Attractiveness of constant states in logistic-type Keller-Segel systems involving subquadratic growth restrictions

Michael Winkler[*]
Institut für Mathematik, Universität Paderborn,
Warburger Str. 100, 33098 Paderborn, Germany

Abstract

The chemotaxis-growth system

$$\begin{cases} u_t = D\Delta u - \chi\nabla\cdot(u\nabla v) + \rho u - \mu u^\alpha, \\ v_t = d\Delta v - \kappa v + \lambda u, \end{cases} \qquad (\star)$$

is considered under homogeneous Neumann boundary conditions in smoothly bounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 1$. For any choice of $\alpha > 1$, the literature provides a comprehensive result on global existence for widely arbitrary initial data within a suitably generalized solution concept, but the regularity properties of such solutions may be rather poor, as indicated by precedent results on the occurrence of finite-time blow-up in corresponding parabolic-elliptic simplifications.

Based on the analysis of a certain eventual Lyapunov-type feature of $(\star)$, the present work shows that whenever

$$\alpha \geq 2 - \frac{2}{n},$$

under an appropriate smallness assumption on χ any such solution at least asymptotically exhibits relaxation by approaching the nontrivial spatially homogeneous steady state $\Big((\frac{\rho}{\mu})^{\frac{1}{\alpha-1}}, \frac{\lambda}{\kappa}(\frac{\rho}{\mu})^{\frac{1}{\alpha-1}} \Big)$ in the large time limit.

Key words: chemotaxis; logistic source; stabilization
MSC (2010): 35B40 (primary); 92C17, 35Q92, 35K55 (secondary)

[*]michael.winkler@math.uni-paderborn.de

1 Introduction

The reduction of regularity belongs to the most intensely studied effects of chemotactic cross-diffusion. In the context of classical Keller-Segel systems, this becomes manifest not only in a comprehensive literature focusing on the detection of blow-up phenomena ([13], [12], [24], [4], [41]), but also in several findings concerned with more detailed analysis of singularity formation ([12], [33]), or extension of solutions beyond blow-up ([3], [23], [54]).

The analysis of corresponding features apparently becomes significantly more challenging in systems which couple chemotactic interaction to further mechanisms. Among accordingly refined variants which attempt to provide more realistic descriptions in situations more complex than those addressed by minimal Keller-Segel systems, of particular importance seem models which account for proliferation and competition-induced death, known as relevant in a noticeably large number of biological contexts such as bacterial pattern formation, self-organization during embryonic development, and tumor invasion. also to self-organization during embryonic development ([48], [29], [6], [34], [37]).

Correspondingly, logistic Keller-Segel systems of the form

$$\begin{cases} u_t = D\Delta u - \chi\nabla\cdot(u\nabla v) + \rho u - \mu u^\alpha, \\ v_t = d\Delta v - \kappa v + \lambda u, \end{cases} \tag{1.1}$$

as well as some close relatives have received considerable interest in the past years, and elaborate methods have been developed to identify conditions on the system parameters therein which ensure tht the joint dissipative action of diffusion and suitably strong degradation rules out the occurrence of blow-up phenomena. In the best understood case $\alpha = 2$ of quadratic absorption, for instance, associated no-flux initial-boundary value problems in n-dimensional bounded domains Ω are known to admit global bounded classical solutions for all suitably regular initial data if either $n \le 2$ ([27], [26]), or $n \ge 3$ and $\mu > \mu_0(\mu_0(D, d, \chi, \rho, \kappa, \lambda, \Omega)$ ([39]); for appropriately large μ, even some results on global asymptotic stability of the corresponding spatialy homogenous equilibria $(\frac{\rho}{\mu}, \frac{\lambda\rho}{\kappa\mu})$ are available ([5], [42]). Further findings in these directions, inter alia focusing at refinements with respect to parameter setting, or generalizations to slightly modified systems, or also qualitative facets such as wave-like behavior, can be found in [8], [14], [19], [18], [25], [49], [50], [55], [30], [31] and [32], for instance.

In the presence of weaker absorption, however, the knowledge in this regard seems significantly sparser: While in two-dimensional domains already some subquadratic death terms involving certain logarithmic corrections have been shown to rule out blow-up ([51]), in the case $n \ge 3$ it yet appears to be unknown whether or not explosions may occur for small values of μ when $\alpha = 2$; for such parameter choices, only global weak solutions have been shown to exist, and results concerned with their qualitative behavior seem limited to statements on eventual smoothness for small ρ, and on asymptotic decay for $\rho \le 0$, in the special case $n = 3$ ([17]; cf. also [36]).

Possible dampening effects of yet weaker degradation have been understood to a rudimentary extent only up to now. Indeed, the knowledge in this regard so far reduces to statements on mere global existence in suitably generalized solution frameworks. In [35], certain global solutions have been constructed under the assumptions that $n \ge 2$ and $\alpha > 2 - \frac{1}{n}$, and in [46] and [53] a relaxation of these hypotheses could be achieved so as to ensure solvability even for any $\alpha > \min\{2 - \frac{2}{n}, \frac{2n+4}{n+4}\}$ when $n \ge 2$. Only recently, in the context of a yet further generalized solution concept it has been found that actually any choice of $\alpha > 1$ is sufficient to ensure global solvability for widely arbitrary intitial data ([47]).

Asymptotics in weakly dampened chemotaxis-growth systems. Analysis beyond blow-up. Beyond quite poor basic regularity features, however, no qualitative information on the behavior of such solutions seems available in such weakly dampened cases. That, in fact, the correspondingly generated dynamics might be considerably complex is indicated by some noticeable caveats contained in the literature. Besides providing numerical evidence that shows remarkably colorful facets in logistic Keller-Segel systems, up to even chaotic behavior ([28]), previous studies have revealed quite drastic phenomena related to the spontaneous emergence of large population densities, possibly at intermediate timescales, partially even in frameworks of bounded solutions ([15], [16], [43], [44], [38]). Yet more drastically, in some parabolic-elliptic simplifications of (1.1) even finite-time blow-up has been detected, e.g. under the hypotheses that $n \in \{3,4\}$ and $\alpha < \frac{7}{6}$ ([45]), or $n \geq 5$ and $\alpha < \frac{3}{2} + \frac{1}{2(n-1)}$ ([40]); recent progress indicates that similar statements are actually available under the mere assumptions that $n \geq 3$, $\alpha = 2$ and $\mu > 0$ is sufficiently small ([10]; cf. also [9] and [21] for further blow-up results in this direction).

Despite these complicating circumstances, the present work attempts to develop a basic qualitative theory for generalized solutions to (1.1) within reasonably large parameter ranges. By addressing arbitrarily large initial data, we especially intend to include situations in which the above precedents suggest to expect the occurrence of finite-time explosions, and in which thus a large time analysis of global solutions amounts to describing *life beyond blow-up*, as having formed the objective, meanwhile quite well-understood, of seminal studies concerned with simpler scalar parabolic problems ([2], [11], [22]).

To make this more precise, let us henceforth consider the full initial-boundary value problem

$$\begin{cases} u_t = D\Delta u - \chi\nabla\cdot(u\nabla v) + \rho u - \mu u^\alpha, & x \in \Omega,\ t > 0, \\ v_t = d\Delta v - \kappa v + \lambda u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

in a smoothly bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with positive parameters $D, d, \chi, \rho, \mu, \kappa$ and λ, with $\alpha > 1$, and with initial data complying with the hypotheses that

$$\begin{cases} u_0 \in C^0(\overline{\Omega}) \text{ such that } u_0 > 0 \text{ in } \overline{\Omega} \quad \text{and} \\ v_0 \in W^{1,\infty}(\Omega) \text{ such that } v_0 \geq 0 \text{ in } \Omega. \end{cases} \tag{1.3}$$

Within this general setting, it follows from the results in [47] that (1.2) indeed admits globally defined solutions in an appropriately generalized sense, and that these can be approximated by solutions to suitably regularized variants of (1.2):

Proposition 1.1 *Let $\alpha > 1$, $D > 0, d > 0, \chi > 0$, $\rho > 0$, $\mu > 0$, $\kappa > 0$ and $\lambda > 0$, and assume (1.3). Then there exists nonnegative functions*

$$\begin{cases} u \in L^\alpha_{loc}(\overline{\Omega} \times [0,\infty)) \quad \textit{and} \\ v \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)) \end{cases} \tag{1.4}$$

such that (u,v) forms a global generalized solution of (1.2) in the sense of Definition 5.5 given in the Appendix, and that (u,v) can be approximated by solutions to the regularized problems (2.1) below in the following sense: For each $\varepsilon \in (0,1)$, (2.1) admits a global classical solution $(u_\varepsilon, v_\varepsilon) \in (C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)))^2$, and there exist $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ and a null set $N \subset (0,\infty)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$, and that

$$u_\varepsilon \to u \quad \text{in } L^1_{loc}(\overline{\Omega} \times [0,\infty)) \text{ and a.e. in } \Omega \times (0,\infty), \tag{1.5}$$

$$v_\varepsilon \to v \quad \text{in } L^1_{loc}(\overline{\Omega} \times [0,\infty)) \text{ and a.e. in } \Omega \times (0,\infty) \qquad \text{and} \tag{1.6}$$

$$v_\varepsilon(\cdot,t) \to v(\cdot,t) \quad \text{in } L^1(\Omega) \text{ and a.e. in } \Omega \qquad \text{for all } t \in (0,\infty) \setminus N \tag{1.7}$$

as $\varepsilon = \varepsilon_j \searrow 0$.

In order to describe the large time behavior of these solutions, we shall examine how far expresions of the form

$$\mathcal{F}(t) := \int_\Omega \Big(u(\cdot,t) - u_\star - u_\star \ln \frac{u(\cdot,t)}{u_\star}\Big) + \frac{b}{2} \int_\Omega \Big(v(\cdot,t) - \overline{v(\cdot,t)}\Big)^2, \qquad t > 0, \tag{1.8}$$

enjoy certain Lyapunov-type properties for (1.2) if the free parameter $b > 0$ therein is chosen appropriately, and if the number $u_\star$ denotes the first component of the associated nontrivial spatially homogeneous equilibrium $(u_\star, v_\star)$ of (1.2) given by

$$u_\star := \Big(\frac{\rho}{\mu}\Big)^{\frac{1}{\alpha-1}} \qquad \text{and} \qquad v_\star := \frac{\lambda u_\star}{\kappa}. \tag{1.9}$$

Here and throughout the sequel, we adapt standard notation by abbreviating

$$\overline{\psi} := \frac{1}{|\Omega|} \int_\Omega \psi \qquad \text{for } \psi \in L^1(\Omega). \tag{1.10}$$

In spite of evident challenges linked to the poor information on regularity and the topological setting in the approximation statements in (1.5)-(1.7), a suitably designed analysis of $\mathcal{F}$, as well as of a natural counterpart $\mathcal{F}_\varepsilon$ thereof at the level of approximate solutions, will reveal that whenever $\alpha \ge 2 - \frac{2}{n}$ and χ is appropriately small, $\mathcal{F}$ plays the role of an *eventual energy functional* in the sense that for each individual trajectory, $\mathcal{F}$ becomes nonincreasing after an adequate waiting time. On the basis of this observation, we shall see that within this framework, any such solution approaches $(u_\star, v_\star)$ in the sense substantiated in the following main result of this work:

Theorem 1.2 *Let $n \ge 1$, $D > 0, d > 0$, $\rho > 0$, $\mu > 0$ and $\lambda > 0$, and suppose that $\alpha > 1$ is such that*

$$\alpha \ge 2 - \frac{2}{n}. \tag{1.11}$$

Then given any bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, one can find $C(\Omega) > 0$ with the property that whenever $\chi > 0$ satisfies

$$\chi^2 \le C(\Omega) \cdot \frac{d^2 D}{\lambda^2} \cdot \rho^{-\frac{3-\alpha}{\alpha-1}} \mu^{\frac{2}{\alpha-1}}, \tag{1.12}$$

for arbitrary initial data fulfilling (1.3) the problem (1.2) possesses a global generalized solution (u,v), *in the sense of Definition 5.5 below, such that with some* $T>0$ *and some null set* $N\subset(T,\infty)$ *we have*

$$u(\cdot,t)\to u_\star \quad \text{in } L^1(\Omega) \qquad \text{as } (T,\infty)\setminus N\ni t\to\infty \tag{1.13}$$

and

$$v(\cdot,t)\to v_\star \quad \text{in } L^2(\Omega) \qquad \text{as } (T,\infty)\setminus N\ni t\to\infty, \tag{1.14}$$

where $u_\star>0$ *and* $v_\star>0$ *are given by (1.9).*

Remark. i) We underline that the key condition $\alpha\ge 2-\frac{2}{n}$ in Theorem 1.2 firstly allows for actually any $\alpha>1$ when $n=2$, but moreover includes some choices of α for which the literature indicates the possibility of finite-time blow-up: In fact, when $n=5$ the assumption $\alpha\ge 2-\frac{2}{n}=\frac{8}{5}$ can simultaneously be fulfilled with the assumption $\alpha<\frac{3}{2}+\frac{1}{2(n-1)}=\frac{13}{8}$ from [40], and corresponding consistency with the hypotheses from the yet unpublished work [10] can be achieved even for any $n\ge 3$.

ii) By essentially asserting relaxation into constant equilibria, Theorem 1.2 reveals that with respect to solution behavior after a possible singularity formation, (1.1) considerably differs from corresponding proliferation-free Keller-Segel systems in which, at least in two-dimensional parabolic-elliptic cases, extensions beyond blow-up seem to reflect eternal persistence of Dirac-type singularities, as typically emerging during explosion processes at some finite time ([3], [23]).

2 Preliminaries

2.1 Approximation of generalized solutions

As already announced, given parameters $\alpha>1$, $D>0, d>0, \chi>0$, $\rho>0$, $\mu>0$, $\kappa>0$ and $\lambda>0$, as well as initial data fulfilling (1.3), we follow the regularization procedure in [47] and hence consider the approximate problems

$$\begin{cases} u_{\varepsilon t}=D\Delta u_\varepsilon-\chi\nabla\cdot(u_\varepsilon\nabla v_\varepsilon)+\rho u_\varepsilon-\mu u_\varepsilon^\alpha, & x\in\Omega,\ t>0,\\ v_{\varepsilon t}=d\Delta v_\varepsilon-\kappa v_\varepsilon+\lambda\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}, & x\in\Omega,\ t>0,\\ \frac{\partial u_\varepsilon}{\partial\nu}=\frac{\partial v_\varepsilon}{\partial\nu}=0, & x\in\partial\Omega,\ t>0,\\ u_\varepsilon(x,0)=u_0(x), \quad v_\varepsilon(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{2.1}$$

for $\varepsilon\in(0,1)$. Each of these indeed admits a globally defined classical solution $(u_\varepsilon,v_\varepsilon)\in(C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty)))^2$ such that, according to (1.3) and the strong maximum principle, $u_\varepsilon>0$ and $v_\varepsilon>0$ in $\overline{\Omega}\times(0,\infty)$ (cf. also [47, Lemma 2.1]), and, in fact, the arguments detailed in [47, Lemma 7.1, Lemma 8.2] for the prototypical choices $D=d=\chi=\kappa=\lambda=1$ show that in the limit of vanishing ε, these solutions approach a solution of (1.2) in the sense documented in Proposition 1.1.

2.2 Basic bounds for u_ε. Absorbing sets in L^1

Let us first apply an essentially straightforward argument to the first equation in (2.1) to achieve the following basic regularity information.

Lemma 2.1 *Let $\alpha > 1$, $D > 0, d > 0, \chi > 0$, $\rho > 0$, $\mu > 0$, $\kappa > 0$ and $\lambda > 0$, and assume (1.3). Then for all $t \ge 0$ and $\varepsilon \in (0,1)$,*

$$\int_\Omega u_\varepsilon(\cdot,t) \le \left\{ \frac{2^{\alpha-1}\mu}{(3|\Omega|)^{\alpha-1}\rho} + \left(\left\{ \int_\Omega u_0 \right\}^{1-\alpha} - \frac{2^{\alpha-1}\mu}{(3|\Omega|)^{\alpha-1}\rho} \right) \cdot e^{-(\alpha-1)\rho t} \right\}^{-\frac{1}{\alpha-1}} \tag{2.2}$$

and

$$\int_\Omega u_\varepsilon(\cdot,t) \le m := \max\left\{ \int_\Omega u_0 \, , \, \frac{3}{2} \cdot \left(\frac{\rho}{\mu}\right)^{\frac{1}{\alpha-1}} |\Omega| \right\} \tag{2.3}$$

as well as

$$\int_t^{t+1} \int_\Omega u_\varepsilon^\alpha \le \frac{(\rho+1)m}{(1-(\frac{2}{3})^{\alpha-1}) \cdot \mu}. \tag{2.4}$$

Proof. We abbreviate $y_0 := \int_\Omega u_0$ and $\theta := (\frac{2}{3})^{\alpha-1}\mu \in (0,\mu)$, and let $y \in C^1([0,\infty))$ denote the solution of

$$\begin{cases} y'(t) = \rho y(t) - \theta|\Omega|^{1-\alpha} y^\alpha(t), & t > 0, \\ y(0) = y_0, \end{cases} \tag{2.5}$$

that is, we let

$$y(t) := \left\{ \frac{\theta|\Omega|^{1-\alpha}}{\rho} + \left(y_0^{1-\alpha} - \frac{\theta|\Omega|^{1-\alpha}}{\rho} \right) \cdot e^{-(\alpha-1)\rho t} \right\}^{-\frac{1}{\alpha-1}}, \qquad t \ge 0.$$

Since in view of the Hölder inequality the first equation in (2.1) ensures that for each $\varepsilon \in (0,1)$,

$$\frac{d}{dt}\int_\Omega u_\varepsilon = \rho \int_\Omega u_\varepsilon - \mu \int_\Omega u_\varepsilon^\alpha \le \rho \int_\Omega u_\varepsilon - \theta|\Omega|^{1-\alpha} \cdot \left\{ \int_\Omega u_\varepsilon \right\}^\alpha - (\mu-\theta)\int_\Omega u_\varepsilon^\alpha \qquad \text{for all } t > 0,$$

and that thus $y_\varepsilon(t) := \int_\Omega u_\varepsilon(\cdot,t)$, $t \ge 0$, and $h_\varepsilon(t) := (\mu-\theta)\int_\Omega u_\varepsilon^\alpha(\cdot,t)$, $t > 0$, satisfy

$$y_\varepsilon'(t) + h_\varepsilon(t) \le \rho y_\varepsilon(t) - \theta|\Omega|^{1-\alpha} y_\varepsilon^\alpha(t) \qquad \text{for all } t > 0, \tag{2.6}$$

by nonnegativity of h_ε a comparison argument shows that

$$y_\varepsilon(t) \le y(t) \le \max\left\{ y_0 \, , \, \left(\frac{\rho}{\theta|\Omega|^{1-\alpha}} \right)^{\frac{1}{\alpha-1}} \right\} = m \qquad \text{for all } t > 0$$

and thereby establishes both (2.2) and (2.3). An integration in (2.6) thereupon warrants that

$$\int_t^{t+1} h_\varepsilon(s)ds \le y_\varepsilon(t) + \rho \int_t^{t+1} y_\varepsilon(s)ds \le m + \rho m \qquad \text{for all } t > 0,$$

which by definition of h_ε entails (2.4). $\Box$

In particular, this entails an absorption feature of suitably large balls in L^1, in the following flavor.

Lemma 2.2 *Let $\alpha > 1$, $D > 0, d > 0, \chi > 0$, $\rho > 0$, $\mu > 0$, $\kappa > 0$ and $\lambda > 0$, and let $u_\star > 0$ be as in (1.9). Then given any (u_0, v_0) fulfilling (1.3) one can find $T_\star = T_\star(u_0, v_0) > 0$ with the property that*

$$\int_\Omega u_\varepsilon(\cdot, t) \le 2u_\star|\Omega| \qquad \textit{for all } t > T_\star \textit{ and } \varepsilon \in (0,1). \tag{2.7}$$

Proof. We only need to employ (2.2) and observe that therein,

$$\left\{ \frac{2^{\alpha-1}\mu}{(3|\Omega|)^{\alpha-1}\rho} + \left(\left\{ \int_\Omega u_0 \right\}^{1-\alpha} - \frac{2^{\alpha-1}\mu}{(3|\Omega|)^{\alpha-1}\rho} \right) \cdot e^{-(\alpha-1)\rho t} \right\}^{-\frac{1}{\alpha-1}} \to \left\{ \frac{2^{\alpha-1}\mu}{(3|\Omega|)^{\alpha-1}\rho} \right\}^{-\frac{1}{\alpha-1}} = \frac{3}{2} u_\star |\Omega|$$

as $t \to \infty$. $\square$

3 Construction of eventual energy functionals at the approximate level

The purpose of this section is to make sure that for appropriately small $\chi > 0$, at the stage of approximate solutions the functional in (1.8) enjoys a genuine Lyapunov property after some relaxation time possibly depending on the initial data. Here in comparison to the case $\alpha = 2$ addressed in several related precedents in the literature ([1], [7], [52]), in the presence especially of subquadratic degradation some further technical efforts seem necessary so as to facilitate an efficient quantitative analysis of how far the steady state $u_\star$ from (1.9) inherits attractiveness features from corresponding taxis-free frameworks.

3.1 The time evolution of $\int_\Omega (u_\varepsilon - u_\star - u_\star \ln \frac{u_\varepsilon}{u_\star})$

Our first objective in this regard is the function appearing in the first integral from (1.8). Some useful of its properties are summarized in the following lemma.

Lemma 3.1 *Let $\alpha > 1$, $D > 0, d > 0, \chi > 0$, $\rho > 0$, $\mu > 0$, $\kappa > 0$ and $\lambda > 0$, and with $u_\star$ taken from (1.9), let*

$$H(\xi) := \xi - u_\star - u_\star \ln \frac{\xi}{u_\star}, \qquad \xi > 0. \tag{3.1}$$

Then $H(\xi) \ge 0$ for all $\xi > 0$, and there exists $C > 0$ such that with $q := \min\{\alpha, 2\}$ we have

$$\int_\Omega H^q(\psi) \quad \le \quad \int_\Omega \psi^\alpha + C \int_\Omega \frac{|\nabla\psi|^2}{\psi^2} + C \cdot \left\{ \int_\Omega |\ln\psi| \right\}^2 + C \qquad \textit{for all } \psi \in L^\alpha(\Omega; (0,\infty))$$
$$\textit{such that } \ln\psi \in W^{1,2}(\Omega). \tag{3.2}$$

Proof. Noting that nonnegativity of H can be seen by elementary analysis, to verify (3.2) we use that according to a Poincaré inequality we can fix $c_1 > 0$ such that

$$\int_\Omega \zeta^2 \le c_1 \int_\Omega |\nabla\zeta|^2 + c_1 \cdot \left\{ \int_\Omega |\zeta| \right\}^2 \qquad \text{for all } \zeta \in W^{1,2}(\Omega).$$

Therefore, given a positive $\psi \in L^\alpha(\Omega)$ such that $\ln\psi \in W^{1,2}(\Omega)$, observing that

$$H(\xi) \le -u_\star \ln\xi + c_2 \qquad \text{for all } \xi \in (0,2u_\star)$$

with $c_2 := u_\star + u_\star|\ln u_\star| > 0$, we can set $\zeta := \ln\psi$ to see that thanks to Young's inequality and the fact that $q \le 2$,

$$\begin{aligned}
\int_{\{\psi<2u_\star\}} H^q(\psi) &\le \int_{\{\psi<2u_\star\}} H^2(\psi) + |\Omega| \\
&\le 2u_\star^2 \int_{\{\psi<2u_\star\}} |\ln\psi|^2 + 2c_2^2|\Omega| + |\Omega| \\
&\le 2c_1u_\star^2 \int_\Omega |\nabla\ln\psi|^2 + 2c_1u_\star^2 \cdot \bigg\{\int_\Omega |\ln\psi|\bigg\}^2 + 2c_2^2|\Omega| + |\Omega|. \qquad (3.3)
\end{aligned}$$

In the corresponding complementary region, however, we may simply use that $\ln\frac{\xi}{u_\star} \ge 0$ for $\xi \ge 2u_\star$, and that hence

$$H(\xi) \le \xi \qquad \text{for all } \xi \ge 2u_\star,$$

to find that for any such ψ, again by Young's inequality, and by the restriction $q \le \alpha$,

$$\int_{\{\psi\ge 2u_\star\}} H^q(\psi) \le \int_{\{\psi\ge 2u_\star\}} H^\alpha(\psi) + |\Omega| \le \int_{\{\psi\ge 2u_\star\}} \psi^\alpha + |\Omega|.$$

Therefore, (3.2) results upon choosing $C := \max\{2c_1u_\star^2\,,\, 2c_2^2|\Omega| + 2|\Omega|\}$, for instance. $\Box$

Let us already here add a second preparation in this regard, albeit only used in our final proof of Theorem 1.2 in Section 5.3, which asserts that $\int_\Omega H(\psi)$ controls differences to $u_\star$ actually with respect to the norm in $L^1(\Omega)$:

Lemma 3.2 *Let $\alpha > 1$, $D > 0, d > 0, \chi > 0$, $\rho > 0$, $\mu > 0$, $\kappa > 0$ and $\lambda > 0$, let $u_\star$ and H be as in (1.9) and (3.1), and let $\psi : \Omega \to (0,\infty)$ be measurable. Then*

$$\int_\Omega |\psi - u_\star| \le \frac{1}{1-\ln 2}\int_\Omega H(\psi) + \sqrt{8u_\star|\Omega|}\cdot\bigg\{\int_\Omega H(\psi)\bigg\}^{\frac12}. \qquad (3.4)$$

Proof. Using that $H(u_\star) = H'(u_\star) = 0$ and that $H''(\xi) = \frac{u_\star}{\xi^2} \ge \frac{1}{4u_\star}$ for all $\xi \in (0,2u_\star)$, we first observe that

$$H(\xi) \ge \frac12 \cdot \bigg\{\inf_{\sigma\in(0,2u_\star)} H''(\sigma)\bigg\}\cdot(\xi-u_\star)^2 \ge \frac{1}{8u_\star}\cdot(\xi-u_\star)^2 \qquad \text{for all } \xi \in (0,2u_\star),$$

so that given any measurable $\psi : \Omega \to (0,\infty)$ we can estimate

$$\begin{aligned}
\int_{\{\psi<2u_\star\}} |\psi - u_\star| &\le \int_{\{\psi<2u_\star\}} \sqrt{8u_\star H(\psi)} \\
&\le \sqrt{8u_\star|\Omega|}\cdot\bigg\{\int_{\{\psi<2u_\star\}} H(\psi)\bigg\}^{\frac12} \qquad (3.5)
\end{aligned}$$

by means of the Cauchy-Schwarz inequality. Apart from that, since writing $c_1 := 1 - \ln 2$ we see that for

$$\varphi(\xi) := H(\xi) - c_1 \cdot (\xi - u_\star), \qquad \xi \ge 2u_\star,$$

we have

$$\varphi(2u_\star) = H(2u_\star) - c_1 u_\star = (1 - c_1)u_\star - u_\star \ln 2 = 0$$

as well as

$$\varphi'(\xi) = H'(\xi) - c_1 = 1 - \frac{u_\star}{\xi} - c_1 \ge 1 - \frac{1}{2} - c_1 = \ln 2 - \frac{1}{2} > 0 \qquad \text{for all } \xi \ge 2u_\star,$$

so that $\varphi(\xi) \ge 0$ for all $\xi \ge 2u_\star$. Accordingly, for any ψ as given above we have

$$(1 - \ln 2) \int_{\{\psi \ge 2u_\star\}} (\psi - u_\star) \le \int_{\{\psi \ge 2u_\star\}} H(\psi),$$

which together with (3.5) readily yields (3.4). $\square$

The role of H in our asymptotic analysis of (1.2) is foreshadowed by the following straightforward observation.

Lemma 3.3 *Let $\alpha > 1$, $D > 0, d > 0, \chi > 0$, $\rho > 0$, $\mu > 0$, $\kappa > 0$ and $\lambda > 0$, and let $u_\star$ and H be as in (1.9) and (3.1). Then for all $t > 0$ and $\varepsilon \in (0,1)$,*

$$\frac{d}{dt} \int_\Omega H(u_\varepsilon) + Du_\star \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} + \mu \int_\Omega (u_\varepsilon^{\alpha-1} - u_\star^{\alpha-1}) \cdot (u_\varepsilon - u_\star) = \chi u_\star \int_\Omega \frac{\nabla u_\varepsilon}{u_\varepsilon} \cdot \nabla v_\varepsilon. \tag{3.6}$$

PROOF. Using that

$$H'(\xi) = 1 - \frac{u_\star}{\xi} \quad \text{and} \quad H''(\xi) = \frac{u_\star}{\xi^2} \qquad \text{for all } \xi > 0,$$

thanks to the positivity of u_ε we may use the first equation in (2.1) to compute

$$\begin{aligned} \frac{d}{dt} \int_\Omega H(u_\varepsilon) &= -\int_\Omega H''(u_\varepsilon)\nabla u_\varepsilon \cdot (D\nabla u_\varepsilon - \chi u_\varepsilon \nabla v_\varepsilon) + \int_\Omega H'(u_\varepsilon) \cdot (\rho u_\varepsilon - \mu u_\varepsilon^\alpha) \\ &= -Du_\star \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} + \chi u_\star \int_\Omega \frac{\nabla u_\varepsilon}{u_\varepsilon} \cdot \nabla v_\varepsilon + \int_\Omega \Big(1 - \frac{u_\star}{u_\varepsilon}\Big) \cdot (\rho u_\varepsilon - \mu u_\varepsilon^\alpha) \end{aligned} \tag{3.7}$$

for $t > 0$ and $\varepsilon \in (0,1)$. Here by (1.9) we may replace $\frac{\rho}{\mu} = u_\star^{\alpha-1}$ in verifying that

$$\begin{aligned} \int_\Omega \Big(1 - \frac{u_\star}{u_\varepsilon}\Big) \cdot (\rho u_\varepsilon - \mu u_\varepsilon^\alpha) &= \mu \int_\Omega (u_\varepsilon - u_\star) \cdot \Big(\frac{\rho}{\mu} - u_\varepsilon^{\alpha-1}\Big) \\ &= -\mu \int_\Omega (u_\varepsilon - u_\star) \cdot (u_\varepsilon^{\alpha-1} - u_\star^{\alpha-1}) \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1), \end{aligned}$$

so that (3.7) indeed is equivalent to (3.6). $\square$

3.2 Estimating $\int_\Omega (u_\varepsilon^{\alpha-1} - u_\star^{\alpha-1}) \cdot (u_\varepsilon - u_\star)$ from below

Now in order to appropriately estimate the last summand on the left of (3.6) from below, we first rely on elementary calculus to verify the following.

Lemma 3.4 *Let $\alpha > 1$. Then*

$$(\xi^{\alpha-1} - 1) \cdot (\xi - 1) \ge K \cdot (\xi - 1)^2 \qquad \text{for all } \xi \in [0,2], \tag{3.8}$$

where

$$K := \begin{cases} (\alpha-1)\cdot 2^{\alpha-2} & \text{if } \alpha \in (1,2), \\ 1 & \text{if } \alpha \ge 2. \end{cases} \tag{3.9}$$

Proof. If $\alpha < 2$, then $\varphi_1(\xi) := 1 - \xi^{\alpha-1} - (\alpha-1)\cdot(1-\xi)$, $\xi \in [0,1]$, satisfies $\varphi_1(1) = 0$ and $\varphi_1'(\xi) = -(\alpha-1)\xi^{\alpha-2} + \alpha - 1 \le 0$ for all $\xi \in (0,1)$, so that $\varphi_1 \ge 0$ on $[0,1]$ and hence

$$(\xi^{\alpha-1} - 1)\cdot(\xi-1) = \Big(\varphi_1(\xi) + (\alpha-1)\cdot(1-\xi)\Big)\cdot(1-\xi) \ge (\alpha-1)\cdot(1-\xi)^2 \qquad \text{for all } \xi \in [0,1]. \tag{3.10}$$

Moreover, for such α we see that $\varphi_2(\xi) := \xi^{\alpha-1} - 1 - (\alpha-1)\cdot 2^{\alpha-2}(\xi-1)$, $\xi \in [1,2]$, has the properties that $\varphi_2(1) = 0$ and $\varphi_2'(\xi) = (\alpha-1)\cdot\xi^{\alpha-2} - (\alpha-1)\cdot 2^{\alpha-2} \ge 0$ for all $\xi \in (1,2)$, whence φ_2 is nonnegative on $[1,2]$. Therefore,

$$\begin{aligned} (\xi^{\alpha-1} - 1)\cdot(\xi-1) &= \Big(\varphi_2(\xi) + (\alpha-1)\cdot 2^{\alpha-2}(\xi-1)\Big)\cdot(\xi-1) \\ &\ge (\alpha-1)\cdot 2^{\alpha-2}(\xi-1)^2 \qquad \text{for all } \xi \in [1,2], \end{aligned}$$

which together with (3.10) proves (3.8) in this case, because $2^{\alpha-2} \le 1$.

If, conversely, $\alpha \ge 2$, then $\xi^{\alpha-1} \le \xi$ for $\xi \in [0,1]$ and thus

$$(\xi^{\alpha-1} - 1)\cdot(\xi-1) = (1-\xi^{\alpha-1})\cdot(1-\xi) \ge (1-\xi)^2 \qquad \text{for all } \xi \in [0,1], \tag{3.11}$$

whereas letting $\varphi_3(\xi) := \xi^{\alpha-1} - 1 - (\alpha-1)\cdot(\xi-1)$, $\xi \in [1,2]$, we obtain $\varphi_3(1) = 0$ and $\varphi_3'(\xi) = (\alpha-1)\cdot\xi^{\alpha-2} - (\alpha-1) \ge 0$ for all $\xi \in (1,2)$, so that also $\varphi_3 \ge 0$ on $[1,2]$ and hence

$$(\xi^{\alpha-1} - 1)\cdot(\xi-1) = \Big(\varphi_3(\xi) + (\alpha-1)\cdot(\xi-1)\Big)\cdot(\xi-1) \ge (\alpha-1)\cdot(\xi-1)^2 \qquad \text{for all } \xi \in [1,2].$$

Along with (3.11), this establishes (3.8) for any such α, because then $\alpha - 1 \ge 1$. $\square$

Combining the latter with the eventual L^1 bound from Lemma 2.2, by means of a suitable splitting of the integration domain we obtain the following important lower bound for the degradation-induced contribution to (3.6), underlining its independence of the initial data.

Lemma 3.5 *Let $\alpha > 1$, and let*

$$p := \begin{cases} \frac{2}{3-\alpha} & \text{if } \alpha \le 2, \\ 2 & \text{if } \alpha > 2. \end{cases} \tag{3.12}$$

Then there exists $K_1 > 0$ with the property that for any choice of $D > 0, d > 0, \chi > 0, \rho > 0, \mu > 0, \kappa > 0$ and $\lambda > 0$, given any (u_0, v_0) fulfilling (1.3) one can find $T_\star > 0$ such that

$$\|u_\varepsilon(\cdot,t) - u_\star\|_{L^p(\Omega)}^2 \le K_1 u_\star^{2-\alpha} \int_\Omega (u_\varepsilon^{\alpha-1} - u_\star^{\alpha-1}) \cdot (u_\varepsilon - u_\star) \qquad \text{for all } t > T_\star \text{ and } \varepsilon \in (0,1), \tag{3.13}$$

where $u_\star > 0$ is taken from (1.9).

Proof. We let $K > 0$ be as introduced in Lemma 3.4 and claim that then for any u_0 and v_0 fulfilling (1.3) we can find $T_\star > 0$ such that (3.13) holds with

$$K_1 := c_1 + c_2, \qquad \text{where} \qquad c_1 := \begin{cases} \frac{2^{\frac{2}{p}}}{1-2^{1-\alpha}} \cdot (2|\Omega|)^{2-\alpha} & \text{if } \alpha \le 2, \\ \frac{2^{\frac{2}{p}}}{1-2^{1-\alpha}} \cdot 2^{2-\alpha} & \text{if } \alpha > 2 \end{cases} \qquad \text{and} \qquad c_2 := \frac{(2|\Omega|)^{\frac{2-p}{p}}}{K}. \tag{3.14}$$

To this end, given any such (u_0, v_0) we first invoke Lemma 2.2 to pick $T_\star = T_\star(u_0, v_0) > 0$ fulfilling

$$\|u_\varepsilon(\cdot,t)\|_{L^1(\Omega)} \le 2u_\star|\Omega| \qquad \text{for all } t > T_\star \text{ and } \varepsilon \in (0,1), \tag{3.15}$$

and use the fact that since $p \ge 2$ we have $(a+b)^{\frac{2}{p}} \le 2^{\frac{2}{p}-1}(a^{\frac{2}{p}} + b^{\frac{2}{p}})$ for $a \ge 0$ and $b \ge 0$,

$$\begin{aligned} \|u_\varepsilon - u_\star\|_{L^p(\Omega)}^2 &= \Big\{ \int_{\{u_\varepsilon < 2u_\star\}} |u_\varepsilon - u_\star|^p + \int_{\{u_\varepsilon \ge 2u_\star\}} (u_\varepsilon - u_\star)^p \Big\}^{\frac{2}{p}} \\ &\le 2^{\frac{2}{p}-1} \cdot \Big\{ \int_{\{u_\varepsilon < 2u_\star\}} |u_\varepsilon - u_\star|^p \Big\}^{\frac{2}{p}} \\ &\quad + 2^{\frac{2}{p}-1} \cdot \Big\{ \int_{\{u_\varepsilon \ge 2u_\star\}} u_\varepsilon^p \Big\}^{\frac{2}{p}} \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1). \end{aligned} \tag{3.16}$$

Here in the latter integral we can make use of (3.15) to see that if $\alpha \le 2$ then due to the Hölder inequality and thanks to our choice of p,

$$\begin{aligned} \Big\{ \int_{\{u_\varepsilon \ge 2u_\star\}} u_\varepsilon^p \Big\}^{\frac{2}{p}} &= \|u_\varepsilon\|^2_{L^{\frac{2}{3-\alpha}}(\{u_\varepsilon \ge 2u_\star\})} \\ &\le \|u_\varepsilon\|^\alpha_{L^\alpha(\{u_\varepsilon \ge 2u_\star\})} \|u_\varepsilon\|^{2-\alpha}_{L^1(\{u_\varepsilon \ge 2u_\star\})} \\ &\le (2u_\star|\Omega|)^{2-\alpha} \int_{\{u_\varepsilon \ge 2u_\star\}} u_\varepsilon^\alpha \qquad \text{for all } t > T_\star \text{ and } \varepsilon \in (0,1), \end{aligned} \tag{3.17}$$

while in the case when $\alpha > 2$ and thus $p = 2$, we can trivially estimate

$$\Big\{ \int_{\{u_\varepsilon \ge 2u_\star\}} u_\varepsilon^p \Big\}^{\frac{2}{p}} = \int_{\{u_\varepsilon \ge 2u_\star\}} u_\varepsilon^2 \le (2u_\star)^{2-\alpha} \int_{\{u_\varepsilon \ge 2u_\star\}} u_\varepsilon^\alpha \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1). \tag{3.18}$$

As, on the other hand, regardless of the sign of $\alpha - 2$ we have

$$\begin{aligned} \int_{\{u_\varepsilon \ge 2u_\star\}} (u_\varepsilon^{\alpha-1} - u_\star^{\alpha-1}) \cdot (u_\varepsilon - u_\star) &\ge \int_{\{u_\varepsilon \ge 2u_\star\}} \Big(u_\varepsilon^{\alpha-1} - \Big(\frac{u_\varepsilon}{2}\Big)^{\alpha-1} \Big) \cdot \Big(u_\varepsilon - \frac{u_\varepsilon}{2} \Big) \\ &= \frac{1 - 2^{1-\alpha}}{2} \int_{\{u_\varepsilon \ge 2u_\star\}} u_\varepsilon^\alpha \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1), \end{aligned}$$

from (3.17) and (3.18) we infer that

$$2^{\frac{2}{p}-1} \cdot \left\{ \int_{\{u_\varepsilon \geq 2u_\star\}} u_\varepsilon^p \right\}^{\frac{2}{p}} \leq c_1 u_\star^{2-\alpha} \int_{\{u_\varepsilon \geq 2u_\star\}} (u_\varepsilon^{\alpha-1} - u_\star^{\alpha-1}) \cdot (u_\varepsilon - u_\star) \qquad \text{for all } t > T_\star \text{ and } \varepsilon \in (0,1) \tag{3.19}$$

with $c_1 > 0$ as defined in (3.14).

Next, in order to control the first summand on the right of (3.16), we note that again by applying the Hölder inequality and relying on the fact that $p \leq 2$, we may utilize Lemma 3.4 to find that for all $t > 0$ and $\varepsilon \in (0,1)$,

$$\begin{aligned}
2^{\frac{2}{p}-1} \cdot \left\{ \int_{\{u_\varepsilon < 2u_\star\}} |u_\varepsilon - u_\star|^p \right\}^{\frac{2}{p}} &\leq 2^{\frac{2}{p}-1} |\Omega|^{\frac{2-p}{p}} \int_{\{u_\varepsilon < 2u_\star\}} (u_\varepsilon - u_\star)^2 \\
&= (2|\Omega|)^{\frac{2-p}{p}} u_\star^2 \int_{\{u_\varepsilon < 2u_\star\}} \Big(\frac{u_\varepsilon}{u_\star} - 1 \Big)^2 \\
&\leq \frac{(2|\Omega|)^{\frac{2-p}{p}} u_\star^2}{K} \int_{\{u_\varepsilon < 2u_\star\}} \bigg(\Big(\frac{u_\varepsilon}{u_\star} \Big)^{\alpha-1} - 1 \bigg) \cdot \Big(\frac{u_\varepsilon}{u_\star} - 1 \Big) \\
&= \frac{(2|\Omega|)^{\frac{2-p}{p}} u_\star^{2-\alpha}}{K} \int_{\{u_\varepsilon < 2u_\star\}} (u_\varepsilon^{\alpha-1} - u_\star^{\alpha-1}) \cdot (u_\varepsilon - u_\star).
\end{aligned}$$

Together with (3.19), this shows that indeed (3.13) is valid if we take $K_1 > 0$ as in (3.14). $\square$

3.3 The evolution of $\int_\Omega (v_\varepsilon - \overline{v_\varepsilon})^2$

Next turning our attention to the second summand making up (1.8), we note the following basic description of its evolution at the level of approximate solutions.

Lemma 3.6 *Let $\alpha > 1$, $D > 0, d > 0, \chi > 0$, $\rho > 0$, $\mu > 0$, $\kappa > 0$ and $\lambda > 0$. Then for any choice of $a \in \mathbb{R}$,*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \int_\Omega \Big(v_\varepsilon(\cdot,t) - \overline{v_\varepsilon(\cdot,t)} \Big)^2 + d \int_\Omega |\nabla v_\varepsilon(\cdot,t)|^2 + \kappa \int_\Omega \Big(v_\varepsilon(\cdot,t) - \overline{v_\varepsilon(\cdot,t)} \Big)^2 \\
&\qquad = \lambda \int_\Omega \Big(\frac{u_\varepsilon(\cdot,t)}{1+\varepsilon u_\varepsilon(\cdot,t)} - a \Big) \cdot \Big(v_\varepsilon(\cdot,t) - \overline{v_\varepsilon(\cdot,t)} \Big) \qquad \textit{for all } t > 0 \textit{ and } \varepsilon \in (0,1).
\end{aligned} \tag{3.20}$$

Proof. As

$$\int_\Omega \Big(v_\varepsilon(\cdot,t) - \overline{v_\varepsilon(\cdot,t)} \Big) = 0 \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1) \tag{3.21}$$

by (1.10), on the basis of the second equation in (2.1) we see that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \int_\Omega \Big(v_\varepsilon(\cdot,t) - \overline{v_\varepsilon(\cdot,t)} \Big)^2 &= \int_\Omega \Big(v_\varepsilon(\cdot,t) - \overline{v_\varepsilon(\cdot,t)} \Big) \cdot \Big(v_{\varepsilon t}(\cdot,t) - \partial_t \overline{v_\varepsilon(\cdot,t)} \Big) \\
&= \int_\Omega \Big(v_\varepsilon(\cdot,t) - \overline{v_\varepsilon(\cdot,t)} \Big) \cdot \Big(d\Delta v_\varepsilon(\cdot,t) - \kappa v_\varepsilon(\cdot,t) + \lambda \frac{u_\varepsilon(\cdot,t)}{1+\varepsilon u_\varepsilon(\cdot,t)} \Big)
\end{aligned}$$

$$
\begin{aligned}
= \quad & -d\int_\Omega |\nabla v_\varepsilon(\cdot,t)|^2 - \kappa\int_\Omega \Big(v_\varepsilon(\cdot,t) - \overline{v_\varepsilon(\cdot,t)}\Big)\cdot v_\varepsilon(\cdot,t) \\
& +\lambda\int_\Omega \Big(v_\varepsilon(\cdot,t) - \overline{v_\varepsilon(\cdot,t)}\Big)\cdot \frac{u_\varepsilon(\cdot,t)}{1+\varepsilon u_\varepsilon(\cdot,t)} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1).
\end{aligned}
$$

Since, again due to (3.21),

$$
-\kappa\int_\Omega \Big(v_\varepsilon(\cdot,t) - \overline{v_\varepsilon(\cdot,t)}\Big)\cdot v_\varepsilon(\cdot,t) = -\kappa\int_\Omega \Big(v_\varepsilon(\cdot,t) - \overline{v_\varepsilon(\cdot,t)}\Big)^2
$$

as well as

$$
\lambda\int_\Omega \Big(v_\varepsilon(\cdot,t) - \overline{v_\varepsilon(\cdot,t)}\Big)\cdot \frac{u_\varepsilon(\cdot,t)}{1+\varepsilon u_\varepsilon(\cdot,t)} = \lambda\int_\Omega \Big(v_\varepsilon(\cdot,t) - \overline{v_\varepsilon(\cdot,t)}\Big)\cdot \Big(\frac{u_\varepsilon(\cdot,t)}{1+\varepsilon u_\varepsilon(\cdot,t)} - a\Big)
$$

for $t>0$ and $\varepsilon\in(0,1)$ whenever $a\in\mathbb{R}$, this already establishes (3.20). $\square$

3.4 An eventual Lyapunov property for small χ

Now it turns out that for adequately small χ, the right-hand side contributions to both (3.6) and (3.20) can simultaneously be absorbed by suitable of the respectively dissipated quantities in such a way that for some choice of b a linear combination of the form in (1.8) indeed eventually plays the role of an energy functional for (2.1):

Lemma 3.7 *Let $\alpha>1$ be such that $\alpha\ge 2-\frac{2}{n}$, let $D>0, d>0$, $\rho>0$, $\mu>0$, $\kappa>0$ and $\lambda>0$, and suppose that $\chi>0$ is such that*

$$
\chi^2 \le \frac{d^2D}{K_1K_2\lambda^2}\rho^{-\frac{3-\alpha}{\alpha-1}}\mu^{\frac{2}{\alpha-1}}, \tag{3.22}
$$

where K_1 is taken from Lemma 3.5, and where $K_2>0$ is such that with $p>1$ taken from (3.12) we have

$$
\|\varphi-\overline{\varphi}\|^2_{L^{\frac{p}{p-1}}(\Omega)} \le K_2\|\nabla\varphi\|^2_{L^2(\Omega)} \qquad \text{for all } \varphi\in W^{1,2}(\Omega). \tag{3.23}
$$

Then there exists $b>0$ such that whenever (u_0,v_0) satisfies (1.3), one can find $C=C(u_0,v_0)>0$ and $T_\star=T_\star(u_0,v_0)>0$ such that writing

$$
\mathcal{F}_\varepsilon(t) := \int_\Omega \Big(u_\varepsilon(\cdot,t) - u_\star - u_\star\ln\frac{u_\varepsilon(\cdot,t)}{u_\star}\Big) + \frac{b}{2}\int_\Omega \Big(v_\varepsilon(\cdot,t) - \overline{v_\varepsilon(\cdot,t)}\Big)^2, \tag{3.24}
$$

and

$$
\mathcal{D}_\varepsilon(t) := C\int_\Omega \frac{|\nabla u_\varepsilon(\cdot,t)|^2}{u_\varepsilon^2(\cdot,t)} + C\|u_\varepsilon(\cdot,t)-u_\star\|^2_{L^p(\Omega)} + C\int_\Omega \Big(v_\varepsilon(\cdot,t) - \overline{v_\varepsilon(\cdot,t)}\Big)^2, \qquad t>0,\ \varepsilon\in(0,1), \tag{3.25}
$$

we have

$$
\mathcal{F}_\varepsilon'(t) + \mathcal{D}_\varepsilon(t) \le 0 \qquad \text{for all } t>T_\star \text{ and } \varepsilon\in(0,1). \tag{3.26}
$$

PROOF. We let

$$b := \frac{\chi^2 u_\star}{dD} \tag{3.27}$$

and recall that $u_\star = (\frac{\rho}{\mu})^{\frac{1}{\alpha-1}}$ to see that according to (3.22) we have

$$\frac{bK_1K_2\lambda^2 u_\star^{2-\alpha}}{d} = \frac{K_1K_2\lambda^2 u_\star^{3-\alpha}\chi^2}{d^2D} = \frac{K_1K_2\lambda^2\rho^{\frac{3-\alpha}{\alpha-1}}\mu^{-\frac{2}{\alpha-1}}\chi^2}{d^2D}\cdot\mu \le \mu. \tag{3.28}$$

Assuming (1.3) and taking $\mathcal{F}_\varepsilon$ as accordingly defined by (3.24), on the basis of Lemma 3.3 and Lemma 3.6, the latter applied to $a := \frac{u_\star}{1+\varepsilon u_\star}$, we then obtain that

$$\begin{aligned}
\mathcal{F}_\varepsilon'(t) \quad &+ \quad Du_\star\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} + bd\int_\Omega |\nabla v_\varepsilon|^2 + \mu\int_\Omega (u_\varepsilon^{\alpha-1} - u_\star^{\alpha-1})\cdot(u_\varepsilon - u_\star) + b\kappa\int_\Omega \Big(v_\varepsilon - \overline{v_\varepsilon(\cdot,t)}\Big)^2 \\
&= \quad \chi u_\star \int_\Omega \frac{\nabla u_\varepsilon}{u_\varepsilon}\cdot\nabla v_\varepsilon \\
&\quad\quad + b\lambda\int_\Omega \Big(\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon} - \frac{u_\star}{1+\varepsilon u_\star}\Big)\cdot\Big(v_\varepsilon - \overline{v_\varepsilon(\cdot,t)}\Big) \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1).
\end{aligned} \tag{3.29}$$

Here by Young's inequality and (3.27),

$$\begin{aligned}
\chi u_\star \int_\Omega \frac{\nabla u_\varepsilon}{u_\varepsilon}\cdot\nabla v_\varepsilon &\le \frac{Du_\star}{2}\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} + \frac{\chi^2 u_\star}{2D}\int_\Omega |\nabla v_\varepsilon|^2 \\
&\le \frac{Du_\star}{2}\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} + \frac{bd}{2}\int_\Omega |\nabla v_\varepsilon|^2 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),
\end{aligned} \tag{3.30}$$

whereas due to the Hölder inequality, (3.23) and Young's inequality,

$$\begin{aligned}
&b\lambda\int_\Omega \Big(\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon} - \frac{u_\star}{1+\varepsilon u_\star}\Big)\cdot\Big(v_\varepsilon - \overline{v_\varepsilon(\cdot,t)}\Big) \\
&\quad \le b\lambda\Big\|\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon} - \frac{u_\star}{1+\varepsilon u_\star}\Big\|_{L^p(\Omega)}\Big\|v_\varepsilon - \overline{v_\varepsilon(\cdot,t)}\Big\|_{L^{\frac{p}{p-1}}(\Omega)} \\
&\quad \le b\lambda\sqrt{K_2}\Big\|\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon} - \frac{u_\star}{1+\varepsilon u_\star}\Big\|_{L^p(\Omega)}\|\nabla v_\varepsilon\|_{L^2(\Omega)} \\
&\quad \le \frac{bd}{2}\|\nabla v_\varepsilon\|_{L^2(\Omega)}^2 + \frac{bK_2\lambda^2}{2d}\Big\|\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon} - \frac{u_\star}{1+\varepsilon u_\star}\Big\|_{L^p(\Omega)}^2 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1).
\end{aligned} \tag{3.31}$$

Since

$$\Big\|\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon} - \frac{u_\star}{1+\varepsilon u_\star}\Big\|_{L^p(\Omega)} \le \|u_\varepsilon - u_\star\|_{L^p(\Omega)} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1)$$

by the mean value theorem, and since thus Lemma 3.5 in conjunction with (3.28) say that with $T_\star > 0$ as provided there we have

$$\frac{bK_2\lambda^2}{2d}\Big\|\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon} - \frac{u_\star}{1+\varepsilon u_\star}\Big\|_{L^p(\Omega)}^2 + \frac{bK_2\lambda^2}{2d}\|u_\varepsilon - u_\star\|_{L^p(\Omega)}^2$$

$$
\begin{aligned}
&\le \frac{bK_2\lambda^2}{d}\|u_\varepsilon - u_\star\|^2_{L^p(\Omega)} \\
&\le \frac{bK_1K_2\lambda^2 u_\star^{2-\alpha}}{d}\int_\Omega (u_\varepsilon^{\alpha-1} - u_\star^{\alpha-1})\cdot(u_\varepsilon - u_\star) \\
&\le \mu\int_\Omega (u_\varepsilon^{\alpha-1} - u_\star^{\alpha-1})\cdot(u_\varepsilon - u_\star) \qquad \text{for all } t > T_\star \text{ and } \varepsilon \in (0,1),
\end{aligned}
$$

from (3.29), (3.30) and (3.31) we thus infer that

$$
\mathcal{F}_\varepsilon'(t) + \frac{Du_\star}{2}\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} + \frac{bK_2\lambda^2}{d}\|u_\varepsilon - u_\star\|^p_{L^p(\Omega)} + b\kappa\int_\Omega \Big(v_\varepsilon - \overline{v_\varepsilon(\cdot,t)}\Big)^2 \le 0
$$

for all $t > T_\star$ and $\varepsilon \in (0,1)$. We therefore readily arrive at (3.26) upon taking $C > 0$ suitably small and then defining $\mathcal{D}_\varepsilon$ through (3.25). $\Box$

Intending to make use of (3.26) for suitably large times only, we next aim at providing bounds for $\mathcal{F}_\varepsilon$ from above, possibly depending on time but not on ε. This will be based on the following elementary evolution property.

Lemma 3.8 *Let $\alpha > 1$, $D > 0, d > 0, \chi > 0$, $\rho > 0$, $\mu > 0$, $\kappa > 0$ and $\lambda > 0$. Then*

$$
-\frac{d}{dt}\int_\Omega \ln u_\varepsilon + D\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} = \chi\int_\Omega \frac{\nabla u_\varepsilon}{u_\varepsilon}\cdot\nabla v_\varepsilon - \rho|\Omega| + \mu\int_\Omega u_\varepsilon^{\alpha-1} \qquad \textit{for all } t > 0 \textit{ and } \varepsilon \in (0,1). \tag{3.32}
$$

Proof. This can be seen by straightforward computation using the first equation in (2.1). $\Box$

In fact, in conjunction with another standard testing procedure the latter enables us to derive the following.

Lemma 3.9 *Let $\alpha > 1$ be such that $\alpha \ge 2 - \frac{2}{n}$, and let $D > 0, d > 0$, $\rho > 0$, $\mu > 0$, $\kappa > 0$ and $\lambda > 0$. Then for all $T > 0$ there exists $C(T) > 0$ such that*

$$
-\int_\Omega \ln u_\varepsilon(\cdot,t) \le C(T) \qquad \textit{for all } t \in (0,T) \textit{ and } \varepsilon \in (0,1), \tag{3.33}
$$

and that

$$
\int_\Omega v_\varepsilon^2(\cdot,t) \le C(T) \qquad \textit{for all } t \in (0,T) \textit{ and } \varepsilon \in (0,1). \tag{3.34}
$$

Proof. We test the second equation in (2.1) against v_ε to see that taking p as in (3.12), due to the Hölder inequality we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega v_\varepsilon^2 + d\int_\Omega |\nabla v_\varepsilon|^2 + \kappa\int_\Omega v_\varepsilon^2 &= \lambda\int_\Omega \frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}v_\varepsilon \\
&\le \lambda\Big\|\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\Big\|_{L^p(\Omega)}\|v_\varepsilon\|_{L^{\frac{p}{p-1}}(\Omega)} \\
&\le \lambda\|u_\varepsilon\|_{L^p(\Omega)}\|v_\varepsilon\|_{L^{\frac{p}{p-1}}(\Omega)} \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1).
\end{aligned} \tag{3.35}
$$

Since $\frac{p}{p-1}$ is finite and satisfies $\frac{p}{p-1} \le \frac{2n}{(n-2)_+}$ thanks to the assumption that $\alpha \ge 2 - \frac{2}{n}$, and since thus $W^{1,2}(\Omega)$ is continuously embedded into $L^{\frac{p}{p-1}}(\Omega)$, we can find $c_1 > 0$ such that $\|\varphi\|^2_{L^{\frac{p}{p-1}}(\Omega)} \le$

$c_1 \cdot (\|\nabla\varphi\|^2_{L^2(\Omega)} + \|\varphi\|^2_{L^2(\Omega)})$ for all $\varphi \in W^{1,2}(\Omega)$, so that letting $c_2 := \min\{\frac{d}{2}, \kappa\} > 0$, from (3.35) we infer that by Young's inequality,

$$\begin{aligned} \frac{d}{dt}\int_\Omega v_\varepsilon^2 + d\int_\Omega |\nabla v_\varepsilon|^2 &\le -2c_2 \cdot \Big\{ \int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega v_\varepsilon^2 \Big\} + 2\sqrt{c_1}\lambda \|u_\varepsilon\|_{L^p(\Omega)} \cdot \Big\{ \int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega v_\varepsilon^2 \Big\}^{\frac{1}{2}} \\ &\le \frac{c_1\lambda^2}{2c_2} \|u_\varepsilon\|^2_{L^p(\Omega)} \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1). \end{aligned} \tag{3.36}$$

Here in the case $\alpha \le 2$ we can use the Hölder inequality along with (2.3) to find $c_3 > 0$ such that

$$\frac{c_1\lambda^2}{2c_2}\|u_\varepsilon\|^2_{L^p(\Omega)} \le \frac{c_1\lambda^2}{2c_2}\|u_\varepsilon\|^\alpha_{L^\alpha(\Omega)}\|u_\varepsilon\|^{2-\alpha}_{L^1(\Omega)} \le c_3\int_\Omega u_\varepsilon^\alpha \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1), \tag{3.37}$$

while if $\alpha > 2$ then, again by Young's inequality,

$$\frac{c_1\lambda^2}{2c_2}\|u_\varepsilon\|^2_{L^p(\Omega)} = \frac{c_1\lambda^2}{2c_2}\int_\Omega u_\varepsilon^2 \le \frac{c_1\lambda^2}{2c_2}\int_\Omega u_\varepsilon^\alpha + \frac{c_1\lambda^2|\Omega|}{2c_2} \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1). \tag{3.38}$$

Therefore, (3.36) entails that with $c_4 := \max\{c_3\,,\,\frac{c_1\lambda^2}{2c_2}\,,\,\frac{c_1\lambda^2|\Omega|}{2c_2}\}$ we have

$$\frac{d}{dt}\int_\Omega v_\varepsilon^2 + d\int_\Omega |\nabla v_\varepsilon|^2 \le c_4\int_\Omega u_\varepsilon^\alpha + c_4 \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1), \tag{3.39}$$

which combined with Lemma 3.8 shows that if we let $c_5 := \frac{\chi^2}{4dD}$, then once more due to Young's inequality,

$$\begin{aligned} \frac{d}{dt}\Big\{ -\int_\Omega \ln u_\varepsilon + c_5\int_\Omega v_\varepsilon^2 \Big\} &\le -D\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} + \chi\int_\Omega \frac{\nabla u_\varepsilon}{u_\varepsilon}\cdot\nabla v_\varepsilon - \rho|\Omega| + \mu\int_\Omega u_\varepsilon^{\alpha-1} \\ &\quad - c_5 d\int_\Omega |\nabla v_\varepsilon|^2 + c_4c_5\int_\Omega u_\varepsilon^\alpha + c_4c_5 \\ &\le -\rho|\Omega| + \mu\int_\Omega u_\varepsilon^{\alpha-1} + c_4c_5\int_\Omega u_\varepsilon^\alpha + c_4c_5 \\ &\le (c_4c_5+\mu)\int_\Omega u_\varepsilon^\alpha + c_4c_5 + \mu|\Omega| \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1) \end{aligned}$$

and hence

$$-\int_\Omega \ln u_\varepsilon(\cdot,t) + c_5\int_\Omega v_\varepsilon^2(\cdot,t) \le -\int_\Omega \ln u_0 + c_5\int_\Omega v_0^2 + (c_4c_5+\mu)\int_0^t\int_\Omega u_\varepsilon^\alpha + (c_4c_5+\mu|\Omega|)\cdot t$$

for all $t > 0$ and $\varepsilon \in (0,1)$. In view of (2.4) and the fact that $-\int_\Omega \ln u_0 < \infty$ by (1.3), this readily yields the claim, because $\int_\Omega \ln u_\varepsilon(\cdot,t) \le \int_\Omega u_\varepsilon(\cdot,t) \le m$ for all $t > 0$ and $\varepsilon \in (0,1)$ thanks to (2.3). $\Box$

We are now in the position to derive the following consequence of (3.26).

Lemma 3.10 *Let $\alpha > 1$ be such that $\alpha \geq 2 - \frac{2}{n}$, let $D > 0, d > 0$, $\rho > 0$, $\mu > 0$, $\kappa > 0$ and $\lambda > 0$, and suppose that $\chi > 0$ satisfies (3.22). Then given (u_0, v_0) fulfilling (1.3), one can find $C = C(u_0, v_0) > 0$ and $T_\star = T_\star(u_0, v_0) > 0$ such that*

$$\int_\Omega \ln u_\varepsilon(\cdot,t) \geq -C \qquad \textit{for all } t > T_\star \textit{ and } \varepsilon \in (0,1) \tag{3.40}$$

and

$$\int_{T_\star}^\infty \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} \leq C \qquad \textit{for all } \varepsilon \in (0,1) \tag{3.41}$$

as well as

$$\int_{T_\star}^\infty \|u_\varepsilon(\cdot,t) - u_\star\|_{L^p(\Omega)}^2 dt \leq C \qquad \textit{for all } \varepsilon \in (0,1) \tag{3.42}$$

and

$$\int_{T_\star}^\infty \|v_\varepsilon(\cdot,t) - \overline{v_\varepsilon(\cdot,t)}\|_{L^2(\Omega)}^2 dt \leq C \qquad \textit{for all } \varepsilon \in (0,1), \tag{3.43}$$

where $p > 1$ is taken from (3.12).

Proof. According to Lemma 3.7, we can fix $b > 0, c_1 = c_1(u_0, v_0) > 0$ and $T_\star = T_\star(u_0, v_0) > 0$ such that with $(\mathcal{F}_\varepsilon)_{\varepsilon\in(0,1)}$ as defined in (3.24) we have

$$\mathcal{F}_\varepsilon'(t) + c_1 \cdot \left\{ \left\| \frac{\nabla u_\varepsilon(\cdot,t)}{u_\varepsilon(\cdot,t)} \right\|_{L^2(\Omega)}^2 + \left\| v_\varepsilon(\cdot,t) - \overline{v_\varepsilon(\cdot,t)} \right\|_{L^2(\Omega)}^2 + \|u_\varepsilon(\cdot,t) - u_\star\|_{L^p(\Omega)}^2 \right\} \leq 0 \tag{3.44}$$

for all $t > T_\star$ and $\varepsilon \in (0,1)$. Apart from that, again explicitly relying on the assumption $\alpha \geq 2 - \frac{2}{n}$ we may invoke Lemma 3.9 along with (2.3) to find $c_i = c_i(u_0, v_0) > 0$, $i \in \{2,3,4\}$, such that

$$-\int_\Omega \ln u_\varepsilon(\cdot,T_\star) \leq c_2, \qquad \int_\Omega v_\varepsilon^2(\cdot,T_\star) \leq c_3 \quad \text{and} \quad \int_\Omega u_\varepsilon(\cdot,T_\star) \leq c_4 \qquad \text{for all } \varepsilon \in (0,1),$$

so that

$$\begin{aligned} \mathcal{F}_\varepsilon(T_\star) &\leq \int_\Omega u_\varepsilon(\cdot,T_\star) - u_\star \int_\Omega \ln u_\varepsilon(\cdot,T_\star) + |\Omega| u_\star |\ln u_\star| + \frac{b}{2} \int_\Omega v_\varepsilon^2(\cdot,T_\star) \\ &\leq c_5 := c_4 + c_2 u_\star + u_\star |\ln u_\star| + \frac{bc_3}{2} \qquad \text{for all } \varepsilon \in (0,1). \end{aligned}$$

An integration of (3.44) therefore shows that for all $t > T_\star$ and $\varepsilon \in (0,1)$,

$$\mathcal{F}_\varepsilon(t) + c_1 \int_{T_\star}^t \left\{ \left\| \frac{\nabla u_\varepsilon(\cdot,s)}{u_\varepsilon(\cdot,s)} \right\|_{L^2(\Omega)}^2 + \left\| v_\varepsilon(\cdot,s) - \overline{v_\varepsilon(\cdot,s)} \right\|_{L^2(\Omega)}^2 + \|u_\varepsilon(\cdot,s) - u_\star\|_{L^p(\Omega)}^2 \right\} ds \leq c_5,$$

and thereby entails (3.40), (3.41), (3.42) and (3.43) due to the fact that $\mathcal{F}_\varepsilon(t) \geq -(u_\star + u_\star |\ln u_\star|)|\Omega| - u_\star \int_\Omega \ln u_\varepsilon(\cdot,t)$ for all $t > 0$ and $\varepsilon \in (0,1)$. $\square$

4 Persistence of energy decrease in the limit problem

We shall next address the question how far the monotonicity property in (3.26) persists in the limit of vanishing ε. Our crucial preparation in this respect relies on the integral estimate from Lemma 3.1.

Corollary 4.1 *Let $\alpha > 1$ be such that $\alpha \geq 2 - \frac{2}{n}$, let $D > 0, d > 0$, $\rho > 0$, $\mu > 0$, $\kappa > 0$, $\lambda > 0$ and $\chi > 0$ be such that (3.22) holds, and suppose that (1.3) is satisfied. Then there exist $T_\star = T_\star(u_0, v_0) > 0$ and $C = C(u_0, v_0) > 0$ such that*

$$\int_t^{t+1} \int_\Omega H^q(u_\varepsilon) \leq C \qquad \textit{for all } t > T_\star \textit{ and } \varepsilon \in (0,1), \tag{4.1}$$

where H is taken from (3.1) and $q := \max\{\alpha, 2\}$.

Proof. We employ Lemma 3.10 to find $T_\star = T_\star(u_0, v_0) > 0$, $c_1 = c_1(u_0, v_0) > 0$ and $c_2 = c_2(u_0, v_0) > 0$ such that

$$\int_\Omega \ln u_\varepsilon(\cdot, t) \geq -c_1 \qquad \text{for all } t > T_\star \text{ and } \varepsilon \in (0,1) \tag{4.2}$$

and

$$\int_{T_\star}^\infty \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} \leq c_2 \qquad \text{for all } \varepsilon \in (0,1), \tag{4.3}$$

and applying Lemma 2.1 yields $c_3 = c_3(u_0, v_0) > 0$ and $c_4 = c_4(u_0, v_0) > 0$ fulfilling

$$\int_\Omega u_\varepsilon(\cdot, t) \leq c_3 \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1) \tag{4.4}$$

as well as

$$\int_t^{t+1} \int_\Omega u_\varepsilon^\alpha \leq c_4 \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1). \tag{4.5}$$

Since Lemma 3.1 provides $c_5 = c_5(u_0, v_0) > 0$ such that

$$\int_\Omega H^q(u_\varepsilon) \leq \int_\Omega u_\varepsilon^\alpha + c_5 \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} + c_5 \cdot \left\{ \int_\Omega |\ln u_\varepsilon| \right\}^2 + c_5 \qquad \text{for all } t > 0,$$

and since using the elementary inequality $\ln \xi \leq \xi$ for $\xi > 1$ we see that herein

$$\begin{aligned} \int_\Omega |\ln u_\varepsilon| &= 2 \int_{\{u_\varepsilon > 1\}} u_\varepsilon - \int_\Omega \ln u_\varepsilon \\ &\leq 2c_3 + c_1 \qquad \text{for all } t > T_\star \text{ and } \varepsilon \in (0,1), \end{aligned}$$

this implies that

$$\begin{aligned} \int_t^{t+1} \int_\Omega H^q(u_\varepsilon) &\leq \int_t^{t+1} \int_\Omega u_\varepsilon^\alpha + c_5 \int_t^{t+1} \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} + c_5 \int_t^{t+1} \left\{ \int_\Omega |\ln u_\varepsilon| \right\}^2 + c_5 \\ &\leq c_4 + c_2 c_5 + (2c_3 + c_1)^2 c_5 + c_5 \qquad \text{for all } t > T_\star \text{ and } \varepsilon \in (0,1), \end{aligned}$$

and hence establishes (4.1). □

Thanks to the latter, passing to the limit in the first integral appearing in (3.24) is therefore possible in the following sense.

Lemma 4.2 *Let $\alpha > 1$ be such that $\alpha \ge 2 - \frac{2}{n}$, let $D > 0, d > 0$, $\rho > 0$, $\mu > 0$, $\kappa > 0$, $\lambda > 0$ and $\chi > 0$ be such that (3.22) holds, and assume (1.3). Then there exist $T_\star = T_\star(u_0, v_0) > 0$ and a null set $N = N(u_0, v_0) \subset (T_\star, \infty)$ such that with $(\varepsilon_j)_{j\in\mathbb{N}}$ and H taken from Proposition 1.1 and (3.1), we have $H(u(\cdot, t)) \in L^1(\Omega)$ for all $t \in (T_\star, \infty) \setminus N$ and*

$$\int_\Omega H(u_\varepsilon(\cdot, t)) \to \int_\Omega H(u(\cdot, t)) \quad \textit{for all } t \in (T_\star, \infty) \setminus N \qquad \textit{as } \varepsilon = \varepsilon_j \searrow 0. \tag{4.6}$$

Proof. We take $T_\star = T_\star(u_0, v_0) > 0$ as given by Corollary 4.1 and hence infer from the latter that for each $T > T_\star$, the family $(H(u_\varepsilon))_{\varepsilon\in(0,1)}$ is bounded in $L^q(\Omega \times (T_\star, T))$ for $q = \min\{\alpha, 2\}$. As $q > 1$, this implies that for any such T, $(H(u_\varepsilon))_{\varepsilon\in(0,1)}$ is uniformly integrable over $\Omega \times (T_\star, T)$, so that the existence of a null set $N = N(u_0, v_0) \subset (T_\star, \infty)$ with the claimed properties readily results upon an application of the Vitali convergence theorem, relying on the fact that $u_\varepsilon \to u$ a.e. in $\Omega \times (T_\star, \infty)$ as $\varepsilon = \varepsilon_j \searrow 0$ according to Proposition 1.1. $\square$

Along with a similar but in fact more straightforward property of the respective second summands, this ensures that we may indeed pass to the limit in (3.24) and (3.26) to achieve the following.

Corollary 4.3 *Let $\alpha > 1$ satisfy $\alpha \ge 2 - \frac{2}{n}$, let $D > 0, d > 0$, $\rho > 0$, $\mu > 0$, $\kappa > 0$, $\lambda > 0$ and $\chi > 0$ be such that (3.22) is valid, let $b > 0$ be as in Lemma 3.7, and assume (1.3). Then there exist $T_\star = T_\star(u_0, v_0) > 0$ and a null set $N_\star = N_\star(u_0, v_0) \subset (T_\star, \infty)$ such that letting*

$$\mathcal{F}(t) := \int_\Omega \Big(u(\cdot, t) - u_\star - u_\star \ln \frac{u(\cdot, t)}{u_\star}\Big) + \frac{b}{2} \int_\Omega \Big(v(\cdot, t) - \overline{v(\cdot, t)}\Big)^2, \qquad t \in (T_\star, \infty) \setminus N_\star, \tag{4.7}$$

defined a function $\mathcal{F} : (T_\star, \infty) \setminus N_\star \to [0, \infty)$ satisfying

$$\mathcal{F}(t) \le \mathcal{F}(t_0) \qquad \textit{for all } t_0 \in (T_\star, \infty) \setminus N_\star \textit{ and each } t \in (t_0, \infty) \setminus N_\star. \tag{4.8}$$

Proof. We combine Lemma 4.2 with Lemma 3.7 to find $T_\star = T_\star(u_0, v_0) > 0$ and a null set $N_1 = N_1(u_0, v_0) \subset (T_\star, \infty)$ such that the function H defined in (3.1) satisfies $H(u(\cdot, t)) \in L^1(\Omega)$ for all $t \in (T_\star, \infty) \setminus N_1$, that taking $(\varepsilon_j)_{j\in\mathbb{N}}$ from Proposition 1.1 we have

$$\int_\Omega H(u_\varepsilon(\cdot, t)) \to \int_\Omega H(u(\cdot, t)) \quad \text{for all } t \in (T_\star, \infty) \setminus N_1 \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0, \tag{4.9}$$

and that moreover

$$\mathcal{F}_\varepsilon(t) \le \mathcal{F}_\varepsilon(t_0) \qquad \text{for all } t_0 \in (T_\star, \infty), \text{ any } t > t_0 \text{ and each } \varepsilon \in (0, 1), \tag{4.10}$$

where $(\mathcal{F}_\varepsilon)_{\varepsilon\in(0,1)}$ is as in (3.24). Apart from that, taking a null set $N_2 = N_2(u_0, v_0) \subset (0, \infty)$ such that in accordance with Proposition 1.1 we have

$$v_\varepsilon(\cdot, t) \to v(\cdot, t) \quad \text{in } L^2(\Omega) \quad \text{for all } t \in (0, \infty) \setminus N_2 \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0, \tag{4.11}$$

and that hence clearly also

$$\overline{v_\varepsilon(\cdot, t)} \to \overline{v(\cdot, t)} \quad \text{for all } t \in (0, \infty) \setminus N_2 \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0, \tag{4.12}$$

we conclude that if we let $N_\star := N_1 \cup N_2$, then for all $t \in (T_\star, \infty) \setminus N_\star$ the definition in (4.7) introduces a real-valued and nonnegative function $\mathcal{F}$ on $(T_\star, \infty) \setminus N_\star$ which due to (4.10), (4.9), (4.11) and (4.12) indeed satisfies (4.8). $\square$

5 Convergence of (u, v) in the large time limit

5.1 Stabilization of $\int_\Omega v$

Next concerned with possible implications of (4.8) on the large time behavior of u and v, we first address the averages $\overline{v}$ appearing therein, and proceed to make sure that these stabilize toward a limit compatible with the claim in (1.14). Our verification of this will utilize the following elementary convergence feature.

Lemma 5.1 *If $g \in L^1_{loc}((0,\infty))$ and $g_\infty \in \mathbb{R}$ are such that*

$$\int_t^{t+1} |g(s) - g_\infty| ds \to 0 \qquad \textit{as } t \to \infty, \tag{5.1}$$

then for each $a > 0$,

$$\int_0^t e^{-a(t-s)} g(s) ds \to \frac{g_\infty}{a} \qquad \textit{as } t \to \infty. \tag{5.2}$$

Proof. We first note that since $\int_0^t e^{-a(t-s)}ds = \frac{1}{a}(1-e^{-at}) \to \frac{1}{a}$ as $t \to \infty$, upon replacing g by $g - g_\infty$ if necessary we may assume that $g_\infty = 0$. Then given $\eta > 0$, we can first pick $t_1 = t_1(\eta) > 1$ large such that

$$\int_{t-1}^t |g(s)| ds \le \frac{(1-e^{-a})\eta}{2} \qquad \text{for all } t > t_1, \tag{5.3}$$

and then choose $t_2 = t_2(\eta) > t_1$ in such a way that

$$\left| \int_0^{t_1} e^{as} g(s) ds \right| \cdot e^{-at_2} \le \frac{\eta}{2}. \tag{5.4}$$

For fixed $t > t_2$, we now rely on (5.4) in estimating

$$\begin{aligned}
\left| \int_0^t e^{-a(t-s)} g(s) ds \right| &= \left| e^{-at} \int_0^{t_1} e^{as} g(s) ds + \int_{t_1}^t e^{-a(t-s)} g(s) ds \right| \\
&\le \frac{\eta}{2} + \int_{t_1}^t e^{-a(t-s)} |g(s)| ds,
\end{aligned} \tag{5.5}$$

where taking $k = k(t) \in \mathbb{N}$ such that $t - k \le t_1 < t - k + 1$ we see that

$$\begin{aligned}
\int_{t_1}^t e^{-a(t-s)} |g(s)| ds &\le \int_{t-k}^t e^{-a(t-s)} |g(s)| ds \\
&= \sum_{j=0}^{k-1} \int_{t-j-1}^{t-j} e^{-a(t-s)} |g(s)| ds \\
&\le \sum_{j=0}^{k-1} e^{-aj} \int_{t-j-1}^{t-j} |g(s)| ds \\
&\le \left\{ \sum_{j=0}^{k-1} e^{-aj} \right\} \cdot \frac{(1-e^{-a})\eta}{2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1-e^{-ka}}{1-e^{-a}} \cdot \frac{(1-e^{-a})\eta}{2} \\
&\le \frac{\eta}{2}.
\end{aligned}
$$

Combined with (5.5), this yields the claim due to the fact that $\eta > 0$ was arbitrary. $\Box$

In fact, due to a basic mass evolution property in (2.1) the latter warrants the following.

Lemma 5.2 *Let $\alpha > 1$ be such that $\alpha \ge 2 - \frac{2}{n}$, let $D > 0, d > 0$, $\rho > 0$, $\mu > 0$, $\kappa > 0$, $\lambda > 0$ and $\chi > 0$ be such that (3.22) holds, and assume (1.3). Then there exists and a null set $N = N(u_0, v_0) \subset (T_\star, \infty)$ such that*

$$
\int_\Omega v(\cdot,t) \to \frac{\lambda u_\star |\Omega|}{\kappa} \qquad \text{as } (0,\infty) \setminus N \ni t \to \infty. \tag{5.6}
$$

PROOF. An integration of the second equation in (2.1) shows that

$$
\frac{d}{dt} \int_\Omega v_\varepsilon + \kappa \int_\Omega v_\varepsilon = \lambda \int_\Omega \frac{u_\varepsilon}{1+\varepsilon u_\varepsilon} \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1), \tag{5.7}
$$

and that hence

$$
\int_\Omega v_\varepsilon(\cdot,t) = e^{-\kappa t} \int_\Omega v_0 + \lambda \int_0^t e^{-\kappa(t-s)} \cdot \Big\{ \int_\Omega \frac{u_\varepsilon(\cdot,s)}{1+\varepsilon u_\varepsilon(\cdot,s)} \Big\} ds \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1). \tag{5.8}
$$

Now according to Proposition 1.1 we can find a null set $N = N(u_0, v_0) \subset (0,\infty)$ such that with $(\varepsilon_j)_{j\in\mathbb{N}}$ as given there we have

$$
u_\varepsilon \to u \quad \text{in } L^1(\Omega \times (0,t)) \text{ and a.e. in } \Omega \times (0,t) \qquad \text{for all } t > 0 \tag{5.9}
$$

and

$$
v_\varepsilon(\cdot,t) \to v(\cdot,t) \quad \text{in } L^1(\Omega) \qquad \text{for all } t \in (0,\infty) \setminus N \tag{5.10}
$$

and $\varepsilon = \varepsilon_j \searrow 0$. Since (5.9) together with the dominated convergence theorem entails that for all $t > 0$ we also have

$$
\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon} = u_\varepsilon - \frac{\varepsilon u_\varepsilon^2}{1+\varepsilon u_\varepsilon} \to u \quad \text{in } L^1(\Omega \times (0,t)) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0
$$

due to the fact that $\frac{\varepsilon u_\varepsilon^2}{1+\varepsilon u_\varepsilon} \to 0$ a.e. in $\Omega \times (0,\infty)$ as $\varepsilon = \varepsilon_j \searrow 0$ and $0 \le \frac{\varepsilon u_\varepsilon^2}{1+\varepsilon u_\varepsilon} \le u_\varepsilon$ for all $\varepsilon \in (0,1)$, we see that on the right-hand side of (5.8),

$$
\begin{aligned}
&\Big| \lambda \int_0^t e^{-\kappa(t-s)} \cdot \Big\{ \int_\Omega \frac{u_\varepsilon(\cdot,s)}{1+\varepsilon u_\varepsilon(\cdot,s)} \Big\} ds - \lambda \int_0^t e^{-\kappa(t-s)} \cdot \Big\{ \int_\Omega u(\cdot,s) \Big\} ds \Big| \\
&\qquad = \lambda \cdot \Big| \int_0^t e^{-\kappa(t-s)} \cdot \Big\{ \int_\Omega \Big(\frac{u_\varepsilon(\cdot,s)}{1+\varepsilon u_\varepsilon(\cdot,s)} - u(\cdot,s) \Big) \Big\} ds \Big| \\
&\qquad \le \lambda \int_0^t \int_\Omega \Big| \frac{u_\varepsilon}{1+\varepsilon u_\varepsilon} - u \Big| \\
&\qquad \to 0 \quad \text{for all } t > 0 \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0.
\end{aligned}
$$

Consequently, (5.8) and (5.10) imply that

$$\int_\Omega v(\cdot,t) = e^{-\kappa t}\int_\Omega v_0 + \lambda\int_0^t e^{-\kappa(t-s)}\cdot\Big\{\int_\Omega u(\cdot,s)\Big\}ds \qquad \text{for all } t\in(0,\infty)\setminus N, \tag{5.11}$$

where clearly

$$e^{-\kappa t}\int_\Omega v_0 \to 0 \qquad \text{as } t\to\infty. \tag{5.12}$$

Furthermore, an application of Lemma 5.1 to $a := \kappa$ and $g(t) := \lambda\int_\Omega u(\cdot,t)$, $t>0$, shows that

$$\lambda\int_0^t e^{-\kappa(t-s)}\cdot\Big\{\int_\Omega u(\cdot,s)\Big\}ds \to \frac{\lambda u_\star|\Omega|}{\kappa} \qquad \text{as } t\to\infty, \tag{5.13}$$

because by the Cauchy-Schwarz inequality,

$$\begin{aligned}
\int_t^{t+1}\big|g(s)-\lambda u_\star|\Omega|\,\big|ds &= \lambda\int_t^{t+1}\Big|\int_\Omega\Big(u(\cdot,s)-u_\star\Big)\Big|ds \\
&\le \lambda\int_t^{t+1}\|u(\cdot,s)-u_\star\|_{L^1(\Omega)}ds \\
&\le \lambda\cdot\Big\{\int_t^{t+1}\|u(\cdot,s)-u_\star\|^2_{L^1(\Omega)}ds\Big\}^{\frac{1}{2}} \\
&\le \lambda\cdot\Big\{\int_t^{\infty}\|u(\cdot,s)-u_\star\|^2_{L^1(\Omega)}ds\Big\}^{\frac{1}{2}} \qquad \text{for all } t>0,
\end{aligned}$$

and because Lemma 3.10 obviously entails that

$$\int_t^\infty \|u(\cdot,s)-u_\star\|^2_{L^1(\Omega)} \to 0 \qquad \text{as } t\to\infty.$$

In summary, from (5.11), (5.12) and (5.13) we obtain (5.6). $\Box$

5.2 Decay of $\mathcal{F}$ along a subsequence

Thanks to the weak convergence properties implied by Lemma 5.2 and Lemma 3.10, the two summands constituting $\mathcal{F}$ become arbitrarily small at least along suitable sequences of times:

Lemma 5.3 *Let $\alpha>1$ be such that $\alpha\ge 2-\frac{2}{n}$, let $D>0, d>0$, $\rho>0$, $\mu>0$, $\kappa>0$, $\lambda>0$ and $\chi>0$ be such that (3.22) holds, and given (u_0,v_0) fulfilling (1.3), let $N_\star=N_\star(u_0,v_0)$ be as in Corollary 4.3. Then there exists $(t_k)_{k\in\mathbb{N}}\subset(T_\star,\infty)\setminus N_\star$ such that $t_k\to\infty$ as $k\to\infty$, and that with H taken from (3.1) we have*

$$H(u(\cdot,t_k))\to 0 \quad \textit{in } L^1(\Omega) \qquad \textit{as } k\to\infty \tag{5.14}$$

and

$$v(\cdot,t_k)\to v_\star \quad \textit{in } L^2(\Omega) \qquad \textit{as } k\to\infty. \tag{5.15}$$

PROOF. We first invoke Lemma 5.2 to identify a null set $N_1 \supset N_\star$ such that

$$\overline{v(\cdot,t)} \to \frac{\lambda u_\star}{\kappa} = v_\star \qquad \text{as } (0,\infty) \setminus N_1 \ni t \to \infty, \tag{5.16}$$

and once more use that the exponent p appearing in Lemma 3.10 satisfies $p \ge 1$ in finding $T_1 = T_1(u_0, v_0) > 0$, $c_1 = c_1(u_0, v_0) > 0$ and $c_2 = c_2(u_0, v_0) > 0$ fulfilling

$$\int_{T_1}^\infty \|u_\varepsilon(\cdot,t) - u_\star\|_{L^1(\Omega)}^2 dt \le c_1 \qquad \text{for all } \varepsilon \in (0,1) \tag{5.17}$$

and

$$\int_{T_1}^\infty \int_\Omega \Big(v_\varepsilon(x,t) - \overline{v_\varepsilon(\cdot,t)} \Big)^2 dxdt \le c_2 \qquad \text{for all } \varepsilon \in (0,1). \tag{5.18}$$

Since Proposition 1.1 ensures that with $(\varepsilon_j)_{j\in\mathbb{N}}$ as given there we have $\|u_\varepsilon(\cdot,t) - u_\star\|_{L^1(\Omega)} \to \|u(\cdot,t) - u_\star\|_{L^1(\Omega)}$ for a.e. $t > T_1$ and $\varepsilon = \varepsilon_j \searrow 0$, through Fatou's lemma we infer from (5.17) that

$$\int_{T_1}^\infty \|u(\cdot,t) - u_\star\|_{L^1(\Omega)}^2 dt < \infty, \tag{5.19}$$

and similarly we conclude from (5.18) that

$$\int_{T_1}^\infty \int_\Omega \Big(v(x,t) - \overline{v(\cdot,t)} \Big)^2 dxdt < \infty, \tag{5.20}$$

because $v_\varepsilon \to v$ a.e. in $\Omega \times (T_1, \infty)$ as $\varepsilon = \varepsilon_j \searrow 0$ by (1.6), and because $\overline{v_\varepsilon(\cdot,t)} \to \overline{v(\cdot,t)}$ for a.e. $t > T_1$ and $\varepsilon = \varepsilon_j \searrow 0$ due to (1.7).

Apart from that, we may employ Corollary 4.1 to obtain $T_2 = T_2(u_0, v_0) > T_1$ and $c_3 = c_3(u_0, v_0) > 0$ such that again with $q = \min\{\alpha, 2\}$ we have

$$\int_t^{t+1} \int_\Omega H^q(u_\varepsilon) \le c_3 \qquad \text{for all } t > T_2,$$

which once more by means of Fatou's lemma entails that

$$\int_t^{t+1} \int_\Omega H^q(u) \le c_3 \qquad \text{for all } t > T_2, \tag{5.21}$$

because $H(u_\varepsilon) \to H(u)$ a.e. in $\Omega \times (T_2, \infty)$ as $\varepsilon = \varepsilon_j \searrow 0$ according to (1.5).

Now a combination of (5.19), (5.20) and (5.21) enables us to pick $(t_k)_{k\in\mathbb{N}} \subset (T_2, \infty) \setminus N_1$ with the properties that as $k \to \infty$ we have $t_k \to \infty$ and

$$\|u(\cdot,t_k) - u_\star\|_{L^1(\Omega)} \to 0 \tag{5.22}$$

as well as

$$\|v(\cdot,t_k) - \overline{v(\cdot,t_k)}\|_{L^2(\Omega)} \to 0, \tag{5.23}$$

and that
$$\int_\Omega H^q(u(\cdot,t_k)) \le c_3 \qquad \text{for all } k \in \mathbb{N}, \tag{5.24}$$
where on the basis of (5.22) we may assume upon extracting a subsequence if necessary that also
$$u(\cdot,t_k) \to u_\star \quad \text{a.e. in } \Omega \qquad \text{as } k \to \infty.$$
Since the latter clearly entails that $H(u(\cdot,t_k)) \to H(u_\star) = 0$ a.e. in Ω as $k \to \infty$, and since (5.24) along with the the inequality $q > 1$ warrants equi-integrability of $(H(u(\cdot,t_k)))_{k\in\mathbb{N}}$ over Ω, an application of the Vitali convergence theorem shows that indeed (5.14) is valid, whereas (5.15) results from (5.14) due to the fact that $\overline{v(\cdot,t_k)} \to v_\star$ as $k \to \infty$ by (5.16). □

5.3 Convergence. Proof of Theorem 1.2

We now only need to make use of the downward monotonicity of $\mathcal{F}$ outside $N_\star$ to obtain its genuine decay in the following flavor.

Corollary 5.4 *Let $\alpha > 1$ be such that $\alpha \ge 2 - \frac{2}{n}$, let $D > 0, d > 0$, $\rho > 0$, $\mu > 0$, $\kappa > 0$, $\lambda > 0$ and $\chi > 0$ be such that (3.22) holds, and given (u_0, v_0) fulfilling (1.3), let $N_\star = N_\star(u_0, v_0)$, $T_\star = T_\star(u_0, v_0) > 0$ and $\mathcal{F}$ be as in Corollary 4.3. Then*
$$\mathcal{F}(t) \to 0 \qquad \text{as } (T_\star, \infty) \setminus N_\star \ni t \to \infty. \tag{5.25}$$

Proof. We only need to observe that according to Lemma 5.3, given any $\eta > 0$ we can choose $t_0 = t_0(\eta) \in (T_\star, \infty) \setminus N_\star$ suitably large such that
$$\int_\Omega H(u(\cdot,t_0)) \le \frac{\eta}{2},$$
and that with b as in Corollary 4.3 we have
$$\int_\Omega \Big(v(\cdot,t) - \overline{v(\cdot,t)}\Big)^2 \le \frac{\eta}{b},$$
and that thus, by (4.7),
$$\mathcal{F}(t_0) \le \frac{\eta}{2} + \frac{b}{2} \cdot \frac{\eta}{b} = \eta.$$
Thefeore, namely, (4.8) ensures that
$$\mathcal{F}(t) \le \eta \qquad \text{for all } t \in (t_0, \infty) \setminus N_\star,$$
and that therefore (5.25) holds. □

Thanks to the auxiliary statement on H from Lemma 3.2, this can be turned into our main result asserting convergence of u and v in the claimed Lebesgue space topologies:

Proof of Theorem 1.2. With $K_1 > 0$ and $K_2 > 0$ taken from Lemma 3.5 and Lemma 3.7, we define $C(\Omega) := \frac{1}{K_1 K_2}$, and assuming (1.12) we let $T_\star > 0$ and $N_\star \subset (T_\star, \infty)$ be as given by Corollary 4.3, to then infer from Corollary 5.4 that then the global generalized solution (u,v) of (1.2) from Proposition 1.1 satisfies

$$\int_\Omega H(u(\cdot,t)) \to 0 \qquad \text{as } (T_\star,\infty)\setminus N_\star \ni t\to\infty \tag{5.26}$$

and

$$\int_\Omega \Big(v(\cdot,t)-\overline{v(\cdot,t)}\Big)^2 \to 0 \qquad \text{as } (T_\star,\infty)\setminus N_\star \ni t\to\infty. \tag{5.27}$$

Apart from that taking a null set $N_1\subset(0,\infty)$ such that in accordance with Lemma 5.2 we have

$$\overline{v(\cdot,t)} \to v_\star \qquad \text{as } (0,\infty)\setminus N_1 \ni t\to\infty,$$

from (5.27) we infer that (1.14) holds if we let $T:=T_\star$ and $N:=N_\star\cup N_1$. Since (5.26) together with Lemma 3.2 ensures that

$$\int_\Omega |u(\cdot,t)-u_\star| \to 0 \qquad \text{as } (T_\star,\infty)\setminus N_\star \ni t\to\infty,$$

we furthermore see that also (1.13) is valid for this choice of T and N. □

Appendix: The underlying solution concept

The following definition essentially adapts the one underlying the existence theory from [47] to the case of arbitrary positive parameters D,d,χ,κ and λ. We accordingly may refrain from detailing a discussion about how far this concept is consistent with that of classical solvability here, and rather refer the reader to ([47] and the related precedent in [20] instead.

Definition 5.5 *Let*

$$\begin{cases} u\in L^\alpha_{loc}(\overline{\Omega}\times[0,\infty)) \qquad \textit{and} \\ v\in L^1_{loc}([0,\infty);W^{1,1}(\Omega)) \end{cases} \tag{5.28}$$

be nonnegative. Then we call (u,v) a global generalized solution of (1.2) *if*

$$-\int_0^\infty\int_\Omega v\varphi_t - \int_\Omega v_0\varphi(\cdot,0) = -d\int_0^\infty\int_\Omega \nabla v\cdot\nabla\varphi - \kappa\int_0^\infty\int_\Omega v\varphi + \lambda\int_0^\infty\int_\Omega u\varphi \tag{5.29}$$

for all $\varphi\in c_0^\infty(\overline{\Omega}\times[0,\infty))$, if

$$\int_\Omega u(\cdot,t) \le \int_\Omega u_0 + \int_0^t\int_\Omega (\rho u-\mu u^\alpha) \qquad \textit{for a.e. } t>0, \tag{5.30}$$

and if there exist functions $\phi\in C^2([0,\infty))$, $\psi\in C^2([0,\infty))$ and $\Phi\in C^2([0,\infty))$ such that

$$\phi'<0, \quad \psi>0 \quad \textit{and} \quad \phi''>0 \qquad \textit{on } [0,\infty), \tag{5.31}$$

that

$$\Phi' = \sqrt{\phi''} \qquad \textit{on } [0,\infty), \tag{5.32}$$

that

$$\Big\{d\phi(u)\psi''(v)-\frac{(D+d)^2}{4D}\frac{\phi'^2(u)}{\phi''(u)}\cdot\frac{\psi'^2(v)}{\psi(v)}-\frac{\chi^2}{4D}u^2\phi''(u)\psi(v)+\frac{(d-D)\chi}{2D}u\phi'(u)\psi'(v)\Big\}|\nabla v|^2,$$

$$u\phi'(u)\psi(v)|\nabla v|,\quad \phi(u)\psi'(v)|\nabla v| \quad \textit{and} \quad \frac{\Phi(u)\phi'(u)}{\sqrt{\phi''(u)}}\psi'(v)|\nabla v| \qquad \textit{as well as}$$

$$u^\alpha\phi'(u)\psi(v),\quad v\phi(u)\psi'(v) \quad \textit{and} \quad u\phi(u)\psi'(v) \quad \textit{belong to } L^1_{loc}(\overline{\Omega}\times[0,\infty)), \tag{5.33}$$

that

$$\Phi(u)\sqrt{\psi(v)}\in L^2_{loc}([0,\infty);W^{1,2}(\Omega)), \tag{5.34}$$

and that for each nonnegative $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$, the inequality

$$\begin{aligned}
&-\int_0^\infty\int_\Omega\phi(u)\psi(v)\varphi_t-\int_\Omega\phi(u_0)\psi(v_0)\varphi(\cdot,0)\\
\le\;& -D\int_0^\infty\int_\Omega\left|\nabla\Big(\Phi(u)\sqrt{\psi(v)}\Big)+\Big\{\frac{D+d}{2D}\frac{\phi'(u)}{\sqrt{\phi''(u)}}\cdot\frac{\psi'(v)}{\sqrt{\psi(v)}}-\frac{1}{2}\Phi(u)\frac{\psi'(v)}{\sqrt{\psi(v)}}-\frac{\chi}{2D}u\sqrt{\phi''(u)}\cdot\sqrt{\psi(v)}\Big\}\nabla v\right|^2\varphi\\
&-\int_0^\infty\int_\Omega\Big\{d\phi(u)\psi''(v)-\frac{(D+d)^2}{4D}\frac{\phi'^2(u)}{\phi''(u)}\cdot\frac{\psi'^2(v)}{\psi(v)}-\frac{\chi^2}{4D}u^2\phi''(u)\psi(v)+\frac{(d-D)\chi}{2D}u\phi'(u)\psi'(v)\Big\}\cdot|\nabla v|^2\varphi\\
&-D\int_0^\infty\int_\Omega\frac{\phi'(u)}{\sqrt{\phi''(u)}}\sqrt{\psi(v)}\nabla\Big(\Phi(u)\sqrt{\psi(v)}\Big)\cdot\nabla\varphi\\
&+\int_0^\infty\int_\Omega\Big\{\chi u\phi'(u)\psi(v)-d\phi(u)\psi'(v)+\frac{D}{2}\frac{\Phi(u)\phi'(u)}{\sqrt{\phi''(u)}}\psi'(v)\Big\}\nabla v\cdot\nabla\varphi\\
&+\int_0^\infty\int_\Omega\Big\{(\rho u-\mu u^\alpha)\phi'(u)\psi(v)-\kappa v\phi(u)\psi'(v)+\lambda u\phi(u)\psi'(v)\Big\}\cdot\varphi
\end{aligned} \tag{5.35}$$

holds.

According to the analysis detailed in [47], with some appropriately chosen $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ fulfilling $\varepsilon_j\searrow 0$ as $j\to\infty$, and with some null set $N\subset(0,\infty)$, the corresponding solutions $(u_\varepsilon,v_\varepsilon)$ of (2.1) indeed satisfy (1.5)-(1.7) with some pair (u,v) which complies with (5.28)-(5.35) if we let

$$\phi(s):=(s+1)^{-r},\quad \Phi(s):=-2\sqrt{\frac{r+1}{r}}(s+1)^{-\frac{r}{2}} \quad\text{and}\quad \psi(\overline{s}):=e^{-\theta\overline{s}},\qquad s\ge 0,\ \overline{s}\ge 0, \tag{5.36}$$

and herein firstly take $r>0$ suitably small and then $\theta>0$ adequately large such that

$$\Big\{d-\frac{(D+d)^2}{4D}\cdot\frac{r}{r+1}\Big\}\cdot\theta^2-\frac{(D-d)\chi r}{2D}\cdot\theta-\frac{r(r+1)}{4D}>0.$$

Acknowledgement. The author is very grateful to the anonymous reviewers, especially one of whom gave numerous useful suggestions that lead to a substantial improvement of this manuscript. The author moreover acknowledges support of the *Deutsche Forschungsgemeinschaft* in the context of the project *Emergence of structures and advantages in cross-diffusion systems* (No. 411007140, GZ: WI 3707/5-1).

References

[1] Bai, X., Winkler, M.: *Equilibration in a fully parabolic two-species chemotaxis system with competitive kinetics.* Indiana Univ. Math. J. **65**, 553-583 (2016)

[2] Baras, P, Cohen, L.: *Complete blow-up after T_{max} for the solution of a semilinear heat equation.* J. Funct. Anal. **71**, 142-174 (1987)

[3] Biler, P.: *Radially symmetric solutions of a chemotaxis model in the plane – the supercritical case.* In: Parabolic and Navier-Stokes equations. Part 1, vol. 81 of *Banach Center Publ.*, pp. 31–42. Polish Acad. Sci. Inst.

[4] Blanchet, A., Dolbeault, J., Perthame, B.: *Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions.* Electron. J. Differential Eq., No. 44, 32 pp. (2006)

[5] Cao, X.: *Large time behavior in the logistic Keller-Segel model via maximal Sobolev regularity.* Discrete Contin. Dyn. Syst. Ser. B **22**, 3369-3378 (2017)

[6] Chaplain, M.A.J., Lolas, G.: *Mathematical modelling of cancer cell invasion of tissue: the role of the urokinase plasminogen activation system.* Math. Mod. Meth. Appl. Sci. **15**, 1685-1734 (2005)

[7] Ding, M., Wang, W., Zhou, S., Zheng, S.: *Asymptotic stability in a fully parabolic quasilinear chemotaxis model with general logistic source and signal production.* J. Differential Eq. **268**, 6729-6777 (2020)

[8] Ding, M., Wang, W., Zhou, S.: *Global existence of solutions to a fully parabolic chemotaxis system with singular sensitivity and logistic source.* Nonlinear Anal. Real World Appl. **49**, 286-311 (2019)

[9] Fuest, M.: *Finite-time blow-up in a two-dimensional Keller-Segel system with an environmental dependent logistic source.* Nonlinear Anal. Real World Appl. **52**, 103022 (2019)

[10] Fuest, M.: *Critical parameters for finite-time blow-up in chemotaxis systems with quadratic dampening.* In preparation

[11] Galaktionov, V.A., Vázquez, J.L.: *Continuation of blow-up solutions of nonlinear heat equations in several space dimensions.* Comm. Pure Appl. Math. **50**, 1-67 (1997)

[12] Herrero, M.A., Velázquez, J.J.L.: *A blow-up mechanism for a chemotaxis model.* Ann. Scu. Norm. Super. Pisa Cl. Sci. **24**, 663-683 (1997)

[13] Jäger, W., Luckhaus, S.: *On explosions of solutions to a system of partial differential equations modelling chemotaxis.* Trans. Amer. Math. Soc. **329**, 819-824 (1992)

[14] Jin, H.-Y., Xiang, T.: *Chemotaxis effect vs. logistic damping on boundedness in the 2-D minimal Keller-Segel model.* C. R. Math. Acad. Sci. Paris **356**, 875-885 (2018)

[15] Kang, K., Stevens, A.: *Blowup and global solutions in a chemotaxis-growth system.* Nonlinear Anal. **135**, 57-72 (2016)

[16] Lankeit, J.: *Chemotaxis can prevent thresholds on population density.* Discrete Contin. Dyn. Syst. B **20**, 1499-1527 (2015)

[17] Lankeit, J.: *Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source.* J. Differential Eq. **258**, 1158-1191 (2015)

[18] Lankeit, E., Lankeit, J.: *Classical solutions to a logistic chemotaxis model with singular sensitivity and signal absorption.* Nonlinear Anal. Real World Appl. **46**, 421-445 (2019)

[19] Lankeit, J., Viglialoro, G.: *Global Existence and Boundedness of Solutions to a Chemotaxis-Consumption Model with Singular Sensitivity.* Acta Appl. Math **167**, 75-97 (2020)

[20] Lankeit, J., Winkler, M.: *A generalized solution concept for the KellerSegel system with logarithmic sensitivity: global solvability for large nonradial data.* Nonlinear Differential Eq. Appl. **24**, 49 (2017)

[21] Lin, K., Mu, C., Zhong, H.: *A blow-up result for a quasilinear chemotaxis system with logistic source in higher dimensions.* J. Math. Anal. Appl. **464**, 435-455 (2018)

[22] Lopez-Gomez, J., Quittner, P.: *Complete and energy blow-up in indefinite superlinear parabolic problems.* Discrete Contin. Dyn. Syst. **14**, 169-186 (2006)

[23] Luckhaus, S., Sugiyama, Y., Velázquez, J.J.L.: *Measure valued solutions of the 2D Keller-Segel system.* Arch. Ration. Mech. Anal. **206**, 31-80 (2012)

[24] Nagai, T.: *Blowup of Nonradial Solutions to Parabolic-Elliptic Systems Modeling Chemotaxis in Two-Dimensional Domains.* J. Inequal. Appl. **6**, 37-55 (2001)

[25] Nakaguchi, E., Osaki, K.: *Global existence of solutions to a parabolic-parabolic system for chemotaxis with weak degradation.* Nonlinear Anal. **74** (1), 286-297 (2011)

[26] Osaki, K., Tsujikawa, T., Yagi, A., Mimura, M.: *Exponential attractor for a chemotaxis-growth system of equations.* Nonlinear Anal. **51**, 119-144 (2002)

[27] Osaki, K., Yagi, A.: *Finite dimensional attractor for one-dimensional Keller-Segel equations.* Funkc. Ekvacioj, Ser. Int. **44** (3), 441-469 (2001)

[28] Painter, K.J., Hillen, T.: *Spatio-Temporal Chaos in a Chemotaxis Model.* Physica D **240**, 363-375 (2011)

[29] Painter, K.J., Maini, P.K., Othmer, H.G.: *Development and applications of a model for cellular response to multiple chemotactic cues.* J. Math. Biol. 41 (4) 285-314 (2000)

[30] Ren, G., Liu, B.: *Global existence of bounded solutions for a quasilinear chemotaxis system with logistic source.* Nonlinear Anal. Real World Appl. **46**, 545-582 (2019)

[31] Salako, R., Shen, W.: *Global existence and asymptotic behavior of classical solutions to a parabolic-elliptic chemotaxis system with logistic source on $\mathbb{R}^N$.* J. Differential Eq. **262**, 5635-5690 (2017)

[32] Salako, R., Shen, W.: *Parabolic-elliptic chemotaxis model with space-time-dependent logistic sources on $\mathbb{R}^n$. I. Persistence and asymptotic spreading.* Math. Mod. Meth. Appl. Sci. **28**, 2237-2273 (2018)

[33] Souplet, Ph., Winkler, M.: *Blow-up profiles for the parabolic-elliptic Keller-Segel system in dimensions $n \geq 3$.* Comm. Math. Phys. **367**, 665-681 (2019)

[34] Szymańska, Z., Morales Rodrigo, C., Lachowicz, M., Chaplain, M.A.: *Mathematical modelling of cancer invasion of tissue: The role and effect of nonlocal interactions.* Math. Mod. Meth. Appl. Sci. **19**, 257-281 (2009)

[35] Viglialoro, G.: *Very weak global solutions to a parabolic-parabolic chemotaxis-system with logistic source.* J. Math. Anal. Appl. **439**, 197-212 (2016)

[36] Viglialoro, G., Woolley, T.E.: *Eventual smoothness and asymptotic behaviour of solutions to a chemotaxis system perturbed by a logistic growth.* Discrete Contin. Dyn. Syst. Ser. B **23**, 3023-3045 (2018)

[37] Waldeland, J.O., Evje, S.: *A multiphase model for exploring tumor cell migration driven by autologous chemotaxis.* Chemical Engineering Science **191**, 268-287 (2018)

[38] Wang, Y., Winkler, M., Xiang, Z.: *The fast signal diffusion limit in Keller-Segel(-fluid) systems.* Calc. Var. Partial Differential Eq. **58**, 196 (2019)

[39] Winkler, M.: *Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source.* Comm. Partial Differential Eq. **35**, 1516-1537 (2010)

[40] Winkler, M.: *Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction.* J. Math. Anal. Appl. **384**, 261-272 (2011)

[41] Winkler, M: *Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system.* J. Math. Pures Appl. **100**, 748-767 (2013), `arXiv:1112.4156v1`

[42] Winkler, M.: *Global asymptotic stability of constant equilibria in a fully parabolic chemotaxis system with strong logistic dampening.* J. Differential Eq. **257**, 1056-1077 (2014)

[43] Winkler, M.: *How far can chemotactic cross-diffusion enforce exceeding carrying capacities?* J. Nonlinear Sci. **24**, 809-855 (2014)

[44] Winkler, M.: *Emergence of large population densities despite logistic growth restrictions in fully parabolic chemotaxis systems.* Discrete Contin. Dyn. Syst. Ser. B **22**, 2777-2793 (2017)

[45] Winkler, M.: *Finite-time blow-up in low-dimensional Keller-Segel systems with logistic-type superlinear degradation.* Z. Angew. Math. Phys. **69**, Art. 69, 40 pp. (2018)

[46] Winkler, M.: *The role of superlinear damping in the construction of solutions to drift-diffusion problems with initial data in L^1.* Adv. Nonlin. Anal. **9**, 526-566 (2020)

[47] Winkler, M.: *L^1 solutions to parabolic Keller-Segel systems involving arbitrary superlinear degradation.* Preprint

[48] Woodward, D.E., Tyson, R., Myerscough, M.R., Murray, J.D., Budrene, E.O., Berg, H.C.: *Spatiotemporal patterns generated by Salmonella typhimurium.* Biophys. J. **68** (5), 2181-2189 (1995)

[49] Xiang, T.: *How strong a logistic damping can prevent blow-up for the minimal KellerSegel chemotaxis system?* J. Math. Anal. Appl. **459**, 1172-1200 (2018)

[50] Xiang, T.: *Chemotactic aggregation versus logistic damping on boundedness in the 3D minimal Keller-Segel model.* SIAM J. Appl. Math. **78**, 2420-2438 (2018)

[51] Xiang, T.: *Sub-logistic source can prevent blow-up in the 2D minimal Keller-Segel chemotaxis system.* J. Math. Phys. **59**, 081502 (2018)

[52] Xiang, T.: *Dynamics in a parabolic-elliptic chemotaxis system with growth source and nonlinear secretion.* Comm. Pure Appl. Anal. **18**, 255-284 (2019)

[53] Yan, J., Fuest, M.: *When do Keller-Segel systems with heterogeneous logistic sources admit generalized solutions?* Preprint

[54] Zhigun, A.: *Generalized global supersolutions with mass control for systems with taxis.* SIAM J. Math. Anal. **51**, 2425-2443 (2019)

[55] Zhuang, M., Wang, W., Zheng, S.: *Boundedness in a fully parabolic chemotaxis system with logistic-type source and nonlinear production.* Nonlinear Anal. Real World Appl. **47**, 473-483 (2019)