Immediate regularization of measure-type population densities in a two-dimensional chemotaxis system with signal consumption

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Abstract

This paper deals with the Neumann initial-boundary value problem for a classical chemotaxis system with signal consumption in a disk. In contrast to previous studies which have established a comprehensive theory of global classical solutions for suitably regular nonnegative initial data, the focus in the present work is on the question to which extent initially prescribed singularities can be regularized despite the presence of the nonlinear cross-diffusive interaction.

The main result in this paper asserts that at least in the framework of radial solutions immediate regularization occurs under an essentially optimal condition on the initial distribution of the population density. More precisely, it will turn out that for any radially symmetric initial data belonging to the space of regular signed Borel measures for the population density and to L^2 for the signal density, there exists a classical solution to the Neumann initial-boundary value problem, which is smooth and approaches the given initial data in an appropriate trace sense.

Key words: chemotaxis; measure valued initial data; instantaneous smoothing; global existence
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1 Introduction

Cross-diffusion vs. dissipation in Keller-Segel systems with regular initial data. Effects of collective behavior resulting from chemotactic motion of individuals are known to be of essential relevance in numerous biological contexts, especially involving pattern formation, at various lengthscales ([14]). At macroscopic levels, processes of this type are commonly modeled by the celebrated Keller-Segel system ([16]) which, when exclusively concentrating on the description of the population density $u = u(x,t)$ and the concentration $v = v(x,t)$ of the corresponding signal, in a general form consists of the two parabolic equations

$$\begin{cases} u_t = \nabla \cdot (D_u(u,v)\nabla u - \chi(u,v)u\nabla v) + H(u,v), \\ v_t = D_v(u,v)\Delta v + K(u,v). \end{cases} \tag{1.1}$$

Here specific application contexts may suggest quite different choices of the diffusion rates D_u and D_v, the chemotactic sensitivity χ and the coefficient functions H and K measuring cell proliferation and signal kinetics ([14]), and an accordingly large literature is concerned with the question how far the cross-diffusive interaction in respectively obtained particular versions of (1.1) supports spatial structures.

Most precedent studies in this direction concentrate on either detecting or ruling out effects of spontaneous *formation* of structures, usually interpreted mathematically as corresponding to blow-up phenomena, that is, to evolution of initially regular solutions into some singular profiles within finite time. Among the relevant versions of (1.1), the one possibly best understood in this regard appears to be the paradigmatic Keller-Segel model accounting for signal production through cells, as given by

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v), \\ v_t = \Delta v - v + u, \end{cases} \tag{1.2}$$

and the literature has in fact revealed a certain ability of this system to describe spontaneous aggregation in two- and higher-dimensional settings: Namely, it is well-known that for all suitably regular initial data (u_0, v_0) an associated Neumann-type initial-boundary value problem, posed in a smooth n-dimensional domain Ω, always admits a global bounded classical solution if either $n = 1$ ([21]), or $n = 2$ and $\int_\Omega u_0$ is small ([20]), or $n \geq 3$ and $\|u_0\|_{L^{\frac{n}{2}}(\Omega)} + \|v_0\|_{W^{1,n}(\Omega)}$ does not exceed some threshold ([6]); this is complemented by corresponding findings on the occurrence of finite-time blow-up of some solutions emanating from smooth but appropriately large initial data ([13, 34]). This particular system, and to a yet larger extent some parabolic-elliptic simplifications thereof, even allows for describing the respective sets of set explosion-enforcing initial data much more precisely ([34, 38]), and for characterizing corresponding blow-up asymptotics in more detail ([25, 26, 27]).

For some close relatives of (1.2) involving different diffusion and cross-diffusion rates, several results indicate similar dichotomies (see e.g. [8] and [28], and also [3] for a survey addressing further versions of (1.1)). A significant weakening of this singularity-supporting feature in chemotaxis-production systems, however, can be observed upon passing from (1.2) to those variants of (1.1) which account for signal consumption, rather than production, of the chemoattractant by individuals in the considered population.

In one of the prototypical representatives of this model class, as given by

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v), \\ v_t = \Delta v - uv, \end{cases} \tag{1.3}$$

the interplay of the dissipative mechanisms of diffusion and absorption is indeed known to suppress any blow-up phenomenon in two-dimensional Neumann problems, for which, namely, global bounded classical solutions are known to exist for all reasonably regular initial data; for the three-dimensional analogue, it is after all possible to construct global weak solutions which eventually become smooth and bounded ([29]). That this tendency toward blow-up inhibition is not limited to (1.3), but rather constitutes a more general property of chemotaxis-consumption models, is indicated by several additional findings on similar effects in related systems coupling the attractant absorption mechanism from (1.3) to various further components and processes, such as interplay with liquid environments ([9, 35, 37]), or multi-species interaction ([15]).

Chemotaxis-consumption interaction in presence of measures as initial data. Still attempting to further understand the structure-supporting potential of the interplay between chemoattraction and the dissipative mechanisms in chemotaxis-consumption models, in slight contrast to that in the above developments the purpose of the present work is to investigate this interaction in the context of a supposedly present singular setting. Thus *prescribing* initial data with accordingly irregular behavior, this study will examine how far the joint action of diffusion and signal absorption is able to overbalance potentially destabilizing effects of nonlinear cross-diffusion even in presence of singularities. Here focusing on the particular system (1.3) throughout, we note that any relaxation of requirements on the initial data should at least be consistent with an evident mass conservation property enjoyed by the first solution component thereof. In consequence, we are led to the ambition to analyze the behavior in (1.3) near initial population distributions merely assumed to satisfy some finite-mass hypothesis. When viewed from a purely mathematical and more general perspective, we thus pursue the goal of describing nonlinearly driven parabolic evolution out of measure-type initial data, as indeed having born considerable fruit in various different contexts (cf. [1, 5, 12, 17], for instance); in frameworks of Keller-Segel type systems, however, the few existing precedents dealing with such very singular data seem to concentrate on production-type chemotaxis models deviating from (1.2), in which smoothing can be expected to occur, if at all, only for small initial data, or in one-dimensional cases, or in presence of additional dissipative mechanisms such as superlinear cell degradation ([2, 19, 24]).

Main results. The goal of the present paper consists in deriving a result which indicates that in the context of the two-dimensional version of (1.3), unlike in (1.2) dissipation dominates over tactic aggregation also near measure-like structures, even of arbitrary size. To make this more precise, we consider

$$\begin{cases} u_t \;=\; \Delta u - \nabla \cdot (u\nabla v), & x \in \Omega,\ t > 0, \\ v_t \;=\; \Delta v - uv, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(\cdot,0) = \mu_0 \quad \text{and} \quad v(\cdot,0) = v_0, & x \in \Omega. \end{cases} \tag{1.4}$$

in the disk $\Omega = B_R(0) \subset \mathbb{R}^2$, $R > 0$, with initial data such that

$$\begin{cases} \mu_0 \in \mathcal{M}(\overline{\Omega}) & \text{is nonnegative and radially symmetric with } \mu_0 \not\equiv 0, \text{and where} \\ v_0 \in L^2(\Omega) & \text{is nonnegative and radially symmetric,} \end{cases} \tag{1.5}$$

where $\mathcal{M}(\overline{\Omega})$ denotes the space of all Radon regular signed Borel measures on $\overline{\Omega}$, and throughout we make use of the identification $\mathcal{M}(\overline{\Omega}) = (C^0(\overline{\Omega}))^\star$ according to the Riesz representation theorem ([4, Theorem 4.31]). Here an element $\mu \in \mathcal{M}(\overline{\Omega})$ will be called radially symmetric if and only if for all $\psi \in C^0(\overline{\Omega})$ we have $\mu(\psi) = \mu(S\psi)$, where the spherical average operator $S : C^0(\overline{\Omega}) \to C^0(\overline{\Omega})$ is defined through the relation

$$(S\psi)(x) := \frac{1}{|\partial B_{|x|}(0)|} \int_{\partial B_{|x|}(0)} \psi \qquad \text{for } \psi \in C^0(\overline{\Omega}) \text{ and } x \in \overline{\Omega}. \tag{1.6}$$

In this setting, we shall see that indeed instantaneous and persistent regularization occurs in that a global classical solution exists which attains (μ_0, v_0) in the apparently best possible topology compatible with (1.5):

Theorem 1.1 *Let $R > 0$ and $\Omega = B_R(0) \subset \mathbb{R}^2$, and suppose that μ_0 and v_0 satisfy (1.5). Then there exists at least one pair of nonnegative functions*

$$\begin{cases} u \in C^{2,1}(\overline{\Omega} \times (0,\infty)) \qquad \textit{and} \\ v \in C^{2,1}(\overline{\Omega} \times (0,\infty)) \cap L^2_{loc}([0,\infty); W^{1,2}(\Omega)) \end{cases} \tag{1.7}$$

such that $u(\cdot,t)$ and $v(\cdot,t)$ are radially symmetric for all $t > 0$, that (u,v) solves the boundary value problem in (1.4) in the classical sense in $\overline{\Omega} \times (0,\infty)$, and that furthermore with $m > 0$ as defined in (2.1) we have

$$\int_\Omega u(\cdot,t) = m \qquad \textit{for all } t > 0 \tag{1.8}$$

as well as

$$u(\cdot,t) \overset{\star}{\rightharpoonup} \mu_0 \quad \textit{in } \mathcal{M}(\overline{\Omega}) \qquad \textit{as } t \searrow 0 \tag{1.9}$$

and

$$v(\cdot,t) \to v_0 \quad \textit{in } L^2(\Omega) \qquad \textit{as } t \searrow 0. \tag{1.10}$$

Main ideas and structure of the paper. The first step toward our construction of a solution via approximation by smooth solutions $(u_\varepsilon, v_\varepsilon)$ emanating from regularized initial data $(u_{0\varepsilon}, v_{0\varepsilon})$, $(0,1) \ni \varepsilon \searrow 0$, will be based on the well-known observation that within the class of smooth solutions the problem (1.4) admits a favorable Lyapunov-type inequality (Lemma 2.5). Although the fact that the energy functional appearing therein may initially be unbounded in the limit $\varepsilon \searrow 0$ impedes straightforward conclusions thereof, an appropriately careful exploitation of this structural property, followed by temporally localized bootstrap arguments, will enable us to derive some ε-independent estimates for $(u_\varepsilon, v_\varepsilon)$ in regions where $t \geq \tau$ with arbitrary but fixed positive τ (Section 2.2), and to accordingly pass to the limit in any such region (Section 2.3).

A major challenge will thereafter be encountered in Section 3 when addressing the question how far the limit function thereby obtained satisfies the initial conditions in (1.4). Our analysis in this direction will firstly concentrate on the second solution component: By means of two quite standard regularity arguments (Lemma 2.2 and Lemma 2.4), namely, v can be seen to actually enjoy some basic temporally global regularity features which are sufficient to warrant fulfillment of its respective initial-boundary sub-problem in (1.4) at least in an adequately weak sense. In conjunction with an additional reasoning that asserts validity of a fundamental energy inequality associated with the evolution of the functional $\int_\Omega v^2$ (Lemma 3.2), this will not only ensure (1.10), but moreover also imply, *a posteriori*, some strong L^2 convergence property of ∇v_ε (Lemma 3.4).

The latter will subsequently play a significant role in our verification of (1.9): Namely, when verifying that also u solves the desired initial-boundary value problem in a certain generalized sense (Lemma 3.7), we shall rely on this as well as on a further fundamental and quite weak but temporally global L^2 estimate for $\frac{\nabla u_\varepsilon}{u_\varepsilon}$ (Lemma 2.3). Here essential use is made of the assumed radial symmetry, which in fact warrants that via one-dimensional embeddings, the latter implies a space-time L^2 estimate for u_ε in arbitrary annuli (Lemma 3.6). On the basis of the obtained integral identity thus satisfied by the limit (u,v), (1.9) will finally be derived in a two-step procedure involving approximation of test functions from $C^0(\overline{\Omega})$ by functions supported in such annuli (Lemma 3.8 and Lemma 3.10).

2 *A priori* estimates for regularized problems

2.1 Regularized problems and their basic properties

In order to suitably approximate solutions to (1.4), let us fix families $(u_{0\varepsilon})_{\varepsilon\in(0,1)} \subset C^0(\overline{\Omega})$ and $(v_{0\varepsilon})_{\varepsilon\in(0,1)} \subset W^{1,\infty}(\Omega)$ such that

$$\begin{cases} u_{0\varepsilon} \text{ is positive and radially symmetric in } \overline{\Omega} \text{ with } \int_\Omega u_{0\varepsilon} = m := \mu_0(\mathbf{1}) \text{ for all } \varepsilon\in(0,1) \text{ and} \\ \qquad u_{0\varepsilon} \overset{\star}{\rightharpoonup} \mu_0 \text{ in } (C^0(\overline{\Omega}))^\star \text{ as } \varepsilon\searrow 0, \text{ and that} \\ v_{0\varepsilon} \text{ is positive and radially symmetric in } \overline{\Omega} \text{ with } v_{0\varepsilon}\to v_0 \text{ in } L^2(\Omega) \text{ as } \varepsilon\searrow 0 \end{cases} \tag{2.1}$$

(see [4, Problem 24] for approximation of measures by smooth functions).

For $\varepsilon\in(0,1)$, we then consider

$$\begin{cases} \partial_t u_\varepsilon = \Delta u_\varepsilon - \nabla\cdot(u_\varepsilon\nabla v_\varepsilon), & x\in\Omega,\ t>0, \\ \partial_t v_\varepsilon = \Delta v_\varepsilon - u_\varepsilon v_\varepsilon, & x\in\Omega,\ t>0, \\ \frac{\partial u_\varepsilon}{\partial\nu} = \frac{\partial v_\varepsilon}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0, \\ u_\varepsilon(x,0)=u_{0\varepsilon}(x), \quad v_\varepsilon(x,0)=v_{0\varepsilon}(x), & x\in\Omega. \end{cases} \tag{2.2}$$

In the context of these regularized problems, global solvability is then asserted by known results:

Lemma 2.1 *Let $\varepsilon\in(0,1)$. Then the problem (2.2) admits a global classical solution $(u_\varepsilon,v_\varepsilon)\in(C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty)))^2$ such that both $u_\varepsilon(\cdot,t)$ and $v_\varepsilon(\cdot,t)$ are positive and radially symmetric in $\overline{\Omega}$ for all $t>0$, and such that moreover*

$$\int_\Omega u_\varepsilon(\cdot,t) = m \qquad \textit{for all } t>0 \tag{2.3}$$

with $m > 0$ as defined in (2.1).

PROOF. According to the regularity and positivity of $(u_{0\varepsilon}, v_{0\varepsilon})$ required in (2.1), a well-known result ([33]) asserts the global existence of a global classical solution, uniquely determined by the inclusions

$$\begin{cases} u_\varepsilon \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \qquad \text{and} \\ v_\varepsilon \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \cap \bigcup_{q>2} L^\infty_{loc}([0,\infty); W^{1,q}(\Omega)), \end{cases} \tag{2.4}$$

which has the claimed positivity features due to the strong maximum principle. The identity in (2.3) thereupon becomes obvious by integrating the first equation in (2.2) over $\Omega \times (0,t)$ for $t > 0$, and the radial symmetry of the obtained solution is a straightforward consequence of the above uniqueness property. □

Beyond (2.3), the problem (2.2) possesses some further basic but important features, the first among which has extensively been used in the literature on this and closely related chemotaxis-consumption systems ([18, 36]):

Lemma 2.2 *The solutions of (2.2) satisfy*

$$\frac{1}{2}\int_\Omega v_\varepsilon^2(\cdot,t) + \int_0^t \int_\Omega |\nabla v_\varepsilon|^2 + \int_0^t \int_\Omega u_\varepsilon v_\varepsilon^2 = \frac{1}{2}\int_\Omega v_{0\varepsilon}^2 \qquad \textit{for all } \varepsilon \in (0,1) \textit{ and each } t > 0. \tag{2.5}$$

In particular,

$$\int_0^\infty \int_\Omega |\nabla v_\varepsilon|^2 + \int_0^\infty \int_\Omega u_\varepsilon v_\varepsilon^2 \le \frac{1}{2}\int_\Omega v_{0\varepsilon}^2 \qquad \textit{for any } \varepsilon \in (0,1). \tag{2.6}$$

PROOF. We multiply the second equation in (2.2) by v_ε and integrate by parts to find that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v_\varepsilon^2 + \int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega u_\varepsilon v_\varepsilon^2 = 0 \qquad \text{for all } t > 0,$$

from which (2.5) and hence also (2.6) directly follow. □

As observed in [36], the L^2 bound for ∇v_ε thereby implied can be used to derive some information on the regularity of ∇u_ε, though yet involving a strongly dampening weight function at this stage.

Lemma 2.3 *We have*

$$\int_0^\infty \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} \le 2m + \frac{1}{2}\int_\Omega v_{0\varepsilon}^2 \qquad \textit{for all } \varepsilon \in (0,1), \tag{2.7}$$

where $m > 0$ is as in (2.1).

PROOF. Following [36], we compute

$$\frac{d}{dt}\int_\Omega \ln(u_\varepsilon+1) + \int_\Omega \frac{u_\varepsilon}{(u_\varepsilon+1)^2}\nabla u_\varepsilon \cdot \nabla v_\varepsilon = \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} \qquad \text{for all } t > 0, \tag{2.8}$$

and use Young's inequality to estimate

$$\left| \int_\Omega \frac{u_\varepsilon}{(u_\varepsilon+1)^2}\nabla u_\varepsilon \cdot \nabla v_\varepsilon \right| \le \frac{1}{2}\int_\Omega \frac{u_\varepsilon^2}{(u_\varepsilon+1)^4}|\nabla u_\varepsilon|^2 + \frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2 \le \frac{1}{2}\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} + \frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2$$

for all $t > 0$. As $0 \le \ln(\xi+1) \le \xi$ for all $\xi \ge 0$, (2.8) therefore implies that

$$\frac{1}{2}\int_0^t \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} \le \int_\Omega \ln\Big(u_\varepsilon(\cdot,t)+1\Big) - \int_\Omega \ln(u_{0\varepsilon}+1) + \frac{1}{2}\int_0^t\int_\Omega |\nabla v_\varepsilon|^2 \le \int_\Omega u_\varepsilon(\cdot,t) + \frac{1}{2}\int_0^t\int_\Omega |\nabla v_\varepsilon|^2$$

for all $t > 0$, whence using (2.3) and Lemma 2.2 we arrive at (2.7). $\square$

Most previous studies on (1.4) and its relatives have been relying in a more or less essential way on the fact that the second equation therein preserves supposedly available L^∞ bounds on v due to the maximum principle ([18, 22, 29, 32, 33]). In the present setting of comparatively weak hypotheses on initial regularity of v, we may accordingly resort to the following somewhat restricted boundedness statement only.

Lemma 2.4 *There exists $C > 0$ such that whenever $\varepsilon \in (0,1)$,*

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C \cdot (t^{-\frac{1}{2}} + 1) \qquad \textit{for all } t > 0. \tag{2.9}$$

PROOF. Relying on a well-known smoothing property of the Neumann heat semigroup $(e^{t\Delta})_{t\ge 0}$ on the two-dimensional domain Ω, we can find $c_1 > 0$ such that

$$\|e^{t\Delta}\psi\|_{L^\infty(\Omega)} \le c_1 \cdot (t^{-\frac{1}{2}} + 1)\|\psi\|_{L^2(\Omega)} \qquad \text{for all } \psi \in C^0(\overline{\Omega}).$$

Since according to the nonnegativity of $u_\varepsilon v_\varepsilon$ the comparison principle ensures the pointwise inequality $v_\varepsilon(\cdot,t) \le e^{t\Delta}v_{0\varepsilon}$ in Ω for all $t > 0$, by nonnegativity of v_ε we can therefore estimate

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|e^{t\Delta}v_{0\varepsilon}\|_{L^\infty(\Omega)} \le c_1 \cdot (t^{-\frac{1}{2}} + 1)\|v_{0\varepsilon}\|_{L^2(\Omega)} \qquad \text{for all } t > 0,$$

so that (2.9) becomes a consequence of (2.1). $\square$

2.2 Estimates away from $t = 0$ via a refined energy analysis

Our analysis of solutions in space-time regions separated from the temporal origin will be based on the following natural energy inequality associated with (1.4). Variants thereof have played essential roles in numerous studies concerned with (1.4) and even with slightly more complex relatives ([9, 10, 29, 31]).

Lemma 2.5 *For arbitrary $\varepsilon \in (0,1)$, we have*

$$\frac{d}{dt}\Big\{\int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}\Big\} + \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \frac{1}{(2+\sqrt{2})^2}\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \le 0 \qquad \textit{for all } t > 0. \tag{2.10}$$

PROOF. Combining the first two equations from (2.2) in a straightforward manner (cf. [33, Lemma 3.2] for details), one can readily verify the identity

$$\frac{d}{dt}\Big\{\int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}\Big\} + \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \int_\Omega v_\varepsilon|D^2\ln v_\varepsilon|^2 + \frac{1}{2}\int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 = \frac{1}{2}\int_{\partial\Omega}\frac{1}{v_\varepsilon}\cdot\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu}$$

for all $t > 0$. Since $\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu} = 0$ throughout $\partial\Omega \times (0,\infty)$ by (2.2) and radial symmetry of v_ε, and since, by a functional inequality established in [33, Lemma 3.3],

$$\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \le (2+\sqrt{2})^2 \int_\Omega v_\varepsilon|D^2\ln v_\varepsilon|^2 \qquad \text{for all } t > 0,$$

this implies (2.10). □

Unlike in most precedent exploitations of inequalities like (2.10), the current situation of lacking information on appropriate initial regularity gives rise to the considerations to be carried out in the next three lemmata, the goal of which is to prepare the identification of small positive times at which the energy functional in (2.10) remains conveniently small.

Our first step in this direction relies on the two-dimensional Moser-Trudinger inequality to conclude from Lemma 2.3 and (2.3) that the first ingredient of the energy can be controlled as follows.

Lemma 2.6 *For all $\tau \in (0,1)$ there exists $C(\tau) > 0$ with the property that for all $\varepsilon \in (0,1)$,*

$$\left| \left\{ t \in (0,\tau) \ \Big| \ \int_\Omega u_\varepsilon(\cdot,t) \ln u_\varepsilon(\cdot,t) \le C(\tau) \right\} \right| \ge \frac{3\tau}{4}. \tag{2.11}$$

PROOF. By means of Lemma 2.3 and (2.1), we may fix $c_1 > 0$ such that

$$\int_0^1 \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} \le c_1 \qquad \text{for all } \varepsilon \in (0,1), \tag{2.12}$$

which we exploit as follows: Thanks to the Moser-Trudinger inequality ([7]) and the Poincaré inequality, there exists $c_2 > 0$ such that

$$\int_\Omega e^{|\psi|} \le c_2 \exp \left\{ c_2 \int_\Omega |\nabla\psi|^2 + c_2 \cdot \Big(\int_\Omega |\psi| \Big)^2 \right\} \qquad \text{for all } \psi \in W^{1,2}(\Omega), \tag{2.13}$$

and therein for $t > 0$ and $\varepsilon \in (0,1)$ we choose

$$\psi := \ln \Big\{ (u_\varepsilon(\cdot,t)+1) \ln (u_\varepsilon(\cdot,t)+e) \Big\}.$$

For that purpose we observe that then $\psi \ge 0$ and

$$\int_\Omega \psi \le \int_\Omega \ln \Big\{ (u_\varepsilon+1)\cdot(u_\varepsilon+e) \Big\} \le 2\int_\Omega \ln(u_\varepsilon+e) \le 2\int_\Omega u_\varepsilon + 2e|\Omega| \le c_3 := 2m + 2e|\Omega| \tag{2.14}$$

due to (2.3) and the rough estimate $\ln(\xi+e) \le \xi+e$ for $\xi \ge 0$. Moreover, noting that

$$\left| \frac{d}{d\xi} \ln \Big\{ (\xi+1)\ln(\xi+e) \Big\} \right| = \frac{\ln(\xi+e) + \frac{\xi+1}{\xi+e}}{(\xi+1)\ln(\xi+e)} \le \frac{\ln(\xi+e)+1}{(\xi+1)\ln(\xi+e)} \le \frac{2}{\xi+1}$$

for all $\xi \ge 0$, we find that

$$\int_\Omega |\nabla\psi|^2 \le \int_\Omega \Big(\frac{2}{u_\varepsilon+1} \Big)^2 |\nabla u_\varepsilon|^2 = 4\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2},$$

so that (2.13) along with (2.14) implies that

$$\ln \left\{ \frac{1}{|\Omega|} \int_\Omega (u_\varepsilon+1)\ln(u_\varepsilon+e) \right\} \le \ln \frac{c_2}{|\Omega|} + 4c_2 \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} + c_2 c_3^2 \qquad \text{for all } \varepsilon \in (0,1) \text{ and } t > 0.$$

In light of (2.12), this entails that if we let $c_4 := |\ln \frac{c_2}{|\Omega|}| + 4c_1c_2 + c_2c_3^2$, then $h_\varepsilon(t) := \ln \Big\{ \frac{1}{|\Omega|} \int_\Omega (u_\varepsilon(\cdot,t)+1)\ln(u_\varepsilon(\cdot,t)+e) \Big\}$, $\varepsilon \in (0,1), t \in (0,1)$, satisfies

$$\int_0^1 h_\varepsilon(t)dt \le c_4 \qquad \text{for all } \varepsilon \in (0,1).$$

As h_ε evidently is nonnegative, if given any $\tau \in (0,1)$ and $\varepsilon \in (0,1)$ we let $S(\tau,\varepsilon) := \{t \in (0,\tau) \mid h_\varepsilon(t) \le \frac{4c_4}{\tau}\}$, then we can estimate

$$c_4 \ge \int_0^1 h_\varepsilon(t)dt \ge \int_{(0,\tau)\setminus S(\tau,\varepsilon)} h_\varepsilon(t)dt \ge |(0,\tau)\setminus S(\tau,\varepsilon)| \cdot \frac{4c_4}{\tau}$$

and thus infer that $|(0,\tau)\setminus S(\tau,\varepsilon)| \le \frac{\tau}{4}$. Since therefore $|S(\tau,\varepsilon)| \ge \tau - \frac{\tau}{4} = \frac{3\tau}{4}$ for all $\tau \in (0,1)$ and $\varepsilon \in (0,1)$, and since for any such τ and ε we trivially have

$$\int_\Omega u_\varepsilon(\cdot,t)\ln u_\varepsilon(\cdot,t) \le \int_\Omega (u_\varepsilon(\cdot,t)+1)\ln\big(u_\varepsilon(\cdot,t)+e\big) \le |\Omega| e^{h_\varepsilon(t)} \le |\Omega| e^{\frac{4c_4}{\tau}} \qquad \text{for all } t \in S(\tau,\varepsilon),$$

the claimed inequality in (2.11) holds if we let $C(\tau) := |\Omega| e^{\frac{4c_4}{\tau}}$, for instance. $\Box$

In order to next derive a similar result for the second contribution $\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}$ to the energy functional in Lemma 2.5, in view of the singular factor $\frac{1}{v_\varepsilon}$ therein we refine our information on gradient regularity of v_ε by means of another testing procedure.

Lemma 2.7 *Let $\varepsilon \in (0,1)$. Then with $m > 0$ taken from (2.1), we have*

$$\int_0^t \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \le \frac{m}{e} \cdot t + \int_\Omega v_{0\varepsilon}^2 + \frac{|\Omega|}{e} \qquad \textit{for all } t > 0. \tag{2.15}$$

PROOF. Using that v_ε is positive in $\overline{\Omega} \times (0,\infty)$ by Lemma 2.1, we may integrate by parts in computing

$$\begin{aligned} \frac{d}{dt}\int_\Omega v_\varepsilon \ln v_\varepsilon &= \int_\Omega \ln v_\varepsilon \cdot (\Delta v_\varepsilon - u_\varepsilon v_\varepsilon) + \int_\Omega (\Delta v_\varepsilon - u_\varepsilon v_\varepsilon) \\ &= -\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} - \int_\Omega u_\varepsilon v_\varepsilon \ln v_\varepsilon - \int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } t > 0. \end{aligned} \tag{2.16}$$

Here the rightmost summand is nonpositive, whereas the validity of $\xi \ln \xi \ge -\frac{1}{e}$ for all $\xi > 0$ warrants that

$$-\int_\Omega u_\varepsilon v_\varepsilon \ln v_\varepsilon \le \frac{1}{e}\int_\Omega u_\varepsilon = \frac{m}{e} \qquad \text{for all } t > 0$$

according to (2.3). Upon integration in time, from (2.16) we thus infer that

$$\int_0^t \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \le \frac{m}{e} \cdot t + \int_\Omega v_{0\varepsilon} \ln v_{0\varepsilon} - \int_\Omega v_\varepsilon(\cdot,t) \ln v_\varepsilon(\cdot,t) \qquad \text{for all } t > 0,$$

which readily results in (2.15) due to the fact that the inequalities $-\frac{1}{e} \le \xi \ln \xi \le \xi^2$ for $\xi > 0$ ensure that

$$\int_\Omega v_{0\varepsilon} \ln v_{0\varepsilon} - \int_\Omega v_\varepsilon(\cdot,t) \ln v_\varepsilon(\cdot,t) \le \int_\Omega v_{0\varepsilon}^2 + \frac{|\Omega|}{e}$$

for any $t > 0$. □

We can thereby establish our counterpart of Lemma 2.6 as follows.

Lemma 2.8 *For all $\tau \in (0,1)$ one can find $C(\tau) > 0$ such that for all $\varepsilon \in (0,1)$,*

$$\left| \left\{ t \in (0,\tau) \ \middle| \ \int_\Omega \frac{|\nabla v_\varepsilon(\cdot,t)|^2}{v_\varepsilon(\cdot,t)} \le C(\tau) \right\} \right| \ge \frac{3\tau}{4}. \tag{2.17}$$

PROOF. This is an evident consequence of Lemma 2.7: Fixing $c_1 > 0$ such that

$$\int_0^1 \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \le c_1 \qquad \text{for all } \varepsilon \in (0,1),$$

and writing $h_\varepsilon(t) := \int_\Omega \frac{|\nabla v_\varepsilon(\cdot,t)|^2}{v_\varepsilon(\cdot,t)}$ as well as $S(\tau,\varepsilon) := \{t \in (0,\tau) \mid h_\varepsilon(t) \le \frac{4c_1}{\varepsilon}\}$ for $\tau \in (0,1)$ and $\varepsilon \in (0,1)$, we see that

$$c_1 \ge \int_0^1 h_\varepsilon(t)dt \ge \Big|(0,\tau) \setminus S(\tau,\varepsilon)\Big| \cdot \frac{4c_1}{\tau} \qquad \text{for all } \tau \in (0,1) \text{ and } \varepsilon \in (0,1).$$

As thus $|S(\tau,\varepsilon)| \ge \frac{3\tau}{4}$ for all $\tau \in (0,1)$ and $\varepsilon \in (0,1)$, (2.17) follows with $C(\tau) := \frac{4c_1}{\tau}$. □

We are now in the position to draw suitable consequences from Lemma 2.5 through integration.

Lemma 2.9 *For all $\tau \in (0,1)$ there exists $C(\tau) > 0$ such that*

$$\int_\tau^\infty \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} \le C(\tau) \qquad \textit{for all } \varepsilon \in (0,1) \tag{2.18}$$

and

$$\int_\tau^\infty \int_\Omega |\nabla v_\varepsilon|^4 \le C(\tau) \qquad \textit{for all } \varepsilon \in (0,1) \tag{2.19}$$

PROOF. Given $\tau \in (0,1)$, we first employ Lemma 2.6 and Lemma 2.8 to fix $c_1(\tau) > 0$ and $c_2(\tau) > 0$ with the properties that for each $\varepsilon \in (0,1)$, the sets $S_1(\varepsilon) := \{t \in (0,\tau) \mid \int_\Omega u_\varepsilon(\cdot,t) \ln u_\varepsilon(\cdot,t) \le c_1(\tau)\}$ and $S_2(\varepsilon) := \{t \in (0,\tau) \mid \int_\Omega \frac{|\nabla v_\varepsilon(\cdot,t)|^2}{v_\varepsilon(\cdot,t)} \le c_2(\tau)\}$ satisfy $|S_1(\varepsilon)| \ge \frac{3\tau}{4}$ and $|S_2(\varepsilon)| \ge \frac{3\tau}{4}$. As thus $|S_1(\varepsilon) \cap S_2(\varepsilon)| \ge \frac{\tau}{2}$, for any such ε we can hence find some $t_0(\varepsilon) \in (0,\tau)$ simultaneously fulfilling $\int_\Omega u_\varepsilon(\cdot,t_0(\varepsilon)) \ln u_\varepsilon(\cdot,t_0(\varepsilon)) \le c_1(\tau)$ and $\int_\Omega \frac{|\nabla v_\varepsilon(\cdot,t_0(\varepsilon))|^2}{v_\varepsilon(\cdot,t_0(\varepsilon))} \le c_2(\tau)$, so that $y_\varepsilon(t) := \int_\Omega u_\varepsilon(\cdot,t) \ln u_\varepsilon(\cdot,t) + \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon(\cdot,t)|^2}{v_\varepsilon(\cdot,t)}$, $\varepsilon \in (0,1)$, $t > 0$, satisfies

$$y_\varepsilon(t_0(\varepsilon)) \le c_1(\tau) + \frac{1}{2}c_2(\tau) \qquad \text{for all } \varepsilon \in (0,1).$$

As, on the other hand, again using that $\xi \ln \xi \ge -\frac{1}{e}$ for all $\xi > 0$ we see that

$$y_\varepsilon(t) \ge -\frac{|\Omega|}{e} \qquad \text{for all } \varepsilon \in (0,1) \text{ and } t > 0,$$

on integrating (2.10) in time we find that for all $\varepsilon \in (0,1)$ and each $t > t_0(\varepsilon)$,

$$\int_{t_0(\varepsilon)}^t \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \frac{1}{(2+\sqrt{2})^2} \int_{t_0(\varepsilon)}^t \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \le y_\varepsilon(t_0(\varepsilon)) - y_\varepsilon(t) \le c_3(\tau) := c_1(\tau) + \frac{1}{2} c_2(\tau) + \frac{|\Omega|}{e}.$$

Since $t_0(\varepsilon) < \tau$, this shows that

$$\int_\tau^t \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} \le c_3(\tau) \qquad \text{for all } \varepsilon \in (0,1) \text{ and } t > \tau,$$

and that moreover

$$\int_\tau^t \int_\Omega |\nabla v_\varepsilon|^4 \le (2+\sqrt{2})^2 c_3(\tau) \cdot \sup_{s>\tau} \|v_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}^3 \qquad \text{for all } \varepsilon \in (0,1) \text{ and } t > \tau,$$

and thereby entails both (2.18) and (2.19). □

A careful refinement of a standard testing procedure next turns the two inequalities from the previous lemma into L^p bounds on u_ε for arbitrary $p > 1$, locally away from $t = 0$.

Lemma 2.10 *Let $p > 1$. Then for all $\tau \in (0,1)$ one can find $C(p,\tau) > 0$ such that whenever $\varepsilon \in (0,1)$,*

$$\int_\Omega u_\varepsilon^p(\cdot,t) \le C(p,\tau) \qquad \textit{for all } t > \tau. \tag{2.20}$$

Proof. On the basis of an integration by parts in the first equation from (2.2) we see that due to Young's inequality and the Hölder inequality,

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt} \int_\Omega u_\varepsilon^p + (p-1) \int_\Omega u_\varepsilon^{p-2} |\nabla u_\varepsilon|^2 \\
&\quad = (p-1) \int_\Omega u_\varepsilon^{p-1} \nabla u_\varepsilon \cdot \nabla v_\varepsilon \\
&\quad \le \frac{p-1}{2} \int_\Omega u_\varepsilon^{p-2} |\nabla u_\varepsilon|^2 + \frac{p-1}{2} \int_\Omega u_\varepsilon^p |\nabla v_\varepsilon|^2 \\
&\quad \le \frac{p-1}{2} \int_\Omega u_\varepsilon^{p-2} |\nabla u_\varepsilon|^2 + \frac{p-1}{2} \cdot \left\{ \int_\Omega u_\varepsilon^{2p} \right\}^{\frac{1}{2}} \cdot \left\{ \int_\Omega |\nabla v_\varepsilon|^4 \right\}^{\frac{1}{2}}
\end{aligned} \tag{2.21}$$

for all $t > 0$. Here by the Gagliardo-Nirenberg inequality, (2.3) and Young's inequality, we can find positive constants $c_1(p), c_2(p)$ and $c_3(p)$ such that whenever $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\frac{p-1}{2} \cdot \left\{ \int_\Omega u_\varepsilon^{2p} \right\}^{\frac{1}{2}} \cdot \left\{ \int_\Omega |\nabla v_\varepsilon|^4 \right\}^{\frac{1}{2}} \\
&\quad = \frac{p-1}{2} \|u_\varepsilon^{\frac{p}{2}}\|_{L^4(\Omega)}^2 \|\nabla v_\varepsilon\|_{L^4(\Omega)}^2
\end{aligned}$$

$$
\begin{aligned}
&\le c_1(p)\cdot\Big\{\|\nabla u_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}\|u_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}+\|u_\varepsilon^{\frac{p}{2}}\|^2_{L^{\frac{2}{p}}(\Omega)}\Big\}\cdot\|\nabla v_\varepsilon\|^2_{L^4(\Omega)}\\
&\le c_2(p)\|\nabla u_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}\|u_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}\|\nabla v_\varepsilon\|^2_{L^4(\Omega)}+c_2(p)\|\nabla v_\varepsilon\|^2_{L^4(\Omega)}\\
&\le \frac{p-1}{p^2}\|\nabla u_\varepsilon^{\frac{p}{2}}\|^2_{L^2(\Omega)}+c_3(p)\|u_\varepsilon^{\frac{p}{2}}\|^2_{L^2(\Omega)}\|\nabla v_\varepsilon\|^4_{L^4(\Omega)}+c_2(p)\|\nabla v_\varepsilon\|^4_{L^4(\Omega)}+c_2(p)\\
&= \frac{p-1}{4}\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2+\Big\{c_3(p)\int_\Omega u_\varepsilon^p+c_2(p)\Big\}\cdot\int_\Omega|\nabla v_\varepsilon|^4+c_2(p)\qquad\text{for all } t>0,
\end{aligned}
$$

and similarly we obtain $c_4(p)>0$ and $c_5(p)>0$ such that for all $\varepsilon\in(0,1)$,

$$
\int_\Omega u_\varepsilon^p=\|u_\varepsilon^{\frac{p}{2}}\|^2_{L^2(\Omega)}\le\frac{p-1}{p^2}\|\nabla u_\varepsilon^{\frac{p}{2}}\|^2_{L^2(\Omega)}+c_4(p)\|u_\varepsilon^{\frac{p}{2}}\|^2_{L^{\frac{2}{p}}(\Omega)}\le\frac{p-1}{4}\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2+c_5(p)
$$

for all $t>0$. Therefore, (2.21) implies that $y_\varepsilon(t):=\int_\Omega u_\varepsilon^p(\cdot,t)$ and $h_\varepsilon(t):=\int_\Omega|\nabla v_\varepsilon(\cdot,t)|^4$, $(t>0)$, satisfy

$$
\frac{1}{p}y_\varepsilon'(t)+y_\varepsilon(t)\le c_3(p)y_\varepsilon(t)h_\varepsilon(t)+c_2(p)h_\varepsilon(t)+c_2(p)+c_5(p)\qquad\text{for all } t>0
$$

and hence

$$
y_\varepsilon'(t)\le\Big\{-p+c_6(p)h_\varepsilon(t)\Big\}\cdot y_\varepsilon(t)+c_6(p)h_\varepsilon(t)+c_6(p)\qquad\text{for all }\varepsilon\in(0,1)\text{ and } t>0 \tag{2.22}
$$

with $c_6(p):=p\cdot\max\{c_3(p),c_2(p)+c_5(p)\}$.

Now for any fixed $\tau\in(0,1)$, Lemma 2.9 provides $c_7(\tau)>0$ such that

$$
\int_{\frac{\tau}{2}}^\infty h_\varepsilon(t)dt\le c_7(\tau)\qquad\text{for all }\varepsilon\in(0,1), \tag{2.23}
$$

and such that moreover

$$
\int_{\frac{\tau}{2}}^\tau\int_\Omega|\nabla\sqrt{u_\varepsilon}|^2\le c_7(\tau)\qquad\text{for all }\varepsilon\in(0,1),
$$

where the latter in conjunction with (2.3) and the continuity of the embedding $W^{1,2}(\Omega)\hookrightarrow L^{2p}(\Omega)$ ensures the existence of $c_8(p,\tau)>0$ such that

$$
\int_{\frac{\tau}{2}}^\tau y_\varepsilon^{\frac{1}{p}}(t)dt=\int_{\frac{\tau}{2}}^\tau\|\sqrt{u_\varepsilon(\cdot,t)}\|^2_{L^{2p}(\Omega)}dt\le c_8(p,\tau)\qquad\text{for all }\varepsilon\in(0,1).
$$

Therefore, given any $\varepsilon\in(0,1)$ we can find $t_0(\varepsilon)\in(\frac{\tau}{2},\tau)$ such that $y_\varepsilon(t_0(\varepsilon))\le c_9(p,\tau):=(\frac{2c_8(p,\tau)}{p})^p$, whence integrating (2.22) using (2.23) we may estimate $y_\varepsilon(t)$ for $t>t_0(\varepsilon)$ according to

$$
\begin{aligned}
y_\varepsilon(t)&\le y_\varepsilon(t_0(\varepsilon))\cdot e^{\int_{t_0(\varepsilon)}^t\{-p+c_6(p)h_\varepsilon(s)\}ds}+c_6(p)\cdot\int_{t_0(\varepsilon)}^t e^{\int_s^t\{-p+c_6(p)h_\varepsilon(\sigma)\}d\sigma}\cdot\Big\{h_\varepsilon(s)+1\Big\}ds\\
&\le c_9(p,\tau)\cdot e^{c_6(p)\int_{\frac{\tau}{2}}^t h_\varepsilon(s)ds}+c_6(p)\cdot\int_{\frac{\tau}{2}}^t e^{-p(t-s)}\cdot e^{c_6(p)\int_s^t h_\varepsilon(\sigma)d\sigma}\cdot\Big\{h_\varepsilon(s)+1\Big\}ds\\
&\le c_9(p,\tau)\cdot e^{c_6(p)c_7(\tau)}+c_6(p)\cdot e^{c_6(p)c_7(\tau)}\cdot\Big\{\int_{\frac{\tau}{2}}^t h_\varepsilon(s)ds+\int_{\frac{\tau}{2}}^t e^{-p(t-s)}ds\Big\}\\
&\le c_9(p,\tau)\cdot e^{c_6(p)c_7(\tau)}+c_6(p)\cdot e^{c_6(p)c_7(\tau)}\cdot\Big\{c_7(\tau)+\frac{1}{p}\Big\}
\end{aligned}
$$

and conclude as intended. □

Based on the latter, we can additionally improve our knowledge on ∇v_ε.

Lemma 2.11 *For all $\tau \in (0,1)$ there exists $C(\tau) > 0$ such that for each $\varepsilon \in (0,1)$,*

$$\int_\Omega |\nabla v_\varepsilon(\cdot,t)|^4 \le C(\tau) \qquad \textit{for all } t > \tau. \tag{2.24}$$

Proof. Following a well-established testing procedure (cf. e.g. [33]), we use the pointwise relations $\nabla v_\varepsilon \cdot \nabla \Delta v_\varepsilon = \frac{1}{2}\Delta|\nabla v_\varepsilon|^2 - |D^2 v_\varepsilon|^2$ and $|\Delta v_\varepsilon| \le \sqrt{2}|D^2 v_\varepsilon|$ as well as, again, the fact that $\frac{\partial |\nabla v_\varepsilon|^2}{\partial \nu} = 0$ on $\partial\Omega \times (0,\infty)$ due to radial symmetry, to derive from (2.2) by means of Young's inequality that

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla v_\varepsilon|^4 + \int_\Omega |\nabla v_\varepsilon|^4 \\
&= \int_\Omega |\nabla v_\varepsilon|^2 \nabla v_\varepsilon \cdot \nabla(\Delta v_\varepsilon - u_\varepsilon v_\varepsilon) + \int_\Omega |\nabla v_\varepsilon|^4 \\
&= \frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2 \Delta |\nabla v_\varepsilon|^2 - \int_\Omega |\nabla v_\varepsilon|^2 |D^2 v_\varepsilon|^2 + \int_\Omega u_\varepsilon v_\varepsilon \cdot \Big\{ 2\nabla v_\varepsilon \cdot (D^2 v_\varepsilon \cdot \nabla v_\varepsilon) + |\nabla v_\varepsilon|^2 \Delta v_\varepsilon \Big\} + \int_\Omega |\nabla v_\varepsilon|^4 \\
&\le -\frac{1}{2}\int_\Omega \Big|\nabla |\nabla v_\varepsilon|^2\Big|^2 - \int_\Omega |\nabla v_\varepsilon|^2 |D^2 v_\varepsilon|^2 + (2+\sqrt{2})\int_\Omega u_\varepsilon v_\varepsilon |\nabla v_\varepsilon|^2 |D^2 v_\varepsilon| + \int_\Omega |\nabla v_\varepsilon|^4 \\
&\le \frac{(2+\sqrt{2})^2}{4}\int_\Omega u_\varepsilon^2 v_\varepsilon^2 |\nabla v_\varepsilon|^2 + \int_\Omega |\nabla v_\varepsilon|^4 \\
&\le 2\int_\Omega |\nabla v_\varepsilon|^4 + \frac{(2+\sqrt{2})^4}{16}\int_\Omega u_\varepsilon^4 v_\varepsilon^4 \qquad \text{for all } t > 0.
\end{aligned}$$

According to Lemma 2.10 and Lemma 2.4, this means that for each $\tau \in (0,1)$ one can find $c_1(\tau) > 0$ such that the functions y_ε and h_ε defined by $y_\varepsilon(t) := \int_\Omega |\nabla v_\varepsilon(\cdot,t)|^4$ and $h_\varepsilon(t) := 8y_\varepsilon(t)$, $t > 0$ satisfy

$$y_\varepsilon'(t) + 4y_\varepsilon(t) \le h_\varepsilon(t) + c_1(\tau) \qquad \text{for all } t > \frac{\tau}{2}, \tag{2.25}$$

and invoking Lemma 2.9 we can furthermore pick $c_2(\tau) > 0$ fulfilling

$$\int_{\frac{\tau}{2}}^\infty y_\varepsilon(t)dt = \frac{1}{8}\int_{\frac{\tau}{2}}^\infty h_\varepsilon(t)dt \le c_2(\tau) \qquad \text{for all } \varepsilon \in (0,1).$$

This, namely, particularly enables us to choose $t_0(\varepsilon) \in (\frac{\tau}{2},\tau)$ such that $y_\varepsilon(t_0(\varepsilon)) \le \frac{2c_2}{\tau}$, and moreover allows us to infer upon integrating (2.25) that indeed

$$\begin{aligned}
y_\varepsilon(t) &\le y_\varepsilon(t_0(\varepsilon)) \cdot e^{-4(t-t_0(\varepsilon))} + \int_{t_0(\varepsilon)}^t e^{-4(t-s)} \cdot \Big\{ h_\varepsilon(s) + c_1(\tau) \Big\} ds \\
&\le \frac{2c_2}{\tau} + \int_{\frac{\tau}{2}}^t h_\varepsilon(s)ds + c_1(\tau)\int_{\frac{\tau}{2}}^t e^{-4(t-s)}ds \\
&\le \frac{2c_2}{\tau} + 8c_2(\tau) + \frac{c_1(\tau)}{4}
\end{aligned}$$

for all $\varepsilon \in (0,1)$ and any $t > \tau > t_0(\varepsilon)$. $\square$

Using that the integrability exponent in (2.24) is large than 2, by localizing an essentially well-established argument we can assert L^∞ bounds for u_ε:

Lemma 2.12 *Let $\tau \in (0,1)$. Then there exists $C(\tau) > 0$ such that for any choice of $\varepsilon \in (0,1)$, we have*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C(\tau) \qquad \text{for all } t > \tau. \tag{2.26}$$

PROOF. We fix any $q \in (2,4)$ and then obtain from known theory of parabolic smoothing estimate([11]) that there exist $c_1 = c_1(q) > 0$ and $c_2 > 0$ such that for the Neumann heat semigroup $(e^{t\Delta})_{t\ge 0}$ in Ω we have

$$\|e^{t\Delta}\nabla\cdot\psi\|_{L^\infty(\Omega)} \le c_1 t^{-\frac12-\frac1q}\|\psi\|_{L^q(\Omega)} \qquad \text{for all } t\in(0,1) \text{ and any } \psi \in C^1(\overline{\Omega};\mathbb{R}^2) \text{ with } \frac{\partial\psi}{\partial\nu}|_{\partial\Omega} = 0 \tag{2.27}$$

and

$$\|e^{t\Delta}\psi\|_{L^\infty(\Omega)} \le c_2 t^{-1}\|\psi\|_{L^1(\Omega)} \qquad \text{for all } t \in (0,1) \text{ and each } \psi \in C^0(\overline{\Omega}). \tag{2.28}$$

Apart from that, given $\tau \in (0,1)$, relying on the fact that $q < 4$ we may use the Hölder inequality and combine Lemma 2.10 with Lemma 2.11 to see that with some $c_3(\tau) > 0$ we have

$$\|u_\varepsilon(\cdot,t)\nabla v_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \le \|u_\varepsilon(\cdot,t)\|_{L^{\frac{4q}{4-q}}(\Omega)}\|\nabla v_\varepsilon(\cdot,t)\|_{L^4(\Omega)} \le c_3(\tau) \qquad \text{for all } \varepsilon\in(0,1) \text{ and } t > \frac{\tau}{2}.$$

In view of (2.3), again taking $m > 0$ from (2.1) we can thus estimate $u_\varepsilon(\cdot,t)$ for arbitrary $t > \tau$ by means of a associated variation-of-constants representation according to

$$\begin{aligned}
\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} &= \left\| e^{\frac{\tau}{2}\Delta}u_\varepsilon\Big(\cdot,t-\frac{\tau}{2}\Big) - \int_{t-\frac{\tau}{2}}^t e^{(t-s)\Delta}\nabla\cdot\Big(u_\varepsilon(\cdot,s)\nabla v_\varepsilon(\cdot,s)\Big)ds\right\|_{L^\infty(\Omega)} \\
&\le c_2\cdot\Big(\frac{\tau}{2}\Big)^{-1}\left\|u_\varepsilon\Big(\cdot,t-\frac{\tau}{2}\Big)\right\|_{L^1(\Omega)} + c_1\int_{t-\frac{\tau}{2}}^t (t-s)^{-\frac12-\frac1q}\|u_\varepsilon(\cdot,s)\nabla v_\varepsilon(\cdot,s)\|_{L^q(\Omega)}ds \\
&\le \frac{2c_2m}{\tau} + c_1c_3(\tau)\int_{t-\frac{\tau}{2}}^t (t-s)^{-\frac12-\frac1q}ds \qquad \text{for all } \varepsilon\in(0,1).
\end{aligned}$$

Since herein $\int_{t-\frac{\tau}{2}}^t (t-s)^{-\frac12-\frac1q}ds = \int_0^{\frac{\tau}{2}}\sigma^{-\frac12-\frac1q}d\sigma = \frac{2q}{q-2}\cdot(\frac{\tau}{2})^{\frac{q-2}{2q}}$ according to the restriction $q > 2$, this establishes (2.26). $\square$

As a final bootstrapping step, in quite a standard manner we achieve higher-order estimates.

Lemma 2.13 *For all $\tau \in (0,1)$ and $T > 1$ there exist $\theta = \theta(\tau,T) \in (0,1)$ and $C(\tau,T) > 0$ such that*

$$\|u_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[\tau,T])} + \|v_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[\tau,T])} \le C(\tau,T) \qquad \text{for all } \varepsilon\in(0,1). \tag{2.29}$$

PROOF. This can be obtained by following a straightforward bootstrap procedure: According to the boundedness results from Lemma 2.12 and Lemma 2.11, for each $\tau \in (0,1)$ and $T > 1$ standard parabolic Hölder regularity theory ([23]) asserts boundedness of $(u_\varepsilon)_{\varepsilon\in(0,1)}$ in $C^{\theta_1,\frac{\theta_1}{2}}(\overline{\Omega}\times[\frac{\tau}{4},4T])$ for some $\theta_1 = \theta_1(\tau,T) \in (0,1)$. Thereupon, classical interior parabolic Schauder theory applies so as to yield bounds firstly for $(v_\varepsilon)_{\varepsilon\in(0,1)}$ in $C^{2+\theta_2,1+\frac{\theta_2}{2}}(\overline{\Omega}\times[\frac{\tau}{2},2T])$, and secondly also for $(u_\varepsilon)_{\varepsilon\in(0,1)}$ in $C^{2+\theta_3,1+\frac{\theta_3}{2}}(\overline{\Omega}\times[\tau,T])$, with some suitably small $\theta_2 = \theta_2(\tau,T) \in (0,1)$ and $\theta_3 = \theta_3(\tau,T) \in (0,1)$. $\square$

2.3 Constructing a limit and identifying its solution properties for $t > 0$

Now a standard extraction process enables us to identify a limit with the desired solution properties for positive times.

Lemma 2.14 *There exists $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$, and such that*

$$u_\varepsilon \to u \qquad in\ C^{2,1}_{loc}(\overline{\Omega} \times (0,\infty)) \tag{2.30}$$

and

$$v_\varepsilon \to v \qquad in\ C^{2,1}_{loc}(\overline{\Omega} \times (0,\infty)) \tag{2.31}$$

as well as

$$\nabla v_\varepsilon \rightharpoonup \nabla v \qquad in\ L^2(\Omega \times (0,\infty)) \tag{2.32}$$

as $\varepsilon = \varepsilon_j \searrow 0$, where u and v are nonnegative radially symmetric functions from $C^{2,1}(\overline{\Omega} \times (0,\infty))$, with $\nabla v \in L^2(\Omega \times (0,\infty))$, which satisfy the boundary value problem in (1.4) in the classical sense in $\overline{\Omega} \times (0,\infty)$, and which moreover have the mass conservation property (1.8).

Proof. In view of the Arzelà-Ascoli theorem and weak sequential precompactness of bounded sets in $L^2(\Omega \times (0,\infty))$, due to (2.2), (2.1) and (2.3) all statements are immediate consequences of Lemma 2.13 and Lemma 2.2. $\square$

3 Solution properties near $t = 0$

The key part of our analysis now consists in verifying that despite the poor regularity information on the initial data, the limit functions u and v obtained in Lemma 2.14 are sufficiently well-behaved near $t = 0$ so as to allow for the claimed statements (1.10) and, especially, (1.9) concerning their initial traces. Our reasoning in this direction will be characterized by some significant interrelation, firstly concentrating on aspects related to the second solution component, and thereafter crucially relying on parts thereof when addressing the first.

3.1 Initial behavior of v

We first make sure that indeed v solves its sub-problem of the initial-boundary value problem in (1.4) in a suitably generalized sense:

Lemma 3.1 *Let u and v be as provided by Lemma 2.14. Then v and uv belong to $L^1_{loc}(\overline{\Omega} \times [0,\infty))$ with*

$$\int_0^\infty \int_\Omega v\varphi_t + \int_\Omega v_0\varphi(\cdot,0) = \int_0^\infty \int_\Omega \nabla v \cdot \nabla\varphi + \int_0^\infty \int_\Omega uv\varphi \qquad for\ all\ \varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty)). \tag{3.1}$$

Proof. The claimed integrability properties are immediate consequences of Lemma 2.14, because Lemma 2.4 along with Fatou's lemma warrants that actually $v \in L^1_{loc}([0,\infty); L^\infty(\Omega))$.

To derive (3.1), we fix $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$ and then obtain from (2.2) that

$$\int_0^\infty \int_\Omega v_\varepsilon\varphi_t + \int_\Omega v_{0\varepsilon}\varphi(\cdot,0) = \int_0^\infty \int_\Omega \nabla v_\varepsilon \cdot \nabla\varphi + \int_0^\infty \int_\Omega u_\varepsilon v_\varepsilon\varphi \qquad \text{for all } \varepsilon \in (0,1), \tag{3.2}$$

and here (2.1) guarantees that

$$\int_\Omega v_{0\varepsilon}\varphi(\cdot,0) \to \int_\Omega v_0\varphi(\cdot,0) \qquad \text{as } \varepsilon \searrow 0. \tag{3.3}$$

Moreover, taking $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ as given by Lemma 2.14 we know from the latter that $\nabla v_\varepsilon \rightharpoonup \nabla v$ in $L^2(\Omega\times(0,\infty))$ and hence infer that

$$\int_0^\infty \int_\Omega \nabla v_\varepsilon \cdot \nabla\varphi \to \int_0^\infty \int_\Omega \nabla v \cdot \nabla\varphi \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0. \tag{3.4}$$

To prepare an appropriate limit procedure in the first and last summand in (3.2), in accordance with Lemma 2.4 we pick $c_1 > 0$ such that

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le c_1 t^{-\frac{1}{2}} \qquad \text{for all } \varepsilon \in (0,1) \text{ and any } t \in (0,1), \tag{3.5}$$

and let $\eta > 0$ be given. Then fixing $\tau \in (0,1)$ in such a way that with $m > 0$ as in (2.1) we have

$$2c_1 m\|\varphi\|_{L^\infty(\Omega\times(0,\infty))}\sqrt{\tau} < \frac{\eta}{3}, \tag{3.6}$$

$$m\|\varphi\|_{L^\infty(\Omega\times(0,\infty))} \int_0^\tau \|v(\cdot,t)\|_{L^\infty(\Omega)}dt < \frac{\eta}{3}, \tag{3.7}$$

$$2c_1\|\varphi_t\|_{L^\infty(\Omega\times(0,\infty))}\sqrt{\tau} < \frac{\eta}{3} \tag{3.8}$$

and

$$\|\varphi_t\|_{L^\infty(\Omega\times(0,\infty))} \int_0^\tau \|v(\cdot,t)\|_{L^\infty(\Omega)}dt < \frac{\eta}{3}, \tag{3.9}$$

we split

$$\int_0^\infty \int_\Omega u_\varepsilon v_\varepsilon \varphi = \int_0^\tau \int_\Omega u_\varepsilon v_\varepsilon\varphi + \int_\tau^\infty \int_\Omega u_\varepsilon v_\varepsilon\varphi \qquad \text{for all } \varepsilon\in(0,1), \tag{3.10}$$

where in view of the compactness of $\operatorname{supp}\varphi$, Lemma 2.14 implies the existence of $\varepsilon_0 \in (0,1)$ such that

$$\left| \int_\tau^\infty \int_\Omega u_\varepsilon v_\varepsilon\varphi - \int_\tau^\infty \int_\Omega uv\varphi \right| < \frac{\eta}{3} \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}} \text{ such that } \varepsilon < \varepsilon_0.$$

Since, apart from that, (2.3) together with (3.5), (3.6) and (3.7) ensures that

$$\begin{aligned} \left| \int_0^\tau \int_\Omega u_\varepsilon v_\varepsilon\varphi \right| &\le m\|\varphi\|_{L^\infty(\Omega\times(0,\infty))} \int_0^\tau \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}dt \\ &\le m\|\varphi\|_{L^\infty(\Omega\times(0,\infty))} \cdot c_1 \int_0^\tau t^{-\frac{1}{2}}dt = m\|\varphi\|_{L^\infty(\Omega\times(0,\infty))}\cdot 2c_1\sqrt{\tau} \\ &< \frac{\eta}{3} \qquad \text{for all } \varepsilon\in(0,1) \end{aligned}$$

and that

$$\left| \int_0^\tau \int_\Omega uv\varphi \right| \le m\|\varphi\|_{L^\infty(\Omega\times(0,\infty))} \int_0^\tau \|v(\cdot,t)\|_{L^\infty(\Omega)}dt < \frac{\eta}{3},$$

and since $\eta > 0$ was arbitrary, it follows from (3.10) that

$$\int_0^\infty \int_\Omega u_\varepsilon v_\varepsilon \varphi \to \int_0^\infty \int_\Omega uv\varphi \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0. \tag{3.11}$$

Relying of (3.8) and (3.9), in quite a similar manner we can verify that also

$$\int_0^\infty \int_\Omega v_\varepsilon \varphi_t \to \int_0^\infty \int_\Omega v\varphi_t \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0,$$

which in conjunction with (3.3), (3.4) and (3.11) shows that (3.2) in fact implies (3.1). □

By adapting an argument from [36], we can next verify that the limit v from lemma 2.14 actually satisfies the expected counterpart of (2.5).

Lemma 3.2 *If u and v are as in Lemma 2.14, then $uv^2 \in L^1(\Omega \times (0,\infty))$, and the identity*

$$\frac{1}{2}\int_\Omega v^2(\cdot,t) + \int_0^t \int_\Omega |\nabla v|^2 + \int_0^t \int_\Omega uv^2 = \frac{1}{2}\int_\Omega v_0^2 \tag{3.12}$$

holds for all $t > 0$.

PROOF. In view of Lemma 2.2, the convergence properties from Lemma 2.14 together with Fatou's lemma entail that indeed uv^2 belongs to $L^1(\Omega \times (0,\infty))$, and that

$$\frac{1}{2}\int_\Omega v^2(\cdot,t) + \int_0^t \int_\Omega |\nabla v|^2 + \int_0^t \int_\Omega uv^2 \le \frac{1}{2}\int_\Omega v_0^2 \qquad \text{for all } t > 0. \tag{3.13}$$

In order to verify that moreover

$$\frac{1}{2}\int_\Omega v^2(\cdot,t) + \int_0^t \int_\Omega |\nabla v|^2 + \int_0^t \int_\Omega uv^2 \ge \frac{1}{2}\int_\Omega v_0^2 \qquad \text{for all } t > 0, \tag{3.14}$$

for $\eta > 0$ we introduce $\Phi_\eta(\xi) := \frac{1}{2} \cdot \frac{\xi^2}{1+\eta\xi}$, $\xi \ge 0$, and compute

$$\Phi_\eta'(\xi) = \frac{1}{2} \cdot \frac{2\xi + \eta\xi^2}{(1+\eta\xi)^2} = \xi - \frac{\eta}{2} \cdot \frac{\xi^2(3+2\eta\xi)}{(1+\eta\xi)^2}, \qquad \xi \ge 0, \tag{3.15}$$

as well as

$$\Phi_\eta''(\xi) = \frac{1}{(1+\eta\xi)^3}, \qquad \xi \ge 0. \tag{3.16}$$

Since thus, in particular, Φ_η is convex on $[0,\infty)$, we may follow a straightforward approximation procedure, as detailed e.g. in [30, 36] in closely related settings, to firstly derive from the weak identity in Lemma 3.1 that for each $\eta > 0$,

$$\int_\Omega \Phi_\eta(v(\cdot,t)) + \int_0^t \int_\Omega \Phi_\eta''(v)|\nabla v|^2 + \int_0^t \int_\Omega uv\Phi_\eta'(v) \ge \int_\Omega \Phi_\eta(v_0) \qquad \text{for a.e. } t > 0, \tag{3.17}$$

which clearly extends so as to remain valid actually for all $t > 0$ due to the continuity of v in $\overline{\Omega} \times (0,\infty)$ asserted by Lemma 2.14. Since using (3.16) we find that for all $\xi \ge 0$ we have $\Phi_\eta(\xi) \nearrow \frac{1}{2}\xi^2$ and $\Phi_\eta''(\xi) \nearrow 1$ as $\eta \searrow 0$, we may invoke Beppo Levi's theorem to infer that for each $t > 0$, we herein have

$$\int_\Omega \Phi_\eta(v(\cdot,t)) \nearrow \frac{1}{2}\int_\Omega v^2(\cdot,t) \qquad \text{and} \qquad \int_\Omega \Phi_\eta(v_0) \nearrow \frac{1}{2}\int_\Omega v_0^2 \tag{3.18}$$

as well as

$$\int_0^t \int_\Omega \Phi_\eta''(v)|\nabla v|^2 \nearrow \int_0^t \int_\Omega |\nabla v|^2 \tag{3.19}$$

as $\eta \searrow 0$. Moreover, (3.15) entails that

$$\int_0^t \int_\Omega uv\Phi_\eta'(v) = \int_0^t \int_\Omega uv^2 - \frac{1}{2}\int_0^t \int_\Omega \frac{\eta v(3+2\eta v)}{(1+\eta v)^2} \cdot uv^2 \qquad \text{for all } t > 0 \text{ and } \eta > 0,$$

where noting that

$$\frac{\eta v(3+2\eta v)}{(1+\eta v)^2} \cdot uv^2 \to 0 \quad \text{a.e. in } \Omega \times (0,\infty) \qquad \text{as } \eta \searrow 0,$$

and that

$$\Big|\frac{\eta v(3+2\eta v)}{(1+\eta v)^2} \cdot uv^2\Big| \le 2uv^2 \quad \text{in } \Omega \times (0,\infty) \qquad \text{for all } \eta > 0,$$

in view of the inclusion $uv^2 \in L^1(\Omega \times (0,\infty))$ we infer from the dominated convergence theorem that for any $t > 0$,

$$\int_0^t \int_\Omega uv\Phi_\eta'(v) \to \int_0^t \int_\Omega uv^2 \qquad \text{as } \eta \searrow 0.$$

In combination with (3.18) and (3.19), this shows that (3.17) indeed implies (3.14), and hence completes the proof. □

A first conclusion thereof already establishes (1.10).

Lemma 3.3 *With v taken from Lemma 2.14, we have*

$$v(\cdot,t) \to v_0 \quad \textit{in } L^2(\Omega) \qquad \textit{as } t \searrow 0. \tag{3.20}$$

PROOF. Since $|\nabla v|^2$ and uv^2 are elements of $L^1(\Omega \times (0,1))$ according to Lemma 3.2, and since thus, in particular,

$$\int_0^t \int_\Omega |\nabla v|^2 \to 0 \quad \text{and} \quad \int_0^t \int_\Omega uv^2 \to 0 \qquad \text{as } t \searrow 0,$$

from (3.12) we obtain that

$$\|v(\cdot,t)\|_{L^2(\Omega)} \to \|v_0\|_{L^2(\Omega)} \qquad \text{as } t \searrow 0. \tag{3.21}$$

Apart from that, fixing $t_0 > 0$ and an arbitrary $\psi \in C^\infty(\overline{\Omega})$ we can readily verify by means of a standard approximation procedure that the weak identity (3.1) remains valid also for $\varphi(x,t) := \zeta_h(t) \cdot \psi(x)$, $(x,t) \in \overline{\Omega} \times [0,\infty)$, where $h \in (0,1)$ is arbitrary and

$$\zeta_h(t) := \begin{cases} 1 & \text{if } t \in [0,t_0], \\ 1 - \frac{t-t_0}{h} & \text{if } t \in (t_0, t_0+h), \\ 0 & \text{if } t \ge t_0 + h. \end{cases} \tag{3.22}$$

Accordingly,

$$-\frac{1}{h}\int_{t_0}^{t_0+h}\int_\Omega v\psi + \int_\Omega v_0\psi = \int_0^{t_0}\int_\Omega \nabla v \cdot \nabla\psi + \int_0^{t_0}\int_\Omega uv\psi + \lambda(h) \qquad \text{for all } h \in (0,1), \tag{3.23}$$

where

$$\lambda(h) := \int_{t_0}^{t_0+h}\int_\Omega \zeta_h(t) \cdot \Big\{\nabla v(x,t) \cdot \nabla\psi(x) + u(x,t)v(x,t)\psi(x)\Big\}dxdt, \qquad h \in (0,1),$$

satisfies

$$|\lambda(h)| \le \|\nabla\psi\|_{L^\infty(\Omega)} \cdot \int_{t_0}^{t_0+h}\int_\Omega |\nabla v| + \|\psi\|_{L^\infty(\Omega)} \cdot \int_{t_0}^{t_0+h}\int_\Omega uv \to 0 \qquad \text{as } h \searrow 0$$

due to, e.g., the inclusions $\nabla v \in L^1(\Omega \times (t_0,t_0+1);\mathbb{R}^2)$ and $uv \in L^1(\Omega \times (t_0,t_0+1))$ asserted by Lemma 3.1. Since furthermore v is continuous on $\overline{\Omega} \times \{t_0\}$ according to Lemma 2.14, we also know that

$$-\frac{1}{h}\int_{t_0}^{t_0+h}\int_\Omega v\psi \to -\int_\Omega v(\cdot,t_0)\psi \qquad \text{as } h \searrow 0,$$

so that (3.23) implies that for any such ψ,

$$\int_\Omega v(\cdot,t_0)\psi - \int_\Omega v_0\psi = -\int_0^{t_0}\int_\Omega \nabla v \cdot \nabla\psi - \int_0^{t_0}\int_\Omega uv\psi \qquad \text{for all } t_0 > 0.$$

Again relying on the local integrability of ∇v and uv on $\overline{\Omega} \times [0,\infty)$, from this we infer in a way similar to the above that indeed

$$\int_\Omega v(\cdot,t_0)\psi - \int_\Omega v_0\psi \to 0 \qquad \text{as } t_0 \searrow 0.$$

Since $C^\infty(\overline{\Omega})$ is dense in $L^2(\Omega)$, and since $(v(\cdot,t))_{t\in(0,1)}$ is bounded in $L^2(\Omega)$ by e.g. (3.12), this ensures that $v(\cdot,t_0) \rightharpoonup v_0$ in $L^2(\Omega)$ as $t_0 \searrow 0$, and thereby, when combined with (3.21), establishes (3.20). □

Apart from that, Lemma 3.2 secondly implies strong L^2 convergence of ∇v_ε along the sequence from Lemma 2.14.

Lemma 3.4 *Let v and $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ be as given by Lemma 2.14. Then for each $T > 0$,*

$$\nabla v_\varepsilon \to \nabla v \quad \textit{in } L^2(\Omega \times (0,T)) \qquad \textit{as } \varepsilon = \varepsilon_j \searrow 0. \tag{3.24}$$

PROOF. By relying on (3.12), our argument follows the basic approach from [36], but in contrast to the situation addressed there, due to lacking information on boundedness of v_0, and hence on possible convergence of $u_\varepsilon v_\varepsilon^2$ in L^1 as $\varepsilon = \varepsilon_j \searrow 0$, we need to adequately cope with the rightmost summand on the left of (3.12).

To achieve this, we start with the observation that thanks to Lemma 2.14 and Fatou's lemma, for each $T > 0$ we separately have

$$\int_0^T \int_\Omega |\nabla v|^2 \le \liminf_{\varepsilon=\varepsilon_j \searrow 0} \int_0^T \int_\Omega |\nabla v_\varepsilon|^2 \qquad \text{and} \qquad \int_0^T \int_\Omega uv^2 \le \liminf_{\varepsilon=\varepsilon_j \searrow 0} \int_0^T \int_\Omega u_\varepsilon v_\varepsilon^2, \tag{3.25}$$

whereas Lemma 2.14 moreover entails that

$$\int_\Omega v_\varepsilon^2(\cdot, T) \to \int_\Omega v^2(\cdot, T) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0.$$

Therefore, by once more using (2.2) we find that due to (2.1) and Lemma 3.2,

$$\begin{aligned}
\int_0^T \int_\Omega |\nabla v|^2 + \int_0^T \int_\Omega uv^2 &\le \liminf_{\varepsilon=\varepsilon_j \searrow 0} \left\{ \int_0^T \int_\Omega |\nabla v_\varepsilon|^2 + \int_0^T \int_\Omega u_\varepsilon v_\varepsilon^2 \right\} \\
&\le \limsup_{\varepsilon=\varepsilon_j \searrow 0} \left\{ \int_0^T \int_\Omega |\nabla v_\varepsilon|^2 + \int_0^T \int_\Omega u_\varepsilon v_\varepsilon^2 \right\} \\
&= \limsup_{\varepsilon=\varepsilon_j \searrow 0} \left\{ \frac{1}{2} \int_\Omega v_{0\varepsilon}^2 - \frac{1}{2} \int_\Omega v_\varepsilon^2(\cdot, T) \right\} \\
&= \frac{1}{2} \int_\Omega v_0^2 - \frac{1}{2} \int_\Omega v^2(\cdot, T) = \int_0^T \int_\Omega |\nabla v|^2 + \int_0^T \int_\Omega uv^2.
\end{aligned}$$

In consequence, both inequalities in (3.25) must actually be identities, whence in particular

$$\int_0^T \int_\Omega |\nabla v_\varepsilon|^2 \to \int_0^T \int_\Omega |\nabla v|^2 \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0,$$

so that (3.24) results from (2.32). □

3.2 Initial behavior of u

Now the verification of (1.9) turns out to be somewhat more subtle, and our reasoning in this context will strongly rely on the assumed radial symmetry, although our first and basic step toward this is yet quite independent of this additional hypothesis, merely combining Lemma 2.3 with (2.3):

Lemma 3.5 *There exists $C > 0$ such that*

$$\int_0^\infty \left\| \nabla \sqrt{u_\varepsilon(\cdot, t) + 1} \right\|_{L^1(\Omega)}^2 dt \le C \qquad \textit{for all } \varepsilon \in (0, 1). \tag{3.26}$$

PROOF. By employing the Cauchy-Schwarz inequality and recalling (2.3) we see that with $m > 0$ from (2.1) we have

$$\begin{aligned}
\int_0^\infty \Big\| \nabla \sqrt{u_\varepsilon(\cdot,t)+1} \Big\|_{L^1(\Omega)}^2 dt &= \frac{1}{4} \int_0^\infty \bigg\{ \int_\Omega \frac{|\nabla u_\varepsilon|}{\sqrt{u_\varepsilon+1}} \bigg\}^2 \\
&\le \frac{1}{4} \int_0^\infty \bigg\{ \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} \bigg\} \cdot \bigg\{ \int_\Omega (u_\varepsilon+1) \bigg\} = \frac{m+|\Omega|}{4} \int_0^\infty \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}
\end{aligned}$$

for all $\varepsilon \in (0,1)$. Therefore, (3.26) results from Lemma 2.3. □

From now on, we make essential use of our assumption on radial symmetry of solutions, and in order to avoid abundant notation during our subsequent arguments in this direction, whenever convenient we switch to the usual radial notation in writing, e.g., $u_\varepsilon(r,t)$ instead of $u_\varepsilon(x,t)$ with $r = |x| \in [0,R]$.

By viewing (3.26) as an integral inequality involving the *de facto* spatially one-dimensional gradient $\partial_r \sqrt{u_\varepsilon+1}$, namely, through suitable interpolation we may exploit the latter in annular regions excluding the origin so as to obtain the following spatio-temporal L^2 bound for u_ε.

Lemma 3.6 *For all $\delta \in (0,R)$ and any $T > 0$ one can find $C(\delta,T) > 0$ such that*

$$\int_0^T \int_{\Omega \setminus B_\delta(0)} u_\varepsilon^2 \le C(\delta,T) \qquad \textit{for all } \varepsilon \in (0,1). \tag{3.27}$$

PROOF. According to the one-dimensional Gagliardo-Nirenberg inequality, given $\delta \in (0,R)$ we can fix $c_1(\delta) > 0$ such that

$$\|\psi\|_{L^4((\delta,R))}^4 \le c_1(\delta) \|\psi_r\|_{L^1((\delta,R))}^2 \|\psi\|_{L^2((\delta,R))}^2 + c_1(\delta) \|\psi\|_{L^2((\delta,R))}^4 \qquad \text{for all } \psi \in W^{1,1}((\delta,R)). \tag{3.28}$$

Apart from that, from Lemma 3.5 and (2.3) we obtain $c_2 > 0$ and $c_3 > 0$ such that

$$\int_0^\infty \bigg\{ \int_0^R r \cdot \Big| \partial_r \sqrt{u_\varepsilon(r,t)+1} \Big| dr \bigg\}^2 dt \le c_2 \qquad \text{for all } \varepsilon \in (0,1)$$

as well as

$$\int_0^R r \cdot \Big| \sqrt{u_\varepsilon(r,t)+1} \Big|^2 dr \le c_3 \qquad \text{for all } \varepsilon \in (0,1) \text{ and any } t > 0.$$

These inequalities in particular imply that whenever $\varepsilon \in (0,1)$,

$$\int_0^\infty \Big\| \partial_r \sqrt{u_\varepsilon(\cdot,t)+1} \Big\|_{L^1((\delta,R))}^2 dt \le \frac{c_2}{\delta^2}$$

and

$$\Big\| \sqrt{u_\varepsilon(\cdot,t)+1} \Big\|_{L^2((\delta,R))} \le \sqrt{\frac{c_3}{\delta}} \qquad \text{for all } t > 0,$$

whence an application of (3.28) shows that

$$\begin{aligned}
\int_0^T \int_{\Omega\setminus B_\delta(0)} (u_\varepsilon+1)^2 &= 2\pi \int_0^T \int_\delta^R r\cdot \Big(u_\varepsilon(r,t)+1\Big)^2 drdt \\
&\le 2\pi R \int_0^T \int_\delta^R \Big(u_\varepsilon(r,t)+1\Big)^2 drdt \;=\; 2\pi R\int_0^T \Big\|\sqrt{u_\varepsilon(\cdot,t)+1}\Big\|^4_{L^4((\delta,R))} dt \\
&\le 2\pi R c_1 \int_0^T \Big\|\partial_r \sqrt{u_\varepsilon(\cdot,t)+1}\Big\|^2_{L^1((\delta,R))} \Big\|\sqrt{u_\varepsilon(\cdot,t)+1}\Big\|^2_{L^2((\delta,R))} dt \\
&\quad +2\pi R c_1 \int_0^T \Big\|\sqrt{u_\varepsilon(\cdot,t)+1}\Big\|^4_{L^2((\delta,R))} dt \\
&\le 2\pi R c_1 \cdot \frac{c_2}{\delta^2}\cdot\frac{c_3}{\delta} + 2\pi R c_1 \cdot \frac{c_3^2}{\delta^2}\cdot T \qquad \text{for all } \varepsilon\in(0,1)
\end{aligned}$$

and thereby entails (3.27). □

Along with the strong convergence statement from Lemma 3.4, the weak compactness property thereby implied enables us to make sure that at least outside the spatial origin, also u satisfies a certain weak formulation of the initial-boundary value problem in (1.4):

Lemma 3.7 *Let u and v be as given by Lemma 2.14. Then the identity*

$$-\int_0^\infty \int_\Omega u\varphi_t - \mu_0(\varphi(\cdot,0)) = \int_0^\infty \int_\Omega u\Delta\varphi + \int_0^\infty \int_\Omega u\nabla v\cdot\nabla\varphi \qquad (3.29)$$

is valid for each $\varphi\in C_0^\infty((\overline{\Omega}\setminus\{0\})\times[0,\infty))$ fulfilling $\frac{\partial\varphi}{\partial\nu}=0$ on $\partial\Omega\times(0,\infty)$.

Proof. According to (2.2), given any such φ we have

$$-\int_0^\infty \int_\Omega u_\varepsilon\varphi_t - \int_\Omega u_{0\varepsilon}\varphi(\cdot,0) = \int_0^\infty \int_\Omega u_\varepsilon\Delta\varphi + \int_0^\infty \int_\Omega u_\varepsilon\nabla v_\varepsilon\cdot\nabla\varphi \qquad \text{for all } \varepsilon\in(0,1), \qquad (3.30)$$

where by (2.1),

$$\int_\Omega u_{0\varepsilon}\varphi(\cdot,0) \to \mu_0(\varphi(\cdot,0)) \qquad \text{as } \varepsilon\searrow 0. \qquad (3.31)$$

Moreover, using that $\operatorname{supp}\varphi\subset(\overline{\Omega}\setminus B_\delta(0))\times[0,T]$ for some $\delta>0$ and $T>0$, we may employ Lemma 3.6, which namely guarantees that with $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ taken from Lemma 2.14,

$$u_\varepsilon \rightharpoonup u \qquad \text{in } L^2\Big((\overline{\Omega}\setminus B_\delta(0))\times(0,T)\Big), \qquad (3.32)$$

and that hence

$$-\int_0^\infty \int_\Omega u_\varepsilon\varphi_t = -\int_0^T \int_{\Omega\setminus B_\delta(0)} u_\varepsilon\varphi_t \to -\int_0^T \int_{\Omega\setminus B_\delta(0)} u\varphi_t = -\int_0^\infty \int_\Omega u\varphi_t \qquad (3.33)$$

and, similarly,

$$\int_0^\infty \int_\Omega u_\varepsilon\Delta\varphi \to \int_0^\infty \int_\Omega u\Delta\varphi \qquad (3.34)$$

as $\varepsilon = \varepsilon_j \searrow 0$. Since furthermore $\nabla v_\varepsilon \to \nabla v$ in $L^2(\Omega \times (0,T))$ as $\varepsilon = \varepsilon_j \searrow 0$ by Lemma 3.4, again by means of (3.32) we conclude that also

$$\int_0^\infty \int_\Omega u_\varepsilon \nabla v_\varepsilon \cdot \nabla \varphi \to \int_0^\infty \int_\Omega u \nabla v \cdot \nabla \varphi \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0,$$

and that therefore (3.29) results from (3.30), (3.31), (3.33) and (3.34). □

Properly evaluating the latter, we can next make sure that u indeed attains μ_0 as its initial trace, albeit yet in a topology weaker than that claimed in Theorem 1.1.

Lemma 3.8 *The function u gained in Lemma 2.14 has the property that for each $\psi \in C_0^\infty(\overline{\Omega} \setminus \{0\})$ satisfying $\frac{\partial \psi}{\partial \nu} = 0$ on $\partial\Omega$,*

$$\int_\Omega u(\cdot,t)\psi \to \mu_0(\psi) \qquad \text{as } t \searrow 0. \tag{3.35}$$

PROOF. Similar to the proof of Lemma 3.1, for $t_0 > 0$ and $h \in (0,1)$ we let $\zeta_h(t) := 1$ for $t \in [0, t_0]$, $\zeta_h(t) := 1 - \frac{t-t_0}{h}$ for $t \in (t_0, t_0+h)$ and $\zeta_h(t) := 0$ for $t \ge t_0 + h$, and then see through a standard approximation that (3.29) remains valid for $\varphi(x,t) := \zeta_h(t) \cdot \psi(x)$, $(x,t) \in (\overline{\Omega} \setminus \{0\}) \times [0,\infty)$, and thus yields the identity

$$\begin{aligned} \frac{1}{h}\int_{t_0}^{t_0+h} \int_\Omega u\psi - \mu_0(\psi) &= \int_0^{t_0}\int_\Omega u\Delta\psi + \int_0^{t_0}\int_\Omega u\nabla v \cdot \nabla\psi \\ &\quad + \int_{t_0}^{t_0+h} \zeta_h(t) \cdot \int_\Omega \Big\{ u\Delta\psi + u\nabla v \cdot \nabla \psi \Big\} \qquad \text{for all } h \in (0,1). \end{aligned} \tag{3.36}$$

Here using the continuity of u and of ∇v on $\overline{\Omega} \times \{t_0\}$, as asserted by Lemma 2.14, we see that as $h \searrow 0$,

$$\frac{1}{h}\int_{t_0}^{t_0+h}\int_\Omega u\psi \to \int_\Omega u(\cdot,t_0)\psi$$

and

$$\int_{t_0}^{t_0+h} \zeta_h(t) \cdot \int_\Omega \Big\{ u\Delta\psi + u\nabla v \cdot \nabla\psi \Big\} \to 0,$$

whence (3.36) implies that

$$\int_\Omega u(\cdot,t_0)\psi - \mu_0(\psi) = \int_0^{t_0}\int_\Omega u\Delta\psi + \int_0^{t_0}\int_\Omega u\nabla v\cdot\nabla\psi \qquad \text{for all } t_0 > 0.$$

But since $\operatorname{supp}\psi \subset \overline{\Omega} \setminus B_\delta(0)$ for some $\delta \in (0,R)$ by hypothesis, and since from Lemma 3.6 and Lemma 2.14 we know that the functions u and $u|\nabla v|$ belong to $L^1((\Omega \setminus B_\delta(0)) \times (0,1))$, this implies that indeed

$$\int_\Omega u(\cdot,t_0)\psi - \mu_0(\psi) \to 0$$

as $t_0 \searrow 0$. □

In order to turn the above into the statement actually intended in (1.9), let us include a brief argument ensuring that any continuous radial function on $\overline{\Omega}$ vanishing at the origin can be suitably approximated by functions complying with the assumptions from Lemma 3.8.

Lemma 3.9 *Let $R > 0$ and $\rho \in C^0([0,R])$ be such that $\rho(0) = 0$. Then for all $\eta > 0$ one can find $\rho_\eta \in C_0^\infty((0,R])$ fulfilling*

$$\|\rho_\eta - \rho\|_{L^\infty((0,R))} < \eta \tag{3.37}$$

as well as

$$\rho_\eta'(R) = 0. \tag{3.38}$$

Proof. Since $\rho \in C^0([0,R])$, for each $\eta > 0$ the Weierstraß approximation theorem provides $\widehat{\rho}_\eta \in C^\infty([0,R])$ such that

$$\|\widehat{\rho}_\eta - \rho\|_{L^\infty((0,R))} < \frac{\eta}{4}. \tag{3.39}$$

As then also $\widehat{\rho}_\eta'$ in particular belongs to $L^1((0,R))$, by density of $C_0^\infty((0,R))$ in $L^1((0,R))$ we can find $P_\eta \in C_0^\infty((0,R))$ satisfying

$$\|P_\eta - \widehat{\rho}_\eta'\|_{L^1((0,R))} < \frac{\eta}{2}. \tag{3.40}$$

Letting

$$\rho_\eta(r) := \int_0^r P_\eta(s)ds, \qquad r \in [0,R],$$

we then immediately see that $\rho_\eta \in C^\infty([0,R])$ vanishes in some neighborhood of $r = 0$ and moreover satisfies $\rho_\eta' = P_\eta \equiv 0$ near $r = R$, which especially warrants (3.38). Moreover, combining (3.39) with (3.40) and our assumption that $\rho(0) = 0$, we infer that indeed

$$\begin{aligned}
|\rho_\eta(r) - \rho(r)| &\le |\rho_\eta(r) - \widehat{\rho}_\eta(r)| + |\widehat{\rho}_\eta(r) - \rho(r)| \\
&= \left| \int_0^r \Big\{ P_\eta(s) - \widehat{\rho}_\eta'(s) \Big\} ds - \widehat{\rho}_\eta(0) \right| + |\widehat{\rho}_\eta(r) - \rho(r)| \\
&\le \int_0^r |P_\eta(s) - \widehat{\rho}_\eta'(s)| ds + |\widehat{\rho}_\eta(0) - \rho(0)| + |\widehat{\rho}_\eta(r) - \rho(r)| \\
&\le \|P_\eta - \widehat{\rho}_\eta'\|_{L^1((0,R))} + 2\|\widehat{\rho}_\eta - \rho\|_{L^\infty((0,R))} \\
&< \frac{\eta}{2} + 2 \cdot \frac{\eta}{4} = \eta
\end{aligned}$$

for all $r \in (0,R)$. □

By means of suitable approximation based on the latter, we can now finally derive the claimed convergence property of the first solution component near the temporal origin.

Lemma 3.10 *For arbitrary $\psi \in C^0(\overline{\Omega})$, the function u obtained in Lemma 2.14 satisfies*

$$\int_\Omega u(\cdot,t)\psi \to \mu_0(\psi) \qquad \textit{as } t \searrow 0. \tag{3.41}$$

Proof. For fixed $\psi \in C^0(\overline{\Omega})$ and $\eta > 0$, we let the spherical average $S\psi \in C^0(\overline{\Omega})$ be as defined through (1.6), and abbreviating $\psi_0 := \psi(0)$ we infer from the continuity of the mapping $\mu_0 : C^0(\overline{\Omega}) \to \mathbb{R}$ at $S\psi - \psi_0$ that there exists $\eta_1(\eta) > 0$ such that

$$\left| \mu_0(\widehat{\psi}) - \mu_0(S\psi - \psi_0) \right| < \frac{\eta}{3} \qquad \text{for all } \widehat{\psi} \in C^0(\overline{\Omega}) \text{ fulfilling } \|\widehat{\psi} - (S\psi - \psi_0)\|_{L^\infty(\Omega)} \le \eta_1(\eta), \tag{3.42}$$

where taking $m > 0$ from (2.1) we may also assume that

$$\eta_1(\eta) < \frac{\eta}{3m}. \tag{3.43}$$

Moreover, using that with the vector $e_1 := (1,0)$, $\rho(r) := (S\psi)(re_1) - \psi_0$, $r \in [0,R]$, defines a function $\rho \in C^0([0,R])$ fulfilling $\rho(0) = 0$, by means of Lemma 3.9 we can find $\rho_\eta \in C_0^\infty((0,R])$ with the properties that $\rho'_\eta(R) = 0$ and

$$\|\rho_\eta - \rho\|_{L^\infty((0,R))} < \frac{\eta_1(\eta)}{3}. \tag{3.44}$$

Therefore,

$$\widehat{\psi}(x) := \rho_\eta(|x|), \qquad x \in \overline{\Omega},$$

belongs to $C_0^\infty(\overline{\Omega} \setminus \{0\})$ and satisfies $\frac{\partial \widehat{\psi}}{\partial \nu}(x) = 0$ for all $x \in \partial\Omega$, whence Lemma 3.8 applies so as to ensure the existence of $t_0(\eta) > 0$ such that

$$\Big| \int_\Omega u(\cdot,t)\widehat{\psi} - \mu_0(\widehat{\psi}) \Big| < \frac{\eta}{3} \qquad \text{for all } t \in (0, t_0(\eta)). \tag{3.45}$$

Now since (3.44) together with the radial symmetry of $S\psi$ guarantees that

$$\|\widehat{\psi} - (S\psi - \psi_0)\|_{L^\infty(\Omega)} = \sup_{r \in (0,R)} \Big| \widehat{\psi}(re_1) - \Big\{ (S\psi)(re_1) - \psi_0 \Big\} \Big| < \frac{\eta_1(\eta)}{3}, \tag{3.46}$$

and since $\mu_0(S\psi) = \mu_0(\psi)$ as well as $\int_\Omega u(\cdot,t)\psi = \int_\Omega u(\cdot,t) \cdot S\psi$ for all $t > 0$ by radial symmetry of μ_0 and u, by using (1.8) and combining (3.46) with (3.45), (3.42) and (3.43) we can estimate

$$\begin{aligned}
\Big| \int_\Omega u(\cdot,t)\psi - \mu_0(\psi) \Big| &= \Big| \int_\Omega u(\cdot,t) \cdot S\psi - \mu_0(S\psi) \Big| \\
&= \Big| \Big\{ \int_\Omega u(\cdot,t) \cdot (S\psi - \psi_0) + \psi_0 \cdot \int_\Omega u(\cdot,t) \Big\} - \Big\{ \mu_0(S\psi - \psi_0) + \mu_0(\psi_0) \Big\} \Big| \\
&= \Big| \int_\Omega u(\cdot,t) \cdot (S\psi - \psi_0) - \mu_0(S\psi - \psi_0) \Big| \\
&\le \Big| \int_\Omega u(\cdot,t) \cdot (S\psi - \psi_0 - \widehat{\psi}) \Big| + \Big| \int_\Omega u(\cdot,t)\widehat{\psi} - \mu_0(\widehat{\psi}) \Big| + \Big| \mu_0(\widehat{\psi}) - \mu_0(S\psi - \psi_0) \Big| \\
&\le \|\widehat{\psi} - (S\psi - \psi_0)\|_{L^\infty(\Omega)} \cdot m + \Big| \int_\Omega u(\cdot,t)\widehat{\psi} - \mu_0(\widehat{\psi}) \Big| + \Big| \mu_0(\widehat{\psi}) - \mu_0(S\psi - \psi_0) \Big| \\
&< \frac{\eta_1(\eta)}{3} \cdot m + \frac{\eta}{3} + \frac{\eta}{3} < \eta \qquad \text{for all } t \in (0, t_0(\eta)),
\end{aligned}$$

and hence conclude that in fact (3.41) holds, for $\eta > 0$ was arbitrary. $\Box$

3.3 Proof of Theorem 1.1

We finally only need to collect suitable among the above results to arrive at our main result:

Proof of Theorem 1.1. Taking u and v as constructed in Lemma 2.14, from the latter we directly obtain that $(u,v) \in (C^{2,1}(\overline{\Omega} \times (0,\infty))^2$ has the claimed symmetry and mass conservation properties and solves the first three lines from (1.4) classically in $\overline{\Omega} \times (0,\infty)$. The identification of its initial trace in (1.9) and (1.10) has been achieved in Lemma 3.10 and Lemma 3.3, respectively, where the latter in conjunction with (2.32) finally ensures that v enjoys the additional regularity property $v \in L^2((0,T);W^{1,2}(\Omega))$ for arbitrary $T > 0$. $\square$

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