Can simultaneous density-determined enhancement of diffusion and cross-diffusion foster boundedness in Keller-Segel type systems involving signal-dependent motilities?

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Abstract

The reaction-(cross-)diffusion system

$$\begin{cases} u_t = \Delta(u^m \phi(v)), & x \in \Omega, \ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, \ t > 0, \end{cases} \tag{0.1}$$

is considered under no-flux boundary conditions in smoothly bounded convex domains $\Omega \subset \mathbb{R}^n$, where $m \geq 1$ and $n \geq 2$, and where ϕ generalizes the prototype obtained on letting

$$\phi(v) = a + b(v+d)^{-\alpha}, \qquad v > 0,$$

with $a \geq 0, b > 0, d \geq 0$ and $\alpha \geq 0$.

In this framework, it is firstly seen that if

$$m > \frac{n}{2}, \qquad \alpha < \frac{nm-2}{n-2} \qquad \text{and} \qquad \alpha < \frac{2(m+1)}{n-2},$$

then finite-time blow-up is excluded in the sense that for all suitably regular initial data an associated initial-boundary value problem admits a globally defined weak solution (u,v) with u being locally bounded in $\overline{\Omega} \times [0,\infty)$. Under the assumption that additionally

$$\alpha < \frac{(n+2)m - n + 2}{2(n-2)},$$

these solutions are moreover shown to be bounded throughout $\Omega \times (0,\infty)$ in both components.

In view of results known for the case $m = 1$, this particularly indicates that increasing m in (0.1) goes along with a certain regularizing effect despite the circumstance that thereby both the diffusion and the cross-diffusion mechanisms implicitly contained in (0.1) are simultaneously enhanced.

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1 Introduction

We consider the parabolic system

$$\begin{cases} u_t = \Delta(u^m \phi(v)), \\ v_t = \Delta v - v + u, \end{cases} \tag{1.1}$$

with $m \geq 1$ and a given parameter function $\phi : (0,\infty) \to (0,\infty)$. In recent literature on biomathematical modeling, systems of this form have been proposed as possible descriptions of the collective behavior within bacterial populations in certain situations in which, according to experimental findings, the ability to perform random diffusive movement is significantly influenced by a signal substance secreted by the cells themselves, in essence particularly leading to reduced motility of individuals in regions of large signal concentrations ([8], [19]).

Accordingly guided by the ambition to understand (1.1) in the presence of functions ϕ which depend on the chemical concentration $v = v(x,t)$ in a nonincreasing manner, e.g. by generalizing simple functional laws of the form

$$\phi(v) = a + b(v+d)^{-\alpha}, \qquad v > 0, \tag{1.2}$$

with $a \geq 0, b > 0, d \geq 0$ and $\alpha \geq 0$, the mathematical literature on (1.1) and close relatives has as yet concentrated on the case $m = 1$ in which with regard to the population density $u = u(x,t)$ the considered diffusion process is essentially Brownian. In such settings and when posed along with homogeneous no-flux boundary conditions in n-dimensional bounded domains, (1.1) indeed has been found to admit global bounded classical solutions for widely arbitrary initial data when $n = 2$ and ϕ is suitably regular and uniformly positive, which in the context of (1.2) essentially reduces to the requirement that a be positive ([28]). In three- and higher-dimensional frameworks, such assumptions on non-degeneracy of diffusion at large values of v are merely known to warrant global existence of weak solutions, with a statement on smooth solvability and boundedness available only under appropriate restrictions on the size of the initial data ([28]). In degenerate situations associated with the choice $a = 0$ in (1.2), results on global existence of classical solutions seem limited to the case when $n = 1$ and $\alpha > 0$ is arbitrary ([7]), whereas in multi-dimensional settings the literature is yet apparently restricted to weak solution frameworks, and to corresponding findings exclusively in the cases when either $n = 2$ and $\alpha < 2$, or $n = 3$ and $\alpha < \frac{4}{3}$ ([7]).

In this case $m = 1$, farther-reaching information especially on boundedness properties seems available only for variants of (1.1) either containing a simplified signal evolution mechanism, or additionally including the dissipative action of logistic-type growth restrictions. For instance, in a simplified variant of (1.1) in which the second equation therein is replaced with the elliptic equation $0 = \Delta v - v + u$, global bounded classical solutions are known to exist for all reasonably regular initial data whenever $n \geq 1$ and ϕ satisfies (1.2) with $a = 0, b > 0, d = 0$ and $\alpha < \frac{2}{(n-2)_+}$ ([1]). If logistic terms of the form $\rho u - \mu u^2$ with $\rho > 0$ and $\mu > 0$ are added to the first equation in the fully parabolic model (1.1) with $m = 1$, actually any choice of $\alpha > 0$ in (1.2) leads to global existence of bounded smooth solutions when $n = 3$ ([15], cf. also [30]), and even some statements on stabilization toward homogeneous equilibria available if μ is suitably large ([15], [20]; cf. also [22], [23] and [21] for similar results on related systems involving superquadratic degradation terms).

The case $m > 1$: Enhancement of diffusion and cross-diffusion in a Keller-Segel system. The purpose of the present study is to provide a first step toward an understanding of how far an increase of the parameter m may regularize (1.1) in the sense of blow-up suppression, especially in situations in which for the case $m = 1$ either only weak and possibly quite irregular solutions are known to exist, or any result on global solvability is lacking at all.

To put this question into a slightly broader perspective, let us observe that (1.1) can be rewritten according to

$$\begin{cases} u_t = \nabla \cdot (D(u,v)\nabla u) - \nabla \cdot (S(u,v)\nabla v), \\ v_t = \Delta v - v + u, \end{cases} \tag{1.3}$$

with

$$D(u,v) := mu^{m-1}\phi(v) \quad \text{and} \quad S(u,v) := -u^m\phi'(v) \qquad \text{for } (u,v) \in [0,\infty)^2, \tag{1.4}$$

and that hence (1.1) formally corresponds to a Keller-Segel type cross-diffusion system with diffusion and cross-diffusion rates which, besides depending on u whenever $m > 1$, are both explicitly influenced by v. Now a rich literature indicates that with regard to the occurrence of singularity phenomena, at least in some subclasses of (1.3) with more general ingredients the behavior of $S(u,v)$ relative to $D(u,v)$ especially at large values of u plays a decisive role. Specifically, when $D = D(u)$ and $S = S(u)$ are independent of the signal concentration, the corresponding Neumann problem for (1.3) is known to admit globally bounded smooth solutions for all suitably regular initial data if D and S, besides complying with some technical assumptions which in substance mainly reduce to the mere requirement that $D(u)$ neither grows nor decays faster than algebraically as $u \to \infty$, satisfy the subcriticality inequality

$$\limsup_{u\to\infty} \left\{ u^{-\frac{2}{n}+\eta} \frac{S(u)}{D(u)} \right\} < \infty \qquad \text{for some } \eta > 0. \tag{1.5}$$

A certain optimality of this result ([27], [13], [14]), which has partially been extended to cases of exponentially decaying D ([6], [35]), is indicated by several findings on the occurrence of some unbounded radial solutions in balls under assumptions on D and S which essentially complement that in (1.4) by supposing that, in different concrete flavors available in the literature, $\frac{S(u)}{D(u)}$ grows somewhat faster than $u^{\frac{2}{n}}$ as $u \to \infty$ ([32], [5], [36], [34]).

In contrast to this quite comprehensive picture, only little seems known with respect to the obvious question to which extent features of the latter dichotomy-like flavor persist also in the presence of signal dependencies in the migration rates entering (1.3), with the few examples available in the literature mainly concentrating on the derivation of global existence and boundedness results in the application-relevant special case when $D \equiv 1$ and $S(u,v) = \chi\frac{u}{v}$ with $\chi > 0$ (cf. e.g. [18], [3], [33] and also [10], for instance). In particular, the literature seems to have left widely unclarified how for the ratio $\frac{D}{S}$ might retain relevance in this regard also when D and S depend on v.

Main results. When viewed against this background, the outcome of this work may be summarized as indicating that in the special setup defined through (1.4), with regard to boundedness properties of solutions the system (1.3) might exhibit features quite different from those known for the signal-independent case in which $(D,S) = (D,S)(u)$. Namely, we shall see that despite the fact that then $\frac{S(u,v)}{D(u,v)} = h(v)u$ with $h(v) = \frac{-\phi'(v)}{m\phi(v)}$ for $(u,v) \in (0,\infty)^2$, and that thus the findings around (1.5) suggest linear and hence explosion-supercritical growth of $\frac{S}{D}$ at large values of u, in the presence of suitably large $m > 1$ the particular system structure in (1.1), and especially the precise link between

D and S in (1.3) via (1.4), allows for the construction of global solutions with favorable boundedness properties for arbitrarily large initial data, and under quite mild assumptions on ϕ which inter alia include degenerate cases. Specifically, we shall consider

$$\begin{cases} u_t = \Delta(u^m \phi(v)), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \frac{\partial(u^m\phi(v))}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.6}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, where $n \geq 2$ and $m > 1$, and where ϕ generalizes the prototypical choice from (1.2) in that

$$\phi \in C^3((0,\infty)) \text{ is such that } \phi > 0 \text{ in } (0,\infty), \tag{1.7}$$

that

$$\sup_{v>s_0} \Big\{ \phi(v) + |\phi'(v)| \Big\} < \infty \quad \text{for all } s_0 > 0, \tag{1.8}$$

and that

$$\inf_{v>1} v^\alpha \phi(v) > 0 \tag{1.9}$$

with a certain number $\alpha \geq 0$.

The first of our main results then asserts global existence of a solution locally bounded in $\overline{\Omega} \times [0,\infty)$, provided that m is suitably large and α is appropriately small:

Theorem 1.1 *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with smooth boundary, and suppose that*

$$m > \frac{n}{2}, \tag{1.10}$$

and that ϕ satisfies (1.7), (1.8) and (1.9) with some $\alpha \geq 0$ satisfying

$$\alpha < \frac{nm-2}{n-2} \tag{1.11}$$

and

$$\alpha < \frac{2(m+1)}{n-2}. \tag{1.12}$$

Then for any pair (u_0, v_0) of initial data $u_0 \in W^{1,\infty}(\Omega)$ and $v_0 \in W^{1,\infty}(\Omega)$ fulfilling $u_0 \geq 0$, $u_0 \not\equiv 0$ and $v_0 > 0$ in $\overline{\Omega}$, the problem (1.6) admits at least one global weak solution (u,v), in the sense of Definition 2.1 below, which has the additional properties that $v \in C^0(\overline{\Omega} \times [0,\infty))$, and that for each $T > 0$ one can find $C(T) > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C(T) \qquad \text{for a.e. } t \in (0,T) \tag{1.13}$$

and

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \leq C(T) \qquad \text{for all } t \in (0,T). \tag{1.14}$$

Under a slightly stronger assumption on the parameter α, each of these solutions is bounded actually throughout $\Omega \times (0,\infty)$.

Theorem 1.2 *Suppose that beyond the assumptions from Theorem 1.1 we have*

$$\alpha < \frac{(n+2)m-n+2}{2(n-2)}. \tag{1.15}$$

Then there exists $C>0$ such that the global weak solution of (1.6) from Theorem 1.1 additionally satisfies

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for a.e. } t>0 \tag{1.16}$$

and

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t>0. \tag{1.17}$$

Remark. i) We emphasize that in the particular case $n=2$, any choice of $m>1$ and $\alpha \ge 0$ is admissible both in Theorem 1.1 and Theorem 1.2, hence implying that in this planar situation, any superlinear porous medium type enhancement of diffusion and cross-diffusion considerably increases the knowledge about global boundedness in comparison to the findings achieved in [7] only for $\alpha<2$ and only in contexts of possibly unbounded weak solutions.
ii) If α satisfies both (1.11) and (1.12), then necessarily $\alpha < \frac{1}{2}\cdot(\frac{nm-2}{n-2}+\frac{2(m+1)}{n-2}) = \frac{(n+2)m}{2(n-2)}$, so that the hypotheses from Theorem 1.2 are indeed stronger than those underlying Theorem 1.1 whenever $n\ge 3$.
iii) In view of the diffusion degeneracy near points where $u=0$ whenever $m>1$, it seems that classical solutions to (1.6) can in general not be expected, and that hence resorting to appropriately generalized frameworks of solvability, such as done in Theorem 1.1, indeed appears in order.

Main ideas. In order to suitably capture the structural information encoded in (1.6) and the particular liaison between diffusion and cross-diffusion therein, in a first step we shall pursue the goal of deriving some fundamental regularity information through a duality-based argument in the style of reasonings which are quite well-established in the context of reaction-diffusion systems free of cross-diffusion ([17], [4]), but quite a simple form of which has also been underlying the analysis performed in [28] for the case $m=1$. For general $m\ge 1$ and ϕ satisfying (1.7) and (1.9) with some $\alpha\ge 0$, this will firstly lead to an inequality of the form

$$\frac{d}{dt}\int_\Omega |A^{-\frac{1}{2}}(u_\varepsilon+1)|^2 + \frac{1}{C}\int_\Omega u_\varepsilon^{m+1}v_\varepsilon^{-\alpha} \le C\int_\Omega |A^{-1}(u_\varepsilon+1)|^{m+1} + C \tag{1.18}$$

with a certain $C=C(m,\phi)>0$, where A denotes the self-adjoint realization of $-\Delta+1$ under homogeneous Neumann boundary conditions in $L^2(\Omega)$, and where $(u_\varepsilon,v_\varepsilon)$ denotes the global solution of a suitably regularized variant of (1.6) for $\varepsilon\in(0,1)$ (Lemma 3.1). By means of suitable interpolation arguments, for $T>0$ this will be seen to imply estimates of the form

$$\int_t^{t+1}\int_\Omega u_\varepsilon^{m+1}v_\varepsilon^{-\alpha} \le K(T) \qquad \text{for all } t\in(0,T) \text{ and } \varepsilon\in(0,1), \tag{1.19}$$

provided that $\alpha\ge 0$ satisfies (1.12) and the condition $\alpha<\frac{nm}{n-2}$ slightly stronger than (1.11), with $K(T)$ actually independent of T if additionally (1.15) holds (Lemma 3.5).

Next relying on the hypotheses on α from Theorem 1.1 and Theorem 1.2 in their full strength, we shall thereafter see that under the additional assumption $m > \frac{n}{2}$ it is possible to apply an iterative L^p estimation procedure to the respective second equation in order to turn (1.19) into bounds for $\int_\Omega v_\varepsilon^p$ with arbitrary $p > 1$ (Corollary 4.4). Using that (1.19) thus actually implies corresponding boundedness properties of u_ε in space-time L^p norms with arbitrary $p \in (1, m+1)$ (Corollary 4.5), we can thereafter derive estimates for v_ε in $W^{1,q}(\Omega)$ with some $q > n$ (Lemma 4.6), which will finally be seen to entail L^∞ bounds for u_ε (Lemma 5.3). Both Theorem 1.1 and Theorem 1.2 can hence be verified on the basis of straightforward extraction procedures.

2 Preliminaries. Global approximate solutions

In order to substantiate our goal, let us begin by specifying the concept of solvability to be pursued in the sequel.

Definition 2.1 *Suppose that $m > 0$, that ϕ satisfies (1.7), and that u_0 and v_0 are nonnegative functions from $L^1(\Omega)$. Then a pair of functions*

$$\begin{cases} u \in L^1_{loc}(\overline{\Omega} \times [0,\infty)) \qquad \textit{and} \\ v \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)) \end{cases} \tag{2.1}$$

will be called a global weak solution of (1.6) *if $u \geq 0$ and $v > 0$ a.e. in $\Omega \times (0,\infty)$, if*

$$u^m \phi(v) \in L^1_{loc}(\overline{\Omega} \times [0,\infty)), \tag{2.2}$$

and if

$$-\int_0^\infty \int_\Omega u\varphi_t - \int_\Omega u_0 \varphi(\cdot,0) = \int_0^\infty \int_\Omega u^m \phi(v) \Delta\varphi \qquad \textit{for all } \varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty)) \textit{ such that } \tfrac{\partial\varphi}{\partial\nu}|_{\partial\Omega} = 0 \tag{2.3}$$

as well as

$$-\int_0^\infty \int_\Omega v\varphi_t - \int_\Omega v_0 \varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla v \cdot \nabla\varphi - \int_0^\infty \int_\Omega v\varphi + \int_0^\infty \int_\Omega u\varphi \quad \textit{for all } \varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty)). \tag{2.4}$$

Our path toward the construction of a solution in this framework will be based on a regularization of (1.6) which for convenience we plan to design in such a way that approaches well-known in the theory of quasilinear Keller-Segel type systems become applicable so as to assert global smooth solvability thereof. To this end, let us fix a number

$$M > m + 1 - \frac{2}{n}, \tag{2.5}$$

and, for $\varepsilon \in (0,1)$, consider the problem

$$\begin{cases} u_{\varepsilon t} = \varepsilon \Delta (u_\varepsilon + 1)^M + \Delta\Big(u_\varepsilon (u_\varepsilon + \varepsilon)^{m-1} \phi(v_\varepsilon)\Big), & x \in \Omega,\ t > 0, \\ v_{\varepsilon t} = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon, & x \in \Omega,\ t > 0, \\ \frac{\partial u_\varepsilon}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x,0) = u_0(x), \quad v_\varepsilon(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{2.6}$$

in which apparently the degeneracy of cell diffusion at vanishing population densities, as present in (1.6) whenever $m > 1$, is removed. Apart from that, due to (2.5) and strength of diffusion enhancement thereby induced, in each of these problems also finite-time blow-up phenomena can be ruled out, as seen in Lemme 2.4 below on the basis of the following essentially well-known global solvability feature of quite general quasilinear chemotaxis systems with subcritical sensitivities.

Lemma 2.2 *Suppose that $D \in C^2([0,\infty)^2)$ and $S \in C^2([0,\infty)^2)$ are such that $S(0,v) = 0$ for all $v \ge 0$, that*

$$k_D(u+1)^\gamma \le D(u,v) \le K_D(u+1)^\Gamma \qquad \text{for all } (u,v) \in [0,\infty)^2$$

with some $k_D > 0, K_D > 0, \gamma \in \mathbb{R}$ and $\Gamma \in \mathbb{R}$, and that there exist $C > 0$ and $\eta > 0$ such that

$$\frac{|S(u,v)|}{D(u,v)} \le C(u+1)^{\frac{2}{n}-\eta} \qquad \text{for all } (u,v) \in [0,\infty)^2.$$

Then for any nonnegative $u_0 \in W^{1,\infty}(\Omega)$ and $v_0 \in W^{1,\infty}(\Omega)$, the problem

$$\begin{cases} u_t = \nabla \cdot \Big(D(u,v)\nabla u\Big) - \nabla \cdot \Big(S(u,v)\nabla v\Big), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{2.7}$$

possesses at least one global classical solution (u,v) with

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \qquad \text{and} \\ v \in \bigcap_{q>n} C^0([0,\infty); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \end{cases} \tag{2.8}$$

and such that both u and v are nonnegative in $\overline{\Omega} \times (0,\infty)$.

Proof. Based on well-established theories of local existence and extensibility, as contained in [2] for rather general second-order cross-diffusion systems and e.g. in [36] for a more specific setting close to that in (2.7), this can be derived by performing evident minor adaptations to the a priori estimation procedure applied to merely u-dependent functions D and S in [27], so that we may refrain from giving details here. $\Box$

In order to make (2.6) accessible to the latter especially in cases when ϕ is irregular near $v = 0$, let us reformulate a favorable a priori positivity property of externally forced linear heat equations even in the presence of linear degradation, as implicitly contained in [37, Lemma 2.2] already but concretized there in a slightly different setting. This is the only place where convexity of Ω is explicitly referred to in this paper.

Lemma 2.3 *There exists $\Lambda(\Omega) > 0$ with the property that if $T > 0$, and if $h \in C^0(\overline{\Omega} \times (0,T))$ and $z \in C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty))$ are nonnegative functions fulfilling*

$$\begin{cases} z_t = \Delta z - z + h(x,t), \qquad x \in \Omega,\ t \in (0,T), \\ \frac{\partial z}{\partial \nu} = 0, \qquad x \in \partial\Omega,\ t \in (0,T), \end{cases}$$

then

$$z(x,t) \geq \Lambda(\Omega) \cdot \min\left\{ \inf_{y\in\Omega} z(y,0)\,,\ \inf_{s\in(0,T)} \int_\Omega h(\cdot,s) \right\} \qquad \textit{for all } x \in \Omega \textit{ and } t \in (0,T). \tag{2.9}$$

Proof. According to the convexity of Ω, we can find $c_1 > 0$ such that for the Neumann heat semigroup $(e^{t\Delta})_{t\geq 0}$ on Ω we have

$$e^{t\Delta}\varphi \geq c_1 \int_\Omega \varphi \qquad \text{for all } t > 1 \text{ and each nonnegative } \varphi \in C^0(\overline{\Omega})$$

([9], [12]). By nonnegativity of h and the comparison principle, in the variation-of-constants representation

$$z(\cdot,t) = e^{t(\Delta-1)}z(\cdot,0) + \int_0^t e^{(t-s)(\Delta-1)}h(\cdot,s)ds, \qquad t \in (0,T), \tag{2.10}$$

whenever $T > 2$ we can therefore estimate

$$\begin{aligned}
\int_0^t e^{(t-s)(\Delta-1)}h(\cdot,s)ds &\geq c_1 \cdot \left\{ \int_1^t e^{-(t-s)}ds \right\} \cdot \inf_{s\in(0,T)} \int_\Omega h(\cdot,s) \\
&= c_1(1-e^{-(t-1)}) \cdot \inf_{s\in(0,T)} \int_\Omega h(\cdot,s) \\
&\geq c_1(1-e^{-1}) \cdot \inf_{s\in(0,T)} \int_\Omega h(\cdot,s) \quad \text{in } \Omega \qquad \text{for all } t \in [2,T).
\end{aligned}$$

As furthermore, again by the comparison principle,

$$\begin{aligned}
e^{t(\Delta-1)}z(\cdot,0) &\geq e^{t(\Delta-1)} \cdot \inf_{y\in\Omega} z(y,0) \\
&\geq e^{-t} \cdot \inf_{y\in\Omega} z(y,0) \\
&\geq e^{-2} \cdot \inf_{y\in\Omega} z(y,0) \quad \text{in } \Omega \qquad \text{for all } t \in (0,\min\{2,T\}),
\end{aligned}$$

from (2.10) we infer (2.9) upon an evident choice of $\Lambda(\Omega)$. □

By suitably combining Lemma 2.3 with Lemma 2.2 we can now make sure that indeed (2.6) is globally solvable for each $\varepsilon \in (0,1)$.

Lemma 2.4 *If (2.5) holds and $u_0 \in W^{1,\infty}(\Omega)$ and $v_0 \in W^{1,\infty}(\Omega)$ are such that $u_0 \geq 0$, $u_0 \not\equiv 0$ and $v_0 > 0$ in $\overline{\Omega}$, then for each $\varepsilon \in (0,1)$ one can find nonnegative functions*

$$\begin{cases} u_\varepsilon \in C^0(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)) \qquad \textit{and} \\ v_\varepsilon \in \bigcap_{q>n} C^0([0,\infty);W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)), \end{cases}$$

such that $(u_\varepsilon, v_\varepsilon)$ solves (2.6) classically in $\Omega\times(0,\infty)$. Moreover,

$$\int_\Omega u_\varepsilon(\cdot,t) = \int_\Omega u_0 \qquad \textit{for all } t > 0, \tag{2.11}$$

and there exists $C > 0$ fulfilling

$$v_\varepsilon(x,t) \geq C \qquad \textit{for all } x \in \Omega \textit{ and any } t > 0. \tag{2.12}$$

PROOF. With $\Lambda(\Omega) > 0$ taken from Lemma 2.3, we let

$$c_1 := \Lambda(\Omega) \cdot \min \left\{ \inf_\Omega v_0 \, , \, \int_\Omega u_0 \right\}$$

and using that c_1 is positive thanks to our hypotheses on u_0 and v_0 we may fix a positive function $\widehat{\phi} \in C^3([0,\infty))$ such that $\widehat{\phi} \equiv \phi$ in $[c_1, \infty)$, observing that then (1.8) ensures that both ϕ and ϕ' are bounded on $[0,\infty)$. Then for fixed $\varepsilon \in (0,1)$,

$$D(u,v) \equiv D_\varepsilon(u,v) := M\varepsilon(u+1)^{M-1} + (mu+\varepsilon)(u+\varepsilon)^{m-2}\widehat{\phi}(v), \qquad (u,v) \in [0,\infty)^2,$$

and

$$S(u,v) \equiv S_\varepsilon(u,v) := u(u+\varepsilon)^{m-1}\widehat{\phi}'(v), \qquad (u,v) \in [0,\infty)^2,$$

define functions D and S which belong to $C^2([0,\infty)^2)$ with $S(0,v) = 0$ for all $v \geq 0$, and since $m \geq 1$ and $M \geq m$ by (2.5), we can estimate

$$\begin{array}{rcl} D(u,v) & \leq & M\varepsilon(u+1)^{M-1} + m(u+\varepsilon)^{m-1}\widehat{\phi}(v) \\ & \leq & M\varepsilon(u+1)^{M-1} + m(u+1)^{m-1}\|\widehat{\phi}\|_{L^\infty((0,\infty))} \\ & \leq & \left\{ M\varepsilon + m\|\widehat{\phi}\|_{L^\infty((0,\infty))} \right\} \cdot (u+1)^{M-1} \qquad \text{for all } (u,v) \in [0,\infty)^2, \end{array}$$

whereas clearly

$$D(u,v) \geq M\varepsilon(u+1)^{M-1} \qquad \text{for all } (u,v) \in [0,\infty)^2.$$

As the latter moreover implies that

$$\begin{array}{rcl} \dfrac{|S(u,v)|}{D(u,v)} & \leq & \dfrac{u(u+\varepsilon)^{m-1}|\widehat{\phi}'(v)|}{M\varepsilon(u+1)^{M-1}} \\ & \leq & \dfrac{\|\widehat{\phi}'\|_{L^\infty((0,\infty))}}{M\varepsilon} \cdot (u+1)^{m-M+1} \qquad \text{for all } (u,v) \in [0,\infty)^2, \end{array}$$

using that (2.5) guarantees that $m - M + 1 < \frac{2}{n}$ we may invoke Lemma 2.2 to infer the existence of a global classical solution $(u,v) = (u_\varepsilon, v_\varepsilon)$ to the accordingly obtained problem (2.7), enjoying the regularity features in (2.8), and nonnegative in both its components.

An integration of the first equation in (2.7) thereafter shows that (2.11) holds, which in turn, through Lemma 2.3, entails that thanks to our definition of c_1 we have $v_\varepsilon \geq c_1$ in $\Omega \times (0,\infty)$. We therefore conclude that actually $\widehat{\phi}(v_\varepsilon) \equiv \phi(v_\varepsilon)$ throughout $\Omega \times (0,\infty)$, and that hence our definitions of D and S warrant that indeed $(u_\varepsilon, v_\varepsilon)$ solves (2.6) and satisfies (2.12) with $C := c_1$. □

Without further explicit mentioning, throughout the sequel we shall assume that (1.7) and (1.8) be satisfied, that u_0 and v_0 satisfy the requirements from Theorem 1.1, and that M is such that (2.5) holds, and let $((u_\varepsilon, v_\varepsilon))_{\varepsilon \in (0,1)}$ denote the family of approximate solutions obtained in Lemma 2.4.

Forming a last preliminary, let us formulate a rather straightforward consequence of (2.11) for a first ε-independent regularity feature of the respective second solution components.

Lemma 2.5 *Let $p \in [0, \frac{n}{n-2})$. Then there exists $C(p) > 0$ such that*

$$\|v_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le C(p) \qquad \textit{for all } t>0 \textit{ and } \varepsilon \in (0,1). \tag{2.13}$$

PROOF. Relying on (2.11), this can be seen by straightforward application of well-known smoothing properties of the Neumann heat semigroup $(e^{t\Delta})_{t\ge 0}$ on Ω: In fact, without loss of generality assuming that $p > 1$ we may invoke standard regularization estimates therefor ([31, Lemma 1.3]) to find $c_1(p) > 0$ such that for all $t > 0$ and $\varepsilon \in (0,1)$,

$$\begin{aligned}
\|v_\varepsilon(\cdot,t)\|_{L^p(\Omega)} &= \left\| e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)}u_\varepsilon(\cdot,s)ds \right\|_{L^p(\Omega)} \\
&\le c_1(p)e^{-t}\|v_0\|_{W^{1,\infty}(\Omega)} + c_1(p)\int_0^t e^{-(t-s)}\Big(1+(t-s)^{-\frac{n}{2}(1-\frac{1}{p})}\Big)\|u_\varepsilon(\cdot,s)\|_{L^1(\Omega)}ds \\
&= c_1(p)e^{-t}\|v_0\|_{W^{1,\infty}(\Omega)} + c_1(p)\|u_0\|_{L^1(\Omega)}\int_0^t e^{-\sigma}(1+\sigma^{-\frac{n}{2}(1-\frac{1}{p})})d\sigma
\end{aligned}$$

because of (2.11). Since $\int_0^\infty e^{-\sigma}(1+\sigma^{-\frac{n}{2}(1-\frac{1}{p})})d\sigma$ is finite due to the fact that $\frac{n}{2}(1-\frac{1}{p}) < 1$ thanks to the assumption $p < \frac{n}{n-2}$, this already establishes (2.13). □

3 Space-time L^1 estimates for $u_\varepsilon^{m+1}v_\varepsilon^{-\alpha}$ via a duality argument

Our next goal will consist in making appropriate use of the particular structure of the first equation in (2.6) and the particular link between the diffusion and cross-diffusion mechanisms contained therein, and our strategy in this regard will follow classical duality-based arguments (cf. e.g. [17] and [4] for related reasonings in more general frameworks, and [28] for a precedent addressing a less degenerate variant of (1.6)). To prepare an appropriate setup for our analysis in this direction, we let A denote the realization of $-\Delta+1$ under homogeneous Neumann boundary conditions in $L^2(\Omega)$, with its domain thus given by $D(A) = \{\psi \in W^{2,2}(\Omega) \mid \frac{\partial\psi}{\partial\nu}|_{\partial\Omega} = 0\}$, and recall that A is self-adjoint and possesses a family $(A^\beta)_{\beta\in\mathbb{R}}$ of corresponding densely defined self-adjoint fractional powers.

Now a rather straightforward pursuit of duality-guided ideas leads to a first observation concerning the time evolution of $A^{-\frac{1}{2}}(u_\varepsilon+1)$ which in its dissipated part contains a functional that with respect to u_ε grows in a considerably superlinear manner.

Lemma 3.1 *Assume that $n \ge 2$ and $m \ge 1$, and that (1.9) holds with some $\alpha \ge 0$. Then there exists $C > 0$ such that*

$$\frac{d}{dt}\int_\Omega |A^{-\frac{1}{2}}(u_\varepsilon+1)|^2 + \frac{1}{C}\int_\Omega u_\varepsilon^{m+1}v_\varepsilon^{-\alpha} \le C\int_\Omega |A^{-1}(u_\varepsilon+1)|^{m+1} + C \qquad \textit{for all } t>0 \textit{ and } \varepsilon\in(0,1). \tag{3.1}$$

PROOF. Taking any $p \in (1, M+1)$ such that $W^{2,p}(\Omega) \hookrightarrow L^{M+1}(\Omega)$, from a corresponding embedding inequality and standard elliptic regularity in $L^p(\Omega)$ ([11]) we obtain $c_1 > 0$ and $c_2 > 0$ such that

$$\|\psi\|_{L^{M+1}(\Omega)}^{M+1} \le c_1\|\psi\|_{W^{2,p}(\Omega)}^{M+1} \le c_2\|A\psi\|_{L^p(\Omega)}^{M+1} \qquad \text{for all } \psi \in W^{2,p}(\Omega) \text{ such that } \frac{\partial\psi}{\partial\nu}|_{\partial\Omega} = 0, \tag{3.2}$$

and thereafter we twice employ Young's inequality to infer the existence of $c_3 > 0$ such that

$$\xi\eta \le \frac{1}{2}\xi^{\frac{M+1}{M}} + c_3\eta^{M+1} \qquad \text{for all } \xi \ge 0 \text{ and } \eta \ge 0, \tag{3.3}$$

and that since $M+1 > p > 1$ we may pick $c_4 > 0$ such that

$$c_2c_3\|\psi\|_{L^p(\Omega)}^{M+1} \le \frac{1}{2}\|\psi\|_{L^{M+1}(\Omega)}^{M+1} + c_4\|\psi\|_{L^1(\Omega)}^{M+1} \qquad \text{for all } \psi \in L^p(\Omega). \tag{3.4}$$

Next, as a consequence of (2.12), we can find $c_5 > 0$ fulfilling

$$v_\varepsilon \ge c_5 \quad \text{in } \Omega \times (0,\infty) \qquad \text{for all } \varepsilon \in (0,1),$$

and note that due to (1.8) this firstly implies that with some $c_6 > 0$ we have

$$\phi(v_\varepsilon) \le c_6 \quad \text{in } \Omega \times (0,\infty) \qquad \text{for all } \varepsilon \in (0,1), \tag{3.5}$$

and that thanks to (1.9) and (1.7) this secondly entails the existence of $c_7 > 0$ satisfying

$$\phi(v_\varepsilon) \ge c_7 v_\varepsilon^{-\alpha} \quad \text{in } \Omega \times (0,\infty) \qquad \text{for all } \varepsilon \in (0,1). \tag{3.6}$$

As a final preparation, we once more draw on Young's inequality to fix $c_8 > 0$ such that

$$\xi\eta \le \frac{1}{2c_6^{\frac{1}{m}}}\xi^{\frac{m+1}{m}} + c_8\eta^{m+1} \qquad \text{for all } \xi \ge 0 \text{ and } \eta \ge 0, \tag{3.7}$$

and now make use of all these selections as follows: Since $\partial_t(u_\varepsilon + 1) = u_{\varepsilon t}$, from (2.6) we obtain the identity

$$\begin{aligned} &\partial_t A^{-1}(u_\varepsilon+1) + \varepsilon(u_\varepsilon+1)^M + u_\varepsilon(u_\varepsilon+\varepsilon)^{m-1}\phi(v_\varepsilon) \\ &\quad = A^{-1}\Big\{\varepsilon(u_\varepsilon+1)^M + u_\varepsilon(u_\varepsilon+\varepsilon)^{m-1}\phi(v_\varepsilon)\Big\} \quad \text{in } \Omega\times(0,\infty) \quad \text{for all } \varepsilon\in(0,1), \end{aligned}$$

which we test by $u_\varepsilon + 1$ to see by self-adjointness of $A^{-\frac{1}{2}}$ and of A^{-1},

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |A^{-\frac{1}{2}}(u_\varepsilon+1)|^2 + \varepsilon\int_\Omega (u_\varepsilon+1)^{M+1} + \int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{m-1}(u_\varepsilon+1)\phi(v_\varepsilon) \\ &\quad = \varepsilon\int_\Omega (u_\varepsilon+1)^M A^{-1}(u_\varepsilon+1) + \int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{m-1}\phi(v_\varepsilon)A^{-1}(u_\varepsilon+1) \end{aligned} \tag{3.8}$$

for all $t > 0$ and $\varepsilon \in (0,1)$. Here due to (3.3), (3.2), (3.4) and (2.11),

$$\begin{aligned} \varepsilon\int_\Omega (u_\varepsilon+1)^M A^{-1}(u_\varepsilon+1) &\le \frac{\varepsilon}{2}\int_\Omega (u_\varepsilon+1)^{M+1} + c_3\varepsilon\int_\Omega |A^{-1}(u_\varepsilon+1)|^{M+1} \\ &\le \frac{\varepsilon}{2}\int_\Omega (u_\varepsilon+1)^{M+1} + c_2c_3\varepsilon\|u_\varepsilon+1\|_{L^p(\Omega)}^{M+1} \\ &\le \frac{\varepsilon}{2}\int_\Omega (u_\varepsilon+1)^{M+1} + \frac{\varepsilon}{2}\|u_\varepsilon+1\|_{L^{M+1}(\Omega)}^{M+1} + c_4\varepsilon\|u_\varepsilon+1\|_{L^1(\Omega)}^{M+1} \\ &= \varepsilon\int_\Omega (u_\varepsilon+1)^{M+1} + c_9\varepsilon \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1) \end{aligned} \tag{3.9}$$

with $c_9 := c_4\|u_0+1\|_{L^1(\Omega)}^{M+1}$. Furthermore, combining (3.7) with (3.5) we see that for all $t>0$ and $\varepsilon\in(0,1)$,

$$\begin{aligned}
&\int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{m-1}\phi(v_\varepsilon)A^{-1}(u_\varepsilon+1)\\
&\qquad\le \frac{1}{2c_6^{\frac{1}{m}}}\int_\Omega \Big\{u_\varepsilon(u_\varepsilon+\varepsilon)^{m-1}\Big\}^{\frac{m+1}{m}}\phi^{\frac{m+1}{m}}(v_\varepsilon)+c_8\int_\Omega |A^{-1}(u_\varepsilon+1)|^{m+1}\\
&\qquad\le \frac12\int_\Omega u_\varepsilon^{\frac{m+1}{m}}(u_\varepsilon+\varepsilon)^{\frac{m^2-1}{m}}\phi(v_\varepsilon)+c_8\int_\Omega |A^{-1}(u_\varepsilon+1)|^{m+1},
\end{aligned}$$

whence (3.8) and (3.9) entail that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |A^{-\frac12}(u_\varepsilon+1)|^2+2\int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{m-1}(u_\varepsilon+1)\phi(v_\varepsilon)-\int_\Omega u_\varepsilon^{\frac{m+1}{m}}(u_\varepsilon+\varepsilon)^{\frac{m^2-1}{m}}\phi(v_\varepsilon)\\
&\qquad\le 2c_8\int_\Omega |A^{-1}(u_\varepsilon+1)|^{m+1}+2c_9\varepsilon \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1).
\end{aligned}$$

Since finally estimating $u_\varepsilon+1\ge\max\{u_\varepsilon+\varepsilon,\varepsilon\}$ and using (3.6) shows that

$$\begin{aligned}
&2\int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{m-1}(u_\varepsilon+1)\phi(v_\varepsilon)-\int_\Omega u_\varepsilon^{\frac{m+1}{m}}(u_\varepsilon+\varepsilon)^{\frac{m^2-1}{m}}\phi(v_\varepsilon)\\
&\qquad\ge \int_\Omega u_\varepsilon^{m+1}\phi(v_\varepsilon)\ge c_7\int_\Omega u_\varepsilon^{m+1}v_\varepsilon^{-\alpha} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),
\end{aligned}$$

this establishes (3.1) if we let $C:=\max\{\frac{1}{c_7}\,,\,2c_8\,,\,2c_9\}$, for instance. □

In order to appropriately estimate the integral on the right of (3.1) in terms of the second summand on the left-hand side therein, to be accomplished in Lemma 3.3, but to furthermore prepare an argument revealing dominance of the latter over the expression $\int_\Omega |A^{-\frac12}(u_\varepsilon+1)|^2$ (see Lemma 3.4), let us state the following consequence of a simple interpolation based on the Hölder inequality.

Lemma 3.2 *If $n\ge2, m\ge1$ and (1.9) holds with some $\alpha\ge0$, then for all $p>0$ fulfilling*

$$p<\frac{n(m+1)}{n+(n-2)\alpha}, \tag{3.10}$$

one can find $C(p)>0$ such that

$$\int_\Omega \Big(u_\varepsilon(\cdot,t)+1\Big)^p\le C(p)\cdot\Big\{\int_\Omega u_\varepsilon^{m+1}(\cdot,t)v_\varepsilon^{-\alpha}(\cdot,t)\Big\}^{\frac{p}{m+1}}+C(p) \qquad \textit{for all } t>0 \textit{ and } \varepsilon\in(0,1). \tag{3.11}$$

Proof. We first note that due to (3.10), we particularly have $p<m+1$, so that setting

$$q:=\frac{p\alpha}{m-p+1}$$

defines a nonnegative number which, by making full use of (3.10) now, can be seen to satisfy

$$(m+1-p)\cdot\Big\{(n-2)q-n\Big\}=(n-2)\cdot p\alpha-n(m+1-p)=\Big\{n+(n-2)\alpha\Big\}\cdot p-n(m+1)<0$$

and hence $q < \frac{n}{n-2}$. Therefore, Lemma 2.5 applies so as to yield $c_1 > 0$ such that

$$\int_\Omega v_\varepsilon^q \le c_1 \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1),$$

which we combine with the Hölder inequality, applicable since, still, $p < m+1$, to estimate

$$\begin{aligned}
\int_\Omega u_\varepsilon^p &= \int_\Omega (u_\varepsilon^{m+1} v_\varepsilon^{-\alpha})^{\frac{p}{m+1}} \cdot v_\varepsilon^{\frac{p\alpha}{m+1}} \\
&\le \left\{ \int_\Omega u_\varepsilon^{m+1} v_\varepsilon^{-\alpha} \right\}^{\frac{p}{m+1}} \cdot \left\{ \int_\Omega v_\varepsilon^{\frac{p\alpha}{m+1-p}} \right\}^{\frac{m+1-p}{m+1}} \\
&\le c_1^{\frac{m+1-p}{m+1}} \cdot \left\{ \int_\Omega u_\varepsilon^{m+1} v_\varepsilon^{-\alpha} \right\}^{\frac{p}{m+1}} \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1).
\end{aligned}$$

Since $\int_\Omega (u_\varepsilon + 1)^p \le 2^p \int_\Omega u_\varepsilon^p + 2^p |\Omega|$ for all $t > 0$ and $\varepsilon \in (0,1)$ by Young's inequality, this already entails (3.11). $\Box$

In fact, under assumptions on α actually less restrictive than those in Theorem 1.1 the latter can be combined with adequate embedding properties to suitably bound the right-hand side in (3.1).

Lemma 3.3 *If $n \ge 2, m \ge 1$ and (1.9) is satisfied with some $\alpha \ge 0$ fulfilling (1.12) as well as*

$$\alpha < \frac{nm}{n-2}, \tag{3.12}$$

then for all $\eta > 0$ there exists $C(\eta) > 0$ such that

$$\int_\Omega \left| A^{-1}\Big(u_\varepsilon(\cdot,t) + 1\Big)\right|^{m+1} \le \eta \int_\Omega u_\varepsilon^{m+1}(\cdot,t) v_\varepsilon^{-\alpha}(\cdot,t) + C(\eta) \qquad \textit{for all } t > 0 \textit{ and } \varepsilon \in (0,1). \tag{3.13}$$

Proof. According to (3.12), we have $(n-2)\alpha < nm$ and hence

$$\frac{n(m+1)}{n + (n-2)\alpha} > 1,$$

whereas (1.12) ensures that $(n-2)\alpha < 2m+2$ and thus

$$\frac{n(m+1)}{n+2m+2} < \frac{n(m+1)}{n+(n-2)\alpha}.$$

Combining these two inequalities we see that we can firstly fix a number $p_0 > 1$ fulfilling

$$\frac{n(m+1)}{n+2m+2} < p_0 < \frac{n(m+1)}{n+(n-2)\alpha}, \tag{3.14}$$

and thereupon take some $p > p_0$ such that still

$$p < \frac{n(m+1)}{n+(n-2)\alpha}. \tag{3.15}$$

Then the left inequality in (3.14) guarantees that $2-\frac{n}{p_0} > -\frac{n}{m+1}$ and that hence $W^{2,p_0}(\Omega)$ is continuously embedded into $L^{m+1}(\Omega)$, whence again relying on standard elliptic regularity theory we obtain $c_1>0$ and $c_2>0$ satisfying

$$\|\psi\|_{L^{m+1}(\Omega)} \le c_1\|\psi\|_{W^{2,p_0}(\Omega)} \le c_2\|A\psi\|_{L^{p_0}(\Omega)} \qquad \text{for all } \psi\in W^{2,p_0}(\Omega) \text{ such that } \tfrac{\partial\psi}{\partial\nu}|_{\partial\Omega}=0. \quad (3.16)$$

Apart from that, from Lemma 3.2 we infer that due to (3.15) there exist $c_3>0$ and $c_4>0$ such that

$$\|u_\varepsilon+1\|_{L^p(\Omega)}^{m+1} \le c_3\int_\Omega u_\varepsilon^{m+1}v_\varepsilon^{-\alpha} + c_4 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1), \quad (3.17)$$

while a simple application of Young's inequality shows that since $1<p_0<p$, given $\eta>0$ we can find $c_5(\eta)>0$ in such a way that

$$c_2^{m+1}\|\psi\|_{L^{p_0}(\Omega)}^{m+1} \le \frac{\eta}{c_3}\|\psi\|_{L^p(\Omega)}^{m+1} + c_5(\eta)\|\psi\|_{L^1(\Omega)}^{m+1} \qquad \text{for all } \psi\in L^p(\Omega). \quad (3.18)$$

We now employ (3.16) with $\psi:=A^{-1}(u_\varepsilon+1)$ to see that thanks to (3.18) and (3.17), for all $t>0$ and $\varepsilon\in(0,1)$ we have

$$\begin{aligned}
\|A^{-1}(u_\varepsilon+1)\|_{L^{m+1}(\Omega)}^{m+1} &\le c_2^{m+1}\|u_\varepsilon+1\|_{L^{p_0}(\Omega)}^{m+1} \\
&\le \frac{\eta}{c_3}\|u_\varepsilon+1\|_{L^p(\Omega)}^{m+1} + c_5(\eta)\|u_\varepsilon+1\|_{L^1(\Omega)}^{m+1} \\
&\le \frac{\eta}{c_3}\cdot\Big\{c_3\int_\Omega u_\varepsilon^{m+1}v_\varepsilon^{-\alpha}+c_4\Big\} + c_5(\eta)\|u_\varepsilon+1\|_{L^1(\Omega)}^{m+1},
\end{aligned}$$

which due to (2.11) indeed leads to (3.13). □

A second application of Lemma 3.2 will eventually enable us to turn (3.1) into an autonomous ODI containing a linear absorption, provided that the stronger hypotheses on α from Theorem 1.2 are met.

Lemma 3.4 *Assume that $n\ge 2$ and $m\ge 1$, and that (1.9) is satisfied with some $\alpha\ge 0$ such that (1.15) holds. Then there exists $C>0$ such that*

$$\int_\Omega \Big|A^{-\frac12}\Big(u_\varepsilon(\cdot,t)+1\Big)\Big|^2 \le \int_\Omega u_\varepsilon^{m+1}(\cdot,t)v_\varepsilon^{-\alpha}(\cdot,t) + C \qquad \textit{for all } t>0 \textit{ and } \varepsilon\in(0,1). \quad (3.19)$$

Proof. We first note that (1.15) ensures that

$$2n+2(n-2)\alpha < 2n + \Big\{(n+2)m-n+2\Big\} = (n+2)(m+1),$$

and that thus $\frac{2n}{n+2} < \frac{n(m+1)}{n+(n-2)\alpha}$, to pick $p>1$ such that

$$\frac{2n}{n+2} < p < \frac{n(m+1)}{n+(n-2)\alpha}. \quad (3.20)$$

Here the first inequality warrants that $p' := \frac{p}{p-1}$ satisfies $-\frac{n}{p'} < 2-\frac{n}{p}$ and that hence $W^{2,p}(\Omega)\hookrightarrow L^{p'}(\Omega)$, so that once more invoking elliptic regularity we can fix $c_1>0$ fulfilling

$$\|\psi\|_{L^{p'}(\Omega)} \le c_1\|A\psi\|_{L^p(\Omega)} \qquad \text{for all } \psi\in W^{2,p}(\Omega) \text{ such that } \tfrac{\partial\psi}{\partial\nu}|_{\partial\Omega}=0. \quad (3.21)$$

Moreover, the second inequality in (3.20) enables us to infer from Lemma 3.2 that with some $c_2 > 0$ we have

$$\|u_\varepsilon + 1\|_{L^p(\Omega)}^2 \le c_2 \cdot \left\{ \int_\Omega u_\varepsilon^{m+1} v_\varepsilon^{-\alpha} \right\}^{\frac{2}{m+1}} + c_2 \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1), \tag{3.22}$$

whereupon we use that $m > 1$ in choosing $c_3 > 0$ such that

$$c_1 c_2 \xi^{\frac{2}{m+1}} \le \xi + c_3 \qquad \text{for all } \xi \ge 0. \tag{3.23}$$

Now by self-adjointness of $A^{-\frac{1}{2}}$, the Hölder inequality and (3.21),

$$\begin{aligned}
\int_\Omega |A^{-\frac{1}{2}}(u_\varepsilon + 1)|^2 &= \int_\Omega (u_\varepsilon + 1) A^{-1}(u_\varepsilon + 1) \\
&\le \|u_\varepsilon + 1\|_{L^p(\Omega)} \|A^{-1}(u_\varepsilon + 1)\|_{L^{p'}(\Omega)} \\
&\le c_1 \|u_\varepsilon + 1\|_{L^p(\Omega)}^2 \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1),
\end{aligned}$$

where due to (3.22) and (3.23),

$$\begin{aligned}
c_1 \|u_\varepsilon + 1\|_{L^p(\Omega)}^2 &\le c_1 c_2 \cdot \left\{ \int_\Omega u_\varepsilon^{m+1} v_\varepsilon^{-\alpha} \right\}^{\frac{2}{m+1}} + c_1 c_2 \\
&\le \int_\Omega u_\varepsilon^{m+1} v_\varepsilon^{-\alpha} + c_3 + c_1 c_2 \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1),
\end{aligned}$$

so that the claim follows. $\Box$

When combined with Lemma 3.1, the previous two lemmata now enable us to derive the main result of this section.

Lemma 3.5 *If $n \ge 2, m \ge 1$ and (1.9) is valid with some $\alpha \ge 0$ fulfilling (1.12) and (3.12), then there exists $K : (0,\infty) \to (0,\infty)$ such that for all $T > 0$,*

$$\int_t^{t+1} \int_\Omega u_\varepsilon^{m+1} v_\varepsilon^{-\alpha} \le K(T) \qquad \textit{for all } t \in (0,T) \textit{ and } \varepsilon \in (0,1), \tag{3.24}$$

and such that

$$\sup_{T>0} K(T) < \infty \quad \textit{if (1.15) holds.} \tag{3.25}$$

Proof. Using Lemma 3.1 as a starting point, we take $c_1 > 0$ and $c_2 > 0$ such that

$$\frac{d}{dt} \int_\Omega |A^{-\frac{1}{2}}(u_\varepsilon + 1)|^2 + c_1 \int_\Omega u_\varepsilon^{m+1} v_\varepsilon^{-\alpha} \le c_2 \int_\Omega |A^{-1}(u_\varepsilon + 1)|^{m+1} + c_2 \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1),$$

and relying on our hypotheses that (1.12) and (3.12) be satisfied, we invoke Lemma 3.3 to find $c_3 > 0$ fulfilling

$$c_2 \int_\Omega |A^{-1}(u_\varepsilon + 1)|^{m+1} \le \frac{c_1}{2} \int_\Omega u_\varepsilon^{m+1} v_\varepsilon^{-\alpha} + c_3 \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1).$$

Therefore,

$$\frac{d}{dt}\int_\Omega |A^{-\frac12}(u_\varepsilon+1)|^2 + \frac{c_1}{2}\int_\Omega u_\varepsilon^{m+1}v_\varepsilon^{-\alpha} \le c_2+c_3 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1), \tag{3.26}$$

so that given $T>0$ we may infer upon an integration that for all $t\in(0,T)$ and $\varepsilon\in(0,1)$,

$$\frac{c_1}{2}\int_t^{t+1}\int_\Omega u_\varepsilon^{m+1}v_\varepsilon^{-\alpha} \le \int_\Omega |A^{-\frac12}(u_0+1)|^2 + (c_2+c_3)\cdot(T+1). \tag{3.27}$$

If additionally (1.15) holds, then Lemma 3.4 provides $c_4>0$ such that

$$\int_\Omega |A^{-\frac12}(u_\varepsilon+1)|^2 \le \int_\Omega u_\varepsilon^{m+1}v_\varepsilon^{-\alpha} + c_4 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),$$

so that in this case from (3.26) we obtain that

$$\begin{aligned}\frac{d}{dt}\int_\Omega |A^{-\frac12}(u_\varepsilon+1)|^2 &+ \frac{c_1}{4}\int_\Omega |A^{-\frac12}(u_\varepsilon+1)|^2 + \frac{c_1}{4}\int_\Omega u_\varepsilon^{m+1}v_\varepsilon^{-\alpha} \\ &\le c_5 := c_2+c_3+\frac{c_1c_4}{4} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1).\end{aligned} \tag{3.28}$$

Through an ODE comparison argument, this firstly entails that then

$$\int_\Omega |A^{-\frac12}(u_\varepsilon+1)|^2 \le c_6 := \max\left\{\int_\Omega |A^{-\frac12}(u_0+1)|^2\,,\,\frac{4c_5}{c_1}\right\} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),$$

whereafter a direct integration of (3.28) shows that for such choices of ϕ we actually have

$$\begin{aligned}\frac{c_1}{4}\int_t^{t+1}\int_\Omega u_\varepsilon^{m+1}v_\varepsilon^{-\alpha} &\le \int_\Omega \Big|A^{-\frac12}\Big(u_\varepsilon(\cdot,t+1)\Big)\Big|^2 + c_5 \\ &\le c_6+c_5 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),\end{aligned}$$

which in conjunction with (3.27) readily yields a positive function K on $(0,\infty)$ fulfilling both (3.24) and (3.25). □

4 Bounds for v_ε in $W^{1,q}(\Omega)$ with some $q>n$

4.1 Estimating v_ε in $L^p(\Omega)$ for arbitrary finite p

We next intend to turn the weighted L^{m+1} estimate for u_ε provided by Lemma 3.5 into bounds for v_ε suitably improving those from Lemma 2.5. In view of the particular structure of the integrand in (3.24), and especially its dependence on v_ε, for an efficient exploitation thereof it seems promising to act, at a first stage, in the context of standard L^p testing procedures. The information thereby gained, consisting in L^p bounds for v_ε in arbitrary L^p spaces not only in the case $n=2$ (see Lemma 2.5) but also when $n\ge3$, will finally enable us to essentially neglect the factor $v_\varepsilon^{-\alpha}$ in (3.24), and to use the resulting version thereof in the derivation of gradient bounds for v_ε through more straightforward semigroup estimates.

The following outcome of [25, Lemma 3.4] will be needed in our first step in this direction, to be established in Lemma 4.2.

Lemma 4.1 *Let $T>1$, and let $y:[0,T)\to[0,\infty)$ be absolutely continuous and such that*

$$y'(t)+\lambda y(t)\le h(t) \qquad \textit{for a.e. } t\in(0,T)$$

with some $\lambda>0$ and some nonnegative $h\in L^1_{loc}([0,T))$ fo which there exists $\kappa>0$ fulfilling

$$\int_t^{t+1} h(s)ds\le\kappa \qquad \textit{for all } t\in(0,T-1).$$

Then

$$y(t)\le\max\Big\{y(0)+\kappa\,,\,\frac{\kappa}{\lambda}+2\kappa\Big\} \qquad \textit{for all } t\in(0,T).$$

By means of the latter and an appropriate testing procedure, we can derive the following core of an iterative step potentially improving our regularity information on v_ε. Here our assumption (4.2) on u_ε is formulated in such a way that both alternatives possible in (3.25) can conveniently be included without explicit reference to requirements on α which are actually not needed in this part.

Lemma 4.2 *Assume that $n\ge3$, that $m\ge1$, and that (1.9) be valid with some $\alpha\ge0$, and suppose that $p_\star\ge1$ and $L_i:(0,\infty)\to(0,\infty)$, $i\in\{1,2\}$, are such that for all $T>0$ we have*

$$\int_\Omega v_\varepsilon^{p_\star}(\cdot,t)\le L_1(T) \qquad \textit{for all } t\in(0,T) \textit{ and } \varepsilon\in(0,1) \tag{4.1}$$

as well as

$$\int_t^{t+1}\int_\Omega u_\varepsilon^{m+1}v_\varepsilon^{-\alpha}\le L_2(T) \qquad \textit{for all } t\in(0,T) \textit{ and } \varepsilon\in(0,1). \tag{4.2}$$

Then given any $p>p_0$ fulfilling

$$(n-2m-2)p<(n-2)(m+1-\alpha) \tag{4.3}$$

and

$$p<m+1-\alpha+\frac{2m}{n}\cdot p_\star, \tag{4.4}$$

one can find $K^{(p)}:(0,\infty)\to(0,\infty)$ with the properties that for all $T>0$,

$$\int_\Omega v_\varepsilon^{p}(\cdot,t)\le K^{(p)}(T) \qquad \textit{for all } t\in(0,T) \textit{ and } \varepsilon\in(0,1), \tag{4.5}$$

and that

$$\sup_{T>0}K^{(p)}(T)<\infty \quad \textit{if} \quad \sup_{T>0}(L_1(T)+L_2(T))<\infty. \tag{4.6}$$

Proof. We abbreviate

$$f_\varepsilon(t):=\int_\Omega u_\varepsilon^{m+1}(\cdot,t)v_\varepsilon^{-\alpha}(\cdot,t) \qquad \text{for } t>0 \text{ and } \varepsilon\in(0,1),$$

and given $p > p_\star$ satisfying (4.3) and (4.4) we first test the second equation in (2.6) by v_ε^{p-1} to see that due to Young's inequality,

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega v_\varepsilon^p + \frac{4(p-1)}{p^2}\int_\Omega |\nabla v_\varepsilon^{\frac{p}{2}}|^2 + \int_\Omega v_\varepsilon^p \\
&\qquad = \int_\Omega u_\varepsilon v_\varepsilon^{p-1} \\
&\qquad = \int_\Omega (u_\varepsilon^{m+1} v_\varepsilon^{-\alpha})^{\frac{1}{m+1}} \cdot v_\varepsilon^{\frac{(m+1)(p-1)+\alpha}{m+1}} \\
&\qquad \le f_\varepsilon(t) + \int_\Omega v_\varepsilon^{\frac{(m+1)(p-1)+\alpha}{m}} \qquad \text{for } t>0 \text{ and } \varepsilon \in (0,1).
\end{aligned} \tag{4.7}$$

Here in the case when incidentally $\frac{(m+1)(p-1)+\alpha}{m} \le p_\star$, again by means of Young's inequality we can utilize (4.1) to obtain that for each fixed $T > 0$,

$$\int_\Omega v_\varepsilon^{\frac{(m+1)(p-1)+\alpha}{m}} \le \int_\Omega v_\varepsilon^{p_\star} + |\Omega| \le L_1(T) + |\Omega| \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1). \tag{4.8}$$

If, conversely,

$$\frac{(m+1)(p-1)+\alpha}{m} > p_\star, \tag{4.9}$$

then the number

$$a := \frac{np \cdot [(m+1)(p-1)+\alpha - mp_\star]}{(np - np_\star + 2p_\star)\cdot[(m+1)(p-1)+\alpha]}, \tag{4.10}$$

clearly well-defined since $p > 1$ and $np - np_\star + 2p_\star > 2p_\star > 0$ due to our restriction $p > p_\star$, satisfies $a > 0$ by (4.9). As furthermore, by (4.3),

$$\begin{aligned}
&np \cdot \Big[(m+1)(p-1)+\alpha - mp_\star\Big] - (np - np_\star + 2p_\star)\cdot\Big[(m+1)(p-1)+\alpha\Big] \\
&\qquad = \Big[(n-2m-2)p - (n-2)(m+1-\alpha)\Big]\cdot p_\star < 0
\end{aligned}$$

and hence $a < 1$, we may invoke the Gagliardo-Nirenberg inequality to find $c_1 > 0$ such that

$$\begin{aligned}
\int_\Omega v_\varepsilon^{\frac{(m+1)(p-1)+\alpha}{m}} &= \|v_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}\cdot\frac{(m+1)(p-1)+\alpha}{m}}(\Omega)}^{\frac{2}{p}\cdot\frac{(m+1)(p-1)+\alpha}{m}} \\
&\le c_1\|\nabla v_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{2}{p}\cdot\frac{(m+1)(p-1)+\alpha}{m}\cdot a} \|v_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2p_\star}{p}}(\Omega)}^{\frac{2}{p}\cdot\frac{(m+1)(p-1)+\alpha}{m}\cdot(1-a)} \\
&\quad + c_1\|v_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2p_\star}{p}}(\Omega)}^{\frac{2}{p}\cdot\frac{(m+1)(p-1)+\alpha}{m}} \qquad \text{for } t>0 \text{ and } \varepsilon\in(0,1).
\end{aligned} \tag{4.11}$$

Since given $T > 0$ we can use (4.1) to estimate

$$\|v_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2p_\star}{p}}(\Omega)}^{\frac{2p_\star}{p}} = \int_\Omega v_\varepsilon^{p_\star} \le L_1(T) \qquad \text{for all } t\in(0,T) \text{ and } \varepsilon\in(0,1),$$

and since (4.10) along with (4.4) asserts that

$$\theta := \frac{2}{\frac{2}{p} \cdot \frac{(m+1)(p-1)+\alpha}{m} \cdot a}$$

satisfies

$$\begin{aligned} \theta - 1 &= \frac{m(np - np_\star + 2p_\star) - n \cdot [(m+1)(p-1) + \alpha - mp_\star]}{n \cdot [(m+1)(p-1) + \alpha - mp_\star]} \\ &= \frac{nmp - (n-2)mp_\star - n(m+1)p + n(m+1-\alpha) + nmp_\star}{n \cdot [(m+1)(p-1) + \alpha - mp_\star]} \\ &= \frac{-np + n(m+1-\alpha) + 2mp_\star}{n \cdot [(m+1)(p-1) + \alpha - mp_\star]} \\ &= \frac{-p + m + 1 - \alpha + \frac{2m}{m} \cdot p_\star}{(m+1)(p-1) + \alpha - mp_\star} \\ &> 0 \end{aligned}$$

and thus $\theta > 1$, by using Young's inequality we infer from (4.11) that there exist $K_i = K_i^{(p)} : (0,\infty) \to (0,\infty)$, $i \in \{1,2\}$, such that for all $T > 0$,

$$\begin{aligned} \int_\Omega v_\varepsilon^{\frac{(m+1)(p-1)+\alpha}{m}} &\le K_1(T) \|\nabla v_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{2}{\theta}} + K_1(T) \\ &\le \frac{4(p-1)}{p^2} \|v_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^2 + K_2(T) \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1), \end{aligned}$$

and that

$$\sup_{T>0} K_2(T) < \infty \quad \text{if} \quad \sup_{T>0} L_1(T) < \infty. \tag{4.12}$$

In conjunction with (4.7), this implies that for all $T > 0$,

$$\frac{1}{p}\frac{d}{dt}\int_\Omega v_\varepsilon^p + \int_\Omega v_\varepsilon^p \le f_\varepsilon(t) + K_2(T) \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1),$$

so that an application of Lemma 4.1 shows that due to (4.2), given any $T > 0$ we have

$$\int_\Omega v_\varepsilon^p \le \max\left\{ \int_\Omega v_0^p + p(K_2(T) + L_2(T)) \, , \, (2p+1)(K_2(T) + L_2(T)) \right\} \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1),$$

which in view of (4.12) comletes the proof also for such values of p. □

Now an iterative argument based on the preceding lemma indeed yields substantial improvement of knowledge if the factor $\frac{2m}{n}$ multiplied to $p_\star$ in (4.4) exceeds 1, and if α is such that at least the admissible upper bound for p in (4.4) is compatible with (4.3). While the former enforces an additional requirement on m, the latter condition is covered by our previous assumptions on α already.

Lemma 4.3 *Assume that $n \geq 3$, that*

$$m > \frac{n}{2}, \tag{4.13}$$

and that (1.9) holds with some $\alpha \geq 0$ satisfying (1.11), and suppose that $L : (0,\infty) \to (0,\infty)$ is such that for all $T > 0$,

$$\int_t^{t+1} \int_\Omega u_\varepsilon^{m+1} v_\varepsilon^{-\alpha} \leq L(T) \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1). \tag{4.14}$$

Then for each $p \geq 1$ one can find $K^{(p)} : (0,\infty) \to (0,\infty)$ such that for all $T > 0$,

$$\int_\Omega v_\varepsilon^p(\cdot,t) \leq K^{(p)}(T) \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1), \tag{4.15}$$

and that

$$\sup_{T>0} K^{(p)}(T) < \infty \quad \text{if} \quad \sup_{T>0} L(T) < \infty. \tag{4.16}$$

Proof. We let

$$p_0 := 0, \quad p_1 := \frac{n}{n-2} \quad \text{and} \quad p_{j+1} := m+1-\alpha + \frac{2m}{n} \cdot p_j \text{ for integers } j \geq 1, \tag{4.17}$$

and observe that our hypothesis that (4.13) and (1.11) be valid ensures that whenever $j \geq 1$ is such that $p_j \geq \frac{n}{n-2}$, we have

$$p_{j+1} - p_j = \frac{2m-n}{n} \cdot p_j + m + 1 - \alpha \geq \frac{2m-n}{n-2} + m + 1 - \alpha > \frac{2m-n}{n-2} + m + 1 - \frac{nm-2}{n-2} = 0,$$

from which it readily follows by induction that $(p_j)_{j\geq 0}$ is strictly increasing, with $p_j \to +\infty$ as $j \to \infty$ according to (4.17).

In order to recursively show that

$$\begin{aligned} &\text{for each } j \geq 0 \text{ and } p \in [p_j, p_{j+1}) \text{ one can find } K^{(p)} : (0,\infty) \to (0,\infty) \\ &\text{such that (4.16) holds and that (4.15) is valid for all } T > 0, \end{aligned} \tag{4.18}$$

we first recall Lemma 2.5 to see that for each $p \in [0, \frac{n}{n-2})$ we can find $c_1(p) > 0$ fulfilling

$$\int_\Omega v_\varepsilon^p \leq c_1(p) \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1),$$

which clearly establishes the claim in (4.18) for $j = 0$ if we let $K^{(p)}(T) := c_1(p)$ for $T > 0$ and $p \in [p_0, p_1) \equiv [0, \frac{n}{n-2})$.

If the property in (4.18) has already been asserted for any integer up to $j-1$ with some $j \geq 1$, however, then we note that by (4.17) and the inequalities $n - 2m - 2 < 0$ and $p_{j+1} > \frac{n}{n-2}$ we have

$$(n-2m-2)p_{j+1} - (n-2)(m+1-\alpha)$$

$$
\begin{aligned}
&= (n-2m-2)\cdot\Big(m+1-\alpha+\frac{2m}{n}\cdot p_j\Big)-(n-2)(m+1-\alpha)\\
&< (n-2m-2)\cdot\Big(m+1-\alpha+\frac{2m}{n-2}\Big)-(n-2)(m+1-\alpha)\\
&= -2m(m+1-\alpha)+(n-2m-2)\cdot\frac{2m}{n-2}\\
&= 2m\cdot\Big(\alpha-\frac{nm}{n-2}\Big)<0,
\end{aligned}
$$

because $\alpha < \frac{nm}{n-2}$ by (1.11). By means of an argument based on continuous dependence, we can therefore pick $\widehat{p}_{j+1} \in (p_j, p_{j+1})$ such that

$$
(n-2m-2)p < (n-2)(m+1-\alpha) \qquad \text{for all } p \in (\widehat{p}_{j+1}, p_{j+1}),
$$

whence in view of the assumed validity of the statement in (4.18) for $j-1$, Lemma 4.2 applies so as to show that for any such p we can find $K^{(p)} : (0,\infty) \to (0,\infty)$ satisfying (4.15) for all $T>0$, as well as (4.16). Since Ω is bounded, this already implies the property claimed in (4.18) throughout the entire interval $[p_j, p_{j+1})$, and hence completes the verification of (4.18).

It remains to observe that $\cup_{j\ge 0}[p_j, p_{j+1}) = [1,\infty)$ to infer that the infinite collection of statements contained in (4.18) entails the intended conclusion. $\square$

In light of Lemma 3.5 and Lemma 2.5, from the latter we obtain the intended main result concerning integral estimates for v_ε.

Corollary 4.4 *Let $n \ge 2$, $m > \frac{n}{2}$ and ϕ be such that (1.9) is valid with some $\alpha \ge 0$ satisfying (1.11) and (1.12). Then for any $p \ge 0$ there exists $K^{(p)} : (0,\infty) \to (0,\infty)$ such that for all $T>0$,*

$$
\int_\Omega v_\varepsilon^p(\cdot,t) \le K^{(p)}(T) \qquad \textit{for all } t \in (0,T) \textit{ and } \varepsilon \in (0,1), \tag{4.19}
$$

and such that

$$
\sup_{T>0} K^{(p)}(T) < \infty \quad \textit{if (1.15) holds.} \tag{4.20}
$$

Proof. If $n \ge 3$, this readily results upon combining Lemma 4.3 with Lemma 3.5, whereas in the case $n=2$ we only need to recall Lemma 2.5. $\square$

4.2 An estimate including ∇v_ε

Having Corollary 4.4 at hand, from Lemma 3.5 we can immediately draw the following conclusion concerning space-time integrability properties of u_ε without the appearance of weight functions.

Corollary 4.5 *Assume that $n \ge 2$, that $m > \frac{n}{2}$, and that (1.9) holds with some $\alpha \ge 0$ such that (1.11) and (1.12) are satisfied. Then for all $p \in [1, m+1)$ one can find $K^{(p)} : (0,\infty) \to (0,\infty)$ such that for all $T>0$,*

$$
\int_t^{t+1}\int_\Omega u_\varepsilon^p(\cdot,t) \le K^{(p)}(T) \qquad \textit{for all } t \in (0,T) \textit{ and } \varepsilon \in (0,1), \tag{4.21}
$$

and that

$$
\sup_{T>0} K^{(p)}(T) < \infty \quad \textit{if (1.15) holds.} \tag{4.22}
$$

Proof. Given $p \in [1, m+1)$, we use Young's inequality to estimate

$$\begin{aligned}\int_t^{t+1}\int_\Omega u_\varepsilon^p &= \int_t^{t+1}\int_\Omega (u_\varepsilon^{m+1} v_\varepsilon^{-\alpha})^{\frac{p}{m+1}} \cdot v_\varepsilon^{\frac{p\alpha}{m+1}} \\ &\le \int_t^{t+1}\int_\Omega u_\varepsilon^{m+1} v_\varepsilon^{-\alpha} + \int_t^{t+1}\int_\Omega v_\varepsilon^{\frac{p\alpha}{m+1-p}} \qquad \text{for all } t>0 \text{ and } \varepsilon \in (0,1).\end{aligned}$$

Therefore, the conclusion is a direct consequence of Lemma 3.5 when followed by an application of Corollary 4.4 to the nonnegative finite integrability exponent $\frac{p\alpha}{m+1-p}$. $\square$

Through the latter, integrability properties of ∇v_ε become amenable to quite well-established arguments from parabolic regularity theory.

Lemma 4.6 *Let $n \ge 2$ and $m > \frac{n}{2}$, and assume (1.9) to be satisfied with some $\alpha \ge 0$ complying with (1.11) and (1.12). Then there exist $q > n$ and $K : (0,\infty) \to (0,\infty)$ such that for all $T > 0$,*

$$\|v_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} \le K(T) \qquad \textit{for all } t \in (0,T) \textit{ and } \varepsilon \in (0,1), \tag{4.23}$$

and that

$$\sup_{T>0} K(T) < \infty \quad \textit{if (1.15) holds.} \tag{4.24}$$

Proof. Again explicitly using that $m > \frac{n}{2}$, we observe that

$$\frac{m+1}{m} \cdot \Big\{\frac{1}{2} - \frac{n}{2} \cdot \Big(\frac{1}{m+1} - \frac{1}{n}\Big)\Big\} = \frac{n}{2m} < 1,$$

whence it is possible to fix $q > n$ suitably close to n such that

$$\frac{m+1}{m} \cdot \Big\{\frac{1}{2} - \frac{n}{2} \cdot \Big(\frac{1}{m+1} - \frac{1}{q}\Big)\Big\} < 1,$$

and thereafter choose some $p \in [1, m+1)$ in an appropriately small neighborhood of $m+1$ such that still

$$\frac{p}{p-1} \cdot \Big\{\frac{1}{2} - \frac{n}{2} \cdot \Big(\frac{1}{p} - \frac{1}{q}\Big)\Big\} < 1, \tag{4.25}$$

where since $q > n \ge 2$ and hence $\frac{q}{q-1} \cdot \frac{1}{2} < 1$, we can clearly achieve that also $p \le q$. Then due to well-known smoothing properties of the Neumann heat semigroup $(e^{t\Delta})_{t\ge 0}$ on Ω ([31, Lemma 1.3]), we can find positive constants c_1, c_2 and c_3 such that

$$\|e^{t\Delta}\psi\|_{W^{1,q}(\Omega)} \le c_1 \|\psi\|_{W^{1,\infty}(\Omega)} \qquad \text{for all } t \in (0,1) \text{ and } \psi \in W^{1,\infty}(\Omega) \tag{4.26}$$

and

$$\|e^{\Delta}\psi\|_{W^{1,q}(\Omega)} \le c_2 \|\psi\|_{L^1(\Omega)} \qquad \text{for all } t \in (0,1) \text{ and } \psi \in C^0(\overline{\Omega}) \tag{4.27}$$

as well as

$$\|e^{t\Delta}\psi\|_{W^{1,q}(\Omega)} \le c_3 t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|\psi\|_{L^p(\Omega)} \qquad \text{for all } t \in (0,1) \text{ and } \psi \in C^0(\overline{\Omega}), \tag{4.28}$$

which we use in the context of a Duhamel representation associated with the second equation in (2.6), according to which we have

$$\begin{aligned}
\|v_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} &= \left\| e^{(t-(t-1)_+)(\Delta-1)} v_\varepsilon\Big(\cdot,(t-1)_+\Big) + \int_{(t-1)_+}^t e^{(t-s)(\Delta-1)} u_\varepsilon(\cdot,s)ds \right\|_{W^{1,q}(\Omega)} \\
&\le \left\| e^{(t-(t-1)_+)\Delta} v_\varepsilon\Big(\cdot,(t-1)_+\Big)\right\|_{W^{1,q}(\Omega)} \\
&\quad + \int_{(t-1)_+}^t \|e^{(t-s)\Delta} u_\varepsilon(\cdot,s)\|_{W^{1,q}(\Omega)} ds \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1). \qquad (4.29)
\end{aligned}$$

Here, namely, if $t \le 1$ then by (4.26),

$$\left\| e^{(t-(t-1)_+)\Delta} v_\varepsilon\Big(\cdot,(t-1)_+\Big)\right\|_{W^{1,q}(\Omega)} = \|e^{t\Delta} v_0\|_{W^{1,q}(\Omega)} \le c_1 \|v_0\|_{W^{1,\infty}(\Omega)} \qquad \text{for all } \varepsilon\in(0,1), \quad (4.30)$$

while if $t>1$, then by (4.27),

$$\begin{aligned}
\left\| e^{(t-(t-1)_+)\Delta} v_\varepsilon\Big(\cdot,(t-1)_+\Big)\right\|_{W^{1,q}(\Omega)} &= \|e^{\Delta} v_\varepsilon(\cdot,t-1)\|_{W^{1,q}(\Omega)} \\
&\le c_2 \|v_\varepsilon(\cdot,t-1)\|_{L^1(\Omega)} \\
&\le c_4 \qquad \text{for all } \varepsilon\in(0,1) \qquad (4.31)
\end{aligned}$$

with $c_4 := c_2 \sup_{\varepsilon\in(0,1)} \sup_{s>0} \|v_\varepsilon(\cdot,s)\|_{L^1(\Omega)}$ being finite due to Lemma 2.5. Furthermore, using (4.28) and the Hölder inequality we see that for all $t>0$ and $\varepsilon\in(0,1)$,

$$\begin{aligned}
&\int_{(t-1)_+}^t \|e^{(t-s)\Delta} u_\varepsilon(\cdot,s)\|_{W^{1,q}(\Omega)} ds \\
&\qquad \le c_3 \int_{(t-1)_+}^t (t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|u_\varepsilon(\cdot,s)\|_{L^p(\Omega)} ds \\
&\qquad \le c_3 \cdot \left\{ \int_{(t-1)_+}^t \|u_\varepsilon(\cdot,s)\|_{L^p(\Omega)}^p ds \right\}^{\frac{1}{p}} \cdot \left\{ \int_{(t-1)_+^t} (t-s)^{-\frac{p}{p-1}\cdot[\frac{1}{2}+\frac{n}{2}(\frac{1}{p}-\frac{1}{q})]} ds \right\}^{\frac{p-1}{p}}.
\end{aligned}$$

Since

$$\int_{(t-1)_+}^t (t-s)^{-\frac{p}{p-1}\cdot[\frac{1}{2}+\frac{n}{2}(\frac{1}{p}-\frac{1}{q})]} ds \le c_5 := \int_0^1 \sigma^{-\frac{p}{p-1}\cdot[\frac{1}{2}+\frac{n}{2}(\frac{1}{p}-\frac{1}{q})]} d\sigma \qquad \text{for all } t>0,$$

and since (4.25) entails finiteness of c_5, in view of the boundedness properties of $0<t\mapsto \int_t^{t+1}\int_\Omega u_\varepsilon^p$ asserted by Corollary 4.5 due to the inequality $p<m+1$ the claim therefore results from (4.29), (4.30) and (4.31). □

5 L^∞ estimates for u_ε

Strongly relying on the possibility to choose the exponent in Lemma 4.6 to satisfy $q>n$, we can next achieve estimates for u_ε with respect to the norms in $L^p(\Omega)$, firstly for arbitrary finite p (Lemma 5.2) and then for $p = \infty$ (Lemma 5.3), on the basis of a rather straightforward and again variational-type procedure. This will be launched by the standard computation underlying the following basic step toward this, which will independently be used also in our deduction of regularity features enjoyed by spatial and temporal derivatives of u_ε (cf. Lemma 6.1 and Lemma 6.2).

Lemma 5.1 *Let $p > 0$ and $\psi \in C^\infty(\overline{\Omega})$. Then*

$$\begin{aligned}
\frac{1}{p}\int_\Omega \partial_t (u_\varepsilon+\varepsilon)^p \cdot \psi \;=\; & -(p-1)M\varepsilon \int_\Omega (u_\varepsilon+\varepsilon)^{p-2}(u_\varepsilon+1)^{M-1}|\nabla u_\varepsilon|^2\psi \\
& -(p-1)\int_\Omega (mu_\varepsilon+\varepsilon)(u_\varepsilon+\varepsilon)^{m+p-4}\phi(v_\varepsilon)|\nabla u_\varepsilon|^2\psi \\
& -(p-1)\int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{m+p-3}\phi'(v_\varepsilon)(\nabla u_\varepsilon\cdot\nabla v_\varepsilon)\psi \\
& -M\varepsilon\int_\Omega (u_\varepsilon+\varepsilon)^{p-1}(u_\varepsilon+1)^{M-1}\nabla u_\varepsilon\cdot\nabla\psi \\
& -\int_\Omega (mu_\varepsilon+\varepsilon)(u_\varepsilon+\varepsilon)^{m+p-3}\phi(v_\varepsilon)\nabla u_\varepsilon\cdot\nabla\psi \\
& -\int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{m+p-2}\phi'(v_\varepsilon)\nabla v_\varepsilon\cdot\nabla\psi \quad \textit{for all } t>0 \textit{ and } \varepsilon\in(0,1). \qquad (5.1)
\end{aligned}$$

Proof. This can be verified by straightforward computation based on several integrations by parts in the first equation from (2.6). □

Due to Lemma 4.6, for arbitrary $p > 1$ the identity obtained from (5.1) on taking $\psi \equiv 1$ can be turned into the following information on L^p regularity of $(u_\varepsilon)_{\varepsilon\in(0,1)}$.

Lemma 5.2 *Let $n \geq 2$ and $m > \frac{n}{2}$, and assume (1.9) with some $\alpha \geq 0$ fulfilling (1.11) and (1.12). Then for all $p > 1$ there exists $K^{(p)} : (0,\infty) \to (0,\infty)$ such that for all $T > 0$,*

$$\int_\Omega u_\varepsilon^p(\cdot,t) \leq K^{(p)}(T) \qquad \textit{for all } t\in(0,T) \textit{ and } \varepsilon\in(0,1), \qquad (5.2)$$

and that

$$\sup_{T>0} K^{(p)}(T) < \infty \quad \textit{if (1.15) holds.} \qquad (5.3)$$

Proof. From Lemma 4.6 we infer the existence of $\theta \in (1, \frac{n}{n-2})$ and $K_1 : (0,\infty) \to (0,\infty)$ such that for all $T > 0$,

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^{\frac{2\theta}{\theta-1}}(\Omega)} \leq K_1(T) \qquad \text{for all } t\in(0,T) \text{ and } \varepsilon\in(0,1), \qquad (5.4)$$

and that

$$\sup_{T>0} K_1(T) < \infty \quad \text{if (1.15) holds,} \qquad (5.5)$$

whereas (2.12) in conjunction with Lemma 4.6 and (1.7) provides $K_i : (0,\infty) \to (0,\infty)$, $i \in \{2,3\}$, such that for all $T > 0$,

$$\phi(v_\varepsilon) \geq K_2(T) \qquad \text{in } \Omega\times(0,T) \text{ for all } \varepsilon\in(0,1) \qquad (5.6)$$

and

$$\frac{\phi'^2(v_\varepsilon)}{\phi(v_\varepsilon)} \le K_3(T) \qquad \text{in } \Omega\times(0,T) \text{ for all } \varepsilon\in(0,1), \tag{5.7}$$

with

$$\sup_{T>0} K_i(T) < \infty \quad \text{for } i\in\{2,3\} \quad \text{if (1.15) holds.} \tag{5.8}$$

Then in the identity

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega (u_\varepsilon+\varepsilon)^p &+ (p-1)M\varepsilon\int_\Omega (u_\varepsilon+\varepsilon)^{p-2}(u_\varepsilon+1)^{M-1}|\nabla u_\varepsilon|^2 \\
&+(p-1)\int_\Omega (mu_\varepsilon+\varepsilon)(u_\varepsilon+\varepsilon)^{m+p-4}\phi(v_\varepsilon)|\nabla u_\varepsilon|^2 + \int_\Omega (u_\varepsilon+\varepsilon)^p \\
&= -(p-1)\int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{m+p-3}\phi'(v_\varepsilon)\nabla u_\varepsilon\cdot\nabla v_\varepsilon + \int_\Omega (u_\varepsilon+\varepsilon)^p,
\end{aligned} \tag{5.9}$$

as implied for $t>0$ and $\varepsilon\in(0,1)$ by Lemma 5.1, we first use that $m\ge 1$ in estimating $mu_\varepsilon+\varepsilon \ge u_\varepsilon+\varepsilon$ and hence

$$\begin{aligned}
(p-1)&\int_\Omega (mu_\varepsilon+\varepsilon)(u_\varepsilon+\varepsilon)^{m+p-4}\phi(v_\varepsilon)|\nabla u_\varepsilon|^2 \\
&\ge (p-1)\int_\Omega (u_\varepsilon+\varepsilon)^{m+p-3}\phi(v_\varepsilon)|\nabla u_\varepsilon|^2 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1).
\end{aligned} \tag{5.10}$$

Next, relying on Young's inequality and the Hölder inequality we can use (5.4) and (5.7) to see that for all $T>0$,

$$\begin{aligned}
-(p-1)&\int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{m+p-3}\phi'(v_\varepsilon)\nabla u_\varepsilon\cdot\nabla v_\varepsilon \\
&\le \frac{p-1}{2}\int_\Omega (u_\varepsilon+\varepsilon)^{m+p-3}\phi(v_\varepsilon)|\nabla u_\varepsilon|^2 + \frac{p-1}{2}\int_\Omega (u_\varepsilon+\varepsilon)^{m+p-1}\frac{\phi'^2(v_\varepsilon)}{\phi(v_\varepsilon)}|\nabla v_\varepsilon|^2 \\
&\le \frac{p-1}{2}\int_\Omega (u_\varepsilon+\varepsilon)^{m+p-3}\phi(v_\varepsilon)|\nabla u_\varepsilon|^2 + \frac{(p-1)K_3(T)}{2}\int_\Omega (u_\varepsilon+\varepsilon)^{m+p-1}|\nabla v_\varepsilon|^2 \\
&\le \frac{p-1}{2}\int_\Omega (u_\varepsilon+\varepsilon)^{m+p-3}\phi(v_\varepsilon)|\nabla u_\varepsilon|^2 \\
&\quad +\frac{(p-1)K_3(T)}{2}\cdot\left\{\int_\Omega (u_\varepsilon+\varepsilon)^{(m+p-1)\theta}\right\}^{\frac{1}{\theta}}\cdot\left\{\int_\Omega |\nabla v_\varepsilon|^{\frac{2\theta}{\theta-1}}\right\}^{\frac{\theta-1}{\theta}} \\
&\le \frac{p-1}{2}\int_\Omega (u_\varepsilon+\varepsilon)^{m+p-3}\phi(v_\varepsilon)|\nabla u_\varepsilon|^2 \\
&\quad +\frac{(p-1)K_3(T)}{2}K_1^2(T)\|(u_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|_{L^{2\theta}(\Omega)}^2 \qquad \text{for all } t\in(0,T) \text{ and } \varepsilon\in(0,1),
\end{aligned} \tag{5.11}$$

and that

$$\int_\Omega (u_\varepsilon+\varepsilon)^p \le \int_\Omega (u_\varepsilon+\varepsilon)^{m+p-1}+|\Omega| = \|(u_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|_{L^2(\Omega)}^2+|\Omega| \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1). \tag{5.12}$$

Here since $2\theta < \frac{2n}{n-2}$ and hence $W^{1,2}(\Omega)$ is continuously embedded into $L^{2\theta}(\Omega)$ and into $L^2(\Omega)$, combining an Ehrling-type inequality with (2.11) we obtain $K_i : (0,\infty) \to (0,\infty)$, $i \in \{4,5\}$, such that

$$\sup_{T>0} K_i(T) < \infty \quad \text{for } i \in \{4,5\} \quad \text{if (1.15) holds,}$$

and that for all $T > 0$,

$$\begin{aligned}
\frac{(p-1)K_3(T)}{2} & K_1^2(T) \|(u_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|_{L^{2\theta}(\Omega)}^2 + \|(u_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|_{L^2(\Omega)}^2 \\
&\le \frac{2(p-1)K_2(T)}{(m+p-1)^2} \|\nabla(u_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|_{L^2(\Omega)}^2 + K_4(T) \|(u_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|_{L^{\frac{2}{m+p-1}}(\Omega)}^2 \\
&\le \frac{2(p-1)K_2(T)}{(m+p-1)^2} \|\nabla(u_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|_{L^2(\Omega)}^2 + K_5(T) \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1).
\end{aligned}$$

As

$$\frac{p-1}{2} \int_\Omega (u_\varepsilon+\varepsilon)^{m+p-3} \phi(v_\varepsilon) |\nabla u_\varepsilon|^2 \ge \frac{2(p-1)K_2(T)}{(m+p-1)^2} \|\nabla(u_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|_{L^2(\Omega)}^2 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1)$$

by (5.6), from (5.9), (5.10), (5.11) and (5.12) we infer on dropping a favorably signed summand that for all $T > 0$,

$$\frac{1}{p}\frac{d}{dt} \int_\Omega (u_\varepsilon+\varepsilon)^p + \int_\Omega (u_\varepsilon+\varepsilon)^p \le K_5(T) + |\Omega| \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1),$$

and that thus, by a comparison argument,

$$\int_\Omega (u_\varepsilon+\varepsilon)^p \le \max\left\{ \int_\Omega (u_0+1)^p \, , \, K_5(T) + |\Omega| \right\} \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1),$$

which yields the claimed conclusion. □

The extension of the latter result to the case $p = \infty$ is now rather straightforward:

Lemma 5.3 *Suppose that $n \ge 2$ and $m > \frac{n}{2}$, and that ϕ is such that (1.9) is valid with some $\alpha \ge 0$ satisfying (1.11) and (1.12). Then there exists $K : (0,\infty) \to (0,\infty)$ such that for all $T > 0$,*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le K(T) \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1), \tag{5.13}$$

and that

$$\sup_{T>0} K(T) < \infty \quad \text{if (1.15) holds.} \tag{5.14}$$

PROOF. By means of well-known gradient estimates for the Neumann heat semigroup ([31, Lemma 1.3]), an application of Lemma 5.2 to suitably large p firstly shows that given any $q > 1$ one can find $K^{(q)} : (0,\infty) \to (0,\infty)$ such that

$$\sup_{T>0} K^{(q)}(T) < \infty \quad \text{if (1.15) holds,}$$

and that for all $T > 0$,

$$\|\nabla v_\varepsilon\|_{L^q(\Omega)} \le K^{(q)}(T) \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1).$$

Since (2.12) together with (1.7) and (1.8) provides $c_1 > 0$ and $c_2 > 0$ such that

$$\phi(v_\varepsilon) \ge c_1 \qquad \text{in } \Omega \times (0,\infty) \text{ for all } \varepsilon \in (0,1)$$

and

$$|\phi'(v_\varepsilon)| \le c_2 \qquad \text{in } \Omega \times (0,\infty) \text{ for all } \varepsilon \in (0,1),$$

the claimed boundedness property can readily be derived on the basis of Lemma 5.2 through a Moser-type iteration ([27, Lemma A.1]). □

6 First-order regularity properties of u_ε

In view of the nonlinear nature of the diffusion process in the first equation from (1.6), and the accordingly nonlinear manner in which u appears in (2.3), it seems in order to supplement the bounds for u_ε provided through Lemma 5.3 by some further ε-independent regularity information capable of implying at least some pointwise convergence properties of $(u_\varepsilon)_{\varepsilon\in(0,1)}$ along subsequences. This will be the objective of the next two lemmata which, again on the basis of Lemma 5.1, prepare an argument based on an application of an Aubin-Lions type lemma to $(u_\varepsilon+\varepsilon)^\beta$ with suitably chosen $\beta > 0$ in Lemma 7.1 below.

We first concentrate on the spatial gradient, for which we obtain the following.

Lemma 6.1 *Let $n \ge 2$ and $m > \frac{n}{2}$, and assume that ϕ satisfies (1.9) with some $\alpha \ge 0$ fulfilling (1.11) and (1.12). Then for all $T > 0$ there exists $C(T) > 0$ such that*

$$\int_0^T \int_\Omega (u_\varepsilon+\varepsilon)^{m+p-3} |\nabla u_\varepsilon|^2 \le C(T) \qquad \textit{for all } \varepsilon \in (0,1) \tag{6.1}$$

and

$$\varepsilon \int_0^T \int_\Omega (u_\varepsilon+\varepsilon)^{p-2} (u_\varepsilon+1)^{M-1} |\nabla u_\varepsilon|^2 \le C(T) \qquad \textit{for all } \varepsilon \in (0,1). \tag{6.2}$$

Proof. Relying on our assumptions of ϕ and α, we may combine the outcomes of Lemma 5.3 and Lemma 4.6 with (1.7) and (1.8) to see that given $T > 0$ we can fix $c_1(T) > 0, c_2 > 0, c_3 > 0$ and $c_4(T) > 0$ such that

$$u_\varepsilon \le c_1(T), \quad \phi(v_\varepsilon) \ge c_2 \quad \text{and} \quad |\phi'(v_\varepsilon)| \le c_3 \qquad \text{in } \Omega \times (0,T) \text{ for all } \varepsilon \in (0,1), \tag{6.3}$$

and that

$$\int_\Omega |\nabla v_\varepsilon|^2 \le c_4(T) \qquad \text{for all } t \in (0,T) \text{ and each } \varepsilon \in (0,1), \tag{6.4}$$

where the first inequality in (6.3) warrants that in order to prove the lemma it will be sufficient to consider the case $p \in (0,1)$ only. For such p, an application of Lemma 5.1 to $\psi \equiv 1$ shows that for all $t > 0$ and $\varepsilon \in (0,1)$,

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega (u_\varepsilon+\varepsilon)^p &= (1-p)M\varepsilon\int_\Omega (u_\varepsilon+\varepsilon)^{p-2}(u_\varepsilon+1)^{M-1}|\nabla u_\varepsilon|^2 \\
&\quad +(1-p)\int_\Omega (mu_\varepsilon+\varepsilon)(u_\varepsilon+\varepsilon)^{m+p-4}\phi(v_\varepsilon)|\nabla u_\varepsilon|^2 \\
&\quad +(1-p)\int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{m+p-3}\phi'(v_\varepsilon)\nabla u_\varepsilon\cdot\nabla v_\varepsilon, \qquad (6.5)
\end{aligned}$$

where by Young's inequality, (6.3) and (6.4),

$$\begin{aligned}
&\left|(1-p)\int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{m+p-3}\phi'(v_\varepsilon)\nabla u_\varepsilon\cdot\nabla v_\varepsilon\right| \\
&\quad\le \frac{1-p}{2}\int_\Omega (mu_\varepsilon+\varepsilon)(u_\varepsilon+\varepsilon)^{m+p-4}\phi(v_\varepsilon)|\nabla u_\varepsilon|^2 \\
&\qquad +\frac{1-p}{2}\int_\Omega \frac{u_\varepsilon^2}{mu_\varepsilon+\varepsilon}(u_\varepsilon+\varepsilon)^{m+p-2}\frac{\phi'^2(v_\varepsilon)}{\phi(v_\varepsilon)}|\nabla v_\varepsilon|^2 \\
&\quad\le \frac{1-p}{2}\int_\Omega (mu_\varepsilon+\varepsilon)(u_\varepsilon+\varepsilon)^{m+p-4}\phi(v_\varepsilon)|\nabla u_\varepsilon|^2 \\
&\qquad +\frac{1-p}{2m}\int_\Omega (u_\varepsilon+\varepsilon)^{m+p-1}\frac{\phi'^2(v_\varepsilon)}{\phi(v_\varepsilon)}|\nabla v_\varepsilon|^2 \\
&\quad\le \frac{1-p}{2}\int_\Omega (mu_\varepsilon+\varepsilon)(u_\varepsilon+\varepsilon)^{m+p-4}\phi(v_\varepsilon)|\nabla u_\varepsilon|^2 \\
&\qquad +\frac{1-p}{2m}\cdot(c_1(T)+1)^{m+p-1}\cdot\frac{c_3^2}{c_2}\cdot c_4(T) \qquad \text{for all } t\in(0,T) \text{ and } \varepsilon\in(0,1).
\end{aligned}$$

Since, as further consequences of (6.3), we have

$$\begin{aligned}
&\frac{1-p}{2}\int_\Omega (mu_\varepsilon+\varepsilon)(u_\varepsilon+\varepsilon)^{m+p-4}\phi(v_\varepsilon)|\nabla u_\varepsilon|^2 \\
&\qquad \ge \frac{(1-p)c_2}{2}\int_\Omega (u_\varepsilon+\varepsilon)^{m+p-3}|\nabla u_\varepsilon|^2 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),
\end{aligned}$$

and

$$\frac{1}{p}\int_\Omega \Big(u_\varepsilon(\cdot,T)+\varepsilon\Big)^p \le \frac{1}{p}\cdot(c_1(T)+1)^p|\Omega| \qquad \text{for all } \varepsilon\in(0,1),$$

an integration of (6.5) over $t\in(0,T)$ yields both claimed estimates. $\square$

Again through the identity from Lemma 5.1, the latter also entails some regularity feature of the corresponding time derivatives:

Lemma 6.2 *Let $n \geq 2$ and $m > \frac{n}{2}$, and let ϕ satisfy (1.9) with some $\alpha \geq 0$ satisfying (1.11) and (1.12). Then for each $k \in \mathbb{N}$ satisfying $k > \frac{n}{2}$, any $p > 0$ and all $T > 0$ one can find $C(k,p,T) > 0$ such that*

$$\int_0^T \left\| \partial_t \Big(u_\varepsilon(\cdot,t) + \varepsilon \Big)^p \right\|_{(W^{k,2}(\Omega))^\star} dt \leq C(k,p,T) \qquad \textit{for all } \varepsilon \in (0,1). \tag{6.6}$$

PROOF. Given $T > 0$, let us once more recall Lemma 5.3, Lemma 4.6, (1.7) and (1.8) to find positive constants $c_1(T), c_2, c_3$ and $c_4(T)$ such that

$$u_\varepsilon \leq c_1(T), \quad \phi(v_\varepsilon) \leq c_2 \quad \text{and} \quad |\phi'(v_\varepsilon)| \leq c_3 \qquad \text{in } \Omega \times (0,T) \text{ for all } \varepsilon \in (0,1) \tag{6.7}$$

and

$$\int_\Omega |\nabla v_\varepsilon|^2 \leq c_4(T) \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1), \tag{6.8}$$

and moreover rely on the continuity of the embedding $W^{k,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ to choose $c_5 > 0$ fulfilling $\|\psi\|_{L^\infty(\Omega)} \leq c_5 \|\psi\|_{W^{k,2}(\Omega)}$ for all $\psi \in C^\infty(\overline{\Omega})$.

For fixed $\psi \in C^\infty(\overline{\Omega})$ satisfying $\|\psi\|_{W^{k,2}(\Omega)} \leq 1$, from Lemma 5.1 we then obtain that due to (6.7),

$$\begin{aligned}
&\left| \frac{1}{p} \int_\Omega \partial_t (u_\varepsilon + \varepsilon)^p \psi \right| \\
&\qquad \leq \ |p-1| M c_5 \varepsilon \int_\Omega (u_\varepsilon + \varepsilon)^{p-2} (u_\varepsilon + 1)^{M-1} |\nabla u_\varepsilon|^2 \\
&\qquad\quad + |p-1| m c_2 c_5 \int_\Omega (u_\varepsilon + \varepsilon)^{m+p-3} |\nabla u_\varepsilon|^2 \\
&\qquad\quad + |p-1| c_3 c_5 \int_\Omega (u_\varepsilon + \varepsilon)^{m+p-2} |\nabla u_\varepsilon| \cdot |\nabla v_\varepsilon| \\
&\qquad\quad + M\varepsilon \int_\Omega (u_\varepsilon + \varepsilon)^{p-1} (u_\varepsilon + 1)^{M-1} |\nabla u_\varepsilon| \\
&\qquad\quad + m c_2 \int_\Omega (u_\varepsilon + \varepsilon)^{m+p-2} |\nabla u_\varepsilon| \\
&\qquad\quad + c_3 \int_\Omega (u_\varepsilon + \varepsilon)^{m+p-1} |\nabla v_\varepsilon| \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1).
\end{aligned} \tag{6.9}$$

Since herein Young's inequality together with (6.7) and (6.8) ensures that for all $t \in (0,T)$ and $\varepsilon \in (0,1)$ we have

$$\begin{aligned}
\int_\Omega (u_\varepsilon + \varepsilon)^{m+p-2} |\nabla u_\varepsilon| \cdot |\nabla v_\varepsilon| \ &\leq \ \int_\Omega (u_\varepsilon + \varepsilon)^{m+p-3} |\nabla u_\varepsilon|^2 + \frac{1}{4} \int_\Omega (u_\varepsilon + \varepsilon)^{m+p-1} |\nabla v_\varepsilon|^2 \\
&\leq \ \int_\Omega (u_\varepsilon + \varepsilon)^{m+p-3} |\nabla u_\varepsilon|^2 + \frac{1}{4} \cdot (c_1(T) + 1)^{m+p-1} c_4(T)
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon \int_\Omega (u_\varepsilon + \varepsilon)^{p-1} (u_\varepsilon + 1)^{M-1} |\nabla u_\varepsilon| \ &\leq \ \varepsilon \int_\Omega (u_\varepsilon + \varepsilon)^{p-2} (u_\varepsilon + 1)^{M-1} |\nabla u_\varepsilon|^2 + \frac{\varepsilon}{4} \int_\Omega (u_\varepsilon + \varepsilon)^p (u_\varepsilon + 1)^{M-1} \\
&\leq \ \varepsilon \int_\Omega (u_\varepsilon + \varepsilon)^{p-2} (u_\varepsilon + 1)^{M-1} |\nabla u_\varepsilon|^2 + \frac{\varepsilon}{4} \cdot (c_1(T) + 1)^{p+M-1} |\Omega|
\end{aligned}$$

as well as

$$\begin{aligned}\int_\Omega (u_\varepsilon+\varepsilon)^{m+p-1}|\nabla v_\varepsilon| &\le \int_\Omega (u_\varepsilon+\varepsilon)^{2(m+p-1)} + \frac{1}{4}\int_\Omega |\nabla v_\varepsilon|^2 \\ &\le (c_1(T)+1)^{2(m+p-1)} + \frac{1}{4}c_4(T),\end{aligned}$$

from (6.9) we infer the existence of $c_6(k,p,T)>0$ fulfilling

$$\begin{aligned}&\Big\|\partial_t(u_\varepsilon+\varepsilon)^p\Big\|_{(W^{k,2}(\Omega))^\star} \\ &\quad\le c_6(k,p,T)\varepsilon\int_\Omega (u_\varepsilon+\varepsilon)^{p-2}(u_\varepsilon+1)^{M-1}|\nabla u_\varepsilon|^2 \\ &\qquad +c_6(k,p,T)\int_\Omega (u_\varepsilon+\varepsilon)^{m+p-3}|\nabla u_\varepsilon|^2 + c_6(p,k,T) \qquad \text{for all } t\in(0,T) \text{ and } \varepsilon\in(0,1).\end{aligned}$$

In view of Lemma 6.1, the claimed statement therefore results upon an integration over $(0,T)$. □

7 Passing to the limit. Proof of Theorem 1.1 and Theorem 1.2

We now only need to appropriately combine our estimates collected above, and the compacntess properties thus implied, to accomplish the main step toward our main results by suitably passing to the limit $\varepsilon\searrow 0$.

Lemma 7.1 *Let $n\ge 2$, $m>\frac{n}{2}$ and ϕ be such that (1.9) holds with some $\alpha\ge 0$ satisfying (1.11) and (1.12). Then there exist $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ as well as nonnegative functions*

$$\begin{cases} u\in L^\infty_{loc}(\overline{\Omega}\times[0,\infty)) & \text{and} \\ v\in C^0(\overline{\Omega}\times[0,\infty))\cap L^2_{loc}([0,\infty);W^{1,2}(\Omega)) \end{cases} \tag{7.1}$$

such that $\varepsilon_j\searrow 0$ as $j\to\infty$, and that as $\varepsilon=\varepsilon_j\searrow 0$ we have

$$u_\varepsilon\to u \quad \text{in } \bigcap_{p\ge 1} L^p_{loc}(\overline{\Omega}\times[0,\infty)) \text{ and a.e. in } \Omega\times(0,\infty), \tag{7.2}$$

$$v_\varepsilon\to v \quad \text{in } C^0_{loc}(\overline{\Omega}\times[0,\infty)) \qquad \text{and} \tag{7.3}$$

$$\nabla v_\varepsilon\rightharpoonup\nabla v \quad \text{in } L^2_{loc}(\overline{\Omega}\times[0,\infty)). \tag{7.4}$$

Moreover, $v>0$ in $\overline{\Omega}\times[0,\infty)$, and (u,v) forms a global weak solution of (1.6) in the sense of Definition 2.1.

Proof. We fix any $\beta>\frac{m-1}{2}$ and $k\in\mathbb{N}$ such that $k>\frac{n}{2}$, and then infer from Lemma 6.1, (2.11) and Lemma 6.2 that for all $T>0$,

$$\Big((u_\varepsilon+\varepsilon)^\beta\Big)_{\varepsilon\in(0,1)} \text{ is bounded in } L^2((0,T);W^{1,2}(\Omega)),$$

and that

$$\Big(\partial_t(u_\varepsilon+\varepsilon)^\beta\Big)_{\varepsilon\in(0,1)} \text{ is bounded in } L^1((0,T);(W^{k,2}(\Omega))^\star).$$

Apart from that, thanks to Lemma 4.6 and Lemma 5.3 we may invoke standard theory on Hölder regularity in scalar parabolic equations ([24]) to see that given any $T>0$ we can find $\vartheta=\vartheta(T)\in(0,1)$ such that

$$(v_\varepsilon)_{\varepsilon\in(0,1)} \text{ is bounded in } C^{\vartheta,\frac{\vartheta}{2}}(\overline{\Omega}\times[0,T]),$$

and that

$$(u_\varepsilon)_{\varepsilon\in(0,1)} \text{ is bounded in } L^\infty(\Omega\times(0,T))$$

and

$$(\nabla v_\varepsilon)_{\varepsilon\in(0,1)} \text{ is bounded in } L^2(\Omega\times(0,T)).$$

An application of an Aubin-Lions type lemma ([29]) and the Arzelà-Ascoli theorem therefore readily yields nonnegative functions u and v fulfilling (7.1), as well as a sequence $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ such that $\varepsilon_j\searrow 0$ as $j\to\infty$ and that (7.2)-(7.4) hold as $\varepsilon=\varepsilon_j\searrow 0$.

Positivity of v in $\overline{\Omega}\times[0,\infty)$ is therefore immediate from (2.12) and (7.3), and a verification of the identities in (2.3) and (2.4) can be achieved in a straightforward manner on the basis of (7.2), (7.3) and (7.4) in view of the fact that due to the continuity of ϕ and ϕ' on $(0,\infty)$, from (7.3) we also know that $\phi(v_\varepsilon)\to\phi(v)$ and $\phi'(v_\varepsilon)\to\phi'(v)$ in $C^0_{loc}(\overline{\Omega}\times[0,\infty))$ as $\varepsilon=\varepsilon_j\searrow 0$. $\square$

Thanks to the additional information on the respective functions K provided by (5.14) and (4.24), under the stronger assumption on α in (1.15) the additional boundedness statement from Theorem 1.2 actually reduces to a by-product:

Lemma 7.2 *Let $n\ge 2$ and $m>\frac{n}{2}$, and assume that (1.9) holds with some $\alpha\ge 0$ which beyond (1.11) and (1.12) also satisfies (1.15) Then the global weak solution of (1.6) obtained in Lemma 7.1 has the additional property that with some $C>0$ we have*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}\le C \qquad \textit{for a.e. } t>0 \tag{7.5}$$

and

$$\|v(\cdot,t)\|_{L^\infty(\Omega)}\le C \qquad \textit{for all } t>0. \tag{7.6}$$

Proof. The boundedness feature in (7.5) is a consequence of (5.13) and (5.14) when combined with (7.2), whereas that in (7.6) can similarly be obtained from (4.23), (4.24) and (7.3), because $W^{1,q}(\Omega)\hookrightarrow L^\infty(\Omega)$ when $q>n$. $\square$

To finally obtain our main results, we only need to summarize:

Proof of Theorem 1.1. All statements have been asserted by Lemma 7.1 already. $\square$

Proof of Theorem 1.2. We only need to apply Lemma 7.2. $\square$

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