Refined regularity and stabilization properties in a degenerate haptotaxis system

Michael Winkler*
Institut für Mathematik, Universität Paderborn,
33098 Paderborn, Germany

Christian Stinner#
Technische Universität Darmstadt, Fachbereich Mathematik,
Schlossgartenstrasse 7, 64289 Darmstadt, Germany

Abstract

We consider the degenerate haptotaxis system

$$\begin{cases} u_t = (d(x)u)_{xx} - (d(x)uw_x)_x, \\ w_t = -ug(w), \end{cases}$$

endowed with no-flux boundary conditions in a bounded open interval $\Omega \subset \mathbb{R}$. It was proposed as a basic model for haptotactic migration in heterogeneous environments. If the diffusion is degenerate in the sense that d is non-negative, has a non-empty zero set and satisfies $\int_\Omega \frac{1}{d} < \infty$, then it has been shown in [12] under appropriate assumptions on the initial data that the system has a global generalized solution satisfying in particular $u(\cdot,t) \rightharpoonup \frac{\mu_\infty}{d}$ weakly in $L^1(\Omega)$ as $t \to \infty$ for some positive constant μ_∞.
We now prove that under the additional restriction $\int_\Omega \frac{1}{d^2} < \infty$ we have the strong convergence $u(\cdot,t) \to \frac{\mu_\infty}{d}$ in $L^p(\Omega)$ as $t \to \infty$ for any $p \in (1,2)$. In addition, with the same restriction on d we obtain improved regularity properties of u, for instance $du \in L^\infty((0,\infty); L^p(\Omega))$ for any $p \in (1,\infty)$.

Keywords: haptotaxis; degenerate diffusion; refined regularity; large time behavior
MSC 2010: 35B40 (primary); 35D30, 35K65, 92C17 (secondary)

1 Introduction

Tumor invasion into the healthy tissue relies on a plethora of processes. However, many types of cancer cells are ony able to move if they adhere to the tissue fibers in the extracellular matrix. Hence, they migrate from places with low densities of the tissue fibers (and corresponding adhesive molecules on the fibers) to places with higher densities. This process is called haptotaxis (see e.g. [4]) and has been present in a growing number of macroscopic models for tumor invasion into the tissue (see e.g. [5] for one of the first models). Consequently, the mathematical analysis of haptotaxis systems has got growing interest during the past decade. Mathematically, these systems usually consist of a cross-diffusive parabolic PDE for the tumor cell density (modeling diffusion and haptotaxis) coupled with

*michael.winkler@math.uni-paderborn.de
#stinner@mathematik.tu-darmstadt.de

an ODE for the density of the tissue fibers (since the latter is a non-diffusive attractant for the tumor cells). In most of these systems the random movement of cancer cells is described by non-degenerate diffusion of Fickian type (see e.g. [1, Section 4.3] for a recent survey), only few containing degenerate diffusion (see e.g. [13]). However, in organs with very heterogeneous tissue (e.g. in the brain) recent modeling approaches suggest that non-Fickian diffusion operators could possibly be more adequate, among them the so-called myopic diffusion (see e.g. [2]).
In [10] a degenerate haptotaxis system involving myopic diffusion has been proposed as a basic model for describing glioma spread in heterogeneous tissue. We will study a particular one-dimensional version thereof, which has been analyzed in [12] concerning the global existence and asymptotic behavior of a generalized solution. Namely, in a bounded open interval $\Omega \subset \mathbb{R}$, we consider the initial-boundary value problem

$$\begin{cases} u_t = \big(d(x)u\big)_{xx} - \big(d(x)uw_x\big)_x, & x \in \Omega,\ t > 0, \\ w_t = -ug(w), & x \in \Omega,\ t > 0, \\ \big(d(x)u\big)_x - d(x)uw_x = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

with given nonnegative functions d, u_0 and w_0 on $\overline{\Omega}$ and g generalizing the prototyoical choice $g(s) = s$, $s \geq 0$. More precisely, we assume that $d \in C^0(\overline{\Omega}) \cap C^1(\{d > 0\})$ is nonnegative and such that

$$\int_\Omega \frac{1}{d^2} < \infty, \tag{1.2}$$

where $\{d > 0\} := \big\{x \in \overline{\Omega} \;\big|\; d(x) > 0\big\}$. Furthermore, let $g \in C^2([0,\infty))$ such that $g(0) = 0$ and that with some positive constants $\underline{\gamma}$ and $\overline{\gamma}$ we have

$$\underline{\gamma} \leq g'(s) \leq \overline{\gamma} \quad \text{and} \quad \underline{\gamma}s \leq g(s) \leq \overline{\gamma}s \qquad \text{for all } s \geq 0. \tag{1.3}$$

Finally, we choose initial data satisfying

$$u_0, w_0 \in C^0(\overline{\Omega}), \quad u_0 \geq 0, u_0 \not\equiv 0, \quad w_0 \geq 0, \sqrt{w_0} \in W^{1,2}(\Omega) \quad \text{and} \quad \int_\Omega \frac{d_x^2}{d} w_0 < \infty. \tag{1.4}$$

In [12] the above assumptions were prescribed, but with $\int_\Omega \frac{1}{d} < \infty$ instead of (1.2), and it was shown that there exists a global generalized solution (u, w) to (1.1) in the sense of [12, Definition 2.1] such that $u \in C^0_w([0,\infty); L^1(\Omega)) \cap L^\infty((0,\infty); L^1(\Omega))$, $w \in C^0(\overline{\Omega}\times[0,\infty)) \cap L^\infty(\Omega\times(0,\infty)) \cap L^1_{loc}([0,\infty); W^{1,1}(\Omega))$, u obeys conservation of mass and the solution has the asymptotic behavior $w(\cdot,t) \to 0$ in $L^\infty(\Omega)$ as well as

$$u(\cdot,t) \rightharpoonup \frac{\mu_\infty}{d} \quad \text{weakly in } L^1(\Omega) \quad \text{as } t \to \infty, \tag{1.5}$$

where $\mu_\infty := \frac{\int_\Omega u_0}{\int_\Omega \frac{1}{d}}$.
It is the purpose of the present paper to establish a strong convergence of $u(\cdot,t)$ to $\frac{\mu_\infty}{d}$ in some space $L^p(\Omega)$. To this end, it turns out that instead of requiring $\frac{1}{d}$ belonging to $L^1(\Omega)$ we need that it belongs also to $L^2(\Omega)$. The latter means an additional restriction of the behavior of d near its zeros as compared to the setting from [12].

Main results: By requiring the generalization (1.6) of (1.2) we have the following main results:

Theorem 1.1 *Let $\Omega \subset \mathbb{R}$ be a bounded interval, and suppose that $d \in C^0(\overline{\Omega}) \cap C^1(\{d > 0\})$ is nonnegative and such that*

$$\int_\Omega \frac{1}{d^\lambda} < \infty \qquad \textit{for some } \lambda \geq 2. \tag{1.6}$$

Moreover, let $g \in C^2([0,\infty))$ be such that $g(0) = 0$ and that (1.3) is valid with some $\underline{\gamma} > 0$ and $\overline{\gamma} > 0$, and assume that the initial data u_0 and w_0 satisfy (1.4). Then the global generalized solution (u, w) of (1.1) from [12, Theorem 1.1] has the additional properties that

$$du \in L^\infty((0,\infty); L^p(\Omega)) \qquad \textit{for all } p \in (1,\infty) \tag{1.7}$$

and

$$u \in L^\infty((0,\infty); L^p(\Omega)) \qquad \textit{for all } p \in (1,\lambda) \tag{1.8}$$

as well as

$$u \in C^0(\{d > 0\} \times (0,\infty)), \tag{1.9}$$

and furthermore with $\mu_\infty := \frac{\int_\Omega u_0}{\int_\Omega \frac{1}{d}}$ we have

$$u(\cdot,t) \to \frac{\mu_\infty}{d} \quad \textit{in } L^p(\Omega) \quad \textit{for all } p \in (1,\lambda) \qquad \textit{as } t \to \infty. \tag{1.10}$$

While the proof of the regularity properties in [12] is mainly based on bounds obtained from an energy-like inequality for regularized approximations of (1.1), our approach to prove (1.7) stems from the observation that the flux term in the first equation of (1.1) has the form

$$(du)_x - duw_x = e^w \left(due^{-w}\right)_x.$$

This idea was established in non-degenerate haptotaxis systems in [6, 7] and leads for a supposedly given smooth solution of (1.1) to the identity

$$\frac{d}{dt}\int_\Omega \frac{1}{d}\left(due^{-w}\right)^p e^w = -p(p-1)\int_\Omega e^w \left(due^{-w}\right)^{p-2}\left(due^{-w}\right)_x^2 + (p-1)\int_\Omega \frac{1}{d^2} g(w) e^{2w}\left(due^{-w}\right)^{p+1}.$$

In order to rigorously prove an appropriate ODI for a regularized approximation of $\int_\Omega \frac{1}{d}\left(due^{-w}\right)^p e^w$, in the regularized version of the above identity we estimate the last term on the right-hand side, where u appears at a high power, by using on the one hand an interpolation inequality of Gagliardo-Nirenberg type which may be viewed as a derivate of an observation originally made in [3] and on the other hand estimates provided by [12] for the approximate problems (2.1), see (2.2)–(2.13) below. Here we will make essential use of our overall assumption that $\frac{1}{d}$ does not only belong to $L^1(\Omega)$ but even to $L^2(\Omega)$. In addition, we will also have to adequately cope with terms stemming from the artificial diffusion introduced in the second equation in (2.1), in this context no longer acting in a dissipative manner, and this will be achieved by substantially relying on the boundedness properties from [12]. These ingredients will then lead to uniform L^p estimates for the regularizations $d_\varepsilon u_\varepsilon$ of du. This we will do in Section 2.1, after having stated the approximate problems (2.1) along with some of their important properties from [12] in the beginning of Section 2.

If in addition to the assumptions from Theorem 1.1 we require $\frac{w_0}{d} \in L^\infty(\Omega)$, then [12, Theorem 1.3] implies for a.e. $t > 0$ the existence of positive constants $C_1(t)$ and $C_2(t)$ such that

$$\frac{C_1(t)}{d(x)} \le u(x,t) \le \frac{C_2(t)}{d(x)} \quad \text{for a.e. } x \in \Omega.$$

Hence, we cannot expect to achieve bounds for u_ε itself in $L^p(\Omega)$ for large p. However, for any compact set $K \subset \{d > 0\}$ we obtain uniform bounds for u_ε in $L^\infty(K)$. This is done in Section 2.2 with the help of a transformation to an inhomogeneous heat equation and the use of well-known estimates for the heat semigroup. These bounds in conjunction with standard parabolic regularity results then yield uniform interior Hölder estimates for u_ε in domains of the form $K \times (\tau, \frac{1}{\sqrt{\varepsilon}})$, see Section 2.3.
As all these estimates are uniform with respect to ε, by taking the limit $\varepsilon \searrow 0$ along an appropriate subsequence we finally conclude in Section 3 the properties of u claimed in Theorem 1.1. A short appendix contains the proof of the announced interpolation inequality of Gagliardo-Nirenberg type.

2 Refined uniform regularity properties for approximating problems

As in [12] we consider the approximating problems

$$\begin{cases} u_{\varepsilon t} = (d_\varepsilon u_\varepsilon)_{xx} - (d_\varepsilon u_\varepsilon w_{\varepsilon x})_x, & x \in \Omega,\ t > 0, \\ w_{\varepsilon t} = \varepsilon \Big(d_\varepsilon \frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \Big)_x - u_\varepsilon g(w_\varepsilon), & x \in \Omega,\ t > 0, \\ u_{\varepsilon x} = w_{\varepsilon x} = 0, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x,0) = u_0(x), \qquad w_\varepsilon(x,0) = w_{0\varepsilon}(x), & x \in \Omega, \end{cases} \tag{2.1}$$

for $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, where $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ with $\varepsilon_j \searrow 0$ as $j \to \infty$ as well as d_ε and $w_{0\varepsilon}$ are defined in [12, Lemma 2.2 and Lemma 2.6]. In view of the latter references, these functions have the following properties, which we will frequently use in the sequel: For all $\varepsilon, \varepsilon' \in (\varepsilon_j)_{j\in\mathbb{N}}$ we have

$$d_\varepsilon \in C^\infty(\bar\Omega), \qquad d_\varepsilon \to d \qquad \text{in } L^\infty(\Omega) \text{ as } \varepsilon = \varepsilon_j \searrow 0, \tag{2.2}$$

$$d_{\varepsilon x} \to d_x \qquad \text{in } L^p_{loc}(\{d > 0\}) \text{ as } \varepsilon = \varepsilon_j \searrow 0 \text{ for all } p \in [1,\infty), \tag{2.3}$$

$$d_\varepsilon > 0 \quad \text{and} \quad d \le d_\varepsilon \le d_{\varepsilon'} \quad \text{in } \bar\Omega \text{ for } \varepsilon \le \varepsilon', \tag{2.4}$$

$$d_{\varepsilon x} = 0 \qquad \text{on } \partial\Omega, \tag{2.5}$$

$$d_\varepsilon \le \|d\|_{L^\infty(\Omega)} + 1 \qquad \text{in } \Omega, \tag{2.6}$$

$$\varepsilon^2 \int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon^3} \le 1 \quad \text{and} \quad \varepsilon^{\frac14} \cdot \frac{1}{\inf_{x\in\Omega} d_\varepsilon(x)} \le 1, \tag{2.7}$$

$$w_{0\varepsilon}(x) := w_{0j}(x) + \varepsilon^{\frac14}, \qquad x \in \overline{\Omega}, \quad \varepsilon = \varepsilon_j \tag{2.8}$$

where $w_{0j} \in L^\infty(\Omega)$ is nonnegative and in particular satisfies $\sqrt{w_{0j}} \in W^{1,2}(\Omega)$ as well as $\operatorname{supp} w_{0j} \subset \{d > 0\}$ and, as $j \to \infty$, $w_{0j} \nearrow w_0$ in Ω.
Furthermore, it was shown in [12, Section 4] that for any $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ there is a global classical solution $(u_\varepsilon, w_\varepsilon)$ to (2.1). According to [12, Lemmas 2.7., 2.8 and 3.5], for any $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ this solution fulfills

$u_\varepsilon \geq 0$, $w_\varepsilon > 0$ in $\bar\Omega \times [0,\infty)$ as well as

$$\int_\Omega u_\varepsilon(\cdot,t) = \int_\Omega u_0 \qquad \text{for all } t>0, \tag{2.9}$$

$$w_\varepsilon(x,t) \leq M := \|w_0\|_{L^\infty(\Omega)} + 1 \qquad \text{for all } x \in \Omega \text{ and } t>0 \tag{2.10}$$

and there exists a constant $C>0$ such that for any $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ we have

$$\int_\Omega u_\varepsilon(\cdot,t)\Big|\ln(d_\varepsilon u_\varepsilon(\cdot,t))\Big| \leq C\cdot(1+\sqrt{\varepsilon}t) \qquad \text{for all } t>0, \tag{2.11}$$

$$\int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2(\cdot,t)}{w_\varepsilon(\cdot,t)} \leq C\cdot(1+\sqrt{\varepsilon}t) \qquad \text{for all } t>0, \tag{2.12}$$

$$\int_0^t \int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{d_\varepsilon u_\varepsilon} \leq C\cdot(1+\sqrt{\varepsilon}t) \qquad \text{for all } t>0. \tag{2.13}$$

2.1 An estimate for $d_\varepsilon u_\varepsilon$ in $L^p(\Omega)$

A crucial step for our asymptotic analysis consists in deriving appropriate ε-independent regularity properties of the solution component u_ε in Lebesgue spaces involving higher integrability powers. In order to prove the desired L^p estimate for $d_\varepsilon u_\varepsilon$ for abitrary large finite p, we will rely on an interpolation using Lemma 4.1 as well as the estimates from [12] and our assumption that $\frac{1}{d}$ belongs to $L^2(\Omega)$ and not only to $L^1(\Omega)$ as described in the introduction.

Lemma 2.1 *Assume that $\int_\Omega \frac{1}{d^2} < \infty$. Then for all $p \in (1,\infty)$ there exists $C(p)>0$ such that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ we have*

$$\|d_\varepsilon u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \leq C(p)e^{C(p)\sqrt{\varepsilon}t} \qquad \textit{for all } t>0. \tag{2.14}$$

Proof. Let us first follow an idea well-established in related non-degenerate frameworks ([6, 7]) to rewrite the flux in the first equation in (2.1) according to

$$(d_\varepsilon u_\varepsilon)_x - d_\varepsilon u_\varepsilon w_{\varepsilon x} = e^{w_\varepsilon}\Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)_x, \qquad x\in\Omega,\ t>0.$$

This, namely, suggests to test the PDE in question by $(d_\varepsilon u_\varepsilon e^{-w_\varepsilon})^{p-1}$ to obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{1}{d_\varepsilon}\Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)^p e^{w_\varepsilon} &= p\int_\Omega \Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)^{p-1}\cdot\Big\{e^{w_\varepsilon}\Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)_x\Big\}_x \\
&\quad -(p-1)\int_\Omega d_\varepsilon^{p-1}u_\varepsilon^p e^{-(p-1)w_\varepsilon}\cdot\Big\{\varepsilon\Big(d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\Big)_x - u_\varepsilon g(w_\varepsilon)\Big\} \\
&= -p(p-1)\int_\Omega e^{w_\varepsilon}\Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)^{p-2}\cdot\Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)_x^2 \\
&\quad +(p-1)\varepsilon\int_\Omega\Big(d_\varepsilon^{p-1}u_\varepsilon^p e^{-(p-1)w_\varepsilon}\Big)_x\cdot d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \\
&\quad +(p-1)\int_\Omega d_\varepsilon^{p-1}u_\varepsilon^{p+1}g(w_\varepsilon)e^{-(p-1)w_\varepsilon} \qquad \text{for all } t>0,
\end{aligned} \tag{2.15}$$

where the second summand on the right can be expanded so as to yield

$$
\begin{aligned}
&(p-1)\varepsilon\int_\Omega \Big(d_\varepsilon^{p-1}u_\varepsilon^p e^{-(p-1)w_\varepsilon}\Big)_x\cdot d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\\
&\quad= (p-1)\varepsilon\int_\Omega\Big\{\frac{1}{d_\varepsilon}e^{w_\varepsilon}\cdot\Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)^p\Big\}_x\cdot d_\varepsilon\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\\
&\quad= p(p-1)\varepsilon\int_\Omega e^{w_\varepsilon}\cdot\Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)^{p-1}\cdot\Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)_x\cdot\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\\
&\qquad+(p-1)\varepsilon\int_\Omega e^{w_\varepsilon}\cdot\Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)^p\cdot\frac{w_{\varepsilon x}^2}{\sqrt{g(w_\varepsilon)}}\\
&\qquad-(p-1)\varepsilon\int_\Omega\frac{d_{\varepsilon x}}{d_\varepsilon}e^{w_\varepsilon}\cdot\Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)^p\cdot\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\qquad\text{for all } t>0.
\end{aligned}
\tag{2.16}
$$

Here by Young's inequality, (2.10) and (1.3),

$$
\begin{aligned}
&p(p-1)\varepsilon\int_\Omega e^{w_\varepsilon}\cdot\Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)^{p-1}\cdot\Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)_x\cdot\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\\
&\quad\le \frac{p(p-1)}{2}\int_\Omega\Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)^{p-2}\Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)_x^2\\
&\qquad+\frac{p(p-1)\varepsilon^2}{2}\int_\Omega e^{2w_\varepsilon}\cdot\Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)^p\cdot\frac{w_{\varepsilon x}^2}{g(w_\varepsilon)}\\
&\quad\le \frac{p(p-1)}{2}\int_\Omega\Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)^{p-2}\Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)_x^2\\
&\qquad+\frac{p(p-1)\varepsilon e^{2M}}{2\underline{\gamma}}\cdot\Big\{\varepsilon\cdot\frac{1}{\inf_{x\in\Omega}d_\varepsilon(x)}\Big\}\cdot\Big\|d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big\|_{L^\infty(\Omega)}^p\cdot\int_\Omega d_\varepsilon\frac{w_{\varepsilon x}^2}{w_\varepsilon}\qquad\text{for all } t>0,
\end{aligned}
$$

so that since

$$
\varepsilon\cdot\frac{1}{\inf_{x\in\Omega}d_\varepsilon(x)}\le 1
$$

by (2.7), and since (2.12) warrants that with some $c_1>0$ we have

$$
\int_\Omega d_\varepsilon\frac{w_{\varepsilon x}^2}{w_\varepsilon}\le c_1\cdot(1+\sqrt{\varepsilon}t)\qquad\text{for all } t>0,
$$

we infer that for any choice of $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ and all $t>0$,

$$
\begin{aligned}
&p(p-1)\varepsilon\int_\Omega e^{w_\varepsilon}\cdot\Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)^{p-1}\cdot\Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)_x\cdot\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}}\\
&\le\frac{p(p-1)}{2}\int_\Omega\Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)^{p-2}\Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)_x^2+c_2\cdot(1+\sqrt{\varepsilon}t)\cdot\Big\|d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big\|_{L^\infty(\Omega)}^p,
\end{aligned}
\tag{2.17}
$$

where $c_2 := \frac{p(p-1)e^{2M}c_1}{2\underline{\gamma}}$.
Similarly, the second last summand in (2.16) can be controlled according to

$$\begin{aligned}
(p-1)\varepsilon \int_\Omega e^{w_\varepsilon} \cdot \Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)^p \cdot \frac{w_{\varepsilon x}^2}{\sqrt{g(w_\varepsilon)}} \\
&\le (p-1)\sqrt{\frac{M}{\underline{\gamma}}} e^M \cdot \Big\{ \varepsilon \cdot \frac{1}{\inf_{x\in\Omega} d_\varepsilon(x)} \Big\} \cdot \Big\| d_\varepsilon u_\varepsilon e^{-w_\varepsilon} \Big\|_{L^\infty(\Omega)}^p \cdot \int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} \\
&\le c_3 \cdot (1+\sqrt{\varepsilon} t) \cdot \Big\| d_\varepsilon u_\varepsilon e^{-w_\varepsilon} \Big\|_{L^\infty(\Omega)}^p \qquad \text{for all } t>0 \qquad (2.18)
\end{aligned}$$

with $c_3 := (p-1)\sqrt{\frac{M}{\underline{\gamma}}} e^M c_1$, whereas for the rightmost term in (2.16) we find on invoking the Cauchy-Schwarz inequality that

$$\begin{aligned}
-(p-1)\varepsilon \int_\Omega \frac{d_{\varepsilon x}}{d_\varepsilon} e^{w_\varepsilon} \cdot \Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)^p \cdot \frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \\
&\le (p-1)e^M \cdot \Big\{ \varepsilon \cdot \Big\{ \int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon^3} \Big\}^{\frac{1}{2}} \Big\} \cdot \Big\| d_\varepsilon u_\varepsilon e^{-w_\varepsilon} \Big\|_{L^\infty(\Omega)}^p \cdot \Big\{ \int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{g(w_\varepsilon)} \Big\}^{\frac{1}{2}} \\
&\le c_4 \cdot (1+\sqrt{\varepsilon} t)^{\frac{1}{2}} \cdot \Big\| d_\varepsilon u_\varepsilon e^{-w_\varepsilon} \Big\|_{L^\infty(\Omega)}^p \\
&\le c_4 \cdot (1+\sqrt{\varepsilon} t) \cdot \Big\| d_\varepsilon u_\varepsilon e^{-w_\varepsilon} \Big\|_{L^\infty(\Omega)}^p \qquad \text{for all } t>0 \qquad (2.19)
\end{aligned}$$

with $c_4 := \frac{(p-1)e^M\sqrt{c_1}}{\sqrt{\underline{\gamma}}}$, because (2.7) asserts that

$$\varepsilon \cdot \Big\{ \int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon^3} \Big\}^{\frac{1}{2}} \le 1.$$

In summary, (2.16)-(2.19) show that writing $c_5 := c_2 + c_3 + c_4$ we obtain

$$\begin{aligned}
(p-1)\varepsilon \int_\Omega \Big(d_\varepsilon^{p-1} u_\varepsilon^p e^{-(p-1)w_\varepsilon}\Big)_x \cdot d_\varepsilon \frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} &\le \frac{p(p-1)}{2} \int_\Omega \Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)^{p-2} \Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)_x^2 \\
&\quad + c_5 \cdot (1+\sqrt{\varepsilon} t) \cdot \Big\| d_\varepsilon u_\varepsilon e^{-w_\varepsilon} \Big\|_{L^\infty(\Omega)}^p \qquad (2.20)
\end{aligned}$$

for all $t>0$, and in order to further estimate the last summand herein, employing the Gagliardo-Nirenberg inequality we pick $c_6 > 0$ such that

$$\|\varphi\|_{L^\infty(\Omega)}^2 \le c_6 \|\varphi_x\|_{L^2(\Omega)}^{\frac{2p}{p+1}} \|\varphi\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{p+1}} + c_6 \|\varphi\|_{L^{\frac{2}{p}}(\Omega)}^2 \qquad \text{for all } \varphi \in W^{1,2}(\Omega)$$

and invoke Young's inequality in fixing $c_7 > 0$ such that

$$ab \le \frac{p-1}{p} a^{\frac{p+1}{p}} + c_7 b^{p+1} \qquad \text{for all } a \ge 0 \text{ and } b \ge 0.$$

Using that by (2.6) and (2.9) we know that

$$\left\| \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^{\frac{p}{2}} \right\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{p}} = \int_\Omega d_\varepsilon u_\varepsilon e^{-w_\varepsilon} \le c_8 := (\|d\|_{L^\infty(\Omega)} + 1) \cdot \int_\Omega u_0 \qquad \text{for all } t > 0, \tag{2.21}$$

we thereby see that for all $t > 0$,

$$\begin{aligned}
& c_5 \cdot (1 + \sqrt{\varepsilon} t) \cdot \left\| d_\varepsilon u_\varepsilon e^{-w_\varepsilon} \right\|_{L^\infty(\Omega)}^p \\
&= c_5 \cdot (1 + \sqrt{\varepsilon} t) \cdot \left\| \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^{\frac{p}{2}} \right\|_{L^\infty(\Omega)}^2 \\
&\le c_5 c_6 \cdot (1 + \sqrt{\varepsilon} t) \cdot \left\| \left\{ \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^{\frac{p}{2}} \right\}_x \right\|_{L^2(\Omega)}^{\frac{2p}{p+1}} \cdot \left\| \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^{\frac{p}{2}} \right\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{p+1}} \\
&\quad + c_5 c_6 \cdot (1 + \sqrt{\varepsilon} t) \cdot \left\| \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^{\frac{p}{2}} \right\|_{L^{\frac{2}{p}}(\Omega)}^2 \\
&\le \frac{p-1}{p} \left\| \left\{ \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^{\frac{p}{2}} \right\}_x \right\|_{L^2(\Omega)}^2 \\
&\quad + c_7 \cdot \left\{ c_5 c_6 \cdot (1 + \sqrt{\varepsilon} t) \right\}^{p+1} \left\| \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^{\frac{p}{2}} \right\|_{L^{\frac{2}{p}}(\Omega)}^2 \\
&\quad + c_5 c_6 \cdot (1 + \sqrt{\varepsilon} t) \cdot \left\| \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^{\frac{p}{2}} \right\|_{L^{\frac{2}{p}}(\Omega)}^2 \\
&\le \frac{p-1}{p} \left\| \left\{ \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^{\frac{p}{2}} \right\}_x \right\|_{L^2(\Omega)}^2 + c_9 \cdot (1 + \sqrt{\varepsilon} t)^{p+1} \\
&= \frac{p(p-1)}{4} \int_\Omega \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^{p-2} \cdot \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)_x^2 + c_9 \cdot (1 + \sqrt{\varepsilon} t)^{p+1}
\end{aligned} \tag{2.22}$$

with $c_9 := c_5^{p+1} c_6^{p+1} c_7 c_8^p + c_5 c_6 c_8^p$.
On the right-hand side of (2.15), we next proceed to use (1.3) and (2.10) as well as (2.4) to estimate

$$\begin{aligned}
(p-1) \int_\Omega d_\varepsilon^{p-1} u_\varepsilon^{p+1} g(w_\varepsilon) e^{-(p-1)w_\varepsilon} &= (p-1) \int_\Omega \frac{1}{d_\varepsilon^2} g(w_\varepsilon) e^{2w_\varepsilon} \cdot \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^{p+1} \\
&\le c_{10} \left\| d_\varepsilon u_\varepsilon e^{-w_\varepsilon} \right\|_{L^\infty(\Omega)}^{p+1} \qquad \text{for all } t > 0,
\end{aligned} \tag{2.23}$$

noting that $c_{10} := (p-1)\overline{\gamma} M e^{2M} \int_\Omega \frac{1}{d^2}$ is finite thanks to our overall assumption on square integrability of $\frac{1}{d}$. Here we recall that (2.11) provides $c_{11} > 0$ such that

$$\int_\Omega u_\varepsilon \left| \ln(d_\varepsilon u_\varepsilon) \right| \le c_{11} \cdot (1 + \sqrt{\varepsilon} t) \qquad \text{for all } t > 0,$$

which by (2.6), (2.9) and (2.10) entails that there exists $c_{12} > 0$ such that

$$
\begin{aligned}
\left\| \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^{\frac{p}{2}} \cdot \left| \ln \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^{\frac{p}{2}} \right|^{\frac{p}{2}} \right\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{p}} &= \frac{p}{2} \int_\Omega d_\varepsilon u_\varepsilon e^{-w_\varepsilon} \cdot \left| \ln \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right) \right| \\
&\le \frac{p}{2} \int_\Omega d_\varepsilon u_\varepsilon \left| \ln(d_\varepsilon u_\varepsilon) \right| e^{-w_\varepsilon} + \frac{p}{2} \int_\Omega d_\varepsilon u_\varepsilon w_\varepsilon e^{-w_\varepsilon} \\
&\le \frac{p}{2}(\|d\|_{L^\infty(\Omega)} + 1) \int_\Omega u_\varepsilon \left| \ln(d_\varepsilon u_\varepsilon) \right| \\
&\quad + \frac{p}{2}(\|d\|_{L^\infty(\Omega)} + 1) M \int_\Omega u_\varepsilon \\
&\le c_{12} \cdot (1 + \sqrt{\varepsilon} t) \qquad \text{for all } t > 0.
\end{aligned}
$$

Therefore, applying Lemma 4.1 to $q := \frac{2}{p}$ and recalling (2.21) we see that with some $c_{13} \ge p + 1$ we have

$$
\begin{aligned}
& c_{10} \left\| d_\varepsilon u_\varepsilon e^{-w_\varepsilon} \right\|_{L^\infty(\Omega)}^{p+1} \\
&= c_{10} \left\| \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^{\frac{p}{2}} \right\|_{L^\infty(\Omega)}^{\frac{2}{p}+2} \\
&\le \frac{p-1}{2p} \cdot \frac{1}{c_{12} \cdot (1 + \sqrt{\varepsilon} t)} \cdot \left\| \left\{ \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^{\frac{p}{2}} \right\}_x \right\|_{L^2(\Omega)}^2 \cdot \left\| \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^{\frac{p}{2}} \left| \ln \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^{\frac{p}{2}} \right|^{\frac{p}{2}} \right\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{p}} \\
&\quad + c_{13} e^{c_{13} \cdot (1 + \sqrt{\varepsilon} t)} + c_{13} \left\| \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^{\frac{p}{2}} \right\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \\
&\le \frac{p-1}{2p} \left\| \left\{ \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^{\frac{p}{2}} \right\}_x \right\|_{L^2(\Omega)}^2 + c_{13} e^{c_{13} \cdot (1 + \sqrt{\varepsilon} t)} + c_{14} \\
&= \frac{p(p-1)}{8} \int_\Omega \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^{p-2} \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)_x^2 + c_{13} e^{c_{13} \cdot (1 + \sqrt{\varepsilon} t)} + c_{14} \qquad \text{for all } t > 0.
\end{aligned}
$$

Together with (2.23), (2.22), (2.20) and (2.15), since $(1 + \sqrt{\varepsilon} t)^{p+1} \le e^{(p+1)\sqrt{\varepsilon} t} \le e^{c_{13} \cdot (1 + \sqrt{\varepsilon} t)}$ for all $t > 0$ this entails that

$$
\begin{aligned}
& \frac{d}{dt} \int_\Omega \frac{1}{d_\varepsilon} \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^p e^{w_\varepsilon} + \frac{p(p-1)}{8} \int_\Omega \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^{p-2} \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)_x^2 \\
&\qquad \le (c_9 + c_{13}) e^{c_{13} \cdot (1 + \sqrt{\varepsilon} t)} + c_{14} \qquad \text{for all } t > 0. \qquad (2.24)
\end{aligned}
$$

Since finally a Sobolev inequality associated with the embedding $W^{1,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ yields $c_{15} > 0$ such that

$$
\|\varphi\|_{L^\infty(\Omega)}^2 \le c_{15} \|\varphi_x\|_{L^2(\Omega)}^2 + c_{15} \|\varphi\|_{L^{\frac{2}{p}}(\Omega)}^2 \qquad \text{for all } \varphi \in W^{1,2}(\Omega),
$$

again by means of (2.21) and (2.4), as $\int_\Omega \frac{1}{d}$ is finite we can find $c_{16} > 0$ and $c_{17} > 0$ such that

$$
\begin{aligned}
\int_\Omega \frac{1}{d_\varepsilon} \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^p e^{w_\varepsilon} &\le e^M \cdot \left\{ \int_\Omega \frac{1}{d_\varepsilon} \right\} \cdot \left\| \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^{\frac{p}{2}} \right\|_{L^\infty(\Omega)}^2 \\
&\le c_{16} \int_\Omega \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)^{p-2} \left(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\right)_x^2 + c_{16} \qquad \text{for all } t > 0
\end{aligned}
$$

and hence

$$\int_\Omega \Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)^{p-2}\Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)_x^2 \ge \frac{1}{c_{16}}\int_\Omega \frac{1}{d_\varepsilon}\Big(d_\varepsilon u_\varepsilon e^{-w_\varepsilon}\Big)^p e^{w_\varepsilon} - 1 \qquad \text{for all } t>0.$$

Therefore, (2.24) shows that for

$$y_\varepsilon(t) := \int_\Omega \frac{1}{d_\varepsilon}\Big(d_\varepsilon u_\varepsilon(\cdot,t)e^{-w_\varepsilon(\cdot,t)}\Big)^p e^{w_\varepsilon(\cdot,t)}, \qquad t\in[0,T_{max,\varepsilon}),$$

and $c_{17} := \frac{p(p-1)}{8c_{16}}$ as well as $c_{18} := (c_9+c_{13})e^{c_{13}} + c_{14} + \frac{p(p-1)}{8}$ we have

$$y_\varepsilon'(t) + c_{17}y_\varepsilon(t) \le c_{18}e^{c_{13}\sqrt{\varepsilon}t} \qquad \text{for all } t>0,$$

which by an ODE comparison argument implies that

$$\begin{aligned} y_\varepsilon(t) &\le y_\varepsilon(0)e^{-c_{17}t} + c_{18}\int_0^t e^{-c_{17}(t-s)}e^{c_{13}\sqrt{\varepsilon}s}ds \\ &= y_\varepsilon(0)e^{-c_{17}t} + \frac{c_{18}}{c_{17}+c_{13}\sqrt{\varepsilon}}e^{-c_{17}t}\cdot\Big\{e^{(c_{17}+c_{13}\sqrt{\varepsilon})t}-1\Big\} \\ &\le y_\varepsilon(0)e^{-c_{17}t} + \frac{c_{18}}{c_{17}}e^{c_{13}\sqrt{\varepsilon}t} \qquad \text{for all } t>0 \end{aligned}$$

and thereby yields the claim, because

$$y_\varepsilon(0) = \int_\Omega d_\varepsilon^{p-1}u_0^p e^{-(p-1)w_{0\varepsilon}} \le (\|d\|_{L^\infty(\Omega)}+1)^{p-1}\|u_0\|_{L^\infty(\Omega)}^p|\Omega|$$

by (2.6). □

2.2 A local L^∞ bound for u_ε in $\{d>0\}$

We first plan to derive a bound for u_ε in $L^\infty(K)$ with arbitrary compact $K\subset\{d>0\}$, which in view of the equation satisfied by the quantity $d_\varepsilon u_\varepsilon e^{-w_\varepsilon}$ apparently does not follow from Lemma 2.1 in a trivial manner upon performing a straightforward Moser-type iteration. Fortunately, in the present one-dimensional situation an alternative approach can be based on a variable transformation which allows for a reduction to a linear inhomogeneous heat equation:

Lemma 2.2 *Let $J\subset\overline{\Omega}$ be an interval and $x_0\in J$, and given $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}$ let*

$$\phi_\varepsilon(x) := \int_{x_0}^x \frac{d\xi}{\sqrt{d_\varepsilon(\xi)}}, \qquad x\in J, \tag{2.25}$$

as well as $\widetilde{J}_\varepsilon := \phi_\varepsilon(J)$. Then $\phi_\varepsilon\in C^\infty(J)$ is strictly increasing, and if for arbitrary $\zeta\in C^2(\widetilde{J}_\varepsilon)$ we let the functions $Z_\varepsilon, W_\varepsilon$ and H_ε be definied on $\widetilde{J}_\varepsilon\times[0,\infty)$ by setting

$$Z_\varepsilon(y,t) := d_\varepsilon(x)u_\varepsilon(x,t) \tag{2.26}$$

and

$$W_\varepsilon(y,t) := w_\varepsilon(x,t) \tag{2.27}$$

as well as

$$H_\varepsilon(y,t) := \zeta(y) Z_\varepsilon(y,t) \tag{2.28}$$

with

$$x := \phi_\varepsilon^{-1}(y) \tag{2.29}$$

for $y \in \widetilde{J}_\varepsilon$ *and* $t \ge 0$*, then*

$$H_{\varepsilon t} = H_{\varepsilon yy} - b^{(1)}_{\varepsilon y} - b^{(2)}_\varepsilon + b^{(3)}_\varepsilon, \qquad y \in \widetilde{J}_\varepsilon,\ t > 0, \tag{2.30}$$

where

$$b^{(1)}_\varepsilon(y,t) := \zeta(y) Z_\varepsilon(y,t) W_{\varepsilon y}(y,t) \tag{2.31}$$

and

$$b^{(2)}_\varepsilon(y,t) := \Big(2\zeta_y(y) + D_\varepsilon(y)\zeta(y)\Big) Z_{\varepsilon y}(y,t) \tag{2.32}$$

as well as

$$b^{(3)}_\varepsilon(y,t) := \Big(- \zeta_{yy}(y) + \zeta_y(y) W_{\varepsilon y}(y,t) + D_\varepsilon(y)\zeta(y) W_{\varepsilon y}(y,t)\Big) Z_\varepsilon(y,t) \tag{2.33}$$

with

$$D_\varepsilon(y) := \frac{d_{\varepsilon x}(x)}{2\sqrt{d_\varepsilon(x)}}, \qquad x = \phi_\varepsilon^{-1}(y), \tag{2.34}$$

for $(y,t) \in \widetilde{J}_\varepsilon \times (0,\infty)$*. Moreover, if* $J \cap \partial\Omega \ne \emptyset$*, and if for some* $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ *we have* $\zeta_y = 0$ *on* $\phi_\varepsilon(J \cap \partial\Omega)$*, then*

$$H_{\varepsilon y} = 0 \qquad \text{on } \phi_\varepsilon(J\cap\partial\Omega). \tag{2.35}$$

Proof. The claimed regularity and monotonicity properties of ϕ_ε are evident from the inclusion $d_\varepsilon \in C^\infty(\overline{\Omega})$ and the positivity of d_ε on $\overline{\Omega}$, as asserted by (2.2) and (2.4). To verify (2.30), we only need to combine (2.25)-(2.29) in computing

$$Z_{\varepsilon y} = \sqrt{d_\varepsilon} \cdot (d_\varepsilon u_\varepsilon)_x \tag{2.36}$$

as well as

$$Z_{\varepsilon yy} = d_\varepsilon \cdot (d_\varepsilon u_\varepsilon)_{xx} + \frac{1}{2} d_{\varepsilon x} \cdot (d_\varepsilon u_\varepsilon)_x = d_\varepsilon \cdot (d_\varepsilon u_\varepsilon)_{xx} + D_\varepsilon Z_{\varepsilon y}$$

and, similarly,

$$\begin{aligned} W_{\varepsilon y} &= \sqrt{d_\varepsilon} w_{\varepsilon x} \qquad \text{as well as} \\ W_{\varepsilon yy} &= d_\varepsilon w_{\varepsilon xx} + \frac{1}{2} d_{\varepsilon x} w_{\varepsilon x} = d_\varepsilon w_{\varepsilon xx} + D_\varepsilon W_{\varepsilon y}, \end{aligned}$$

so that by (2.1),

$$\begin{aligned} Z_{\varepsilon t} &= d_\varepsilon u_{\varepsilon t} \\ &= d_\varepsilon \cdot (d_\varepsilon u_\varepsilon)_{xx} - d_\varepsilon \cdot (d_\varepsilon u_\varepsilon)_x w_{\varepsilon x} - d_\varepsilon \cdot (d_\varepsilon u_\varepsilon) w_{\varepsilon xx} \\ &= \Big\{ Z_{\varepsilon yy} - D_\varepsilon Z_{\varepsilon y} \Big\} - Z_{\varepsilon y} W_{\varepsilon y} - Z_\varepsilon \cdot \Big\{ W_{\varepsilon yy} - D_\varepsilon W_{\varepsilon y} \Big\} \\ &= Z_{\varepsilon yy} - (Z_\varepsilon W_{\varepsilon y})_y - D_\varepsilon Z_{\varepsilon y} + D_\varepsilon Z_\varepsilon W_{\varepsilon y} \end{aligned}$$

in $\widetilde{J}_\varepsilon \times (0,\infty)$. By furthermore using the identities

$$H_{\varepsilon y} = \zeta Z_{\varepsilon y} + \zeta_y Z_\varepsilon \tag{2.37}$$

and

$$H_{\varepsilon yy} = \zeta Z_{\varepsilon yy} + 2\zeta_y Z_{\varepsilon y} + \zeta_{yy} Z_\varepsilon,$$

from this we readily derive (2.30). Finally, (2.35) is a direct consequence of (2.37) and the fact that due to (2.36), the boundary condition for u_ε in (2.1) together with the property $d_{\varepsilon x}|_{\partial\Omega} = 0$ achieved in (2.5) warrants that $Z_{\varepsilon y} = 0$ on $\phi_\varepsilon(J \cap \partial\Omega)$. □

In consequence, deriving bounds of the desired type essentially reduces to suitably estimating the inhomogeneities in (2.30) on the basis of [12, Lemma 3.5] and Lemma 2.1. Indeed, this will form the core of the otherwise mainly technical reasoning in the following.

Lemma 2.3 *Suppose that $\int_\Omega \frac{1}{d^2} < \infty$, and let $K \subset \{d > 0\}$ be compact. Then there exists $C(K) > 0$ such that whenever $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(K)} \le C(K) e^{C(K)\sqrt{\varepsilon} t} \qquad \textit{for all } t > 0. \tag{2.38}$$

PROOF. We evidently only need to consider the situation when $K \subset \{d > 0\}$ is a compact interval, and hence contained in a single connected component $J \subset \overline{\Omega}$ of $\{d > 0\}$. Here we first concentrate on the case when $J \subset \Omega$, in which since then J is open, by compactness of K we can find points a_0, a, b_0 and b in Ω fulfilling

$$K \subset (a_0, b_0) \subset [a_0, b_0] \subset (a, b) \subset [a, b] \subset J. \tag{2.39}$$

Now fixing any $x_0 \in J$, for $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ we let ϕ_ε and $\widetilde{J}_\varepsilon$ be taken from Lemma 2.2, and observe that since $d_\varepsilon \to d$ in $L^\infty(\Omega)$ by (2.2) and hence $\phi_\varepsilon(x) \to \int_{x_0}^x \frac{d\xi}{\sqrt{d(\xi)}}$ uniformly with respect to $x \in [a,b] \subset \{d > 0\}$ as $\varepsilon = \varepsilon_j \searrow 0$, due to (2.39) it is possible to choose real numbers $\widetilde{a}_0, \widetilde{a}, \widetilde{b}_0$ and $\widetilde{b}$ as well as $\varepsilon^{(1)} \in (0,1)$ such that for any $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ fulfilling $\varepsilon < \varepsilon^{(1)}$ we have

$$\phi_\varepsilon(K) \subset [\widetilde{a}_0, \widetilde{b}_0] \subset (\widetilde{a}, \widetilde{b}) \subset [\widetilde{a}, \widetilde{b}] \subset \phi_\varepsilon([a,b]). \tag{2.40}$$

Now writing $G_0 := (\widetilde{a}_0, \widetilde{b}_0)$ and $G := (\widetilde{a}, \widetilde{b})$, we fix a cut-off function $\zeta \in C_0^\infty(G)$ satisfying $0 \le \zeta \le 1$ in G as well as $\zeta \equiv 1$ in G_0, and thereupon take $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ such that $\varepsilon < \varepsilon^{(1)}$ as well as $Z_\varepsilon, W_\varepsilon, H_\varepsilon, D_\varepsilon$ and $b_\varepsilon^{(i)}$, $i \in \{1,2,3\}$, as introduced in Lemma 2.2. Then letting A denote the realization of the operator $-(\cdot)_{yy}$ under homogeneous Neumann boundary conditions in the fixed domain G, we may use (2.30) to represent H_ε according to

$$\begin{aligned} H_\varepsilon(\cdot,t) &= e^{-\min\{1,t\}A} H_\varepsilon(\cdot,(t-1)_+) - \int_{(t-1)_+}^t e^{-(t-s)A} b_{\varepsilon y}^{(1)}(\cdot,s)ds \\ &\quad - \int_{(t-1)_+}^t e^{-(t-s)A} b_\varepsilon^{(2)}(\cdot,s)ds + \int_{(t-1)_+}^t e^{-(t-s)A} b_\varepsilon^{(3)}(\cdot,s)ds \end{aligned} \tag{2.41}$$

for $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}} \cap (0,\varepsilon^{(1)})$ and $t>0$, where by the maximum principle, (2.28), (2.26) and (2.6),

$$\begin{aligned}
\Big\| e^{-\min\{1,t\}A} H_\varepsilon(\cdot,(t-1)_+)\Big\|_{L^\infty(G)} &= \Big\| e^{-tA} H_\varepsilon(\cdot,0)\Big\|_{L^\infty(G)} \\
&\le \|H_\varepsilon(\cdot,0)\|_{L^\infty(G)} \\
&\le \|d_\varepsilon u_0\|_{L^\infty(\Omega)} \\
&\le c_1 := (\|d\|_{L^\infty(\Omega)}+1)\|u_0\|_{L^\infty(\Omega)} \qquad \text{for all } t\le 1, \qquad (2.42)
\end{aligned}$$

and where by a known regularization property of $(e^{-\tau A})_{\tau\ge 0}$ ([11]), there exists $c_2>0$ such that

$$\begin{aligned}
\Big\| e^{-\min\{1,t\}A} H_\varepsilon(\cdot,(t-1)_+)\Big\|_{L^\infty(G)} &= \Big\| e^{-A} H_\varepsilon(\cdot,t-1)\Big\|_{L^\infty(G)} \\
&\le c_2 \|H_\varepsilon(\cdot,t-1)\|_{L^1(G)} \\
&\le c_2 \|Z_\varepsilon(\cdot,t-1)\|_{L^1(G)} \qquad \text{for all } t>1. \qquad (2.43)
\end{aligned}$$

To further estimate the latter, we substitute $y=\phi_\varepsilon(x)$ and recall (2.9) in deriving

$$\begin{aligned}
\|Z_\varepsilon(\cdot,t-1)\|_{L^1(G)} &= \int_G Z_\varepsilon(y,t-1)dy \\
&= \int_{\phi_\varepsilon^{-1}(G)} d_\varepsilon(x) u_\varepsilon(x,t-1) \frac{dx}{\sqrt{d_\varepsilon(x)}} \\
&\le (\|d\|_{L^\infty(\Omega)}+1)^{\frac{1}{2}} \int_\Omega u_\varepsilon(x,t-1)dx \\
&= c_3 := (\|d\|_{L^\infty(\Omega)}+1)^{\frac{1}{2}} \int_\Omega u_0 \qquad \text{for all } t>1,
\end{aligned}$$

whence (2.42) and (2.43) imply that

$$\Big\| e^{-\min\{1,t\}A} H_\varepsilon(\cdot,(t-1)_+)\Big\|_{L^\infty(G)} \le \max\{c_1, c_2c_3\} \qquad \text{for all } t>0. \qquad (2.44)$$

We next fix an arbitrary $p\in(\frac{3}{2},2)$ and employ further known smoothing estimates for the Neumann heat semigroup ([8], [11]) to obtain $c_4>0$ and $c_5>0$ with the property that for all $\tau\in(0,1)$ we have

$$\|e^{-\tau A}\varphi_y\|_{L^\infty(G)} \le c_4 \tau^{-\frac{1}{2}-\frac{1}{2p}} \|\varphi\|_{L^p(G)} \qquad \text{for all } \varphi\in C^1(\bar G) \text{ such that } \varphi|_{\partial G}=0 \qquad (2.45)$$

and

$$\|e^{-\tau A}\varphi\|_{L^\infty(G)} \le c_5 \tau^{-\frac{1}{2p}} \|\varphi\|_{L^p(G)} \qquad \text{for all } \varphi\in C^0(\bar G). \qquad (2.46)$$

Therefore,

$$\begin{aligned}
\Big\| \int_{(t-1)_+}^t e^{-(t-s)A} b_{\varepsilon y}^{(1)}(\cdot,s)ds \Big\|_{L^\infty(G)} &\le c_4 \int_{(t-1)_+}^t (t-s)^{-\frac{1}{2}-\frac{1}{2p}} \|b_\varepsilon^{(1)}(\cdot,s)\|_{L^p(\Omega)} ds \\
&\le c_4 c_6 \|b_\varepsilon^{(1)}\|_{L^\infty((0,t);L^p(G))} \qquad \text{for all } t>0 \qquad (2.47)
\end{aligned}$$

with $c_6 := \int_0^1 \sigma^{-\frac{1}{2}-\frac{1}{2p}} d\sigma = \frac{2p}{p-1}$, and by means of the Hölder inequality we see that

$$
\begin{aligned}
\Big\| \int_{(t-1)_+}^t & e^{-(t-s)A} b_\varepsilon^{(2)}(\cdot,s)ds \Big\|_{L^\infty(G)} \\
&\le c_5 \int_{(t-1)_+}^t (t-s)^{-\frac{1}{2p}} \|b_\varepsilon^{(2)}(\cdot,s)\|_{L^p(G)} ds \\
&\le c_5 \cdot \Big\{ \int_{(t-1)_+}^t (t-s)^{-\frac{1}{2(p-1)}} ds \Big\}^{\frac{p-1}{p}} \cdot \Big\{ \int_{(t-1)_+}^t \|b_\varepsilon^{(2)}(\cdot,s)\|_{L^p(G)}^p ds \Big\}^{\frac{1}{p}} \\
&\le c_5 c_7 \|b_\varepsilon^{(2)}\|_{L^p(G\times((t-1)_+,t))} \qquad \text{for all } t>0
\end{aligned}
\tag{2.48}
$$

with

$$
c_7 := \Big\{ \int_0^1 \sigma^{-\frac{1}{2(p-1)}} d\sigma \Big\}^{\frac{p-1}{p}} = \Big(\frac{2(p-1)}{2p-3}\Big)^{\frac{p-1}{p}},
$$

and

$$
\begin{aligned}
\Big\| \int_{(t-1)_+}^t e^{-(t-s)A} b_\varepsilon^{(3)}(\cdot,s)ds \Big\|_{L^\infty(G)} &\le c_5 \int_{(t-1)_+}^t (t-s)^{-\frac{1}{2p}} \|b_\varepsilon^{(3)}(\cdot,s)\|_{L^p(G)} ds \\
&\le c_5 c_8 \|b_\varepsilon^{(3)}\|_{L^\infty((0,t);L^p(G))} \qquad \text{for all } t>0,
\end{aligned}
\tag{2.49}
$$

where $c_8 := \int_0^1 \sigma^{-\frac{1}{2p}} d\sigma = \frac{2p}{2p-1}$.
To prepare an appropriate further estimation of the right-hand sides in (2.47), (2.48) and (2.49), using that $\phi_\varepsilon^{-1}(G) \subset [a,b]$ we infer from the uniform positivity of d in $[a,b]$ and the fact that $d_\varepsilon \ge d$ for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ that there exists $\delta>0$ such that for each $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,

$$
d_\varepsilon(x) \ge \delta \qquad \text{for all } x \in [a,b]. \tag{2.50}
$$

For arbitrary $q \in (1,\infty)$, we thus obtain that

$$
\begin{aligned}
\int_G Z_\varepsilon^q(y,t)dy &= \int_{\phi_\varepsilon^{-1}(G)} \Big(d_\varepsilon(x)u_\varepsilon(x,t)\Big)^q \frac{dx}{\sqrt{d_\varepsilon(x)}} \\
&\le \frac{1}{\sqrt{\delta}} \int_a^b (d_\varepsilon u_\varepsilon)^q \qquad \text{for all } t>0,
\end{aligned}
$$

so that from Lemma 2.1 it follows that for any such q we can find $c_9(q)>0$ fulfilling

$$
\|Z_\varepsilon(\cdot,t)\|_{L^q(G)} \le c_9(q) e^{c_9(q)\sqrt{\varepsilon}t} \qquad \text{for all } t>0. \tag{2.51}
$$

Moreover, if $q \in (1,2)$, then by (2.6) and the Hölder inequality we see that for all $t>0$,

$$
\begin{aligned}
\int_t^{t+1}\int_G |Z_{\varepsilon y}(y,s)|^q dyds &= \int_t^{t+1} \int_{\phi_\varepsilon^{-1}(G)} d_\varepsilon^{\frac{q}{2}}(x) \Big|(d_\varepsilon u_\varepsilon)_x(x,s)\Big|^q \frac{dx}{\sqrt{d_\varepsilon(x)}} ds \\
&\le \int_t^{t+1}\int_a^b d_\varepsilon^{\frac{q-1}{2}} \cdot \Big\{ \frac{(d_\varepsilon u_\varepsilon)_x^2}{d_\varepsilon u_\varepsilon} \Big\}^{\frac{q}{2}} \cdot (d_\varepsilon u_\varepsilon)^{\frac{q}{2}} \\
&\le (\|d\|_{L^\infty(\Omega)}+1)^{\frac{q-1}{2}} \cdot \Big\{ \int_t^{t+1}\int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{d_\varepsilon u_\varepsilon} \Big\}^{\frac{q}{2}} \cdot \Big\{ \int_t^{t+1}\int_\Omega (d_\varepsilon u_\varepsilon)^{\frac{q}{2-q}} \Big\}^{\frac{2-q}{2}},
\end{aligned}
$$

where (2.13) yields $c_{10} > 0$ such that

$$\int_t^{t+1} \int_\Omega \frac{(d_\varepsilon u_\varepsilon)_x^2}{d_\varepsilon u_\varepsilon} \le c_{10} \cdot (1 + \sqrt{\varepsilon} t) \qquad \text{for all } t > 0.$$

Again employing Lemma 2.1, we thus conclude that for each $q \in (1,2)$ there exists $c_{11}(q) > 0$ satisfying

$$\|Z_{\varepsilon y}\|_{L^q(G\times(t,t+1))} \le c_{11}(q) e^{c_{11}(q)\sqrt{\varepsilon}t} \qquad \text{for all } t > 0. \tag{2.52}$$

Proceeding similarly, using (2.10) and (2.50) we estimate

$$\begin{aligned}
\int_G W_{\varepsilon y}^2(y,t)dy &= \int_{\phi_\varepsilon^{-1}(G)} \sqrt{d_\varepsilon} w_{\varepsilon x}^2 dx \\
&\le \int_a^b \Big\{ d_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} \Big\} \cdot \frac{w_\varepsilon}{\sqrt{d_\varepsilon}} \\
&\le \frac{M}{\sqrt{\delta}} \int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} \qquad \text{for all } t > 0,
\end{aligned}$$

so that from (2.12) we readily infer the existence of $c_{12} > 0$ satisfying

$$\|W_{\varepsilon y}(\cdot,t)\|_{L^2(G)} \le c_{12} e^{\sqrt{\varepsilon}t} \qquad \text{for all } t > 0. \tag{2.53}$$

Finally, relying on the local convergence properties of $(d_{\varepsilon x})_{\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}}$ expressed in (2.3), given any $q \in (1,\infty)$ we can find $\varepsilon^{(2)}(q) \in (0,\varepsilon^{(1)})$ and $c_{13}(q) > 0$ such that

$$\begin{aligned}
\int_G |D_\varepsilon(y)|^q dy &= \int_{\phi_\varepsilon^{-1}(G)} \frac{1}{\sqrt{d_\varepsilon}} \Big| \frac{d_{\varepsilon x}}{2\sqrt{d_\varepsilon}} \Big|^q dx \\
&\le \frac{1}{2^q \sqrt{\delta}^{q+1}} \int_a^b |d_{\varepsilon x}|^q \\
&\le c_{13}(q) \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}} \text{ such that } \varepsilon < \varepsilon^{(2)}(q).
\end{aligned} \tag{2.54}$$

Now going back to (2.47), using (2.51), (2.53) and the fact that $|\zeta| \le 1$, by the Hölder inequality we find that therein for each $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ such that $\varepsilon < \varepsilon^{(1)}$ we have

$$\begin{aligned}
\|b_\varepsilon^{(1)}(\cdot,t)\|_{L^p(G)} &= \|\zeta Z_\varepsilon W_{\varepsilon y}\|_{L^p(G)} \\
&\le \|Z_\varepsilon\|_{L^{\frac{2p}{2-p}}(G)} \|W_{\varepsilon y}\|_{L^2(G)} \\
&\le c_9\Big(\frac{2p}{2-p}\Big) e^{c_9(\frac{2p}{2-p})\sqrt{\varepsilon}t} \cdot c_{12} e^{\sqrt{\varepsilon}t} \qquad \text{for all } t > 0,
\end{aligned}$$

and that hence in view of (2.47) there exists $c_{14} > 0$ such that

$$\Big\| \int_{(t-1)_+}^t e^{-(t-s)A} b_{\varepsilon y}^{(1)}(\cdot,s)ds \Big\|_{L^\infty(G)} \le c_{14} e^{c_{14}\sqrt{\varepsilon}t} \qquad \text{for all } t > 0. \tag{2.55}$$

Likewise, fixing any $r \in (p,2)$ we may combine (2.52) with (2.54) to see that whenever $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ is such that $\varepsilon < \varepsilon^{(2)}(\frac{pr}{r-p})$,

$$\begin{aligned}
\|b_\varepsilon^{(2)}\|_{L^p(G\times(t,t+1))} &= \|2\zeta_y Z_{\varepsilon y} + D_\varepsilon \zeta Z_{\varepsilon y}\|_{L^p(G\times(t,t+1))} \\
&\le 2\|\zeta_y\|_{L^\infty(G)}\|Z_{\varepsilon y}\|_{L^p(G\times(t,t+1))} + \|D_\varepsilon\|_{L^{\frac{pr}{r-p}}(G)}\|Z_{\varepsilon y}\|_{L^r(G\times(t,t+1))} \\
&\le 2\|\zeta_y\|_{L^\infty(G)}\cdot c_{11}(p)e^{c_{11}(p)\sqrt{\varepsilon}t} + c_{13}\Big(\frac{pr}{r-p}\Big)\cdot c_{11}(r)e^{c_{11}(r)\sqrt{\varepsilon}t} \qquad \text{for all } t>0
\end{aligned}$$

and thus, by (2.48),

$$\Big\|\int_{(t-1)_+}^t e^{-(t-s)A}b_\varepsilon^{(2)}(\cdot,s)ds\Big\|_{L^\infty(G)} \le c_{15}e^{c_{15}\sqrt{\varepsilon}t} \qquad \text{for all } t>0 \tag{2.56}$$

with some suitably large $c_{15}>0$. Finally, (2.51), (2.53) and (2.54) imply that for $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ fulfilling $\varepsilon < \varepsilon^{(2)}(\frac{4p}{2-p})$ we have

$$\begin{aligned}
\|b_\varepsilon^{(3)}(\cdot,t)\|_{L^p(G)} &= \|-\zeta_{yy}Z_\varepsilon + \zeta_y W_{\varepsilon y}Z_\varepsilon + D_\varepsilon \zeta W_{\varepsilon y}Z_\varepsilon\|_{L^p(G)} \\
&\le \|\zeta_{yy}\|_{L^\infty(G)}\|Z_\varepsilon\|_{L^p(G)} + \|\zeta_y\|_{L^\infty(G)}\|W_{\varepsilon y}\|_{L^2(G)}\|Z_\varepsilon\|_{L^{\frac{2p}{2-p}}(G)} \\
&\quad +\|D_\varepsilon\|_{L^{\frac{4p}{2-p}}(G)}\|W_{\varepsilon y}\|_{L^2(G)}\|Z_\varepsilon\|_{L^{\frac{4p}{2-p}}(G)} \\
&\le \|\zeta_{yy}\|_{L^\infty(G)}\cdot c_9(p)e^{c_9(p)\sqrt{\varepsilon}t} + \|\zeta_y\|_{L^\infty(G)}\cdot c_{12}e^{\sqrt{\varepsilon}t}\cdot c_9\Big(\frac{2p}{2-p}\Big)e^{c_9(\frac{2p}{2-p})\sqrt{\varepsilon}t} \\
&\quad +c_{13}^{\frac{2-p}{4p}}\Big(\frac{4p}{2-p}\Big)\cdot c_{12}e^{\sqrt{\varepsilon}t}\cdot c_9\Big(\frac{4p}{2-p}\Big)e^{c_9(\frac{4p}{2-p})\sqrt{\varepsilon}t} \qquad \text{for all } t>0,
\end{aligned}$$

by (2.49) meaning that there exists $c_{16}>0$ with the property that for any such ε,

$$\Big\|\int_{(t-1)_+}^t e^{-(t-s)A}b_\varepsilon^{(3)}(\cdot,s)ds\Big\|_{L^\infty(G)} \le c_{16}e^{c_{16}\sqrt{\varepsilon}t} \qquad \text{for all } t>0. \tag{2.57}$$

In summary, (2.41), (2.44), (2.55), (2.56) and (2.57) entail the existence of $c_{17}>0$ such that writing $\varepsilon^{(3)} := \min\{\varepsilon^{(2)}(\frac{pr}{r-p}), \varepsilon^{(2)}(\frac{4p}{2-p})\}$, for each $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ satisfying $\varepsilon < \varepsilon^{(3)}$ we have

$$\|H_\varepsilon(\cdot,t)\|_{L^\infty(G)} \le c_{17}e^{c_{17}\sqrt{\varepsilon}t} \qquad \text{for all } t>0$$

and hence

$$\|Z_\varepsilon(\cdot,t)\|_{L^\infty(G_0)} \le c_{17}e^{c_{17}\sqrt{\varepsilon}t} \qquad \text{for all } t>0$$

due to the fact that $\zeta \equiv 1$ in G_0. Transforming back to the original variables, again by (2.50) we thus see that for any such ε and each $t>0$,

$$\delta\|u_\varepsilon(\cdot,t)\|_{L^\infty(K)} \le \|d_\varepsilon u_\varepsilon(\cdot,t)\|_{L^\infty(K)} \le \|d_\varepsilon u_\varepsilon(\cdot,t)\|_{L^\infty(\phi_\varepsilon^{-1}(G_0))} = \|Z_\varepsilon(\cdot,t)\|_{L^\infty(G_0)} \le c_{17}e^{c_{17}\sqrt{\varepsilon}t},$$

because $K \subset \phi_\varepsilon^{-1}(G_0) \subset [a,b]$ by (2.40).

This proves (2.38) in the case when $J \subset \Omega$, whereas the situation when $J \cap \partial\Omega \neq \emptyset$ can be dealt with quite similarly, by e.g. fixing, for convenience, $x_0 \in J \cap \partial\Omega$ in (2.25), and choosing ζ to be identically equal to 1 near the boundary point $0 = \phi_\varepsilon(x_0)$. $\square$

2.3 Local Hölder regularity of u_ε in $\{d>0\}$

Now with the above boundedness information at hand, we may invoke standard parabolic regularity to obtain the announced interior Hölder regularity property.

Lemma 2.4 *Assume that $\int_\Omega \frac{1}{d^2} < \infty$. Then for each compact $K \subset \{d>0\}$ and any $\tau \in (0,1)$ there exist $\theta(K,\tau) \in (0,1)$ and $C(K,\tau)>0$ with the property that for any $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ we have*

$$\|u_\varepsilon\|_{C^{\theta(K,\tau),\frac{\theta(K,\tau)}{2}}(K\times[t,t+1])} \le C(K,\tau) \qquad \textit{for all } t \in \Big(\tau, \frac{1}{\sqrt{\varepsilon}}\Big). \tag{2.58}$$

PROOF. We write the first equation in (2.1) in the form

$$u_{\varepsilon t} = \Big(a_\varepsilon(x,t,u_{\varepsilon x})\Big)_x, \qquad x\in\Omega,\ t>0,$$

where

$$a_\varepsilon(x,t,\xi) := d_\varepsilon(x)\xi + d_{\varepsilon x}(x)u_\varepsilon(x,t) - d_\varepsilon(x)u_\varepsilon(x,t)w_{\varepsilon x}(x,t), \qquad x\in\Omega,\ t>0,\ \xi\in\mathbb{R}. \tag{2.59}$$

Once again assuming without loss of generality that K is an interval, we can fix an open interval $\Omega_0 \subset \Omega$ such that $K \subset \Omega_0 \subset \overline{\Omega}_0 \subset \{d>0\}$, so that since $\overline{\Omega}_0$ still is a compact subinterval of $\{d>0\}$, using (2.3) and (2.4) we obtain positive constants c_1, c_2 and c_3 such that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,

$$c_1 \le d_\varepsilon(x) \le c_2 \qquad \text{for all } x\in\Omega_0 \tag{2.60}$$

and

$$\int_{\Omega_0} d_{\varepsilon x}^2 \le c_3. \tag{2.61}$$

Moreover, employing Lemma 2.3 and (2.12) we see that for some $c_4>0$ and $c_5>0$ and any $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ we have

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega_0)} \le c_4 \qquad \text{for all } t \in \Big(0,\frac{1}{\sqrt{\varepsilon}}\Big) \tag{2.62}$$

as well as

$$\int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} \le c_5 \qquad \text{for all } t\in\Big(0,\frac{1}{\sqrt{\varepsilon}}\Big), \tag{2.63}$$

where the latter along with (2.60) and (2.10) entails that

$$\int_{\Omega_0} w_{\varepsilon x}^2 \le \frac{M}{c_1}\int_{\Omega_0} d_\varepsilon \frac{w_{\varepsilon x}^2}{w_\varepsilon} \le \frac{Mc_5}{c_1} \qquad \text{for all } t\in\Big(0,\frac{1}{\sqrt{\varepsilon}}\Big). \tag{2.64}$$

Writing

$$\psi_\varepsilon^{(0)}(x,t) := \frac{d_{\varepsilon x}^2(x)}{d_\varepsilon(x)} u_\varepsilon^2(x,t) + d_\varepsilon(x)u_\varepsilon^2(x,t)w_{\varepsilon x}^2(x,t)$$

and

$$\psi_\varepsilon^{(1)}(x,t) := |d_{\varepsilon x}(x)|u_\varepsilon(x,t) + d_\varepsilon(x)u_\varepsilon(x,t)|w_{\varepsilon x}(x,t)|$$

for $x \in \Omega, t > 0$ and $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$, from (2.60), (2.61), (2.62) and (2.64) we thus infer that for any such ε we have

$$\int_{\Omega_0} |\psi_\varepsilon^{(0)}(\cdot,t)| \le \frac{c_3 c_4^2}{c_1} + c_2 c_4^2 \cdot \frac{M c_5}{c_1} \qquad \text{for all } t \in \Big(0, \frac{1}{\sqrt{\varepsilon}}\Big)$$

and

$$\int_{\Omega_0} |\psi_\varepsilon^{(1)}(\cdot,t)|^2 \le 2 \cdot \Big\{ c_3 c_4^2 + c_2^2 c_4^2 \cdot \frac{M c_5}{c_1} \Big\} \qquad \text{for all } t \in \Big(0, \frac{1}{\sqrt{\varepsilon}}\Big).$$

Since by Young's inequality and (2.60),

$$a_\varepsilon(x,t,\xi) \cdot \xi \ge \frac{1}{2} d_\varepsilon(x) \xi^2 - \psi_\varepsilon^{(0)}(x,t) \ge \frac{c_1}{2} \xi^2 - \psi_\varepsilon^{(0)}(x,t) \qquad \text{for all } x \in \Omega_0, t > 0 \text{ and } \xi \in \mathbb{R},$$

and since (2.60) moreover warrants that

$$|a_\varepsilon(x,t,\xi)| \le d_\varepsilon(x)|\xi| + \psi_\varepsilon^{(1)}(x,t) \le c_2 |\xi| + \psi_\varepsilon^{(1)}(x,t) \qquad \text{for all } x \in \Omega_0, t > 0 \text{ and } \xi \in \mathbb{R},$$

in view of the boundedness property (2.62) the inequality in (2.58) follows from a standard result on interior Hölder regularity of bounded solutions to scalar parabolic equations ([9, Theorem 1.1]). □

3 Proof of Theorem 1.1

The uniform estimates of the solutions to the approximate problems (2.1) proved in Section 2 imply the following refined regularity properties of the generalized solution to (1.1).

Lemma 3.1 *Suppose that $\int_\Omega \frac{1}{d^2} < \infty$, and let (u, w) denote the global generalized solution of (1.1) from [12, Theorem 1.1]. Then*

$$du \in L^\infty((0,\infty); L^p(\Omega)) \qquad \text{for all } p \in (1,\infty) \tag{3.1}$$

and

$$u \in C^0(\{d > 0\} \times (0,\infty)). \tag{3.2}$$

Moreover, for all $p \in (1,\infty)$ there exists $C(p) > 0$ such that

$$\|du(\cdot,t)\|_{L^p(\Omega)} \le C(p) \qquad \text{for all } t > 0, \tag{3.3}$$

and given any compact $K \subset \{d > 0\}$ and $\tau > 0$ one can find $\theta(K,\tau) \in (0,1)$ and $C(K,\tau) > 0$ such that

$$\|u(\cdot,t)\|_{C^{\theta(K,\tau)}(K)} \le C(K,\tau) \qquad \text{for all } t > \tau. \tag{3.4}$$

Proof. In view of the pointwise approximation property of $(u_{\varepsilon_{j_k}})_{k \in \mathbb{N}}$ stated in [12, Lemma 6.1], i.e. $u_{\varepsilon_{j_k}} \to u$ a.e. in $\Omega \times (0,\infty)$, both (3.1) and (3.3) directly result on taking $\varepsilon = \varepsilon_{j_k} \searrow 0$ in Lemma 2.1, whereas (3.2) and (3.4) can easily be deduced from Lemma 2.4. □

Finally, we are in the position to prove our main result.

Proof of Theorem 1.1. The regularity properties (1.7) and (1.9) have precisely been established in Lemma 3.1 already, as (1.6) implies (1.2) in view of $\lambda \ge 2$ and the boundedness of Ω. Hence, it remains to show that (1.6) implies (1.8) and (1.10). To achieve this, we let $p \in (1, \lambda)$ be given and then infer from (1.7) that with some $c_1 > 0$ we have

$$\int_\Omega (du)^{\frac{p\lambda}{\lambda-p}} \le c_1 \qquad \text{for all } t > 0. \tag{3.5}$$

By means of the Hölder inequality, this firstly implies that

$$\int_\Omega u^p = \int_\Omega \frac{1}{d^p} \cdot (du)^p \le \Big\{ \int_\Omega \frac{1}{d^\lambda} \Big\}^{\frac{p}{\lambda}} \cdot \Big\{ \int_\Omega (du)^{\frac{p\lambda}{\lambda-p}} \Big\}^{\frac{\lambda-p}{\lambda}} \le \Big\{ \int_\Omega \frac{1}{d^\lambda} \Big\}^{\frac{p}{\lambda}} \cdot c_1^{\frac{\lambda-p}{\lambda}} \qquad \text{for all } t > 0$$

and thereby, thanks to (1.6), already establishes (1.8). Secondly, given any $\eta > 0$ we may use (1.6) along with the fact that $p < \lambda$ in choosing a relatively open subset Ω_0 of $\overline{\Omega}$ satisfying $\{d = 0\} \subset \Omega_0$ and

$$2^{p-1} c_1^{\frac{\lambda-p}{\lambda}} \cdot \Big\{ \int_{\Omega_0} \frac{1}{d^\lambda} \Big\}^{\frac{p}{\lambda}} \le \frac{\eta^p}{3} \tag{3.6}$$

as well as

$$2^{p-1} \mu_\infty^p \int_{\Omega_0} \frac{1}{d^p} \le \frac{\eta^p}{3}. \tag{3.7}$$

Then since $K := \overline{\Omega} \setminus \Omega_0$ is compact, Lemma 3.1 applies so as to show that in view of the Arzelà-Ascoli theorem the semi-orbit $(u(\cdot,t))_{t>1}$ is relatively compact in $C^0(K)$ and that hence, thanks to the outcome of [12, Theorem 1.2], namely (1.5), we have

$$u(\cdot,t) \to \frac{\mu_\infty}{d} \quad \text{in } L^\infty(K) \qquad \text{as } t \to \infty,$$

so that we can fix $t_0 > 1$ such that

$$\Big\| u(\cdot,t) - \frac{\mu_\infty}{d} \Big\|_{L^\infty(\Omega\setminus\Omega_0)} \le \frac{\eta^p}{3|\Omega|} \qquad \text{for all } t > t_0. \tag{3.8}$$

Now in the inequality

$$\begin{aligned} \int_\Omega \Big| u(\cdot,t) - \frac{\mu_\infty}{d} \Big|^p &= \int_{\Omega_0} \Big| u(\cdot,t) - \frac{\mu_\infty}{d} \Big|^p + \int_{\Omega\setminus\Omega_0} \Big| u(\cdot,t) - \frac{\mu_\infty}{d} \Big|^p \\ &\le 2^{p-1} \int_{\Omega_0} u^p + 2^{p-1} \mu_\infty^p \int_{\Omega_0} \frac{1}{d^p} + \int_{\Omega\setminus\Omega_0} \Big| u(\cdot,t) - \frac{\mu_\infty}{d} \Big|^p, t > 0, \end{aligned} \tag{3.9}$$

according to (3.6) and again the Hölder inequality we have

$$\begin{aligned} 2^{p-1} \int_{\Omega_0} u^p &\le 2^{p-1} \cdot \Big\{ \int_{\Omega_0} \frac{1}{d^\lambda} \Big\}^{\frac{p}{\lambda}} \cdot \Big\{ \int_{\Omega_0} (du)^{\frac{p\lambda}{\lambda-p}} \Big\}^{\frac{\lambda-p}{\lambda}} \\ &\le 2^{p-1} \cdot \Big\{ \int_{\Omega_0} \frac{1}{d^\lambda} \Big\}^{\frac{p}{\lambda}} \cdot c_1^{\frac{\lambda-p}{\lambda}} \\ &\le \frac{\eta^p}{3} \qquad \text{for all } t > 0, \end{aligned}$$

whereas (3.7) warrants that

$$2^{p-1}\mu_\infty^p \int_{\Omega_0} \frac{1}{d^p} \le \frac{\eta^p}{3}.$$

As (3.8) ensures that apart from that we have

$$\int_{\Omega\setminus\Omega_0} \Big| u(\cdot,t) - \frac{\mu_\infty}{d}\Big|^p \le \Big\| u(\cdot,t) - \frac{\mu_\infty}{d}\Big\|_{L^\infty(\Omega\setminus\Omega_0)}^p |\Omega\setminus\Omega_0| \le \frac{\eta^p}{3} \qquad \text{for all } t>t_0,$$

it follows from (3.9) that

$$\Big\| u(\cdot,t) - \frac{\mu_\infty}{d}\Big\|_{L^p(\Omega)} \le \Big(\frac{\eta^p}{3} + \frac{\eta^p}{3} + \frac{\eta^p}{3}\Big)^{\frac{1}{p}} = \eta \qquad \text{for all } t>t_0,$$

thereby verifying (1.10). □

4 Appendix: A refined interpolation inequality

We prove the following interpolation inequality of Gagliardo-Nirenberg type which is based on an observation originally made in [3].

Lemma 4.1 *Let $q>0$. Then there exist $C(q)>0$ and $\Lambda(q)>0$ such that for any choice of $\eta\in(0,1)$ we have*

$$\|\varphi\|_{L^\infty(\Omega)}^{q+2} \le \eta \|\varphi_x\|_{L^2(\Omega)}^2 \cdot \Big\| \varphi \Big| \ln|\varphi| \Big|^{\frac{1}{q}} \Big\|_{L^q(\Omega)}^q + C(q)\|\varphi\|_{L^q(\Omega)}^{q+2} + C(q) e^{\frac{\Lambda(q)}{\eta}} \qquad \textit{for all } \varphi\in W^{1,2}(\Omega). \quad (4.1)$$

Proof. Following the argument in [3], we first invoke the Gagliardo-Nirenberg inequality to find $c_1\ge 1$ such that

$$\|\psi\|_{L^\infty(\Omega)}^{q+2} \le c_1 \|\psi_x\|_{L^2(\Omega)}^2 \|\psi\|_{L^q(\Omega)}^q + c_1\|\psi\|_{L^q(\Omega)}^{q+2} \qquad \text{for all } \psi\in W^{1,2}(\Omega). \quad (4.2)$$

For fixed $\eta\in(0,1)$, we then let

$$N := \exp\Big\{\frac{2^{q+3}c_1}{\eta}\Big\} > 1$$

and introduce $\zeta\in W^{1,\infty}_{loc}(\mathbb{R})$ by defining $\zeta(\xi):=0$ for $\xi\in[-N,N]$, $\zeta(\xi):=|\xi|$ for $|\xi|\ge 2N$ and $\zeta(\xi):=2(|\xi|-N)$ for $N<|\xi|<2N$. Then given $\varphi\in W^{1,2}(\Omega)$, we evidently have

$$\| |\varphi| - \zeta(\varphi)\|_{L^\infty(\Omega)} \le 2N$$

and furthermore

$$\|\zeta(\varphi)\|_{L^q(\Omega)}^q \le \int_{\{|\varphi|\ge N\}} |\varphi|^q \le \frac{1}{\ln N}\int_\Omega |\varphi|^q \Big|\ln|\varphi|\Big| = \frac{1}{\ln N}\Big\|\varphi\Big|\ln|\varphi|\Big|^{\frac{1}{q}}\Big\|_{L^q(\Omega)}^q.$$

Since $(a+b)^{q+2} \le 2^{q+1}(a^{q+2}+b^{q+2})$ for all $a \ge 0$ and $b \ge 0$, (4.2) thus entails that

$$\begin{aligned} \|\varphi\|_{L^\infty(\Omega)}^{q+2} &\le 2^{q+1}\|\zeta(\varphi)\|_{L^\infty(\Omega)}^{q+2} + 2^{q+1}\||\varphi|-\zeta(\varphi)\|_{L^\infty(\Omega)}^{q+2} \\ &\le 2^{q+1}c_1\|(\zeta(\varphi))_x\|_{L^2(\Omega)}^2\|\zeta(\varphi)\|_{L^q(\Omega)}^q + 2^{q+1}c_1\|\zeta(\varphi)\|_{L^q(\Omega)}^{q+2} + 2^{q+1}\cdot(2N)^{q+2} \\ &\le \frac{2^{q+3}c_1}{\ln N}\|\varphi_x\|_{L^2(\Omega)}^2\Big\||\varphi|\ln|\varphi|\Big|^{\frac{1}{q}}\Big\|_{L^q(\Omega)}^q + 2^{q+1}c_1\|\varphi\|_{L^q(\Omega)}^{q+2} + 2^{2q+3}N^{q+2}, \end{aligned}$$

because $\|\zeta'\|_{L^\infty(\mathbb{R})} = 2$ and $|\zeta(\xi)| \le |\xi|$ for all $\xi \in \mathbb{R}$. In view of our definition of N, this proves (4.1) with $C(q) := \max\{2^{q+1}c_1\,,\,2^{2q+3}\}$ and $\Lambda(q) := 2^{q+3}c_1(q+2)$. $\square$

Acknowledgement. M. Winkler acknowledges support of the *Deutsche Forschungsgemeinschaft* in the context of the project *Analysis of chemotactic cross-diffusion in complex frameworks.*

References

[1] Bellomo, N., Bellouquid, A., Tao, Y., Winkler, M.: *Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues.* Math. Models Methods Appl. Sci. **25**, 1663–1763 (2015)

[2] Belmonte-Beitia, J., Woolley, T.E., Scott, J.G., Maini, P.K., Gaffney, E.A.: *Modelling biological invasions: Individual to population scales at interfaces.* J. Theoret. Biol. **334**, 1–12 (2013)

[3] Biler, P., Hebisch, W., Nadzieja, T.: *The Debye system: Existence and large time behavior of solutions.* Nonlinear Anal. **23**, 1189–1209 (1994)

[4] Carter, S.B.: *Haptotaxis and the mechanism of cell motility,* Nature **213**, 256–260 (1967)

[5] Chaplain, M.A.J., Lolas, G.: *Mathematical modelling of cancer invasion of tissue: dynamic heterogeneity,* Netw. Heterog. Media **1**, 399–439 (2006)

[6] Corrias, L., Perthame, B., Zaag, H: *Global solutions of some chemotaxis and angiogenesis systems in high space dimensions.* Milan J. Math. **72**, 1–28 (2004)

[7] Fontelos, M.A., Friedman, A., Hu, B.: *Mathematical analysis of a model for the initiation of angiogenesis.* SIAM J. Math. Anal. **33**, 1330–1355 (2002)

[8] Fujie, K., Ito, A., Winkler, M., Yokota, T.: *Stabilization in a chemotaxis model for tumor invasion.* Discrete Cont. Dyn. Syst. **36**, 151–169 (2016)

[9] Porzio, M.M., Vespri, V.: *Holder estimates for local solutions of some doubly nonlinear degenerate parabolic equations.* J. Differential Equations **103**, 146–178 (1993)

[10] Surulescu, C., Winkler, M.: *Global weak solutions to a strongly degenerate haptotaxis model.* Commun. Math. Sci. **15**, 1581–1616 (2017)

[11] Winkler, M: *Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model.* J. Differential Equations **248**, 2889–2905 (2010)

[12] Winkler, M.: *Singular structure formation in a degenerate haptotaxis model involving myopic diffusion.* J. Math. Pures Appl. **112**, 118–169 (2018)

[13] Zhigun, A., Surulescu, C., Uatay, A.: *Global existence for a degenerate haptotaxis model of cancer invasion.* Z. Angew. Math. Phys. **67**, 146 (2016)