A critical virus production rate for blow-up suppression in a haptotatxis model for oncolytic virotherapy

Youshan Tao*
School of Mathematical Sciences, Shanghai Jiao Tong University,
Shanghai 200240, P.R. China

Michael Winkler#
Institut für Mathematik, Universität Paderborn,
33098 Paderborn, Germany

Abstract

In a two-dimensional smoothly bounded domain, we consider the oncolytic virotherapy model given by

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v) - \rho uz, \\ v_t = -(u+w)v, \\ w_t = D_w \Delta w - w + uz, \\ z_t = D_z \Delta z - z - uz + \beta w, \end{cases}$$

with parameters $D_w > 0, D_z > 0, \beta > 0$ and $\rho \geq 0$.

According to previous findings, a corresponding Neumann initial-boundary value problem possesses a globally defined classical solution whenever the prescribed initial data are suitably regular and satisfy appropriate positivity assumptions. In the present study it is shown that if $\beta < 1$, then any such solution is uniformly bounded. This complements a recent result which in the case $\rho = 0$ has asserted the existence of unbounded solutions whenever $\beta > 1$.

Key words: haptotaxis; critical parameter; boundedness
MSC (2020): 35B33, 35B40, 35K57, 35Q92, 92C17

*taoys@sjtu.edu.cn
#michael.winkler@math.uni-paderborn.de

1 Introduction

In this work we consider the haptotactic cross-diffusion initial-boundary value problem

$$\begin{cases} u_t &= \Delta u - \nabla\cdot(u\nabla v) - \rho uz, & x\in\Omega,\ t>0,\\ v_t &= -(u+w)v, & x\in\Omega,\ t>0,\\ w_t &= D_w\Delta w - w + uz, & x\in\Omega,\ t>0,\\ z_t &= D_z\Delta z - z - uz + \beta w, & x\in\Omega,\ t>0,\\ (\nabla u - u\nabla v)\cdot\nu = \frac{\partial w}{\partial\nu} = \frac{\partial z}{\partial\nu} = 0, & & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), \quad w(x,0)=w_0(x), \quad z(x,0)=z_0(x), & & x\in\Omega, \end{cases} \tag{1.1}$$

in a smoothly bounded domain $\Omega\subset\mathbb{R}^2$, with positive parameters β, D_w and D_z and with a given number $\rho\geq 0$. This haptotaxis model has been introduced in [1] to mathematically describe oncolytic virotherapy that has become a new promising treatment against cancer, due to the specific ability of genetically designed oncolytic viruses to replicate inside cancer cells and kill them. In this context of oncolytic viral therapy, the unknown variables u, w, z and v in (1.1) represent the densities of uninfected and infected cancer cells, of virus particles, and of the extracellular matrix (ECM), respectively. Here, following biological evidence ([7], [21]), it is assumed that uninfected cancer cells not only move randomly but also migrate toward ECM gradients in a haptotactic manner, and that this cell population may decrease due to infection by oncolytic viruses. Infected cancer cells also move randomly and they can die owing to infection; at the same time, the infected cell population can increase as a result of infection with the oncolytic virus. As the infected cancer cells die, the virus particles inside them are released. These free viruses diffuse randomly and undergo natural decay, and their number is further reduced while infecting cancer cells. Finally, (1.1) presupposes that the ECM does not move nor diffuse, but can be degraded by both type of cancer cells upon contact.

From the viewpoint of mathematical analysis, models of this type with haptotactic interaction considerably differ from classical reaction-diffusion equations, predominantly due to the cross-diffusive coupling of the key variable u to the quantity v which due to absence of diffusion apparently lacks any significant regularization during evolution. Accordingly, existing analytical works concerning haptotaxis systems have mainly been focusing on issues from basic global solvability theory ([16], [22], [20], [9], [8], [10], [6], [12]), and only a few rigorous results have addressed aspects related to qualitative behavior or structure of solutions ([4], [5], [11], [19]).

In particular, methodological limitations seem to widely restrict analytical access to phenomena related to singularity formation, and hence also to the detection of key parameters in this regard, and to the identification of possibly critical values thereof. A partial exception has recently been achieved in [14], where a certain critical mass phenomenon for (1.1) has been discovered in the case when $\rho=0$ and the virus production rate β satisfies $\beta>1$; more precisely, within this parameter regime any reasonably regular initial data satisfying $\frac{1}{|\Omega|}\int_\Omega u_0 > \frac{1}{\beta-1}$ has been found in [14] to evolve into a classical solution which is global in time but blows up in the large time limit in the sense that $\limsup_{t\to\infty}(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} + \|z(\cdot,t)\|_{L^\infty(\Omega)}) = \infty$. In said study, this unboundedness detection is complemented by a result on boundedness in the case when $\frac{1}{|\Omega|}\int_\Omega u_0 < \frac{1}{(\beta-1)_+}$, thus inter alia covering arbitrarily large data when $\beta\leq 1$, under the essential hypothesis that $v_0\equiv 0$.

Main results. The purpose of the present work consists in extending the latter statement on globally regular solution behavior to cases genuinely involving haptotactic interaction, thus allowing for nontrivial choices of v_0, and to the biologically most relevant constellation in which the parameter ρ is allowed to attain arbitrary nonnegative values. To achieve this in contexts free of size restrictions on the initial data, we shall concentrate on the case of suitably small β by assuming that $\beta < 1$, and henceforth accordingly consider (1.1) under the mere assumptions that

$$\begin{cases} u_0, v_0 \text{ and } w_0 \text{ are nonnegative functions from } C^{2+\vartheta}(\overline{\Omega}) \text{ for some } \vartheta > 0, \\ \text{with } u_0 > 0,\, w_0 \not\equiv 0,\, z_0 \not\equiv 0,\, \sqrt{v_0} \in W^{1,2}(\Omega) \text{ and } \frac{\partial u_0}{\partial \nu} = \frac{\partial v_0}{\partial \nu} = \frac{\partial w_0}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.2}$$

Then the outcome of [13] asserts the existence of a unique classical solution which is global in time. The main objective of this study is to make sure that in the considered range of parameters, this solution in fact is globally bounded and approaches a semi-trivial homogeneous equilibrium in the large time limit. Our derivation of this will be launched by the observation that when $\beta < 1$, w and z enjoy some basic boundedness and decay features (Lemma 3.1), and by a result on uniform positivity of u obtained in [15] (Lemma 2.2). At several stages relying on a favorable decay property of v implied by the latter (Lemma 2.3), namely, through a series of bootstrap arguments based on parabolic regularization, both in the semilinear equations for w and z from (1.1) and in a quasilinear problem satisfied by ue^{-v} (cf. (4.2) below), we shall see that these features indeed entail globally smooth behavior and asymptotic relaxation in the following sense:

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and let*

$$\beta \in (0,1).$$

Then for any choice of (u_0, v_0, w_0, z_0) fulfilling (1.2), the solution (u, v, w, z) of (1.1) is bounded in the sense that

$$\sup_{t>0} \Big\{ \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} + \|z(\cdot,t)\|_{L^\infty(\Omega)} \Big\} < \infty, \tag{1.3}$$

and apart from that there exists $u_\infty > 0$ such that

$$u(\cdot,t) \to u_\infty \quad \text{in } L^p(\Omega) \qquad \text{for all } p \ge 1$$

and

$$v(\cdot,t) \to 0 \qquad \text{in } L^\infty(\Omega)$$

as well as

$$w(\cdot,t) \to 0 \qquad \text{in } L^\infty(\Omega)$$

and

$$z(\cdot,t) \to 0 \qquad \text{in } L^\infty(\Omega)$$

as $t \to \infty$.

2 Preliminaries

Let us first recall that the outcome of [13] warrants global smooth solvability, as well as two basic solution properties, in the following sense.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, let $\beta > 0$, and suppose that (u_0, v_0, w_0, z_0) satisfies (1.2). Then the problem (1.1) possesses a uniquely determined classical solution $(u, v, w, z) \in (C^{2,1}(\overline{\Omega} \times [0, \infty)))^4$ for which v is nonnegative, and for which u, w and z are positive in $\overline{\Omega} \times (0, \infty)$. Moreover,*

$$\int_\Omega u(\cdot, t) \le \int_\Omega u_0 \qquad \textit{for all } t > 0, \tag{2.1}$$

and for any choice of $t_0 \ge 0$ we have

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \le \|v(\cdot, t_0)\|_{L^\infty(\Omega)} \qquad \textit{for all } t > t_0. \tag{2.2}$$

From now on assuming the hypotheses of Theorem 1.1 to hold, without explicit further mentioning we let (u, v, w, z) denote the correspondingly obtained solution of (1.1) from Lemma 2.1.

Thus in particular supposing henceforth that $\beta < 1$, we may subsequently rely on a pointwise lower bound for u that has already been derived in [15, Theorem 1.1], and that will be of fundamental importance for our subsequent analysis:

Lemma 2.2 *There exists $C > 0$ such that*

$$u(x, t) \ge C \qquad \textit{for all } x \in \Omega \textit{ and } t > 0. \tag{2.3}$$

In view of the second equation in (1.1), this information immediately implies the following uniform decay property of the haptoattractant concentration:

Lemma 2.3 *We have*

$$v(\cdot, t) \to 0 \quad \textit{in } L^\infty(\Omega) \qquad \textit{as } t \to \infty. \tag{2.4}$$

Proof. Taking $c_1 > 0$ such that in accordance with Lemma 2.2 we have $u \ge c_1$ in $\Omega \times (0, \infty)$, from (1.1) and the nonnegativity of v and w we obtain that

$$v_t = -(u + w)v \le -c_1 v \qquad \text{in } \Omega \times (0, \infty).$$

Therefore,

$$v(x, t) \le v_0(x) e^{-c_1 t} \le \|v_0(\cdot)\|_{L^\infty(\Omega)} e^{-c_1 t} \qquad \text{for all } x \in \Omega \text{ and } t > 1,$$

from which (2.4) follows. □

3 Boundedness and decay of w and z in L^1. L^p bounds for z

Once more making explicit use of our smallness assumption on β, independently of the above one can readily derive some elementary L^1 boundedness and relaxation properties of the solution components w and z:

Lemma 3.1 *We have*

$$w(\cdot,t) \to 0 \quad \textit{in } L^1(\Omega) \quad \textit{and} \quad z(\cdot,t) \to 0 \quad \textit{in } L^1(\Omega) \qquad \textit{as } t \to \infty \tag{3.1}$$

as well as

$$\int_0^\infty \int_\Omega z < \infty. \tag{3.2}$$

Proof. We use (1.1) to compute

$$\begin{aligned} \frac{d}{dt}\Big\{ \int_\Omega w + \int_\Omega z \Big\} &= \Big\{ -\int_\Omega w + \int_\Omega uz \Big\} + \Big\{ -\int_\Omega z - \int_\Omega uz + \beta \int_\Omega w \Big\} \\ &= -(1-\beta)\int_\Omega w - \int_\Omega z \qquad \text{for all } t>0, \end{aligned}$$

so that

$$\frac{d}{dt}\Big\{ \int_\Omega w + \int_\Omega z \Big\} + (1-\beta)\cdot \Big\{ \int_\Omega w + \int_\Omega z \Big\} = -\beta \int_\Omega z \le 0 \qquad \text{for all } t>0.$$

Upon integration, this readily implies both (3.1) and (3.2) due to the fact that $\beta<1$. $\Box$

Now parabolic smoothing estimates in the two-dimensional domain Ω ensure that the L^1 boundedness information implicitly contained in (3.1) entails bounds for z in L^p actually for arbitrarily large finite p:

Lemma 3.2 *Let $p\in[1,\infty)$. Then there exists $C(p)>0$ such that*

$$\|z(\cdot,t)\|_{L^p(\Omega)} \le C(p) \qquad \textit{for all } t>0. \tag{3.3}$$

Proof. According to well-known smoothing properties of the Neumann heat semigroup $(e^{\sigma\Delta})_{\sigma\ge 0}$ ([17]), given $p\ge 1$ we can find $c_1=c_1(p)>0$ such that

$$\|e^{D_z\sigma\Delta}\varphi\|_{L^p(\Omega)} \le c_1\cdot (1+\sigma^{-1+\frac{1}{p}})\|\varphi\|_{L^1(\Omega)} \qquad \text{for all } t>0 \text{ and } \varphi\in C^0(\overline{\Omega}), \tag{3.4}$$

whereas Lemma 3.1 provides $c_2>0$ fulfilling

$$\|w(\cdot,t)\|_{L^1(\Omega)} \le c_2 \qquad \text{for all } t>0. \tag{3.5}$$

By nonnegativity of uz and z, in view of the order preserving property of $e^{\sigma\Delta}$ for $\sigma\ge 0$ we firstly obtain that

$$\begin{aligned} z(\cdot,t) &= e^{t(D_z\Delta-1)}z_0 - \int_0^t e^{(t-s)(D_z\Delta-1)}u(\cdot,s)z(\cdot,s)ds + \beta\int_0^t e^{(t-s)(D_z\Delta-1)}w(\cdot,s)ds \\ &\le e^{tD_z\Delta}z_0 + \beta\int_0^t e^{(t-s)(D_z\Delta-1)}w(\cdot,s)ds \quad \text{in } \Omega \qquad \text{for all } t>0, \end{aligned}$$

and hence, by (3.4) and (3.5), secondly conclude that

$$\begin{aligned}\|z(\cdot,t)\|_{L^p(\Omega)} &\le \|e^{tD_z\Delta}z_0\|_{L^p(\Omega)} + c_1\beta\int_0^t \Big(1+(t-s)^{-1+\frac{1}{p}}\Big)e^{-(t-s)}\|w(\cdot,s)\|_{L^1(\Omega)}ds \\ &\le \|z_0\|_{L^p(\Omega)} + c_1c_2\beta\int_0^t \Big(1+(t-s)^{-1+\frac{1}{p}}\Big)e^{-(t-s)}ds \qquad \text{for all } t>0.\end{aligned}$$

As

$$\int_0^t \Big(1+(t-s)^{-1+\frac{1}{p}}\Big)e^{-(t-s)}ds \le \int_0^\infty (1+\sigma^{-1+\frac{1}{p}})e^{-\sigma}d\sigma \qquad \text{for all } t>0,$$

this already establishes (3.3). □

4 A basic evolution property of $a := ue^{-v}$

Following a variable change widely used in the analysis of haptotaxis-related systems ([2], [3], [16] and [9]), we introduce

$$a := ue^{-v} \tag{4.1}$$

and then observe that according to (1.1),

$$\begin{cases} a_t = e^{-v}\nabla\cdot(e^v\nabla a) + a(ae^v+w)v - \rho az, & x\in\Omega,\ t>0,\\ v_t = -(ae^v+w)v, & x\in\Omega,\ t>0,\\ \frac{\partial a}{\partial\nu}=0, & x\in\partial\Omega,\ t>0,\\ a(x,0)=u_0(x)e^{-v_0(x)} =: a_0(x), \quad v(x,0)=v_0(x), & x\in\Omega. \end{cases} \tag{4.2}$$

A straightforward testing procedure therefore yields the following basic inequality that will be relied on in Lemma 5.3, Lemma 6.2, Lemma 6.4 and Lemma 6.5 below.

Lemma 4.1 *Let $p>0$ be such that $p\ne 1$. Then*

$$\frac{d}{dt}\int_\Omega e^v a^p \le -p(p-1)\int_\Omega e^v a^{p-2}|\nabla a|^2 + (p-1)\int_\Omega e^v a^p(ae^v+w)v \qquad \textit{for all } t>0. \tag{4.3}$$

Proof. Since (4.2) implies that $(ae^v+w)v = -v_t$ in $\Omega\times(0,\infty)$ and

$$\begin{aligned}\frac{d}{dt}\int_\Omega e^v a^p &= p\int_\Omega e^v a^{p-1}\cdot\Big\{e^{-v}\nabla\cdot(e^v\nabla a) - av_t - \rho az\Big\} + \int_\Omega e^v a^p v_t \\ &= -p(p-1)\int_\Omega e^v a^{p-2}|\nabla a|^2 - (p-1)\int_\Omega e^v a^p v_t - \rho p\int_\Omega e^v a^p z \qquad \text{for all } t>0,\end{aligned}$$

this is evident upon discarding the rightmost nonpositive summand herein. □

5 A space-time L^r bound for a with arbitrary $r < 2$

As a preparation for establishing an a priori estimate for a in L^∞, this section aims at providing a bound for the space-time integral $\int_t^{t+1}\int_\Omega a^r$ with arbitrary $r < 2$ (Lemma 5.4). In a first step toward this, we firstly enlarge our knowledge on boundedness features of w by means of a corresponding L^p testing procedure, which due to the uniform L^p boundedness of w implied by Lemma 3.1 yields the following.

Lemma 5.1 *For all $p \in (0,1)$, one can find $C(p) > 0$ such that*

$$\int_t^{t+1}\int_\Omega |\nabla w^{\frac{p}{2}}|^2 \le C(p) \qquad \textit{for all } t > 0. \tag{5.1}$$

PROOF. Using that w is positive in $\overline{\Omega} \times (0,\infty)$, we may integrate by parts in (1.1) to see that

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega w^p &= (1-p)D_w\int_\Omega w^{p-2}|\nabla w|^2 - \int_\Omega w^p + \int_\Omega uw^{p-1}z \\
&\ge (1-p)D_w\int_\Omega w^{p-2}|\nabla w|^2 - \int_\Omega w^p \qquad \text{for all } t > 0.
\end{aligned}$$

Therefore, by Young's inequality,

$$\begin{aligned}
\frac{4(1-p)D_w}{p^2}\int_t^{t+1}\int_\Omega |\nabla w^{\frac{p}{2}}|^2 &\le \frac{1}{p}\int_\Omega w^p(\cdot,t+1) - \frac{1}{p}\int_\Omega w^p(\cdot,t) + \int_t^{t+1}\int_\Omega w^p \\
&\le \frac{1}{p}\int_\Omega \Big(w(\cdot,t+1)+1\Big) + \int_t^{t+1}\int_\Omega (w+1) \qquad \text{for all } t > 0,
\end{aligned}$$

so that (5.1) is a consequence of Lemma 3.1. $\square$

By means of straightforward interpolation, this readily entails a space-time L^r bound for w with any given $r < 2$.

Lemma 5.2 *If $p \in (0,1)$, then with some $C(p) > 0$ we have*

$$\int_t^{t+1}\int_\Omega w^{p+1} \le C(p) \qquad \textit{for all } t > 0. \tag{5.2}$$

PROOF. By means of the Gagliardo-Nirenberg inequality, we can find $c_1 = c_1(p) > 0$ such that

$$\int_\Omega w^{p+1} = \|w^{\frac{p}{2}}\|_{L^{\frac{2(p+1)}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \le c_1\|\nabla w^{\frac{p}{2}}\|_{L^2(\Omega)}^2 \|w^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{p}} + c_1\|w^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \qquad \text{for all } t > 0.$$

Since $\|w^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{p}} = \int_\Omega w$ for all $t > 0$, in view of Lemma 5.1 and Lemma 3.1 this implies (5.2) upon an integration in time. $\square$

Along with the decay property of v identified in Lemma 2.3, the latter can be used to enrich our information on regularity of a, firstly quite in the style of that from Lemma 5.1.

Lemma 5.3 *Given any $p \in (0,1)$, one can find $C(p) > 0$ such that*

$$\int_t^{t+1} \int_\Omega |\nabla a^{\frac{p}{2}}|^2 \le C(p) \qquad \text{for all } t > 0. \tag{5.3}$$

PROOF. We abbreviate $c_1 := \|v_0\|_{L^\infty(\Omega)}$ and again invoke the Gagliardo-Nirenberg inequality to see that since $\|a^{\frac{p}{2}}\|^{\frac{2}{p}}_{L^{\frac{2}{p}}(\Omega)} = \int_\Omega a = \int_\Omega ue^{-v} \le \int_\Omega u = \int_\Omega u_0$ for all $t > 0$ by (4.1) and (2.1), there exist $c_2 = c_2(p) > 0$ and $c_3 = c_3(p) > 0$ such that

$$\begin{aligned}
\int_\Omega a^{p+1} &= \|a^{\frac{p}{2}}\|^{\frac{2(p+1)}{p}}_{L^{\frac{2(p+1)}{p}}(\Omega)} \\
&\le c_2 \|\nabla a^{\frac{p}{2}}\|^2_{L^2(\Omega)} \|a^{\frac{p}{2}}\|^{\frac{2}{p}}_{L^{\frac{2}{p}}(\Omega)} + c_2 \|a^{\frac{p}{2}}\|^{\frac{2(p+1)}{p}}_{L^{\frac{2}{p}}(\Omega)} \\
&\le c_3 \|\nabla a^{\frac{p}{2}}\|^2_{L^2(\Omega)} + c_3 \\
&= \frac{p^2 c_3}{4} \int_\Omega a^{p-2} |\nabla a|^2 + c_3 \qquad \text{for all } t > 0.
\end{aligned} \tag{5.4}$$

We thereupon make use of Lemma 2.3 to pick $t_0 = t_0(p) > 0$ suitably large such that

$$(e^{2c_1} + e^{c_1}) \|v(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{2}{pc_3} \qquad \text{for all } t > t_0, \tag{5.5}$$

and recall that by Lemma 4.1,

$$-\frac{1}{1-p} \frac{d}{dt} \int_\Omega e^v a^p + p \int_\Omega e^v a^{p-2} |\nabla a|^2 \le \int_\Omega e^{2v} a^{p+1} v + \int_\Omega e^v a^p vw \qquad \text{for all } t > 0, \tag{5.6}$$

where clearly

$$p \int_\Omega e^v a^{p-2} |\nabla a|^2 \ge p \int_\Omega a^{p-2} |\nabla a|^2 \qquad \text{for all } t > 0. \tag{5.7}$$

Furthermore, by Young's inequality,

$$\begin{aligned}
\int_\Omega e^v a^p vw &\le \int_\Omega e^v a^{p+1} v + \int_\Omega e^v v w^{p+1} \\
&\le e^{c_1} \int_\Omega a^{p+1} v + c_1 e^{c_1} \int_\Omega w^{p+1} \qquad \text{for all } t > 0,
\end{aligned}$$

so that

$$\int_\Omega e^{2v} a^{p+1} v + \int_\Omega e^v a^p vw \le (e^{2c_1} + e^{c_1}) \int_\Omega a^{p+1} v + c_1 e^{c_1} \int_\Omega w^{p+1} \qquad \text{for all } t > 0. \tag{5.8}$$

Here we apply (5.4) along with (5.5) to estimate

$$\begin{aligned}
(e^{2c_1} + e^{c_1}) \int_\Omega a^{p+1} v &\le (e^{2c_1} + e^{c_1}) \|v\|_{L^\infty(\Omega)} \cdot \left\{ \frac{p^2 c_3}{4} \int_\Omega a^{p-2} |\nabla a|^2 + c_3 \right\} \\
&\le \frac{2}{pc_3} \cdot \left\{ \frac{p^2 c_3}{4} \int_\Omega a^{p-2} |\nabla a|^2 + c_3 \right\} \\
&= \frac{p}{2} \int_\Omega a^{p-2} |\nabla a|^2 + \frac{2}{p} \qquad \text{for all } t > t_0,
\end{aligned}$$

so that (5.6), (5.7) and (5.8) entail that

$$-\frac{1}{1-p}\frac{d}{dt}\int_\Omega e^v a^p + \frac{p}{2}\int_\Omega a^{p-2}|\nabla a|^2 \le \frac{2}{p} + c_1 e^{c_1}\int_\Omega w^{p+1} \qquad \text{for all } t > t_0.$$

Thus,

$$\begin{aligned}\frac{p}{2}\int_t^{t+1}\int_\Omega a^{p-2}|\nabla a|^2 \quad &\le \quad \frac{1}{1-p}\int_\Omega e^{v(\cdot,t+1)}a^p(\cdot,t+1) - \frac{1}{1-p}\int_\Omega e^{v(\cdot,t)}a^p(\cdot,t) \\ &\quad + \frac{2}{p} + c_1 e^{c_1}\int_t^{t+1}\int_\Omega w^{p+1} \qquad \text{for all } t > t_0,\end{aligned}$$

so that thanks to Lemma 5.2, (5.3) results upon observing that once more due to Young's inequality, (4.1) and (2.1),

$$\int_\Omega e^v a^p \le \int_\Omega e^v \cdot (a+1) = \int_\Omega (u + e^v) \le \int_\Omega u_0 + e^{c_1}|\Omega|$$

for all $t > 0$. □

The main goal of this section can now be achieved by once more using a simple interpolation argument.

Lemma 5.4 *For all $p \in (0,1)$ there exists $C(p) > 0$ satisfying*

$$\int_t^{t+1}\int_\Omega a^{p+1} \le C(p) \qquad \textit{for all } t > 0. \tag{5.9}$$

Proof. In much the same manner as in the proof of Lemma 5.2, this can be derived by combining Lemma 5.3 with (4.1) and (2.1), because a is already known to be bounded in $\overline{\Omega} \times [0, t_0]$ due to Lemma 2.1. □

6 L^∞ estimates. Proof of Theorem 1.1

Forming a next step in our bootstrap procedure, the following combines Lemma 5.4 with Lemma 3.2 and standard parabolic regularity theory to improve the topological framework within we know w to remain bounded:

Lemma 6.1 *Let $p \in (2,\infty)$. Then there exists $C(p) > 0$ such that*

$$\|w(\cdot,t)\|_{L^p(\Omega)} \le C(p) \qquad \textit{for all } t > 0. \tag{6.1}$$

Proof. Given $p > 2$, we fix any $r \in (1,2)$ such that $r > \frac{2p}{p+1}$, which ensures that $\frac{(p+1)r-p}{pr} > \frac{1}{r}$ and that hence we can choose some $q > 1$ suitably close to r such that

$$\frac{(p+1)r-p}{pr} > \frac{1}{q} > \frac{1}{r}. \tag{6.2}$$

We next recall known smoothing estimates for the Neumann heat semigroup on Ω ([17]) to pick positive constants $c_1 = c_1(p)$ and $c_2 = c_2(p)$ fulfilling

$$\|e^{D_w\Delta}\varphi\|_{L^p(\Omega)} \le c_1\|\varphi\|_{L^1(\Omega)} \qquad \text{for all } \varphi \in C^0(\overline{\Omega}) \tag{6.3}$$

and

$$\|e^{\sigma D_w\Delta}\varphi\|_{L^p(\Omega)} \le c_2\sigma^{-\kappa}\|\varphi\|_{L^q(\Omega)} \qquad \text{for all } \sigma\in(0,1) \text{ and each } \varphi\in C^0(\overline{\Omega}), \tag{6.4}$$

and note that the number $\kappa := \frac{1}{q}-\frac{1}{p}$ herein satisfies

$$\frac{r}{r-1}\cdot\kappa < \frac{r}{r-1}\cdot\Big(\frac{(p+1)r-p}{pr}-\frac{1}{p}\Big) = 1 \tag{6.5}$$

due to the left inequality in (6.2).

Now relying on a variation-of-constants representation of w related to the third equation in (1.1), we can utilize (6.3) and (6.4) to estimate

$$\begin{aligned} \|w(\cdot,t)\|_{L^p(\Omega)} &= \Big\|e^{D_w\Delta-1}w(\cdot,t-1)+\int_{t-1}^t e^{(t-s)(D_w\Delta-1)}u(\cdot,s)z(\cdot,s)ds\Big\|_{L^p(\Omega)} \\ &\le e^{-1}\cdot c_1\|w(\cdot,t-1)\|_{L^1(\Omega)} + c_2\int_{t-1}^t (t-s)^{-\kappa}\|u(\cdot,s)z(\cdot,s)\|_{L^q(\Omega)}ds \end{aligned} \tag{6.6}$$

for all $t>1$, where according to the Hölder inequality, Lemma 3.2 and (6.5), with some $c_3=c_3(p)>0$ and $c_4=c_4(p)>0$ we have

$$\begin{aligned} c_2\int_{t-1}^t (t-s)^{-\kappa}\|u(\cdot,s)z(\cdot,s)\|_{L^q(\Omega)}ds &\le c_2\int_{t-1}^t (t-s)^{-\kappa}\|u(\cdot,s)\|_{L^r(\Omega)}\|z(\cdot,s)\|_{L^{\frac{qr}{r-q}}(\Omega)}ds \\ &\le c_3\int_{t-1}^t (t-s)^{-\kappa}\|u(\cdot,s)\|_{L^r(\Omega)}ds \\ &\le c_3\cdot\Big\{\int_{t-1}^t (t-s)^{-\frac{r}{r-1}\cdot\kappa}ds\Big\}^{\frac{r-1}{r}}\cdot\Big\{\int_{t-1}^t \|u(\cdot,s)\|_{L^r(\Omega)}^r ds\Big\}^{\frac{1}{r}} \\ &\le c_4\cdot\Big\{\int_{t-1}^t \|u(\cdot,s)\|_{L^r(\Omega)}^r ds\Big\}^{\frac{1}{r}} \qquad \text{for all } r>1. \end{aligned}$$

Since

$$\int_{t-1}^t \|u(\cdot,s)\|_{L^r(\Omega)}^r ds = \int_{t-1}^t\int_\Omega (ae^v)^r \le e^{r\|v_0\|_{L^\infty(\Omega)}}\int_{t-1}^t\int_\Omega a^r \qquad \text{for all } t>1$$

by (4.1) and (2.2), and since

$$\sup_{t>1}\|w(\cdot,t-1)\|_{L^1(\Omega)}<\infty \qquad \text{and} \qquad \sup_{t>1}\int_{t-1}^t\int_\Omega a^r<\infty$$

according to Lemma 3.1, Lemma 5.4 and our restriction $r<2$, (6.6) implies (6.1), because w is bounded in $\overline{\Omega}\times[0,1]$ by Lemma 2.1. □

Once more thanks to the uniform decay of v, the latter can now be used to establish an L^2 bound for a on the basis of Lemma 4.1.

Lemma 6.2 *There exists $C > 0$ fulfilling*

$$\|a(\cdot,t)\|_{L^2(\Omega)} \le C \qquad \textit{for all } t > 0. \tag{6.7}$$

Proof. We first employ the Gagliardo-Nirenberg inequality along with (2.1) to see that since $a \le u$, with some $c_1 > 0$ and $c_2 > 0$ we have

$$\begin{aligned} \int_\Omega a^3 &\le c_1 \|\nabla a\|_{L^2(\Omega)}^2 \|a\|_{L^1(\Omega)} + c_1 \|a\|_{L^1(\Omega)}^3 \\ &\le c_2 \int_\Omega |\nabla a|^2 + c_2 \qquad \text{for all } t > 0, \end{aligned} \tag{6.8}$$

and abbreviating $c_3 := \|v_0\|_{L^\infty(\Omega)}$ we may rely on a Poincaré-type inequality in fixing $c_4 > 0$ fulfilling

$$\frac{1}{2} e^{c_3} \int_\Omega a^2 \le \frac{1}{2} \int_\Omega |\nabla a|^2 + c_4 \qquad \text{for all } t > 0. \tag{6.9}$$

We moreover recall Lemma 2.3 to find $t_0 > 0$ such that

$$(e^{2c_3} + 1) \|v(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{1}{2c_2} \qquad \text{for all } t > t_0, \tag{6.10}$$

and apply Lemma 6.1 to see that there exists $c_5 > 0$ satisfying

$$\int_\Omega w^3(\cdot,t) \le c_5 \qquad \text{for all } t > 0. \tag{6.11}$$

Now going back to Lemma 4.1, we obtain the inequality

$$\frac{1}{2}\frac{d}{dt} \int_\Omega e^v a^2 + \int_\Omega e^v |\nabla a|^2 + \frac{1}{2} \int_\Omega e^v a^2 \le \int_\Omega e^{2v} a^3 v + \int_\Omega e^v a^2 v w + \frac{1}{2} \int_\Omega e^v a^2 \qquad \text{for all } t > 0, \tag{6.12}$$

where by Young's inequality, (6.11) and (6.8),

$$\begin{aligned} \int_\Omega e^{2v} a^3 v + \int_\Omega e^v a^2 v w &\le \int_\Omega e^{2v} a^3 v + \int_\Omega a^3 v + \int_\Omega e^{3v} v w^3 \\ &\le (e^{2c_3} + 1) \|v\|_{L^\infty(\Omega)} \cdot \int_\Omega a^3 + c_3 e^{3c_3} \int_\Omega w^3 \\ &\le \frac{1}{2c_2} \int_\Omega a^3 + c_3 c_5 e^{3c_3} \\ &\le \frac{1}{2} \int_\Omega |\nabla a|^2 + c_6 \qquad \text{for all } t > t_0 \end{aligned}$$

with $c_6 := \frac{1}{2} + c_3 c_5 e^{3c_3}$. As (6.9) ensures that

$$\frac{1}{2} \int_\Omega e^v a^2 \le \frac{1}{2} e^{c_3} \int_\Omega a^2 \le \frac{1}{2} \int_\Omega |\nabla a|^2 + c_4 \qquad \text{for all } t > 0,$$

estimating $\int_\Omega e^v|\nabla a|^2 \ge \int_\Omega |\nabla a|^2$ we thus infer from (6.12) that

$$\frac{d}{dt}\int_\Omega e^v a^2 + \int_\Omega e^v a^2 \le 2(c_4+c_6) \qquad \text{for all } t>t_0,$$

upon an ODE comparison implying that

$$\int_\Omega e^v a^2 \le \max\left\{ \int_\Omega e^{v(\cdot,t_0)} a^2(\cdot,t_0)\,,\, 2(c_4+c_6)\right\} \qquad \text{for all } t>t_0.$$

Again since $e^v \ge 1$, and since a is bounded in $\overline{\Omega}\times[0,t_0]$ by Lemma 2.1, the proof thereby becomes complete. $\square$

Again through regularization in semilinear heat equations, this already implies Hölder bounds for w and z, and in conjunction with the weak decay information in (3.1) hence also uniform decay of both these quantities.

Lemma 6.3 *There exist $\theta\in(0,1)$ and $C>0$ such that*

$$\|w(\cdot,t)\|_{C^\theta(\overline{\Omega})} \le C \qquad \textit{for all } t>0 \tag{6.13}$$

and

$$\|z(\cdot,t)\|_{C^\theta(\overline{\Omega})} \le C \qquad \textit{for all } t>0, \tag{6.14}$$

and moreover

$$w(\cdot,t)\to 0 \quad \textit{and} \quad z(\cdot,t)\to 0 \quad \textit{in } L^\infty(\Omega) \qquad \textit{as } t\to\infty. \tag{6.15}$$

PROOF. To derive (6.13), we take any $p>2$ and $q\in(1,2)$ such that $q>\frac{2p}{p+2}$, and combine Lemma 6.2, Lemma 3.1 and Lemma 3.2 with (4.1) and (2.2) to fix positive constants c_1, c_2 and c_3 fulfilling

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le c_1, \quad \|w(\cdot,t)\|_{L^1(\Omega)} \le c_2 \quad \text{and} \quad \|z(\cdot,t)\|_{L^{\frac{2q}{2-q}}(\Omega)} \le c_3 \qquad \text{for all } t>0.$$

Since well-known theory of the Neumann problem for the heat equation ([17]) provides $c_4>0$ and $c_5>0$ such that

$$\|e^{D_w\Delta}\varphi\|_{W^{1,p}(\Omega)} \le c_4\|\varphi\|_{L^1(\Omega)} \qquad \text{for all } \varphi\in C^0(\overline{\Omega})$$

and

$$\|e^{\sigma D_w\Delta}\varphi\|_{W^{1,p}(\Omega)} \le c_5\sigma^{-\kappa}\|\varphi\|_{L^q(\Omega)} \qquad \text{for all } \sigma\in(0,1) \text{ and any } \varphi\in C^0(\overline{\Omega})$$

with $\kappa := \frac{1}{2}+\frac{1}{q}-\frac{1}{p}$, by means of a Duhamel representation of w we thus obtain that due to the Hölder inequality,

$$\begin{aligned}
\|w(\cdot,t)\|_{W^{1,p}(\Omega)} &= \left\| e^{D_w\Delta-1}w(\cdot,t-1) + \int_{t-1}^t e^{(t-s)(D_w\Delta-1)}u(\cdot,s)z(\cdot,s)ds\right\|_{W^{1,p}(\Omega)} \\
&\le c_4e^{-1}\|w(\cdot,t-1)\|_{L^1(\Omega)} + c_5\int_{t-1}^t (t-s)^{-\kappa}\|u(\cdot,s)z(\cdot,s)\|_{L^q(\Omega)}ds \\
&\le c_4e^{-1}\|w(\cdot,t-1)\|_{L^1(\Omega)} + c_5\int_{t-1}^t (t-s)^{-\kappa}\|u(\cdot,s)\|_{L^2(\Omega)}\|z(\cdot,s)\|_{L^{\frac{2q}{2-q}}(\Omega)}ds \\
&\le c_2c_4e^{-1} + c_1c_3c_5\int_{t-1}^t (t-s)^{-\kappa}ds \qquad \text{for all } t>1.
\end{aligned}$$

Using that $\kappa < \frac{1}{2} + \frac{p+2}{2p} - \frac{1}{p} = 1$, from this together with Lemma 2.1 we hence infer the existence of $c_6 > 0$ such that

$$\|w(\cdot,t)\|_{W^{1,p}(\Omega)} \le c_6 \qquad \text{for all } t>0, \tag{6.16}$$

which for any choice of $\theta \in (0, 1-\frac{2}{p})$ entails (6.13) due to the fact that then $W^{1,p}(\Omega) \hookrightarrow C^\theta(\overline{\Omega})$. Apart from that, noting that the Gagliardo-Nirenberg inequality yields $c_7 > 0$ such that

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} \le c_7 \|w(\cdot,t)\|_{W^{1,p}(\Omega)}^{\frac{2p}{3p-2}} \|w(\cdot,t)\|_{L^1(\Omega)}^{\frac{p-2}{3p-2}} \qquad \text{for all } t>0,$$

we moreover conclude from (6.16) that as a consequence of (3.1) we then in fact have

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } t\to\infty.$$

The statements concerning z can be derived in quite a similar manner. $\square$

On the basis of a Moser-type iteration procedure, launched at the starting point information provided by Lemma 6.2 and again relying on Lemma 4.1, we can finally make sure that a in fact is bounded:

Lemma 6.4 *There exists $C>0$ such that*

$$\|a(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t>0. \tag{6.17}$$

PROOF. For nonnegative integers k, we let $p_k := 2^{k+1}$ and

$$M_k(T) := \max\Big\{1\,,\ \sup_{t\in(0,T)} \int_\Omega a^{p_k}(\cdot,t)\Big\}, \qquad T>0,$$

as well as $\overline{M}_k := \sup_{T>0} M_k(T)$, so that $\overline{M}_0$ is finite by Lemma 6.2. To estimate $M_k(T)$ for $k \ge 1$, given any such k we use (4.3) with $p := p_k$ to see that since $p(p-1)e^v \ge p \cdot \frac{3p}{4}$,

$$\begin{aligned}
\frac{d}{dt}\int_\Omega e^v a^p + 3\int_\Omega |\nabla a^{\frac{p}{2}}|^2 + \int_\Omega e^v a^p &\le \frac{d}{dt}\int_\Omega e^v a^p + p(p-1)\int_\Omega e^v a^{p-2}|\nabla a|^2 + \int_\Omega e^v a^p \\
&\le (p-1)\int_\Omega e^{2v} a^{p+1} v + (p-1)\int_\Omega e^v a^p v w + \int_\Omega e^v a^p
\end{aligned} \tag{6.18}$$

for all $t>0$. Here in accordance with (2.2) and Lemma 6.3, we pick $c_1>0$ and $c_2>0$ such that $v \le c_1$ and $w \le c_2$ on $\Omega\times(0,\infty)$, and use Young's inequality to see that

$$(p-1)\int_\Omega e^{2v} a^{p+1} v \le p c_1 e^{2c_1} \int_\Omega a^{p+1} \qquad \text{for all } t>0$$

and

$$\begin{aligned}
(p-1)\int_\Omega e^v a^p v w &\le p c_1 c_2 e^{c_1} \int_\Omega a^p \\
&\le p c_1 c_2 e^{c_1} \cdot \Big\{ \frac{p}{p+1}\int_\Omega a^{p+1} + \frac{|\Omega|}{p+1} \Big\} \\
&\le p c_1 c_2 e^{c_1} \int_\Omega a^{p+1} + c_1 c_2 e^{c_1} |\Omega| \qquad \text{for all } t>0
\end{aligned}$$

as well as

$$\int_\Omega e^v a^p \le e^{c_1} \int_\Omega a^p \le e^{c_1} \int_\Omega a^{p+1} + e^{c_1}|\Omega| \qquad \text{for all } t>0,$$

so that since $p \ge 1$, (6.18) implies that

$$\frac{d}{dt} \int_\Omega e^v a^p + 3\int_\Omega |\nabla a^{\frac{p}{2}}|^2 + \int_\Omega e^v a^p \le c_3 p \int_\Omega a^{p+1} + c_4 \qquad \text{for all } t>0 \tag{6.19}$$

with $c_3 := c_1 e^{2c_1} + c_1 c_2 e^{c_1} + e^{c_1}$ and $c_4 := c_1 c_2 e^{c_1}|\Omega| + e^{c_1}|\Omega$. Now to estimate the integral on the right hand side herein by an interpolation appropriately revealing possible dependences on $p = p_k$, let us first invoke the Gagliardo-Nirenberg inequality to pick $c_5 \ge \frac{1}{2}$ such that

$$\|\varphi\|_{L^4(\Omega)} \le c_5 \|\nabla\varphi\|_{L^2(\Omega)}^{\frac{3}{4}} \|\varphi\|_{L^1(\Omega)}^{\frac{1}{4}} + c_5\|\varphi\|_{L^1(\Omega)} \qquad \text{for all } \varphi \in W^{1,2}(\Omega).$$

Applying this together with the Hölder inequality and Young's inequality, we obtain that

$$\begin{aligned}
c_3 p \int_\Omega a^{p+1} &= c_3 p \|a^{\frac{p}{2}}\|_{L^{\frac{2(p+1)}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \\
&\le c_3 p \|a^{\frac{p}{2}}\|_{L^4(\Omega)}^{\frac{4(p+2)}{3p}} \|a^{\frac{p}{2}}\|_{L^1(\Omega)}^{\frac{2(p-1)}{3p}} \\
&\le c_3 p \cdot \Big\{ c_5 \|\nabla a^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{3}{4}} \|a^{\frac{p}{2}}\|_{L^1(\Omega)}^{\frac{1}{4}} + c_5 \|a^{\frac{p}{2}}\|_{L^1(\Omega)} \Big\}^{\frac{4(p+2)}{3p}} \cdot \|a^{\frac{p}{2}}\|_{L^1(\Omega)}^{\frac{2(p-1)}{3p}} \\
&\le c_3 p \cdot (2c_5)^{\frac{4(p+2)}{3p}} \cdot \Big\{ \|\nabla a^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{p+2}{p}} \|a^{\frac{p}{2}}\|_{L^1(\Omega)}^{\frac{p+2}{3p}} + \|a^{\frac{p}{2}}\|_{L^1(\Omega)}^{\frac{4(p+2)}{3p}} \Big\} \cdot \|a^{\frac{p}{2}}\|_{L^1(\Omega)}^{\frac{2(p-1)}{3p}} \\
&\le c_6 p \|\nabla a^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{p+2}{p}} \|a^{\frac{p}{2}}\|_{L^1(\Omega)} + c_6 p \|a^{\frac{p}{2}}\|_{L^1(\Omega)}^{\frac{2(p+1)}{p}} \qquad \text{for all } t>0
\end{aligned}$$

with $c_6 := c_3 \cdot (2c_5)^2$, because $\frac{4(p+2)}{3p} \le \frac{4\cdot(4+2)}{3\cdot 4} = 2$ due to the fact that $p \ge 4$, and because $2c_5 \ge 1$.

Now given $T>0$, we may recall the definition of $(M_j(T))_{j\ge 0}$ to see that $\|a^{\frac{p}{2}}\|_{L^1(\Omega)} = \int_\Omega a^{p_{k-1}} \le M_{k-1}(T)$ for all $t \in (0,T)$, so that once more thanks to Young's inequality we infer that for all $t \in (0,T)$,

$$\begin{aligned}
c_3 p \int_\Omega a^{p+1} &\le c_6 p \|\nabla a^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{p+2}{p}} M_{k-1}(T) + c_6 p M_{k-1}^{\frac{2(p+1)}{p}}(T) \\
&= \Big\{ 3\|\nabla a^{\frac{p}{2}}\|_{L^2(\Omega)}^2 \Big\}^{\frac{p+2}{2p}} \cdot \Big\{ 3^{-\frac{p+2}{2p}} c_6 p M_{k-1}(T) \Big\} + c_6 p M_{k-1}^{\frac{2(p+1)}{p}}(T) \\
&\le 3\|\nabla a^{\frac{p}{2}}\|_{L^2(\Omega)}^2 + \Big\{ 3^{-\frac{p+2}{2p}} c_6 p M_{k-1}(T) \Big\}^{\frac{2p}{p-2}} + c_6 p M_{k-1}^{\frac{2(p+1)}{p}}(T) \\
&= 3\|\nabla a^{\frac{p}{2}}\|_{L^2(\Omega)}^2 + 3^{-\frac{p+2}{p-2}} c_6^{\frac{2p}{p-2}} p^{\frac{2p}{p-2}} M_{k-1}^{\frac{2p}{p-2}}(T) + c_6 p M_{k-1}^{\frac{2(p+1)}{p}}(T),
\end{aligned} \tag{6.20}$$

where again using that $p \ge 4$, we can easily estimate

$$3^{-\frac{p+2}{p-2}} c_6^{\frac{2p}{p-2}} \le c_7 := \max\{1\,,\, c_6^4\}$$

and $p^{\frac{2p}{p-2}} \le p^4$. As furthermore

$$c_6 p M_{k-1}^{\frac{2(p+1)}{p}}(T) \le c_6 p^4 M_{k-1}^{\frac{2p}{p-2}}(T) \qquad \text{and} \qquad c_4 \le c_4 p^4 M_{k-1}^{\frac{2p}{p-2}}(T)$$

due to the evident inequalities $p \ge 1$, $\frac{2(p+1)}{p} \le \frac{2p}{p-2}$ and $M_{k-1}(T) \ge 1$, inserting (6.20) into (6.19) thus shows that writing $c_8 := c_4 + c_6 + c_7$ we have

$$\frac{d}{dt}\int_\Omega e^v a^p + \int_\Omega e^v a^p \le c_8 p^4 M_{k-1}^{\frac{2p}{p-2}}(T) \qquad \text{for all } t \in (0,T)$$

and hence, by means of an ODE comparison and (4.1),

$$\int_\Omega e^v a^p \le \max\left\{ \int_\Omega u_0^p e^{-(p-1)v_0}\,,\, c_8 p^4 M_{k-1}^{\frac{2p}{p-2}}(T)\right\} \qquad \text{for all } t \in (0,T).$$

Consequently,

$$M_k(T) \le \max\left\{1\,,\, \int_\Omega u_0^{p_k}\,,\, c_8 p_k^4 M_{k-1}^{\frac{2p_k}{p_k-2}}(T)\right\} \qquad \text{for all } T>0 \text{ and } k \ge 1,$$

so that, in fact, $\overline{M}_k$ is finite for all $k \ge 1$, with

$$\overline{M}_k \le \max\left\{1\,,\, \int_\Omega u_0^{p_k}\,,\, c_8 p_k^4 \overline{M}_{k-1}^{\frac{2p_k}{p_k-2}}\right\} \qquad \text{for all } k \ge 1. \tag{6.21}$$

The remaining part is quite standard: If there exists $(k_j)_{j\in\mathbb{N}}$ such that $k_j \to \infty$ as $j \to \infty$ and

$$\overline{M}_{k_j} \le \max\left\{1\,,\, \int_\Omega u_0^{p_{k_j}}\right\} \qquad \text{for all } j \in \mathbb{N},$$

then clearly

$$\|a(\cdot,t)\|_{L^\infty(\Omega)} = \lim_{j\to\infty}\left\{\int_\Omega a^{p_{k_j}}(\cdot,t)\right\}^{\frac{1}{p_{k_j}}} \le \limsup_{j\to\infty}\left\{\int_\Omega a^{p_{k_j}}(\cdot,t)\right\}^{\frac{1}{p_{k_j}}} \le \max\left\{1\,,\, \|u_0\|_{L^\infty(\Omega)}\right\}$$

for all $t>0$, while otherwise we infer from (6.21) that there exists $b>1$ fulfilling

$$\overline{M}_k \le b^k \overline{M}_{k-1}^{\frac{2p_k}{p_k-2}} \qquad \text{for all } k \ge 1.$$

As herein

$$0 \le \frac{2p_k}{p_k-2} = 2\Big(1+\frac{2}{p_k-2}\Big) = 2\Big(1+\frac{1}{2^k-1}\Big) \le 2\Big(1+\frac{2}{2^k}\Big) \qquad \text{for all } k \ge 1,$$

an application of [18, Lemma 4.3] shows that in this case,

$$\overline{M}_k \le b^{k+e^2\cdot 2^{k+1}} \overline{M}_0^{e^2\cdot 2^k} \qquad \text{for all } k \ge 1$$

and hence

$$\overline{M}_k^{\frac{1}{p_k}} \le b^{k\cdot 2^{-k-1}+e^2} \cdot \overline{M}_0^{\frac{e^2}{2}} \qquad \text{for all } k \ge 1,$$

which implies (6.17) also in this case, because both $\sup_{k\ge1}\{k\cdot 2^{-k-1}\}$ and $\overline{M}_0$ are finite. $\Box$

Since thus (a,v,w,z) is bounded in $\Omega\times(0,\infty)$, and since $\int_0^\infty\int_\Omega z<\infty$ by (3.2), an argument similar to a related precedent reasoning from [14] applies so as to warrant that the deviation of a from its spatial average decays with respect to the norm in $L^2(\Omega)$.

Lemma 6.5 *We have*

$$\|a(\cdot,t)-\overline{a(\cdot,t)}\|_{L^2(\Omega)} \to 0 \qquad \textit{as } t\to\infty. \tag{6.22}$$

Proof. According to Lemma 6.4, (2.2) and Lemma 6.3, we can fix positive constants $c_i, i \in \{1,...,4\}$, such that

$$a(x,t)\le c_1, \quad v(x,t)\le c_2, \quad w(x,t)\le c_3 \quad \text{and} \quad z(x,t)\le c_4 \qquad \text{for all } x\in\Omega \text{ and } t>0, \tag{6.23}$$

and (3.2) warrants that

$$c_5 := \int_0^\infty\int_\Omega z$$

is finite. To derive (6.22) from this, we proceed in three steps.

Step 1: We first claim that

$$\int_0^\infty \|a(\cdot,t)-\overline{a(\cdot,t)}\|^2_{L^2(\Omega)}dt<\infty. \tag{6.24}$$

Indeed, once again setting $p=2$ in (4.3) we see by now making use of (6.23) that since $(ae^v+w)v=-v_t\ge0$,

$$\begin{aligned}
\frac{d}{dt}\int_\Omega e^v a^2 &\le -2\int_\Omega e^v|\nabla a|^2-\int_\Omega e^v a^2 v_t \\
&\le -2\int_\Omega e^v|\nabla a|^2+c_1^2e^{c_2}\int_\Omega|v_t| \\
&= -2\int_\Omega e^v|\nabla a|^2-c_1^2e^{c_2}\int_\Omega v_t \qquad \text{for all } t>0.
\end{aligned}$$

Integrating this over time shows that

$$\begin{aligned}
\int_\Omega e^{v(\cdot,t)}a^2(\cdot,t)+2\int_0^t\int_\Omega e^v|\nabla a|^2 &\le \int_\Omega e^{v_0}a_0^2-c_1^2e^{c_2}\cdot\Big\{\int_\Omega v(\cdot,t)-\int_\Omega v_0\Big\} \\
&\le \int_\Omega e^{v_0}a_0^2+c_1^2e^{c_2}\cdot\int_\Omega v_0 \qquad \text{for all } t>0
\end{aligned}$$

and that hence

$$\int_0^\infty\int_\Omega|\nabla a|^2<\infty,$$

because $e^v \geq 1$. The property (6.24) results from this by means of a Poincaré inequality.

Step 2: Let us next make sure that

$$\int_0^\infty \int_\Omega a_t^2 < \infty. \tag{6.25}$$

and

$$\sup_{t>0} \int_\Omega |\nabla a(\cdot,t)|^2 < \infty. \tag{6.26}$$

To verify this, we multiply the identity $a_t = e^{-v}\nabla\cdot(e^v\nabla a) - av_t - \rho az$, as implied by (4.2), by $e^v a_t$ and integrate by parts to obtain that

$$\begin{aligned}
\int_\Omega e^v a_t^2 &= \int_\Omega a_t \nabla\cdot(e^v\nabla a) - \int_\Omega e^v a a_t v_t - \rho\int_\Omega e^v a a_t z \\
&= -\int_\Omega e^v \nabla a\cdot\nabla a_t - \int_\Omega e^v a a_t v_t - \rho\int_\Omega e^v a a_t z \qquad \text{for all } t>0,
\end{aligned}$$

where, once again by nonpositivity of v_t,

$$\begin{aligned}
-\int_\Omega e^v\nabla a\cdot\nabla a_t &= -\frac{1}{2}\int_\Omega e^v\partial_t|\nabla a|^2 \\
&= -\frac{1}{2}\frac{d}{dt}\int_\Omega e^v|\nabla a|^2 + \frac{1}{2}\int_\Omega e^v|\nabla a|^2 v_t \\
&\leq -\frac{1}{2}\frac{d}{dt}\int_\Omega e^v|\nabla a|^2 \qquad \text{for all } t>0.
\end{aligned}$$

Moreover, invoking Young's inequality we infer that, again due the identity $v_t = -(ae^v + w)v$,

$$\begin{aligned}
-\int_\Omega e^v a a_t v_t &\leq \frac{1}{4}\int_\Omega e^v a_t^2 + \int_\Omega e^v a^2 v_t^2 \\
&= \frac{1}{4}\int_\Omega e^v a_t^2 + \frac{1}{2}\int_\Omega e^v a^2(ae^v+w)|\partial_t v^2| \\
&\leq \frac{1}{4}\int_\Omega e^v a_t^2 - \frac{1}{2}e^{c_2}c_1^2(c_1e^{c_2}+c_3)\cdot\frac{d}{dt}\int_\Omega v^2,
\end{aligned}$$

and that furthermore

$$\begin{aligned}
-\rho\int_\Omega e^v a a_t z &\leq \frac{1}{4}\int_\Omega e^v a_t^2 + \rho^2\int_\Omega e^v a^2 z^2 \\
&= \frac{1}{4}\int_\Omega e^v a_t^2 + \rho^2 e^{c_2}c_1^2 c_4\int_\Omega z,
\end{aligned}$$

we therefore arrive at the inequality

$$\frac{1}{2}\int_\Omega e^v a_t^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega e^v|\nabla a|^2 \leq -c_6\frac{d}{dt}\int_\Omega v^2 + c_7\int_\Omega z \qquad \text{for all } t>0$$

with $c_6 := \frac{1}{2}e^{c_2}c_1^2(c_1e^{c_2}+c_3)$ and $c_7 := \rho^2 e^{c_2}c_1^2 c_4$. After a time integration this leads to

$$\begin{aligned}
&\frac{1}{2}\int_0^t\int_\Omega e^v a_t^2 + \frac{1}{2}\int_\Omega e^{v(\cdot,t)}|\nabla a(\cdot,t)|^2 \\
&\qquad \le \frac{1}{2}\int_\Omega e^{v_0}|\nabla a_0|^2 - c_6\int_\Omega v^2(\cdot,t) + c_6\int_\Omega v_0^2 + c_7\int_0^t\int_\Omega z \\
&\qquad \le \frac{1}{2}\int_\Omega e^{v_0}|\nabla a_0|^2 + c_6\int_\Omega v_0^2 + c_7\cdot c_5 \qquad \text{for all } t>0,
\end{aligned}$$

and thereby which implies both (6.25) and (6.26) thanks to the fact that $e^v \ge 1$.

Step 3: We finally claim that (6.22) holds.

In fact, this can be achieved in quite a straightforward manner by relying on combining the basic decay information contained in (6.24) and (6.25) with the compactness properties of trajectories implied through (6.26); for details in a closely related setting, we may refer to [14, Lemma 3.5]. □

Due to the uniform convergence statement from Lemma 2.3, this entails a similar feature of u:

Lemma 6.6 *The solution of (1.1) satisfies*

$$u(\cdot,t) - \overline{u(\cdot,t)} \to 0 \quad \textit{in } L^2(\Omega) \qquad \textit{as } t\to\infty. \tag{6.27}$$

Proof. We first note that according to Lemma 2.3 we have

$$e^{-v(\cdot,t)} - 1 \to 0 \quad \text{in } L^\infty(\Omega) \qquad \text{as } t\to\infty,$$

which when combined with (2.1) implies that

$$\left|\overline{a(\cdot,t)} - \overline{u(\cdot,t)}\right| = \frac{1}{|\Omega|}\cdot\left|\int_\Omega u(e^{-v}-1)\right| \le \frac{1}{|\Omega|}\|u\|_{L^1(\Omega)}\|e^{-v}-1\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } t\to\infty, \tag{6.28}$$

and that furthermore also

$$\left\|\overline{u(\cdot,t)} - e^{-v}\overline{u(\cdot,t)}\right\|_{L^\infty(\Omega)} = \overline{u(\cdot,t)}\|1-e^{-v}\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } t\to\infty. \tag{6.29}$$

Since recalling (2.2) we see that writing $c_1 := e^{\|v_0\|_{L^\infty(\Omega)}}$ we have

$$\begin{aligned}
\|u(\cdot,t) - \overline{u(\cdot,t)}\|_{L^2(\Omega)} &= \left\|e^v\left(a - e^{-v}\overline{u(\cdot,t)}\right)\right\|_{L^2(\Omega)} \\
&\le c_1\left\|a - e^{-v}\overline{u(\cdot,t)}\right\|_{L^2(\Omega)} \\
&\le c_1\|a - \overline{a(\cdot,t)}\|_{L^2(\Omega)} + c_1|\Omega|^{\frac{1}{2}}\left|\overline{a(\cdot,t)} - \overline{u(\cdot,t)}\right| + c_1|\Omega|^{\frac{1}{2}}\left\|\overline{u(\cdot,t)} - e^{-v}\overline{u(\cdot,t)}\right\|_{L^\infty(\Omega)}
\end{aligned}$$

for all $t>0$, from Lemma 6.5 in conjunction with (6.28) and (6.29) we already obtain (6.27). □

By monotonicity of the spatial average appearing in (6.27), thanks to a final simple interpolation the latter can in fact be turned into a convergence statement of the flavor announced in Theorem 1.1:

Lemma 6.7 *There exists $u_\infty > 0$ such that*

$$u(\cdot,t) \to u_\infty \quad \textit{in } L^p(\Omega) \quad \textit{for all } p \in [1,\infty) \qquad \textit{as } t \to \infty. \tag{6.30}$$

Proof. Since $0 \ni t \mapsto \overline{u(\cdot,t)}$ is nonincreasing according to (1.1), the number $u_\infty := \lim_{t\to\infty} \overline{u(\cdot,t)}$ is well-defined, so that from Lemma 6.6 it follows that $u(\cdot,t) \to u_\infty$ in $L^2(\Omega)$ as $t \to \infty$. According to the boundedness of u in $\Omega \times (0,\infty)$ asserted by Lemma 6.4, due to the Hölder inequality this readily implies that actually (6.30) holds, whereas, finally, the claimed positivity of u_∞ is an obvious consequence of Lemma 2.2. $\square$

All statements in Theorem 1.1 have thereby been brought in:

Proof of Theorem 1.1. We only need to collect the outcomes of Lemma 6.4, Lemma 2.3, Lemma 6.3 and Lemma 6.7. $\square$

Acknowledgement. Youshan Tao was supported by the *National Natural Science Foundation of China (No. 11861131003).* Michael Winkler acknowledges support of the *Deutsche Forschungsgemeinschaft* in the context of the project *Emergence of structures and advantages in cross-diffusion systems* (No. 411007140, GZ: WI 3707/5-1).

References

[1] Alzahrani, T., Raluca Eftimie, R., Dumitru Trucu, D.: *Multiscale modelling of cancer response to oncolytic viral therapy.* Math. Biosci. **310**, 76-95 (2019)

[2] Fontelos, M.A., Friedman, A., Hu, B.: *Mathematical analysis of a model for the initiation of angiogenesis.* SIAM J. Math. Anal. **33**, 1330-1355 (2002)

[3] Friedman, A., Tello, J.I.: *Stability of solutions of chemotaxis equations in reinforced random walks.* J. Math. Anal. Appl. **272**, 138-163 (2002)

[4] Liţcanu, G., Morales-Rodrigo, C.: *Asymptotic behavior of global solutions to a model of cell invasion.* Math. Models Methods Appl. Sci. **20**, 1721-1758 (2010)

[5] Morales-Rodrigo, C., Tello, J.I.: *Global existence and asymptotic behavior of a tumor angiogenesis model with chemotaxis and haptotaxis.* Math. Models Methods Appl. Sci. **24**, 427-464 (2014)

[6] Pang, P.Y.H., Wang, Y.: *Global boundedness of solutions to a chemotaxis-haptotaxis model with tissue remodeling.* Math. Mod. Meth. Appl. Sci. **28**, 2211-2235 (2018)

[7] Petrie, R.J., Doyle, A.D., Yamada, K.M.: *Random versus directionally persistent cell migration.* Nature Reviews Molecular Cell Biology **10**, 538-549 (2009)

[8] Stinner, C., Surulescu, C., Winkler, M.: *Global weak solutions in a PDE-ODE system modeling multiscale cancer cell invasion.* SIAM J. Math. Anal. **46**, 1969-2007 (2014)

[9] Tao, Y., Winkler, M.: *Energy-type estimates and global solvability in a two-dimensional chemotaxis-haptotaxis model with remodeling of non-diffusible attractant.* J. Differential Eq. **257**, 784-815 (2014)

[10] Tao, Y., Winkler, M.: *Dominance of chemotaxis in a chemotaxis-haptotaxis model.* Nonlinearity **27** (6), 1225-1239 (2014)

[11] Tao, Y., Winkler, M.: *Large time behavior in a mutidimensional chemotaxis-haptotaxis model with slow signal diffusion.* SIAM J. Math. Anal. **47**, 4229-4250 (2015)

[12] Tao, Y., Winkler, M.: *A chemotaxis-haptotaxis system with haptoattractant remodeling: boundedness enforced by mild saturation of signal production.* Commun. Pure Appl. Anal. **18**, 2047-2067 (2019)

[13] Tao, Y., Winkler, M.: *Global classical solutions to a doubly haptotactic cross-diffusion system modeling oncolytic virotherapy.* J. Differential Equations, DOI: 10.1016/j.jde.2019.10.046

[14] Tao, Y., Winkler, M.: *Critical mass for infinite-time blow-up in a haptotaxis system with nonlinear zero-order interaction.* Preprint

[15] Tao, Y., Winkler, M.: *A critical virus production rate for efficiency of oncolytic virotherapy* Preprint

[16] Walker, C., Webb, G.F.: *Global existence of classical solutions for a haptotaxis model.* SIAM J. Math. Anal. **38**, 1694-1713 (2007)

[17] Winkler, M.: *Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model.* J. Differential Eq. **248**, 2889-2905 (2010)

[18] Winkler, M.: *Global existence and slow grow-up in a quasilinear Keller-Segel system with exponentially decaying diffusivity.* Nonlinearity **30**, 735-764 (2017)

[19] Winkler, M.: *Singular structure formation in a degenerate haptotaxis model involving myopic diffusion.* J. Math. Pures Appl. **112**, 118-169 (2018)

[20] Winkler, M., Surulescu, C.: *A global weak solutions to a strongly degenerate haptotaxis model.* Commun. Math. Sci. **15**, 1581-1616 (2017)

[21] Wu, P.-H., Giri, A., Sun, S.X., Wirtz, D.: *Three-dimensional cell migration does not follow a random walk.* Proceedings of the National Academy of Sciences **111**, 3949-3954 (2014)

[22] Zhigun, A., Surulescu, C., Uatay, A.: *Global existence for a degenerate haptotaxis model of cancer invasion.* Z. Angew. Math. Phys. **67**, Art. 146, 29 pp (2016)