# Boundedness in a two-dimensional Keller-Segel-Navier-Stokes system involving a rapidly diffusing repulsive signal

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**Abstract**

This paper is concerned with the Keller-Segel-Navier-Stokes system

$$\begin{cases} n_t + u\cdot\nabla n &= \Delta n + \nabla\cdot(n\nabla c), & & x\in\Omega,\ t>0,\\ u\cdot\nabla c &= \Delta c - c + n, & & x\in\Omega,\ t>0,\\ u_t + (u\cdot\nabla)u &= \Delta u + \nabla P + n\nabla\Phi, & \nabla\cdot u = 0, & x\in\Omega,\ t>0, \end{cases} \qquad (\star)$$

with a given smooth gravitational potential $\Phi$.

It is shown that for all suitably regular initial data, a corresponding no-flux/no-flux/Dirichlet initial-boundary value problem posed in a smoothly bounded planar domain admits a uniquely determined global classical solution $(n,c,u,P)$ which has the additional property that $n$ remains uniformly bounded.

This partially goes beyond a recent result asserting global classical solvability, but without any boundedness information, in a related slightly more complex variant of $(\star)$ accounting for parabolic evolution of the quantity $c$. In particular, the obtained outcome thereby provides further evidence indicating that the considered fluid interaction does not substantially reduce a certain explosion-avoiding character of the Keller-Segel-type chemorepulsion mechanisms, as known to form an essential feature of corresponding fluid-free analogues.

The reasoning at its core relies on the use of a quasi-Lyapunov inequality which operates at regularity levels that seem rather unusual in this and related contexts, but which in the considered two-dimensional setting can be seen to serve as a starting point sufficient for a bootstrap-type series of arguments finally providing global boundedness.

**Key words:** chemotaxis; repulsion; Navier-Stokes; global existence; boundedness
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## 1 Introduction

The interaction between populations of chemotactically migrating individuals and liquid environments has been the objective of considerable developments in the mathematical literature during the past decade. With to date persisting motivation originating from experimental findings that report nontrivial effects of such types of interplay, including support of pattern generation but also quantitative influences on spreading properties ([39], [6], [7], [29], [38]), analytical studies in this field on the one hand have been concerned with widely nontrivial questions related to corresponding existence theories, but beyond this, on the other hand, have partially been addressing aspects of qualitative solution behavior. Apart from some simulation-endorsed predictions on possibly blow-up preventing effects of fluid flows on Keller-Segel type models ([28]), in certain particular cases characterized by suitably weak coupling, in which only the motion of the liquid influences migration of cells and signals but not vice versa, some recent analytical results have rigorously confirmed nontrivial consequences of suitably chosen given fluid velocity fields on aggregation and spreading behavior in chemotaxis systems ([20], [21], [22], [17]).

In situations characterized by a genuinely mutual interaction in the sense that additionally also cells influence the fluid flow through buoyancy, as forming a core assumption in the modeling approaches e.g. from [39] and [1], in line with an accordingly increased complexity the knowledge is yet considerably sparser, with a predominant focus on aspects from mere existence theory in many cases. Indeed, already for a model coupling a chemotaxis-*consumption* process to the Navier-Stokes equations in the context of the system

$$\begin{cases} n_t + u\cdot\nabla n &= \Delta n - \nabla\cdot(n\nabla c), \\ c_t + u\cdot\nabla c &= \Delta c - nc, \\ u_t + (u\cdot\nabla)u &= \Delta u + \nabla P + n\nabla\Phi, \qquad \nabla\cdot u = 0, \end{cases} \tag{1.1}$$

for the cell population density $n$, the signal concentration $c$ and the fluid variables $u$ and $P$, establishing a reasonably comprehensive theory of global weak solvability has only been achieved quite recently, and only after a series of partial results relying on various types of simplifications or restrictions on sizes of either the initial data or some key model ingredients ([9], [3], [23], [44], [47]), although the corresponding fluid-free nutrient taxis system is known to admit quite a straightforward existence theory according to a substantially dissipative character going along with the considered signal absorption mechanism ([33]).

Whereas beyond deriving similar existence statements for several more general chemotaxis-consumption-fluid systems ([10], [40], [8], [36]), the literature for (1.1) and some close relatives meanwhile even provides some results on large time stabilization toward homogeneous equilibria ([54], [46], [48], [25], [51]), the knowledge seems yet at a significantly more rudimentary level in cases when unlike in (1.1), the respective signal evolution is determined by *production* through cells, rather than consumption. In fact, rigorous conclusions concerned with accordingly obtained Keller-Segel-fluid systems of the form

$$\begin{cases} n_t + u\cdot\nabla n &= \nabla\cdot(D(n)\nabla n) - \nabla\cdot(nS(n)\nabla c) + f(u), \\ c_t + u\cdot\nabla c &= \Delta c - c + n, \\ u_t + \kappa(u\cdot\nabla)u &= \Delta u + \nabla P + n\nabla\Phi, \qquad \nabla\cdot u = 0, \end{cases} \tag{1.2}$$

have so far mainly been restricted to statements on global solvability, with information on further properties such as boundedness limited to a few respective subcases, under appropriate hypotheses on smallness of initial data when $f \equiv 0, D \equiv S \equiv 1$ and $\kappa = 1$ ([23]), on suitably strong growth restrictions implied by considering logistic-type proliferation terms $f$ of the form $f(n) = \rho n - \mu n^2$ with $\rho \in \mathbb{R}$ and $\mu > 0$ when $D \equiv S \equiv 1$ and either $\kappa = 0$ or $\kappa = 1$ ([34], [35]), or on adequately efficient enhancement of diffusion or saturation of taxis at large cell denisties when $f \equiv 0$ ([27], [42], [43], [55], [2], [41], [49]). Clearly, this lack of comprehensiveness with regard to unconditionally smooth solution behavior partially reflects the circumstance that in comparison to the respective fluid-free version of (1.1), the signal production mechanism in the Keller-Segel chemotaxis-only subsystem of (1.2) is known to enforce singularity formation in the sense of finite-time blow-up throughout various ranges of its ingredients ([19], [45], [5]).

Now in contrast to the latter, Keller-Segel-production systems are known to exhibit significantly more regular solution behavior in cases when instead of attractive cross-diffusive mechanisms, as characterized by choices of positive $S$, rather *chemorepulsion* processes are accounted for. Fluid-free variants of (1.2) obtained on letting $S \equiv -1$, for instance, have been found to admit global classical and bounded solutions in two-dimensional domains in the paradigmatic setting when $D \equiv 1$ and $f \equiv 0$ for which the corresonding chemoattraction system with $S \equiv 1$ possesses some exploding solutions; even in its three-dimensional version, this repulsive Keller-Segel system still allows at least for global weak solvability, regardless of the size of the respective initial data ([4]). Also in the presence of more general parameter functions $D$ and $S$, some significantly relaxing effects of repulsion in comparison to attraction have been identified in the literature ([37], [11], [24]).

**Absence of blow-up in two-dimensional chemorepulsion-Navier-Stokes systems. Main results.** In such contexts of chemorepulsion, possible effects of fluid coupling in the style of (1.2) have recently been studied in [53], and the main outcome thereof indicated that some unconditionality of finite-time blow-up suppression in fact persists also in the presence of buoyancy-induced interaction with the Navier-Stokes system in spatially two-dimensional frameworks. Indeed, that main results from [53] assert global classical solvability in an initial-boundary value problem for the fully parabolic system (1.2) in bounded planar domains in the apparently most prototypical situation determined by the choices $D \equiv 1$, $f \equiv 0$ and $\kappa = 1$.

The present work intends to provide one step further in this direction by investigating the question how far apart from this, also *infinite-time blow-up*, well-known as a possible feature of various attractive Keller-Segel type systems ([5], [26], [50]) can be ruled out in two-dimensional chemorepulsion-Navier-Stokes systems. In order to make this advanced problem accessible to our analysis, we shall address this issue in the framework of a simplified variant of (1.2) in which the signal evolution is governed by an associated elliptic equation, and which hence reflects the assumption that the considered chemical diffuses much faster than cells and fluid particles, as having formed a technically essential fundament already in numerous precedents from the chemotaxis(-fluid) literature (cf. e.g. [30] and [22] for two representative examples).

Throughout the sequel, in a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary we shall accordingly be concerned with the initial-boundary value problem

$$\begin{cases} n_t + u\cdot\nabla n &= \Delta n + \nabla\cdot(n\nabla c), & & x\in\Omega,\ t>0,\\ u\cdot\nabla c &= \Delta c - c + n, & & x\in\Omega,\ t>0,\\ u_t + (u\cdot\nabla)u &= \Delta u + \nabla P + n\nabla\Phi, & \nabla\cdot u = 0, & x\in\Omega,\ t>0,\\ \frac{\partial n}{\partial\nu} = \frac{\partial c}{\partial\nu} = 0, \quad u = 0, & & & x\in\partial\Omega,\ t>0,\\ n(x,0) = n_0(x), \quad u(x,0) = u_0(x), & & & x\in\Omega, \end{cases} \tag{1.3}$$

where our standing hypothesis on the gravitational potential $\Phi$ will be that

$$\Phi \in W^{2,\infty}(\Omega), \tag{1.4}$$

and where concerning the initial data we shall assume that

$$\begin{cases} n_0 \in C^0(\overline{\Omega}) \quad \text{is nonnegative with } \overline{n}_0 > 0, \qquad \text{and that}\\ u_0 \in W^{2,2}(\Omega;\mathbb{R}^2) \cap W^{1,2}_{0,\sigma}(\Omega). \end{cases} \tag{1.5}$$

Here and below, as usual we write $\overline{\varphi} := \frac{1}{|\Omega|}\int_\Omega \varphi$ for $\varphi \in L^1(\Omega)$, and let $W^{1,2}_{0,\sigma}(\Omega) := W^{1,2}_0(\Omega;\mathbb{R}^2)\cap L^2_\sigma(\Omega)$, with $L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega;\mathbb{R}^2) \mid \nabla\cdot\varphi = 0 \text{ in } \mathcal{D}(\Omega)\}$ denoting the space of all solenoidal vector fields in $L^2(\Omega;\mathbb{R}^2)$. To complete our collection of notational conventions in this respect, let us furthermore announce that throughout the sequel, we let $A = -\mathcal{P}\Delta$ denote the realization of the Stokes operator in $L^2(\Omega;\mathbb{R}^2)$, with its domain given by $D(A) = W^{2,2}(\Omega;\mathbb{R}^2)\cap W^{1,2}_{0,\sigma}(\Omega)$, and with $\mathcal{P}$ denoting the Helmholtz projection on $L^2(\Omega;\mathbb{R}^2)$, and for $\alpha > 0$ we let $A^\alpha$ represent the corresponding sectorial fractional powers.

In this form, (1.3) will turn out to enjoy a certain basic quasi-Lyapunov inequality which, though operating at yet quite low regularity levels, will enable us to adequately cope with the key challenge to establish some first time-independent estimates suitably going beyond the $L^1$ boundedness information gained from an evident mass conservation feature in (1.3). Specifically, our analysis will be based on the observation that thanks to some rudimentary but time-independent basic regularity features of $c$, available due to ellipticity of its sub-problem in (1.3) (Section 3), for appropriately chosen $a > 0$ and $C > 0$ the quantities

$$\mathcal{F}(t) := -\int_\Omega \ln(n+1) + a\int_\Omega |u|^2 \qquad \text{and} \qquad \mathcal{D}(t) := \int_\Omega n\ln(n+1) + \int_\Omega |\nabla u|^2 \tag{1.6}$$

satisfy

$$\mathcal{F}'(t) + \frac{1}{C}\mathcal{F}(t) + \frac{1}{C}\mathcal{D}(t) \le C$$

throughout the life span of a smooth solution to (1.3) (Lemma 4.4), the existence of which, as well as an associated extensibility criterion, is asserted in the preliminary Section 2. Accordingly implied a priori estimates for the dissipation rate functional $\mathcal{D}$ (Lemma 4.5) will thereafter be seen to constitute a starting point for a series of arguments acting at successively higher stages of regularity, inter alia addressing the time evolution of the functionals $\int_\Omega n\ln\frac{n}{\overline{n}_0} + b\int_\Omega |\nabla u|^2 + 1$, for some $b > 0$ (Lemma

5.4), and $\int_\Omega n^2$ (Lemma 5.5), as seeming to be more closely related to approaches pursued in precedent studies.

This will finally reveal the following main result which indeed warrants boundedness of solutions irrespective of the size of their initial data, and hence excludes any possibility of infinite-time blow-up in (1.3):

**Theorem 1.1** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and assume that (1.4) holds and that $n_0$ and $u_0$ comply with (1.5). Then there exist functions*

$$\begin{cases} n \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ c \in C^{2,0}(\overline{\Omega} \times (0,\infty)), \\ u \in \bigcap_{\alpha \in (\frac{1}{2},1)} C^0([0,\infty); D(A^\alpha)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty); \mathbb{R}^2) \qquad \text{and} \\ P \in C^{1,0}(\Omega \times (0,\infty)), \end{cases}$$

*uniquely determined up to addition of constants to $P$, such that $n > 0$ and $c > 0$ in $\overline{\Omega} \times (0,\infty)$, that $(n,c,u,P)$ solves (1.3) classically in $\Omega \times (0,\infty)$, and that*

$$\sup_{t>0} \|n(\cdot,t)\|_{L^\infty(\Omega)} < \infty. \tag{1.7}$$

## 2 Local existence and extensibility

A theory of local existence in (1.3) can be obtained by adapting basically well-established arguments involving a contraction-type reasoning in a suitable function space framework. As we could not find a reference reasonably close to the situation addressed here, however, we include at least the essential part of a corresponding proof here.

Our approach will address the main difference in comparison to fully parabolic relatives, such as e.g. the class of problems considered in detail in [44], through the following statement on unique solvability in a variant of the elliptic sub-problem contained in (1.3).

**Lemma 2.1** *Let $\alpha \in (\frac{1}{2},1)$. Then there exists $C > 0$ with the property that whenever $T > 0$, for each $\widehat{n} \in C^0(\overline{\Omega} \times [0,T])$ and $\widehat{u} \in C^0([0,T]; D(A^\alpha))$ the problem*

$$\begin{cases} -\Delta c + c = \widehat{n} - \widehat{u} \cdot \nabla c, & x \in \Omega, \ t \in [0,T], \\ \frac{\partial c}{\partial \nu} = 0, & x \in \partial\Omega, \ t \in [0,T], \end{cases} \tag{2.1}$$

*possesses a unique weak solution $c \in C^0([0,T]; W^{2,2}(\Omega))$, and such that this solution satisfies*

$$\|c(\cdot,t)\|_{W^{2,2}(\Omega)} \le C \cdot \sup_{s\in(0,T)} \|\widehat{n}(\cdot,s)\|_{L^\infty(\Omega)} \cdot \Big\{ 1 + \sup_{s\in(0,T)} \|A^\alpha \widehat{u}(\cdot,s)\|_{L^2(\Omega)} \Big\} \qquad \textit{for all } t \in [0,T]. \tag{2.2}$$

PROOF. Given any such $T, \widehat{n}$ and $\widehat{u}$, recalling that

$$D(A^\alpha) \hookrightarrow C^0(\overline{\Omega}; \mathbb{R}^2) \tag{2.3}$$

([14], [18]) we see that for each fixed $t \in [0,T]$ letting

$$B(\varphi,\psi) := \int_\Omega \nabla\varphi \cdot \nabla\psi + \int_\Omega \varphi\psi + \int_\Omega (\widehat{u}(\cdot,t)\cdot\nabla\varphi)\psi, \qquad \varphi \in W^{1,2}(\Omega),\ \psi \in W^{1,2}(\Omega),$$

introduces a well-defined and continuous bilinear form on $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$. By solenoidality of $\widehat{u}(\cdot,t)$ and the fact that $\widehat{u}(\cdot,t) = 0$ on $\partial\Omega$, $B$ moreover is corecive, because

$$\begin{aligned} B(\varphi,\varphi) &= \int_\Omega |\nabla\varphi|^2 + \int_\Omega \varphi^2 + \frac{1}{2}\int_\Omega \widehat{u}(\cdot,t)\cdot\nabla\varphi^2 \\ &= \int_\Omega |\nabla\varphi|^2 + \int_\Omega \varphi^2 \qquad \text{for all } \varphi \in W^{1,2}(\Omega), \end{aligned} \tag{2.4}$$

whence according to the Lax-Milgram lemma there exists a unique $c(\cdot,t) \in W^{1,2}(\Omega)$ such that

$$\int_\Omega \nabla c(\cdot,t)\cdot\nabla\varphi + \int_\Omega c(\cdot,t)\varphi = \int_\Omega \widehat{n}(\cdot,t)\varphi - \int_\Omega \Big(\widehat{u}(\cdot,t)\cdot\nabla c(\cdot,t)\Big)\varphi \qquad \text{for all } \varphi \in W^{1,2}(\Omega). \tag{2.5}$$

As a first estimate for $c(\cdot,t)$, in view of (2.4) we obtain from (2.5) and Young's inequality that

$$\int_\Omega |\nabla c(\cdot,t)|^2 + \int_\Omega c^2(\cdot,t) = \int_\Omega \widehat{n}(\cdot,t)c(\cdot,t) \le \frac{1}{2}\int_\Omega c^2(\cdot,t) + \frac{|\Omega|}{2}\|\widehat{n}(\cdot,t)\|^2_{L^\infty(\Omega)},$$

which implies that

$$\int_\Omega |\nabla c(\cdot,t)|^2 + \int_\Omega c^2(\cdot,t) \le |\Omega|\cdot R_1^2, \tag{2.6}$$

where $R_1 := \sup_{s\in(0,T)} \|n(\cdot,s)\|_{L^\infty(\Omega)}$.

Apart from that, (2.4) and (2.5) ensure that for arbitrary $t \in [0,T]$ and $s \in [0,T]$, $d := c(\cdot,t) - c(\cdot,s)$ satisfies

$$\begin{aligned} \int_\Omega |\nabla d|^2 + \int_\Omega d^2 &= \int_\Omega \Big(\widehat{n}(\cdot,t) - \widehat{n}(\cdot,s)\Big)d - \int_\Omega \Big\{\Big(\widehat{u}(\cdot,t) - \widehat{u}(\cdot,s)\Big)\cdot\nabla c(\cdot,t)\Big\}\,d - \int_\Omega \Big(\widehat{u}(\cdot,s)\cdot\nabla d\Big)d \\ &= \int_\Omega \Big(\widehat{n}(\cdot,t) - \widehat{n}(\cdot,s)\Big)d - \int_\Omega \Big\{\Big(\widehat{u}(\cdot,t) - \widehat{u}(\cdot,s)\Big)\cdot\nabla c(\cdot,t)\Big\}\,d \\ &\le \frac{1}{2}\int_\Omega d^2 + \|\widehat{n}(\cdot,t) - \widehat{n}(\cdot,s)\|^2_{L^2(\Omega)} + \|\widehat{u}(\cdot,t) - \widehat{u}(\cdot,s)\|^2_{L^\infty(\Omega)}\|\nabla c(\cdot,t)\|^2_{L^2(\Omega)}, \end{aligned}$$

which together with (2.6) entails that

$$c \in C^0([0,T]; W^{1,2}(\Omega)), \tag{2.7}$$

because our assumptions in particular guarantee that

$$\widehat{n} \in C^0([0,T]; L^2(\Omega)) \qquad \text{and} \qquad \widehat{u} \in C^0(\overline{\Omega}\times[0,T]; \mathbb{R}^2) \tag{2.8}$$

due to (2.3).

We next recall standard elliptic regularity theory ([16]) to fix $C_1 > 0$ with the property that if

$f \in L^2(\Omega)$ and $v \in W^{1,2}(\Omega)$ are such that $\int_\Omega \nabla v \cdot \nabla \varphi = \int_\Omega f\varphi$ for all $\varphi \in W^{1,2}(\Omega)$, then actually $v \in W^{2,2}(\Omega)$ with

$$\|v\|_{W^{2,2}(\Omega)} \le C_1 \|f\|_{L^2(\Omega)}. \tag{2.9}$$

In view of (2.6), a first application of this yields, as our second estimate for $c$, the inequality

$$\begin{aligned} \|c(\cdot,t)\|_{W^{2,2}(\Omega)} &\le C_1 \Big\| \widehat{n}(\cdot,t) - \widehat{u}(\cdot,t)\cdot\nabla c(\cdot,t)\Big\|_{L^2(\Omega)} \\ &\le C_1\|\widehat{n}(\cdot,t)\|_{L^2(\Omega)} + C_1\|\widehat{u}(\cdot,t)\|_{L^\infty(\Omega)}\|\nabla c(\cdot,t)\|_{L^2(\Omega)} \\ &\le C_1|\Omega|^{\frac{1}{2}}R_1 + C_1C_2R_1R_2 \qquad \text{for all } t\in[0,T], \end{aligned} \tag{2.10}$$

with $R_2 := \sup_{s\in(0,T)} \|A^\alpha \widehat{u}(\cdot,s)\|_{L^2(\Omega)}$, and with $C_2 := |\Omega|^{\frac{1}{2}} \cdot \sup_{0\ne\varphi\in D(A^\alpha)} \frac{\|\varphi\|_{L^\infty(\Omega)}}{\|A^\alpha\varphi\|_{L^2(\Omega)}}$ being finite due to (2.3). This establishes (2.2), so that it remains to verify the claimed continuity property. To this end, we note that as a second consequence of (2.9) we obtain that for all $t\in[0,T]$ and $s\in[0,T]$,

$$\begin{aligned} \|c(\cdot,t)-c(\cdot,s)\|_{W^{2,2}(\Omega)} &\le C_1\Big\|\widehat{n}(\cdot,t)-\widehat{n}(\cdot,s)-\widehat{u}(\cdot,t)\cdot\nabla c(\cdot,t)+\widehat{u}(\cdot,s)\cdot\nabla c(\cdot,s)\Big\|_{L^2(\Omega)} \\ &\le C_1\|\widehat{n}(\cdot,t)-\widehat{n}(\cdot,s)\|_{L^2(\Omega)} \\ &\quad +C_1\|\widehat{u}(\cdot,t)-\widehat{u}(\cdot,s)\|_{L^\infty(\Omega)}\|\nabla c(\cdot,s)\|_{L^2(\Omega)} \\ &\quad +C_1\|\widehat{u}(\cdot,t)\|_{L^\infty(\Omega)}\|\nabla c(\cdot,t)-\nabla c(\cdot,s)\|_{L^2(\Omega)}, \end{aligned}$$

so that relying on (2.7) and once more on our assumption we conclude that indeed $c$ is continuous on $[0,T]$ as a $W^{2,2}(\Omega)$-valued function. $\square$

We can thereby establish a local theory by means of a rather straightforward reasoning based on Banach's fixed point theorem.

**Lemma 2.2** *Suppose that (1.4) and (1.5) hold. Then there exist $T_{max}\in(0,\infty]$ and functions*

$$\begin{cases} n\in C^0(\overline{\Omega}\times[0,T_{max}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{max})), \\ c\in C^{2,0}(\overline{\Omega}\times(0,T_{max})), \\ u\in \bigcap_{\alpha\in(\frac{1}{2},1)} C^0([0,T_{max});D(A^\alpha))\cap C^{2,1}(\overline{\Omega}\times(0,T_{max});\mathbb{R}^2) \qquad \textit{and} \\ P\in C^{1,0}(\Omega\times(0,T_{max})), \end{cases} \tag{2.11}$$

*uniquely determined up to addition of constants to $P$, such that $n>0$ and $c>0$ in $\overline{\Omega}\times(0,T_{max})$, that $(n,c,u,P)$ forms a classical solution of (1.3) in $\Omega\times(0,T_{max})$, and that*

$$\textit{if } T_{max}<\infty, \textit{ then } \limsup_{t\nearrow T_{max}} \Big\{\|n(\cdot,t)\|_{L^\infty(\Omega)}+\|A^\alpha u(\cdot,t)\|_{L^2(\Omega)}\Big\} \textit{ for all } \alpha\in(\tfrac{1}{2},1). \tag{2.12}$$

*Moreover,*

$$\int_\Omega n(\cdot,t)=\int_\Omega n_0 \qquad \textit{for all } t\in(0,T_{max}) \tag{2.13}$$

*and*

$$\int_\Omega c(\cdot,t)=\int_\Omega n_0 \qquad \textit{for all } t\in(0,T_{max}). \tag{2.14}$$

PROOF. As our reasoning follows quite a well-established line of arguments, we restrict ourselves to sketching the main aspects. For fixed $\alpha \in (\frac{1}{2}, 1)$ and with $T \in (0,1)$ to be chosen below and

$$R := \|n_0\|_{L^\infty(\Omega)} + \|A^\alpha u_0\|_{L^2(\Omega)} + 1,$$

on the closed subset

$$S := \Big\{ (n,u) \in X \ \Big| \ \|n(\cdot,t)\|_{L^\infty(\Omega)} + \|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} \le R \ \text{ for all } t \in [0,T] \Big\}$$

of the Banach space $X := C^0([0,T]; C^0(\overline{\Omega}) \times D(A^\alpha))$, equipped with the norm defined by setting $\|(n,u)\|_X := \sup_{t\in(0,T)} \Big\{ \|n(\cdot,t)\|_{L^\infty(\Omega)} + \|A^\alpha u(\cdot,t)\|_{L^\infty(\Omega)} \Big\}$ for $(u,v) \in X$, we introduce a mapping $\Psi = (\Psi_1, \Psi_2)$ by letting

$$(\Psi_1[\widehat{n},\widehat{u}])(\cdot,t) := e^{t\Delta} n_0 + \int_0^t e^{(t-s)\Delta} \nabla \cdot \Big\{ \widehat{n}(\cdot,s) \Big( \nabla c(\cdot,s) - \widehat{u}(\cdot,s) \Big) \Big\} ds$$

and

$$(\Psi_2[\widehat{n},\widehat{u}])(\cdot,t) := e^{-tA} u_0 - \int_0^t e^{-(t-s)A} \mathcal{P} \Big\{ (\widehat{u}(\cdot,s)\cdot\nabla)\widehat{u}(\cdot,s) - \widehat{n}(\cdot,s)\nabla\Phi \Big\} ds$$

for $(\widehat{n},\widehat{u}) \in S$ and $t \in [0,T]$, where $(e^{t\Delta})_{t\ge0}$ and $(e^{-tA})_{t\ge0}$ represent the Neumann heat semigroup and the Dirichlet Stokes semigroup on $\Omega$, and where according to Lemma 2.1, $c \in C^0([0,T]; W^{2,2}(\Omega))$ denotes the correspondingly obtained weak solution of (2.1).

To see that $\Psi$ in fact maps $S$ into itself, we fix any $r > 2$ and observe that by well-known regularization properties of $(e^{t\Delta})_{t\ge0}$ ([13]), by (2.2) and by continuity of the embeddings $W^{2,2}(\Omega) \hookrightarrow W^{1,r}(\Omega)$ and $D(A^\alpha) \hookrightarrow L^r(\Omega;\mathbb{R}^2)$, we can find $C_1 > 0$ and $C_2 = C_2(R) > 0$ such that regardless of our subsequent choice of $T \in (0,1)$,

$$\begin{aligned}
\Big\| (\Psi_1[\widehat{n},\widehat{u}])(\cdot,t) \Big\|_{L^\infty(\Omega)} &\le \|n_0\|_{L^\infty(\Omega)} + C_1 \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{r}} \Big\| \widehat{n}(\cdot,t) \Big( \nabla c(\cdot,s) - \widehat{u}(\cdot,s) \Big) \Big\|_{L^r(\Omega)} ds \\
&\le \|n_0\|_{L^\infty(\Omega)} \\
&\quad + C_1 \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{r}} \|\widehat{n}(\cdot,s)\|_{L^\infty(\Omega)} \cdot \Big\{ \|\nabla c(\cdot,s)\|_{L^r(\Omega)} + \|\widehat{u}(\cdot,s)\|_{L^r(\Omega)} \Big\} ds \\
&\le \|n_0\|_{L^\infty(\Omega)} + C_2 T^{\frac{1}{2}-\frac{1}{r}} \qquad \text{for all } t \in [0,T].
\end{aligned} \tag{2.15}$$

Likewise, the smoothing action of the semigroup $(e^{-tA})_{t\ge0}$ ([12]) along with the continuity of the embeddings $D(A^\alpha) \hookrightarrow L^\infty(\Omega;\mathbb{R}^2)$ and $D(A^\alpha) \hookrightarrow W^{1,2}(\Omega;\mathbb{R}^2)$ as well as the boundedness of $\nabla\Phi$ ensures the existence of $C_3 > 0$ and $C_4 = C_4(R) > 0$, again independent of $T \in (0,1)$, such that

$$\begin{aligned}
&\Big\| (A^\alpha \Psi_2[\widehat{n},\widehat{u}])(\cdot,t) \Big\|_{L^2(\Omega)} \\
&\le \|A^\alpha u_0\|_{L^2(\Omega)} + C_3 \int_0^t (t-s)^{-\alpha} \Big\| \mathcal{P} \Big\{ (\widehat{u}(\cdot,s)\cdot\nabla)\widehat{u}(\cdot,s) - \widehat{n}(\cdot,s)\nabla\Phi \Big\} \Big\|_{L^2(\Omega)} ds \\
&\le \|A^\alpha u_0\|_{L^2(\Omega)} + C_3 \int_0^t (t-s)^{-\alpha} \Big\{ \|\widehat{u}(\cdot,s)\|_{L^\infty(\Omega)} \|\nabla\widehat{u}(\cdot,s)\|_{L^2(\Omega)} + \|\widehat{n}(\cdot,s)\|_{L^2(\Omega)} \|\nabla\Phi\|_{L^\infty(\Omega)} \Big\} ds \\
&\le \|A^\alpha u_0\|_{L^2(\Omega)} + C_4 T^{1-\alpha} \qquad \text{for all } t \in [0,T].
\end{aligned} \tag{2.16}$$

Therefore, if we fix $T_1 \in (0,1)$ small enough fulfilling $\max\{C_2 T^{\frac{1}{2}-\frac{1}{r}}\, , \, C_4 T^{1-\alpha}\} \le \frac{1}{2}$, then for any choice of $T \in (0, T_1)$ we infer from (2.15) and (2.16) that indeed

$$\|\Psi[\widehat{n}, \widehat{u}]\|_X \le \Big\{ \|n_0\|_{L^\infty(\Omega)} + \frac{1}{2} \Big\} + \Big\{ \|A^\alpha u_0\|_{L^2(\Omega)} + \frac{1}{2} \Big\} = R \qquad \text{for all } (\widehat{n}, \widehat{u}) \in S.$$

By quite straightforward modification of the above procedure, it can furthermore readily be shown that there exists $T_2 \in (0, T_1)$ such that whenever $T \in (0, T_2)$, $\Psi$ even becomes a contraction on $S$, meaning that fixing any such $T$ we obtain a unique $(n,u) \in S$ such that $(n,u) = \Psi[n,u]$. Standard prolongation and regularity arguments thereafter show that $(n,u)$ can be extended up to some $T_{max} \in (0,\infty]$ fulfilling (2.12), that actually $n, c$ and $u$ satisfy the inclusions in (2.11), and that $(n,c,u,P)$ solves (1.3) classically in $\Omega \times (0, T_{max})$ with some appropriately chosen $P \in C^{1,0}(\Omega \times (0,T_{max}))$ enjoying the claimed uniqueness property ([31]).

The statements on positivity and the mass conservation features in (2.13) and (2.14), finally, result from applications of the parabolic and elliptic strong maximum principles, and from integration of the first two equations in (1.3). $\Box$

## 3 Making use of ellipticity: Basic time-independent estimates for $c$

The following lemma documents an observation of quite basic character, but of essential importance for our approach, by stating a regularity feature of $c$ which is implied by nonnegativity of $n$ and $c$ and by solenoidality of $u$, and which due to ellipticity of the second equation in (1.3) is independent with respect to time.

**Lemma 3.1** *Let $p \in (0,1)$. Then there exists $C(p) > 0$ such that*

$$\int_\Omega \Big| \nabla \Big( c(\cdot,t) + 1 \Big)^{\frac{p}{2}} \Big|^2 \le C(p) \qquad \textit{for all } t \in (0, T_{max}). \tag{3.1}$$

Proof. We multiply the second equation in (1.3) by $(c+1)^{p-1}$ to see that since $\nabla \cdot u = 0$ as well as $n \ge 0$ and $c \ge 0$, due to (2.14) we have

$$\begin{aligned}
(1-p)\int_\Omega (c+1)^{p-2}|\nabla c|^2 &= \int_\Omega c(c+1)^{p-1} - \int_\Omega n(c+1)^{p-1} + \frac{1}{p}\int_\Omega u \cdot \nabla (c+1)^p \\
&= \int_\Omega c(c+1)^{p-1} - \int_\Omega n(c+1)^{p-1} + \frac{1}{p}\int_\Omega u \cdot \nabla (c+1)^p \\
&\le \int_\Omega c(c+1)^{p-1} \\
&\le \int_\Omega c \\
&= \int_\Omega n_0 \qquad \text{for all } t \in (0, T_{max}),
\end{aligned}$$

from which (3.1) follows. $\Box$

By straightforward interpolation relying on (2.14), this implies time-independent bounds for $c$ in any $L^q$ space with finite $q$.

**Lemma 3.2** *For all $q \in (1,\infty)$ there exists $C(q)>0$ such that*

$$\|c(\cdot,t)\|_{L^q(\Omega)} \le C(q) \qquad \text{for all } t\in(0,T_{max}). \tag{3.2}$$

Proof. Since the two-dimensional Gagliardo-Nirenberg inequality provides $C_1 = C_1(q) > 0$ fulfilling

$$\begin{aligned}
\|c\|_{L^q(\Omega)}^{\frac{q}{2(q-1)}} &\le \|c+1\|_{L^q(\Omega)}^{\frac{q}{2(q-1)}} \\
&= \Big\|(c+1)^{\frac14}\Big\|_{L^{4q}(\Omega)}^{\frac{2q}{q-1}} \\
&\le C_1\Big\|\nabla(c+1)^{\frac14}\Big\|_{L^2(\Omega)}^2 \Big\|(c+1)^{\frac14}\Big\|_{L^4(\Omega)}^{\frac{2}{q-1}} + C_1\Big\|(c+1)^{\frac14}\Big\|_{L^4(\Omega)}^{\frac{2q}{q-1}} \qquad \text{for all } t\in(0,T_{max}),
\end{aligned}$$

and since

$$\Big\|(c+1)^{\frac14}\Big\|_{L^4(\Omega)}^4 = \int_\Omega (c+1) = \int_\Omega n_0 + |\Omega| \qquad \text{for all } t\in(0,T_{max})$$

by (2.14), this is a consequence of Lemma 3.1 when applied to $p := \frac12$. □

Again combined with Lemma 3.1, this in fact enables us to derive bounds for $\nabla c$, without any weight functions, with respect to the norm in $L^r(\Omega)$ for arbitrary $r<2$.

**Lemma 3.3** *For each $r\in[1,2)$ one can find $C(r)>0$ with the property that*

$$\|\nabla c(\cdot,t)\|_{L^r(\Omega)} \le C(r) \qquad \text{for all } t\in(0,T_{max}). \tag{3.3}$$

Proof. Using that $r<2$, we may employ Young's inequality to estimate

$$\begin{aligned}
\int_\Omega |\nabla c|^r &= \int_\Omega \Big\{(c+1)^{-\frac32}|\nabla c|^2\Big\}^{\frac r2}\cdot (c+1)^{\frac{3r}{4}} \\
&\le \int_\Omega (c+1)^{-\frac32}|\nabla c|^2 + \int_\Omega (c+1)^{\frac{3r}{2(2-r)}} \\
&\le \int_\Omega (c+1)^{-\frac32}|\nabla c|^2 + 2^{\frac{3r}{2(2-r)}}\int_\Omega c^{\frac{3r}{2(2-r)}} + 2^{\frac{3r}{2(2-r)}}|\Omega| \qquad \text{for all } t\in(0,T_{max}),
\end{aligned}$$

whence another application of Lemma 3.1 to $p := \frac12$ entails (3.3). □

# 4 Basic boundedness features relying on a quasi-energy inequality

Next approaching the core of our analysis, let us state the direct outcome of a testing procedure which addresses the time evolution of the first contribution to the functional $\mathcal{F}$ in (1.6), and which again relies on the elliptic character of the second equation in (1.3).

**Lemma 4.1** *We have*

$$\begin{aligned}
-\frac{d}{dt}\int_\Omega \ln(n+1) + \int_\Omega \frac{|\nabla n|^2}{(n+1)^2} + \int_\Omega n\ln(n+1) + \int_\Omega \frac{n}{n+1} \\
= \int_\Omega \Big\{\ln(n+1) + \frac{1}{n+1}\Big\}\cdot(c+u\cdot\nabla c) \qquad \text{for all } t\in(0,T_{max}).
\end{aligned} \tag{4.1}$$

PROOF. According to the first equation in (1.3) and the identity $\nabla \cdot u = 0$, we see that

$$
\begin{aligned}
-\frac{d}{dt}\int_\Omega \ln(n+1) + \int_\Omega \frac{|\nabla n|^2}{(n+1)^2} &= -\int_\Omega \frac{n}{(n+1)^2}\nabla n \cdot \nabla c + \int_\Omega u \cdot \nabla \ln(n+1) \\
&= -\int_\Omega \frac{n}{(n+1)^2}\nabla n \cdot \nabla c \qquad \text{for all } t \in (0, T_{max}), \qquad (4.2)
\end{aligned}
$$

where rewriting

$$
\frac{n}{(n+1)^2}\nabla n = \nabla \ln(n+1) + \nabla \frac{1}{n+1} \qquad \text{in } \Omega \times (0, T_{max})
$$

we may integrate by parts once more and use the second equation from (1.3) to obtain that

$$
\begin{aligned}
\int_\Omega \frac{n}{(n+1)^2}\nabla n \cdot \nabla c &= \int_\Omega \Big\{ \ln(n+1) + \frac{1}{n+1} \Big\} \cdot \Delta c \\
&= \int_\Omega \Big\{ \ln(n+1) + \frac{1}{n+1} \Big\} \cdot (-n + c + u \cdot \nabla c) \qquad \text{for all } t \in (0, T_{max}).
\end{aligned}
$$

Inserted into (4.2) this yields (4.1). □

As for the evolution of the respective second integral in (1.6), we may import the following basic information from the literature concerned with the fully parabolic analogue of (1.3), without repeating details of a proof which, according to the identical form of the respective Navier-Stokes-subproblem, can in fact be copied word by word from the considered precedent.

**Lemma 4.2** *There exists $C > 0$ such that*

$$
\frac{d}{dt}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 \le C \int_\Omega n \ln \frac{n}{\overline{n}_0} + C \qquad \textit{for all } t \in (0, T_{max}). \qquad (4.3)
$$

PROOF. On the basis of a functional inequality from [52, Lemma 2.2], this can be seen by precisely repeating the argument from [53, Lemma 3.4]. □

Now thanks to the a priori information on $c$ and $\nabla c$ from Lemma 3.2 and Lemma 3.3, the right-hand side in (4.1) can be estimated in terms of a sublinear power of the dissipation rate appearing in (4.3).

**Lemma 4.3** *There exists $C > 0$ such that*

$$
\int_\Omega \Big\{ \ln(n+1) + \frac{1}{n+1} \Big\} \cdot (c + u \cdot \nabla c) \le C\|\nabla u\|_{L^2(\Omega)} + C \qquad \textit{for all } t \in (0, T_{max}). \qquad (4.4)
$$

PROOF. We use the elementary inequality $\ln^6 \xi \le C_1 \xi$, valid for all $\xi > 0$ with $C_1 := (\frac{6}{e})^6$, to see that thanks to (2.13),

$$
\begin{aligned}
\int_\Omega \Big\{ \ln(n+1) + \frac{1}{n+1} \Big\}^6 &\le 32 \int_\Omega \ln^6(n+1) + 32 \int_\Omega \frac{1}{(n+1)^6} \\
&\le 32C_1 \int_\Omega (n+1) + 32|\Omega| \\
&= C_2 := 32C_1 \int_\Omega (n_0 + 1) + 32|\Omega| \qquad \text{for all } t \in (0, T_{max}).
\end{aligned}
$$

Therefore, the Hölder inequality implies that

$$\begin{aligned}
\int_\Omega \Big\{ \ln(n+1) + \frac{1}{n+1} \Big\} \cdot (c + u\cdot\nabla c) &\le \Big\{ \int_\Omega \Big\{ \ln(n+1) + \frac{1}{n+1} \Big\}^6 \Big\}^{\frac{1}{6}} \cdot \|c + u\cdot\nabla c\|_{L^{\frac{6}{5}}(\Omega)} \\
&\le C_2^{\frac{1}{6}} \|c + u\cdot\nabla c\|_{L^{\frac{6}{5}}(\Omega)} \\
&\le C_2^{\frac{1}{6}} \|c\|_{L^{\frac{6}{5}}(\Omega)} + C_2^{\frac{1}{6}} \|u\|_{L^6(\Omega)} \|\nabla c\|_{L^{\frac{3}{2}}(\Omega)} \\
&\le C_2^{\frac{1}{6}} C_3 + C_2^{\frac{1}{6}} C_4 \|u\|_{L^6(\Omega)} \qquad \text{for all } t\in(0,T_{max})
\end{aligned}$$

with $C_3 := \sup_{t\in(0,T_{max})} \|c(\cdot,t)\|_{L^{\frac{6}{5}}(\Omega)}$ and $C_4 := \sup_{t\in(0,T_{max})} \|\nabla c(\cdot,t)\|_{L^{\frac{3}{2}}(\Omega)}$ being finite due to Lemma 3.2 and Lemma 3.3. Since clearly $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, this entails (4.4) with some appropriately large $C > 0$. $\square$

Therefore, adding a suitably small multiple of the inequality (4.3) to (4.1) reveals the announced property of the functional in (1.6):

**Lemma 4.4** *There exist $a > 0$ and $C > 0$ such that*

$$\frac{d}{dt}\Big\{ -\int_\Omega \ln(n+1) + a\int_\Omega |u|^2 \Big\} + \frac{1}{C}\cdot\Big\{ -\int_\Omega \ln(n+1) + a\int_\Omega |u|^2 \Big\} + \frac{1}{2}\int_\Omega n\ln(n+1) + \frac{a}{2}\int_\Omega |\nabla u|^2 \le C \qquad \textit{for all } t\in(0,T_{max}). \tag{4.5}$$

Proof. On the basis of Lemma 4.1, Lemma 4.3, Lemma 4.2 and (2.13), let us pick $C_1 > 0$, $C_2 > 0$ and $C_3 > 0$ such that

$$-\frac{d}{dt}\int_\Omega \ln(n+1) + \int_\Omega n\ln(n+1) \le C_1\|\nabla u\|_{L^2(\Omega)} + C_1 \qquad \text{for all } t\in(0,T_{max}) \tag{4.6}$$

and

$$\frac{d}{dt}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 \le C_2\int_\Omega n\ln n + C_3 \qquad \text{for all } t\in(0,T_{max}), \tag{4.7}$$

and thereupon let

$$a := \frac{1}{2C_2}, \qquad C_4 := \frac{C_1^2}{a} + C_1 + aC_3 \qquad \text{and} \qquad C_5 := \max\{4C_6\,,\,C_3\}, \tag{4.8}$$

where according to the Poincaré inequality, $C_6 > 0$ is such that

$$\int_\Omega |\varphi|^2 \le C_6\int_\Omega |\nabla\varphi|^2 \qquad \text{for all } \varphi\in W_0^{1,2}(\Omega;\mathbb{R}^2). \tag{4.9}$$

Then combining (4.6) with (4.7) and using Young's inequality along with (4.9) and our definition of $C_4$ shows that since $\ln(n+1) \ge \max\{0,\ln n\}$,

$$\frac{d}{dt}\Big\{ -\int_\Omega \ln(n+1) + a\int_\Omega |u|^2 \Big\} + \frac{1}{C_5}\cdot\Big\{ -\int_\Omega \ln(n+1) + a\int_\Omega |u|^2 \Big\} + \int_\Omega n\ln(n+1) + a\int_\Omega |\nabla u|^2$$

$$
\begin{aligned}
&\le -\frac{1}{C_5}\int_\Omega \ln(n+1) + \frac{a}{C_5}\int_\Omega |u|^2 \\
&\quad +C_1\|\nabla u\|_{L^2(\Omega)} + C_1 + aC_2\int_\Omega n\ln n + aC_3 \\
&\le \frac{a}{C_5}\int_\Omega |u|^2 + \frac{a}{4}\int_\Omega |\nabla u|^2 + \frac{C_1^2}{a} + C_1 + aC_2\int_\Omega n\ln(n+1) + aC_3 \\
&= \frac{a}{C_5}\int_\Omega |u|^2 + \frac{a}{4}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega n\ln(n+1) + C_4 \\
&\le \frac{aC_6}{C_5}\int_\Omega |\nabla u|^2 + \frac{a}{4}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega n\ln(n+1) + C_4 \\
&= \frac{a}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega n\ln(n+1) + C_4 \qquad \text{for all } t\in(0,T_{max}), \qquad (4.10)
\end{aligned}
$$

because (4.8) inter alia says that $\frac{aC_6}{C_5} \le \frac{a}{4}$. As (4.8) furthermore implies that $C_4 \le C_5$, (4.10) immediately leads to (4.5) with $C := C_5$. □

By straightforward integration, the latter entails the following estimates, the second and third ones among which will be of predominant importance for our subsequent analysis.

**Lemma 4.5** *There exists $C > 0$ such that*

$$\int_\Omega |u(\cdot,t)|^2 \le C \qquad \textit{for all } t\in(0,T_{max}) \tag{4.11}$$

*and*

$$\int_t^{t+\tau}\int_\Omega n\ln(n+1) \le C \qquad \textit{for all } t\in[0,T_{max}-\tau) \tag{4.12}$$

*as well as*

$$\int_t^{t+\tau}\int_\Omega |\nabla u|^2 \le C \qquad \textit{for all } t\in[0,T_{max}-\tau), \tag{4.13}$$

*where $\tau := \min\{1, \frac{1}{2}T_{max}\}$.*

Proof. From Lemma 4.4 we obtain $a > 0, C_1 > 0$ and $C_2 > 0$ such that writing

$$\mathcal{F}(t) := -\int_\Omega \ln\Big(u(\cdot,t)+1\Big) + a\int_\Omega |u(\cdot,t)|^2, \qquad t\in[0,T_{max}),$$

and

$$g(t) := \frac{1}{2}\int_\Omega n(\cdot,t)\ln\Big(n(\cdot,t)+1\Big) + \frac{a}{2}\int_\Omega |\nabla u(\cdot,t)|^2, \qquad t\in(0,T_{max}),$$

we have

$$\mathcal{F}'(t) + C_1\mathcal{F}(t) + g(t) \le C_2 \qquad \text{for all } t\in(0,T_{max}). \tag{4.14}$$

Since $g \ge 0$, through an ODE comparison this firstly implies that

$$\mathcal{F}(t) \le C_3 := \max\Big\{\mathcal{F}(0)\,,\,\frac{C_2}{C_1}\Big\} \qquad \text{for all } t\in(0,T_{max}) \tag{4.15}$$

and thus entails (4.11) due to the fact that

$$\begin{aligned}
\mathcal{F}(t) &\geq -\int_\Omega \Big(n(\cdot,t)+1\Big) + a\int_\Omega |u(\cdot,t)|^2 \\
&= -\int_\Omega (n_0+1) + a\int_\Omega |u(\cdot,t)|^2 \qquad \text{for all } t\in(0,T_{max})
\end{aligned} \tag{4.16}$$

thanks to the validity of $\ln(\xi+1)\leq\xi$ for all $\xi\geq 0$, and to (2.13).

As (4.16) moreover means that $-\mathcal{F}(t)\leq C_4 := \int_\Omega (n_0+1)$ for all $t\in(0,T_{max})$, an integration of (4.14) finally shows that

$$\begin{aligned}
\int_t^{t+\tau} g(s)ds &\leq \mathcal{F}(t)-\mathcal{F}(t+\tau) - C_1\int_t^{t+\tau}\mathcal{F}(s)ds + C_2\tau \\
&\leq C_3+C_4+C_1C_4\tau+C_2\tau \qquad \text{for all } t\in[0,T_{max}-\tau)
\end{aligned}$$

and that thus also (4.13) and (4.12) are valid, because $\tau\leq 1$. □

# 5 Higher regularity information. Proof of Theorem 1.1

With the information from Lemma 4.5 at hand, we can now proceed to derive improved regularity properties on the basis of more standard testing procedures. Our first step in this respect is concernd with a functional of classical logarithmic entropy type.

**Lemma 5.1** *We have*

$$\frac{d}{dt}\int_\Omega n\ln\frac{n}{\overline{n}_0} + \int_\Omega \frac{|\nabla n|^2}{n} + \frac{1}{2}\int_\Omega n^2 \leq \int_\Omega c^2 + \int_\Omega |u\cdot\nabla c|^2 \qquad \textit{for all } t\in(0,T_{max}). \tag{5.1}$$

Proof. Once more using that $\nabla\cdot u=0$, from (2.13) and the first two equations in (1.3) we obtain that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega n\ln\frac{n}{\overline{n}_0} + \frac{|\nabla n|^2}{n} &= -\int_\Omega \nabla n\cdot\nabla c - \int_\Omega u\cdot\nabla(n\ln n) \\
&= \int_\Omega n\Delta c \\
&= -\int_\Omega n^2 + \int_\Omega nc - \int_\Omega n(u\cdot\nabla c) \qquad \text{for all } t\in(0,T_{max}).
\end{aligned}$$

This implies (5.1) upon two applications of Young's inequality, which namely show that

$$\int_\Omega nc \leq \frac{1}{4}\int_\Omega n^2 + \int_\Omega c^2 \qquad \text{and} \qquad -\int_\Omega n(u\cdot\nabla c)\leq \frac{1}{4}\int_\Omega n^2 + \int_\Omega |u\cdot\nabla c|^2$$

for all $t\in(0,T_{max})$. □

In order to appropriately absorb the rightmost contribution therein, we shall linearly combine (5.1) with the following outcome of a standard testing procedure performed to the Navier-Stokes subsystem of (1.3).

**Lemma 5.2** *There exists $C > 0$ such that*

$$\frac{d}{dt}\int_\Omega |\nabla u|^2 + \int_\Omega |Au|^2 \le C \cdot \left\{ \int_\Omega |\nabla u|^2 \right\}^2 + C\int_\Omega n^2 \qquad \textit{for all } t \in (0, T_{max}). \tag{5.2}$$

PROOF. As a proof of this can be built on (4.11) through a rather standard argument, we may confine ourselves to referring to e.g. [44, Proof of Theorem 1.1] and [53, Lemma 4.2] for details in two closely related situations. □

In fact, the crucial term on the right of (5.1), even when augmented by some second-order expression containing $c$, can be estimated in terms of a quadratic power of the functional on the left of (5.2), and of an integral involving an essentially arbitrary subquadratic power of $n$:

**Lemma 5.3** *For all $p \in (1, 2)$ there exists $C(p) > 0$ such that*

$$\int_\Omega |u\cdot\nabla c|^2 + \|c\|^p_{W^{2,p}(\Omega)} \le C(p)\int_\Omega n^p + C(p)\cdot\left\{\int_\Omega |\nabla u|^2\right\}^2 + C(p) \qquad \textit{for all } t \in (0, T_{max}). \tag{5.3}$$

PROOF. Given $p \in (1, 2)$, using that then $4 > 4 - p + 2 - p$ we can pick $r \in (1, 2)$ suitably close to 2 such that $4 > \frac{8}{r} - p + 2 - \frac{2p}{r}$. This in turn enables us to fix $s > 2$ close enough to 2 such that

$$\frac{8}{s} > \frac{8}{r} - p + 2 - \frac{2p}{r}, \tag{5.4}$$

and we thereupon let

$$\lambda := \frac{\frac{2}{r} - \frac{2}{s}}{1 - \frac{2}{p} + \frac{2}{r}}.$$

Then $\lambda > 0$ since $r < 2 < s$, and (5.4) asserts that moreover

$$\left(1 - \frac{2}{p} + \frac{2}{r}\right)\cdot(4\lambda - p) = \frac{8}{r} - \frac{8}{s} - p + 2 - \frac{2p}{r} < 0$$

and thus $4\lambda < p$. In consequence, after employing the Gagliardo-Nirenberg inequality and relying on the fact that $r < 2$ in applying Lemma 3.3, we may use Young's inequality to see that with some positive constants $C_1 = C_1(p), C_2 = C_2(p)$ and $C_3 = C_3(p)$ we have

$$\begin{aligned} \|\nabla c\|^4_{L^s(\Omega)} &\le C_1\|c\|^{4\lambda}_{W^{2,p}(\Omega)}\|\nabla c\|^{4(1-\lambda)}_{L^r(\Omega)} \\ &\le C_2\|c\|^{4\lambda}_{W^{2,p}(\Omega)} \\ &\le \frac{1}{2}\|c\|^p_{W^{2,p}(\Omega)} + C_3 \qquad \text{for all } t \in (0, T_{max}). \end{aligned}$$

Thanks to the continuity of the embedding $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2s}{s-2}}(\Omega)$, by once more utilizing Young's inequality we therefore obtain $C_4 = C_4(p) > 0$ such that

$$\begin{aligned} \int_\Omega |u\cdot\nabla c|^2 &\le \|u\|^4_{L^{\frac{2s}{s-2}}(\Omega)} + \|\nabla c\|^4_{L^s(\Omega)} \\ &\le C_4\|\nabla u\|^4_{L^2(\Omega)} + \frac{1}{2}\|c\|^p_{W^{2,p}(\Omega)} + C_4 \qquad \text{for all } t \in (0, T_{max}). \end{aligned} \tag{5.5}$$

Independently, we next employ standard elliptic regularity theory to the second equation in (1.3) to find $C_5 = C_5(p) > 0$ such that due to Young's inequality and the restriction that $p < 2$,

$$\begin{aligned}
2\|c\|^p_{W^{2,p}(\Omega)} &\le C_5\|n - u\cdot\nabla c\|^p_{L^p(\Omega)} \\
&\le 2^{p-1}C_5\|n\|^p_{L^p(\Omega)} + 2^{p-1}C_5\|u\cdot\nabla c\|^p_{L^p(\Omega)} \\
&\le 2^{p-1}C_5\|n\|^p_{L^p(\Omega)} + \|u\cdot\nabla c\|^2_{L^2(\Omega)} + C_6 \qquad \text{for all } t \in (0, T_{max})
\end{aligned}$$

with some $C_6 = C_6(p) > 0$. When added to (5.5), this shows that for all $t \in (0, T_{max})$,

$$\begin{aligned}
\int_\Omega |u\cdot\nabla c|^2 + 2\|c\|^p_{W^{2,p}(\Omega)} &\le 2\int_\Omega |u\cdot\nabla c|^2 + 2^{p-1}C_5\int_\Omega n^p + C_6 \\
&\le 2C_4\cdot\Big\{\int_\Omega |\nabla u|^2\Big\}^2 + \|c\|^p_{W^{2,p}(\Omega)} + 2C_3 + 2^{p-1}C_5\int_\Omega n^p + C_6,
\end{aligned}$$

and thereby implies (5.3) with $C(p) := \max\{2^{p-1}C_5(p)\,,\,2C_4(p)\,,\,2C_3(p) + C_6(p)\}$. □

Thanks to the spatio-temporal $L^2$ bound for $\nabla u$ available due to Lemma 4.5, in view of Lemma 5.3 a suitable combination of the inequalities from Lemma 5.1 and Lemma 5.2 yields the following improvement on our knowledge about the regularity of the fluid flow.

**Lemma 5.4** *There exists $C > 0$ such that*

$$\int_\Omega |\nabla u(\cdot,t)|^2 \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{5.6}$$

Proof. We apply Lemma 5.2 and Lemma 5.3 to fix $C_1 > 0$ and $C_2 > 0$ such that

$$\frac{d}{dt}\int_\Omega |\nabla u|^2 + \int_\Omega |Au|^2 \le C_1\cdot\Big\{\int_\Omega |\nabla u|^2\Big\}^2 + C_1\int_\Omega n^2 \qquad \text{for all } t \in (0, T_{max}) \tag{5.7}$$

and

$$\int_\Omega |u\cdot\nabla c|^2 \le C_2\int_\Omega n^{\frac{3}{2}} + C_2\cdot\Big\{\int_\Omega |\nabla u|^2\Big\}^2 + C_2 \qquad \text{for all } t \in (0, T_{max}), \tag{5.8}$$

and let $b := \frac{1}{4C_1}$. Then taking an appropriate linear combination of the inequality from Lemma 5.1 with (5.7) shows that

$$\begin{aligned}
&\frac{d}{dt}\Big\{\int_\Omega n\ln\frac{n}{\overline{n}_0} + b\int_\Omega |\nabla u|^2 + 1\Big\} + \frac{1}{2}\int_\Omega n^2 \\
&\quad\le \int_\Omega c^2 + \int_\Omega |u\cdot\nabla c|^2 + bC_1\cdot\Big\{\int_\Omega |\nabla u|^2\Big\}^2 + bC_1\int_\Omega n^2 \\
&\quad\le \int_\Omega c^2 + (C_2 + bC_1)\cdot\Big\{\int_\Omega |\nabla u|^2\Big\}^2 + C_2\int_\Omega n^{\frac{3}{2}} + C_2 + \frac{1}{4}\int_\Omega n^2 \qquad \text{for all } t \in (0, T_{max}),
\end{aligned}$$

and that thus $y(t) := \int_\Omega n(\cdot,t)\ln\frac{n(\cdot,t)}{\overline{n}_0} + b\int_\Omega |\nabla u(\cdot,t)|^2 + 1$, $t \in [0, T_{max})$, satisfies

$$y'(t) + \frac{1}{4}\int_\Omega n^2 \le (C_2 + bC_1)\cdot\Big\{\int_\Omega |\nabla u|^2\Big\}\cdot y(t) + C_2\int_\Omega n^{\frac{3}{2}} + \int_\Omega c^2 + C_2 \qquad \text{for all } t \in (0, T_{max}).$$

Since here Young's inequality provides $C_4 > 0$ such that

$$C_2 \int_\Omega n^{\frac{3}{2}} \le \frac{1}{4} \int_\Omega n^2 + C_4 \qquad \text{for all } t \in (0, T_{max}),$$

and since $C_5 := \sup_{t\in(0,T_{max})} \int_\Omega c^2(\cdot,t)$ is finite by Lemma 3.2, this entails that

$$y'(t) \le h(t)y(t) \qquad \text{for all } t \in (0, T_{max}) \tag{5.9}$$

with $h(t) := (C_2 + bC_1) \int_\Omega |\nabla u(\cdot,t)|^2 + C_2 + C_4 + C_5$, $t \in (0, T_{max})$, because $y \ge 1$.

Now Lemma 4.5 ensures the existence of $C_6 > 0$ and $C_7 > 0$ such that again writing $\tau := \min\{1, \frac{1}{2}T_{max}\}$ we have

$$\int_{(t-\tau)_+}^t h(s)ds \le C_6 \qquad \text{for all } t \in [0, T_{max}),$$

and that for any choice of $t \in (0, T_{max})$ we can fix $t_0(t) \in (t-\tau, t) \cap [0, T_{max})$ fulfilling

$$y(t_0(t)) \le C_7.$$

Integrating in (5.9) hence shows that

$$\begin{aligned} y(t) &\le y(t_0(t)) \cdot e^{\int_{t_0(t)}^t h(s)ds} \\ &\le C_7 e^{C_6} \qquad \text{for all } t \in (0, T_{max}) \end{aligned}$$

and thereby establishes (5.6), because $t \in (0, T_{max})$ was arbitrary. $\square$

The latter in turn provides bounds sufficient for estimating the contribution of $u$ to the forcing terms arising in the course of a standard $L^2$ testing procedure applied to the first equation in (1.3):

**Lemma 5.5** *There exists $C > 0$ such that*

$$\int_\Omega n^2(\cdot,t) \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{5.10}$$

Proof. We integrate by parts in the first equation from (1.3) and again rely on the solenoidality of $u$ to obtain that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega n^2 + \int_\Omega |\nabla n|^2 &= -\int_\Omega n\nabla n \cdot \nabla c \\ &= \frac{1}{2}\int_\Omega n^2 \Delta c \\ &= \frac{1}{2}\int_\Omega n^2 c - \frac{1}{2}\int_\Omega n^3 + \frac{1}{2}\int_\Omega n^2 (u\cdot\nabla c) \\ &= \frac{1}{2}\int_\Omega n^2 c - \frac{1}{2}\int_\Omega n^3 - \int_\Omega nc(u\cdot\nabla n) \qquad \text{for all } t \in (0, T_{max}), \end{aligned} \tag{5.11}$$

and using Young's inequality we find $C_1 > 0, C_2 > 0$ and $C_3 > 0$ such that

$$\int_\Omega n^2 \le \frac{1}{6}\int_\Omega n^3 + C_1 \qquad \text{for all } t \in (0, T_{max})$$

and

$$\frac{1}{2}\int_\Omega n^2 c \le \frac{1}{6}\int_\Omega n^3 + C_2\int_\Omega c^3 \qquad \text{for all } t\in(0,T_{max})$$

as well as

$$\begin{aligned}
-\int_\Omega nc(u\cdot\nabla n) &\le \int_\Omega |\nabla n|^2 + \frac{1}{4}\int_\Omega n^2 + \frac{1}{4}\int_\Omega n^2c^2|u|^2 \\
&\le \int_\Omega |\nabla n|^2 + \frac{1}{6}\int_\Omega n^3 + C_3\int_\Omega c^6|u|^6 \\
&\le \int_\Omega |\nabla n|^2 + \frac{1}{6}\int_\Omega n^3 + \frac{C_3}{2}\int_\Omega c^{12} + \frac{C_3}{2}\int_\Omega |u|^{12} \qquad \text{for all } t\in(0,T_{max}).
\end{aligned}$$

Observing that the constants $C_4 := \sup_{t\in(0,T_{max})}\int_\Omega c^3(\cdot,t)$, $C_5 := \sup_{t\in(0,T_{max})}\int_\Omega c^{12}(\cdot,t)$ and $C_6 := \sup_{t\in(0,T_{max})}\int_\Omega |u(\cdot,t)|^{12}$ are all finite thanks to Lemma 3.2, Lemma 5.4 and the continuity of the embedding $W^{1,2}(\Omega)\hookrightarrow L^{12}(\Omega)$, from (5.11) we thus infer that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega n^2 + \int_\Omega n^2 \le C_1 + C_2C_4 + \frac{C_3C_5}{2} + \frac{C_3C_6}{2} \qquad \text{for all } t\in(0,T_{max})$$

and that therefore (5.10) follows by means of an ODE comparison argument. □

We can thereby easily improve our knowledge about regularity of the signal gradient:

**Lemma 5.6** *Let $r>2$. Then there exists $C(r)>0$ fulfilling*

$$\|\nabla c(\cdot,t)\|_{L^r(\Omega)} \le C(r) \qquad \textit{for all } t\in(0,T_{max}). \tag{5.12}$$

Proof. Using that $p := \frac{2r}{r+2}$ satisfies $p\in(1,2)$ by assumption on $r$, we see that Lemma 5.3 in conjunction with Lemma 5.5 and Lemma 5.4 ensures the existence of $C_1 = C_1(r) > 0$ such that $\|c\|_{W^{2,p}(\Omega)} \le C_1$ for all $t\in(0,T_{max})$. Since $W^{2,p}(\Omega)\hookrightarrow W^{1,r}(\Omega)$, this implies (5.12). □

The ultimate step within our bootstrap process yields bounds not only sufficient to assert global extensibility via Lemma 2.2, but moreover also covering the intended boundedness feature in (1.7):

**Lemma 5.7** *Given any $\alpha\in(\frac{1}{2},1)$, one can find $C(\alpha)>0$ such that*

$$\|A^\alpha u(\cdot,t)\|^2_{L^2(\Omega)} \le C(\alpha) \qquad \textit{for all } t\in(0,T_{max}). \tag{5.13}$$

*Moreover,*

$$\sup_{t\in(0,T_{max})}\|n(\cdot,t)\|_{L^\infty(\Omega)} < \infty. \tag{5.14}$$

Proof. The inequality in (5.13) can be derived in quite a standard manner from the regularity properties obtained in Lemma 5.4 and Lemma 5.5 (see e.g. [44, Proof of Theorem 1.1] or also [53, Lemma 4.3] for closely related precedents). As $D(A^\alpha)\hookrightarrow L^{12}(\Omega;\mathbb{R}^2)$ for any such $\alpha$, this particularly entails the existence of $C_1>0$ such that $\|u\|_{L^{12}(\Omega)}\le C_1$ for all $t\in(0,T_{max})$, so that according to

Lemma 5.6 we can find $C_2 > 0$ such that $h := \nabla c - u$ satisfies $\|h\|_{L^{12}(\Omega)} \le C_2$ for all $t \in (0, T_{max})$. Therefore, using the Hölder inequality along with (2.13) we can estimate

$$\|nh\|_{L^3(\Omega)} \le \|n\|_{L^4(\Omega)} \|h\|_{L^{12}(\Omega)} \le \|n\|_{L^\infty(\Omega)}^{\frac{3}{4}} \|n\|_{L^1(\Omega)}^{\frac{1}{4}} \|h\|_{L^{12}(\Omega)} \le C_3 \|u\|_{L^\infty(\Omega)}^{\frac{3}{4}} \qquad \text{for all } t \in (0, T_{max})$$

with $C_3 := C_2 \|n_0\|_{L^1(\Omega)}^{\frac{1}{4}}$, so that employing known smoothing properties of the Neumann heat semigroup $(e^{t\Delta})_{t\ge 0}$ on $\Omega$ ([13]), we see that for some $C_4 > 0$ and all $t \in (0, T_{max})$ we have

$$\begin{aligned}
\|n(\cdot,t)\|_{L^\infty(\Omega)} &= \left\| e^{t(\Delta-1)} n_0 + \int_0^t e^{(t-s)(\Delta-1)} \nabla \cdot \Big( n(\cdot,s) h(\cdot,s) \Big) ds + \int_0^t e^{(t-s)(\Delta-1)} n(\cdot,s) ds \right\|_{L^\infty(\Omega)} \\
&\le e^{-t} \|n_0\|_{L^\infty(\Omega)} + C_4 \int_0^t \Big( 1 + (t-s)^{-\frac{5}{6}} \Big) e^{-(t-s)} \|n(\cdot,s) h(\cdot,s)\|_{L^3(\Omega)} ds \\
&\quad + C_4 \int_0^t \Big( 1 + (t-s)^{-\frac{1}{2}} \Big) e^{-(t-s)} \|n(\cdot,s)\|_{L^2(\Omega)} ds \\
&\le \|n_0\|_{L^\infty(\Omega)} + C_3 C_4 C_5 \sup_{s \in (0,t)} \|n(\cdot,s)\|_{L^\infty(\Omega)}^{\frac{3}{4}} + C_4 C_6 \sup_{s \in (0,t)} \|n(\cdot,s)\|_{L^2(\Omega)}
\end{aligned}$$

with $C_5 := \int_0^\infty (1+\sigma^{-\frac{5}{6}}) e^{-\sigma} d\sigma$ and $C_6 := \int_0^\infty (1+\sigma^{-\frac{1}{2}}) e^{-\sigma} d\sigma$. Now since $C_7 := \sup_{t\in(0,T_{max})} \|n(\cdot,t)\|_{L^2(\Omega)}$ is finite due to Lemma 5.5, this shows that writing $C_8 := \max\{\|n_0\|_{L^\infty(\Omega)} + C_4 C_6 C_7 \,,\, C_3 C_4 C_5\}$ and $M(T) := \sup_{t\in(0,T)} \|n(\cdot,t)\|_{L^\infty(\Omega)}$, $T \in (0, T_{max})$, we have

$$M(T) \le C_8 + C_8 M^{\frac{3}{4}}(T) \qquad \text{for all } T \in (0, T_{max}).$$

This implies that $M(T) \le \max\{1 \,,\, (2C_8)^4\}$ for all $T \in (0, T_{max})$ and hence completes the proof. □

Our main goal has thereby been accomplished:

Proof of Theorem 1.1. The claim directly results from Lemma 5.7 and Lemma 5.6 upon an application of Lemma 2.2. □

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