Approaching critical decay in a strongly degenerate parabolic equation

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Abstract

The Cauchy problem in $\mathbb{R}^n$, $n \geq 1$, for the parabolic equation

$$u_t = u^p \Delta u \qquad (\star)$$

is considered in the strongly degenerate regime $p \geq 1$.

The focus is firstly on the case of positive continuous and bounded initial data, in which it is known that a minimal positive classical solution exists, and that this solution satisfies

$$t^{\frac{1}{p}} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \to \infty \qquad \text{as } t \to \infty. \tag{0.1}$$

The first result of this study complements this by asserting that given any positive $f \in C^0([0,\infty))$ fulfilling $f(t) \to +\infty$ as $t \to \infty$ one can find a positive nondecreasing function $\phi \in C^0([0,\infty))$ such that whenever $u_0 \in C^0(\mathbb{R}^n)$ is radially symmetric with $0 < u_0 < \phi(|\cdot|)$, the corresponding minimal solution u satisfies

$$\frac{t^{\frac{1}{p}} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}}{f(t)} \to 0 \qquad \text{as } t \to \infty.$$

Secondly, $(\star)$ is considered along with initial conditions involving nonnegative but not necessarily strictly positive bounded and continuous initial data u_0. It is shown that if the connected components of $\{u_0 > 0\}$ comply with a condition reflecting some uniform boundedness property, then a corresponding uniquely determined continuous weak solution to $(\star)$ satisfies

$$0 < \liminf_{t\to\infty} \Big\{ t^{\frac{1}{p}} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \Big\} \leq \limsup_{t\to\infty} \Big\{ t^{\frac{1}{p}} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \Big\} < \infty.$$

Under a somewhat complementary hypothesis, particularly fulfilled if $\{u_0 > 0\}$ contains components with arbitrarily small principal eigenvalues of the associated Dirichlet Laplacian, it is finally seen that (0.1) continues to hold also for such not everywhere positive weak solutions.

Key words: degenerate parabolic equation; decay rates of solutions
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1 Introduction

The dynamical features of the nonlinear parabolic equation

$$u_t = u^p \Delta u \tag{1.1}$$

are known to depend quite crucially on the exponent $p > 0$ that quantifies the strength of diffusion degeneracies in regions where the solution is small; indeed, a considerable literature has rigorously revealed various parabolictiy-diminishing effects going along with an increase of p. Among the most comprehensively understood aspects in this regard seem to be phenomena related to propagation of positivity: In striking difference to the borderline case $p = 0$ of the linear heat equation, throughout the range $p \in (0, 1)$ in which (1.1) is equivalent to the porous medium equation $v_t = \Delta v^m$ with $m = \frac{1}{1-p} > 1$, compactly supported initial data evolve into continuous solutions ([8]) which at each point in the considered domain do eventually become positive, but the spatial positivity set of which propagates at finite speed ([15] and [5]; see also [13], [7], [9] and [2] for more detailed information, and [1] or [16] for an overview).

In this respect, a second sharp transition in behavior can be observed when further increasing p: Yet more drastically, namely, the support of solutions remains constant in time whenever $p \geq 1$ ([21]; cf. also Proposition 3.1 below), and in the case $p > 2$ there even exist classical solutions to an associated homogeneous Dirichlet problem in domains $\Omega \subset \mathbb{R}^n$ which satisfy $u(\cdot, t) \in C_0^\infty(\Omega)$ for all $t > 0$ ([22]). Two examples addressing a relative of (1.1) with $p \geq 3$ in such Dirichlet problems, augmented by the zero-oder source term u^{p+1}, have unveiled that in such very strongly degenerate cases, the global behavior may be influenced quite substantially, up to an enforcement of repeated oscillations between vanishing and everywhere infinite profiles, by the particular manner in which the boundary value 0 is approached by the initial data ([20], [19]).

Beyond this, however, increasing the degeneracy in (1.1) may considerably affect the dynamics even of solutions which are strictly positive, and for which (1.1) hence actually is non-degenerate near each fixed point (x, t). In the context of the Cauchy problem

$$\begin{cases} u_t = u^p \Delta u, & x \in \mathbb{R}^n, \ t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

for instance, the large time asymptotics of positive classical solutions emanating from positive and sufficiently fast decaying initial data u_0 in general differs from that in the heat equation by some quantitative corrections already in the porous medium regime: When $p \in (0, 1)$, namely, any such solution with $u_0^{1-p} \in L^1(\mathbb{R}^n)$ satisfies $\frac{1}{C} t^{-\frac{n}{2+(n-2)p}} \leq \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-\frac{n}{2+(n-2)p}}$ for all $t > 1$ with some $C > 0$ ([16, Theorem I.2.5], [12]), meaning that temporal decay properties of widely arbitrary solutions with rapidly decreasing initial data rather closely parallel those of the particular explicit self-similar solutions that form the celebrated family of so-called Barenblatt solutions ([3], [1]).

As some more recent findings have been indicating, however, outside the range $p \in (0, 1)$ within which such Barenblatt solutions are available, some yet more subtle facets in the dependence of large time decay on spatial asymptotics need to be expected. When $p \geq 1$, namely, decay rates of strictly positive solutions have some common lower bound which can be approached up to errors with arbitrarily small algebraic asymptotics, but which can never be attained exactly by any such solution. More precisely, the following has been shown in [10]:

Proposition A *Let $n \geq 1, p \geq 1$ and $u_0 \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be such that $u_0(x) > 0$ for all $x \in \mathbb{R}^n$. Then (1.2) possesses a classical solution $u \in C^0(\mathbb{R}^n \times [0,\infty)) \cap C^{2,1}(\mathbb{R}^n \times (0,\infty))$ which is such that $u(x,t) > 0$ for all $x \in \mathbb{R}^n$ and $t > 0$, and which is minimal in the sense that whenever $T \in (0,\infty]$ and $\widetilde{u} \in C^0(\mathbb{R}^n \times [0,T)) \cap C^{2,1}(\mathbb{R}^n \times (0,T))$ are such that $\widetilde{u}$ is positive and solves (1.2) classically in $\mathbb{R}^n \times (0,T)$, we have $u \leq \widetilde{u}$ in $\mathbb{R}^n \times (0,T)$.*
Moreover,

$$\liminf_{t\to\infty} \left\{ t^{\frac{1}{p}} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \right\} = \infty$$

and if in addition $u_0 \in \bigcap_{q>0} L^q(\mathbb{R}^n)$, then

$$\limsup_{t\to\infty} \left\{ t^{\frac{1}{p}-\delta} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \right\} < \infty \qquad \textit{for all } \delta > 0.$$

Apart from this, in [11] respective classes of suitably fast decreasing initial data have been identified within which actually any logarithmic, and even doubly logarithmic, corrections to the algebraic decay of $t^{-\frac{1}{p}}$ is essentially attained by corresponding positive solutions to (1.2) (see [11, Corollaries 1.5, 1.6, 1.8 and 1.9]).

Main results I: Arbitrarily slow increase of $t^{\frac{1}{p}} \|u(\cdot,t)\|_{L^\infty(\Omega)}$**.** The first objective of the present study now consists in examining whether beyond the latter particular examples, arbitrarily small deviations of the borderline decay rate indicated in Proposition A can be undercut by some positive solutions to (1.2). Our main results in this direction show that this indeed is possible in the following flavor that seems to be the least restricitive conceivable in this regard:

Theorem 1.1 *Let $n \geq 1$ and $p \geq 1$, and suppose that $f \in C^0([0,\infty))$ is positive and such that $f(t) \to +\infty$ as $t \to \infty$. Then there exists a positive nonincreasing function $\phi \in C^0([0,\infty))$ with the property that whenever $u_0 \in C^0(\mathbb{R}^n)$ is radially symmetric and such that*

$$0 < u_0(x) < \phi(|x|) \qquad \textit{for all } x \in \mathbb{R}^n, \tag{1.3}$$

the corresponding minimal solution u of (1.2) satisfies

$$\frac{t^{\frac{1}{p}} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}}{f(t)} \to 0 \qquad \textit{as } t \to \infty. \tag{1.4}$$

Main results II: Attaining vs. remaining away from critical decay for solutions with $\{u_0 > 0\} \neq \mathbb{R}^n$**.** We shall next address the question whether critical decay can be observed at least when the initial data are not strictly positive throughout $\mathbb{R}^n$. Here we note that already at the level of basic solution theories, the strong diffusion degeneracies present in the considered range $p \geq 1$ give rise to significant challenges, for the caveat documented in [14] indicates that within straightforward and seemingly natural adaptations of weak solution concepts to the framework of (1.2), uniqueness of solutions can not even be expected for initial data from $C_0^\infty(\mathbb{R}^n)$. We accordingly resort to a slightly modified notion of solvability, to be substantiated in Proposition 3.1 below, which inter alia requires continuity of the considered solution, and for which we thus, in accordance with known results on discontinuous solution behavior in the presence of initially isolated zeros ([4], [6]), impose some restrictions on the regularity of the positivity set $\{u_0 > 0\}$ in order to assert the mere existence of such solutions. More precisely, in this part we shall assume that

$$\begin{cases} 0 \not\equiv u_0 \in C^0(\mathbb{R}^n) \text{ is nonnegative such that} \\ \{u_0 > 0\} \text{ coincides with the interior of } \operatorname{supp} u_0, \text{ and that} \\ \text{each } \Omega \in \mathcal{C}(u_0) \text{ is a bounded domain with Lipschitz boundary,} \end{cases} \tag{1.5}$$

where for $0 \le \varphi \in C^0(\mathbb{R}^n)$ we have set

$$\mathcal{C}(\varphi) := \Big\{ \Omega \subset \mathbb{R}^n \ \Big| \ \Omega \text{ is a connected component of } \{\varphi > 0\} \Big\}, \tag{1.6}$$

and note that under these hypotheses, a uniquely determined continous weak solution can indeed be found (Proposition 3.1).

Now our intention in this part is to relate the possibility of exhibiting critical decay to some properties exclusively referring to features of the connected components of $\{u_0 > 0\}$, rather than to the size of u_0 nor its overall decay in space. Specifically, our first result in this respect reads as follows.

Proposition 1.2 *Let $n \ge 1$ and $p \ge 1$, and suppose that u_0 satisfies (1.5) with*

$$\inf_{\Omega \in \mathcal{C}(u_0)} \ \sup_{\substack{0 \le \varphi \in C^0(\overline{\Omega}) \cap C^2(\Omega) \\ \|\varphi\|_{L^\infty(\Omega)} = 1}} \ \inf_{x \in \Omega} \Big\{ -\varphi^{p-1}(x) \Delta \varphi(x) \Big\} > 0. \tag{1.7}$$

Then the continous weak solution u of (1.2) has the property that

$$0 < \liminf_{t \to \infty} \Big\{ t^{\frac{1}{p}} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \Big\} \le \limsup_{t \to \infty} \Big\{ t^{\frac{1}{p}} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \Big\} < \infty. \tag{1.8}$$

Indeed, the following consequence thereof establishes a link to the maximum size of all the members from $\mathcal{C}(u_0)$:

Corollary 1.3 *Let $n \ge 1$ and $p \ge 1$, and suppose that u_0 is such that (1.5) holds, and that there exists $K > 0$ with the property that each $\Omega \in \mathcal{C}(u_0)$ lies between two parallel hyperplanes with distance K, that is, for any such Ω one can find $x_0 \in \mathbb{R}^n$ and $A \in SO(n)$ such that*

$$\Omega \subset x_0 + AS \qquad \textit{with} \quad S := \Big\{ x = (x_1, ..., x_n) \in \mathbb{R}^n \ \Big| \ 0 < x_1 < K \Big\}.$$

Then the continuous weak solution u of (1.2) satisfies (1.8). In particular, this conclusion holds if

$$\sup_{\Omega \in \mathcal{C}(u_0)} \operatorname{diam} \Omega < \infty.$$

We shall next identify a criterion, partially complementary to that from Proposition 1.2, as sufficient to ensure absence of critical decay speeds also for some not strictly positive initial data. In formulating this, for notational convenience we abbreviate $C_0^0(\overline{\Omega}) := \{\varphi \in C^0(\overline{\Omega}) \mid \varphi|_{\partial\Omega} = 0\}$ for open sets $\Omega \subset \mathbb{R}^n$.

Proposition 1.4 *Let $n \ge 1$ and $p \ge 1$, and let u_0 be such that (1.5) holds, and that*

$$\inf_{\Omega \in \mathcal{C}(u_0)} \inf_{\substack{0 \le \varphi \in C_0^0(\overline{\Omega}) \cap C^2(\{\varphi>0\}) \\ \|\varphi\|_{L^\infty(\Omega)}=1}} \sup_{x \in \{\varphi>0\}} \Big\{ -\varphi^{p-1}(x)\Delta\varphi(x) \Big\} = 0. \tag{1.9}$$

Then for the continous weak solution u of (1.2) we have

$$\liminf_{t\to\infty} \Big\{ t^{\frac{1}{p}} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \Big\} = \infty. \tag{1.10}$$

To finally indicate that here the requirement (1.9) again is in close relationship to the component sizes of $\{u_0 > 0\}$, let us adopt the standard notation

$$\lambda_1(\Omega) := \inf_{0 \not\equiv \varphi \in W_0^{1,2}(\Omega)} \frac{\int_\Omega |\nabla\varphi|^2}{\int_\Omega \varphi^2} \tag{1.11}$$

for the principal Dirichlet eigenvalue of $-\Delta$ in a bounded domain $\Omega \subset \mathbb{R}^n$. In fact, we shall see that the conclusion of Proposition 1.4 holds whenever $\{u_0 > 0\}$ contains components with arbitrarily small values of these eigenvalues, and hence whenever $\{u_0 > 0\}$ has infinite inradius:

Corollary 1.5 *Let $n \ge 1$ and $p \ge 1$, and let u_0 be such that (1.5) holds, and that*

$$\inf_{\Omega \in \mathcal{C}(u_0)} \lambda_1(\Omega) = 0. \tag{1.12}$$

Then the continuous weak solution u of (1.2) satisfies (1.10). This especially follows if

$$\sup \Big\{ R > 0 \ \Big| \ \textit{There exist } \Omega \in \mathcal{C}(u_0) \textit{ and } x_0 \in \mathbb{R}^n \textit{ such that } B_R(x_0) \subset \Omega \Big\} = \infty. \tag{1.13}$$

2 Slow increase of $t^{\frac{1}{p}} \|u(\cdot,t)\|_{L^\infty(\Omega)}$. Proof of Theorem 1.1

2.1 Specifying the objective

Our approach toward the derivation of Theorem 1.1 will be based on the following fundamental observation made in [11, Theorem 1.3].

Theorem B *Assume that $n \ge 1$ and $p \ge 1$, that $s_0 > 0$, and that $\mathcal{L} \in C^0([0,\infty)) \cap L^\infty((0,\infty)) \cap C^2((0,s_0))$ is positive and nondecreasing on $(0,\infty)$ and such that $\mathcal{L}(0) = 0$, that there exist $a > 0$ and $\lambda_0 > 0$ fulfilling*

$$\mathcal{L}(s) \le (1 + a\lambda)\mathcal{L}(s^{1+\lambda}) \qquad \textit{for all } s \in (0,s_0) \textit{ and } \lambda \in (0,\lambda_0), \tag{2.1}$$

and that furthermore

$$s\mathcal{L}''(s) \geq -\frac{3p+q_0-2}{p+q_0}\mathcal{L}'(s) \qquad \text{for all } s \in (0,s_0) \tag{2.2}$$

with a certain $q_0 > 0$*. Then whenever* $u_0 \in C^0(\mathbb{R}^n)$ *is positive, radially symmetric and nonincreasing with respect to* $|x|$ *and such that* $u_0 < \min\left\{s_0^{\frac{2}{p}}, s_0^{\frac{2}{p+q_0}}\right\}$ *in* $\mathbb{R}^n$ *as well as*

$$\int_{\mathbb{R}^n} \mathcal{L}(u_0) < \infty,$$

there exist $t_0 > 0$ *and* $C > 0$ *such that the minimal classical solution* u *of (1.2) satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{1}{p}}\mathcal{L}^{-\frac{2}{np}}\left(\frac{1}{t}\right) \qquad \text{for all } t \geq t_0. \tag{2.3}$$

Now in order to appropriately prepare a construction of a function $\mathcal{L}$ which on the one hand satisfies the requirements in Theorem B, and especially the inequalities (2.1) and (2.2), but for which, on the other hand, the correction factor $\mathcal{L}^{-\frac{2}{np}}(\frac{1}{t})$ in (2.3) remains small relative to a given divergent function f in the style of Theorem 1.1, let us firstly derive a handy criterion sufficient for (2.1).

Lemma 2.1 *Let* $s_0 \in (0,1]$ *and* $a \in (0,1]$*, and suppose that* $\mathcal{L} \in C^1((0,s_0))$ *is positive and such that*

$$\mathcal{L}'(s) \leq a \cdot \frac{\mathcal{L}(s)}{s \ln \frac{s_0}{s}} \qquad \text{for all } s \in (0,s_0). \tag{2.4}$$

Then

$$\mathcal{L}(s) \leq (1+a\lambda) \cdot \mathcal{L}(s^{1+\lambda}) \qquad \text{for all } s \in (0,s_0) \text{ and } \lambda > 0. \tag{2.5}$$

Proof. Since $\mathcal{L}$ is positive, letting

$$H(s) := \ln \frac{1}{\mathcal{L}(s)}, \qquad s \in (0,s_0),$$

we obtain a well-defined element H of $C^1((0,s_0))$ which according to (2.4) satisfies

$$H'(s) = -\frac{\mathcal{L}'(s)}{\mathcal{L}(s)} \geq -\frac{a}{s\ln\frac{s_0}{s}} \qquad \text{for all } s \in (0,s_0).$$

Using that $s_0 \leq 1$, and that thus $s^{a+\lambda} \leq s$ for all $s \in (0,s_0)$ and $\lambda > 0$, we can therefore estimate

$$\begin{aligned} H(s) - H(s^{1+\lambda}) &= \int_{s^{1+\lambda}}^s H'(\sigma)d\sigma \\ &\geq -a\int_{s^{1+\lambda}}^s \frac{d\sigma}{\sigma \ln\frac{s_0}{\sigma}} \\ &= -a\int_{(\frac{s}{s_0})^{1+\lambda}}^{\frac{s}{s_0}} \frac{d\xi}{\xi\ln\frac{1}{\xi}} \qquad \text{for all } s \in (0,s_0) \text{ and } \lambda > 0. \end{aligned} \tag{2.6}$$

Since

$$\begin{aligned}
\int_{\Sigma^{1+\lambda}}^{\Sigma} \frac{d\xi}{\xi \ln \frac{1}{\xi}} &= -\left[\ln\ln\frac{1}{\xi}\right]_{\xi=\Sigma^{1+\lambda}}^{\xi=\Sigma} \\
&= -\ln\left(\frac{\ln\frac{1}{\Sigma}}{(1+\lambda)\ln\frac{1}{\Sigma}}\right) \\
&= \ln(1+\lambda) \qquad \text{for all } \Sigma \in (0,1) \text{ and } \lambda > 0,
\end{aligned}$$

and since

$$a\ln(1+\lambda) = \ln\Big\{(1+\lambda)^a\Big\} \le \ln(1+a\lambda) \qquad \text{for all } \lambda > 0$$

due to the fact that $a \le 1$ ensures that $(1+\lambda)^a \le 1 + a\lambda$ for all $\lambda > 0$, from (2.6) we thus obtain that

$$H(s) - H(s^{1+\lambda}) \ge -\ln(1+a\lambda) \qquad \text{for all } s \in (0,s_0) \text{ and } \lambda > 0.$$

According to the definition of H, this implies that

$$\ln\left(\frac{\mathcal{L}(s)}{(1+a\lambda)\mathcal{L}(s^{1+\lambda})}\right) = H(s^{1+\lambda}) - H(s) - \ln(1+a\lambda) \le 0 \qquad \text{for all } s \in (0,s_0) \text{ and } \lambda > 0$$

and hence establishes (2.5). $\square$

Fortunately, both this condition (2.4) and (2.2) can be reformulated in a rather convenient manner after a simple variable transformation:

Lemma 2.2 *Let $h \in C^2([0,\infty))$. Then for any choice of $s_0 > 0$, writing*

$$z \equiv z(s) := \ln\frac{s_0}{s} \quad \textit{and} \quad \mathcal{L}(s) := e^{-h(z)}, \qquad s \in (0,s_0], \tag{2.7}$$

defines a positive function $\mathcal{L} \in C^2((0,s_0])$ which is such that whenever $\kappa \in \mathbb{R}$, for all $s \in (0,s_0)$ we have

$$s\ln\frac{s_0}{s}\cdot\frac{\mathcal{L}'(s)}{\mathcal{L}(s)} = zh'(z) \tag{2.8}$$

and

$$s\mathcal{L}''(s) + \kappa\mathcal{L}'(s) = \frac{1}{s}\cdot e^{-h(z)}\cdot\Big\{-h''(z) + (\kappa-1)h'(z) + h'^2(z)\Big\} \tag{2.9}$$

with $z = z(s)$.

Proof. On the basis of (2.7), for $s \in (0,s_0)$ we compute $z'(s) = -\frac{1}{s}$ and

$$\mathcal{L}'(s) = \frac{1}{s}\cdot e^{-h(z)}h'(z)$$

and

$$\mathcal{L}''(s) = -\frac{1}{s^2}\cdot e^{-h(z)}h''(z) + \frac{1}{s^2}\cdot e^{-h(z)}h'^2(z) - \frac{1}{s^2}\cdot e^{-h(z)}h'(z)$$

with $z = z(s)$, so that both (2.8) and (2.9) readily follow. $\square$

In line with Theorem B, Lemma 2.1 and Lemma 2.2, we will thus subsequently intend to make sure that given any function $f = f(t)$ exhibiting arbitrarily slow unbounded growth, after transformation to a positive divergent function $F = F(z)$ on $[0, \infty)$ in the style suggested by (2.3) and (2.7), we can find a yet unbounded minorant h for which the correspondingly translated version $\mathcal{L}$, as defined through (2.7), satisfies the requirement in (2.1) in the sharpened sense expressed in Lemma 2.1 and (2.8), and which simultaneously complies with (2.2) via (2.9). Here we observe that since fortunately the rightmost summand in (2.9) is nonnegative, and since the factor appearing on the right of (2.2) satisfies $\frac{3p+q_0-2}{p+q_0} \geq 1$ for any choice of $p \geq 1$ and $q_0 > 0$, with regard to (2.2) it will be sufficient to construct h in such a way that $h' \geq 0$ and $h'' \leq 0$.

2.2 Construction of slowly increasing minorants

To accomplish the first among two major steps in our design of a smooth minorant with the desired properties, let us construct a piecewise linear but already concave preliminary candidate.

Lemma 2.3 *Let $F \in C^0([0, \infty))$ be positive and such that $F(z) \to +\infty$ as $z \to \infty$. Then there exist a strictly increasing sequence $(z_j)_{j \in \mathbb{N}} \subset [0, \infty)$ and a positive concave function $h_0 \in W^{1,\infty}_{loc}([0, \infty))$ such that $z_1 = 0$ and $z_j \to \infty$ as $j \to \infty$, that for all $j \in \mathbb{N}$ we have*

$$h_0 \in C^2((z_j, z_{j+1})) \qquad \textit{with} \qquad h_0''(z) = 0 \qquad \textit{for all } z \in (z_j, z_{j+1}) \tag{2.10}$$

and

$$0 < h_0'(z) \leq \frac{1}{z+1} \qquad \textit{for all } z \in (z_j, z_{j+1}), \tag{2.11}$$

and that

$$h(z) \leq F(z) \qquad \textit{for all } z > 0 \tag{2.12}$$

and

$$h_0(z) \to +\infty \qquad \textit{as } z \to \infty. \tag{2.13}$$

Proof. We pick $b \in (0, 1)$ in such a way that $F(z) \geq 2b$ for all $z \geq 0$, and construct $(z_j)_{j \in \mathbb{N}}$ and h_0 recursively as follows: Taking $z_1 := 0$, and for notational convenience also introducing $z_0 := -b$, we assume that for some $j \geq 1$ we already have found $(z_i)_{i \in \{0,...,j\}} \subset \mathbb{R}$ such that $z_{i+1} > z_i$ for all $i \in \{0, ..., j-1\}$ and

$$F(z) \geq (i+1)b \qquad \text{for all } z \geq z_i \text{ and } i \in \{1, ..., j\}, \tag{2.14}$$

and that letting

$$h_0(z) := m_i \cdot (z - z_i) + i \cdot b, \qquad z \in (z_i, z_{i+1}], \quad i \in \{0, ..., j-1\}, \tag{2.15}$$

with

$$m_i := \frac{b}{z_{i+1} - z_i}, \qquad i \in \{0, ..., j-1\}, \tag{2.16}$$

defines a continuous and concave function h_0 on $(z_0, z_j]$ which satisfies

$$0 < h_0'(z) \leq \frac{1}{z+1} \qquad \text{for all } z \in (z_0, z_j) \setminus \Big\{ z_i \ \Big| \ i \in \{1, ..., j-1\} \Big\}. \tag{2.17}$$

Now since $F(z) \to +\infty$ as $z \to \infty$, and since $b < 1$, we can fix $z_{j+1} > z_j$ large enough such that

$$F(z) \geq (j+2)b \qquad \text{for all } z \geq z_{j+1}, \tag{2.18}$$

and that furthermore

$$z_{j+1} > 2z_j - z_{j-1} \tag{2.19}$$

as well as

$$(1-b)z_{j+1} \geq z_j + b. \tag{2.20}$$

Then letting

$$m_j := \frac{b}{z_{j+1} - z_j} \qquad \text{and} \qquad h_0(z) := m_j \cdot (z - z_j) + j \cdot b, \quad z \in (z_j, z_{j+1}], \tag{2.21}$$

evidently extends h_0 to a function defined on all of $(z_0, z_{j+1}]$, in a manner consistent with (2.15) and (2.16), which is continuous on $(z_0, z_{j+1}]$ due to the fact that according to (2.21) and (2.15) we have $h_0(z) \to j \cdot b = h_0(z_j)$ as $(z_j, z_{j+1}] \ni z \searrow z_j$. To see that h_0 is concave on $(0, z_{j+1}]$, in view of (2.15) and (2.21) it is sufficient to observe that thanks to the definition of $(m_i)_{i \in \{0,\dots,j\}}$ in (2.16) and (2.21), the requirement in (2.19) guarantees that

$$\frac{1}{m_j} - \frac{1}{m_{j-1}} = \frac{z_{j+1} - 2z_j + z_{j-1}}{b} > 0,$$

which namely asserts that

$$\lim_{z \searrow z_j} h_0'(z) = m_j < m_{j-1} = \lim_{z \nearrow z_j} h_0'(z).$$

We next make use of (2.20) to confirm that

$$\begin{aligned}
(z+1) \cdot h_0'(z) &= (z+1)m_j \\
&\leq (z_{j+1}+1)m_j \\
&= \frac{b(z_{j+1}+1)}{z_{j+1} - z_j} \\
&= b \cdot \left\{ 1 + \frac{z_j + 1}{z_{j+1} - z_j} \right\} \\
&\leq b \cdot \left\{ 1 + \frac{z_j + 1}{\frac{z_j + b}{1-b} - z_j} \right\} \\
&= 1 \qquad \text{for all } z \in (z_j, z_{j+1}),
\end{aligned}$$

which together with (2.17) implies that

$$0 < h_0'(z) \leq \frac{1}{z+1} \qquad \text{for all } z \in (z_0, z_{j+1}) \setminus \left\{ z_i \;\middle|\; i \in \{1, \dots, j\} \right\}.$$

Since (2.18) along with (2.14) clearly ensures that

$$F(z) \geq (i+1)b \qquad \text{for all } z \geq z_i \text{ and any } i \in \{1, \dots, j+1\},$$

this completes our inductive construction of a strictly increasing sequence $(z_j)_{j\ge 0} \subset \mathbb{R}$ which satisfies $z_1 = 0$ and is such that (2.14) holds for all $j \in \mathbb{N}$, in particular meaning that necessarily $z_j \to \infty$ as $j \to \infty$.

In view of the fact that (2.15), (2.16) and (2.17) are valid for all $j \in \mathbb{N}$, we therefore moreover obtain a function $h_0 : (z_0, \infty) \to \mathbb{R}$ which when restricted to $[0,\infty)$ belongs to $W^{1,\infty}([0,\infty))$ and satisfies (2.10) and (2.11) for all $j \in \mathbb{N}$, which is concave by construction, and for which (2.13) holds thanks to the circumstance that $h_0(z_j) = j \cdot b \to +\infty$ as $j \to \infty$.

In order to finally verify (2.12), given $z > 0$ we fix $j \in \mathbb{N}$ such that $z \in (z_j, z_{j+1}]$, and use the definition of h_0 in $(z_j, z_{j+1}]$ implied by (2.15) in estimating

$$h_0(z) = b \cdot \frac{z - z_j}{z_{j+1} - z_j} + j \cdot b \le b + j \cdot b,$$

because $z \le z_{j+1}$. Since, on the other hand, the inequality $z \ge z_j$ enables us to conclude from (2.14) that

$$F(z) \ge (j+1)b,$$

this completes the proof. □

We next prepare an appropriate smoothing procedure, to be finally performed near each discontinuity point of the function h_0 from Lemma 2.3, by means of an explicit construction concentrating on cases in which only one point of such nonsmooth behavior is present.

Lemma 2.4 *Let $z_\star \in \mathbb{R}$ and $h_\star \in C^0(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{z_\star\})$ be concave and such that $h_\star''(z) = 0$ for all $z \in \mathbb{R} \setminus \{z_\star\}$. Then for any $\varepsilon > 0$ there exists $h_\varepsilon \in C^2(\mathbb{R})$ such that*

$$h_\varepsilon(z) = h_\star(z) \qquad \text{for all } z \in \mathbb{R} \setminus (z_\star - \varepsilon, z_\star + \varepsilon) \tag{2.22}$$

and

$$h_\varepsilon(z) \le h_\star(z) \qquad \text{for all } z \in \mathbb{R} \tag{2.23}$$

as well as

$$h_\varepsilon''(z) \le 0 \qquad \text{for all } z \in \mathbb{R}. \tag{2.24}$$

PROOF. Our hypotheses precisely mean that there exist $c_1 \in \mathbb{R}, \underline{m} \in \mathbb{R}$ and $\overline{m} \le \underline{m}$ such that

$$h_\star(z) = \begin{cases} \underline{m} \cdot (z - z_\star) + c_1 & \text{for all } z \le z_\star, \\ \overline{m} \cdot (z - z_\star) + c_1 & \text{for all } z > z_\star, \end{cases}$$

and we may assume that actually $\overline{m} < \underline{m}$, for otherwise choosing $h_\varepsilon \equiv h_\star$ clearly warrants validity of (2.22)-(2.24).

For fixed $\varepsilon > 0$, we then let

$$A_1 := \frac{\varepsilon}{2}(\underline{m} - \overline{m}), \quad A_2 := \frac{\underline{m} + \overline{m}}{2} \quad \text{and} \quad A_3 := c_1 - \frac{\varepsilon}{2}(\underline{m} - \overline{m}) - \frac{1}{2}(\underline{m} + \overline{m}) z_\star$$

and observe that, as can easily be verified, these selections ensure that

$$h_\star(z) = A_1 \cdot \widehat{h}_\star\Big(\frac{z-z_\star}{\varepsilon}\Big) + A_2 z + A_3 \qquad \text{for all } z \in \mathbb{R}, \tag{2.25}$$

where

$$\widehat{h}_\star(\xi) := 1 - |\xi|, \qquad \xi \in \mathbb{R}.$$

To see that the normalized situation thus obtained can be coped with by means of an explicit construction, we introduce

$$\widehat{h}_1(\xi) := \begin{cases} 1+\xi, & \xi \in (-\infty,-1], \\ -\frac{1}{3}\xi^3 - \xi^2 + \frac{2}{3}, & \xi \in (-1,0], \\ \frac{1}{3}\xi^3 - \xi^2 + \frac{2}{3}, & \xi \in (0,1], \\ 1-\xi, & \xi \in (1,\infty). \end{cases}$$

Then straightforward computation shows that $\widehat{h}_1$ belongs to $C^2(\mathbb{R})$ and satisfies

$$\widehat{h}_1(\xi) = \widehat{h}_\star(\xi) \qquad \text{for all } \xi \in \mathbb{R} \setminus (-1,1) \tag{2.26}$$

and

$$\begin{aligned} \widehat{h}_1(\xi) - \widehat{h}_\star(\xi) &= \Big\{\frac{1}{3}|\xi|^3 - \xi^2 + \frac{2}{3}\Big\} - \Big\{1 - |\xi|\Big\} \\ &\le \frac{1}{3} \cdot (|\xi|-1)^3 \\ &\le 0 \qquad \text{for all } \xi \in (-1,1) \end{aligned} \tag{2.27}$$

as well as

$$\widehat{h}_1''(\xi) = 2(|\xi|-1) \le 0 \qquad \text{for all } \xi \in (-1,1). \tag{2.28}$$

Therefore, if in reminiscence of (2.25) we let

$$h_\varepsilon(z) := A_1 \cdot \widehat{h}_1\Big(\frac{z-z_\star}{\varepsilon}\Big) + A_2 z + A_3, \qquad z \in \mathbb{R}, \tag{2.29}$$

then (2.22) and (2.23) directly result from (2.26), (2.27) and (2.25), whereas (2.24) is a consequence of (2.28), because

$$h_\varepsilon''(z) = \frac{A_1}{\varepsilon} \cdot \widehat{h}_1''\Big(\frac{z-z_\star}{\varepsilon^2}\Big) \qquad \text{for all } z \in (z_\star - \varepsilon, z_\star + \varepsilon)$$

by (2.29). □

Now suitable application of the latter to the function gained in Lemma 2.3 yields the following.

Lemma 2.5 *Let $F \in C^0([0,\infty))$ be such that $F(z) > 0$ for all $z \ge 0$ and $F(z) \to +\infty$ as $z \to \infty$. Then there exists $h \in C^2([0,\infty))$ such that*

$$0 < h(z) \le F(z) \qquad \textit{for all } z \ge 0 \tag{2.30}$$

and

$$0 < h'(z) \le \frac{1}{z} \qquad \text{for all } z > 0, \tag{2.31}$$

and such that moreover

$$h''(z) \le 0 \qquad \text{for all } z > 0 \tag{2.32}$$

and

$$h(z) \to +\infty \qquad \text{as } z \to \infty. \tag{2.33}$$

Proof. We take $(z_j)_{j\in\mathbb{N}} \subset [0,\infty)$ and $h_0 \in W^{1,\infty}_{loc}([0,\infty))$ as provided by Lemma 2.3, and for $j \in \mathbb{N}$ with $j \ge 2$ we then obtain from the linearity of h_0 on $[z_{j-1}, z_j]$ and on $[z_j, z_{j+1}]$ that

$$h_0(z) = h_\star^{(j)}(z) \qquad \text{for all } z \in [z_{j-1}, z_{j+1}], \tag{2.34}$$

where

$$h_\star^{(j)}(z) := \begin{cases} \underline{m}_j \cdot (z - z_j) + b_j, & z \in (-\infty, z_j], \\ \overline{m}_j \cdot (z - z_j) + b_j, & z \in (z_j, \infty), \end{cases}$$

with $b_j := h_0(z_j)$, and with $\underline{m}_j \in \mathbb{R}$ and $\overline{m}_j \in \mathbb{R}$ being the well-defined constants fulfilling $h_0' \equiv \underline{m}_j$ in (z_{j-1}, z_j) and $h_0' \equiv \overline{m}_j$ in (z_j, z_{j+1}). As the concavity of h_0 requires that $\underline{m}_j \ge \overline{m}_j$ and that thus also $h_\star^{(j)}$ is concave for any such j, fixing any $\varepsilon_j > 0$ such that

$$\varepsilon_j < \min\left\{ \frac{z_j - z_{j-1}}{2}, \frac{z_{j+1} - z_j}{2}, \frac{1}{2} \right\}, \tag{2.35}$$

we may employ Lemma 2.4 to find

$$h^{(j)} \equiv h^{(j)}_{\varepsilon_j} \in C^2(\mathbb{R}) \tag{2.36}$$

such that

$$h^{(j)}(z) = h_\star^{(j)}(z) \qquad \text{for all } z \in \mathbb{R} \setminus (z_j - \varepsilon_j, z_j + \varepsilon_j), \tag{2.37}$$

that

$$h^{(j)}(z) \le h_\star^{(j)}(z) \qquad \text{for all } z \in \mathbb{R}, \tag{2.38}$$

and that

$$(h^{(j)})''(z) \le 0 \qquad \text{for all } z \in \mathbb{R}. \tag{2.39}$$

Then since (2.35) ensures that for all $j \ge 2$ we have

$$z_j + \varepsilon_j < z_j + \frac{z_{j+1} - z_j}{2} = \frac{z_j + z_{j+1}}{2} = z_{j+1} - \frac{z_{j+1} - z_j}{2} < z_{j+1} - \varepsilon_{j+1},$$

it follows that

$$[z_j - \varepsilon_j, z_j + \varepsilon_j] \cap [z_k - \varepsilon_k, z_k + \varepsilon_k] = \emptyset \qquad \text{for all } j \in \mathbb{N} \text{ and } k \in \mathbb{N} \text{ such that } j \ge 2, k \ge 2 \text{ and } j \ne k, \tag{2.40}$$

and that thus

$$h(z) := \begin{cases} h^{(j)}(z) & \text{if } z \in [z_j - \varepsilon_j, z_j + \varepsilon_j] \text{ for some } j \ge 2, \\ h_0(z) & \text{if } z \in [0,\infty) \setminus \bigcup_{j=2}^\infty [z_j - \varepsilon_j, z_j + \varepsilon_j], \end{cases} \tag{2.41}$$

introduces a well-defined function h on $[0,\infty)$ which due to (2.36), (2.37) and (2.40) belongs to $C^2([0,\infty))$, and for which from (2.39) and the piecewise linearity of h_0 we know that (2.32) holds.

This concavity property also entails the left inequality in (2.31) as a particular consequence, because given any $z>0$ we can rely on the unboundedness of $(z_j)_{j\in\mathbb{N}}$ to find $j\ge 2$ such that $z\le z_j+\varepsilon_j$, so that by (2.32), (2.41) and the left inequality in (2.11),

$$h'(z)\ge h'(z_j+\varepsilon_j)=h_0'(z_j+\varepsilon_j)>0.$$

Likewise, combining (2.32) with the right inequality in (2.11) we see that if $z>0$ is such that $z\in[z_j-\varepsilon_j,z_j+\varepsilon_j]$ for some $j\ge 2$, then due to the rightmost restriction expressed in (2.35),

$$\begin{aligned} zh'(z)-1 &\le zh'(z_j-\varepsilon_j)-1 \\ &= zh_0'(z_j-\varepsilon_j)-1 \\ &\le \frac{z}{(z_j-\varepsilon_j)+1}-1 \\ &\le \frac{z_j+\varepsilon_j}{z_j-\varepsilon_j+1}-1 \\ &= \frac{2\varepsilon_j-1}{z_j-\varepsilon_j+1} \\ &\le 0, \end{aligned}$$

whereas if $z\in[0,\infty)\setminus\bigcup_{j\ge 2}[z_j-\varepsilon_j,z_j+\varepsilon_j]$, then clearly $zh'(z)\le\frac{z}{z+1}\le 1$ by (2.11).

Having thereby asserted both inequalities in (2.31) for all $z\ge 0$, we proceed to observe that again thanks to (2.40), we may draw on (2.13) to infer that

$$\limsup_{z\to\infty} h(z)\ge \limsup_{j\to\infty} h(z_j+\varepsilon_j)=\limsup_{j\to\infty} h_0(z_j+\varepsilon_j)=+\infty,$$

whence (2.33) becomes a consequence of the upward monotonicity of h guaranteed by (2.31).

Thus left with the verification of (2.30), we first note that for all $z\in[0,\infty)\setminus\bigcup_{j=2}^\infty[z_j-\varepsilon_j,z_j+\varepsilon_j]$ it directly follows from (2.12) that

$$h(z)=h_0(z)\le F(z),$$

while if $z\in[z_j-\varepsilon_j,z_j+\varepsilon_j]$ for some $j\ge 2$, then (2.38) and (2.34) enable us to again invoke (2.12) when concluding that

$$h(z)=h^{(j)}(z)\le h_\star^{(j)}(z)=h_0(z)\le F(z).$$

Finally, the left inequality in (2.30) also results from the nonnegativity of h' when combined with the observation that since $\varepsilon_2<\frac{z_2-z_1}{2}=\frac{z_2}{2}$ by (2.35), and since thus $z_2-\varepsilon_2>\frac{z_2}{2}>0$, the definition (2.41) ensures that $h(0)=h_0(0)$ and that hence $h(0)>0$ due to the positivity of h_0, as warranted by Lemma 2.3. $\square$

We can now return to Lemma 2.1 and Lemma 2.2 to verify that indeed for essentially any given f diverging to $+\infty$ we can find a function $\mathcal{L}$ that simultaneously possesses all the intended properties.

Lemma 2.6 *Let $n \geq 1$ and $p \geq 1$, and suppose that $f \in C^0([1,\infty))$ is such that $f(t) > 1$ for all $t \geq 1$, and that $f(t) \to +\infty$ as $t \to \infty$. Then one can find $\mathcal{L} \in C^0([0,\infty)) \cap C^2((0,1))$ with the properties that*

$$\mathcal{L}(0) = 0, \qquad \mathcal{L}(s) > 0 \quad \textit{for all } s \in (0,1] \qquad \textit{and} \qquad \mathcal{L}(s) = \mathcal{L}(1) \quad \textit{for all } s > 1, \tag{2.42}$$

that

$$0 < \mathcal{L}'(s) \leq \frac{\mathcal{L}(s)}{s \ln \frac{1}{s}} \qquad \textit{for all } s \in (0,1), \tag{2.43}$$

that

$$s\mathcal{L}''(s) \geq -\mathcal{L}'(s) \qquad \textit{for all } s \in (0,1), \tag{2.44}$$

and that

$$\mathcal{L}(s) \geq f^{-\frac{np}{4}}\Big(\frac{1}{s}\Big) \qquad \textit{for all } s \in (0,1). \tag{2.45}$$

In particular,

$$\frac{\mathcal{L}^{-\frac{2}{np}}\big(\frac{1}{t}\big)}{f(t)} \to 0 \qquad \textit{as } 1 < t \to \infty. \tag{2.46}$$

Proof. We let

$$F(z) := \frac{np}{4} \ln f(e^z), \qquad z \geq 0, \tag{2.47}$$

and observe that our assumptions on f ensure that F is continuous and positive on $[0,\infty)$ with $F(z) \to +\infty$ as $z \to \infty$. We may therefore employ Lemma 2.5 to obtain a function $h \in C^2([0,\infty))$ which satisfies (2.30)-(2.33), and thereupon define

$$\mathcal{L}(s) := \begin{cases} 0 & \text{if } s = 0, \\ e^{-h(z)}, \quad z \equiv z(s) := \ln \frac{1}{s}, & \text{if } s \in (0,1], \\ e^{-h(0)} & \text{if } s > 1. \end{cases} \tag{2.48}$$

Then since $h(z) \to +\infty$ as $z \to \infty$, it follows that $\mathcal{L}$ is continuous, whereas the inclusion $h \in C^2([0,\infty))$ clearly implies that $\mathcal{L}$ moreover belongs to $C^2((0,1))$. All three properties in (2.42) and the left inequality in (2.43) are evident from (2.48) and the strict positivity of h', and the right inequality in (2.43) results from the identity in (2.8), applied to $s_0 := 1$, and the fact that $zh'(z) \leq 1$ for all $z > 0$ by (2.31). To verify (2.44), we only need to invoke (2.9) with $\kappa := 1$ and use that $h''(z) \leq 0$ for all $z > 0$, and (2.45) can be seen by combining (2.48) with (2.47), which thanks to the right inequality in (2.30), namely, guarantees that

$$\begin{aligned} \ln \mathcal{L}(s) &= -h(z(s)) \\ &\geq -F(z(s)) \\ &= -\frac{np}{4} \ln f(e^{z(s)}) \\ &= \ln f^{-\frac{np}{4}}\Big(\frac{1}{s}\Big) \qquad \text{for all } s \in (0,1). \end{aligned}$$

Finally, since $f(t) \to +\infty$ as $t \to \infty$, this indeed entails (2.46) as a particular consequence, for by (2.45),

$$\frac{\mathcal{L}^{-\frac{2}{np}}\left(\frac{1}{t}\right)}{f(t)} \le \frac{f^{\frac{1}{2}}(t)}{f(t)} = f^{-\frac{1}{2}}(t) \to 0$$

as $1 < t \to \infty$. $\Box$

The derivation of our main result on arbitrarily small deviations from the critical decay rate thereupon becomes quite straightforward:

Proof of Theorem 1.1. We take $\mathcal{L}$ as given by Lemma 2.6, and note that as a strictly increasing function, $\mathcal{L}|_{[0,1]}$ possesses a strictly increasing inverse Λ defined on $[0, \mathcal{L}(1)]$. Fixing any nonincreasing $\psi \in C^0([0,\infty))$ such that $0 < \psi(r) < \mathcal{L}(1)$ for all $r \ge 0$ and

$$\int_0^\infty r^{n-1}\psi(r)dr < \infty, \tag{2.49}$$

we then see that letting

$$\phi(r) := \Lambda(\psi(r)), \qquad r \ge 0, \tag{2.50}$$

introduces a well-defined $\phi \in C^0([0,\infty))$ which is positive and nonincreasing according to the monotonicity properties of Λ.

Now (2.44) together with the nonnegativity of $\mathcal{L}'$ ensures that if we pick any $q_0 > 0$, then

$$\begin{aligned} s\mathcal{L}''(s) + \frac{3p+q_0-2}{p+q_0} \cdot \mathcal{L}'(s) &\ge -\mathcal{L}'(s) + \frac{3p+q_0-2}{p+q_0} \cdot \mathcal{L}'(s) \\ &= \frac{2(p-1)}{p+q_0} \cdot \mathcal{L}'(s) \qquad \text{for all } s \in (0,1), \end{aligned}$$

whereas (2.43) in conjunction with Lemma 2.1 warrants that

$$\mathcal{L}(s) \le (1+\lambda)\mathcal{L}(s^{1+\lambda}) \qquad \text{for all } s \in (0,1) \text{ and } \lambda > 0.$$

We may therefore employ Theorem B to conclude that whenever $u_0 \in C^0(\mathbb{R}^n)$ is radially symmetric and such that (1.3) holds, then since especially also $u_0(x) < \Lambda(\mathcal{L}(1)) = 1 = \max\{1^{\frac{2}{p}}, 1^{\frac{2}{p+q_0}}\}$ for all $x \in \mathbb{R}^n$ by (2.50), and since

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{L}(u_0) &\le \int_{\mathbb{R}^n} \mathcal{L}(\phi(|x|))dx \\ &= n|B_1(0)| \int_0^\infty r^{n-1}\mathcal{L}(\phi(r))dr \\ &= n|B_1(0)| \int_0^\infty r^{n-1}\psi(r)dr \\ &< \infty \end{aligned}$$

due to (2.49), we can find $t_0 \ge 1$ and $c_1 > 0$ such that

$$t^{\frac{1}{p}}\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le c_1\mathcal{L}^{-\frac{2}{np}}\left(\frac{1}{t}\right) \qquad \text{for all } t \ge t_0.$$

In view of (2.46), however, this already establishes (1.4). $\Box$

3 Continuous weak solutions with nontrivial zero sets

The basis for our investigation of solutions emanating from initial data u_0 with $\{u_0 > 0\} \neq \mathbb{R}^n$ will be formed by the following statement on existence and uniqueness of continuous weak solutions, as essentially contained already in the literature, together with a basic lower bound for their temporal decay.

Proposition 3.1 *Given $p \geq 1$, let $\Xi(s) := \int_1^s \frac{d\sigma}{\sigma^p}$, $\sigma > 0$, and assume that $n \geq 1$ and that $u_0 \in C^0(\mathbb{R}^n)$ is nonnegative and such that $\{u_0 > 0\}$ coincides with the interior of* $\operatorname{supp} u_0$*, and that each connected component of $\{u_0 > 0\}$ is a bounded domain with Lipschitz boundary. Then there exists a nonnegative function $u \in C^0(\mathbb{R}^n \times [0,\infty)) \cap L^\infty(\mathbb{R}^n \times (0,\infty))$, uniquely determined by the additional regularity requirements that*

$$(u-\eta)_+ \in W^{1,2}(\mathbb{R}^n \times (t_1,t_2)) \quad \textit{for any } \eta > 0,\ t_1 > 0 \textit{ and } t_2 > 0,$$

and that for all bounded domains $\Omega \subset \mathbb{R}^n$ and any $\varphi \in C^2(\overline{\Omega})$ with $\varphi > 0$ in Ω and $\varphi|_{\partial\Omega} = 0$,

$$0 \leq t \mapsto \int_\Omega \Xi(u(\cdot,t))\varphi \quad \textit{is continuous as a } [-\infty,\infty)\textit{-valued mapping,}$$

such that u forms a continuous weak solution of (1.2) *in the sense that $u|_{t=0} = u_0$ and that whenever $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary and $\varphi \in C^2(\overline{\Omega})$ satisfies $\varphi > 0$ in Ω with $\varphi|_{\partial\Omega} = 0$,*

$$\int_\Omega \Xi(u(\cdot,t_2))\varphi = \int_{t_1}^{t_2}\int_\Omega u\Delta\varphi - \int_{t_1}^{t_2}\int_{\partial\Omega} u\frac{\partial\varphi}{\partial\nu} + \int_\Omega \Xi(u(\cdot,t_1))\varphi$$

holds as an identity in $[-\infty,\infty)$ for any $t_1 \geq 0$ and $t_2 > t_1$.

In addition, this solution satisfies

$$u(x,t) = 0 \qquad \textit{for all } x \in \mathbb{R}^n \setminus \{u_0 > 0\} \textit{ and } t > 0, \tag{3.1}$$

and for each connected component Ω_0 of $\{u_0 > 0\}$, u belongs to $C^{2,1}(\Omega_0 \times (0,\infty))$ with $u > 0$ in $\Omega_0 \times (0,\infty)$. Furthermore, there exists $C > 0$ such that

$$t^{\frac{1}{p}}\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \geq C \qquad \textit{for all } t > 1. \tag{3.2}$$

Proof. Except for (3.2), all statements can be obtained by means of an almost verbatim transfer of the arguments from [18, Theorem 1.2.4], as detailed there for homogeneous Dirichlet problems in bounded domains, to the present Cauchy problem situation (cf. also [21, Theorem 2.1] for a slightly simpler close relative involving marginally stronger regularity classes).

To derive (3.2), we fix any ball $B \subset \mathbb{R}^n$ such that $\overline{B} \subset \{u_0 > 0\}$, and let $\Theta \in C^2(\overline{B})$ denote the principal Dirichlet eigenfunction of $-\Delta$ in B with $\max_{x\in\overline{B}} \Theta(x) = 1$. Then defining

$$\underline{u}(x,t) := y(t)\Theta(x), \quad x \in \overline{B},\ t \geq 0, \qquad \text{where} \qquad y(t) := \Big\{y_0^{-p} + p\lambda_1(B)t\Big\}^{-\frac{1}{p}}, \quad t \geq 0, \tag{3.3}$$

with $y_0 := \frac{1}{2}\min_{x\in\overline{B}} u_0(x)$ being positive by continuity of u_0, we immediately see that $\underline{u}(x,0) = y_0\Theta(x) < u_0(x)$ for all $x \in \overline{B}$ and $\underline{u}(x,t) = 0 < u(x,t)$ for all $x \in \overline{B}$ and $t \geq 0$ by positivity of u in $\{u_0 > 0\} \times [0,\infty)$. As moreover the identities $-\Delta\Theta = \lambda_1(B)\Theta$ and $y' = -\lambda_1 y^{p+1}$ along with the inequalities $0 \leq \Theta \leq 1$ ensure that

$$\underline{u}_t - \underline{u}^p\Delta\underline{u} = \Theta\cdot\Big\{y'(t) + \lambda_1(B)y^{p+1}(t)\Theta^p\Big\} \leq \Theta\cdot\Big\{y'(t) + \lambda_1(B)y^{p+1}(t)\Big\} = 0$$

in $B \times (0,\infty)$, due to the fact that u classically solves $u_t = u^p\Delta u$ in $B \times (0,\infty)$ we may conclude by a comparison argument ([17, Sect. 3.1]) that $\underline{u}(x,t) \leq u(x,t)$ for all $x \in B$ and $t > 0$. Since $\|\underline{u}(\cdot,t)\|_{L^\infty(B)} = y(t)$ for all $t \geq 0$, and since y is positive with $t^{\frac{1}{p}}y(t) \to (p\lambda_1(B))^{-\frac{1}{p}}$ as $t \to \infty$ by (3.3), this immediately yields (3.2) with suitably small $C > 0$. □

3.1 Attaining critical decay. Proof of Proposition 1.2 and of Corollary 1.3

Now our general criterion ensuring attainment of critical speed is based on a comparison argument involving separated supersolutions:

PROOF of Proposition 1.2. According to (1.7), there exists $c_1 > 0$ with the property that for any $\Omega \in \mathcal{C}(u_0)$ one can find $\varphi \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$0 \leq \varphi(x) \leq 1 \qquad \text{for all } x \in \Omega \tag{3.4}$$

and $-\varphi^{p-1}(x)\Delta\varphi(x) \geq c_1$ for all $x \in \Omega$, where the latter clearly entails that actually $\varphi(x) > 0$ for all $x \in \Omega$ and

$$-\frac{1}{c_1}\varphi^{p-1}(x)\Delta\varphi(x) \geq 1 \qquad \text{for all } x \in \Omega. \tag{3.5}$$

For any such Ω, we now define $\overline{u} \equiv \overline{u}_\Omega$ by letting

$$\overline{u}(x,t) := y(t)\cdot\big(\varphi(x)+1\big), \qquad x \in \overline{\Omega},\ t \geq 0, \tag{3.6}$$

where

$$y(t) := \Big\{y_0^{-p} + pc_1 t\Big\}^{-\frac{1}{p}}, \qquad t \geq 0, \tag{3.7}$$

with

$$y_0 := \|u_0\|_{L^\infty(\mathbb{R}^n)}, \tag{3.8}$$

and observe that then

$$\overline{u}(x,0) = y_0\cdot\big(\varphi(x)+1\big) > y_0 \geq u(x,0) \qquad \text{for all } x \in \overline{\Omega} \tag{3.9}$$

by (3.6), (3.7) and (3.8), and that

$$\overline{u}(x,t) > u(x,t) \qquad \text{for all } x \in \partial\Omega \text{ and } t \geq 0 \tag{3.10}$$

due to the fact that $u|_{\partial\Omega\times[0,\infty)} = 0$ thanks to Proposition 3.1. Apart from that, using that $y'(t) = -c_1 y^{p+1}(t)$ for all $t > 0$ by (3.7), from (3.5) we obtain that

$$\begin{aligned}
\overline{u}_t - \overline{u}^p\Delta\overline{u} &= y'(t)\cdot(\varphi+1) - y^{p+1}(t)\cdot(\varphi+1)^p\Delta\varphi \\
&= (\varphi+1)\cdot\Big\{y'(t) - y^{p+1}(t)\cdot(\varphi+1)^p\Delta\varphi\Big\} \\
&= c_1 y^{p+1}(t)\cdot(\varphi+1)\cdot\Big\{-1-\frac{1}{c_1}(\varphi+1)^p\Delta\varphi\Big\} \\
&\ge c_1 y^{p+1}(t)\cdot(\varphi+1)\cdot\Big\{-1+\frac{(\varphi+1)^{p-1}}{\varphi^{p-1}}\Big\} \\
&> 0 \qquad \text{in } \Omega\times(0,\infty).
\end{aligned}$$

Relying on the strictness of the inequalities both in (3.9) and (3.10), we may therefore employ the comparison principle from [17, Sect. 3.1] to conclude that whenever $\Omega \in \mathcal{C}(u_0)$,

$$u(x,t) \le \overline{u}_\Omega(x,t) \qquad \text{for all } x\in\Omega \text{ and } t>0,$$

which again due to Proposition 3.1 implies that

$$\begin{aligned}
\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} &= \sup_{\Omega\in\mathcal{C}(u_0)} \|u(\cdot,t)\|_{L^\infty(\Omega)} \\
&\le \sup_{\Omega\in\mathcal{C}(u_0)} \|\overline{u}_\Omega(\cdot,t)\|_{L^\infty(\Omega)} \\
&\le 2(pc_1)^{-\frac{1}{p}} t^{-\frac{1}{p}} \qquad \text{for all } t>0,
\end{aligned} \tag{3.11}$$

because obviously $y(t) \le (pc_1t)^{-\frac{1}{p}}$ for all $t > 0$ by (3.7), and because $1 \le \varphi + 1 \le 2$ in Ω by (3.4). As c_1 was positive, (1.8) thus results from (3.11) when combined with the lower estimate provided by (3.2). $\square$

Indeed, the requirement on boundedness in one direction made in Corollary 1.3 can readily be seen by means of an explicit construction to ensure a uniform elliptic inequality in the flavor of that required in (1.7):

Proof of Corollary 1.3. It is sufficient to verify that

$$\sup_{\substack{0\le\varphi\in C^0(\overline{\Omega})\cap C^2(\Omega)\\ \|\varphi\|_{L^\infty(\Omega)}=1}} \inf_{x\in\Omega}\Big\{-\varphi^{p-1}(x)\Delta\varphi(x)\Big\} \ge c_1 := \frac{\pi^2}{2^{\frac{p+4}{2}}K^2} \qquad \text{for all } \Omega\in\mathcal{C}(u_0), \tag{3.12}$$

and to achieve this, we let any $\Omega \in \mathcal{C}(u_0)$ be given and first note that upon translating and rotating that $\Omega \subset S$. Then

$$\varphi_0(x) := \cos\frac{\pi\cdot(2x_1-K)}{4K}, \qquad x=(x_1,...,x_n)\in\overline{\Omega},$$

defines a function $\varphi_0 \in C^2(\overline{\Omega})$ which satisfies

$$\varphi_0(x) \ge \cos\frac{\pi}{4} = 2^{-\frac{1}{2}} \qquad \text{for all } x\in\overline{\Omega},$$

because

$$-\frac{\pi}{4} = \frac{\pi \cdot (-K)}{4K} \le \frac{\pi \cdot (2x_1 - K)}{4K} \le \frac{\pi \cdot (2K - K)}{4K} = \frac{\pi}{4} \qquad \text{for all } x_1 \in [0, K].$$

Since clearly $\Delta\varphi_0(x) = -(\frac{\pi}{2K})^2\varphi_0(x)$ for all $x \in \Omega$, we therefore obtain that

$$-\varphi_0^{p-1}(x)\Delta\varphi_0(x) = \frac{\pi^2}{4K^2}\varphi_0^p(x) \ge c_1 \qquad \text{for all } x \in \Omega, \tag{3.13}$$

so that (3.12) results upon observing that

$$\varphi(x) := c_2\varphi_0(x), \quad x \in \overline{\Omega}, \qquad \text{with } c_2 := \frac{1}{\|\varphi_0\|_{L^\infty(\Omega)}} \ge 1,$$

thus defines a nonnegative function $\varphi \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ with $\|\varphi\|_{L^\infty(\Omega)} = 1$ and

$$-\varphi^{p-1}(x)\Delta\varphi(x) = c_2^p \cdot \Big\{\varphi_0^{p-1}(x)\Delta\varphi_0(x)\Big\} \ge c_2^p c_1 \ge c_1 \qquad \text{for all } x \in \Omega$$

by (3.13). Based on the inequality (3.12) thus derived, an application of Proposition 1.2 hence completes the proof. $\square$

3.2 Decay slower than critical. Proof of Proposition 1.4 and of Corollary 1.5

Conversely, the framework created in the formulation of Proposition 1.4 enables us to derive the claimed unboundedness feature through comparison from below with separated subsolutions, refining the corresponding procedure from the proof of Proposition 3.1 so as to yield suitably large lower bounds.

Proof of Proposition 1.4. Given $M > 0$, we let

$$\eta \equiv \eta_M := \frac{1}{2^{p+1}pM^p}, \tag{3.14}$$

and then may rely on (1.9) in choosing $\Omega \in \mathcal{C}(u_0)$ and a nonnegative $\varphi \in C_0^0(\overline{\Omega}) \cap C^2(\{\varphi > 0\})$ such that

$$\max_{x\in\overline{\Omega}} \varphi(x) = 1 \tag{3.15}$$

and $-\varphi^{p-1}(x)\Delta\varphi(x) < \eta$ for all $x \in \{\varphi > 0\}$, that is,

$$-\frac{1}{\eta}\Delta\varphi(x) < \varphi^{1-p}(x) \qquad \text{for all } x \in \{\varphi > 0\}. \tag{3.16}$$

Now since φ is continuous in $\overline{\Omega}$ with $\varphi = 0$ on $\partial\Omega$, the open set $\Omega_0 := \{\varphi > \frac{1}{2}\}$ satisfies $\overline{\Omega}_0 \subset \Omega$, and therefore the positivity of the continuous function u_0 on $\overline{\Omega}_0$ ensures the existence of $y_0 > 0$ such that

$$\frac{1}{2}y_0 < u_0(x) \qquad \text{for all } x \in \overline{\Omega}_0. \tag{3.17}$$

Moreover, (3.15) guarantees that

$$\underline{u}(x,t) := y(t)\cdot\Big(\varphi(x)-\frac{1}{2}\Big), \qquad x\in\overline{\Omega}_0,\ t\ge 0, \tag{3.18}$$

with

$$y(t) := \Big\{y_0^{-p}+p\eta t\Big\}^{-\frac{1}{p}}, \qquad t\ge 0, \tag{3.19}$$

satisfies

$$\underline{u}(x,0) = y_0\cdot\Big(\varphi(x)-\frac{1}{2}\Big) \le \frac{1}{2}y_0 < u_0(x) \qquad \text{for all } x\in\overline{\Omega}_0 \tag{3.20}$$

due to (3.17), and

$$\underline{u}(x,t) = 0 < u(x,t) \qquad \text{for all } x\in\partial\Omega_0 \text{ and } t\ge 0 \tag{3.21}$$

according to the definition of Ω_0 and the positivity of u inside $\Omega\times[0,\infty)$, as asserted by Proposition 3.1. Forthermore, since $y'(t) = -\eta y^{p+1}(t)$ for all $t>0$ by (3.19), using (3.16) we see that

$$\begin{aligned}
\underline{u}_t - \underline{u}^p\Delta\underline{u} &= y'(t)\cdot\Big(\varphi-\frac{1}{2}\Big) - \Big(\varphi-\frac{1}{2}\Big)^p\Delta\varphi\cdot y^{p+1}(t) \\
&= \eta\cdot\Big(\varphi-\frac{1}{2}\Big)\cdot y^{p+1}(t)\cdot\Big\{-1-\frac{1}{\eta}\cdot\Big(\varphi-\frac{1}{2}\Big)^{p-1}\Delta\varphi\Big\} \\
&< \eta\cdot\Big(\varphi-\frac{1}{2}\Big)\cdot y^{p+1}(t)\cdot\Big\{-1+\frac{(\varphi-\frac{1}{2})^{p-1}}{\varphi^{p-1}}\Big\} \\
&\le 0 \qquad \text{in } \Omega_0\times(0,\infty),
\end{aligned}$$

whence on the basis of (3.20) and (3.21) we may once more employ the comparison principle from [17, Sect. 3.1] to infer that

$$u(x,t)\ge\underline{u}(x,t) \qquad \text{for all } x\in\Omega_0 \text{ and } t>0,$$

and that thus

$$\begin{aligned}
\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} &\ge \|\underline{u}(\cdot,t)\|_{L^\infty(\Omega_0)} \\
&= y(t)\cdot\Big\|\varphi-\frac{1}{2}\Big\|_{L^\infty(\Omega_0)} \\
&= \frac{1}{2}y(t) \qquad \text{for all } t>0
\end{aligned}$$

thanks to (3.18) and (3.15). Since (3.19) implies that

$$y(t)\ge(2p\eta_M t)^{-\frac{1}{p}} \qquad \text{for all } t\ge t_M := \frac{1}{p\eta_M y_0^p},$$

and since (3.14) says that

$$\frac{1}{2}\cdot(2p\eta_M)^{-\frac{1}{p}} = M,$$

this means that

$$t^{\frac{1}{p}}\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \ge M \qquad \text{for all } t \ge t_M$$

and thereby establishes (1.10), for $M > 0$ was arbitrary. $\square$

Now in the presence of arbitrarily small principal eigenvalues within $\mathcal{C}(u_0)$, the validity of (1.9) can be verified by simply using appropriate eigenfunctions of $-\Delta$:

PROOF of Corollary 1.5. Given $\varepsilon > 0$, due to (1.12) we can find $\Omega \in \mathcal{C}(u_0)$ such that $\lambda_1(\Omega) \le \frac{\varepsilon}{2}$. Then taking $\Theta \in W_0^{1,2}(\Omega)$ such that $0 < \lambda_1(\Omega)\int_\Omega \Theta^2 = \int_\Omega |\nabla\Theta|^2$, by definition of $W_0^{1,2}(\Omega)$ we can pick $(\varphi_j)_{j\in\mathbb{N}} \subset C_0^\infty(\Omega)\setminus\{0\}$ such that $\varphi_j \to \Theta$ in $W_0^{1,2}(\Omega)$ as $j \to \infty$ and hence

$$\frac{\int_\Omega |\nabla\varphi_j|^2}{\int_\Omega \varphi_j^2} \to \frac{\int_\Omega |\nabla\Theta|^2}{\int_\Omega \Theta^2} = \lambda_1(\Omega)$$

as $j \to \infty$. We can therefore fix $j_0 \in \mathbb{N}$ such that

$$\frac{\int_\Omega |\nabla\varphi_{j_0}|^2}{\int_\Omega \varphi_{j_0}^2} \le \varepsilon, \tag{3.22}$$

and use that then $\overline{\{\varphi_{j_0} > 0\}}$ is a compact subset of Ω to construct a smoothly bounded subdomain $\Omega_0 \subset \Omega$ such that $\overline{\{\varphi_{j_0} > 0\}} \subset \Omega_0$. Since φ_{j_0} clearly belongs to $W_0^{1,2}(\Omega_0)$, relying on the variational characterization of $\lambda_1(\Omega_0)$ we thus infer from (3.22) that $\lambda_1(\Omega_0) \le \frac{\int_{\Omega_0}|\nabla\varphi_{j_0}|^2}{\int_{\Omega_0}\varphi_{j_0}^2} = \frac{\int_{\Omega}|\nabla\varphi_{j_0}|^2}{\int_{\Omega}\varphi_{j_0}^2} \le \varepsilon$, and since Ω_0 has smooth boundary, standard elliptic regularity theory applies so as to ensure the existence of a function $\Theta_0 \in C^2(\overline{\Omega}_0)$ fulfilling $-\Delta\Theta_0(x) = \lambda_1(\Omega_0)\Theta_0(x)$ for all $x \in \Omega_0$, $\Theta_0(x) = 0$ for all $x \in \partial\Omega_0$ and $0 \le \Theta_0(x) \le 1 = \max_{y\in\overline{\Omega}_0}\Theta_0(y)$ for all $x \in \Omega_0$. Therefore, the nonnegative function $\varphi \in C_0^0(\overline{\Omega}) \cap C^2(\{\varphi > 0\})$ defined by

$$\varphi(x) := \begin{cases} \Theta(x), & x \in \Omega_0, \\ 0 & x \in \overline{\Omega}\setminus\Omega_0, \end{cases}$$

satisfies

$$-\varphi^{p-1}(x)\Delta\varphi(x) = -\,\Theta_0^{p-1}(x)\Delta\Theta_0(x) = \lambda_1(\Omega_0)\Theta_0^p(x) \le \lambda_1(\Omega_0) \le \varepsilon \qquad \text{for all } x \in \{\varphi > 0\},$$

so that since $\varepsilon > 0$ was arbitrary, we conclude that (1.9) holds, and that hence (1.12) implies (1.10) as a consequence of Proposition 1.4.

Finally, assuming (1.13) to be satisfied, for arbitrary $\eta > 0$ we can take $R > 0$ large enough such that $\frac{\lambda_1(B_1(0))}{R^2} < \eta$, and then use (1.13) to choose $\Omega \in \mathcal{C}(u_0)$ fulfilling $\Omega \supset B_R(x_0)$ for some $x_0 \in \mathbb{R}^n$. Then, by evident monotonicity and scaling properties of $\lambda_1(\cdot)$, it follows that

$$\lambda_1(\Omega) \le \lambda_1(B_R(x_0)) = \frac{\lambda_1(B_1(0))}{R^2} < \eta,$$

and that therefore (1.12) and hence the claimed conclusion holds. $\square$

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