frac{2m-3}{2m-1}}(T)$. Now since $\frac{2}{m+p-1} < \frac{2}{m+p-1} \cdot \frac{(2m-1)(-m+p+1)}{2m-3}$ due to the fact that $-m+p+1 > \frac{2m-3}{2m-1}$ by assumption on p, the Gagliardo-Nirenberg inequality applies so as to say that with

$$a := \frac{(2m-1)(-m+p+1) - 2m + 3}{(2m-1)(-m+p+1)} \in (0,1) \tag{6.5}$$

and some $c_1 = c_1(p) > 0$ we have

$$\begin{aligned} \left\{ \int_\Omega u_\varepsilon^{\frac{(2m-1)(-m+p-1)}{2m-3}} \right\}^{\frac{2m-3}{2m-1}} &= \|u_\varepsilon^{\frac{m+p-1}{2}}\|_{L^{\frac{2}{m+p-1}\cdot\frac{2m-1)(-m+p+1)}{2m-3}}(\Omega)}^{\frac{2(-m+p+1)}{m+p-1}} \\ &\le c_1 \|\nabla u_\varepsilon^{\frac{m+p-1}{2}}\|_{L^2(\Omega)}^{\frac{2(-m+p+1)a}{m+p-1}} \|u_\varepsilon^{\frac{m+p-1}{2}}\|_{L^{\frac{2}{m+p-1}}(\Omega)}^{\frac{2(-m+p+1)(1-a)}{m+p-1}} \\ &\quad + \|u_\varepsilon^{\frac{m+p-1}{2}}\|_{L^{\frac{2}{m+p-1}}(\Omega)}^{\frac{2(-m+p+1)}{m+p-1}} \end{aligned} \tag{6.6}$$

for all $t > 0$ and $\varepsilon \in (0,1)$. Here we recall that Lemma 2.3 provides $k_6 : (0,\infty) \to (0,\infty)$ such that (K) holds and that for all $T > 0$,

$$\|u_\varepsilon^{\frac{m+p-1}{2}}\|_{L^{\frac{2}{m+p-1}}(\Omega)}^{\frac{2}{m+p-1}} = \int_\Omega u_\varepsilon \le k_6 T) \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1),$$

and furthermore we note that according to (6.5),

$$\begin{aligned}
\frac{(-m+p+1)a}{m+p-1} - 1 &= \frac{(2m-1)(-m+p+1) - 2m+3}{(2m-1)(m+p-1)} - 1 \\
&= \frac{(2m-1)(-2m+2) - 2m+3}{(2m-1)(m+p-1)} \\
&= -\frac{2(m-1)}{m+p-1} - \frac{2m-3}{(2m-1)(m+p-1)} \\
&< 0,
\end{aligned}$$

so that $\theta := \frac{m+p-1}{(-m+p+1)a}$ satisfies $\theta > 1$. An application of Young's inequality to (6.6) therefore yields functions $k_i : (0,\infty) \to (0,\infty)$, $i \in \{7,8\}$, for which (K) is valid, and which are such that for all $T > 0$,

$$\begin{aligned}
& k_5(T) \cdot \left\{ \int_\Omega u_\varepsilon^{\frac{(2m-1)(-m+p-1)}{2m-3}} \right\}^{\frac{2m-3}{2m-1}} \\
&\quad \le k_7(T) \|\nabla u_\varepsilon^{\frac{m+p-1}{2}}\|_{L^2(\Omega)}^{\frac{2(-m+p+1)a}{m+p-1}} + k_7(T) \\
&\quad \le \frac{2(p-1)}{(m+p-1)^2} \int_\Omega |\nabla u_\varepsilon^{\frac{m+p-1}{2}}|^2 + k_8(T) \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1).
\end{aligned}$$

Together with (6.3) and (6.4) inserted into (6.2), this shows that for each $T > 0$ we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{k_1(T)}{2}\int_\Omega u_\varepsilon^p \le k_3(T) + k_8(T) \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1),$$

which results in (6.1) by means of an evident ODE comparison argument. □

This in turn improves our knowledge on the second solution component:

Lemma 6.2. *Let $m > \frac{3}{2}$ and $q > 2$. Then one can find $K \equiv K^{(q)} : (0,\infty) \to (0,\infty)$ such that (K) holds, and that given any $T > 0$ we have*

$$\|v_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} \le K(T) \qquad \textit{for all } t \in (0,T) \textit{ and } \varepsilon \in (0,1). \tag{6.7}$$

Proof. As $W^{1,2m-1}(\Omega) \hookrightarrow L^\infty(\Omega)$ due to the hypothesis $m > \frac{3}{2}$, Lemma 5.1 together with Lemma 6.1 and (B) in particular yields $k_1 : (0,\infty) \to (0,\infty)$ fulfilling (K) and such that writing $f_\varepsilon(x,t) := \frac{u_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon}(x,t) + B_2(x,t)$, $(x,t) \in \Omega \times (0,\infty)$, $\varepsilon \in (0,1)$, for all $T > 0$ we have

$$\|f_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le k_1(T) \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1).$$

Therefore, (6.7) can be derived by straightforward application of well-known regularization estimates for the Neumann heat semigroup ([34]) to the inhomogeneous linear heat equation $v_{\varepsilon t} = \Delta v_\varepsilon + f_\varepsilon$. □

When combined with Lemma 6.1, through a standard argument the latter in fact asserts a boundedness feature of u_ε even with respect to the norm in $L^\infty(\Omega)$.

Lemma 6.3. *Let $m > \frac{3}{2}$. Then there exists $K : (0,\infty) \to (0,\infty)$ fulfilling (K) such that for all $T > 0$,*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le K(T) \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1).$$

Proof. This can readily be obtained from the bounds provided by Lemma 6.1 and Lemma 6.2 through a standard application of a Moser-type recursive argument (cf. e.g. [30]). □

7. Further compactness properties and regularity in time

For our mere existence statement in Theorem 1.1, tracking a possible dependence of estimates on the asymptotic behavior of B_1 and B_2 seems unnecessary; the next three statements preparing our limit procedure $\varepsilon \searrow 0$ will therefore not involve our hypothesis (K), but rather exclusively provide information on arbitrary but fixed time intervals. Our first observation in this regard is an essentially immediate consequence Lemma 4.1 when combined with the boundedness information from Lemma 6.3.

Lemma 7.1. *Let $m > \frac{3}{2}$ and*

$$\alpha \ge \begin{cases} \dfrac{m+1}{2} & \text{if } m \in \left(\frac{3}{2}, 2\right], \\ m-1 & \text{if } m > 2. \end{cases} \tag{7.1}$$

Then for all $T > 0$ there exists $C(\alpha,T) > 0$ such that

$$\int_0^T \int_\Omega |\nabla(u_\varepsilon+\varepsilon)^\alpha|^2 \le C(\alpha,T) \qquad \text{for all } \varepsilon \in (0,1). \tag{7.2}$$

Proof. In view of Lemma 6.3, given $T > 0$ we can fix $c_1(T) > 0$ fulfilling

$$u_\varepsilon \le c_1(T) \quad \text{in } \Omega \times (0,T) \qquad \text{for all } \varepsilon \in (0,1). \tag{7.3}$$

Therefore, in the case $m \in (\frac{3}{2}, 2]$ we can use that then (7.1) requires that $2\alpha \ge m+1$ to estimate

$$\begin{aligned} \frac{1}{\alpha^2}|\nabla(u_\varepsilon+\varepsilon)^\alpha|^2 &= (u_\varepsilon+\varepsilon)^{2\alpha-2}|\nabla u_\varepsilon|^2 \\ &= \Big\{(u_\varepsilon+\varepsilon)^{m-1}(u_\varepsilon+1)^{m-3}|\nabla u_\varepsilon|^2\Big\} \cdot (u_\varepsilon+\varepsilon)^{2\alpha-m-1}(u_\varepsilon+1)^{3-m} \\ &\le \Big\{(u_\varepsilon+\varepsilon)^{m-1}(u_\varepsilon+1)^{m-3}|\nabla u_\varepsilon|^2\Big\} \cdot (c_1(T)+1)^{2\alpha-m-1}(c_1(T)+1)^{3-m} \end{aligned}$$

in $\Omega \times (0,T)$ for all $\varepsilon \in (0,1)$, so that in light of Lemma 4.1, (7.2) results upon an integration over $\Omega \times (0,T)$.

Similarly, if $m > 2$ then $2\alpha \ge 2m-2$ by (7.1), and thus

$$\begin{aligned} \frac{1}{\alpha^2}|\nabla(u_\varepsilon+\varepsilon)^\alpha|^2 &= \Big\{(u_\varepsilon+\varepsilon)^{2m-4}|\nabla u_\varepsilon|^2\Big\} \cdot (u_\varepsilon+\varepsilon)^{2\alpha-2m-2} \\ &\le \Big\{(u_\varepsilon+\varepsilon)^{2m-4}|\nabla u_\varepsilon|^2\Big\} \cdot (c_1(T)+1)^{2\alpha-2m-2} \quad \text{in } \Omega \times (0,T) \end{aligned}$$

for all $\varepsilon \in (0,1)$, again implying (7.2) due to Lemma 4.1. $\square$

Now for suitably large α, the expressions appearing in (7.2) enjoy some favorable time regularity feature:

Lemma 7.2. *Let $m > \frac{3}{2}$ and*

$$\alpha \ge \begin{cases} 2 & \text{if } m \in \left(\frac{3}{2}, 2\right], \\ m-1 & \text{if } m > 2. \end{cases} \tag{7.4}$$

Then for all $T > 0$ there exists $C(\alpha, T) > 0$ such that

$$\int_0^T \left\| \partial_t \Big(u_\varepsilon(\cdot,t) + \varepsilon \Big) \right\|_{(W^{2,2}(\Omega))^\star} dt \le C(\alpha,T) \qquad \text{for all } \varepsilon \in (0,1). \tag{7.5}$$

Proof. Using (2.1), for fixed $t > 0$ and $\varphi \in C^\infty(\overline{\Omega})$ we compute

$$\begin{aligned}
&\frac{1}{\alpha} \int_\Omega \partial_t (u_\varepsilon + \varepsilon)^\alpha \varphi \\
&= \int_\Omega (u_\varepsilon + \varepsilon)^{\alpha-1} \varphi \cdot \Big\{ \nabla \cdot \Big((u_\varepsilon + \varepsilon)^{m-1} \nabla u_\varepsilon \Big) - \chi \nabla \cdot \Big(\frac{u_\varepsilon}{v_\varepsilon} \nabla v_\varepsilon \Big) - u_\varepsilon v_\varepsilon + B_1 \Big\} \\
&= - \int_\Omega \Big\{ (\alpha-1)(u_\varepsilon+\varepsilon)^{\alpha-2} \varphi \nabla u_\varepsilon + (u_\varepsilon+\varepsilon)^{\alpha-1} \nabla\varphi \Big\} \cdot \Big\{ (u_\varepsilon+\varepsilon)^{m-1} \nabla u_\varepsilon - \chi \frac{u_\varepsilon}{v_\varepsilon} \nabla v_\varepsilon \Big\} \\
&\quad - \int_\Omega u_\varepsilon (u_\varepsilon+\varepsilon)^{\alpha-1} v_\varepsilon \varphi + \int_\Omega (u_\varepsilon+\varepsilon)^{\alpha-1} \varphi \\
&= -(\alpha-1) \int_\Omega (u_\varepsilon+\varepsilon)^{m+\alpha-3} |\nabla u_\varepsilon|^2 \varphi + (\alpha-1)\chi \int_\Omega u_\varepsilon (u_\varepsilon+\varepsilon)^{\alpha-2} \Big(\nabla u_\varepsilon \cdot \frac{\nabla v_\varepsilon}{v_\varepsilon} \Big) \varphi \\
&\quad - \int_\Omega (u_\varepsilon+\varepsilon)^{m+\alpha-2} \nabla u_\varepsilon \cdot \nabla\varphi + \chi \int_\Omega u_\varepsilon (u_\varepsilon+\varepsilon)^{\alpha-1} \frac{\nabla v_\varepsilon}{v_\varepsilon} \cdot \nabla\varphi \\
&\quad - \int_\Omega u_\varepsilon (u_\varepsilon+\varepsilon)^{\alpha-1} v_\varepsilon \varphi + \int_\Omega (u_\varepsilon+\varepsilon)^{\alpha-1} \varphi \qquad \text{for all } \varepsilon \in (0,1).
\end{aligned} \tag{7.6}$$

Here given $T > 0$, we note that Lemma 6.3, Lemma 2.2, Lemma 5.1 and (B) ensure the existence of positive constants $c_i(T)$, $i \in \{1,2,3,4\}$, such that for all $\varepsilon \in (0,1)$,

$$u_\varepsilon \le c_1(T), \quad c_2(T) \le v_\varepsilon \le c_3(T) \quad \text{and} \quad B_1 \le c_4(T) \qquad \text{in } \Omega \times (0,T). \tag{7.7}$$

Since (7.4) especially requires that $\alpha \ge 1$, by using Young's inequality we thus obtain that whenever $t \in (0,T)$ and $\varepsilon \in (0,1)$,

$$\begin{aligned}
\left| \chi \int_\Omega u_\varepsilon (u_\varepsilon+\varepsilon)^{\alpha-1} \frac{\nabla v_\varepsilon}{v_\varepsilon} \cdot \nabla\varphi \right| &\le \int_\Omega v_\varepsilon^{-\frac{3}{2}} |\nabla v_\varepsilon|^2 + \frac{\chi^2}{4} \int_\Omega \frac{u_\varepsilon^2 (u_\varepsilon+\varepsilon)^{2\alpha-2}}{v_\varepsilon^{\frac{1}{2}}} |\nabla\varphi|^2 \\
&\le \int_\Omega v_\varepsilon^{-\frac{3}{2}} |\nabla v_\varepsilon|^2 + \frac{\chi^2}{4} \cdot \frac{c_1^2(T) \cdot (c_1(T)+1)^{2\alpha-2}}{c_2^{\frac{1}{2}}(T)} \cdot \|\nabla\varphi\|_{L^2(\Omega)}^2
\end{aligned} \tag{7.8}$$

and

$$\left| - \int_\Omega u_\varepsilon (u_\varepsilon + \varepsilon)^{\alpha-1} v_\varepsilon \varphi \right| \le c_1(T) \cdot (c_1(T)+1)^{\alpha-1} c_3(T)|\Omega| \cdot \|\varphi\|_{L^\infty(\Omega)} \tag{7.9}$$

as well as

$$\left| \int_\Omega (u_\varepsilon + \varepsilon)^{\alpha-1} B_1 \varphi \right| \le (c_1(T)+1)^{\alpha-1} c_4(T)|\Omega| \cdot \|\varphi\|_{L^\infty(\Omega)}. \tag{7.10}$$

Now in the case $m \in (\frac{3}{2}, 2]$ in which (7.4) asserts that $\alpha \ge 2 \ge \max\{\frac{m+1}{2}, \frac{3-m}{2}\}$, the first three summand on the right of (7.6) can similarly be estimated according to

$$\begin{aligned} &\left| -(\alpha-1) \int_\Omega (u_\varepsilon+\varepsilon)^{m+\alpha-3} |\nabla u_\varepsilon|^2 \varphi \right| \\ &\le (\alpha-1)(c_1(T)+\varepsilon)^{\alpha-2}(c_1(T)+1)^{3-m} \cdot \left\{ \int_\Omega (u_\varepsilon+\varepsilon)^{m-1}(u_\varepsilon+1)^{m-3}|\nabla u_\varepsilon|^2 \right\} \cdot \|\varphi\|_{L^\infty(\Omega)} \end{aligned} \tag{7.11}$$

and

$$\begin{aligned} &\left| (\alpha-1)\chi \int_\Omega u_\varepsilon (u_\varepsilon+\varepsilon)^{\alpha-2} \Big(\nabla u_\varepsilon \cdot \frac{\nabla v_\varepsilon}{v_\varepsilon} \Big) \varphi \right| \\ &\le v_\varepsilon^{-\frac{3}{2}} |\nabla v_\varepsilon|^2 + \frac{(\alpha-1)^2\chi^2}{4} \cdot \left\{ \int_\Omega \frac{u_\varepsilon^2 (u_\varepsilon+\varepsilon)^{2\alpha-4}}{v_\varepsilon^{\frac{1}{2}}} |\nabla u_\varepsilon|^2 \right\} \cdot \|\varphi\|_{L^\infty(\Omega)} \\ &\le v_\varepsilon^{-\frac{3}{2}} |\nabla v_\varepsilon|^2 \\ &\quad + \frac{(\alpha-1)^2\chi^2}{4} \cdot \frac{(c_1(T)+\varepsilon)^{-m+2\alpha-1}(c_1(T)+1)^{3-m}}{c_2^{\frac{1}{2}}(T)} \times \\ &\qquad \times \left\{ \int_\Omega (u_\varepsilon+\varepsilon)^{m-1}(u_\varepsilon+1)^{3-m}|\nabla u_\varepsilon|^2 \right\} \cdot \|\varphi\|_{L^\infty(\Omega)} \end{aligned} \tag{7.12}$$

and

$$\begin{aligned} &\left| - \int_\Omega (u_\varepsilon+\varepsilon)^{m+\alpha-2} \nabla u_\varepsilon \cdot \nabla \varphi \right| \\ &\le \int_\Omega (u_\varepsilon+\varepsilon)^{2m+2\alpha-4} |\nabla u_\varepsilon|^2 + \frac{1}{4} \|\nabla \varphi\|_{L^2(\Omega)}^2 \\ &\le (c_1(T)+\varepsilon)^{m+2\alpha-3}(c_1(T)+1)^{3-m} \int_\Omega (u_\varepsilon+\varepsilon)^{m-1}(u_\varepsilon+1)^{m-3}|\nabla u_\varepsilon|^2 + \frac{1}{4}\|\nabla\varphi\|_{L^2(\Omega)}^2 \end{aligned} \tag{7.13}$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$. Since $W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$, from (7.6) and (7.8)-(7.13) we thus infer that if $m \in (\frac{3}{2}, 2]$ and α satisfies (7.4), then for each $T > 0$ there exists $c_5(T) > 0$ such that for all $t \in (0,T)$ and $\varepsilon \in (0,1)$,

$$\|\partial_t (u_\varepsilon+\varepsilon)\|_{(W^{2,2}(\Omega))^\star} \le c_5(T) \cdot \left\{ \int_\Omega (u_\varepsilon+\varepsilon)^{m-1}(u_\varepsilon+1)^{m-3}|\nabla u_\varepsilon|^2 + \int_\Omega v_\varepsilon^{-\frac{3}{2}}|\nabla v_\varepsilon|^2 + 1 \right\},$$

so that (7.5) results from Lemma 4.1 and Lemma 3.1 upon an integration in time for any such m and α.

If $m > 2$, then in view of the accordingly modified form of the estimate in Lemma 4.1, given $T > 0$ we rely on the hypothesis $\alpha \ge m - 1$ in replacing (7.11)-(7.13) with the inequalities

$$\left| -(\alpha-1)\int_\Omega (u_\varepsilon+\varepsilon)^{m+\alpha-3}|\nabla u_\varepsilon|^2\varphi\right| \le (\alpha-1)\cdot(c_1(T)+\varepsilon)^{-m+\alpha+1}\cdot\left\{\int_\Omega (u_\varepsilon+\varepsilon)^{2m-4}|\nabla u_\varepsilon|^2\right\}\cdot\|\varphi\|_{L^\infty(\Omega)}$$

and

$$\left|(\alpha-1)\chi\int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{\alpha-2}\Big(\nabla u_\varepsilon\cdot\frac{\nabla v_\varepsilon}{v_\varepsilon}\Big)\varphi\right| \le v_\varepsilon^{-\frac{3}{2}}|\nabla v_\varepsilon|^2 + \frac{(\alpha-1)^2\chi^2}{4}\,\frac{(c_1(T)+\varepsilon)^{-2m+2\alpha+2}}{c_2^{\frac12}(T)}\left\{\int_\Omega (u_\varepsilon+\varepsilon)^{2m-4}|\nabla u_\varepsilon|^2\right\}\|\varphi\|_{L^\infty(\Omega)}$$

as well as

$$\left|-\int_\Omega (u_\varepsilon+\varepsilon)^{m+\alpha-2}\nabla u_\varepsilon\cdot\nabla\varphi\right| \le (c_1(T)+\varepsilon)^{2\alpha}\int_\Omega (u_\varepsilon+\varepsilon)^{2m-4}|\nabla u_\varepsilon|^2+\frac14\|\nabla\varphi\|^2_{L^2(\Omega)}$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$, and conclude as before. $\square$

Independently from the latter two lemmata, the estimates from Lemma 6.3 and Lemma 6.2 entail a Hölder regularity property of the second solution component as follows.

Lemma 7.3. *Let $m > \frac{3}{2}$. Then for all $T > 0$ there exist $\vartheta = \vartheta(T) \in (0,1)$ and $C(T) > 0$ such that*

$$\|v_\varepsilon\|_{C^{\vartheta,\frac{\vartheta}{2}}(\overline{\Omega}\times[0,T])} \le C(T) \qquad \textit{for all } \varepsilon \in (0,1).$$

Proof. Once more letting $f_\varepsilon := \frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} + B_2$ in $\Omega \times (0,\infty)$ for $\varepsilon \in (0,1)$, from Lemma 6.3 and, e.g., Lemma 6.2 we especially know that $(f_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^\infty_{loc}(\overline{\Omega}\times[0,\infty))$. As v_0 is Hölder continuous in $\overline{\Omega}$ thanks to (1.4), the claimed estimate therefore directly follows from standard theory on Hölder regularity in scalar parabolic equations ([22]). $\square$

8. Passing to the limit. Proof of Theorem 1.1 and Theorem 1.2

We are now prepared to construct a solution of (1.2) by means of appropriate compactness arguments, where following quite standard precedents, our concept of solvability will be as specified in the following.

Definition 8.1. Assume that $m \geq 1$, that $\chi \in \mathbb{R}$, and that (B) and (1.4) hold. Then a pair (u, v) of functions

$$\begin{cases} u \in L^m_{loc}(\overline{\Omega} \times [0,\infty)) \qquad \text{and} \\ v \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)) \end{cases} \tag{8.1}$$

will be called a *global weak solution of (1.2)* if $u \geq 0$ and $v > 0$ a.e. in $\Omega \times (0,\infty)$, if

$$\frac{u}{v}\nabla v \text{ belongs to } L^1_{loc}(\overline{\Omega} \times [0,\infty); \mathbb{R}^2) \tag{8.2}$$

and

$$uv \text{ lies in } L^1_{loc}(\overline{\Omega} \times [0,\infty)), \tag{8.3}$$

and if for each $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$ fulfilling $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega \times (0,\infty)$, and for any $\phi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$, the identities

$$\begin{aligned} -\int_0^\infty \int_\Omega u\varphi_t - \int_\Omega u_0\varphi(\cdot,0) = \frac{1}{m}\int_0^\infty \int_\Omega u^m \Delta\varphi + \chi \int_0^\infty \int_\Omega \frac{u}{v}\nabla v \cdot \nabla\varphi \\ - \int_0^\infty \int_\Omega uv\varphi + \int_0^\infty \int_\Omega B_1\varphi \end{aligned} \tag{8.4}$$

and

$$-\int_0^\infty\int_\Omega v\phi_t - \int_\Omega v_0\phi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla v \cdot \nabla\phi - \int_0^\infty\int_\Omega v\phi + \int_0^\infty\int_\Omega uv\phi + \int_0^\infty\int_\Omega B_2\phi \tag{8.5}$$

are valid.

We are now prepared to construct a solution of (1.2) by means of appropriate compactness arguments.

Lemma 8.1. *Let $m > \frac{3}{2}$. Then there exist $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ as well as functions*

$$\begin{cases} u \in L^\infty_{loc}(\overline{\Omega} \times [0,\infty)) \qquad \textit{and} \\ v \in C^0(\overline{\Omega} \times [0,\infty)) \cap \bigcap_{q>2} L^\infty_{loc}([0,\infty); W^{1,q}(\Omega)) \end{cases} \tag{8.6}$$

such that $\varepsilon_j \searrow 0$ as $j \to \infty$, that $u \geq 0$ a.e. in $\Omega \times (0,\infty)$ and and $v > 0$ in $\overline{\Omega} \times [0,\infty)$, that as $\varepsilon = \varepsilon_j \searrow 0$ we have

$$u_\varepsilon \to u \qquad \text{in } \bigcap_{p\geq 1} L^p_{loc}(\overline{\Omega} \times [0,\infty)) \text{ and a.e. in } \Omega \times (0,\infty), \tag{8.7}$$

$$v_\varepsilon \to v \qquad \text{in } C^0_{loc}(\overline{\Omega} \times [0,\infty)) \qquad \text{and} \tag{8.8}$$

$$\nabla v_\varepsilon \overset{\star}{\rightharpoonup} \nabla v \qquad \text{in } \bigcap_{q>2} L^\infty_{loc}([0,\infty); L^q(\Omega)), \tag{8.9}$$

and that (u, v) form a global weak solution of (1.2) in the sense of Definition 8.1.

Proof. We take any $\alpha > 0$ such that

$$\alpha \ge \begin{cases} \max\left\{\frac{m+1}{2}\,,2\right\} \equiv 2 & \text{if } m \in \left(\frac{3}{2}, 2\right], \\ m-1 & \text{if } m > 2, \end{cases}$$

and note that then Lemma 7.1 and Lemma 7.2 may simultaneously be applied so as to show that thanks to Lemma 2.3,

$$\Big((u_\varepsilon+\varepsilon)^\alpha\Big)_{\varepsilon\in(0,1)} \text{ is bounded in } L^2((0,T);W^{1,2}(\Omega)) \quad \text{for all } T>0$$

and that

$$\Big(\partial_t(u_\varepsilon+\varepsilon)^\alpha\Big)_{\varepsilon\in(0,1)} \text{ is bounded in } L^1\big((0,T);(W^{2,2}(\Omega))^\star\big) \quad \text{for all } T>0.$$

Therefore, employing an Aubin-Lions lemma ([32]) yields $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ and a nonnegative function u on $\Omega\times(0,\infty)$ such that $\varepsilon_j \searrow 0$ as $j\to\infty$, and that as $\varepsilon=\varepsilon_j \searrow 0$ we have $(u_\varepsilon+\varepsilon)^\alpha \to u^\alpha$ in $L^2_{loc}(\overline{\Omega}\times[0,\infty))$ and a.e. in $\Omega\times(0,\infty)$, whence in particular also $u_\varepsilon \to u$ a.e. in $\Omega\times(0,\infty)$. Since furthermore Lemma 6.3 warrants boundedness of $(u_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^\infty(\Omega\times(0,T))$ for all $T>0$, (8.7) as well as the inclusion $u\in L^\infty_{loc}(\overline{\Omega}\times[0,\infty))$ result from this due to the Vitali convergence theorem.

As, apart from that, given $T>0$ we know from Lemma 7.3 and Lemma 6.2 that $(v_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $C^{\vartheta,\frac{\vartheta}{2}}(\overline{\Omega}\times[0,T])$ and in $L^\infty((0,T);W^{1,q}(\Omega))$ for some $\vartheta=\vartheta(T)\in(0,1)$ and each $q>2$, in view of the Arzelá-Ascoli theorem and the Banach-Alaoglu theorem we may assume upon passing to a subsequence if necessary that, in fact, $(\varepsilon_j)_{j\in\mathbb{N}}$ is such that with some function v complying with (8.6) we also have (8.8) and (8.9) as $\varepsilon=\varepsilon_j\searrow 0$. The positivity of v in $\overline{\Omega}\times[0,\infty)$ therefore is a consequence of Lemma 2.2, whereas, finally, the integral inequalities in (8.4) and (8.5) can be verified in a straightforward manner by relying on (8.7)-(8.9) when taking $\varepsilon=\varepsilon_j\searrow 0$ in the corresponding weak formulations associated with (2.1). □

Our main result on global solvability has thereby actually been established already:

Proof. (c of Theorem 1.1). All statements have actually been covered by Lemma 8.1 already. □

According to our preparations, and especially due to our efforts to control the dependence of our estimates from Lemma 6.3 and Lemma 6.2 on T through (K), also the claimed boundedness features can now be obtained as simple consequences:

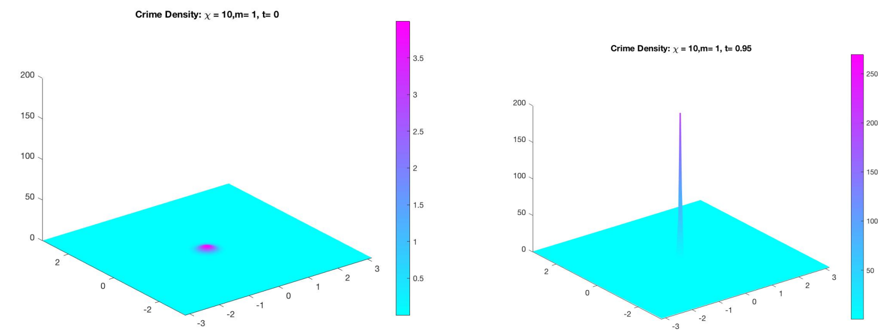
Proof. (c of Theorem 1.2). Again taking the global weak solution of (1.2) obtained in Lemma 8.1, we only need to observe that thanks to the hypotheses (B1) and (B2), Lemma 6.3 and Lemma 6.2 in conjunction with our notational convention concerning the property (K) guarantee boundedness of $(u_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^\infty(\Omega\times(0,\infty))$ and of $(v_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^\infty((0,\infty);W^{1,q}(\Omega))$ for each $q>2$. Therefore, namely, the additional features (1.5) and (1.6) directly result from (8.7) and (8.9). $\square$

9. Numerical Experiments

The purpose of this section is thee-fold: (1) to illustrate how the overcrowding effect included in (1.2) results in the relaxation of solutions, (2) to provide some comparison of this to the situation corresponding to the linear diffusion case $m=1$, which was not addressed by our previous analysis, and (3) to study the effect that the parameter χ has on the potential concentration of the solution in the linear diffusion case. To this end, we consider the associated evolution problems (1.2) under initial conditions involving the mildly concentrated data given by: $u_0(x) = v_0(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{|x|^2}{2\sigma^2}}$, for $x\in\Omega$, with some small σ on the square $\Omega := (-3,3)^2$.

We first solve the (1.2) numerically with $m=1$ (leading to linear diffusion) and $m=3$ (leading to porous medium type diffusion) with $\sigma=1/4$. We illustrate our results for $\chi=10$ in both simulations, but all other terms are as in the original model proposed in [28] with $B_1=1$ and $B_2=1$.

The initial condition for u is illustrated in Figure 1a. In the case when $m=1$, we see a concentration of mass around $t=.95$ – see Figure 1b. Here there is a real possibility that blow-up happens in finite time, although to make this more precise, more thorough numerical experiments need to be run, which goes beyond the scope of the present work. What is evident is the concentration around the origin (even if there were eventual relaxation) in finite time. On the other hand, the porous medium type diffusion suppresses this concentration entirely as can be observed in Figure 2, which illustrates the solution to (1.2) with the same initial data and $m=3$. We clearly see that there is never a concentration of density, and that by time $t=10$, solution comfortably reaches an equilibrium. This may be interpreted as describing crime hotspots that have spontaneously emerged due to the reaction-cross-diffusion interplay in (1.2). Videos of the full simulations can be found in the supplementary material. These preliminary results lead us to believe that there is blow-up when χ is sufficiently large in the presence of linear diffusion, even for some initial data that are only mildly concentrated. However, the considered nonlinear diffusion enhancement suppresses this potential blow-up.



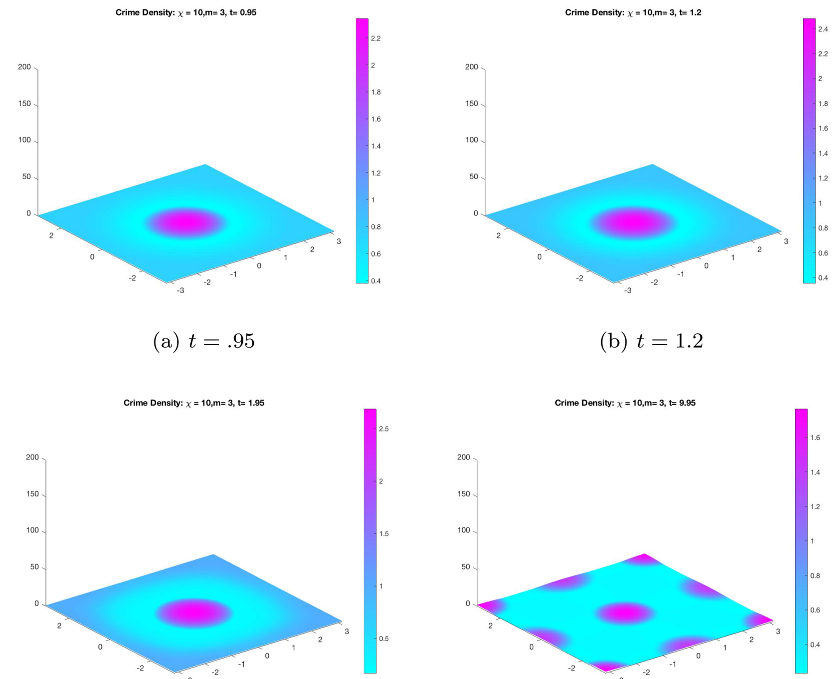
(a) $t = 0$

(b) $t = .95$

Fig. 1: Numerical solutions with $m = 1,\ \chi = 10$ and $u_0(x) = v_0(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{|x|^2}{2\sigma^2}}$ with $\sigma = 1/4$.

From more general numerical experiments, we observe that the smaller χ is, the more concentrated the initial data needs to be in order for a potential blow-up to occur in the $m = 1$ case. Moreover, for each χ there are initial data which are not sufficiently concentrated to lead to potential blow-up, but concentrated enough to see some initial growth. However, this initial growth is suppressed by the overcrowding effect from (1.2). This is shown in Figure 3, where the top row illustrates the linear diffusion case ($m = 1$) and the bottom row illustrates the non-linear diffusion with $m = 3$. In the top row, we observe the initial growth of the solution in Figure 3b. This growth does not last for very long and the solution is already decaying at time $t = .5$ as illustrated in Figure 3c. Note that in the $m = 3$ case, this initial growth never occurs, see Figure 3e. However, we do see some numerical instabilities for the case $m = 3$ on the boundary of the concentration. We expect that this is due to the degeneracy of the diffusion and more sophisticated numerical methods need to be used to deal with potential contact lines.

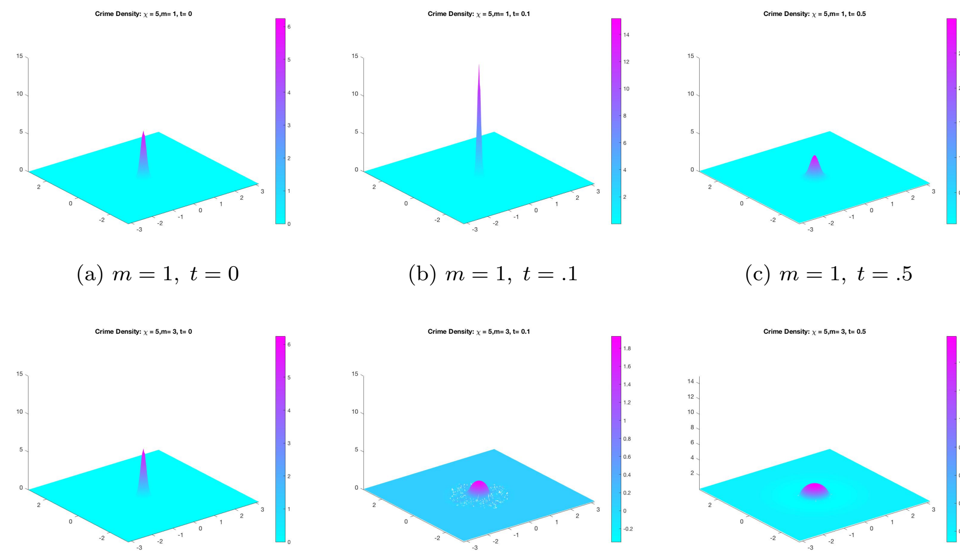
In conclusion, the numerical experiments presented here provide evidence of a potential blow-up of the solution to the original model proposed in [28], when the initial data is sufficiently concentrated. Note that the case of linear diffusion is not covered by our theoretical analysis. At the same time, we observe that this blow-up, or initial growth, are suppressed by replacing the linear diffusion with a porous medium type diffusion.



(a) $t = .95$ (b) $t = 1.2$

(c) $t = 1.95$ (d) $t = 9.95$

Fig. 2: Numerical solutions with $m = 3$, $\chi = 10$ and $u = v = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{|x|^2}{2\sigma^2}}$ with $\sigma = 1/4$.



(a) $m = 1,\ t = 0$ (b) $m = 1,\ t = .1$ (c) $m = 1,\ t = .5$

(d) $m = 3,\ t = 0$ (e) $m = 3,\ t = .1$ (f) $m = 3,\ t = .5$

Fig. 3: Numerical solutions comparing $m = 1$ and $m = 3$ with $\chi = 5$ and $u_0(x) = v_0(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{|x|^2}{2\sigma^2}}$ with $\sigma = .16$.

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