Global smooth solutions in a two-dimensional cross-diffusion system modeling propagation of urban crime

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Abstract

We consider the spatially two-dimensional version of a cross-diffusion system, as originally proposed by Short et al. in [27] to describe the evolution of urban crime. Although sharing some basic structure elements with the well-studied classical Keller-Segel chemotaxis model, this system contains an essential difference to the latter by accounting for a certain *nonlinear* mechanism of attractant production, potentially yet increasing explosion-supporting properties.

The intention of this paper is to make sure that despite this, a theory of global smooth solutions can be established after all within certain small-data settings which can be described in an essentially explicit manner. The main results in this direction firstly identify hypotheses on smallness of the initial data, and of some given external production terms, as sufficient to ensure global existence and uniqueness of smooth and bounded solutions. Secondly, any such bounded solution is shown to asymptotically approach some steady state, provided that the prescribed sources comply with appropriate additional assumptions on their stabilization in the large time limit. Finally, a statement on asymptotic stability of certain steady states is derived as a by-product.

Keywords: crime propagation; chemotaxis; global existence; large time behavior; stability
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1 Introduction

This manuscript is concerned with the parabolic system

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot\left(\frac{u}{v}\nabla v\right) - uv + B_1(x,t), \\ v_t = \Delta v - v + uv + B_2(x,t), \end{cases} \tag{1.1}$$

which for the particular value $\chi = 2$ was proposed in [27] and [26] as a model for the spatio-temporal evolution of urban crime. Here, besides assuming criminal agents to be continuously distributed in space and time, with population density $u = u(x,t)$, and to adjust their motion toward increasing concentrations of an abstract so-called attractiveness value $v = v(x,t)$, key hypotheses underlying the model development in [27] rest on an inclusion of fundamental and statistically affirmed behavioral strategies, as expounded in the context of the *broken windows theory* ([15]) and also becoming manifest in what is commonly referred to as *repeat and near-repeat victimization effect* in criminology ([13]).

As a consequence of apparently noticeable mathematical delicacy, according to the modeling approach in [27] an appropriate incorporation of such self-exciting tendencies in criminal activity is reflected in the nonlinear signal production term $+uv$ in the second equation from (1.1). In particular, this seems to constitute a further significant complexification in comparison to related cross-diffusion systems from mathematical biology, which even despite such additional nonlinear ingredients are yet lacking a complete understanding. Indeed, already when interacting with a linearly produced signal, the potentially destabilizing action of the cross-diffusive mechanism from (1.1) seems to substantially limit existence theories, as indicated by a considerable literature on the classical Keller-Segel model

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot\left(\frac{u}{v}\nabla v\right), \\ v_t = \Delta v - v + u, \end{cases} \tag{1.2}$$

for chemotactic migration ([14]): Results on global classical solvability in n-dimensional domains with $n \ge 2$ seem available only under smallness conditions on the key parameter χ, in general requiring $\chi < \sqrt{\frac{2}{n}}$ ([2], [32]), where in the case $n = 2$ the slightly weaker assumption $\chi < \chi_0$ with some $\chi_0 > 1.015$ is known to be sufficient ([18]); in the presence of larger values of χ, global solutions so far have been constructed only within appropriately generalized frameworks ([32], [28], [19]). Certain parabolic-elliptic simplifications of (1.2) admit slightly more comprehensive existence results, especially in radially symmetric settings ([9], [3]), but on the other hand moreover allow for the rigorous detection of exploding solutions in some three- or higher-dimensional domains ([22]).

In line with this, available existence results for the more complex system (1.1) involving the relevant value $\chi = 2$ seem limited to spatially one-dimensional settings ([25]), and to local-in-time classical ([24]) or some global generalized solvability, at least in radial cases ([34]), in two-dimensional frameworks; findings on global classical solvability in two-dimensional domains seem yet to require the smallness condition $\chi < 1$ ([7]). Considerably more comprehensive insight, inter alia addressing qualitative features such as the possibility of stable spatially heterogeneous behavior resembling crime hotspot formation, could be gained only in related stationary systems ([4], [5], [10], [16], [29]).

Main results. The intention of this work is to develop an analytical approach capable of adequately coping with the challenges linked to the interplay of taxis-type cross-diffusion in the above flavor on the one hand, and the considered nonlinear signal production mechanism on the other, in contexts of smooth solutions at least within certain ranges of suitably small initial data. This will be substantiated in the framework of the initial-boundary value problem

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot\Big(\frac{u}{v}\nabla v\Big) - uv + B_1(x,t), & x\in\Omega,\ t>0,\\ v_t = \Delta v - v + uv + B_2(x,t), & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0 & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), & x\in\Omega. \end{cases} \tag{1.3}$$

in a bounded convex domain $\Omega\subset\mathbb{R}^2$ with smooth boundary, where $\chi>0$, and where throughout the sequel we shall assume that the given source terms B_1 and B_2 for criminal agents and attractiveness are suitably regular in the sense that

$$\begin{cases} B_1\in C^1(\overline{\Omega}\times[0,\infty)) \text{ is nonnegative and bounded, and that}\\ B_2\in C^2(\overline{\Omega}\times[0,\infty)) \text{ is nonnegative and bounded,} \end{cases} \tag{1.4}$$

and that the initial data in (1.3) are such that

$$\begin{cases} u_0\in C^0(\overline{\Omega}) \quad \text{is nonnegative, and that}\\ v_0\in W^{1,\infty}(\Omega) \quad \text{is positive in } \overline{\Omega}. \end{cases} \tag{1.5}$$

In order to formulate our main results, we recall that, firstly, due to the Gagliardo-Nirenberg inequality there exists $K>0$ with the property that

$$\int_\Omega \varphi^4 \le K\Big\{\int_\Omega|\nabla\varphi|^2\Big\}\cdot\Big\{\int_\Omega\varphi^2\Big\} + K\Big\{\int_\Omega\varphi^2\Big\}^2 \qquad \text{for all } \varphi\in W^{1,2}(\Omega), \tag{1.6}$$

that with some $L\in(0,6+4\sqrt{2}]$, as a consequence of [33, Lemma 3.3] we secondly have

$$\int_\Omega \frac{|\nabla\varphi|^4}{\varphi^3} \le L\int_\Omega \varphi|D^2\ln\varphi|^2 \qquad \text{for all } \varphi\in C^2(\overline{\Omega}) \text{ such that } \varphi>0 \text{ in } \overline{\Omega} \text{ and } \frac{\partial\varphi}{\partial\nu}=0 \text{ on } \partial\Omega, \tag{1.7}$$

and that, thirdly, according to the convexity of Ω we can find $\Gamma\in(0,1]$ such that the Neumann heat semigroup $(e^{t\Delta})_{t\ge0}$ satisfies

$$e^{t\Delta}\varphi \ge \Gamma\int_\Omega\varphi \quad \text{in } \Omega \qquad \text{for all } t>1 \text{ and any nonnegative } \varphi\in C^0(\overline{\Omega}) \tag{1.8}$$

(see e.g. [8] and, for a one-dimensional version, also [11]).

Referring to these constants, our main results on global solvability in contexts of suitably small data can be formulated as follows.

Theorem 1.1 *Let $\chi>0$ and $\Omega\subset\mathbb{R}^2$ be a bounded convex domain with smooth boundary, and let $K>0, L>0$ and $\Gamma\in(0,1]$ be such that (1.6), (1.7) and (1.8) are valid. For arbitrary $\delta>0$, let $\lambda:=\min\{1,\frac{\Gamma\delta}{4e}\}$, and suppose that B_1 and B_2 are such that (1.4) holds and such that moreover*

$$\int_\Omega B_1^2(\cdot,t) \le \frac{\Gamma^2}{4KL(L+4)\chi^4e^2}\cdot\delta^2\cdot\lambda^2 \qquad \textit{for all } t>0 \tag{1.9}$$

and

$$\int_\Omega B_2(\cdot,t) \ge \delta \qquad \textit{for all } t>0 \tag{1.10}$$

as well as

$$\int_\Omega |\nabla\sqrt{B_2(\cdot,t))}|^2 \le \frac{\Gamma}{8KL\chi^4 e}\cdot\delta\cdot\lambda^2 \qquad \textit{for all } t>0. \tag{1.11}$$

Then for any u_0 and v_0 fulfilling (1.5) and

$$v_0(x) \ge \delta \qquad \textit{for all } x\in\Omega \tag{1.12}$$

as well as

$$\int_\Omega u_0^2 \le \frac{\Gamma}{2KL(L+4)\chi^4 e}\cdot\delta \tag{1.13}$$

and

$$\int_\Omega |\nabla\sqrt{v_0}|^2 \le \frac{\Gamma}{16KL\chi^4 e}\cdot\delta, \tag{1.14}$$

the problem (1.3) possesses a global classical solution (u,v) which for each $T>0$ and $p>2$ is uniquely determined by the inclusions

$$\begin{cases} u\in C^0(\overline{\Omega}\times[0,T])\cap C^{2,1}(\overline{\Omega}\times(0,T)), \\ v\in C^0([0,T];W^{1,p}(\Omega))\cap C^{2,1}(\overline{\Omega}\times(0,T)), \end{cases} \tag{1.15}$$

and which is such that $u\ge 0$ and $v>0$ in $\overline{\Omega}\times[0,\infty)$. Moreover, this solution is bounded in the sense that

$$\sup_{t>0}\|u(\cdot,t)\|_{L^\infty(\Omega)}<\infty \qquad \textit{and} \qquad \sup_{t>0}\|v(\cdot,t)\|_{W^{1,p}(\Omega)}<\infty \quad \textit{for all } p>1. \tag{1.16}$$

We emphasize that the explicit character of the right-hand sides in (1.9), (1.11), (1.13) and (1.14) particularly enables us to conclude from Theorem 1.1 that given *arbitrary* functions B_1, B_2, u_0 and v_0 fulfilling (1.4) and (1.5) as well as $\inf_{t>0}\int_\Omega B_2(\cdot,t)>0$, one can find $\chi_0=\chi_0(B_1,B_2,u_0,v_0,\Omega)>0$ with the property that whenever $\chi\in(0,\chi_0)$, the claimed statement on global existence and boundedness holds, ; and that hence any blow-up phenomenon is precluded in such situations. From the point of view of In the framework of the intended application, the latter indicates that appropriate smallness of the initial data, or alternatively an insufficient sensitivity of offenders to bias the attractiveness field, may limit the system potential to spontaneously generate crime hotspots in the sense of aggregate-type and spatially localized distributions of criminal activity. An interesting question going beyond the scope of this study is whether such phenomena of locally large density formation, as numerically observed already in [26], can rigorously be constructed in suitable settings involving either large initial data or suitably supercritical χ.

Next, in order to secondly identify some genuinely diffusion-enforced large-time relaxation property of (1.3), at least within the set of all global bounded solutions, in our following result on large time asymptotics we rely on additional assumptions on decay and stabilization of the external ingredients B_1 and B_2 which essentially resemble the hypotheses introduced in [25] to derive a similar conclusion in the one-dimensional version of (1.3).

Theorem 1.2 *Let $\chi > 0$ and $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and suppose that B_1 and B_2 are such that beyond (1.4),*

$$\int_t^{t+1} \int_\Omega B_1 \to 0 \qquad as\ t \to \infty \tag{1.17}$$

and

$$B_2(\cdot,t) \to B_{2,\infty} \quad a.e.\ in\ \Omega \qquad as\ t \to \infty \tag{1.18}$$

with some $0 \not\equiv B_{2,\infty} \in C^0(\overline{\Omega})$. Then whenever (1.5) holds and (u,v) is a global classical solution of (1.3) that satisfies (1.15) and possesses the boundedness features in (1.16), we have

$$u(\cdot,t) \to 0 \qquad in\ L^\infty(\Omega) \qquad as\ t \to \infty, \tag{1.19}$$

and

$$v(\cdot,t) \to v_\infty \qquad in\ L^\infty(\Omega) \qquad as\ t \to \infty, \tag{1.20}$$

where v_∞ denotes the solution of the boundary value problem

$$\begin{cases} -\Delta v_\infty + v_\infty = B_{2,\infty}, & x \in \Omega, \\ \frac{\partial v_\infty}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{1.21}$$

We shall finally see that this especially entails the following result on asymptotic stability of equilibria in the presence of time-independent pairs (B_1, B_2) of system ingredients suitably close to the homogeneous sources $(0, \eta)$ with prescribed $\eta > 0$.

Theorem 1.3 *Let $\chi > 0$ and $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and let $\eta > 0$. Then there exists $\varepsilon = \varepsilon(\chi, \eta, \Omega) > 0$ with the property that whenever $\phi \in C^2(\overline{\Omega}), u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ are such that*

$$\|\phi\|_{L^\infty(\Omega)} \le \varepsilon \tag{1.22}$$

and

$$\|\nabla\phi\|_{L^2(\Omega)} \le \varepsilon, \tag{1.23}$$

and that (1.5) holds with

$$\|u_0\|_{L^2(\Omega)} \le \varepsilon \tag{1.24}$$

as well as

$$\|v_0 - v_\infty\|_{L^\infty(\Omega)} \le \varepsilon \tag{1.25}$$

and

$$\|\nabla v_0 - \nabla v_\infty\|_{L^2(\Omega)} \le \varepsilon, \tag{1.26}$$

the problem (1.3) with

$$B_1(x,t) := 0 \quad and \quad B_2(x,t) := \eta + \phi(x), \qquad (x,t) \in \overline{\Omega} \times [0,\infty), \tag{1.27}$$

admits a global classical solution (u,v) fulfilling (1.19) and (1.20), where v_∞ solves (1.21) with

$$B_{2,\infty}(x) := \eta + \phi(x), \qquad x \in \overline{\Omega}. \tag{1.28}$$

Theorem 1.3 signifies that if only very few criminal agents are introduced as a source, any small pertubation of even a high level of constant attractiveness is insufficient to induce the formation of crime patterns or clusters. Apart from that, we emphasize that both Theorem 1.2 and Theorem 1.3 do not include the simple prototypical case when besides B_2 also B_1 is a positive constant, especially representing settings in which offenders are added at a constant rate. In fact, the simulations performed in [26] and [27] suggest that in the latter case the system may possibly support the formation of crime hotsopts for large χ, indicating that stabilization results in the flavor of those from Theorem 1.2 and Theorem 1.3 may not be expected in such situations, and that hence the spatio-temporal dynamics of (1.3) in the presence of persistent external sources of criminal agents may contain much more colorful facets deserving further investigation beyond this study.

Organization of the paper. After stating a basic result on local existence and extensibility in Section 2, we shall build our existence theory on the observation, to be detailed in Section 3, that for suitably chosen $b > 0$ the quantity

$$\int_\Omega u^2 + b \int_\Omega \frac{|\nabla v|^2}{v} \tag{1.29}$$

satisfies a superlinearly forced differential inequality, provided that the smallness assumptions from Theorem 1.1 are fulfilled. In Section 4, accordingly obtained a priori estimates will successively be improved in order to indeed warrant global extensibility. The proof of Theorem 1.2 and of Theorem 1.3 will thereafter be achieved in Section 5 on the basis of ideas from [25], augmented by an additional observation on relative compactness of any bounded trajectory in $(C^0(\overline{\Omega}))^2$.

2 Local existence and basic estimates

Local solvability, as well as a suitable criterion for a solution to be extensible, can be obtained by following standard arguments.

Lemma 2.1 *Let $\chi > 0$ and $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and suppose that (1.4) and (1.5) hold. Then there exist $T_{max} \in (0, \infty]$ and a pair of functions u and v, for each $T \in (0, T_{max})$ and $p > 2$ uniquely determined by (1.15), which solve (1.3) classically in $\overline{\Omega} \times [0, T_{max})$, and which are such that $u \geq 0$ and $v > 0$ in $\overline{\Omega} \times [0, T_{max})$, and that*

$$\text{either } T_{max} = \infty, \quad \text{or}$$
$$\limsup_{t \nearrow T_{max}} \left\{ \|u(\cdot,t)\|_{L^\infty(\Omega)} + \left\| \frac{1}{v(\cdot,t)} \right\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,p}(\Omega)} \right\} = \infty \quad \text{for all } p > 2. \tag{2.1}$$

Proof. All statements can be derived by straightforward adaptation of existence and extensibility arguments well-established in the context of chemotaxis-type parabolic systems ([1], [12]). □

As a first qualitative feature of this solution, let us state the following pointwise lower bound for its second component that will play an essential role throughout our analysis, inter alia as a means to provide appropriate control of the singular ingredient $\frac{1}{v}$ to the cross-diffusive flux in (1.3).

Lemma 2.2 *Let B_1, B_2, u_0 and v_0 satisfy (1.4) and (1.5). Then with $\Gamma \in (0, 1]$ satisfying (1.8), the solution of (1.3) satisfies*

$$v(x,t) \geq \frac{\Gamma}{2e} \cdot \min \left\{ \inf_{y \in \Omega} v_0(y) \, , \, \inf_{s > 0} \int_\Omega B_2(\cdot, s) \right\} \qquad \text{for all } x \in \Omega \text{ and } t \in (0, T_{max}). \tag{2.2}$$

PROOF. Using the order-preserving property of the Neumann heat semigroup along with the nonnegativity of uv and (1.8), on the basis of a variation-of-constants representation of v we obtain

$$\begin{aligned} v(\cdot,t) &= e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)}u(\cdot,s)v(\cdot,s)ds + \int_0^t e^{(t-s)(\Delta-1)}B_2(\cdot,s)ds \\ &\geq e^{-t}\inf_{y\in\Omega} v_0(y) + \int_0^{(t-1)_+} e^{-(t-s)}\cdot\Gamma\int_\Omega B_2(\cdot,s)ds \quad \text{in } \Omega \qquad \text{for all } t\in(0,T_{max}). \end{aligned}$$

Since for $t \leq 1+\ln 2$ we have

$$e^{-t}\inf_{y\in\Omega} v_0(y) \geq \frac{1}{2e}\inf_{y\in\Omega} v_0(y) \geq \frac{\Gamma}{2e}\inf_{y\in\Omega} v_0(y)$$

due to the fact that $\Gamma \leq 1$, and since for $t > 1+\ln 2$,

$$\begin{aligned} \int_0^{(t-1)_+} e^{-(t-s)}\cdot\Gamma\int_\Omega B_2(\cdot,s)ds &\geq \Gamma\Big\{\inf_{s>0}\int_\Omega B_2(\cdot,s)\Big\}\cdot\int_0^{t-1} e^{-(t-s)}ds \\ &= \Gamma\Big\{\inf_{s>0}\int_\Omega B_2(\cdot,s)\Big\}\cdot(e^{-1}-e^{-t}) \\ &\geq \Gamma\Big\{\inf_{s>0}\int_\Omega B_2(\cdot,s)\Big\}\cdot\frac{1}{2e} \qquad \text{in } \Omega, \end{aligned}$$

this readily leads to (2.2). □

3 Construction of an energy functional

Now the crucial part of our analysis is approached through the following result of a straightforward L^2 testing procedure performed to the first sub-problem of (1.3), where an element of decisive importance for our strategy will be formed by the zero-order dissipative contribution $\int_\Omega u^2 v$ to the following inequality.

Lemma 3.1 *Let $\eta > 0$. Then the solution of (1.3) has the property that for all $t \in (0,T_{max})$,*

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^2 + \Big\{1-\frac{K\chi^4}{4\eta\gamma}\int_\Omega u^2\Big\}\cdot\int_\Omega|\nabla u|^2 + 2\int_\Omega u^2 v &\leq \eta\int_\Omega\frac{|\nabla v|^4}{v^3} + \frac{K\chi^4}{4\eta\gamma}\Big\{\int_\Omega u^2\Big\}^2 \\ &\quad +2\|B_1(\cdot,t)\|_{L^2(\Omega)}\cdot\Big\{\int_\Omega u^2\Big\}^{\frac{1}{2}}, \end{aligned} \tag{3.1}$$

where $K > 0$ is taken from (1.6), and where $\gamma := \inf_{(x,t)\in\Omega\times(0,T_{max})} v(x,t)$.

PROOF. Using u as a test function for the first equation in (1.3), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 + \int_\Omega|\nabla u|^2 = \chi\int_\Omega\frac{u}{v}\nabla u\cdot\nabla v - \int_\Omega u^2 v + \int_\Omega B_1 u \qquad \text{for all } t\in(0,T_{max}), \tag{3.2}$$

where to applications of Young's inequality show that

$$\begin{aligned} \chi\int_\Omega\frac{u}{v}\nabla u\cdot\nabla v &\leq \frac{1}{2}\int_\Omega|\nabla u|^2 + \frac{\chi^2}{2}\int_\Omega\frac{u^2}{v^2}|\nabla v|^2 \\ &\leq \frac{1}{2}\int_\Omega|\nabla u|^2 + \frac{\eta}{2}\int_\Omega\frac{|\nabla v|^4}{v^3} + \frac{\chi^4}{8\eta}\int_\Omega\frac{u^4}{v} \qquad \text{for all } t\in(0,T_{max}). \end{aligned} \tag{3.3}$$

Since $v \ge \gamma$ in $\Omega \times (0, T_{max})$, herein we may invoke (1.6) to further estimate

$$\begin{aligned}
\frac{\chi^4}{8\eta} \int_\Omega \frac{u^4}{v} &\le \frac{\chi^4}{8\eta\gamma} \int_\Omega u^4 \\
&\le \frac{\chi^4 K}{8\eta\gamma} \Big\{ \int_\Omega |\nabla u|^2 \Big\} \cdot \Big\{ \int_\Omega u^2 \Big\} + \frac{\chi^4 K}{8\eta\gamma} \Big\{ \int_\Omega u^2 \Big\}^2 \qquad \text{for all } t \in (0, T_{max}). \qquad (3.4)
\end{aligned}$$

As moreover

$$\int_\Omega B_1 u \le \|B_1\|_{L^2(\Omega)} \Big\{ \int_\Omega u^2 \Big\}^{\frac{1}{2}} \qquad \text{for all } t \in (0, T_{max})$$

due to the Cauchy-Schwarz inequality, by straightforward rearrangement we thus infer from (3.2)-(3.4) that indeed (3.1) is valid. □

Indeed, the expression $\int_\Omega u^2 v$ encountered above precisely appears as an ill-signed term in the course of a second testing process which now operates on the second equation in (1.3):

Lemma 3.2 *Let $L > 0$ be such that (1.7) holds. Then*

$$\frac{d}{dt} \int_\Omega \frac{|\nabla v|^2}{v} + \int_\Omega \frac{|\nabla v|^2}{v} + \frac{1}{2L} \int_\Omega \frac{|\nabla v|^4}{v^3} \le \frac{L+4}{2} \int_\Omega u^2 v + 4 \int_\Omega |\nabla \sqrt{B_2}|^2 \qquad \textit{for all } t \in (0, T_{max}). \quad (3.5)$$

PROOF. Using the identity $\nabla v \cdot \nabla \Delta v = \frac{1}{2} \Delta |\nabla v|^2 - |D^2 v|^2$, from the second equation in (1.3) we obtain that

$$\begin{aligned}
\frac{d}{dt} \int_\Omega \frac{|\nabla v|^2}{v} &= 2 \int_\Omega \frac{1}{v} \nabla v \cdot \nabla (\Delta v - v + uv + B_2) - \int_\Omega \frac{1}{v^2} |\nabla v|^2 (\Delta v - v + uv + B_2) \\
&= \int_\Omega \frac{1}{v} \Delta |\nabla v|^2 - 2 \int_\Omega \frac{1}{v} |D^2 v|^2 - 2 \int_\Omega \frac{1}{v} |\nabla v|^2 + 2 \int_\Omega \nabla u \cdot \nabla v + 2 \int_\Omega \frac{u}{v} |\nabla v|^2 + 2 \int_\Omega \frac{1}{v} \nabla v \cdot \nabla B_2 \\
&\quad - \int_\Omega \frac{1}{v^2} |\nabla v|^2 \Delta v + \int_\Omega \frac{1}{v} |\nabla v|^2 - \int_\Omega \frac{u}{v} |\nabla v|^2 - \int_\Omega \frac{B_2}{v^2} |\nabla v|^2 \\
&= \int_\Omega \frac{1}{v^2} \nabla v \cdot \nabla |\nabla v|^2 - 2 \int_\Omega \frac{1}{v} |D^2 v|^2 - \int_\Omega \frac{1}{v^2} |\nabla v|^2 \Delta v + \int_{\partial\Omega} \frac{1}{v} \frac{\partial |\nabla v|^2}{\partial \nu} \\
&\quad - \int_\Omega \frac{1}{v} |\nabla v|^2 + 2 \int_\Omega \nabla u \cdot \nabla v + \int_\Omega \frac{u}{v} |\nabla v|^2 \\
&\quad + 2 \int_\Omega \frac{1}{v} \nabla v \cdot \nabla B_2 - \int_\Omega \frac{B_2}{v^2} |\nabla v|^2 \qquad \text{for all } t \in (0, T_{max}), \qquad (3.6)
\end{aligned}$$

where due to the convexity of Ω, we know that $\frac{\partial |\nabla v|^2}{\partial \nu} \le 0$ on $\partial\Omega \times (0, \infty)$ ([20]) and hence

$$\int_{\partial\Omega} \frac{1}{v} \frac{\partial |\nabla v|^2}{\partial \nu} \le 0 \qquad \text{for all } t \in (0, T_{max}).$$

Since by straightforward computation one can verify that

$$\int_\Omega \frac{1}{v^2} \nabla v \cdot \nabla |\nabla v|^2 - 2 \int_\Omega \frac{1}{v} |D^2 v|^2 - \int_\Omega \frac{1}{v^2} |\nabla v|^2 \Delta v = -2 \int_\Omega v |D^2 \ln v|^2 \qquad \text{for all } t \in (0, T_{max})$$

(cf. e.g. [33, Lemma 3.2] for details), since we again integrate by parts and use Young's inequality as well as the pointwise $|\Delta \ln v|^2 \le 2|D^2 \ln v|^2$ to estimate

$$
\begin{aligned}
2\int_\Omega \nabla u\cdot\nabla v &= 2\int_\Omega v\nabla u\cdot\nabla \ln v \\
&= -2\int_\Omega uv\Delta \ln v - 2\int_\Omega u\nabla v\cdot\nabla\ln v \\
&= -2\int_\Omega uv\Delta \ln v - 2\int_\Omega \frac{u}{v}|\nabla v|^2 \\
&\le -2\int_\Omega uv\Delta\ln v \\
&\le \frac{1}{2}\int_\Omega v|\Delta \ln v|^2 + 2\int_\Omega u^2 v \\
&\le \int_\Omega v|D^2\ln v|^2 + 2\int_\Omega u^2 v \qquad \text{for all } t\in(0,T_{max}),
\end{aligned}
$$

and since (1.7) entails that

$$
\int_\Omega v|D^2 \ln v|^2 \ge \frac{1}{L}\int_\Omega \frac{|\nabla v|^4}{v^3} \qquad \text{for all } t\in(0,T_{max}),
$$

from (3.6) it thus follows that

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{|\nabla v|^2}{v} &+ \int_\Omega \frac{|\nabla v|^2}{v} + \frac{1}{L}\int_\Omega \frac{|\nabla v|^4}{v^3} \\
&\le \int_\Omega \frac{u}{v}|\nabla v|^2 + 2\int_\Omega u^2 v \\
&\quad +2\int_\Omega \frac{1}{v}\nabla v\cdot\nabla B_2 - \int_\Omega \frac{B_2}{v^2}|\nabla v|^2 \qquad (3.7)
\end{aligned}
$$

for all $t\in(0,T_{max})$. Here two applications of Young's inequality show that

$$
\int_\Omega \frac{u}{v}|\nabla v|^2 \le \frac{1}{2L}\int_\Omega \frac{|\nabla v|^4}{v^3} + \frac{L}{2}\int_\Omega u^2 v \qquad \text{for all } t\in(0,T_{max})
$$

and

$$
2\int_\Omega \frac{1}{v}\nabla v\cdot\nabla B_2 - \int_\Omega \frac{B_2}{v^2}|\nabla v|^2 \le \int_\Omega \frac{|\nabla B_2|^2}{B_2} = 4\int_\Omega |\nabla\sqrt{B_2}|^2 \qquad \text{for all } t\in(0,T_{max}).
$$

Therefore, (3.7) easily leads to (3.5). □

Favorably, by suitable linear combination of (3.1) and (3.5) the first expression on the right-hand side of (3.5) can be eliminated, hence leading to an already quite propitious inequality for the quantity from (1.29).

Lemma 3.3 *Let (u,v) denote the solution of (1.3). Then the function y defined by*

$$y(t) := \int_\Omega u^2(\cdot,t) + b\int_\Omega \frac{|\nabla v(\cdot,t)|^2}{v(\cdot,t)}, \qquad t\in[0,T_{max}), \tag{3.8}$$

with

$$b := \frac{2}{L+4} \tag{3.9}$$

satisfies

$$\begin{aligned} y'(t) + \Big\{1 - \frac{KL(L+4)\chi^4}{4\gamma}\cdot y(t)\Big\}\cdot\int_\Omega |\nabla u(\cdot,t)|^2 + \lambda y(t) \\ \le \frac{KL(L+4)\chi^4}{4\gamma}\cdot y^2(t) + \frac{2}{\gamma}\sup_{s>0}\int_\Omega B_1^2(\cdot,s) \\ +\frac{8}{L+4}\cdot\sup_{s>0}\int_\Omega |\nabla\sqrt{B_2(\cdot,s)}|^2 \qquad \textit{for all } t\in(0,T_{max}), \end{aligned} \tag{3.10}$$

where

$$\gamma := \inf_{(x,t)\in\Omega\times(0,T_{max})} v(x,t) \qquad \textit{and} \qquad \lambda := \min\Big\{1,\frac{\gamma}{2}\Big\}.$$

Proof. We take an appropriate linear combination of the inequalities from Lemma 3.1 and Lemma 3.2, choosing $\eta := \frac{1}{L(L+4)}$ in the former, to see that writing $c_1 := \frac{KL(L+4)\chi^4}{4\gamma}$ we have

$$\begin{aligned} y'(t) + \Big\{1 - c_1\int_\Omega u^2\Big\}\cdot\int_\Omega|\nabla u|^2 + 2\int_\Omega u^2 v \\ +b\int_\Omega\frac{|\nabla v|^2}{v} + \frac{b}{2L}\int_\Omega\frac{|\nabla v|^4}{v^3} \\ \le \frac{1}{L(L+4)}\int_\Omega\frac{|\nabla v|^4}{v^3} + c_1\Big\{\int_\Omega u^2\Big\}^2 + 2\|B_1\|_{L^2(\Omega)}\Big\{\int_\Omega u^2\Big\}^{\frac12} \\ +\frac{(L+4)b}{2}\int_\Omega u^2v + 4b\int_\Omega|\nabla\sqrt{B_2}|^2 \qquad \text{for all } t\in(0,T_{max}), \end{aligned} \tag{3.11}$$

where the first summand on the right is precisely cancelled by the last one on the left due to our choice (3.9) of b. Since (3.9) moreover says that $\frac{(L+4)b}{2} = 1$, from (3.11) we thus obtain that

$$\begin{aligned} y'(t) + \Big\{1 - c_1\int_\Omega u^2\Big\}\cdot\int_\Omega|\nabla u|^2 + \int_\Omega u^2v + b\int_\Omega\frac{|\nabla v|^2}{v} \\ \le c_1\Big\{\int_\Omega u^2\Big\}^2 + 2\|B_1\|_{L^2(\Omega)}\Big\{\int_\Omega u^2\Big\}^{\frac12} + 4b\int_\Omega|\nabla\sqrt{B_2}|^2 \qquad \text{for all } t\in(0,T_{max}). \end{aligned} \tag{3.12}$$

Here we use the definition of γ to estimate

$$\int_\Omega u^2 v \ge \gamma\int_\Omega u^2 \qquad \text{for all } t\in(0,T_{max}),$$

and then employ Young's inequality to find that

$$2\|B_1\|_{L^2(\Omega)}\Big\{\int_\Omega u^2\Big\}^{\frac{1}{2}} \le \frac{\gamma}{2}\int_\Omega u^2 + \frac{2}{\gamma}\int_\Omega B_1^2 \qquad \text{for all } t\in(0,T_{max}).$$

Since

$$\frac{\gamma}{2}\int_\Omega u^2 + b\int_\Omega \frac{|\nabla v|^2}{v} \ge \lambda y(t) \qquad \text{for all } t\in(0,T_{max})$$

by definition of λ, and since

$$\int_\Omega u^2 \le y(t) \qquad \text{for all } t\in(0,T_{max}),$$

from (3.12) we easily derive (3.10). $\square$

Now smallness assumptions in the style of those in Theorem 1.1 warrant that both the initial value of y and the y-independent forcing terms in (3.10) remain conveniently small, hence enforcing bounds for y through a simple ODE comparison.

Lemma 3.4 *Suppose that for some $\delta>0$, the functions B_1, B_2, u_0 and v_0 are such that beyond (1.4) and (1.5), the assumptions (1.9)-(1.14) from Theorem 1.1 are satisfied with $K>0, L>0$ and $\Gamma\in(0,1]$ such that (1.6), (1.7) and (1.8) hold, and with $\lambda:=\min\{1,\frac{\Gamma\delta}{4e}\}$. Then there exists $C>0$ such that for the solution of (1.3) we have*

$$\int_\Omega u^2(\cdot,t)\le C \qquad \textit{for all } t\in(0,T_{max}) \tag{3.13}$$

and

$$\int_\Omega \frac{|\nabla v(\cdot,t)|^2}{v(\cdot,t)} \le C \qquad \textit{for all } t\in(0,T_{max}). \tag{3.14}$$

PROOF. Writing $\gamma:=\frac{\Gamma\delta}{2e}$, in view of Lemma 2.2 we obtain from the assumptions (1.10) and (1.12) that

$$v\ge\gamma \qquad \text{in } \Omega\times(0,T_{max}),$$

whereas (1.9) and (1.11) warrant that abbreviating $c_1:=\frac{KL(L+4)\chi^4}{4\gamma}$ and $c_2:=\frac{2}{\gamma}\sup_{s>0}\int_\Omega B_1^2(\cdot,s)+\frac{8}{L+4}\cdot\sup_{s>0}\int_\Omega|\nabla\sqrt{B_2(\cdot,s)}|^2$, we have

$$\begin{aligned}
4c_1c_2 &= \frac{KL(L+4)\chi^4 e}{2\Gamma\delta}\cdot\Big\{\frac{4e}{\Gamma\delta}\cdot\sup_{s>0}\int_\Omega B_1^2(\cdot,s)+\frac{8}{L+4}\cdot\sup_{s>0}\int_\Omega|\nabla\sqrt{B_2(\cdot,s)}|^2\Big\}\\
&\le \frac{KL(L+4)\chi^4 e}{2\Gamma\delta}\cdot\Big\{\frac{4e}{\Gamma\delta}\cdot\frac{\Gamma^2}{4KL(L+4)\chi^4e^2}\cdot\delta^2+\frac{8}{L+4}\cdot\frac{\Gamma}{8KL\chi^4e}\cdot\delta\Big\}\cdot\lambda^2\\
&= \frac{KL(L+4)\chi^4 e}{2\Gamma\delta}\cdot\Big\{\frac{\Gamma\delta}{KL(L+4)\chi^4e}+\frac{\Gamma\delta}{KL(L+4)\chi^4e}\Big\}\cdot\lambda^2\\
&= \lambda^2.
\end{aligned} \tag{3.15}$$

Now defining y as in (3.8) with b taken from (3.9), invoking (3.10) we see that

$$y'(t) + \Big\{1 - c_1 y(t)\Big\} \cdot \int_\Omega |\nabla u|^2 + \lambda y(t) \le c_1 y^2(t) + c_2 \qquad \text{for all } t \in (0, T_{max}), \tag{3.16}$$

while (1.13) and (1.14) ensure that

$$\begin{aligned} y(0) &= \int_\Omega u_0^2 + \frac{8}{L+4} \int_\Omega |\nabla \sqrt{v_0}|^2 \\ &\le \frac{\Gamma}{2KL(L+4)\chi^4 e} \cdot \delta + \frac{8}{L+4} \cdot \frac{\Gamma}{16KL\chi^4 e} \cdot \delta \\ &= \frac{1}{4c_1} + \frac{1}{4c_1} = \frac{1}{2c_1} \end{aligned} \tag{3.17}$$

and thus, in particular, $y(0) < \frac{1}{c_1}$. Therefore,

$$T := \sup \Big\{ \widetilde{T} \in (0, T_{max}) \ \Big| \ y(t) < \frac{1}{c_1} \qquad \text{for all } t \in [0, \widetilde{T}) \Big\}$$

is a well-defined element of $(0, T_{max}]$, and to verify that actually $T = T_{max}$ we assume that this be false and then obtain from (3.16) that since $1 - c_1 y$ is nonnegative on $(0, T)$, we have

$$y'(t) + \lambda y(t) \le c_1 y^2(t) + c_2 \qquad \text{for all } t \in (0, T).$$

Since

$$\overline{y}(t) := \frac{\lambda}{2c_1}, \qquad t \ge 0,$$

satisfies

$$\overline{y}'(t) + \lambda \overline{y}(t) - c_1 \overline{y}^2(t) - c_2 = \frac{\lambda^2}{2c_1} - \frac{\lambda^2}{4c_1} - c_2 = \frac{\lambda^2}{4c_1} - c_2 \ge 0 \qquad \text{for all } t > 0$$

due to (3.15), in view of (3.17) we may employ an ODE comparison argument to see that $y \le \overline{y}$ throughout $[0, T]$ and thus, in particular, $y(T) \le \overline{y}(T) < \frac{1}{c_1}$. This contradiction to the definition of T shows that indeed $T = T_{max}$ and thereby implies both (3.13) and (3.14) upon an evident choice of C. $\Box$

4 Improved regularity estimates. Proof of Theorem 1.1

Forming a second place in which we explicitly rely on the presence of the absorptive term $-uv$ in the first equation from (1.3), the following lemma asserts a basic L^1 boundedness property of v which will pave our way toward an appropriate further exploitation of the estimates from Lemma 3.4.

Lemma 4.1 *Suppose that the assumptions of Theorem 1.1 hold. Then there exists $C > 0$ such that the solution of (1.3) satisfies*

$$\int_\Omega v(\cdot, t) \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{4.1}$$

PROOF. We integrate the first two equations in (1.3) over Ω to see that

$$\begin{aligned}\frac{d}{dt}\int_\Omega u+\int_\Omega u &= -\int_\Omega uv+\int_\Omega B_1+\int_\Omega u \\ &\le -\int_\Omega uv+c_1 \qquad \text{for all } t\in(0,T_{max}) \end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\frac{d}{dt}\int_\Omega v+\int_\Omega v &= \int_\Omega uv+\int_\Omega B_2 \\ &\le \int_\Omega uv+c_2 \qquad \text{for all } t\in(0,T_{max}), \end{aligned} \tag{4.3}$$

respectively, where

$$c_1 := \sup_{t>0}\int_\Omega B_1(\cdot,t)+\sup_{t\in(0,T_{max})}\int_\Omega u(\cdot,t)$$

and

$$c_2 := \sup_{t>0}\int_\Omega B_2(\cdot,t)$$

are finite according to (1.4) and Lemma 3.4. On adding (4.3) to (4.2) we therefore obtain

$$\frac{d}{dt}\left\{\int_\Omega u+\int_\Omega v\right\}+\left\{\int_\Omega u+\int_\Omega v\right\}\le c_1+c_2 \qquad \text{for all } t\in(0,T_{max}),$$

so that by means of an ODE comparison argument we conclude that

$$\int_\Omega u(\cdot,t)+\int_\Omega v(\cdot,t)\le \max\left\{\int_\Omega u_0+\int_\Omega v_0\,,\,c_1+c_2\right\} \qquad \text{for all } t\in(0,T_{max}),$$

which implies (4.1). □

In fact, interpolating between (4.1) and (3.14) yields L^p bounds for v with arbitrary finite p.

Lemma 4.2 *Under the assumptions of Theorem 1.1, for each $p>1$ one can find $C(p)>0$ such that*

$$\int_\Omega v^p(\cdot,t)\le C(p) \qquad \textit{for all } t\in(0,T_{max}).$$

PROOF. Since the Gagliardo-Nirenberg inequality implies the existence of $c_1>0$ such that

$$\begin{aligned}\int_\Omega v^p=\|\sqrt{v}\|_{L^{2p}(\Omega)}^{2p} &\le c_1\|\nabla\sqrt{v}\|_{L^2(\Omega)}^{2(p-1)}\|\sqrt{v}\|_{L^2(\Omega)}^2+c_1\|\sqrt{v}\|_{L^2(\Omega)}^{2p} \\ &= \frac{c_1}{4^{p-1}}\left\{\int_\Omega\frac{|\nabla v|^2}{v}\right\}^{p-1}\cdot\int_\Omega v+c_1\left\{\int_\Omega v\right\}^p \qquad \text{for all } t\in(0,T_{max}),\end{aligned}$$

this follows by combining Lemma 3.4 with Lemma 4.1. □

Along with the L^2 estimate for u from Lemma 3.4, the latter information warrants integrability features of the source term uv in the second equation from (1.3) which are sufficient to assert bounds even for ∇v in L^p spaces with arbitrarily large $p>1$.

Lemma 4.3 *Suppose that the assumptions of Theorem 1.1 are met. Then for any $p > 1$ there exists $C(p) > 0$ with the property that*

$$\int_\Omega |\nabla v(\cdot,t)|^p \le C(p) \qquad \textit{for all } t \in (0, T_{max}). \tag{4.4}$$

Proof. We pick any $q > \frac{2p}{p+2}$ such that $q < 2$ and then obtain from Lemma 4.2 that with some $c_1 > 0$ we have

$$\|v(\cdot,t)\|_{L^{\frac{2q}{2-q}}(\Omega)} \le c_1 \qquad \text{for all } t \in (0, T_{max}).$$

Since in view of our hypotheses we may invoke Lemma 3.4 to find $c_2 > 0$ fulfilling

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le c_2 \qquad \text{for all } t \in (0, T_{max}),$$

by means of the Hölder inequality we can estimate

$$\|u(\cdot,t)v(\cdot,t)\|_{L^q(\Omega)} \le \|u(\cdot,t)\|_{L^2(\Omega)} \|v(\cdot,t)\|_{L^{\frac{2q}{2-q}}(\Omega)} \le c_1 c_2 \qquad \text{for all } t \in (0, T_{max}). \tag{4.5}$$

We now recall known regularization features of the Neumann heat semigroup ([30, Lemma Lemma 1.3 (ii) and (iii)]) to see that there exists $c_3 > 0$ satisfying

$$\begin{aligned}
\|\nabla v(\cdot,t)\|_{L^p(\Omega)} &= \left\| \nabla e^{t(\Delta-1)} v_0 + \int_0^t \nabla e^{(t-s)(\Delta-1)} u(\cdot,s)v(\cdot,s)ds + \int_0^t \nabla e^{(t-s)(\Delta-1)} B_2(\cdot,s)ds \right\|_{L^p(\Omega)} \\
&\le c_3 \|\nabla v_0\|_{L^p(\Omega)} + c_3 \int_0^t (t-s)^{-\frac{1}{2}-(\frac{1}{q}-\frac{1}{p})} e^{-(t-s)} \|u(\cdot,s)v(\cdot,s)\|_{L^q(\Omega)} ds \\
&\quad + c_3 \int_0^t (t-s)^{-\frac{1}{2}} e^{-(t-s)} \|B_2(\cdot,s)\|_{L^p(\Omega)} ds \qquad \text{for all } t \in (0, T_{max}).
\end{aligned}$$

Hence, using (4.5) we infer that with $c_4 := \sup_{t>0} \|B_2(\cdot,t)\|_{L^p(\Omega)}$ we have

$$\begin{aligned}
\|\nabla v(\cdot,t)\|_{L^p(\Omega)} &\le c_3 \|\nabla v_0\|_{L^p(\Omega)} + c_1 c_2 c_3 \int_0^t (t-s)^{-\frac{1}{2}-(\frac{1}{q}-\frac{1}{p})} e^{-(t-s)} ds \\
&\quad + c_3 c_4 \int_0^t (t-s)^{-\frac{1}{2}} e^{-(t-s)} ds \\
&\le c_3 \|\nabla v_0\|_{L^p(\Omega)} + c_1 c_2 c_3 c_5 + c_3 c_4 c_6 \qquad \text{for all } t \in (0, T_{max}),
\end{aligned}$$

where $c_5 := \int_0^\infty \sigma^{-\frac{1}{2}-(\frac{1}{q}-\frac{1}{p})} e^{-\sigma} d\sigma$ and $c_6 := \int_0^\infty \sigma^{-\frac{1}{2}} e^{-\sigma} d\sigma$ are finite thanks to our restriction that $q > \frac{2p}{p+2}$. $\square$

Through the latter and Lemma 2.2 thus able to suitably control the cross-diffusive flux in (1.3), we can proceed to derive an estimate for u with respect to the norm in $L^\infty(\Omega)$.

Lemma 4.4 *Under the assumptions of Theorem 1.1, we can fix $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{4.6}$$

PROOF. Given $T \in (0, T_{max})$, we estimate

$$M(T) := \sup_{t \in (0,T)} \|u(\cdot,t)\|_{L^\infty(\Omega)}$$

by using a variation-of-constants representation associated with the identity $u_t = (\Delta-1)u - \chi\nabla\cdot(\frac{u}{v}\nabla v) - uv + B_1 + u$ and employing three comparison arguments and two well-known smoothing properties of the Neumann heat semigroup ([30, Lemma 1.3 (ii) and (iv)]) to firstly obtain that

$$\begin{aligned} u(\cdot,t) &= e^{t(\Delta-1)}u_0 - \chi\int_0^t e^{(t-s)(\Delta-1)}\nabla\cdot\Big(\frac{u(\cdot,s)}{v(\cdot,s)}\nabla v(\cdot,s)\Big)ds - \int_0^t e^{(t-s)(\Delta-1)}u(\cdot,s)v(\cdot,s)ds \\ &\quad + \int_0^t e^{(t-s)(\Delta-1)}B_1(\cdot,s)ds + \int_0^t e^{(t-s)(\Delta-1)}u(\cdot,s)ds \\ &\le \|u_0\|_{L^\infty(\Omega)} + c_1\int_0^t (t-s)^{-\frac{3}{4}}e^{-(t-s)}\Big\|\frac{u(\cdot,s)}{v(\cdot,s)}\nabla v(\cdot,s)\Big\|_{L^4(\Omega)}ds \\ &\quad + \int_0^t e^{-(t-s)}\|B_1(\cdot,s)\|_{L^\infty(\Omega)}ds + c_1\int_0^t (t-s)^{-\frac{1}{2}}e^{-(t-s)}\|u(\cdot,s)\|_{L^2(\Omega)}ds \quad \text{in } \Omega \end{aligned} \tag{4.7}$$

for some $c_1 > 0$ and all $t \in (0, T_{max})$. Here since the assumption (1.12) together with Lemma 2.2 warrants the existence of $c_2 > 0$ fulfilling

$$v(x,t) \ge c_2 \qquad \text{for all } x \in \Omega \text{ and } t \in (0, T_{max}),$$

twice invoking the Hölder inequality we can estimate

$$\begin{aligned} \Big\|\frac{u(\cdot,t)}{v(\cdot,t)}\nabla v(\cdot,t)\Big\|_{L^4(\Omega)} &\le \frac{1}{c_2}\|u(\cdot,t)\|_{L^8(\Omega)}\|\nabla v(\cdot,t)\|_{L^8(\Omega)} \\ &\le \frac{1}{c_2}\|u(\cdot,t)\|_{L^\infty(\Omega)}^{\frac{3}{4}}\|u(\cdot,t)\|_{L^2(\Omega)}^{\frac{1}{4}}\|\nabla v(\cdot,t)\|_{L^8(\Omega)} \qquad \text{for all } t \in (0, T_{max}). \end{aligned}$$

As Lemma 3.4 and Lemma 4.3 yield $c_3 > 0$ and $c_4 > 0$ such that

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le c_3 \qquad \text{for all } t \in (0, T_{max}) \tag{4.8}$$

and

$$\|\nabla v(\cdot,t)\|_{L^8(\Omega)} \le c_4 \qquad \text{for all } t \in (0, T_{max}),$$

it thus follows that

$$\begin{aligned} c_1\int_0^t (t-s)^{-\frac{3}{4}}e^{-(t-s)}\Big\|\frac{u(\cdot,s)}{v(\cdot,s)}\nabla v(\cdot,s)\Big\|_{L^4(\Omega)}ds &\le \frac{c_1c_3^{\frac{1}{4}}c_4}{c_2}M^{\frac{3}{4}}(T)\int_0^t (t-s)^{-\frac{3}{4}}e^{-(t-s)}ds \\ &\le \frac{c_1c_3^{\frac{1}{4}}c_4c_5}{c_2}M^{\frac{3}{4}}(T) \qquad \text{for all } t \in (0,T), \end{aligned}$$

with $c_5 := \int_0^\infty \sigma^{-\frac{3}{4}}e^{-\sigma}d\sigma$. Since (4.8) furthermore entails that writing $c_6 := \int_0^\infty \sigma^{-\frac{1}{2}}e^{-\sigma}d\sigma$ we have

$$\int_0^t (t-s)^{-\frac{1}{2}}e^{-(t-s)}\|u(\cdot,s)\|_{L^2(\Omega)}ds \le c_3c_6 \qquad \text{for all } t \in (0, T_{max}),$$

from (4.7) we therefore conclude that there exists $c_7 > 0$ such that

$$M(T) \le c_7 + c_7 M^{\frac{3}{4}}(T) \qquad \text{for all } T \in (0, T_{max})$$

and that hence

$$M(T) \le \max\Big\{1\,,\,(2c_7)^4\Big\} \qquad \text{for all } T \in (0, T_{max}),$$

which on taking $T \nearrow T_{max}$ results in (4.6). □

Our main result on global existence and boundedness has thereby actually been completed already:

PROOF of Theorem 1.1. In view of the assumptions (1.9)-(1.14), we may apply Lemma 4.4 and Lemma 4.3 which together with Lemma 2.1 firstly imply the statement on global existence. Thereupon, the boundedness properties in (1.16) result from (4.6) and (4.4) when combined with the outcome of Lemma 4.1. □

5 Large time behavior. Proof of Theorems 1.2 and 1.3

In order to prepare our analysis of large time features enjoyed by global bounded smooth solutions in the flavor of Theorem 1.2, let us first make sure that bounded trajectories in fact are eventually bounded in Hölder spaces.

Lemma 5.1 *Let B_1 and B_2 satisfy (1.4), (1.17) and (1.18), and suppose that with some u_0 and v_0 fulfilling (1.5), (u, v) is a global and bounded solution of (1.3) in the sense that (1.15) and (1.16) hold. Then there exist $t_0 \ge 1$, $\gamma \in (0, 1)$ and $C > 0$ such that*

$$\|u(\cdot, t)\|_{C^\gamma(\overline{\Omega})} + \|v(\cdot, t)\|_{C^\gamma(\overline{\Omega})} \le C \qquad \textit{for all } t > t_0, \tag{5.1}$$

and that

$$v(x, t) \ge \frac{1}{C} \qquad \textit{for all } x \in \Omega \textit{ and } t > t_0. \tag{5.2}$$

PROOF. When combined with the dominated convergence theorem, (1.18) entails that $\int_\Omega B_2(\cdot, t) \to \int_\Omega B_{2,\infty}$ as $t \to \infty$, so that since $B_{2,\infty}$ is continuous in $\overline{\Omega}$ with $0 \le B_{2,\infty} \not\equiv 0$, for some $t_1 > 0$ we obtain $\inf_{t>t_1} \int_\Omega B_2(\cdot, t) > 0$. Replacing $(B_1, B_2, u, v)(\cdot, t)$ with $(B_1, B_2, u, v)(\cdot, t_1 + t)$ if necessary, in view of the uniqueness statement in Theorem 1.1 and Lemma 2.2 we may henceforth assume that $\inf_{t>0} \int_\Omega B_2(\cdot, t) > 0$ and that

$$v \ge c_1 \qquad \text{in } \Omega \times (0, \infty) \tag{5.3}$$

with some $c_1 > 0$.

Now thanks to (1.4) and the assumed boundedness properties of u and v in (1.15), by referring to standard regularity theory for linear parabolic equations ([17]) we find $\gamma \in (0, 1)$ and $c_2 > 0$ such that

$$\|v(\cdot, t)\|_{C^\gamma(\overline{\Omega})} \le c_2 \qquad \text{for all } t > 1 \tag{5.4}$$

and that

$$\|\nabla v(\cdot, t)\|_{L^\infty(\Omega)} \le c_2 \qquad \text{for all } t > 1. \tag{5.5}$$

Relying on (5.5), (5.3), (1.4) and again the boundedness assumptions on (u, v), from well-known Hölder estimates for bounded solution of semilinear parabolic equations ([23, Theorem 1.3]), we obtain $c_3 > 0$ satisfying

$$\|u(\cdot, t)\|_{C^\gamma(\overline{\Omega})} \leq c_3 \qquad \text{for all } t > 2,$$

which along with (5.4) leads to (5.1), while (5.2) is implied by (5.3). $\square$

In appropriately making use of the latter by means of compactness arguments, we shall utilize the following elementary result on decay in a linear ODE, a proof of which can be found e.g. in [6, Lemma 4.6].

Lemma 5.2 *Let $y \in C^1([0, \infty))$ and $h \in L^1_{loc}([0, \infty))$ be nonnegative functions satisfying*

$$y'(t) + ay(t) \leq h(t) \qquad \textit{for all } t > 0$$

with some $a > 0$. Then if

$$\int_t^{t+1} h(s)ds \to 0 \qquad \textit{as } t \to \infty,$$

we have

$$y(t) \to 0 \qquad \textit{as } t \to \infty.$$

We can thereby indeed turn the hypothesis (1.17) into a statement on uniform decay of u in the intended form:

Lemma 5.3 *Assume that (1.4), (1.17) and (1.18) hold, and that (u, v) is as in Lemma 5.1. Then*

$$u(\cdot, t) \to 0 \quad \textit{in } L^\infty(\Omega) \qquad \textit{as } t \to \infty. \tag{5.6}$$

PROOF. As Lemma 5.1 provides $t_0 > 0$ and $c_1 > 0$ such that $v \geq c_1$ in $\Omega \times (t_0, \infty)$, integrating the first equation in (1.3) we see that

$$\frac{d}{dt}\int_\Omega u = -\int_\Omega uv + \int_\Omega B_1 \leq -c_1 \int_\Omega u + \int_\Omega B_1 \qquad \text{for all } t > t_0.$$

Since $\int_t^{t+1} \int_\Omega B_1(x, s)dxds \to 0$ as $t \to \infty$ according to (1.17), an application of Lemma 5.2 shows that

$$\int_\Omega u(\cdot, t) \to 0 \qquad \text{as } t \to \infty.$$

The claimed property (5.6) now results from this and the precompactness of $(u(\cdot, t))_{t>1}$ in $L^\infty(\Omega)$, as guaranteed by Lemma 5.1 and the Arzelà-Ascoli theorem. $\square$

By means of an energy method, we can next verify the convergence property of the second solution component claimed in Theorem 1.2.

Lemma 5.4 *Let B_1 and B_2 satisfy (1.4), (1.17), and let (u, v) be as in Lemma 5.1. Then*

$$v(\cdot, t) \to v_\infty \quad \text{in } L^\infty(\Omega) \qquad \text{as } t \to \infty, \tag{5.7}$$

where v_∞ denotes the solution of (1.21).

PROOF. Following an idea from [25], in contrast to the approach in the latter we shall here use Lemma 5.3 and the assumed boundedness of the solution to appropriately cope with with the integral $\int_\Omega (uv)^2$ related to the nonlinear production term of signal in the second equation of (1.3). In particular, on the basis of (1.3) and (1.21) we compute

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (v - v_\infty)^2 &= \int_\Omega (v - v_\infty) \cdot (\Delta v - v + uv + B_2) \\
&= \int_\Omega (v - v_\infty) \cdot \Big\{\Delta(v - v_\infty) - (v - v_\infty) + uv + (B_2 - B_{2,\infty})\Big\} \\
&= -\int_\Omega |\nabla(v - v_\infty)|^2 - \int_\Omega (v - v_\infty)^2 + \int_\Omega (v - v_\infty) \cdot \Big\{uv + (B_2 - B_{2,\infty})\Big\}
\end{aligned}$$

for all $t > 0$, where the first term on the right is nonpositive, and where the rightmost integral can be estimated by Young's inequality according to

$$\begin{aligned}
\int_\Omega (v - v_\infty) \cdot \Big\{uv + (B_2 - B_{2,\infty})\Big\} &\le \frac{1}{2}\int_\Omega (v - v_\infty)^2 + \frac{1}{2}\int_\Omega \Big\{uv + (B_2 - B_{2,\infty})\Big\}^2 \\
&\le \frac{1}{2}\int_\Omega (v - v_\infty)^2 + \int_\Omega (uv)^2 + \int_\Omega (B_2 - B_{2,\infty})^2
\end{aligned}$$

for all $t > 0$. Therefore, $y(t) := \int_\Omega (v - v_\infty)^2$ and $h(t) := 2\int_\Omega (u(\cdot, t)v(\cdot, t))^2 + 2\int_\Omega (B_2(\cdot, t) - B_{2,\infty})^2$, $t \ge 0$, satisfy

$$y'(t) + y(t) \le h(t) \qquad \text{for all } t > 0,$$

so that since Lemma 5.3 along with the boundedness of u and v yields $c_1 > 0$ such that

$$\int_t^{t+1} \int_\Omega (uv)^2 \le c_1 \int_t^{t+1} \int_\Omega u \to 0 \qquad \text{as } t \to \infty$$

and that thus

$$\int_t^{t+1} h(s)ds \to 0 \qquad \text{as } t \to \infty$$

thanks to (1.18), an application of Lemma 5.2 enables us to conclude that $y(t) \to 0$ and hence

$$v(\cdot, t) \to v_\infty \quad \text{in } L^2\Omega) \qquad \text{as } t \to \infty.$$

In conjunction with the relative compactness of $(v(\cdot, t))_{t > t_0}$ in $C^0(\overline{\Omega})$, as asserted by Lemma 5.1 with some suitably large $t_0 > 0$, this yields (5.7). $\square$

Our proof of Theorem 1.2 thereby becomes complete:

PROOF of Theorem 1.2. The conclusion immediately results from Lemma 5.3 and Lemma 5.4. □

Our main result concerning asymptotic stability of equilibria finally reduces to suitably dealing with the explicit requirements made in Theorem 1.1 in its essential part.

PROOF of Theorem 1.3. Given $\eta > 0$, we fix $\delta > 0$ such that

$$\delta \le \frac{\eta}{2} \tag{5.8}$$

and

$$\delta \le \frac{\eta|\Omega|}{2}, \tag{5.9}$$

and thereupon let

$$c_1 := \frac{\Gamma}{8KL\chi^4 e} \cdot \delta \cdot \lambda^2, \qquad c_2 := \frac{\Gamma}{2KL(L+4)\chi^4 e} \cdot \delta \qquad \text{and} \qquad c_3 := \frac{\Gamma}{16KL\chi^4 e} \cdot \delta, \tag{5.10}$$

with $\lambda := \min\{1, \frac{\Gamma\delta}{4e}\}$. We then take $\varepsilon = \varepsilon(\eta, \chi, \Omega) > 0$ small enough such that

$$\varepsilon \le \frac{\eta}{2(c_4+1)} \tag{5.11}$$

and

$$\varepsilon^2 \le \frac{c_3\eta}{c_5^2+1} \tag{5.12}$$

as well as

$$\varepsilon^2 \le c_2, \tag{5.13}$$

where in accordance with standard elliptic regularity theory, $c_4 > 0$ and $c_5 > 0$ are such that whenever $f \in C^1(\overline{\Omega})$ and $\psi \in C^2(\overline{\Omega})$ satisfy

$$\begin{cases} -\Delta\psi + \psi = f(x), & x \in \Omega, \\ \frac{\partial\psi}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases}$$

we have

$$\|\psi\|_{L^\infty(\Omega)} \le c_4 \|f\|_{L^\infty(\Omega)} \tag{5.14}$$

and

$$\|\nabla\psi\|_{L^2(\Omega)} \le c_5 \|f\|_{L^\infty(\Omega)}. \tag{5.15}$$

Now assuming that $\phi \in C^2(\overline{\Omega})$ is a given function fulfilling (1.22) and (1.23), we let $B_1, B_2, B_{2,\infty}$ and v_∞ be as defined through (1.27), (1.28) and (1.21), and suppose that u_0 and v_0 comply with (1.5) and moreover satisfy (1.24)-(1.26).

We then firstly note that $\psi := v_\infty - \eta$ satisfies $-\Delta\psi + \psi = -\Delta v_\infty + v_\infty - \eta = B_{2,\infty} - \eta = \phi$ in Ω with $\frac{\partial\psi}{\partial\nu} = 0$ on $\partial\Omega$, whence (5.14) and (5.15) apply so as to warrant that thanks to (1.22),

$$\|\psi\|_{L^\infty(\Omega)} \le c_4 \|\phi\|_{L^\infty(\Omega)} \le c_4\varepsilon \tag{5.16}$$

and

$$\int_\Omega |\nabla\psi|^2 \le c_5^2 \|\phi\|_{L^\infty(\Omega)}^2 \le c_5^2 \varepsilon^2. \tag{5.17}$$

Therefore, (1.25) and (1.26) especially entail that due to (5.11) and (5.8),

$$\begin{aligned} v_0 &\ge v_\infty - \|v_0 - v_\infty\|_{L^\infty(\Omega)} \\ &= \eta + \psi - \|v_0 - v_\infty\|_{L^\infty(\Omega)} \\ &\ge \eta - c_4\varepsilon - \varepsilon \\ &\ge \frac{\eta}{2} \ge \delta \qquad \text{in } \Omega, \end{aligned} \tag{5.18}$$

and that thus also

$$\begin{aligned} \int_\Omega |\nabla\sqrt{v_0}|^2 &= \frac{1}{4}\int_\Omega \frac{|\nabla v_0|^2}{v_0} \\ &\le \frac{1}{2\eta}\int_\Omega |\nabla v_0|^2 \\ &\le \frac{1}{\eta}\int_\Omega |\nabla v_\infty|^2 + \frac{1}{\eta}\int_\Omega |\nabla(v_0 - v_\infty)|^2 \\ &= \frac{1}{\eta}\int_\Omega |\nabla\psi|^2 + \frac{1}{\eta}\int_\Omega |\nabla(v_0 - v_\infty)|^2 \\ &\le \frac{c_5^2\varepsilon^2}{\eta} + \frac{\varepsilon^2}{\eta} \\ &\le c_3 \end{aligned} \tag{5.19}$$

by (5.12). Moreover, using that (5.11) in particular entails that $\varepsilon \le \frac{\eta}{2}$, from (1.28) and (1.22) we infer that

$$B_{2,\infty} = \eta + \phi \ge \eta - \varepsilon \ge \frac{\eta}{2} \qquad \text{in } \Omega,$$

which firstly guarantees that

$$\int_\Omega B_2(\cdot,t) = \int_\Omega B_{2,\infty} \ge \frac{\eta|\Omega|}{2} \ge \delta \qquad \text{for all } t>0 \tag{5.20}$$

according to (5.9), and which secondly, in conjunction with (1.23), implies that

$$\begin{aligned} \int_\Omega |\nabla\sqrt{B_2(\cdot,t)}|^2 &= \frac{1}{4}\int_\Omega \frac{|\nabla B_{2,\infty}|^2}{B_{2,\infty}} \\ &\le \frac{1}{2\eta}\int_\Omega |\nabla B_{2,\infty}|^2 \\ &= \frac{1}{2\eta}\int_\Omega |\nabla\phi|^2 \\ &\le \frac{\varepsilon^2}{2\eta} \\ &\le c_1 \qquad \text{for all } t>0. \end{aligned} \tag{5.21}$$

Now since (5.20) and (5.21) ensure validity of (1.10) and (1.11), and since (1.12), (1.13) and (1.14) are fulfilled thanks to (5.18), (1.24), (5.13) and (5.19), noting that (1.9), (1.17) and (1.18) are trivially satisfied due to (1.27) and (1.28) we may employ Theorem 1.1 and Theorem 1.2 to see that that indeed a global classical solution with the claimed asymptotic properties exists. □

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