Does spatial homogeneity ultimately prevail in nutrient taxis systems? A paradigm for structure support by rapid diffusion decay in an autonomous parabolic flow

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Abstract

This work is concerned with the doubly degenerate cross-diffusion system

$$\begin{cases} u_t = (uvu_x)_x - (u^2vv_x)_x + uv, \\ v_t = v_{xx} - uv, \end{cases} \tag{0.1}$$

that has been proposed as a model for experimentally observable quite complex pattern formation phenomena in bacterial populations.

It is shown that for any initial data satisfying adequate regularity and positivity assumptions, a no-flux initial-boundary value problem for (0.1) in a bounded real interval possesses a global weak solution which is continuous in its first and essentially smooth in its second component.

This solution is seen to asymptotically stabilize in the sense that

$$u(\cdot,t) \to u_\infty \quad \text{and} \quad v(\cdot,t) \to 0 \qquad \text{as } t \to \infty \tag{0.2}$$

with some nonnegative $u_\infty \in C^0(\overline{\Omega})$ which can be obtained as the evaluation of a weak solution $z \in C^0(\overline{\Omega} \times [0,1])$ to a porous medium-type parabolic problem at the finite time 1.

It is moreover revealed that for each suitably regular nonnegative function $u_\star$ on Ω, the pair $(u_\star, 0)$, formally constituting an equilibrium of (0.1), is stable in an appropriate sense. This finally implies a sufficient criterion for the limit u_∞ in (0.2) to be spatially heterogeneous.

The latter properties are in sharp contrast to known asymptotic features of corresponding nutrient taxis systems involving linear non-degenerate diffusion, as for which the literature appears to exclusively provide results on solutions which approach spatially constant states in the large time limit.

Key words: chemotaxis; boundedness; degenerate diffusion; pattern formation; stability
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1 Introduction

The tendency to evolve toward equilibrium belongs to the fundamental features naturally inherent to large classes of parabolic flows. Beyond a virtually inexhaustible multitude of findings concerned with convergence to steady states in numerous particular cases, the literature in fact provides considerably comprehensive rigorous evidence for corresponding stabilization properties of bounded solutions in quite general problem classes ([25], [12], [16], [24]). As a common characteristic, however, most among the systems well-understood in this regard share the peculiarity that their equilibria are either trivial or reflect an appropriate balance of diffusion and the action of reaction-type sources; as the richness especially of the stable among such steady state constellations is usually limited, this goes along with evident confinements with regard to the ability of adequately describing systems in which significant trends to support large varieties of structures are to be expected.

One purpose of the present work consists in rigorously confirming that when suitably accounting for possible subtleties with respect to particle motility, even some quite artless parabolic systems may nevertheless well be appropriate models also under such circumstances of increased complexity. This will be substantiated in the context of a two-component reaction-(cross-)diffusion system which has been proposed to describe bacterial patterning in particular physical frameworks, and a main mathematical feature of which lies in a certain density-dependent limitation of the incorporated diffusion mechanisms. According to a correspondingly included degeneracy of parabolicity, this system admits an abundantly rich – and especially uncountable – set of steady states, and the analysis to be developed in this work will, inter alia, reveal that *each* of these equilibria enjoys some stability property, and that any solution to an associated initial value problem stabilizes toward one among these states.

The challenge of detecting nontrivial asymptotics in nutrient taxis systems. Describing the emergence and evolution of structures appears to be among the most challenging topics in the analysis of models for chemotaxis processes. Indeed, in the context of reinforced taxis mechanisms such as addressed in the classical Keller-Segel system

$$\begin{cases} u_t = \Delta u + \nabla \cdot (u \nabla v), \\ v_t = \Delta v - v + u, \end{cases} \tag{1.3}$$

and some of its close relatives, the literature of the past years could rigorously detect quite a few among the colorful dynamical features of such types of interplay which have been predicted by studies based on either formal analysis or numerical simulations ([5], [28], [31]). Beyond the description of aggregation phenomena in the mathematically extreme sense of finite-time blow-up in appropriate settings ([14], [27], [37]) the existence of spatially heterogeneous steady states ([29], [30], [18], [8], [1]) as well as the role of equilibria in the large time asymptotics ([9], [13]) have been the objectives of numerous contributions in this direction; recent findings have furthermore revealed certain facets of colorful solution behavior at intermediate time scales ([20], [39]).

The corresponding state of knowledge is much weaker in situations when instead of signal production mechanisms as modeled in (1.3), chemotactic migration is directed by a signal substance which is not produced but rather consumed by cells. At the level of biological experiments, observations in such nutrient taxis systems witness the formation of quite strongly structured patterns, exhibiting up to fractal-like complexity, even in very simple settings such as determined by bacteria of the species *Bacillus subtilis* grown on the surface of thin agar plates, especially in presence of sparse nutrient ([6], [11], [10], [26]). However, to the best of our knowledge virtually none of these findings could so far appropriately be captured by any result from mathematical analysis concerned with models for such processes which are evidently lacking any external driving force as well as any self-enhancing effect on cell migration such as described in (1.3) by the signal production term $+u$ in its second equation: in nutrient taxis systems of the form

$$\begin{cases} u_t = \Delta u + \nabla \cdot (uS(u,v)\nabla v) + f(u,v), \\ v_t = \Delta v - uv, \end{cases} \tag{1.4}$$

various choices of the sensitivity and proliferation coefficient functions S and f have been proposed in the literature, but no nontrivial large time behavior could be discovered in any of the previously studied cases. When posed along with no-flux boundary conditions in bounded convex domains $\Omega \subset \mathbb{R}^n$, in the prototypical setting obtained on letting $S \equiv 1$ and $f \equiv 0$, for instance, (1.4) apparently enforces asymptotics exclusively characterized by spatial homogenization: namely, corresponding initial value problems involving reasonably regular but arbitrarily large initial data are known to possess global bounded classical solutions when $n = 2$, whereas global weak and eventually smooth solutions can be constructed when $n = 3$, but in both cases each of these solutions approaches one of the spatially homogeneous equilibria given by $u \equiv a$ and $v \equiv 0$ with appropriate $a \geq 0$ ([35]); even additionally accounting for nutrient-induced proliferation by choosing $f(u,v) = uv$ does not essentially change this asymptotic property of (1.4) ([41]). Together with further results of a similar flavor for the case when $S(u,v) = \frac{1}{v}$ and $f \equiv 0$ ([4]), or when even couplings to surrounding liquid media are included ([21], [38], [40]), these findings suggest that in the context of (1.4), the most colorful large-time dynamics that can at all be expected consists in wave-like propagation phenomena, which in fact have been found to occur for the latter specific version of (1.4), but which according to their particular nature eventually lead to spatial homogeneity in each bounded spatial region ([17]).

Nutrient taxis involving signal-dependent degenerate diffusion. In the present work we shall see that this situation may become substantially different when unlike in (1.4), cell diffusion nonlinearly degenerates at small signal densities. Specifically, we shall be concerned with the nutrient taxis system

$$\begin{cases} u_t = \nabla \cdot (uv\nabla u) - \chi \nabla \cdot (u^2 v \nabla v) + uv, \\ v_t = \Delta v - uv, \end{cases} \tag{1.5}$$

for $\chi > 0$, which has been proposed in [23], and recently also rigorously been derived by means of parabolic limits in [32], as a model for the bacterial pattern formation phenomena reported in [11], [10] and [26]. In fact, numerical simulations performed in [23] for the spatially two-dimensional version of (1.5) indicate that essential experimental observations such as e.g. quite complex forms of structure formation, including branch-like patterning, are quite precisely reflected in the corresponding behavior of solutions, with the level of this conformity apparently being significantly higher in (1.5) than in the corresponding taxis-free simplification thereof obtained on letting $\chi = 0$ (cf. the detailed discussion in [23, Sections 4 and 5]).

Main results. The main goal of this work will be to rigorously capture part of these features, and our results will indicate that in stark contrast to the situation in (1.4) discussed above, already in the spatially one-dimensional version of (1.5) a considerable dynamical complexity can be observed also at large time scales, and that due to the signal-dependent diffusion degeneracy this is possible even despite the fact that each individual trajectory approaches a steady state. Indeed, equilibria (u,v) of (1.5) with $u \not\equiv 0$ yet vanish identically in their second component, as do those of (1.4), but unlike in the latter system there is virtually unlimited freedom in the first component in the sense that any reasonably regular nontrivial nonnegative function u at least formally defines a steady state $(u,0)$ of (1.5). Now constituting one of the probably most striking properties of (1.5), it will turn out that within this large continuum of equilibria, in fact each individual one is stable in an appropriate sense, thus reflecting the ability of (1.5) to support a large variety of arbitrarily complex asymptotic profiles.

Our particular analysis will address the initial-boundary value problem for the spatially one-dimensional normalized version of (1.5) given by

$$\begin{cases} u_t = (uvu_x)_x - (u^2vv_x)_x + uv, & x\in\Omega,\ t>0,\\ v_t = v_{xx} - uv, & x\in\Omega,\ t>0,\\ uvu_x - u^2vv_x = 0, \quad v_x = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{1.6}$$

in a bounded open interval $\Omega\subset\mathbb{R}$, where the initial data in (1.6) will be assumed to be such that

$$\begin{cases} u_0\in C^\vartheta(\overline{\Omega}) \text{ for some } \vartheta\in(0,1), \text{ with } u_0\ge 0 \text{ and } \int_\Omega \ln u_0 > -\infty, \qquad \text{and that}\\ v_0\in W^{1,\infty}(\Omega) \text{ satisfies } v_0>0 \text{ in } \overline{\Omega}. \end{cases} \tag{1.7}$$

In this framework, we shall firstly address the basic issue of global solvability. Here due to the degeneracy of diffusion in (1.6), which can actually be regarded stronger than that in the associated porous medium equation $u_t = (uu_x)_x$, in view of known results on limitations of smoothness in the latter we cannot expect the component u to possess regularity properties substantially beyond continuity. We therefore believe that the following global existence result is essentially optimal in this respect.

Theorem 1.1 *Let $\Omega\subset\mathbb{R}$ be a bounded open interval, and suppose that (1.7) holds. Then there exists at least one pair of nonnegative functions u and v which form a global weak solution of (1.6) in the sense of Definition 2.1 below, and which moreover satisfy*

$$\begin{cases} u\in C^0(\overline{\Omega}\times[0,\infty))\cap L^2_{loc}([0,\infty);W^{1,2}(\Omega)) \qquad \textit{and}\\ v\in C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty)). \end{cases} \tag{1.8}$$

Next addressing the asymptotic behavior of these solutions, we first make sure that their second components must vanish in the large time limit, and that this decay occurs in quite a regular manner: namely, besides the quantity v also the corresponding spatial gradient uniformly approaches zero, and moreover v satisfies the temporally uniform Harnack-type inequality (1.11) which inter alia implies that the corresponding decay rate is spatially uniform in the sense that each of the scaled trajectories $\left(\frac{v(\cdot,t)}{\|v(\cdot,t)\|_{L^\infty(\Omega)}}\right)_{t>0}$ forms a set of functions with values in $[C,1]$ with some $C>0$.

Theorem 1.2 *If (1.7) holds, then the global weak solution (u,v) obtained in Theorem 1.1 has the additional properties that v decays in the sense that*

$$\int_0^\infty \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)}dt < \infty \tag{1.9}$$

as well as

$$v(\cdot,t)\to 0 \quad \textit{in } W^{1,\infty}(\Omega) \qquad \textit{as } t\to\infty. \tag{1.10}$$

Furthermore, there exists $C>0$ such that

$$\min_{x\in\overline{\Omega}} v(x,t) \ge C\cdot \max_{x\in\overline{\Omega}} v(x,t) \qquad \textit{for all } t>0. \tag{1.11}$$

It may now be considered intriguing that despite these highly uniform decay properties of the chemoattractant, the asymptotics of the respective first solution components are determined by substantially more subtle mechanisms: namely, we shall see that for each individual solution the corresponding quantity u also stabilizes toward a continuous limit function u_∞, but that this final profile need no longer be trivial nor even only independent of the choice of the initial data; in fact, it turns out that u_∞ coincides with the spatial profile of a solution to a scalar parabolic equation, evaluated at some *finite* time, where this parabolic equation is essentially of porous medium type, with the degeneracy of the diffusion process therein thus being of rather well-understood type.

We find it worth underlining here that to the best of our knowledge, the literature only contains very few precedents detecting such finite-time evaluations as relevant to the final-time asymptotics in parabolic equations, and in each of these cases the respective phenomenon can only be observed upon an appropriate rescaling of the solution in amplitude ([3], [36]).

Theorem 1.3 *Assume (1.7), and let (u,v) denote the corresponding solution of (1.6) from Theorem 1.1. Then u is bounded in $\Omega\times(0,\infty)$ with*

$$u(\cdot,t)>0 \qquad a.e.\ in\ \Omega\ for\ all\ t>0, \tag{1.12}$$

and there exists $u_\infty\in C^0(\overline{\Omega})$ such that

$$u(\cdot,t)\to u_\infty \quad in\ L^\infty(\Omega) \qquad as\ t\to\infty. \tag{1.13}$$

Moreover, this limit function satisfies $u_\infty=z(\cdot,1)$, with $z\in C^0(\overline{\Omega}\times[0,1])\cap L^2_{loc}([0,1);W^{1,2}(\Omega))$ being a weak solution, in the sense specified in Lemma 10.4 below, of

$$\begin{cases} z_\tau = \Big(a(x,\tau)zz_x\Big)_x - \Big(b(x,\tau)z^2\Big)_x + a(x,\tau)z, & x\in\Omega,\ \tau\in(0,1),\\ z_x=0, & x\in\partial\Omega,\ \tau\in(0,1),\\ z(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.14}$$

where

$$a(x,\tau):=J\cdot\frac{v(x,t)}{\|v(\cdot,t)\|_{L^\infty(\Omega)}} \quad and \quad b(x,\tau):=J\cdot\frac{v(x,t)v_x(x,t)}{\|v(\cdot,t)\|_{L^\infty(\Omega)}} \quad for\ x\in\Omega,\ \ \tau\in(0,1)\ and\ t=\rho^{-1}(\tau), \tag{1.15}$$

with

$$J:=\int_0^\infty \|v(\cdot,t)\|_{L^\infty(\Omega)}dt \qquad and \qquad \rho(t):=\frac{1}{J}\cdot\int_0^t \|v(\cdot,s)\|_{L^\infty(\Omega)}ds, \quad t\ge 0, \tag{1.16}$$

are such that there exists $C>0$ fulfilling

$$\frac{1}{C}\le a(x,\tau)\le C \quad and \quad |b(x,\tau)|\le C \qquad for\ all\ x\in\Omega\ and\ \tau\in(0,1). \tag{1.17}$$

Natural next questions consist in determining which functions from $C^0(\overline{\Omega})$ do appear as limits in (1.13), and more generally, in characterizing stability properties of the functions $(u_\star,0)$, with suitably regular but otherwise arbitrary nonnegative $u_\star$, when considered as a steady state of (1.6). Among the possibly most striking properties of (1.6), the following result asserts that actually each of these equilibria is stable in an appropriate sense.

Theorem 1.4 *Let $p>\frac{8}{3}, q>\frac{p}{p-2}$ and $K>0$. Then for all $\eta>0$ there exists $\delta>0$ with the following property: if $u_\star\in(W_0^{2,q}(\Omega))^\star$ is nonnegative and u_0 and v_0 are such that beyond (1.7) we have*

$$\|u_0\|_{L^p(\Omega)}\le K \qquad and \qquad \int_\Omega \ln u_0\ge -K \tag{1.18}$$

and

$$\|v_0^{\frac{3}{2(p+1)}}\|_{W^{1,\frac{2(p+1)(p+2)}{p+4}}(\Omega)}\le K \tag{1.19}$$

as well as

$$\|u_0 - u_\star\|_{(W_0^{2,q}(\Omega))^\star} \le \delta \qquad \textit{and} \qquad \|v_0\|_{L^1(\Omega)} \le \delta, \tag{1.20}$$

then the corresponding solution (u,v) of (1.6) from Theorem 1.1 satisfies

$$\|u(\cdot,t) - u_\star\|_{(W_0^{2,q}(\Omega))^\star} \le \eta \quad \textit{and} \quad \|v(\cdot,t)\|_{L^1(\Omega)} \le \eta \qquad \textit{for all } t > 0. \tag{1.21}$$

Not surprisingly, the latter can finally be seen to imply an at least partial answer to the question how far Theorem 1.3 indeed describes pattern formation in the sense of stabilization toward spatially heterogeneous states:

Corollary 1.5 *Let $u_0 \in \bigcup_{\vartheta\in(0,1)} C^\vartheta(\overline{\Omega})$ be nonnegative with $\int_\Omega \ln u_0 > -\infty$, and suppose that $u_0 \not\equiv const.$ Then for all $K > 0$ there exists $\delta > 0$ such that whenever $v_0 \in W^{1,\infty}(\Omega)$ is positive in $\overline{\Omega}$ with*

$$\|v_0^{\frac{3}{2(p+1)}}\|_{W^{1,\frac{2(p+1)(p+2)}{p+4}}(\Omega)} \le K \tag{1.22}$$

and

$$\|v_0\|_{L^1(\Omega)} \le \delta, \tag{1.23}$$

for the solution (u,v) of (1.6) from Theorem 1.1 we have

$$u(\cdot,t) \to u_\infty \quad \textit{in } L^\infty(\Omega) \qquad \textit{and} \qquad v(\cdot,t) \to 0 \quad \textit{in } W^{1,\infty}(\Omega) \tag{1.24}$$

as $t \to \infty$, where

$$u_\infty \not\equiv const. \tag{1.25}$$

Main ideas. As a fundamental starting point, our approach will make use of the gradient-like structure of (1.6) formally expressed in the energy identity

$$\frac{d}{dt}\Big\{ -\int_\Omega \ln u + \frac{1}{2}\int_\Omega v_x^2 \Big\} = -\int_\Omega \frac{v}{u} u_x^2 - \int_\Omega v_{xx}^2 - \int_\Omega u v_x^2 - \int_\Omega v. \tag{1.26}$$

A first key step will consist in turning a bound for v in $L^1(\Omega\times(0,\infty))$, as resulting from a rigorous counterpart of (1.26) for solutions to appropriately regularized variants of (1.6) in a rather direct manner, into an estimate for

$$\int_0^\infty \|v(\cdot,t)\|_{L^\infty(\Omega)} dt. \tag{1.27}$$

This will be achieved in Section 3 by using further regularity information implied by (1.26) in the course of an analysis of the time evolution of $\int_\Omega \frac{v_x^2}{v}$.
As a first application, in Section 4 this estimate will be utilized in our derivation of bounds for u in $L^p(\Omega)$ with arbitrary $p \ge 2$, which at its core will result from a differential inequality of the form

$$\frac{d}{dt}\Big\{ \int_\Omega u^p + \int_\Omega v^{-\alpha}|v_x|^q \Big\} \le C\|v\|_{L^\infty(\Omega)} \cdot \Big\{ \int_\Omega u^p + 1 \Big\}$$

with suitably chosen $q = q(p) > 1$ and $\alpha = \alpha(p) > 0$ (Lemma 4.7). Section 5 will thereafter make use of this for appropriately large p to derive a pointwise bound for the quantity $(\ln v)_x$, which will not only form an essential basis for the decay properties of v from Theorem 1.2, but which together with the estimate for the expression in (1.27) will moreover serve as a crucial ingredient in our derivation of an L^∞ bound for u through a newly developed Moser-type iterative procedure in Section 6. By means of these and some further higher regularity properties documented in Section 7, the Sections 8 and 9 will assert the statements on global existence and decay of v from Theorem 1.1 and Theorem 1.2, respectively. Our collection of estimates will moreover turn out to be sufficient to derive the stabilization result from Theorem 1.3 in Section 10 through the analysis of (1.14) and an approximate counterpart, whereas the stability property in Theorem 1.4 and its consequence from Corollary 1.5 will be proved in Section 11.

2 Preliminaries. Global classical solutions to regularized problems

2.1 A weak solution concept and a family of approximate problems

In view of the fact that the diffusion mechanism in (1.6) inter alia contains a degeneracy of porous medium type, our existence theory will be carried out in the framework of the natural generalized solution concept specified as follows.

Definition 2.1 *Let u and v be nonnegative functions defined on $\Omega \times (0,\infty)$ such that*

$$\begin{cases} u \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)) \qquad \textit{and} \\ v \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)) \end{cases} \tag{2.1}$$

and that

$$uvv_x, \qquad u^2vv_x \quad \textit{and} \quad uv \qquad \textit{belong to } L^1_{loc}(\overline{\Omega} \times [0,\infty)). \tag{2.2}$$

Then (u,v) will be called a global weak solution *of (1.6) if*

$$-\int_0^\infty \int_\Omega u\varphi_t - \int_\Omega u_0\varphi(\cdot,0) = -\int_0^\infty \int_\Omega uvv_x\varphi_x + \int_0^\infty \int_\Omega u^2vv_x\varphi_x + \int_0^\infty \int_\Omega uv\varphi \tag{2.3}$$

and

$$\int_0^\infty \int_\Omega v\varphi_t + \int_\Omega v_0\varphi(\cdot,0) = \int_0^\infty \int_\Omega v_x\varphi_x + \int_0^\infty \int_\Omega uv\varphi \tag{2.4}$$

are valid for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$.

In order to obtain weak solutions in this sense through a convenient regularization process, let us fix an arbitrary number

$$m > \frac{9}{4} \tag{2.5}$$

and consider the approximate variants of (1.6) given by

$$\begin{cases} u_{\varepsilon t} = \varepsilon\Big((u_\varepsilon+1)^{m-1}u_{\varepsilon x}\Big)_x + (u_\varepsilon v_\varepsilon u_{\varepsilon x})_x - (u_\varepsilon^2 v_\varepsilon v_{\varepsilon x})_x + u_\varepsilon v_\varepsilon, & x \in \Omega,\ t>0, \\ v_{\varepsilon t} = v_{\varepsilon xx} - u_\varepsilon v_\varepsilon, & x \in \Omega,\ t>0, \\ u_{\varepsilon x} = v_{\varepsilon x} = 0, & x \in \partial\Omega,\ t>0, \\ u_\varepsilon(x,0) = u_0(x), \quad v_\varepsilon(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{2.6}$$

for $\varepsilon \in (0,1)$. Since the diffusion processes in each of these problems are non-degenerate for nonnegative solutions, standard theory from cross-diffusive parabolic systems, particularly of taxis type, becomes applicable ([2], [7], [22]) so as to assert local existence of solutions to (2.6):

Lemma 2.1 *Assume (1.7). Then for each $\varepsilon \in (0,1)$, there exist $T_{max,\varepsilon} \in (0,\infty]$ and at least one pair $(u_\varepsilon, v_\varepsilon)$ of functions*

$$\begin{cases} u_\varepsilon \in C^0(\overline{\Omega} \times [0,T_{max,\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{max,\varepsilon})), \\ v_\varepsilon \in \bigcap_{p>1} C^0([0,T_{max,\varepsilon}); W^{1,p}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,T_{max,\varepsilon})), \end{cases} \tag{2.7}$$

which are such that $u_\varepsilon > 0$ in $\overline{\Omega} \times (0,T_{max,\varepsilon})$ and $v_\varepsilon > 0$ in $\overline{\Omega} \times [0,T_{max,\varepsilon})$ and that $(u_\varepsilon, v_\varepsilon)$ solves (2.6) in the classical sense in $\Omega \times (0,T_{max,\varepsilon})$, and that

$$\textit{if } T_{max,\varepsilon} < \infty, \quad \textit{then} \quad \limsup_{t \nearrow T_{max,\varepsilon}} \Big\{ \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} \Big\} = \infty \quad \textit{for all } q>1. \tag{2.8}$$

Without explicit further mentioning, throughout the sequel we shall let $(u_\varepsilon, v_\varepsilon)$ denote the solution of (2.6) obtained in Lemma 2.1 for $\varepsilon \in (0,1)$, and we shall consistently make use notation such as $u_{\varepsilon x}$, $u_{\varepsilon xx}$ and $u_{\varepsilon t}$ when referring to corresponding spatial or temporal derivatives.

Let us first collect some basic properties of these solutions which in our subsequent analysis will play important roles not only by providing some useful fundamental regularity features, but also by establishing the first quantitative information (2.12) on large time behavior. Indeed, the latter will turn out to be crucial in asserting that the solution components v_ε exhibit a certain decay property which is uniform with respect to $\varepsilon \in (0,1)$ (Lemma 10.3).

Lemma 2.2 *If (1.7) holds, then for all $\varepsilon \in (0,1)$,*

$$\int_\Omega u_\varepsilon(\cdot,t) + \int_\Omega v_\varepsilon(\cdot,t) = \int_\Omega u_0 + \int_\Omega v_0 \qquad \textit{for all } t \in (0, T_{max,\varepsilon}) \tag{2.9}$$

and

$$\int_\Omega u_\varepsilon(\cdot,t) \ge \int_\Omega u_\varepsilon(\cdot,t_0) \qquad \textit{for all } t_0 \in [0, T_{max,\varepsilon}) \textit{ and any } t \in (t_0, T_{max,\varepsilon}) \tag{2.10}$$

as well as

$$\int_\Omega v_\varepsilon(\cdot,t) \le \int_\Omega v_\varepsilon(\cdot,t_0) \qquad \textit{for all } t_0 \in [0, T_{max,\varepsilon}) \textit{ and any } t \in (t_0, T_{max,\varepsilon}). \tag{2.11}$$

Moreover,

$$\int_{t_0}^\infty \int_\Omega u_\varepsilon v_\varepsilon \le \int_\Omega v_\varepsilon(\cdot,t_0) \qquad \textit{for all } t_0 \in [0, T_{max,\varepsilon}) \tag{2.12}$$

and

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|v_\varepsilon(\cdot,t_0)\|_{L^\infty(\Omega)} \qquad \textit{for all } t_0 \in [0, T_{max,\varepsilon}) \textit{ and any } t \in (t_0, T_{max,\varepsilon}). \tag{2.13}$$

Proof. Since integrating the first two equations in (2.6) shows that

$$\frac{d}{dt}\int_\Omega u_\varepsilon = \int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \tag{2.14}$$

and

$$\frac{d}{dt}\int_\Omega v_\varepsilon = -\int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } t \in (0, T_{max,\varepsilon}), \tag{2.15}$$

on adding we directly obtain (2.9). By nonnegativity of both u_ε and v_ε, (2.14) furthermore implies (2.10), whereas (2.15) entails both (2.11) and (2.12). Finally, (2.13) is a consequence of the maximum principle applied to the second equation in (2.6). $\Box$

Beyond implying the above local solvability property, the condition (2.5) guarantees that the regularizing effect of nonlinear diffusion enhancement at large densities is sufficiently effective so as to let these solutions become globally extensible:

Lemma 2.3 *For each $\varepsilon \in (0,1)$, we have $T_{max,\varepsilon} = \infty$; that is, the solution $(u_\varepsilon, v_\varepsilon)$ of (2.6) from Lemma 2.1 is global in time.*

Proof. Let us assume for contradiction that $T_{max,\varepsilon} < \infty$. We then firstly observe that since

$$\|u_\varepsilon(\cdot,t)\|_{L^1(\Omega)} \le c_1 := \int_\Omega u_0 + \int_\Omega v_0 \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \tag{2.16}$$

by (2.9), and since thus, according to (2.13),

$$\|u_\varepsilon(\cdot,t) v_\varepsilon(\cdot,t)\|_{L^1(\Omega)} \le c_1 c_2 \qquad \text{for all } t \in (0, T_{max,\varepsilon})$$

with $c_2 := \|v_0\|_{L^\infty(\Omega)}$, a standard regularity argument applied to the second equation in (2.6) ([15]) shows that for all $q \in (1,\infty)$ we can find $c_3(q) > 0$ such that

$$\|v_{\varepsilon x}(\cdot,t)\|_{L^q(\Omega)} \le c_3(q) \qquad \text{for all } t \in (0, T_{max,\varepsilon}). \tag{2.17}$$

Moreover, for arbitrary $p \ge 2$ we can integrate by parts in the first equation from (2.6) and neglect the second among the diffusive contributions therein to see that due to Young's inequality, the Cauchy-Schwarz inequality and (2.17), for all $t \in (0, T_{max,\varepsilon})$ we have

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u_\varepsilon^p &= -(p-1)\varepsilon \int_\Omega u_\varepsilon^{p-2}(u_\varepsilon+1)^{m-1}u_{\varepsilon x}^2 - (p-1)\int_\Omega u_\varepsilon^{p-1} v_\varepsilon u_{\varepsilon x}^2 \\
&\quad +(p-1)\int_\Omega u_\varepsilon^p v_\varepsilon u_{\varepsilon x} v_{\varepsilon x} + \int_\Omega u_\varepsilon^p v_\varepsilon \\
&\le -(p-1)\varepsilon \int_\Omega u_\varepsilon^{p+m-3} u_{\varepsilon x}^2 + (p-1)\int_\Omega u_\varepsilon^p v_\varepsilon u_{\varepsilon x} v_{\varepsilon x} + \int_\Omega u_\varepsilon^p v_\varepsilon \\
&\le -\frac{(p-1)\varepsilon}{2}\int_\Omega u_\varepsilon^{p+m-3}u_{\varepsilon x}^2 + \frac{(p-1)c_2^2}{2\varepsilon}\int_\Omega u_\varepsilon^{p-m+3} v_{\varepsilon x}^2 + c_2 \int_\Omega u_\varepsilon^p \\
&\le \frac{2(p-1)\varepsilon}{(p+m-1)^2}\int_\Omega (u_\varepsilon^{\frac{p+m-1}{2}})_x^2 + c_4(p,\varepsilon)\cdot\Big\{\int_\Omega u_\varepsilon^{2(p-m+3)}\Big\}^{\frac{1}{2}} + c_2\int_\Omega u_\varepsilon^p,
\end{aligned} \tag{2.18}$$

where $c_4(p,\varepsilon) := \frac{(p-1)c_2^2 c_3^2(4)}{2\varepsilon}$. Now writing $a := \frac{(p+m-1)(2p-2m+5)}{2(p+m-2)(p-m+3)} \in (0,1)$ and noting that

$$\frac{2(p-m+3)a}{p+m-1} - 2 = \frac{-4m+9}{p+m-2} < 0$$

according to our restriction $m > \frac{9}{4}$, by using the Gagliardo-Nirenberg inequality (2.16) and again Young's inequality we obtain positive constants $c_5(p,\varepsilon), c_6(p,\varepsilon)$ and $c_7(p,\varepsilon)$ such that

$$\begin{aligned}
c_4(p,\varepsilon)\cdot\Big\{\int_\Omega u_\varepsilon^{2(p-m+3)}\Big\}^{\frac{1}{2}} &= c_4(p,\varepsilon)\Big\|u_\varepsilon^{\frac{p+m-1}{2}}\Big\|_{L^{\frac{4(p-m+3)}{p+m-1}}(\Omega)}^{\frac{2(p-m+3)}{p+m-1}} \\
&\le c_5(p,\varepsilon)\Big\|(u_\varepsilon^{\frac{p+m-1}{2}})_x\Big\|_{L^2(\Omega)}^{\frac{2(p-m+3)a}{p+m-1}} \Big\|u_\varepsilon^{\frac{p+m-1}{2}}\Big\|_{L^{\frac{2}{p+m-1}}(\Omega)}^{\frac{2(p-m+3)(1-a)}{p+m-1}} \\
&\quad + c_5(p,\varepsilon)\Big\|u_\varepsilon^{\frac{p+m-1}{2}}\Big\|_{L^{\frac{2}{p+m-1}}(\Omega)}^{\frac{2(p-m+3)}{p+m-1}} \\
&\le c_6(p,\varepsilon)\Big\|(u_\varepsilon^{\frac{p+m-1}{2}})_x\Big\|_{L^2(\Omega)}^{\frac{2(p-m+3)a}{p+m-1}} + c_6(p,\varepsilon) \\
&\le \frac{2(p-1)\varepsilon}{(p+m-1)^2}\int_\Omega (u_\varepsilon^{\frac{p+m-1}{2}})_x^2 + c_7(p,\varepsilon) \qquad \text{for all } t \in (0, T_{max,\varepsilon}),
\end{aligned}$$

so that (2.18) implies that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u_\varepsilon^p \le c_6(p,\varepsilon) + c_2\int_\Omega u_\varepsilon^p \qquad \text{for all } t \in (0, T_{max,\varepsilon}).$$

Since thus

$$\int_\Omega u_\varepsilon^p \le \Big\{\int_\Omega u_0^p\Big\}\cdot e^{pc_2 T_{max,\varepsilon}} + \frac{c_6(p,\varepsilon)}{c_2} \qquad \text{for all } t \in (0, T_{max,\varepsilon}),$$

and since $p \ge 2$ was arbitrary, we may utilize the outcome of a standard Moser-type iteration ([34]) to see that actually

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le c_7(\varepsilon) \qquad \text{for all } t \in (0, T_{max,\varepsilon})$$

with some $c_7(\varepsilon) > 0$, which together with (2.17) contradicts (2.8) and thereby completes the proof. □

3 Energy dissipation enforcing integrability of $\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$ in time

Let us next make sure that the dissipative structure formally expressed in (1.26) indeed possesses a rigorous counterpart in (2.6) that is essentially unaffected by the additional regularization term therein. Among several possible natural conclusions, let us focus on formulating the following estimates which, besides asserting a certain positivity property of the first solution component through (3.3), provide the further decay information given in (3.4) and (3.5) which will form the basis for our derivation of the crucial estimate (3.12) for v_ε in $L^1((0,\infty);L^\infty(\Omega))$.

In order to make our results accessible to both our arguments leading to global existence and stabilization of solutions to fixed initial data in Theorem 1.1-Theorem 1.3, but also to our analysis of the stability and heterogeneity properties addressed in Theorem 1.4 and Corollary 1.5, throughout this and the subsequent section we shall emphasize the respective dependence of the obtained bounds on appropriate quantities determined by the initial data.

Lemma 3.1 *Let $K>0$. Then there exists $C(K)>0$ such that whenever u_0 and v_0 satisfy (1.7) and are such that*

$$\int_\Omega \ln u_0 \ge -K \qquad \text{and} \qquad \int_\Omega u_0 \le K \tag{3.1}$$

as well as

$$\int_\Omega v_0 \le K \qquad \text{and} \qquad \int_\Omega v_{0x}^2 \le K, \tag{3.2}$$

for all $\varepsilon\in(0,1)$ we have

$$\int_\Omega \ln u_\varepsilon(\cdot,t) \ge -C(K) \qquad \text{for all } t>0 \tag{3.3}$$

and

$$\int_0^\infty \int_\Omega \frac{v_\varepsilon}{u_\varepsilon} u_{\varepsilon x}^2 \le C(K) \tag{3.4}$$

as well as

$$\int_0^\infty \int_\Omega v_\varepsilon \le C(K). \tag{3.5}$$

Proof. By using (2.6) and several integrations by parts, on dropping three nonpositive summands and making use of a favorable cancellation we obtain

$$\begin{aligned}
\frac{d}{dt}\Big\{-\int_\Omega \ln u_\varepsilon + \frac{1}{2}\int_\Omega v_{\varepsilon x}^2\Big\} &= -\int_\Omega \frac{1}{u_\varepsilon}\cdot\Big\{\Big(\varepsilon(u_\varepsilon+1)^{m-1}u_{\varepsilon x} + u_\varepsilon v_\varepsilon u_{\varepsilon x} - u_\varepsilon^2 v_\varepsilon v_{\varepsilon x}\Big)_x + u_\varepsilon v_\varepsilon\Big\} \\
&\quad + \int_\Omega v_{\varepsilon x}\cdot(v_{\varepsilon xx} - u_\varepsilon v_\varepsilon)_x \\
&= -\varepsilon\int_\Omega \frac{(u_\varepsilon+1)^{m-1}}{u_\varepsilon^2}u_{\varepsilon x}^2 - \int_\Omega \frac{v_\varepsilon}{u_\varepsilon}u_{\varepsilon x}^2 + \int_\Omega v_\varepsilon u_{\varepsilon x} v_{\varepsilon x} - \int_\Omega v_\varepsilon \\
&\quad - \int_\Omega v_{\varepsilon xx}^2 - \int_\Omega v_\varepsilon u_{\varepsilon x} v_{\varepsilon x} - \int_\Omega u_\varepsilon v_{\varepsilon x}^2 \\
&\le -\int_\Omega \frac{v_\varepsilon}{u_\varepsilon}u_{\varepsilon x}^2 - \int_\Omega v_\varepsilon \qquad \text{for all } t>0.
\end{aligned} \tag{3.6}$$

On integration in time, in view of (3.1) and (3.2) this implies that

$$\begin{aligned}
-\int_\Omega \ln u_\varepsilon(\cdot,t) + \int_0^t\int_\Omega \frac{v_\varepsilon}{u_\varepsilon}u_{\varepsilon x}^2 + \int_0^t\int_\Omega v_\varepsilon &\le -\int_\Omega \ln u_0 + \frac{1}{2}\int_\Omega v_{0x}^2 \\
&\le \frac{3}{2}K \qquad \text{for all } t>0
\end{aligned} \tag{3.7}$$

and thereby firstly entails (3.3). As the validity of $\ln\xi \le \xi$ for all $\xi > 0$ warrants that

$$\int_\Omega \ln u_\varepsilon(\cdot,t) \le \int_\Omega u_\varepsilon(\cdot,t) \le \int_\Omega u_0 + \int_\Omega v_0 \le 2K$$

according to (2.9), (3.1) and (3.2), from (3.7) we moreover infer that

$$\int_0^t \int_\Omega \frac{v_\varepsilon}{u_\varepsilon} u_{\varepsilon x}^2 + \int_0^t \int_\Omega v_\varepsilon \le \frac{7}{2}K \qquad \text{for all } t > 0$$

and that thus also (3.4) and (3.5) hold. □

Let us remark here that when aiming at a further development of (3.5) into an estimate for the quantity $\int_0^\infty \|v_\varepsilon(\cdot,t)\|_{L^p(\Omega)}^q dt$ with some $p \in (1,\infty]$ and $q \ge 1$, one might consider interpolating between (3.5) and an additional bound on $\int_0^\infty \int_\Omega v_{\varepsilon xx}^2$ that could as well be derived from (3.6). Since the latter integrability property involves a temporal norm in L^2 rather than L^1, however, any effort in this direction seems limited to certain $q > 1$, with $q = \frac{5p}{4p+1}$ constituting the apparently smallest choice possible without further external information. Since for our subsequent analysis it will be crucial to include the exponent $q = 1$ here, most conveniently combined with the choice $p = \infty$, independently from the above in Lemma 3.3 we shall additionally analyze the time evolution of the weighted functional $\int_\Omega \frac{v_{\varepsilon x}^2}{v_\varepsilon}$ which, unlike the expression $\int_\Omega v_{\varepsilon x}^2$ considered in Lemma 3.1, exhibits linear growth, rather than a quadratic one, with respect to the parameter $\lambda > 0$ in the scaling $v_\varepsilon \mapsto \lambda v_\varepsilon$. Indeed, the corresponding dissipative contribution will turn out to dominate a multiple of $\int_\Omega \frac{v_{\varepsilon x}^4}{v_\varepsilon^3}$ which, again due to its essentially linear growth in terms of the unknown, will enable us to gain the desired L^1 decay information for $\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$ by means of suitable interpolation.

The following basic calculus inequality will firstly allow us to control some ill-signed contributions arising in Lemma 3.3, and it will secondly turn out to be useful in the course of another testing procedure in Lemma 4.1 below.

Lemma 3.2 *Let $q > 0, \alpha > -1$ and $\phi \in C^2(\overline{\Omega})$ be a positive function satisfying $\phi_x = 0$ on $\partial\Omega$. Then*

$$\int_\Omega \phi^{-\alpha-2}|\phi_x|^{q+2} \le \Big(\frac{q+1}{\alpha+1}\Big)^2 \int_\Omega \phi^{-\alpha}|\phi_x|^{q-2}\phi_{xx}^2. \tag{3.8}$$

Proof. We integrate by parts and use the Cauchy-Schwarz inequality to see that

$$\begin{aligned}
\int_\Omega \phi^{-\alpha-2}|\phi_x|^{q+2} &= -\frac{1}{\alpha+1}\int_\Omega (\phi^{-\alpha-1})_x |\phi_x|^q\phi_x \\
&= \frac{q+1}{\alpha+1}\int_\Omega \phi^{-\alpha-1}|\phi_x|^q\phi_{xx} \\
&\le \frac{q+1}{\alpha+1}\cdot\Big\{\int_\Omega \phi^{-\alpha-1}|\phi_x|^{q+2}\Big\}^{\frac12}\cdot\Big\{\int_\Omega \phi^{-\alpha}|\phi_x|^{q-2}\phi_{xx}^2\Big\}^{\frac12},
\end{aligned}$$

which yields (3.8). □

We can now make use of the estimate (3.4) to derive the following additional information on decay in the second solution component.

Lemma 3.3 *For all $K > 0$ there exists $C(K) > 0$ such that if u_0 and v_0 satisfy (1.7) as well as (3.1), (3.2) and*

$$\int_\Omega \frac{v_{0x}^2}{v_0} \le K, \tag{3.9}$$

then

$$\int_0^\infty \int_\Omega \frac{v_{\varepsilon x}^4}{v_\varepsilon^3} \le C(K) \qquad \textit{for all } \varepsilon \in (0,1). \tag{3.10}$$

PROOF. From the regularity properties of u_ε and v_ε asserted by Lemma 2.1, in view of standard parabolic Schauder theory ([19]) it follows that actually also $v_{\varepsilon x}$ belongs to $C^{2,1}(\overline{\Omega} \times (0,\infty))$ and satisfies the accordingly differentiated version of the second equation in (2.6). Thanks to the strict positivity of v_ε in $\overline{\Omega} \times (0,\infty)$, we may therefore integrate by parts to compute

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{v_{\varepsilon x}^2}{v_\varepsilon} &= 2\int_\Omega \frac{v_{\varepsilon x}}{v_\varepsilon} v_{\varepsilon xt} - \int_\Omega \frac{v_{\varepsilon x}^2}{v_\varepsilon^2} v_{\varepsilon t} \\
&= 2\int_\Omega \frac{v_{\varepsilon x}}{v_\varepsilon} \cdot (v_{\varepsilon xxx} - u_{\varepsilon x} v_\varepsilon - u_\varepsilon v_{\varepsilon x}) - \int_\Omega \frac{v_{\varepsilon x}^2}{v_\varepsilon^2} \cdot (v_{\varepsilon xx} - u_\varepsilon v_\varepsilon) \\
&= -2\int_\Omega \frac{v_{\varepsilon xx}^2}{v_\varepsilon} + 2\int_\Omega \frac{v_{\varepsilon x}^2}{v_\varepsilon^2} v_{\varepsilon xx} - 2\int_\Omega u_{\varepsilon x} v_{\varepsilon x} - 2\int_\Omega \frac{u_\varepsilon}{v_\varepsilon} v_{\varepsilon x}^2 - \int_\Omega \frac{v_{\varepsilon x}^2}{v_\varepsilon^2} v_{\varepsilon xx} + \int_\Omega \frac{u_\varepsilon}{v_\varepsilon} v_{\varepsilon x}^2 \\
&= -2\int_\Omega \frac{v_{\varepsilon xx}^2}{v_\varepsilon} + \int_\Omega \frac{v_{\varepsilon x}^2}{v_\varepsilon^2} v_{\varepsilon xx} - 2\int_\Omega u_{\varepsilon x} v_{\varepsilon x} - \int_\Omega \frac{u_\varepsilon}{v_\varepsilon} v_{\varepsilon x}^2 \qquad \text{for all } t > 0.
\end{aligned} \tag{3.11}$$

Here another integration by parts shows that

$$\int_\Omega \frac{v_{\varepsilon x}^2}{v_\varepsilon^2} v_{\varepsilon xx} = \frac{2}{3}\int_\Omega \frac{v_{\varepsilon x}^4}{v_\varepsilon^3} \qquad \text{for all } t > 0,$$

and in order to compensate this we invoke Lemma 3.2 to see that

$$-2\int_\Omega \frac{v_{\varepsilon xx}^2}{v_\varepsilon} \le -\frac{8}{9}\int_\Omega \frac{v_{\varepsilon x}^4}{v_\varepsilon^3} \qquad \text{for all } t > 0.$$

As moreover, by Young's inequality,

$$-2\int_\Omega u_{\varepsilon x} v_{\varepsilon x} \le \int_\Omega \frac{u_\varepsilon}{v_\varepsilon} v_{\varepsilon x}^2 + \int_\Omega \frac{v_\varepsilon}{u_\varepsilon} u_{\varepsilon x}^2 \qquad \text{for all } t > 0,$$

from (3.11) we obtain that

$$\frac{d}{dt}\int_\Omega \frac{v_{\varepsilon x}^2}{v_\varepsilon} + \frac{2}{9}\int_\Omega \frac{v_{\varepsilon x}^4}{v_\varepsilon^3} \le \int_\Omega \frac{v_\varepsilon}{u_\varepsilon} u_{\varepsilon x}^2 \qquad \text{for all } t > 0,$$

which after an integration in time, relying on the $W^{1,2}$-valued continuity of v_ε asserted by Lemma 2.1, and on positivity of v_ε now throughout $\overline{\Omega} \times [0,\infty)$, yields

$$\int_\Omega \frac{v_{\varepsilon x}^2(\cdot,T)}{v_\varepsilon(\cdot,T)} + \frac{2}{9}\int_0^T \int_\Omega \frac{v_{\varepsilon x}^4}{v_\varepsilon^3} \le \int_\Omega \frac{v_{0x}^2}{v_0} + \int_0^T \int_\Omega \frac{v_\varepsilon}{u_\varepsilon} u_{\varepsilon x}^2 \qquad \text{for all } T > 0$$

and thereby proves (3.10) due to Lemma 3.1 and (3.9). □

By means of a straightforward interpolation, a combination of the latter with (3.5) leads to the main outcome of this section.

Lemma 3.4 *For any $K > 0$ one can find $C(K) > 0$ such that whenever (1.7), (3.1), (3.2) and (3.9) hold, we have*

$$\int_0^\infty \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} dt \le C(K) \qquad \textit{for all } \varepsilon \in (0,1). \tag{3.12}$$

Proof. According to the Gagliardo-Nirenberg inequality, there exists $c_1 > 0$ such that

$$\|\phi\|_{L^\infty(\Omega)}^4 \le c_1 \|\phi_x\|_{L^4(\Omega)} \|\phi\|_{L^4(\Omega)}^3 + c_1 \|\phi\|_{L^4(\Omega)}^4 \qquad \text{for all } \phi \in W^{1.4}(\Omega),$$

which applied to $\phi := v_\varepsilon^{\frac{1}{4}}(\cdot, t)$ for $\varepsilon \in (0, 1)$ and $t > 0$ shows that due to Young's inequality we have

$$\begin{aligned}
\int_0^T \|v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} dt &= \int_0^T \|v_\varepsilon^{\frac{1}{4}}(\cdot, t)\|_{L^\infty(\Omega)}^4 dt \\
&\le c_1 \int_0^T \Big\|(v_\varepsilon^{\frac{1}{4}})_x(\cdot, t)\Big\|_{L^4(\Omega)} \|v_\varepsilon^{\frac{1}{4}}(\cdot, t)\|_{L^4(\Omega)}^3 dt + c_1 \int_0^T \|v_\varepsilon^{\frac{1}{4}}(\cdot, t)\|_{L^4(\Omega)}^4 dt \\
&\le c_1 \int_0^T \Big\|(v_\varepsilon^{\frac{1}{4}})_x(\cdot, t)\Big\|_{L^4(\Omega)}^4 dt + 2c_1 \int_0^T \|v_\varepsilon^{\frac{1}{4}}(\cdot, t)\|_{L^4(\Omega)}^4 dt \\
&= \frac{c_1}{256} \int_0^T \int_\Omega \frac{v_{\varepsilon x}^4}{v_\varepsilon^3} + 2c_1 \int_0^T \int_\Omega v_\varepsilon \qquad \text{for all } T > 0.
\end{aligned}$$

Therefore, (3.12) results from Lemma 3.3 and Lemma 3.1. □

4 Deriving L^p bounds for u_ε via further quasi-energy functionals

In deriving appropriate bounds for u_ε with respect to the norm in $L^p(\Omega)$ for large $p > 1$ on the basis of a standard testing procedure in the first sub-problem of (2.6), we evidently need to appropriately cope with the degeneracy of the diffusion mechanism therein with regard to the asymptotically decaying component v_ε. In order to nevertheless make appropriate use of the corresponding dissipation mechanism, we will rather consider the time evolution of a coupled functional additionally containing a weighted L^q norm of the cross-diffusive gradient, thus being concerned with expressions of the form

$$\int_\Omega u_\varepsilon^p + \int_\Omega v_\varepsilon^{-\alpha} |v_{\varepsilon x}|^q \tag{4.1}$$

for conveniently large $p > 1$ and suitably chosen $q > 1$ and $\alpha > 0$. Basic differential inequalities for two summands herein will first be obtained separately in Section 4.1, whereafter Section 4.2 will provide appropriate estimates for the respective right-hand sides. In Section 4.3 these will be combined so as to detect a quasi-energy property of the functional in (4.1) under adequate assumptions on p, q and α, in particular implying an estimate for u_ε in $L^\infty((0, \infty); L^p(\Omega))$ for arbitrary $p \ge 2$ in Lemma 4.8.

4.1 Further testing procedures

Let us first derive a basic information on the time evolution of the second summand in (4.1). We emphasize that through the use of the precise quantitative form of the inequality from Lemma 3.2, our analysis here strongly relies on the fact that the spatial setting is one-dimensional, which especially enables us to allow for values $\alpha \in (0, q)$ in (4.2) which are arbitrarily close to the critical value $\alpha = q$ (cf. e.g. the choice of α in the proof of Lemma 4.7).

Lemma 4.1 *Let $q > 2$ and $\alpha \in (0, q)$. Then there exists $C > 0$ such that for any choice of $\varepsilon \in (0, 1)$ we have*

$$\frac{d}{dt} \int_\Omega v_\varepsilon^{-\alpha} |v_{\varepsilon x}|^q + \frac{1}{C} \int_\Omega v_\varepsilon^{-\alpha-2} |v_{\varepsilon x}|^{q+2} \le C \int_\Omega u_\varepsilon^{\frac{q+2}{2}} v_\varepsilon^{q-\alpha} \qquad \textit{for all } t > 0. \tag{4.2}$$

PROOF. Using the second equation in (2.6) and integrating by parts, we compute

$$\begin{aligned}
\frac{d}{dt}\int_\Omega v_\varepsilon^{-\alpha}|v_{\varepsilon x}|^q &= -\alpha\int_\Omega v_\varepsilon^{-\alpha-1}|v_{\varepsilon x}|^q v_{\varepsilon t} + q\int_\Omega v_\varepsilon^{-\alpha}|v_{\varepsilon x}|^{q-2}v_{\varepsilon x}v_{\varepsilon xt} \\
&= (q-1)\alpha\int_\Omega v_\varepsilon^{-\alpha-1}|v_{\varepsilon x}|^q v_{\varepsilon t} - q(q-1)\int_\Omega v_\varepsilon^{-\alpha}|v_{\varepsilon x}|^{q-2}v_{\varepsilon xx}v_{\varepsilon t} \\
&= (q-1)\alpha\int_\Omega v_\varepsilon^{-\alpha-1}|v_{\varepsilon x}|^q v_{\varepsilon xx} - (q-1)\alpha\int_\Omega u_\varepsilon v_\varepsilon^{-\alpha}|v_{\varepsilon x}|^q \\
&\quad -q(q-1)\int_\Omega v_\varepsilon^{-\alpha}|v_{\varepsilon x}|^{q-2}v_{\varepsilon xx}^2 + q(q-1)\int_\Omega u_\varepsilon v_\varepsilon^{1-\alpha}|v_{\varepsilon x}|^{q-2}v_{\varepsilon xx} \quad \text{for all } t>0. \qquad (4.3)
\end{aligned}$$

Here by means of the Cauchy-Schwarz inequality and Lemma 3.2, we see that

$$\begin{aligned}
(q-1)\alpha\int_\Omega v_\varepsilon^{-\alpha-1}|v_{\varepsilon x}|^q v_{\varepsilon xx} &\le (q-1)\alpha\cdot\left\{\int_\Omega v_\varepsilon^{-\alpha-2}|v_{\varepsilon x}|^{q+2}\right\}^{\frac12}\cdot\left\{v_\varepsilon^{-\alpha}|v_{\varepsilon x}|^{q-2}v_{\varepsilon xx}^2\right\}^{\frac12} \\
&\le (q-1)\alpha\cdot\frac{q+1}{\alpha+1}\cdot\int_\Omega v_\varepsilon^{-\alpha}|v_{\varepsilon x}|^{q-2}v_{\varepsilon xx}^2 \qquad \text{for all } t>0,
\end{aligned}$$

so that

$$(q-1)\alpha\int_\Omega v_\varepsilon^{-\alpha-1}|v_{\varepsilon x}|^q v_{\varepsilon xx} - q(q-1)\int_\Omega v_\varepsilon^{-\alpha}|v_{\varepsilon x}|^{q-2}v_{\varepsilon xx}^2 \le -c_1\int_\Omega v_\varepsilon^{-\alpha}|v_{\varepsilon x}|^{q-2}v_{\varepsilon xx}^2 \quad \text{for all } t>0, \qquad (4.4)$$

where

$$c_1 := q(q-1) - (q-1)\alpha\cdot\frac{q+1}{\alpha+1} = \frac{(q-1)(q-\alpha)}{\alpha+1}$$

is positive due to our restriction $\alpha < q$. As Young's inequality states that the rightmost summand in (4.3) can be estimated according to

$$q(q-1)\int_\Omega u_\varepsilon v_\varepsilon^{1-\alpha}|v_{\varepsilon x}|^{q-2}v_{\varepsilon xx} \le \frac{c_1}{2}\int_\Omega v_\varepsilon^{-\alpha}|v_{\varepsilon x}|^{q-2}v_{\varepsilon xx}^2 + \frac{q^2(q-1)^2}{2c_1}\int_\Omega u_\varepsilon^2 v_\varepsilon^{2-\alpha}|v_{\varepsilon x}|^{q-2} \qquad \text{for all } t>0,$$

and that herein with some $c_2 > 0$ we have

$$\begin{aligned}
\frac{q^2(q-1)^2}{2c_1}\int_\Omega u_\varepsilon^2 v_\varepsilon^{2-\alpha}|v_{\varepsilon x}|^{q-2} &= \frac{q^2(q-1)^2}{2c_1}\int_\Omega \Big(u_\varepsilon v_\varepsilon^{-\alpha}|v_{\varepsilon x}|^q\Big)^{\frac{q-2}{q}}\cdot u_\varepsilon^{\frac{q+2}{q}}\, v_\varepsilon^{\frac{2(q-\alpha)}{q}} \\
&\le \frac{(q-1)\alpha}{2}\int_\Omega u_\varepsilon v_\varepsilon^{-\alpha}|v_{\varepsilon x}|^q + c_2\int_\Omega u_\varepsilon^{\frac{q+2}{2}} v_\varepsilon^{q-\alpha} \qquad \text{for all } t>0,
\end{aligned}$$

it follows from (4.3) and (4.4) that

$$\frac{d}{dt}\int_\Omega v_\varepsilon^{-\alpha}|v_{\varepsilon x}|^q + \frac{c_1}{2}\int_\Omega v_\varepsilon^{-\alpha}|v_{\varepsilon x}|^{q-2}v_{\varepsilon xx}^2 + \frac{(q-1)\alpha}{2}\int_\Omega u_\varepsilon v_\varepsilon^{-\alpha}|v_{\varepsilon x}|^q \le c_2\int_\Omega u_\varepsilon^{\frac{q+2}{2}} v_\varepsilon^{q-\alpha} \qquad \text{for all } t>0.$$

Since again Lemma 3.2 shows that

$$\frac{c_1}{4}\int_\Omega v_\varepsilon^{-\alpha}|v_{\varepsilon x}|^{q-2}v_{\varepsilon xx}^2 \ge \frac{c_1}{4}\cdot\Big(\frac{\alpha+1}{q+1}\Big)^2\int_\Omega v_\varepsilon^{-\alpha-2}|v_{\varepsilon x}|^{q+2} \qquad \text{for all } t>0,$$

this implies (4.2) upon an evident choice of C. □

In a corresponding testing procedure associated with the analysis of the first summand in (4.1), the particular form of the second summand on the right of (4.2) suggests how to estimate $v_{\varepsilon x}$ at a first stage, thus leading to the following preliminary differential inequality.

Lemma 4.2 *Let $p>1, q>1$ and $\alpha>0$. Then for all $\eta>0$ one can pick $C(\eta)>0$ such that if $\varepsilon\in(0,1)$, then*

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon u_{\varepsilon x}^2 + \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p+1}v_\varepsilon v_{\varepsilon x}^2 + p(p-1)\varepsilon\int_\Omega u_\varepsilon^{p+m-3}u_{\varepsilon x}^2 \\ \le \ p\|v_\varepsilon\|_{L^\infty(\Omega)}\int_\Omega u_\varepsilon^p + \eta\int_\Omega v_\varepsilon^{-\alpha-2}|v_{\varepsilon x}|^{q+2} + C(\eta)\int_\Omega u_\varepsilon^{\frac{(p+1)(q+2)}{q}} v_\varepsilon^{\frac{q+2\alpha+6}{q}} \qquad \textit{for all } t>0. \qquad (4.5)\end{aligned}$$

Proof. From the first equation in (2.6) we obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u_\varepsilon^p + (p-1)\varepsilon\int_\Omega u_\varepsilon^{p-2}(u_\varepsilon+1)^{m-1}u_{\varepsilon x}^2 + (p-1)\int_\Omega u_\varepsilon^{p-1}v_\varepsilon u_{\varepsilon x}^2 \\ = \ (p-1)\int_\Omega u_\varepsilon^p v_\varepsilon u_{\varepsilon x}v_{\varepsilon x} + \int_\Omega u_\varepsilon^p v_\varepsilon \qquad \text{for all } t>0, \qquad (4.6)\end{aligned}$$

where by Young's inequality,

$$(p-1)\int_\Omega u_\varepsilon^p v_\varepsilon u_{\varepsilon x}v_{\varepsilon x} \le \frac{p-1}{2}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon u_{\varepsilon x}^2 + \frac{p-1}{2}\int_\Omega u_\varepsilon^{p+1}v_\varepsilon v_{\varepsilon x}^2 \qquad \text{for all } t>0,$$

and where clearly

$$\int_\Omega u_\varepsilon^p v_\varepsilon \le \|v_\varepsilon\|_{L^\infty(\Omega)}\int_\Omega u_\varepsilon^p \qquad \text{for all } t>0.$$

Trivially estimating $(u_\varepsilon+1)^{m-1}\ge u_\varepsilon^{m-1}$ in the second summand on the left-hand side therein, from (4.6) we thus infer that for all $t>0$,

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{p-1}{2}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon u_{\varepsilon x}^2 + \frac{p-1}{2}\int_\Omega u_\varepsilon^{p+1}v_\varepsilon v_{\varepsilon x}^2 + (p-1)\varepsilon\int_\Omega u_\varepsilon^{p-2}(u_\varepsilon+1)^{m-1}u_{\varepsilon x}^2 \\ \le \ (p-1)\int_\Omega u_\varepsilon^{p+1}v_\varepsilon v_{\varepsilon x}^2 + \|v_\varepsilon\|_{L^\infty(\Omega)}\int_\Omega u_\varepsilon^p,\end{aligned}$$

which already implies (4.5) due to the fact that for each $\eta>0$, Young's inequality provides $c_1=c_1(\eta)>0$ fulfilling

$$\begin{aligned}(p-1)\int_\Omega u_\varepsilon^{p+1}v_\varepsilon v_{\varepsilon x}^2 &= (p-1)\int_\Omega \Big(v_\varepsilon^{-\alpha-2}|v_{\varepsilon x}|^{q+2}\Big)^{\frac{2}{q+2}}\cdot u_\varepsilon^{p+1}v_\varepsilon^{\frac{q+2\alpha+6}{q+2}} \\ &\le \eta\int_\Omega v_\varepsilon^{-\alpha-2}|v_{\varepsilon x}|^{q+2} + c_1\int_\Omega u_\varepsilon^{\frac{(p+1)(q+2)}{q}} v_\varepsilon^{\frac{q+2\alpha+6}{q}}\end{aligned}$$

for all $t>0$. □

4.2 Estimating the right-hand sides in (4.2) and (4.5)

We shall next face the yet open challenge how to make appropriate use of the degenerate diffusive action in the first equation from (2.6), in the form expressed in (4.5). A key observation in this direction consists in the following functional inequality which may be viewed as a particular Gagliardo-Nirenberg-type interpolation involving certain products of functions. We underline that for our applications thereof in Lemma 4.5 and Lemma 4.6 the appearance of the spatial L^∞ norm on the left of (4.7) seems to be of crucial importance, and that hence also in this part our analysis strongly relies on our resorting to the the spatially one-dimensional case.

Lemma 4.3 *Let $p > 0$. Then there exists $C > 0$ such that*

$$\|\phi\psi^{\frac{3}{p+1}}\|_{L^\infty(\Omega)}^{p+2} \le C\|\phi\|_{L^1(\Omega)}\|\psi\|_{L^\infty(\Omega)}^{\frac{3}{p+1}} \cdot \Big\{ \|\psi\|_{L^\infty(\Omega)}^2 \int_\Omega \phi^{p-1}\psi\phi_x^2 + \int_\Omega \phi^{p+1}\psi\psi_x^2 + \|\phi\|_{L^1(\Omega)}^{p+1}\|\psi\|_{L^\infty(\Omega)}^3 \Big\} \quad (4.7)$$

is valid for arbitrary positive functions $\phi \in C^1(\overline{\Omega})$ and $\psi \in C^1(\overline{\Omega})$.

Proof. We employ the Gagliardo-Nirenberg inequality to find $c_1 > 0$ fulfilling

$$\begin{aligned} \|\phi\psi^{\frac{3}{p+1}}\|_{L^\infty(\Omega)}^{p+2} &= \|\phi^{\frac{p+1}{2}}\psi^{\frac{3}{2}}\|_{L^\infty(\Omega)}^{\frac{2(p+2)}{p+1}} \\ &\le c_1\Big\|(\phi^{\frac{p+1}{2}}\psi^{\frac{3}{2}})_x\Big\|_{L^2(\Omega)}^2 \|\phi^{\frac{p+1}{2}}\psi^{\frac{3}{2}}\|_{L^{\frac{2}{p+1}}(\Omega)}^{\frac{2}{p+1}} + c_1\|\phi^{\frac{p+1}{2}}\psi^{\frac{3}{2}}\|_{L^{\frac{2}{p+1}}(\Omega)}^{\frac{2(p+2)}{p+1}}, \end{aligned} \quad (4.8)$$

where we note that

$$\|\phi^{\frac{p+1}{2}}\psi^{\frac{3}{2}}\|_{L^{\frac{2}{p+1}}(\Omega)}^{\frac{2}{p+1}} = \int_\Omega \phi\psi^{\frac{3}{p+1}} \le \|\phi\|_{L^1(\Omega)}\|\psi\|_{L^\infty(\Omega)}^{\frac{3}{p+1}}.$$

Since by Young's inequality we can moreover estimate

$$\begin{aligned} \Big\|(\phi^{\frac{p+1}{2}}\psi^{\frac{3}{2}})_x\Big\|_{L^2(\Omega)}^2 &= \int_\Omega \Big\{\frac{p+1}{2}\phi^{\frac{p-1}{2}}\psi^{\frac{3}{2}}\phi_x + \frac{3}{2}\phi^{\frac{p+1}{2}}\psi^{\frac{1}{2}}\psi_x\Big\}^2 \\ &\le \frac{(p+1)^2}{2}\int_\Omega \phi^{p-1}\psi^3\phi_x^2 + \frac{9}{2}\int_\Omega \phi^{p+1}\psi\psi_x^2 \\ &\le \frac{(p+1)^2}{2}\|\psi\|_{L^\infty(\Omega)}^2\int_\Omega \phi^{p-1}\psi\phi_x^2 + \frac{9}{2}\int_\Omega \phi^{p+1}\psi\psi_x^2, \end{aligned}$$

from (4.8) we thus obtain that

$$\begin{aligned} \|\phi\psi^{\frac{3}{p+1}}\|_{L^\infty(\Omega)}^{p+2} \le\ & \frac{(p+1)^2}{2}c_1\|\phi\|_{L^1(\Omega)}\|\psi\|_{L^\infty(\Omega)}^{\frac{3}{p+1}+2}\int_\Omega \phi^{p-1}\psi\phi_x^2 + \frac{9}{2}c_1\|\phi\|_{L^1(\Omega)}\|\psi\|_{L^\infty(\Omega)}^{\frac{3}{p+1}}\int_\Omega \phi^{p+1}\psi\psi_x^2 \\ & + c_1\|\phi\|_{L^1(\Omega)}^{p+2}\|\psi\|_{L^\infty(\Omega)}^{\frac{3(p+2)}{p+1}}, \end{aligned}$$

which directly implies (4.7) due to the fact that $\frac{3(p+2)}{p+1} = \frac{3}{p+1} + 3$. □

In light of our basic information from Lemma 2.2, when applied to $(\phi, \psi) := (u_\varepsilon, v_\varepsilon)$ this entails the following more concrete preparation for Lemma 4.5 and Lemma 4.6.

Lemma 4.4 *Let $p > 0$ and $r > 0$ be such that*

$$\frac{(p+1)(p+2)}{p+4} \le r < p+2.$$

Then for all $\eta > 0$ and $K > 0$ there exists $C(\eta, K) > 0$ such that if besides (1.7) we have

$$\int_\Omega u_0 \le K \quad (4.9)$$

and

$$\|v_0\|_{L^\infty(\Omega)} \le K, \quad (4.10)$$

then

$$\|u_\varepsilon v_\varepsilon^{\frac{3}{p+1}}\|_{L^\infty(\Omega)}^r \le \eta\int_\Omega u_\varepsilon^{p-1}v_\varepsilon u_{\varepsilon x}^2 + \eta\int_\Omega u_\varepsilon^{p+1}v_\varepsilon v_{\varepsilon x}^2 + C(\eta, K)\|v_\varepsilon\|_{L^\infty(\Omega)} \qquad \textit{for all } t > 0. \quad (4.11)$$

Proof. According to (4.9) and (4.10), the inequalities in (2.9) and (2.13) entail that

$$\|u_\varepsilon\|_{L^1(\Omega)} \le \int_\Omega u_0 + \int_\Omega v_0 \le K + |\Omega| \cdot K \qquad \text{for all } t > 0$$

and

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le K \qquad \text{for all } t > 0. \tag{4.12}$$

In particular, Lemma 4.3 therefore implies the existence of $c_1 = c_1(K) > 0$ such that

$$\|u_\varepsilon v_\varepsilon^{\frac{3}{p+1}}\|_{L^\infty(\Omega)}^{p+2} \le c_1 \|v_\varepsilon\|_{L^\infty(\Omega)}^{\frac{3}{p+1}} I(t) + c_1 \|v_\varepsilon\|_{L^\infty(\Omega)}^{\frac{3(p+2)}{p+1}} \qquad \text{for all } t > 0,$$

where we have abbreviated

$$I(t) := \int_\Omega u_\varepsilon^{p-1} v_\varepsilon u_{\varepsilon x}^2 + \int_\Omega u_\varepsilon^{p+1} v_\varepsilon v_{\varepsilon x}^2 \qquad \text{for } t > 0.$$

For any fixed $r \in [\frac{(p+1)(p+2)}{p+4}, p+2)$, this implies that with $c_2 \equiv c_2(K) := (2c_1)^{\frac{r}{p+2}}$ we have

$$\|u_\varepsilon v_\varepsilon^{\frac{3}{p+1}}\|_{L^\infty(\Omega)}^{r} \le c_2 \|v_\varepsilon\|_{L^\infty(\Omega)}^{\frac{3r}{(p+1)(p+2)}} I^{\frac{r}{p+2}}(t) + c_2 \|v_\varepsilon\|_{L^\infty(\Omega)}^{\frac{3r}{p+1}} \qquad \text{for all } t > 0, \tag{4.13}$$

where thanks to the fact that $r < p+2$, for each $\eta > 0$ we may invoke Young's inequality to find $c_3 = c_3(\eta, K) > 0$ such that

$$c_2 \|v_\varepsilon\|_{L^\infty(\Omega)}^{\frac{3r}{(p+1)(p+2)}} I^{\frac{r}{p+2}}(t) \le \eta I(t) + c_3 \|v_\varepsilon\|_{L^\infty(\Omega)}^{\frac{3r}{(p+1)(p+2-r)}} \qquad \text{for all } t > 0. \tag{4.14}$$

Here since our assumption $r \ge \frac{(p+1)(p+2)}{p+4}$ warrants that the difference

$$a := \frac{3r}{(p+1)(p+2-r)} - 1 = \frac{(p+4)r - (p+1)(p+2)}{(p+1)(p+2-r)}$$

is nonnegative, again by (4.12) we may estimate

$$c_3 \|v_\varepsilon\|_{L^\infty(\Omega)}^{\frac{3r}{(p+1)(p+2-r)}} = c_3 \|v_\varepsilon\|_{L^\infty(\Omega)}^{a} \|v_\varepsilon\|_{L^\infty(\Omega)} \le c_3 K^a \|v_\varepsilon\|_{L^\infty(\Omega)} \qquad \text{for all } t > 0.$$

As this hypothesis on r moreover entails that

$$b := \frac{3r}{p+1} - 1 \ge \frac{3(p+2)}{p+4} - 1 = \frac{2(p+1)}{p+4}$$

is positive, we similarly obtain that

$$c_2 \|v_\varepsilon\|_{L^\infty(\Omega)}^{\frac{3r}{p+1}} \le c_2 K^b \|v_\varepsilon\|_{L^\infty(\Omega)} \qquad \text{for all } t > 0,$$

so that combining (4.13) with (4.14) yields (4.11). □

We can thereby estimate the right-hand side in Lemma 4.1 as follows.

Lemma 4.5 *Suppose that $p > \frac{1}{2}$, $q \in (1, 2(p+2))$ and $\alpha > 0$ satisfy*

$$\alpha \le \frac{(2p-1)q}{2(p+1)} \tag{4.15}$$

and

$$\alpha \le q - \frac{3(p+2)}{p+4} \tag{4.16}$$

as well as

$$\alpha > q - \frac{3(p+2)}{p+1}. \tag{4.17}$$

Then for all $\eta > 0$ and any $K > 0$ one can find $C(\eta, K) > 0$ such that whenever (1.7), (4.9) and (4.10) hold, for arbitrary $\varepsilon \in (0,1)$ we have

$$\int_\Omega u_\varepsilon^{\frac{q+2}{2}} v_\varepsilon^{q-\alpha} \le \eta \int_\Omega u_\varepsilon^{p-1} v_\varepsilon u_{\varepsilon x}^2 + \eta \int_\Omega u_\varepsilon^{p+1} v_\varepsilon v_{\varepsilon x}^2 + C(\eta, K)\|v_\varepsilon\|_{L^\infty(\Omega)} \qquad \text{for all } t > 0. \tag{4.18}$$

Proof. We first estimate

$$\begin{aligned} \int_\Omega u_\varepsilon^{\frac{q+2}{2}} v_\varepsilon^{q-\alpha} &= \int_\Omega \Big(u_\varepsilon v_\varepsilon^{\frac{3}{p+1}}\Big)^{\frac{(p+1)(q-\alpha)}{3}} \cdot u_\varepsilon^{\frac{q+2}{2} - \frac{(p+1)(q-\alpha)}{3}} \\ &\le \|u_\varepsilon v_\varepsilon^{\frac{3}{p+1}}\|_{L^\infty(\Omega)}^{\frac{(p+1)(q-\alpha)}{3}} \cdot \int_\Omega u_\varepsilon^{\frac{q+2}{2} - \frac{(p+1)(q-\alpha)}{3}} \qquad \text{for all } t > 0 \end{aligned} \tag{4.19}$$

and note that here

$$\frac{q+2}{2} - \frac{(p+1)(q-\alpha)}{3} = 1 - \frac{(2p-1)q - 2(p+1)\alpha}{6} \le 1$$

due to (4.15). Accordingly, (2.9) along with the Hölder inequality as well as (4.9) and (4.10) yields $c_1 = c_1(K) > 0$ such that

$$\int_\Omega u_\varepsilon^{\frac{q+2}{2} - \frac{(p+1)(q-\alpha)}{3}} \le c_1 \qquad \text{for all } t > 0. \tag{4.20}$$

Apart from that, on the right of (4.19) we may use (4.16) and (4.17) to estimate

$$\frac{(p+1)(q-\alpha)}{3} \ge \frac{(p+1)\cdot \frac{3(p+2)}{p+4}}{3} = \frac{(p+1)(p+2)}{p+4}$$

and

$$\frac{(p+1)(q-\alpha)}{3} < \frac{(p+1)\cdot \frac{3(p+2)}{p+1}}{3} = p+2.$$

Therefore, Lemma 4.4 applies so as to say that given $\eta > 0$ we can find $c_2 = c_2(\eta, K) > 0$ such that

$$c_1 \|u_\varepsilon v_\varepsilon^{\frac{3}{p+1}}\|_{L^\infty(\Omega)}^{\frac{(p+1)(q-\alpha)}{3}} \le \eta \int_\Omega u_\varepsilon^{p-1} v_\varepsilon u_{\varepsilon x}^2 + \eta \int_\Omega u_\varepsilon^{p+1} v_\varepsilon v_{\varepsilon x}^2 + c_2 \|v_\varepsilon\|_{L^\infty(\Omega)} \qquad \text{for all } t > 0,$$

which together with (4.19) and (4.20) establishes (4.18). □

In much the same manner, we can derive a similar inequality for the right-hand side appearing in Lemma 4.2.

Lemma 4.6 *Let $p > \frac{1}{2}, q > 1$ and $\alpha > 0$ be such that*

$$\alpha \ge \frac{(2p-1)q}{2(p+1)} \tag{4.21}$$

and

$$\alpha \ge \frac{3(p+2)q}{2(p+4)} - \frac{q+6}{2} \tag{4.22}$$

as well as

$$\alpha < \frac{3(p+2)q}{2(p+1)} - \frac{q+6}{2}. \tag{4.23}$$

Then for all $\eta > 0$ and $K > 0$ there exists $C(\eta, K) > 0$ with the property that if (1.7), (4.9) and (4.10) holds, then for any $\varepsilon \in (0,1)$,

$$\int_\Omega u_\varepsilon^{\frac{(p+1)(q+2)}{q}} v_\varepsilon^{\frac{q+2\alpha+6}{q}} \le \eta \int_\Omega u_\varepsilon^{p-1} v_\varepsilon u_{\varepsilon x}^2 + \eta \int_\Omega u_\varepsilon^{p+1} v_\varepsilon v_{\varepsilon x}^2 + C(\eta, K) \|v_\varepsilon\|_{L^\infty(\Omega)} \qquad \text{for all } t > 0. \tag{4.24}$$

Proof. Proceeding as in Lemma 4.5, on the right-hand side of the inequality

$$\begin{aligned} \int_\Omega u_\varepsilon^{\frac{(p+1)(q+2)}{q}} v_\varepsilon^{\frac{q+2\alpha+6}{q}} &= \int_\Omega \Big(u_\varepsilon v_\varepsilon^{\frac{3}{p+1}}\Big)^{\frac{(p+1)(q+2\alpha+6)}{3q}} \cdot u_\varepsilon^{\frac{2(p+1)(q-\alpha)}{3q}} \\ &\le \|u_\varepsilon v_\varepsilon^{\frac{3}{p+1}}\|_{L^\infty(\Omega)}^{\frac{(p+1)(q+2\alpha+6)}{3q}} \cdot \int_\Omega u_\varepsilon^{\frac{2(p+1)(q-\alpha)}{3q}}, \qquad t > 0, \end{aligned} \tag{4.25}$$

we use (4.21) to see that

$$\frac{2(p+1)(q-\alpha)}{3q} \le \frac{2(p+1) \cdot \Big(q - \frac{(2p-1)q}{2(p+1)}\Big)}{3q} = 1$$

to conclude from (2.9), (4.9) and (4.10) that

$$\int_\Omega u_\varepsilon^{\frac{2(p+1)(q-\alpha)}{3q}} \le c_1 \qquad \text{for all } t > 0 \tag{4.26}$$

with some $c_1 = c_1(K) > 0$. As furthermore

$$\frac{(p+1)(q+2\alpha+6)}{3q} \ge \frac{(p+1) \cdot \Big\{q + 2 \cdot \Big(\frac{3(p+2)q}{2(p+4)} - \frac{q+6}{2}\Big) + 6\Big\}}{3q} = \frac{(p+1)(p+2)}{p+4}$$

by (4.22) and

$$\frac{(p+1)(q+2\alpha+6)}{3q} < \frac{(p+1) \cdot \Big\{q + 2 \cdot \Big(\frac{3(p+2)q}{2(p+1)} - \frac{q+6}{2}\Big) + 6\Big\}}{3q} = p+2$$

due to (4.23), in view of (4.26) we readily infer (4.24) from (4.25) and Lemma 4.4. □

4.3 An L^p bound for u_ε

Fortunately, the conditions on α from Lemma 4.5 and Lemma 4.6, and in particular the inequalities (4.15) and (4.21), can simultaneously be fulfilled by some $\alpha \in (0, q)$ if $p \ge 2$ and the exponent q is taken from an appropriate intermediate range determined by p. For such choices, the coupled functional in (4.1) satisfies a differential inequality which in view of Lemma 3.4 indeed reflects a certain energy-like feature.

Lemma 4.7 *Let $p \geq 2$ and $q \geq 4$ be such that*

$$\frac{2(p+1)(p+2)}{p+4} \leq q < 2(p+2).$$

Then for all $K > 0$ there exists $C(K) > 0$ such that if (1.7), (4.9) and (4.10) are satisfied, then

$$\frac{d}{dt}\left\{ \int_\Omega u_\varepsilon^p + \int_\Omega v_\varepsilon^{-\frac{2p-1}{2(p+1)}\cdot q} |v_{\varepsilon x}|^q \right\} + \frac{\varepsilon}{C(K)} \int_\Omega u_\varepsilon^{p+m-3} u_{\varepsilon x}^2 \leq C(K) \|v_\varepsilon\|_{L^\infty(\Omega)} \cdot \left\{ 1 + \int_\Omega u_\varepsilon^p \right\} \qquad \text{for all } t > 0 \tag{4.27}$$

whenever $\varepsilon \in (0,1)$.

Proof. We let

$$\alpha := \frac{2p-1}{2(p+1)} \cdot q \tag{4.28}$$

and first invoke Lemma 4.1 to find $c_1 > 0$ and $c_2 > 0$ such that

$$\frac{d}{dt} \int_\Omega v_\varepsilon^{-\alpha} |v_{\varepsilon x}|^q + c_1 \int_\Omega v_\varepsilon^{-\alpha-2} |v_{\varepsilon x}|^{q+2} \leq c_2 \int_\Omega u_\varepsilon^{\frac{q+2}{2}} v_\varepsilon^{q-\alpha} \qquad \text{for all } t > 0, \tag{4.29}$$

whereupon Lemma 4.2 says that with some $c_3 > 0$ and $c_4 > 0$ we have

$$\begin{aligned} \frac{d}{dt} \int_\Omega u_\varepsilon^p + c_3 \int_\Omega u_\varepsilon^{p-1} v_\varepsilon u_{\varepsilon x}^2 + c_3 \int_\Omega u_\varepsilon^{p+1} v_\varepsilon v_{\varepsilon x}^2 + c_3 \varepsilon \int_\Omega u_\varepsilon^{p+m-3} u_{\varepsilon x}^2 & \\ \leq \ c_4 \|v_\varepsilon\|_{L^\infty(\Omega)} \int_\Omega u_\varepsilon^p + c_1 \int_\Omega v_\varepsilon^{-\alpha-2} |v_{\varepsilon x}|^{q+2} + c_4 \int_\Omega u_\varepsilon^{\frac{(p+1)(q+2)}{q}} v_\varepsilon^{\frac{q+2\alpha+6}{q}} & \qquad \text{for all } t > 0. \end{aligned} \tag{4.30}$$

We next observe that according to (4.28), our assumption $p \geq 2$ in particular warrants that

$$\alpha - \left\{ \frac{3(p+2)q}{2(p+4)} - \frac{q+6}{2} \right\} = \frac{3(p-2)q}{2(p+1)(p+4)} + 3 > 0, \tag{4.31}$$

while

$$\alpha - \left\{ \frac{3(p+2)q}{2(p+1)} - \frac{q+6}{2} \right\} = \frac{3(p+1-q)}{p+1} < 0 \tag{4.32}$$

thanks to the fact that $q \geq \frac{2(p+1)(p+2)}{p+4}$ especially ensures that $q \geq \frac{4}{3}(p+1) > p+1$ for any p in the considered range. The hypothesis $q \geq \frac{2(p+1)(p+2)}{p+4}$ moreover entails that

$$\alpha - \left\{ q - \frac{3(p+2)}{p+4} \right\} = 3 \cdot \frac{2(p+1)(p+2) - (p+4)q}{2(p+1)(p+4)} \leq 0, \tag{4.33}$$

whereas the restriction $q < 2(p+2)$ guarantees that

$$\alpha - \left\{ q - \frac{3(p+2)}{p+1} \right\} = 3 \cdot \frac{-q + 2(p+2)}{2(p+1)} > 0. \tag{4.34}$$

Now in view of (4.28), (4.31) and (4.32), Lemma 4.6 becomes applicable so as to yield $c_5 = c_5(K) > 0$ fulfilling

$$c_4 \int_\Omega u_\varepsilon^{\frac{(p+1)(q+2)}{q}} v_\varepsilon^{\frac{q+2\alpha+6}{q}} \leq \frac{c_3}{2} \int_\Omega u_\varepsilon^{p-1} v_\varepsilon u_{\varepsilon x}^2 + \frac{c_3}{2} \int_\Omega u_\varepsilon^{p+1} v_\varepsilon v_{\varepsilon x}^2 + c_5 \|v_\varepsilon\|_{L^\infty(\Omega)} \qquad \text{for all } t > 0, \tag{4.35}$$

and similarly (4.28), (4.33) and (4.34) enable us to infer from Lemma 4.5 that there exists $c_6 = c_6(K) > 0$ such that

$$c_2 \int_\Omega u_\varepsilon^{\frac{q+2}{2}} v_\varepsilon^{q-\alpha} \le \frac{c_3}{2} \int_\Omega u_\varepsilon^{p-1} v_\varepsilon u_{\varepsilon x}^2 + \frac{c_3}{2} \int_\Omega u_\varepsilon^{p+1} v_\varepsilon v_{\varepsilon x}^2 + c_6 \|v_\varepsilon\|_{L^\infty(\Omega)} \qquad \text{for all } t > 0. \tag{4.36}$$

When combined with (4.35) and (4.36), (4.30) and (4.29) thus show that

$$\begin{aligned} \frac{d}{dt}\Big\{ \int_\Omega u_\varepsilon^p + \int_\Omega v_\varepsilon^{-\alpha} |v_{\varepsilon x}|^q \Big\} + c_3 \varepsilon \int_\Omega u_\varepsilon^{p+m-3} u_{\varepsilon x}^2 & \\ \le\ c_4 \|v_\varepsilon\|_{L^\infty(\Omega)} \int_\Omega u_\varepsilon^p + (c_5 + c_6) \|v_\varepsilon\|_{L^\infty(\Omega)} & \qquad \text{for all } t > 0 \end{aligned}$$

and thereby establish (4.27). $\square$

A time integration in (4.27) finally in fact leads to an L^p bound for the first solution component in the following form.

Lemma 4.8 *Let $p \ge 2$. Then for all $K > 0$ there exists $C(K) > 0$ such that if u_0 and v_0 satisfy (1.7), (3.1), (3.2) and (3.9) as well as*

$$\int_\Omega u_0^p \le K \tag{4.37}$$

and

$$\int_\Omega \left| \left(v_0^{\frac{3}{2(p+1)}} \right)_x \right|^{\frac{2(p+1)(p+2)}{p+4}} \le K, \tag{4.38}$$

then for any choice of $\varepsilon \in (0,1)$ we have

$$\int_\Omega u_\varepsilon^p(\cdot,t) \le C \qquad \textit{for all } t > 0 \tag{4.39}$$

and

$$\varepsilon \int_0^T \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}^{2p+m-1} dt \le C \cdot (1 + \varepsilon T) \qquad \textit{for all } t > 0. \tag{4.40}$$

Proof. We apply Lemma 4.7 to $q := \frac{2(p+1)(p+2)}{p+4} \ge 4$ to find $c_1 = c_1(K) > 0$ and $c_2 = c_2(K) > 0$ such that

$$\frac{d}{dt}\Big\{ \int_\Omega u_\varepsilon^p + \int_\Omega v_\varepsilon^{-\frac{2p-1}{2(p+1)} \cdot q} |v_{\varepsilon x}|^q \Big\} + c_1 \varepsilon \int_\Omega (u_\varepsilon^{\frac{p+m-1}{2}})_x^2 \le c_2 \|v_\varepsilon\|_{L^\infty(\Omega)} \cdot \Big\{ \int_\Omega u_\varepsilon^p + 1 \Big\} \qquad \text{for all } t > 0.$$

Therefore,

$$y_\varepsilon(t) := \int_\Omega u_\varepsilon^p(\cdot,t) + \int_\Omega v_\varepsilon^{-\frac{2p-1}{2(p+1)} \cdot q}(\cdot,t) |v_{\varepsilon x}(\cdot,t)|^q, \qquad t \ge 0,$$

and

$$g_\varepsilon(t) := c_1 \varepsilon \int_\Omega (u_\varepsilon^{\frac{p+m-1}{2}})_x^2 \qquad \text{as well as} \qquad h_\varepsilon(t) := c_2 \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}, \qquad t > 0,$$

satisfy

$$y_\varepsilon'(t) + g_\varepsilon(t) \le h_\varepsilon(t) y_\varepsilon(t) \qquad \text{for all } t > 0, \tag{4.41}$$

which upon a first integration shows that

$$y_\varepsilon(t) \le y_\varepsilon(0) \cdot e^{\int_0^t h_\varepsilon(s) ds} \qquad \text{for all } t \ge 0.$$

Now since (3.1), (3.2), (3.9) and (1.7) hold, we may invoke Lemma 3.4 to find $c_3 = c_3(K) > 0$ such that

$$\int_0^t h_\varepsilon(s)ds \le c_3 \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1), \tag{4.42}$$

which implies that writing $c_4 = c_4(K) := K + (\frac{2(p+1)}{3})^{\frac{2(p+1)(p+2)}{p+4}} K$ we have

$$y_\varepsilon(t) = \int_\Omega u_\varepsilon^p + \int_\Omega v_\varepsilon^{-\frac{2p-1}{2(p+1)}\cdot q} |v_{\varepsilon x}|^q \le c_4 e^{c_3} \qquad \text{for all } t > 0, \tag{4.43}$$

because

$$y_\varepsilon(0) = \int_\Omega u_0^p + \Big(\frac{2(p+1)}{3}\Big)^{\frac{2(p+1)(p+2)}{p+4}} \int_\Omega \Big| \Big(v_0^{\frac{3}{2(p+1)}}\Big)_x \Big|^{\frac{2(p+1)(p+2)}{p+4}}$$

due to our definition of q.
Having thereby particularly established (4.39) in order to derive (4.40) we go back to (4.41) to see that a second integration thereof entails that due to (4.43) and again (4.42), we have

$$\int_0^T g_\varepsilon(t)dt \le y_\varepsilon(0) + \int_0^T h_\varepsilon(s) y_\varepsilon(s)ds \le c_4 + c_4 e^{c_3} \qquad \text{for all } T > 0$$

and hence

$$\varepsilon \int_0^T \int_\Omega (u_\varepsilon^{\frac{p+m-1}{2}})_x^2 \le c_5 = c_5(K) := \frac{c_4 + c_4 e^{e_3}}{c_1} \qquad \text{for all } T > 0. \tag{4.44}$$

As the Gagliardo-Nirenberg inequality provides $c_6 > 0$ such that

$$\|\varphi\|_{L^\infty(\Omega)}^{\frac{2(2p+m-1)}{p+m-1}} \le c_6 \|\varphi_x\|_{L^2(\Omega)}^2 \|\varphi\|_{L^{\frac{2p}{p+m-1}}(\Omega)}^{\frac{2p}{p+m-1}} + c_6 \|\varphi\|_{L^{\frac{2p}{p+m-1}}(\Omega)}^{\frac{2(2p+m-1)}{p+m-1}} \qquad \text{for all } \varphi \in W^{1,2}(\Omega),$$

from (4.43) and (4.44) we infer that

$$\begin{aligned}
\varepsilon \int_0^T \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}^{2p+m-1} dt &= \varepsilon \int_0^T \Big\| u_\varepsilon^{\frac{p+m-1}{2}}(\cdot,t) \Big\|_{L^\infty(\Omega)}^{\frac{2(2p+m-1)}{p+m-1}} dt \\
&\le c_6 \varepsilon \int_0^T \Big\| (u_\varepsilon^{\frac{p+m-1}{2}})_x(\cdot,t) \Big\|_{L^2(\Omega)}^2 \Big\| u_\varepsilon^{\frac{p+m-1}{2}}(\cdot,t) \Big\|_{L^{\frac{2p}{p+m-1}}(\Omega)}^{\frac{2p}{p+m-1}} dt \\
&\quad + c_6 \varepsilon \int_0^T \Big\| u_\varepsilon^{\frac{p+m-1}{2}}(\cdot,t) \Big\|_{L^{\frac{2p}{p+m-1}}(\Omega)}^{\frac{2(2p+m-1)}{p+m-1}} dt \\
&\le c_4 c_5 c_6 e^{c_3} + c_6 (c_4 e^{c_3})^{\frac{2p+m-1}{p}} \cdot \varepsilon T \qquad \text{for all } T > 0
\end{aligned}$$

and thus conclude that also (4.40) holds. □

5 Uniform boundedness of $(\ln v_\varepsilon)_x$

Again in view of the v-dependent degeneracy of diffusion in (1.6), deriving L^∞ bounds from the previously gained L^p estimates seems to require additional efforts when compared with well-established approaches from the theory of quasilinear parabolic equations and systems. In fact, it seems that any straightforward application of the standard Moser-type approach to e.g. the differential inequality from Lemma 4.7 leads to bounds for u_ε in $L^p(\Omega)$ which unfavorably depend on p due to the nature of our above methods of estimating the respective cross-diffusive contributions in (2.6).

In order to nevertheless develop a modified Moser-type approach toward corresponding L^∞ estimates in the next section, we shall now make use of the information from Lemma 4.8 for suitably large but fixed p to firstly obtain uniform pointwise bounds for the quantity $(\ln v_\varepsilon)_x$. In particular, these will enable us to estimate, wherever convenient, the gradient $|v_{\varepsilon x}|$ by some multiple of the expression v_ε which we already know to decay conveniently fast e.g. in the sense specified in Lemma 3.4. Moreover, as a by-product this will lead to the Harnack-type property (1.11) which, at the level of approximate solutions (Corollary 5.3), will also ensure a certain uniformity of the decay property from Lemma 3.4 with respect to the parameter ε (Lemma 10.3).

As our analysis in this direction will not be referred to in Theorem 1.4 and Corollary 1.5, we do no longer pursue nor stress the quantitative dependence of the subsequently obtained estimates on the initial data, hence assuming u_0 and v_0 to be fixed functions satisfying (1.7).

For convenience in notation during this section, for $\varepsilon \in (0,1)$ let us introduce the function w_ε given by

$$w_\varepsilon(x,t) := \ln v_\varepsilon(x,t), \qquad x \in \overline{\Omega},\ t \ge 0, \tag{5.1}$$

which according to the regularity and positivity properties of v_ε asserted by Lemma 2.1 is a classical solution of

$$\begin{cases} w_{\varepsilon t} = w_{\varepsilon xx} + w_{\varepsilon x}^2 - u_\varepsilon, & x \in \Omega,\ t > 0, \\ w_{\varepsilon x} = 0, & x \in \partial\Omega,\ t > 0, \\ w_\varepsilon(x,0) = \ln v_0(x), & x \in \Omega, \end{cases} \tag{5.2}$$

Then once again due to the one-dimensional structure of our problem, the spatial gradients of these solutions enjoy a further family of energy-like properties which entail the following boundedness feature.

Lemma 5.1 *Suppose that (1.7) holds, and let $q \ge 2$. Then there exists $C > 0$ such that for all $\varepsilon \in (0,1)$,*

$$\int_\Omega \left| \frac{v_{\varepsilon x}(\cdot,t)}{v_\varepsilon(\cdot,t)} \right|^q \le C \qquad \text{for all } t > 0. \tag{5.3}$$

Proof. As Ω is bounded, in view of the Hölder inequality we may assume without loss of generality that $q \ge 2$ is an even integer. We moreover note that again thanks to parabolic Schauder theory, $w_{\varepsilon x}$ lies in $C^{2,1}(\overline{\Omega} \times (0,\infty))$ and solves the differentiated version of (5.2) classically, so that upon testing the resulting identity by the quantity $w_{\varepsilon x}^{q-1}$, well-defined since $q-1$ is a positive integer, we obtain that

$$\begin{aligned} \frac{1}{q}\frac{d}{dt}\int_\Omega w_{\varepsilon x}^q &= \int_\Omega w_{\varepsilon x}^{q-1} \cdot (w_{\varepsilon xxx} + 2w_{\varepsilon x}w_{\varepsilon xx} - u_{\varepsilon x}) \\ &= -(q-1)\int_\Omega w_{\varepsilon x}^{q-2}w_{\varepsilon xx}^2 + 2\int_\Omega w_{\varepsilon x}^q w_{\varepsilon xx} - \int_\Omega u_{\varepsilon x}w_{\varepsilon x}^{q-1} \qquad \text{for all } t > 0. \end{aligned} \tag{5.4}$$

Once more due to the boundary condition, the second last summand herein satisfies

$$2\int_\Omega w_\varepsilon^q w_{\varepsilon xx} = \frac{2}{q+1}\int_\Omega (w_{\varepsilon x}^{q+1})_x = 0 \qquad \text{for all } t > 0,$$

while in the last we again integrate by parts and useYoung's inequality to see that

$$\begin{aligned} -\int_\Omega u_{\varepsilon x}w_{\varepsilon x}^{q-1} &= (q-1)\int_\Omega u_\varepsilon w_{\varepsilon x}^{q-2}w_{\varepsilon xx} \\ &\le \frac{q-1}{2}\int_\Omega w_{\varepsilon x}^{q-2}w_{\varepsilon xx}^2 + \frac{q-1}{2}\int_\Omega u_\varepsilon^2 w_{\varepsilon x}^{q-2} \\ &\le \frac{q-1}{2}\int_\Omega w_{\varepsilon x}^{q-2}w_{\varepsilon xx}^2 + c_1\|w_{\varepsilon x}\|_{L^\infty(\Omega)}^{q-2} \qquad \text{for all } t > 0, \end{aligned} \tag{5.5}$$

where thanks to Lemma 4.8,

$$c_1 := \frac{q-1}{2} \cdot \sup_{\varepsilon\in(0,1)} \sup_{t>0} \int_\Omega u_\varepsilon^2(\cdot,t)$$

is finite. We now make use of the fact that in the considered one-dimensional setting we have

$$\|\phi\|_{L^\infty(\Omega)} \le c_2 \|\phi_x\|_{L^2(\Omega)} \qquad \text{for all } \phi\in W_0^{1,2}(\Omega) \tag{5.6}$$

with e.g. $c_2 := |\Omega|^{\frac{1}{2}}$, which when applied to $\phi := w_{\varepsilon x}^{\frac{q}{2}}$ shows that due to Young's inequality,

$$\begin{aligned} c_1\|w_{\varepsilon x}\|_{L^\infty(\Omega)}^{q-2} &= c_1 \|w_{\varepsilon x}^{\frac{q}{2}}\|_{L^\infty(\Omega)}^{\frac{2(q-2)}{q}} \\ &\le c_1 c_2^{\frac{2(q-2)}{q}} \Big\|(w_{\varepsilon x}^{\frac{q}{2}})_x\Big\|_{L^2(\Omega)}^{\frac{2(q-2)}{q}} \\ &\le \frac{q-1}{q^2}\cdot \Big\|(w_{\varepsilon x}^{\frac{q}{2}})_x\Big\|_{L^2(\Omega)}^2 + c_3 \\ &= \frac{q-1}{4}\int_\Omega w_{\varepsilon x}^{q-2} w_{\varepsilon xx}^2 + c_3 \qquad \text{for all } t>0 \end{aligned} \tag{5.7}$$

with $c_3 := (\frac{q-1}{q^2})^{\frac{2-q}{2}} c_1^{\frac{q}{2}} c_2^{q-2}$. As once more by (5.6) we can estimate

$$\begin{aligned} \int_\Omega w_{\varepsilon x}^q &\le |\Omega|\cdot \|w_{\varepsilon x}^{\frac{q}{2}}\|_{L^\infty(\Omega)}^2 \\ &\le |\Omega|\cdot c_2 \Big\|(w_{\varepsilon x}^{\frac{q}{2}})_x\Big\|_{L^2(\Omega)}^2 \\ &= \frac{q^2|\Omega|c_2}{4}\int_\Omega w_{\varepsilon x}^{q-2} w_{\varepsilon xx}^2 \qquad \text{for all } t>0, \end{aligned}$$

from (5.4), (5.5) and (5.7) we thus infer that writing $c_4 := \frac{q-1}{q^2|\Omega|c_2}$ we have

$$\frac{1}{q}\frac{d}{dt}\int_\Omega w_{\varepsilon x}^q + c_4 \int_\Omega w_{\varepsilon x}^q \le c_3 \qquad \text{for all } t>0$$

and hence, thanks to the inclusion $w_{\varepsilon x}\in C^0([0,\infty);L^q(\Omega))$ guaranteed by (2.7) in conjunction with the positivity of v_ε on $\overline{\Omega}\times[0,\infty)$,

$$\int_\Omega w_{\varepsilon x}^q \le \max\left\{ \int_\Omega \frac{v_{0x}^q}{v_0^q}\, , \, \frac{c_4}{c_3} \right\} \qquad \text{for all } t>0,$$

which precisely yields (5.3). □

In particular, together with Lemma 4.8 this allows us to view the inhomogeneity $h_\varepsilon := w_{\varepsilon x}^2 - u_\varepsilon$ in (5.2) as a perturbation uniformly bounded with respect to the norm in $L^p(\Omega)$ for arbitrary $p>1$. Therefore, applying straightforward regularity arguments to the heat equation $w_{\varepsilon t} = w_{\varepsilon xx} + h_\varepsilon(x,t)$ yields the desired pointwise estimate for $w_{\varepsilon x}$.

Lemma 5.2 *Assume (1.7). Then there exists $C>0$ such that for all $\varepsilon\in(0,1)$,*

$$|v_{\varepsilon x}(x,t)| \le C v_\varepsilon(x,t) \qquad \textit{for all } x\in\Omega \textit{ and } t>0. \tag{5.8}$$

PROOF. We fix any $p > 1$ and then obtain from Lemma 5.1 and Lemma 4.8 that there exists $c_1 > 0$ such that for all $\varepsilon \in (0,1)$,

$$\int_\Omega |w_{\varepsilon x}|^{2p} \le c_1 \quad \text{and} \quad \int_\Omega u_\varepsilon^p \le c_1 \qquad \text{for all } t > 0. \tag{5.9}$$

Letting $(e^{-t\mathcal{A}_N})_{t\ge 0}$ and $(e^{-t\mathcal{A}_D})_{t\ge 0}$ denote the heat semigroups over Ω under homogeneous Neumann and Dirichlet boundary conditions, respectively, by means of a Duhamel formula associated with (5.2) and a known smoothing property of $(e^{-t\mathcal{A}_N})_{t\ge 0}$ we can find $c_2 > 0$ such that

$$\begin{aligned}
\|w_{\varepsilon x}(\cdot,t)\|_{L^\infty(\Omega)} &= \Big\| \partial_x e^{-\min\{1,t\}\mathcal{A}_N} w_\varepsilon(\cdot,(t-1)_+) + \int_{(t-1)_+}^t \partial_x e^{(t-s)\mathcal{A}_N} \Big\{ w_{\varepsilon x}^2(\cdot,s) - u_\varepsilon(\cdot,s) \Big\} ds \Big\|_{L^\infty(\Omega)} \\
&\le \Big\| \partial_x e^{-\min\{1,t\}\mathcal{A}_N} w_\varepsilon(\cdot,(t-1)_+) \Big\|_{L^\infty(\Omega)} \\
&\quad + c_2 \int_{(t-1)_+}^t (t-s)^{-\frac12 - \frac{1}{2p}} \Big\| w_{\varepsilon x}^2(\cdot,s) - u_\varepsilon(\cdot,s) \Big\|_{L^p(\Omega)} ds \qquad \text{for all } t > 0,
\end{aligned} \tag{5.10}$$

where thanks to (5.9),

$$\Big\| w_{\varepsilon x}^2(\cdot,s) - u_\varepsilon(\cdot,s) \Big\|_{L^p(\Omega)} \le \|w_{\varepsilon x}(\cdot,s)\|_{L^{2p}(\Omega)}^2 + \|u_\varepsilon(\cdot,s)\|_{L^p(\Omega)} \le 2c_1^{\frac1p} \qquad \text{for all } s > 0,$$

so that

$$c_2 \int_{(t-1)_+}^t (t-s)^{-\frac12 - \frac{1}{2p}} \Big\| w_{\varepsilon x}^2(\cdot,s) - u_\varepsilon(\cdot,s) \Big\|_{L^p(\Omega)} ds \le \frac{2c_1^{\frac1p} c_2}{\frac12 - \frac{1}{2p}} \tag{5.11}$$

due to our restriction $p > 1$.
We now note that in the considered one-dimensional framework the identity

$$\partial_x e^{-t\mathcal{A}_N} \phi = e^{-t\mathcal{A}_D} \phi_x$$

can readily be verified to hold in Ω for all $t > 0$ and any $\phi \in C^1(\overline{\Omega})$ fulfilling $\phi_x = 0$ on $\partial\Omega$. Therefore, by means of the comparison principle we find that for small t,

$$\begin{aligned}
\Big\| \partial_x e^{-\min\{1,t\}\mathcal{A}_N} w_\varepsilon(\cdot,(t-1)_+) \Big\|_{L^\infty(\Omega)} &= \|e^{-t\mathcal{A}_D} w_{\varepsilon x}(\cdot,0)\|_{L^\infty(\Omega)} \\
&\le \|w_{\varepsilon x}(\cdot,0)\|_{L^\infty(\Omega)} \\
&= \Big\| \frac{v_{0x}}{v_0} \Big\|_{L^\infty(\Omega)} \qquad \text{for all } t \in (0,1],
\end{aligned} \tag{5.12}$$

whereas for larger times a known regularization feature of $(e^{-t\mathcal{A}_D})_{t\ge 0}$ provides $c_3 > 0$ fulfilling

$$\begin{aligned}
\Big\| \partial_x e^{-\min\{1,t\}\mathcal{A}_N} w_\varepsilon(\cdot,(t-1)_+) \Big\|_{L^\infty(\Omega)} &= \Big\| e^{-\mathcal{A}_D} w_{\varepsilon x}(\cdot,t-1) \Big\|_{L^\infty(\Omega)} \\
&\le c_3 \|w_{\varepsilon x}(\cdot,t-1)\|_{L^{2p}(\Omega)} \\
&\le c_1^{\frac{1}{2p}} c_3 \qquad \text{for all } t > 1
\end{aligned}$$

according to (5.9). Together with (5.12) and (5.11) inserted into (5.10), this establishes (5.8). □

The approximate counterpart of (1.11) is now obvious.

Corollary 5.3 *Suppose that (1.7) holds. Then there exists $C > 0$ such that whenever $\varepsilon \in (0,1)$,*

$$v_\varepsilon(x,t) \ge C \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \qquad \textit{for all } x \in \Omega \textit{ and } t > 0. \tag{5.13}$$

PROOF. According to Lemma 5.2, we can find $c_1 > 0$ such that

$$|v_{\varepsilon x}(x,t)| \le c_1 v_\varepsilon(x,t) \qquad \text{for all } x \in \Omega,\ t > 0 \text{ and } \varepsilon \in (0,1). \tag{5.14}$$

Then for fixed $\varepsilon \in (0,1)$ and $t > 0$ we may choose $x_0 \in \overline{\Omega}$ such that $v_\varepsilon(x_0,t) = \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$, and use (5.14) to estimate

$$\begin{aligned} \ln v_\varepsilon(x,t) &= \ln v_\varepsilon(x_0,t) + \int_{x_0}^x (\ln v_\varepsilon)_x(y,t)dy \\ &\ge \ln v_\varepsilon(x_0,t) - c_1|\Omega| \qquad \text{for all } x \in \overline{\Omega}, \end{aligned}$$

which clearly implies (5.13) with $C := e^{-c_1|\Omega|}$. □

6 Uniform boundedness of u_ε

We are now prepared to perform an iterative argument of Moser type in order to derive an L^∞ bound for u_ε independent of both ε and t. Our procedure will particularly involve a recursive inequality of the form (6.1) which, beyond a quadratic nonlinearity of standard type, contains a certain inhomogeneity with fast but yet digestible growth with respect to the sequence index. Since we could not find a precise reference treating such inequalities in the literature, for completeness we include a short proof of the following elementary statement.

Lemma 6.1 *Let $a \ge 1$, $b \ge 1$, $M_0 \ge 1$ and $(M_k)_{k\in\mathbb{N}} \subset [0,\infty)$ be such that*

$$M_k \le a^k M_{k-1}^2 + b^{2^k} \qquad \textit{for all } k \ge 1. \tag{6.1}$$

Then

$$\liminf_{k\to\infty} M_k^{\frac{1}{2^k}} \le b M_0 e^C, \tag{6.2}$$

where

$$C := \sum_{j=1}^\infty \frac{\ln 2 + j \ln a}{2^j}. \tag{6.3}$$

PROOF. For convenience in notation, we introduce $M_{-1} := 0$ and then observe that the set

$$S := \left\{ k \ge 0 \ \middle|\ b^{2^k} \ge M_{k-1}^2 \right\}$$

is not empty with $0 \in S$, and that inside this set we can trivially estimate

$$M_{k-1}^{\frac{1}{2^{k-1}}} \le \left(b^{2^k}\right)^{\frac{1}{2\cdot 2^{k-1}}} = b \qquad \text{for all } k \in S. \tag{6.4}$$

In particular, this directly entails (6.2) in the case when S contains inifinitely many elements, because $M_0 \ge 1$ and C is positive by (6.3) and our assumption $a \ge 1$.
Thus left with the case when S is finite and hence $k_0 := \max S$ well-defined, using that then

$$b^{2^k} < M_{k-1}^2 \le a^k M_{k-1}^2 \qquad \text{for all } k > k_0,$$

we obtain from (6.1) that

$$M_k \le 2a^k M_{k-1}^2 \qquad \text{for all } k > k_0,$$

which is equivalent to saying that for $z_k := \ln M_k^{\frac{1}{2^k}}$, $k \ge 0$, we have

$$z_k \le \frac{\ln 2 + k \ln a}{2^k} + z_{k-1} \qquad \text{for all } k > k_0.$$

Therefore,

$$z_k \le z_{k_0} + \sum_{j=k_0+1}^{k} \frac{\ln 2 + j \ln a}{2^j} \le z_{k_0} + C \qquad \text{for all } k > k_0$$

and thus

$$M_k^{\frac{1}{2^k}} \le e^{z_{k_0}} \cdot e^C \qquad \text{for all } k > k_0. \tag{6.5}$$

Now in the exceptional case $k_0 = 0$ we have

$$e^{z_{k_0}} = e^{z_0} = M_0 \le bM_0,$$

while if $k_0 > 0$ then its definition along with (6.4) warrants that

$$e^{z_{k_0}} = M_{k_0-1}^{\frac{1}{2^{k_0-1}}} \le b \le bM_0.$$

In consequence, (6.5) hence ensures that (6.2) also holds when S is finite. □

We can now proceed to the verification of the announced boundedness result. As usual in the context of Moser-type iterations, our starting point will consist in a testing procedure of the form in Lemma 4.2, but in contrast to the latter we will now interpret the dissipative expression $\int_\Omega u_\varepsilon^{p-1} v_\varepsilon u_{\varepsilon x}^2$ appearing therein, up to multiplicative constants, as part of the full Dirichlet integral of the coupled quantity $u_\varepsilon^{\frac{p-1}{2}} v_\varepsilon^{\frac{1}{2}}$ (see (6.10)), and estimate both the respective error thereby made, as well as the corresponding cross-diffusive contribution, by making use of the pointwise inequality from Lemma 5.2 (cf. (6.12) and (6.15)). In each of the quantities to be estimated from above, this will enable us to retain the quantity $\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$ as a factor which decays in the sense of Lemma 3.4 and can thus be estimated after an integration in time (see (6.16) and (6.17)). By means of Lemma 6.1 this will entail the following.

Lemma 6.2 *Assume (1.7). Then there exists $C > 0$ with the property that for all $\varepsilon \in (0,1)$ we have*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t > 0. \tag{6.6}$$

PROOF. Writing

$$p_k := 2^k + 1, \qquad k \ge 0, \tag{6.7}$$

we observe that in view of Lemma 4.8, for each $\varepsilon \in (0,1)$ any of the numbers

$$M_{k,\varepsilon} := \max\left\{1\,,\, \sup_{t>0} \int_\Omega u_\varepsilon^{p_k}(\cdot,t)\right\}, \qquad k \in \mathbb{N}_0, \tag{6.8}$$

is finite. To estimate $M_{k,\varepsilon}$ for $k \ge 1$ and $\varepsilon \in (0,1)$, fixing any such k we abbreviate $p := p_k$ and once more use the first equation in (2.6) to see on employing Young's inequality that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u_\varepsilon^p + (p-1)\varepsilon \int_\Omega u_\varepsilon^{p-2}(u_\varepsilon+1)^{m-1}u_{\varepsilon x}^2 + (p-1)\int_\Omega u_\varepsilon^{p-1} v_\varepsilon u_{\varepsilon x}^2 \\
&\quad = (p-1)\int_\Omega u_\varepsilon^p v_\varepsilon u_{\varepsilon x} v_{\varepsilon x} + \int_\Omega u_\varepsilon^p v_\varepsilon \\
&\quad \le \frac{p-1}{2}\int_\Omega u_\varepsilon^{p-1} v_\varepsilon u_{\varepsilon x}^2 + \frac{p-1}{2}\int_\Omega u_\varepsilon^{p+1} v_\varepsilon v_{\varepsilon x}^2 + \int_\Omega u_\varepsilon^p v \qquad \text{for all } t > 0,
\end{aligned}$$

so that since $\frac{p}{4} \le \frac{p-1}{2} \le \frac{p}{2}$ due to the fact that $p \ge p_0 = 2$, we obtain

$$\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{p^2}{4}\int_\Omega u_\varepsilon^{p-1} v_\varepsilon u_{\varepsilon x}^2 \le \frac{p^2}{2}\int_\Omega u_\varepsilon^{p+1} v_\varepsilon v_{\varepsilon x}^2 + p\int_\Omega u_\varepsilon^p v_\varepsilon \qquad \text{for all } t>0. \qquad (6.9)$$

In order to take appropriate advantage of the second summand on the left-hand side herein, we again use Young's inequality along with the fact that $p+1 \le 2p$ to find that

$$\begin{aligned}
\int_\Omega (u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12})_x^2 &= \int_\Omega \Big\{ \frac{p+1}{2} u_\varepsilon^{\frac{p-1}{2}} v_\varepsilon^{\frac12} u_{\varepsilon x} + \frac12 u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{-\frac12} v_{\varepsilon x}\Big\}^2 \\
&\le \frac{(p+1)^2}{2}\int_\Omega u_\varepsilon^{p-1} v_\varepsilon u_{\varepsilon x}^2 + \frac12 \int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{-1} v_{\varepsilon x}^2 \\
&\le 2p^2 \int_\Omega u_\varepsilon^{p-1} v_\varepsilon u_{\varepsilon x}^2 + \frac12 \int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{-1} v_{\varepsilon x}^2 \qquad \text{for all } t>0
\end{aligned}$$

and that hence

$$\frac{p^2}{4}\int_\Omega u_\varepsilon^{p-1} v_\varepsilon u_{\varepsilon x}^2 \ge \frac18 \int_\Omega (u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12})_x^2 - \frac1{16}\int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{-1} v_{\varepsilon x}^2 \qquad \text{for all } t>0. \qquad (6.10)$$

Here the rightmost summand can be controlled by using Lemma 5.2, which namely provides $c_1 > 0$ such that

$$v_{\varepsilon x}^2(x,t) \le c_1 v_\varepsilon^2(x,t) \qquad \text{for all } x\in\Omega, t>0 \text{ and } \varepsilon\in(0,1), \qquad (6.11)$$

so that invoking the Gagliardo-Nirenberg inequality and Young's inequality we infer that with some $c_2 > 0$ and $c_3 > 0$ we have

$$\begin{aligned}
\frac1{16}\int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{-1} v_{\varepsilon x}^2 &\le \frac{c_1}{16}\int_\Omega u_\varepsilon^{p+1} v_\varepsilon \\
&= \frac{c_1}{16} \|u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12}\|_{L^2(\Omega)}^2 \\
&\le c_2 \Big\| (u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12})_x \Big\|_{L^2(\Omega)}^{\frac23} \|u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12}\|_{L^1(\Omega)}^{\frac43} + c_2 \|u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12}\|_{L^1(\Omega)}^2 \\
&\le \frac1{16}\Big\| (u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12})_x \Big\|_{L^2(\Omega)}^2 + c_3 \|u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12}\|_{L^1(\Omega)}^2 \qquad \text{for all } t>0. \qquad (6.12)
\end{aligned}$$

As (6.7) warrants that $\frac{p+1}{2} = \frac{p_k+1}{2} = p_{k-1}$ and hence

$$\|u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12}\|_{L^1(\Omega)}^2 \le \|v_\varepsilon\|_{L^\infty(\Omega)} \cdot \Big\{ \int_\Omega u_\varepsilon^{\frac{p+1}{2}} \Big\}^2 \le \|v_\varepsilon\|_{L^\infty(\Omega)} \cdot M_{k-1,\varepsilon}^2 \qquad \text{for all } t>0 \qquad (6.13)$$

according to (6.8), this means that

$$\frac1{16}\int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{-1} v_{\varepsilon x}^2 \le \frac1{16}\int_\Omega (u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12})_x^2 + c_3 \|v_\varepsilon\|_{L^\infty(\Omega)} M_{k-1,\varepsilon}^2 \qquad \text{for all } t>0,$$

whence combining (6.9) with (6.10) shows that

$$\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac1{16}\int_\Omega (u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12})_x^2 \le \frac{p^2}{2}\int_\Omega u_\varepsilon^{p+1} v_\varepsilon v_{\varepsilon x}^2 + p\int_\Omega u_\varepsilon^p v_\varepsilon + c_3 \|v_\varepsilon\|_{L^\infty(\Omega)} M_{k-1,\varepsilon}^2 \qquad \text{for all } t>0. \quad (6.14)$$

Here the first summand on the right can be treated by arguments quite similar to those used in (6.12) and (6.13): indeed, using (6.11), (2.13), the Gagliardo-Nirenberg inequality and Young's inequality we can find positive constants c_4, c_5 and c_6 such that

$$
\begin{aligned}
\frac{p^2}{2}\int_\Omega u_\varepsilon^{p+1} v_\varepsilon v_{\varepsilon x}^2 &\le \frac{c_1 p^2}{2}\int_\Omega u_\varepsilon^{p+1} v_\varepsilon^3 \\
&\le c_4 p^2 \int_\Omega u_\varepsilon^{p+1} v_\varepsilon \\
&\le c_5 p^2 \Big\| (u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12})_x \Big\|_{L^2(\Omega)}^{\frac23} \| u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12} \|_{L^1(\Omega)}^{\frac43} + c_5 p^2 \| u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12} \|_{L^1(\Omega)}^2 \\
&\le \frac{1}{32} \Big\| (u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12})_x \Big\|_{L^2(\Omega)}^2 + c_6 p^3 \| u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12} \|_{L^1(\Omega)}^2 + c_5 p^2 \| u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12} \|_{L^1(\Omega)}^2 \\
&\le \frac{1}{32}\int_\Omega (u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12})_x^2 + (c_5+c_6) p^3 \|v_\varepsilon\|_{L^\infty(\Omega)} M_{k-1,\varepsilon}^2 \qquad \text{for all } t>0, \qquad (6.15)
\end{aligned}
$$

because $p \ge 1$.
Finally, in the second last term in (6.14) we again invoke the Gagliardo-Nirenberg inequality, Young's inequality and (6.13) to find $c_7 > 0$ and $c_8 > 0$ fulfilling

$$
\begin{aligned}
p\int_\Omega u_\varepsilon^p v_\varepsilon &= p \int_\Omega (u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12})^{\frac{2p}{p+1}} \cdot v_\varepsilon^{\frac{1}{p+1}} \\
&\le p \|v_\varepsilon\|_{L^\infty(\Omega)}^{\frac{1}{p+1}} \| u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12} \|_{L^{\frac{2p}{p+1}}(\Omega)}^{\frac{2p}{p+1}} \\
&\le c_7 p \|v_\varepsilon\|_{L^\infty(\Omega)}^{\frac{1}{p+1}} \Big\| (u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12})_x \Big\|_{L^2(\Omega)}^{\frac{2(p-1)}{3(p+1)}} \| u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12} \|_{L^1(\Omega)}^{\frac{2(2p+1)}{3(p+1)}} + c_7 p \|v_\varepsilon\|_{L^\infty(\Omega)}^{\frac{1}{p+1}} \| u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12} \|_{L^1(\Omega)}^{\frac{2p}{p+1}} \\
&\le \frac{1}{32} \Big\| (u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12})_x \Big\|_{L^2(\Omega)}^2 + c_8 p^{\frac{3(p+1)}{2(p+2)}} \|v_\varepsilon\|_{L^\infty(\Omega)}^{\frac{3}{2(p+2)}} \| u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12} \|_{L^1(\Omega)}^{\frac{2p+1}{p+2}} + c_7 p \|v_\varepsilon\|_{L^\infty(\Omega)}^{\frac{1}{p+1}} \| u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12} \|_{L^1(\Omega)}^{\frac{2p}{p+1}} \\
&\le \frac{1}{32}\int_\Omega (u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12})_x^2 + c_8 p^{\frac{3(p+1)}{2(p+2)}} \|v_\varepsilon\|_{L^\infty(\Omega)}^{\frac{3}{2(p+2)}} \cdot \Big\{ \|v_\varepsilon\|_{L^\infty(\Omega)}^{\frac{2p+1}{2(p+2)}} M_{k-1,\varepsilon}^{\frac{2p+1}{p+2}} \Big\} \\
&\quad + c_7 p \|v_\varepsilon\|_{L^\infty(\Omega)}^{\frac{1}{p+1}} \cdot \Big\{ \|v_\varepsilon\|_{L^\infty(\Omega)}^{\frac{p}{p+1}} M_{k-1,\varepsilon}^{\frac{2p}{p+1}} \Big\} \\
&= \frac{1}{32}\int_\Omega (u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12})_x^2 + c_8 p^{\frac{3(p+1)}{2(p+2)}} \|v_\varepsilon\|_{L^\infty(\Omega)} M_{k-1,\varepsilon}^{\frac{2p+1}{p+2}} + c_7 p \|v_\varepsilon\|_{L^\infty(\Omega)} M_{k-1,\varepsilon}^{\frac{2p}{p+1}} \qquad \text{for all } t>0.
\end{aligned}
$$

Since evidently $\frac{3(p+1)}{2(p+2)} \le 3$, $\frac{2p+1}{p+2} \le 2$ and $\frac{2p}{p+1} \le 2$, we may use the inequalities $p \ge 1$ and $M_{k-1,\varepsilon} \ge 1$ here to obtain

$$
p\int_\Omega u_\varepsilon^p v_\varepsilon \le \frac{1}{32}\int_\Omega (u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^{\frac12})_x^2 + (c_7+c_8) p^3 \|v_\varepsilon\|_{L^\infty(\Omega)} M_{k-1,\varepsilon}^2 \qquad \text{for all } t>0,
$$

which in conjunction with (6.15) and (6.14) shows that

$$
\frac{d}{dt}\int_\Omega u_\varepsilon^p \le c_9 p^3 \|v_\varepsilon\|_{L^\infty(\Omega)} M_{k-1,\varepsilon}^2 \qquad \text{for all } t>0 \qquad (6.16)
$$

with $c_9 := c_3 + c_5 + c_6 + c_7 + c_8$.
We now rely on the fact that Lemma 3.4 yields $c_{10} > 0$ satisfying

$$
\int_0^\infty \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le c_{10} \qquad \text{for all } \varepsilon \in (0,1),
$$

as a consequence of which we infer upon an integration in (6.16) that

$$\begin{aligned} \int_\Omega u_\varepsilon^{p_k}(\cdot,t) &\le \int_\Omega u_0^{p_k} + c_9 p_k^3 M_{k-1,\varepsilon}^2 \int_0^t \|v_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)} ds \\ &\le \int_\Omega u_0^{p_k} + c_9 c_{10} p_k^3 M_{k-1,\varepsilon}^2 \qquad \text{for all } t>0. \end{aligned} \tag{6.17}$$

Writing $c_{11} := \max\{1, c_9 c_{10}\}$, after maximizing over $t \in (0,\infty)$ we thus obtain that in both cases $\|u_\varepsilon\|_{L^\infty((0,\infty);L^{p_k}(\Omega))} < 1$ and $\|u_\varepsilon\|_{L^\infty((0,\infty);L^{p_k}(\Omega))} \ge 1$,

$$M_{k,\varepsilon} \le \int_\Omega u_0^{p_k} + c_{11} p_k^3 M_{k-1,\varepsilon}^2.$$

By means of the obvious estimates

$$c_{11} p_k^3 = c_{11} \cdot (2^k+1)^3 \le c_{11} \cdot (2\cdot 2^k)^3 = 8c_{11} \cdot 8^k \le a^k$$

with $a := 64 c_{11} \ge 1$ and

$$\begin{aligned} \int_\Omega u_0^{p_k} \le |\Omega| \cdot \|u_0\|_{L^\infty(\Omega)}^{p_k} &= \Big(|\Omega| \cdot \|u_0\|_{L^\infty(\Omega)}\Big) \cdot \|u_0\|_{L^\infty(\Omega)}^{2^k} \\ &\le \bigg\{ \max\Big\{1\,,\, |\Omega|\cdot\|u_0\|_{L^\infty(\Omega)}\Big\}\bigg\}^{2^k} \cdot \|u_0\|_{L^\infty(\Omega)}^{2^k} \\ &\le b^{2^k} \end{aligned}$$

with

$$b := \max\bigg\{1\,,\, \max\Big\{1\,,\,|\Omega|\cdot\|u_0\|_{L^\infty(\Omega)}\Big\} \cdot \|u_0\|_{L^\infty(\Omega)}\bigg\} \ge 1,$$

this entails that

$$M_{k,\varepsilon} \le a^k M_{k-1,\varepsilon}^2 + b^{2^k} \qquad \text{for all } k \ge 1 \text{ and } \varepsilon \in (0,1),$$

whence Lemma 6.1 applies so as to show that with some $c_{12} > 0$ we have

$$\liminf_{k\to\infty} M_{k,\varepsilon}^{\frac{1}{2^k}} \le c_{12} \qquad \text{for all } \varepsilon \in (0,1).$$

As $M_{k,\varepsilon} \ge 1$ and $p_k \ge 2^k$, this implies that also

$$\liminf_{k\to\infty} M_{k,\varepsilon}^{\frac{1}{p_k}} \le \liminf_{k\to\infty} M_{k,\varepsilon}^{\frac{1}{2^k}} \le c_{12} \qquad \text{for all } \varepsilon \in (0,1),$$

which by (6.8) entails that

$$c_{12} \ge \liminf_{k\to\infty} \bigg\{ \int_\Omega u_\varepsilon^{p_k}(\cdot,t) \bigg\}^{\frac{1}{p_k}} = \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \qquad \text{for all } t>0 \text{ and any } \varepsilon \in (0,1)$$

and thereby proves the lemma. □

7 Further temporally local estimates

The uniform boundedness property of u_ε obtained in Lemma 6.2 has some straightforward consequences on further regularity properties of both solution components. Once more due to the structure of the diffusion degeneracy in (1.6), these will substantially depend on appropriate positivity features of v_ε and thereby essentially remain local in time by relying on the following immediate by-product of Lemma 6.2.

Lemma 7.1 *There exist $\kappa > 0$ and $C > 0$ such that for all $\varepsilon \in (0,1)$,*

$$v_\varepsilon(x,t) \ge C \cdot e^{-\kappa t} \qquad \textit{for all } x \in \Omega \textit{ and } t > 0. \tag{7.1}$$

PROOF. Since Lemma 6.2 yields $c_1 > 0$ such that $u_\varepsilon \le c_1$ in $\Omega \times (0,\infty)$ and hence

$$v_{\varepsilon t} \ge v_{\varepsilon xx} - c_1 v_\varepsilon \qquad \text{in } \Omega \times (0,\infty),$$

the inequality in (7.1) with $\kappa := c_1$ and $C := \min_{x\in\overline{\Omega}} v_0(x) > 0$ immediately results from a straightforward comparison argument. $\square$

We can thereby extract from our previously gained weighted estimates for $u_{\varepsilon x}$ a corresponding local-in-time integral bound which no longer involves any weight function.

Lemma 7.2 *There exists $C > 0$ such that for any $\varepsilon \in (0,1)$,*

$$\int_0^T \int_\Omega u_{\varepsilon x}^2 \le Ce^{\kappa T} \qquad \textit{for all } T > 0, \tag{7.2}$$

where $\kappa > 0$ is as in Lemma 7.1.

PROOF. According to Lemma 3.1, we can find $c_1 > 0$ such that

$$\int_0^\infty \int_\Omega \frac{v_\varepsilon}{u_\varepsilon} u_{\varepsilon x}^2 \le c_1 \qquad \text{for all } \varepsilon \in (0,1), \tag{7.3}$$

whereas Lemma 6.2 and Lemma 7.1 provide positive constants c_2 and c_3 such that whenever $\varepsilon \in (0,1)$,

$$u_\varepsilon(x,t) \le c_2 \qquad \text{for all } x \in \Omega \text{ and } t > 0$$

and

$$v_\varepsilon(x,t) \ge c_3 e^{-\kappa t} \qquad \text{for all } x \in \Omega \text{ and each } t > 0.$$

Therefore, (7.3) implies that

$$c_1 \ge \int_0^T \int_\Omega \frac{v_\varepsilon}{u_\varepsilon} u_{\varepsilon x}^2 \ge \frac{c_1 c_3 e^{-\kappa T}}{c_2} \int_0^T \int_\Omega u_{\varepsilon x}^2 \qquad \text{for all } T > 0 \text{ and } \varepsilon \in (0,1)$$

and thereby entails (7.2). $\square$

Independently from the preceding, the degeneracy control provided by Lemma 7.1 allows us to invoke standard regularity theory for porous medium type scalar parabolic equations ([33]) in order to obtain Hölder estimates for both solution components.

Lemma 7.3 *Let $T > 0$. Then there exist $\theta = \theta(T) \in (0,1)$ and $C(T) > 0$ with the property that*

$$\|u_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[0,T])} \le C(T) \qquad \textit{for all } \varepsilon \in (0,1) \tag{7.4}$$

and

$$\|v_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[0,T])} \le C(T) \qquad \textit{for all } \varepsilon \in (0,1). \tag{7.5}$$

Proof. We write the first equation in (2.6) in the form

$$u_{\varepsilon t} = \partial_x A_\varepsilon(x,t,u_\varepsilon,u_{\varepsilon x}) + B_\varepsilon(x,t,u_\varepsilon), \qquad x\in\Omega,\ t>0,$$

with

$$A_\varepsilon(x,t,z,\xi) := \varepsilon(z+1)^{m-1}\xi + v_\varepsilon(x,t)z\xi - v_\varepsilon(x,t)v_{\varepsilon x}(x,t)z^2, \qquad (x,t,z,\xi)\in\Omega\times(0,\infty)\times[0,\infty)\times\mathbb{R}$$

and

$$B_\varepsilon(x,t,z) := v_\varepsilon(x,t)z, \qquad (x,t,z)\in\Omega\times(0,\infty)\times[0,\infty).$$

Here using that Lemma 7.1,(2.13) and Lemma 5.2 provide $c_1 = c_1(T), c_2 > 0$ and $c_3 > 0$ such that

$$c_1 \le v_\varepsilon(x,t) \le c_2 \quad \text{and} \quad |v_{\varepsilon x}(x,t)| \le c_3 \qquad \text{for all } x\in\Omega, t>0 \text{ and } \varepsilon\in(0,1),$$

we may use Young's inequality to estimate

$$\begin{aligned} A_\varepsilon(x,t,z,\xi)\cdot\xi &\ge c_1 z\xi^2 - v_\varepsilon v_{\varepsilon x} z^2\xi \\ &\ge \frac{1}{2}c_1 z\xi^2 - \frac{c_2^2c_3^2}{2}z^3 \qquad \text{for all } (x,t,z,\xi)\in\Omega\times(0,\infty)\times[0,\infty)\times\mathbb{R}, \end{aligned}$$

whereas evidently

$$|A_\varepsilon(x,t,z,\xi)| \le \Big\{(z+1)^{m-1} + c_2 z\Big\}\cdot|\xi| + c_2c_3z^2 \qquad \text{for all } (x,t,z,\xi)\in\Omega\times(0,\infty)\times[0,\infty)\times\mathbb{R}$$

and

$$|B_\varepsilon(x,t,z)| \le c_2 z \qquad \text{for all } (x,t,z)\in\Omega\times(0,\infty)\times[0,\infty).$$

As $(u_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^\infty(\Omega\times(0,\infty))$ by Lemma 6.2 and u_0 is Hölder continuous in $\overline{\Omega}$ according to (1.7), the estimate in (7.4) therefore becomes a consequence of a well-known result on Hölder regularity in quasilinear degenerate parabolic equations ([33, Theorem 1.3, Remark 1.4]). Likewise, (7.5) can be derived e.g. from the boundedness of $(-u_\varepsilon v_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^\infty(\Omega\times(0,\infty))$ and the Hölder continuity of v_0 implied by (1.7). □

Based on the latter, the second solution component can be seen to actually possess the following higher-order smoothness feature, due to possibly lacking regularity of v_0 naturally restricted to time intervals away from $t=0$.

Lemma 7.4 *Let $T>0$ and $\tau\in(0,T)$. Then there exist $\theta=\theta(\tau,T)\in(0,1)$ and $C(\tau,T)>0$ such that*

$$\|v_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[\tau,T])} \le C(\tau,T) \qquad \textit{for all } \varepsilon\in(0,1).$$

Proof. As $(-u_\varepsilon v_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $C^{\theta_1,\frac{\theta_1}{2}}(\overline{\Omega}\times[0,T])$ with some $\theta_1=\theta_1(T)\in(0,1)$ according to Lemma 7.3, this is an immediate consequence of standard parabolic Schauder estimates ([19]). □

8 Passing to the limit. Proof of Theorem 1.1

Our construction of a weak solution to (1.6) through an appropriate extraction procedure is now rather straightforward.

Lemma 8.1 *There exist* $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ *and nonnegative functions*

$$\begin{cases} u \in C^0(\overline{\Omega}\times[0,\infty)) \cap L^\infty(\Omega\times(0,\infty)) \qquad \textit{and} \\ v \in C^0(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)) \cap L^\infty(\Omega\times(0,\infty)) \end{cases} \tag{8.1}$$

satisfying

$$u_x \in L^2_{loc}(\overline{\Omega}\times[0,\infty)) \tag{8.2}$$

and

$$v_x \in L^\infty(\Omega\times(0,\infty)) \tag{8.3}$$

which are such that $\varepsilon_j \searrow 0$ *as* $j\to\infty$ *and*

$$u_\varepsilon \to u \qquad \textit{in } C^0_{loc}(\overline{\Omega}\times[0,\infty)), \tag{8.4}$$

$$u_{\varepsilon x} \rightharpoonup u_x \qquad \textit{in } L^2_{loc}(\overline{\Omega}\times[0,\infty)), \tag{8.5}$$

$$v_\varepsilon \to v \qquad \textit{in } C^0_{loc}(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}_{loc}(\overline{\Omega}\times(0,\infty)) \qquad \textit{and} \tag{8.6}$$

$$v_{\varepsilon x} \overset{\star}{\rightharpoonup} v_x \qquad \textit{in } L^\infty(\Omega\times(0,\infty)) \tag{8.7}$$

as $\varepsilon=\varepsilon_j\searrow 0$*, and that* (u,v) *is a global weak solution of (1.6) in the sense of Definition 2.1.*

Proof. According to Lemma 6.2 and Lemma 7.2,

$$(u_\varepsilon)_{\varepsilon\in(0,1)} \quad \text{is bounded in } L^\infty(\Omega\times(0,\infty))$$

and

$$(u_{\varepsilon x})_{\varepsilon\in(0,1)} \quad \text{is bounded in } L^2_{loc}(\overline{\Omega}\times[0,\infty)),$$

whereas due to (2.13) and Lemma 5.2,

$$(v_\varepsilon)_{\varepsilon\in(0,1)} \text{ and } (v_{\varepsilon x})_{\varepsilon\in(0,1)} \quad \text{are bounded in } L^\infty(\Omega\times(0,\infty)).$$

Moreover, Lemma 7.3 and Lemma 7.4 in conjunction with the Arzelá-Ascoli theorem ensure that

$$(u_\varepsilon)_{\varepsilon\in(0,1)} \quad \text{is relatively compact in } C^0_{loc}(\overline{\Omega}\times[0,\infty))$$

and that

$$(v_\varepsilon)_{\varepsilon\in(0,1)} \quad \text{is relatively compact in } C^0_{loc}(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}_{loc}(\overline{\Omega}\times(0,\infty)).$$

By means of a standard extraction procedure we thus infer the existence of $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_j\searrow 0$ as $j\to\infty$ and that (8.4)-(8.7) hold as $\varepsilon=\varepsilon_j\searrow 0$ with some nonnegative functions u and v fulfilling (8.1)-(8.3). The latter properties evidently entail the regularity requirements in (2.1) and (2.2) in Definition 2.1, while the identities (2.3) and (2.4) result from (8.4)-(8.7) in quite a straigtforward manner: given $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$, using (2.6) we see that

$$\begin{aligned} -\int_0^\infty\int_\Omega u_\varepsilon\varphi_t - \int_\Omega u_0\varphi(\cdot,0) &= -\varepsilon\int_0^\infty\int_\Omega (u_\varepsilon+1)^{m-1}u_{\varepsilon x}\varphi_x - \int_0^\infty\int_\Omega u_\varepsilon v_\varepsilon u_{\varepsilon x}\varphi_x \\ &\quad + \int_0^\infty\int_\Omega u_\varepsilon^2 v_\varepsilon v_{\varepsilon x}\varphi_x + \int_0^\infty\int_\Omega u_\varepsilon v_\varepsilon\varphi \end{aligned} \tag{8.8}$$

for all $\varepsilon\in(0,1)$. Here since $\operatorname{supp}\varphi$ is bounded, (8.4) and (8.5) apply to show that

$$-\int_0^\infty\int_\Omega u_\varepsilon\varphi_t \to -\int_0^\infty\int_\Omega u\varphi_t$$

as well as

$$-\int_0^\infty \int_\Omega (u_\varepsilon+1)^{m-1} u_{\varepsilon x}\varphi_x \to -\int_0^\infty \int_\Omega (u+1)^{m-1} u_x\varphi_x$$

and hence

$$-\varepsilon \int_0^\infty \int_\Omega (u_\varepsilon+1)^{m-1} u_{\varepsilon x}\varphi_x \to 0$$

as $\varepsilon=\varepsilon_j \searrow 0$, while combining (8.4) with (8.5), (8.6) and (8.7) yields

$$-\int_0^\infty \int_\Omega u_\varepsilon v_\varepsilon u_{\varepsilon x}\varphi_x \to -\int_0^\infty \int_\Omega uvu_x\varphi_x$$

as well as

$$\int_0^\infty \int_\Omega u_\varepsilon^2 v_\varepsilon v_{\varepsilon x}\varphi_x \to \int_0^\infty \int_\Omega u^2 vv_x\varphi_x$$

and

$$\int_0^\infty \int_\Omega u_\varepsilon v_\varepsilon\varphi \to \int_0^\infty \int_\Omega uv\varphi$$

as $\varepsilon=\varepsilon_j \searrow 0$. Therefore, (8.8) implies (2.3), and (2.4) can be verified similarly. □

Inter alia, this establishes our main result on global existence in (1.6).

PROOF of Theorem 1.1. The statement actually is a by-product of Lemma 8.1. □

9 Asymptotics of v. Proof of Theorem 1.2

The above preparations also immediately entail the claimed qualitative properties of v:

PROOF of Theorem 1.2. From Lemma 3.4 and Lemma 5.2 we know that

$$\int_0^\infty \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}dt \le c_1 \qquad \text{for all } \varepsilon\in(0,1) \tag{9.1}$$

and

$$|v_{\varepsilon x}(x,t)| \le c_2 v_\varepsilon(x,t) \qquad \text{for all } x\in\Omega, t>0 \text{ and } \varepsilon\in(0,1) \tag{9.2}$$

and that thus also

$$\int_0^\infty \|v_{\varepsilon x}(\cdot,t)\|_{L^\infty(\Omega)}dt \le c_1c_2 \qquad \text{for all } \varepsilon\in(0,1). \tag{9.3}$$

In view of (8.6) and Fatou's lemma, combining (9.1) with (9.3) establishes (1.9) and thereby also entails (1.10), because (2.13) and (8.6) imply that

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le \|v(\cdot,t_0)\|_{L^\infty(\Omega)} \qquad \text{for all } t_0\ge 0 \text{ and } t>t_0$$

and hence

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le \int_{t-1}^t \|v(\cdot,s)\|_{L^\infty(\Omega)}ds \to 0 \qquad \text{as } 1\le t\to\infty$$

by (1.9), and because thus (9.2) and again (8.6) imply that also

$$\|v_x(\cdot,t)\|_{L^\infty(\Omega)} \le c_2\|v(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } t\to\infty.$$

Finally, once more thanks to (8.6) the Harnack-type inequality (1.11) is a direct consequence of Corollary 5.3. □

10 Convergence of u. Proof of Theorem 1.3

Next concerned with the asymptotic properties of u, to achieve a first observation in this respect let us perform a substitution of the time variable in the style of that in Theorem 1.3 but yet at the stage of the solutions to the approximate problems (2.6). Here of crucial importance will be the observation that thanks to the Harnack-type inequality (1.11), unlike in (1.6) the degeneracy of the diffusion mechanism in the accordingly transformed problem is, up to a bounded and uniformly positive (x,t)-dependent coefficient and a correction vanishing as $\varepsilon \searrow 0$, essentially of porous medium type.

Lemma 10.1 *Suppose that (1.7) holds, and for $\varepsilon \in (0,1)$ let*

$$t_\varepsilon := \frac{1}{\kappa} \ln \frac{2}{\varepsilon} \tag{10.1}$$

and

$$J_\varepsilon := \int_0^{t_\varepsilon} \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} dt \tag{10.2}$$

as well as

$$z_\varepsilon(x,\tau) := u_\varepsilon(x, \rho_\varepsilon^{-1}(\tau)), \qquad x \in \overline{\Omega},\ \tau \in [0,1], \tag{10.3}$$

where

$$\rho_\varepsilon(t) := \frac{1}{J_\varepsilon} \cdot \int_0^t \|v_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)} ds, \qquad t \in [0,t_\varepsilon], \tag{10.4}$$

where $\kappa > 0$ is as provided by Lemma 7.1, and where $(u_\varepsilon, v_\varepsilon)$ denotes the solution of (2.6) from Lemma 2.1. Then

$$\begin{cases} z_{\varepsilon\tau} = \Big(\widehat{a}_\varepsilon(x,\tau) z_{\varepsilon x}\Big)_x + \Big(a_\varepsilon(x,\tau) z_\varepsilon z_{\varepsilon x}\Big)_x - \Big(b_\varepsilon(x,\tau) z_\varepsilon^2\Big)_x + a_\varepsilon(x,\tau) z_\varepsilon, & x \in \Omega,\ \tau \in (0,1), \\ z_{\varepsilon x} = 0, & x \in \partial\Omega,\ \tau \in (0,1), \\ z_\varepsilon(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{10.5}$$

with

$$\widehat{a}_\varepsilon(x,\tau) := J_\varepsilon \cdot \frac{\varepsilon}{\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}} \cdot \Big(u_\varepsilon(x,t)+1\Big)^{m-1} \tag{10.6}$$

as well as

$$a_\varepsilon(x,\tau) := J_\varepsilon \cdot \frac{v_\varepsilon(x,t)}{\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}} \qquad \text{and} \qquad b_\varepsilon(x,\tau) := J_\varepsilon \cdot \frac{v_\varepsilon(x,t) v_{\varepsilon x}(x,t)}{\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}} \tag{10.7}$$

for $x \in \Omega$, $\tau \in (0,1)$ and $t := \rho_\varepsilon^{-1}(\tau)$. Moreover, there exists $C > 0$ such that for all $\varepsilon \in (0,1)$,

$$0 \le \widehat{a}_\varepsilon(x,\tau) \le C \qquad \text{for all } x \in \Omega \text{ and } \tau \in (0,1) \tag{10.8}$$

and

$$\frac{1}{C} \le a_\varepsilon(x,\tau) \le C \qquad \text{for all } x \in \Omega \text{ and } \tau \in (0,1) \tag{10.9}$$

as well as

$$|b_\varepsilon(x,\tau)| \le C \qquad \text{for all } x \in \Omega \text{ and } \tau \in (0,1). \tag{10.10}$$

PROOF. Observing that z_ε indeed is well-defined due to the fact that ρ_ε is strictly increasing with $\rho_\varepsilon(0) = 0$ and $\rho_\varepsilon(t_\varepsilon) = 1$, from the regularity properties of u_ε and v_ε it is clear that z_ε belongs to $C^{2,1}(\overline{\Omega} \times (0,1))$ with

$$z_{\varepsilon\tau}(x,\tau) = u_{\varepsilon t}(x, \rho_\varepsilon^{-1}(\tau)) \cdot \frac{1}{\rho_\varepsilon'(\rho_\varepsilon^{-1}(\tau))} = u_{\varepsilon t}(x,\rho_\varepsilon^{-1}(\tau)) \cdot \frac{J_\varepsilon}{\|v_\varepsilon(\cdot,\rho_\varepsilon^{-1}(\tau))\|_{L^\infty(\Omega)}}$$

for all $x \in \Omega$ and $\tau \in (0,1)$. Therefore, (10.5) readily results upon multiplying the first equation in (2.6) by $\frac{J_\varepsilon}{\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}}$ and making use of the definitions in (10.6) and (10.7).

Next, to derive (10.8), (10.9) and (10.10) we recall that Lemma 3.4 and Lemma 7.1 provide $c_1 > 0$ and $c_2 > 0$ such that

$$J_\varepsilon \le c_1 \qquad \text{for all } \varepsilon \in (0,1) \tag{10.11}$$

and

$$v_\varepsilon(x,t) \ge c_2 e^{-\kappa t} \qquad \text{for all } x \in \Omega,\ t > 0 \text{ and } \varepsilon \in (0,1), \tag{10.12}$$

and that Corollary 5.3, Lemma 5.2 and Lemma 6.2 in conjunction with (2.13) yield positive constants c_3, c_4 and c_5 fulfilling

$$v_\varepsilon(x,t) \ge c_3 \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \qquad \text{for all } x \in \Omega,\ t > 0 \text{ and } \varepsilon \in (0,1) \tag{10.13}$$

and

$$|v_{\varepsilon x}(x,t)| \le c_4 \qquad \text{for all } x \in \Omega,\ t > 0 \text{ and } \varepsilon \in (0,1) \tag{10.14}$$

as well as

$$u_\varepsilon(x,t) \le c_5 \qquad \text{for all } x \in \Omega,\ t > 0 \text{ and } \varepsilon \in (0,1). \tag{10.15}$$

Therefore, namely, thanks to (10.11) we immediately obtain that for all $\varepsilon \in (0,1)$,

$$a_\varepsilon(x,\tau) \le J_\varepsilon \le c_1 \qquad \text{for all } x \in \Omega \text{ and } \tau \in (0,1),$$

whereas combining (10.13) with (10.12) shows that for all $\varepsilon \in (0,1)$ we have

$$\begin{aligned} a_\varepsilon(x,\tau) &\ge c_3 J_\varepsilon = c_3 \int_0^{t_\varepsilon} \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} dt \ge c_2 c_3 \int_0^{t_\varepsilon} e^{-\kappa t} dt \\ &= \frac{c_2 c_3}{\kappa} \cdot (1 - e^{-\kappa t_\varepsilon}) = \frac{c_2 c_3}{\kappa} \cdot \Big(1 - \frac{\varepsilon}{2}\Big) \ge \frac{c_2 c_3}{2\kappa} \qquad \text{for all } x \in \Omega \text{ and } \tau \in (0,1). \end{aligned}$$

Since furthermore

$$|b_\varepsilon(x,\tau)| = a_\varepsilon(x,\tau) \cdot |v_{\varepsilon x}(x, \rho_\varepsilon^{-1}(\tau))| \le c_4 a_\varepsilon(x,\tau) \qquad \text{for all } x \in \Omega,\ \tau \in (0,1) \text{ and } \varepsilon \in (0,1),$$

and since the definition of t_ε along with (10.11), (10.12) and (10.15) guarantees that

$$\begin{aligned} 0 \le \widehat{a}_\varepsilon(x,\tau) &\le c_1 \cdot \frac{\varepsilon}{c_2 e^{-\kappa \rho_\varepsilon^{-1}(\tau)}} \cdot (c_5+1)^{m-1} \\ &\le c_1 \cdot \frac{\varepsilon}{c_2 e^{-\kappa t_\varepsilon}} \cdot (c_5+1)^{m-1} \\ &= c_6 := \frac{2c_1(c_5+1)^{m-1}}{c_2} \qquad \text{for all } x \in \Omega,\ \tau \in (0,1) \text{ and } \varepsilon \in (0,1), \end{aligned}$$

we thus conclude that indeed (10.8), (10.9) and (10.10) are valid if we let $C := \max\{c_6\,,\, c_1\,,\, \frac{\kappa}{2c_2c_3}\,,\, c_1c_4\}$, for instance. □

As a consequence of the inequalities in (10.8), (10.9) and (10.10), these rescaled solutions are uniformly Hölder continuous, again due to standard regularity theory for porous medium type equations.

Lemma 10.2 *Assume (1.7). Then there exist $\theta \in (0,1)$ and $C > 0$ such that*

$$\|z_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[0,1])} \le C \qquad \textit{for all } \varepsilon \in (0,1). \tag{10.16}$$

PROOF. According to (10.3) and Lemma 6.2, $(z_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^\infty(\Omega\times(0,\tau))$. In view of the upper bounds in (10.8), (10.9) and (10.10), the nonnegativity of $\widehat{a}_\varepsilon$ and the uniform lower estimate for a_ε in (10.9), and due to the Hölder continuity of u_0 asserted by (1.7), the claimed estimate can be seen by straightforward application of Hölder regularity theory for the quasilinear parabolic problem (10.5) in quite a similar fashion as demonstrated in Lemma 7.3. $\Box$

Now an important preparation for an appropriate passage to the limit in (10.5) addresses the corresponding limit behavior in the expressions J_ε from (10.2). In our verification of the expected behavior we shall essentially rely on the integrability property from (2.12) which can be used to provide some ε-independent control of the tail integrals $\int_{t_0}^\infty \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}dt$ for large $t_0>0$ by means of the positivity feature of u_ε from Lemma 3.1 and, again, the Harnack inequality (1.11).

Lemma 10.3 *Assume (1.7), and let $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ and $t_\varepsilon>0$ be as in Lemma 8.1 and (10.1), respectively. Then*

$$\int_0^{t_\varepsilon} \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}dt \to \int_0^\infty \|v(\cdot,t)\|_{L^\infty(\Omega)}dt \qquad \textit{as } \varepsilon=\varepsilon_j\searrow 0. \tag{10.17}$$

PROOF. In order to appropriately estimate the difference of the expressions in (10.17), we first apply Corollary 5.3 and Lemma 3.1 to fix $c_1>0$ and $c_2>0$ such that for all $\varepsilon\in(0,1)$,

$$v_\varepsilon(x,t)\ge c_1\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \qquad \text{for all } x\in\Omega \text{ and } t>0 \tag{10.18}$$

and

$$\int_\Omega \ln u_\varepsilon(\cdot,t)\ge -c_2 \qquad \text{for all } t>0.$$

Here the latter implies that for all $\varepsilon\in(0,1)$ and $t>0$ we have

$$c_2\ge\int_\Omega \ln\frac{1}{u_\varepsilon(\cdot,t)} \ge \int_{\{\ln\frac{1}{u_\varepsilon(\cdot,t)}>\frac{2c_2}{|\Omega|}\}} \ln\frac{1}{u_\varepsilon(\cdot,t)} \ge \frac{2c_2}{|\Omega|}\cdot\left|\left\{\ln\frac{1}{u_\varepsilon(\cdot,t)}>\frac{2c_2}{|\Omega|}\right\}\right|$$

and hence $\left|\left\{\ln\frac{1}{u_\varepsilon(\cdot,t)}>\frac{2c_2}{|\Omega|}\right\}\right|\le\frac{|\Omega|}{2}$, meaning that if we abbreviate $c_3:=e^{-\frac{2c_2}{|\Omega|}}$, then

$$\Big|\{u_\varepsilon(\cdot,t)\ge c_3\}\Big| = \left|\left\{\ln\frac{1}{u_\varepsilon(\cdot,t)}\le\frac{2c_2}{|\Omega|}\right\}\right|\ge\frac{|\Omega|}{2} \qquad \text{for all } t>0 \text{ and any } \varepsilon\in(0,1). \tag{10.19}$$

Now given $\eta>0$ we make use of Theorem 1.2 in finding $t_0>0$ such that

$$\|v(\cdot,t_0)\|_{L^1(\Omega)}\le\frac{c_1c_3|\Omega|\eta}{16},$$

which in view of the approximation property (8.6) warrants the existence of $\varepsilon_\star>0$ fulfilling

$$\|v_\varepsilon(\cdot,t_0)\|_{L^1(\Omega)}\le\frac{c_1c_3|\Omega|\eta}{8} \qquad \text{for all } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}} \text{ with } \varepsilon<\varepsilon_\star. \tag{10.20}$$

Here again due to (8.6), and due to the fact that $t_\varepsilon\to\infty$ as $\varepsilon\searrow0$ by (10.1), diminishing $\varepsilon_\star$ if necessary we can achieve that moreover

$$|v_\varepsilon(x,t)-v(x,t)|\le\frac{\eta}{2t_0} \qquad \text{for all } x\in\Omega \text{ and } t\in(0,t_0) \text{ and each } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}} \text{ with } \varepsilon<\varepsilon_\star \tag{10.21}$$

and that

$$t_\varepsilon\ge t_0 \qquad \text{for all } \varepsilon<\varepsilon_\star. \tag{10.22}$$

Now according to Lemma 2.2, (10.20) entails that

$$\int_{t_0}^{\infty}\int_\Omega u_\varepsilon v_\varepsilon \le \int_\Omega v_\varepsilon(\cdot,t_0) \le \frac{c_1c_3|\Omega|\eta}{8} \qquad \text{for all } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}} \text{ with } \varepsilon<\varepsilon_\star, \tag{10.23}$$

where we firstly make use of (10.18) in estimating

$$\begin{aligned}\int_\Omega u_\varepsilon(\cdot,t)v_\varepsilon(\cdot,t) &\ge \int_{\{u_\varepsilon(\cdot,t)\ge c_3\}} u_\varepsilon(\cdot,t)v_\varepsilon(\cdot,t)\\ &\ge c_3\int_{\{u_\varepsilon(\cdot,t)\ge c_3\}} v_\varepsilon(\cdot,t)\\ &\ge c_1c_3\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\cdot\Big|\{u_\varepsilon(\cdot,t)\ge c_3\}\Big| \qquad \text{for all } t>0 \text{ and any } \varepsilon\in(0,1),\end{aligned}$$

so that thanks to (10.19) we obtain

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{2}{c_1c_3|\Omega|}\int_\Omega u_\varepsilon(\cdot,t)v_\varepsilon(\cdot,t) \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1).$$

Therefore, (10.23) implies that

$$\int_{t_0}^\infty \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}dt \le \frac{2}{c_1c_3|\Omega|}\cdot\frac{c_1c_3|\Omega|\eta}{8} = \frac{\eta}{4} \qquad \text{for all } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}} \text{ with } \varepsilon<\varepsilon_\star$$

and thus clearly also

$$\int_{t_0}^\infty \|v(\cdot,t)\|_{L^\infty(\Omega)}dt \le \frac{\eta}{4}$$

by Lemma 8.1 and Fatou's lemma, whence invoking (10.21) and making use of (10.22) we infer that

$$\begin{aligned}&\Big|\int_0^{t_\varepsilon}\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}dt - \int_0^\infty \|v(\cdot,t)\|_{L^\infty(\Omega)}dt\Big|\\ &\quad\le \int_0^{t_0}\|v_\varepsilon(\cdot,t)-v(\cdot,t)\|_{L^\infty(\Omega)}dt + \int_{t_0}^\infty\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}dt + \int_{t_0}^\infty\|v(\cdot,t)\|_{L^\infty(\Omega)}dt\\ &\quad\le t_0\cdot\frac{\eta}{2t_0}+\frac{\eta}{4}+\frac{\eta}{4}=\eta \qquad \text{for all } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}} \text{ with } \varepsilon<\varepsilon_\star.\end{aligned}$$

As $\eta>0$ was arbitrary, this establishes (10.17). □

This enables us to take $\varepsilon\searrow 0$ along the sequence from Lemma 8.1, and thus to achieve the following.

Lemma 10.4 *Suppose that (1.7) holds, and let $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ be as in Lemma 8.1. Then with $(z_\varepsilon)_{\varepsilon\in(0,1)}\subset C^0(\overline{\Omega}\times[0,1])$ as defined in (10.3), we have*

$$z_\varepsilon\to z \quad \text{in } C^0(\overline{\Omega}\times[0,1]) \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0 \tag{10.24}$$

with some $z\in C^0(\overline{\Omega}\times[0,1])$ satisfying

$$z(x,\tau)=u(x,\rho^{-1}(\tau)), \quad x\in\overline{\Omega},\ \tau\in[0,1), \tag{10.25}$$

where ρ is as determined by (1.16) and u is taken from Lemma 8.1. Moreover, z is a solution of (1.14), with a and b as defined in (1.15) and fulfilling (1.17), in the sense that $z_x\in L^2_{loc}(\overline{\Omega}\times[0,1))$ and that

$$-\int_0^1\int_\Omega z\phi_\tau - \int_\Omega u_0\phi(\cdot,0) = -\int_0^1\int_\Omega a(x,\tau)zz_x\phi_x + \int_0^1\int_\Omega b(x,\tau)z^2\phi_x + \int_0^1\int_\Omega a(x,\tau)z \tag{10.26}$$

is valid for all $\phi\in C_0^\infty(\overline{\Omega}\times[0,1))$.

PROOF. From Lemma 8.1 we know that with ρ_ε as in (10.4), for all $t \ge 0$ we have $\rho_\varepsilon(t) \to \rho(t)$ as $\varepsilon = \varepsilon_j \searrow 0$, while Lemma 10.3 asserts that the numbers from (10.2) satisfy $J_\varepsilon \to J$ as $\varepsilon = \varepsilon_j \searrow 0$. It is therefore clear from (8.4), (8.6) and Lemma 7.1 that $z_\varepsilon(x,\tau) \to z(x,\tau), \widehat{a}_\varepsilon(x,\tau) \to 0, a_\varepsilon(x,\tau) \to a(x,\tau)$ and $b_\varepsilon(x,\tau) \to b(x,\tau)$ for all $x \in \Omega$ and $\tau \in (0,1)$ as $\varepsilon = \varepsilon_j \searrow 0$, so that (10.24) and (1.17) become a consequence of Lemma 10.2 and the Arzelà-Ascoli theorem. The claimed regularity property of z_x as well as the validity of (10.26) can thereafter readily be verified on the basis of (8.2) and (2.3). $\square$

As z trivially possesses a limit as $\tau \nearrow 1$, however, the latter means that u must stabilize in the large time limit.

Lemma 10.5 *Assume (1.7). Then*

$$u(\cdot,t) \to z(\cdot,1) \quad \textit{in } L^\infty(\Omega) \qquad \textit{as } t \to \infty, \tag{10.27}$$

where $z \in C^0(\overline{\Omega} \times [0,1])$ is as obtained in Lemma 10.4.

PROOF. In view of the identity (10.25), the claim is an immediate consequence of the continuity of z in $\overline{\Omega} \times \{1\}$. $\square$

We can thus pass to our main result on qualitative behavior in the first solution component.

PROOF of Theorem 1.3. The convergence property (1.13) as well as the characterization of u_∞ and the claimed upper and lower bounds in (1.17) have precisely been asserted by Lemma 10.5 and Lemma 10.4, whereupon the boundedness of u becomes an evident consequence. To verify (1.12), we note that from Lemma 3.1 and Lemma 6.2 it follows that with some $c_1 > 0$ we have

$$\begin{aligned}
\int_\Omega \Big(\ln \frac{1}{u_\varepsilon(\cdot,t)}\Big)_+ &= -\int_\Omega \ln u_\varepsilon(\cdot,t) + \int_{\{u_\varepsilon(\cdot,t)>1\}} \ln u_\varepsilon(\cdot,t) \\
&\le -\int_\Omega \ln u_\varepsilon(\cdot,t) + |\Omega| \cdot \ln \|u_\varepsilon\|_{L^\infty(\Omega\times(0,\infty))} \\
&\le c_1 \qquad \text{for all } t>0 \text{ and each } \varepsilon \in (0,1),
\end{aligned}$$

so that (8.4) together with Fatou's lemma ensures that

$$\int_\Omega \Big(\ln \frac{1}{u(\cdot,t)}\Big)_+ \le c_1 \qquad \text{for all } t > 0$$

and that thus indeed (1.12) is valid. $\square$

11 Stability of arbitrary steady states. Convergence to nonconstant equilibria. Proofs of Theorem 1.4 and Corollary 1.5

We finally turn our attention to the statements on stability and nontrivial stabilization from Theorem 1.4 and Corollary 1.5. Our analysis in this direction is based on the following observation which at its core consists in appropriately controlling the time derivative $u_{\varepsilon t}$ in suitable dual spaces by means of the previously gained estimates.

Lemma 11.1 *Let $p > \frac{8}{3}, q > \frac{p}{p-2}$ and $K > 0$. Then there exist $\gamma(p,q) \in (0,1)$ and $C(p,q,K) > 0$ such that whenever u_0 and v_0 are such that (1.7), (1.18) and (1.19) hold, for all $\varepsilon \in (0,1)$ we have*

$$\|u_\varepsilon(\cdot,t) - u_0\|_{(W_0^{2,q}(\Omega))^\star} \le C(p,q,K)\|v_0\|_{L^1(\Omega)}^{\gamma(p,q)} \qquad \textit{for all } t \in \Big(0, \frac{1}{\varepsilon}\Big). \tag{11.1}$$

PROOF. We multiply the first equation in (2.6) by an arbitrary $\psi \in C_0^\infty(\Omega)$ to see on several integrations by parts that

$$\begin{aligned}
\Big|\int_\Omega u_{\varepsilon t}\psi\Big| &= \Big| -\varepsilon\int_\Omega (u_\varepsilon+\varepsilon)^{m-1}u_{\varepsilon x}\psi_x - \int_\Omega u_\varepsilon v_\varepsilon u_{\varepsilon x}\psi_x + \int_\Omega u_\varepsilon^2 v_\varepsilon v_{\varepsilon x}\psi_x + \int_\Omega u_\varepsilon v_\varepsilon \psi\Big| \\
&= \Big|\frac{\varepsilon}{m}\int_\Omega (u_\varepsilon+1)^m\psi_{xx} + \frac12\int_\Omega u_\varepsilon^2 v_{\varepsilon x}\psi_x + \frac12\int_\Omega u_\varepsilon^2 v_\varepsilon\psi_{xx} + \int_\Omega u_\varepsilon^2 v_\varepsilon v_\varepsilon \psi_x + \int_\Omega u_\varepsilon v_\varepsilon\psi\Big| \\
&\le \frac{\varepsilon}{m}\int_\Omega (u_\varepsilon+1)^m|\psi_{xx}| \\
&\quad + \int_\Omega u_\varepsilon^2\cdot\Big(\frac12+v_\varepsilon\Big)\cdot|v_{\varepsilon x}|\cdot|\psi_x| + \frac12\int_\Omega u_\varepsilon^2 v_\varepsilon|\psi_{xx}| + \int_\Omega u_\varepsilon v_\varepsilon|\psi| \quad \text{for all } t>0. \qquad (11.2)
\end{aligned}$$

In order to appropriately estimate the expressions on the right-hand side herein, we observe that since $p > \frac83$, the number $\gamma_1 := \frac{p-\frac83}{p-1}$ satisfies $\gamma_1 \in (0,1)$, whereas the inequality $q > \frac{p}{p-2}$ warrants that also $\gamma_2 := \frac{pq-p-2q}{(p-1)q}$ belongs to $(0,1)$. Now by means of the Hölder inequality, we obtain that

$$\begin{aligned}
\int_\Omega u_\varepsilon^2\cdot\Big(\frac12+v_\varepsilon\Big)\cdot|v_{\varepsilon x}|\cdot|\psi_x| &= \int_\Omega \Big(\frac12+v_\varepsilon\Big)\cdot\Big(\frac{v_{\varepsilon x}^4}{v_\varepsilon^3}\Big)^{\frac14}\cdot u_\varepsilon^2 v_\varepsilon^{\frac34}\cdot|\psi_x| \\
&\le \Big(\frac12+\|v_\varepsilon\|_{L^\infty(\Omega)}\Big)\cdot\Big\{\int_\Omega \frac{v_{\varepsilon x}^4}{v_\varepsilon^3}\Big\}^{\frac14}\cdot\Big\{\int_\Omega u_\varepsilon^{\frac83}v_\varepsilon\Big\}^{\frac34}\cdot\|\psi_x\|_{L^\infty(\Omega)} \qquad (11.3)
\end{aligned}$$

for all $t>0$, and that thanks to the fact that $\frac{\frac83-\gamma_1}{1-\gamma_1} = p$,

$$\begin{aligned}
\Big\{\int_\Omega u_\varepsilon^{\frac83}v_\varepsilon\Big\}^{\frac34} &= \Big\{\int_\Omega u_\varepsilon^{\frac83-\gamma_1}\cdot v_\varepsilon^{1-\gamma_1}\cdot(u_\varepsilon v_\varepsilon)^{\gamma_1}\Big\}^{\frac34} \\
&\le \Big\{\int_\Omega u_\varepsilon^{\frac{\frac83-\gamma_1}{1-\gamma_1}}\Big\}^{\frac{3(1-\gamma_1)}{4}}\cdot\|v_\varepsilon\|_{L^\infty(\Omega)}^{\frac{3(1-\gamma_1)}{4}}\cdot\Big\{\int_\Omega u_\varepsilon v_\varepsilon\Big\}^{\frac{3\gamma_1}{4}} \\
&\le c_1^{\frac{3(1-\gamma_1)}{4}}\cdot\|v_\varepsilon\|_{L^\infty(\Omega)}^{\frac{3(1-\gamma_1)}{4}}\cdot\Big\{\int_\Omega u_\varepsilon v_\varepsilon\Big\}^{\frac{3\gamma_1}{4}} \qquad \text{for all } t>0, \qquad (11.4)
\end{aligned}$$

where

$$c_1 := \sup_{\varepsilon\in(0,1)}\sup_{t>0}\int_\Omega u_\varepsilon^p(\cdot,t)$$

is a finite number indeed depending on K only thanks to Lemma 4.8 and our assumptions (1.18) and (1.19). Since, by continuity of the embedding $W^{1,\frac{2(p+1)(p+2)}{p+4}}(\Omega)\hookrightarrow L^\infty(\Omega)$, (2.13) along with (1.19) warrants the existence of $c_2 = c_2(K) > 0$ such that

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|v_0\|_{L^\infty(\Omega)} \le c_2 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1), \qquad (11.5)$$

from (11.3) and (11.4) we thus infer that

$$\int_\Omega u_\varepsilon^2\cdot\Big(\frac12+v_\varepsilon\Big)\cdot|v_{\varepsilon x}|\cdot|\psi_x| \le c_1^{\frac{3(1-\gamma_1)}{4}}\cdot\Big(\frac12+c_2\Big)\cdot\|v_\varepsilon\|_{L^\infty(\Omega)}^{\frac{3(1-\gamma_1)}{4}}\cdot\Big\{\int_\Omega \frac{v_{\varepsilon x}^4}{v_\varepsilon^3}\Big\}^{\frac14}\cdot\Big\{\int_\Omega u_\varepsilon v_\varepsilon\Big\}^{\frac{3\gamma_1}{4}}\cdot\|\psi_x\|_{L^\infty(\Omega)} \quad (11.6)$$

for all $t>0$. Similarly, observing that our definition of γ_2 ensures that $\frac{2-\gamma_2}{p}+\gamma_2+\frac{1}{q}=1$, by means of the Hölder inequality we find that the second last summand in (11.2) satisfies

$$\begin{aligned}
\frac{1}{2}\int_\Omega u_\varepsilon^2 v_\varepsilon|\psi_{xx}| &= \frac{1}{2}\int_\Omega u_\varepsilon^{2-\gamma_2}\cdot v_\varepsilon^{1-\gamma_2}\cdot(u_\varepsilon v_\varepsilon)^{\gamma_2}\cdot|\psi_{xx}| \\
&\le \frac{1}{2}\cdot\Big\{\int_\Omega u_\varepsilon^p\Big\}^{\frac{2-\gamma_2}{p}}\cdot\|v_\varepsilon\|_{L^\infty(\Omega)}^{1-\gamma_2}\cdot\Big\{\int_\Omega u_\varepsilon v_\varepsilon\Big\}^{\gamma_2}\cdot\Big\{\int_\Omega|\psi_{xx}|^q\Big\}^{\frac{1}{q}} \\
&\le \frac{1}{2}c_1^{\frac{2-\gamma_2}{p}}\|v_\varepsilon\|_{L^\infty(\Omega)}^{1-\gamma_2}\cdot\Big\{\int_\Omega u_\varepsilon v_\varepsilon\Big\}^{\gamma_2}\cdot\|\psi_{xx}\|_{L^q(\Omega)} \qquad \text{for all } t>0. \qquad (11.7)
\end{aligned}$$

Since finally

$$\int_\Omega u_\varepsilon v_\varepsilon|\psi| \le \Big\{\int_\Omega u_\varepsilon v_\varepsilon\Big\}\cdot\|\psi\|_{L^\infty(\Omega)} \qquad \text{for all } t>0$$

and

$$\begin{aligned}
\frac{\varepsilon}{m}\int_\Omega(u_\varepsilon+1)^m|\psi_{xx}| &\le \frac{\varepsilon}{m}\|u_\varepsilon+1\|_{L^\infty(\Omega)}^m\int_\Omega|\psi_{xx}| \\
&\le \frac{2^{m-1}}{m}\varepsilon\cdot\Big\{\|u_\varepsilon\|_{L^\infty(\Omega)}^m+1\Big\}\cdot\|\psi_{xx}\|_{L^1(\Omega)} \qquad \text{for all } t>0,
\end{aligned}$$

and since also the embeddings $W_0^{2,q}(\Omega)\hookrightarrow W^{1,\infty}(\Omega)$ and $W_0^{2,q}(\Omega)\hookrightarrow W^{2,1}(\Omega)$ are continuous, from (11.2), (11.6) and (11.7) we obtain that there exists $c_3=c_3(p,q,K)>0$ such that

$$\begin{aligned}
\|u_{\varepsilon t}(\cdot,t)\|_{(W_0^{2,q}(\Omega))^\star} &\le c_3\cdot\|v_\varepsilon\|_{L^\infty(\Omega)}^{\frac{3(1-\gamma_1)}{4}}\cdot\Big\{\int_\Omega\frac{v_{\varepsilon x}^4}{v_\varepsilon^3}\Big\}^{\frac{1}{4}}\cdot\Big\{\int_\Omega u_\varepsilon v_\varepsilon\Big\}^{\frac{3\gamma_1}{4}} \\
&\quad +c_3\cdot\|v_\varepsilon\|_{L^\infty(\Omega)}^{1-\gamma_2}\cdot\Big\{\int_\Omega u_\varepsilon v_\varepsilon\Big\}^{\gamma_2}+c_3\int_\Omega u_\varepsilon v_\varepsilon \\
&\quad +c_3\varepsilon\|u_\varepsilon\|_{L^\infty(\Omega)}^m+c_3\varepsilon
\end{aligned}$$

for all $t>0$. On integrating this with respect to $t\in(0,\infty)$ and once more employing the Hölder inequality several times, we see that thus for all $\varepsilon\in(0,1)$,

$$\begin{aligned}
\int_0^T\|u_{\varepsilon t}(\cdot,t)\|_{(W_0^{2,q}(\Omega))^\star}dt &\le c_3\cdot\Big\{\int_0^\infty\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}dt\Big\}^{\frac{3(1-\gamma_1)}{4}}\cdot\Big\{\int_0^\infty\int_\Omega\frac{v_{\varepsilon x}^4}{v_\varepsilon^3}\Big\}^{\frac{1}{4}}\cdot\Big\{\int_0^\infty\int_\Omega u_\varepsilon v_\varepsilon\Big\}^{\frac{3\gamma_1}{4}} \\
&\quad +c_3\cdot\Big\{\int_0^\infty\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}dt\Big\}^{1-\gamma_2}\cdot\Big\{\int_0^\infty\int_\Omega u_\varepsilon v_\varepsilon\Big\}^{\gamma_2} \\
&\quad +c_3\int_0^\infty\int_\Omega u_\varepsilon v_\varepsilon \\
&\quad +c_3\varepsilon\int_0^T\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}^m dt+c_3\varepsilon T \qquad \text{for all } T>0. \qquad (11.8)
\end{aligned}$$

Here from Lemma 3.4 and Lemma 3.3 we know that with some $c_4=c_4(K)>0$ and $c_5=c_5(K)>0$ we have

$$\int_0^\infty\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}dt\le c_4 \qquad \text{for all } \varepsilon\in(0,1),$$

and

$$\int_0^\infty\int_\Omega\frac{v_{\varepsilon x}^4}{v_\varepsilon^3}\le c_5 \qquad \text{for all } \varepsilon\in(0,1)$$

where we note that our hypothesis (1.19), when viewed together with (11.5), can be seen to guarantee that

$$\int_\Omega \frac{v_{0x}^2}{v_0} \le c_6 \qquad \text{and} \qquad \int_\Omega v_{0x}^2 \le c_7$$

with some $c_6 = c_6(p,K) > 0$ and $c_7 = c_7(p,K) > 0$ due to the fact that the exponents in (1.19) satisfy $\frac{3}{2(p+1)} \le \frac{1}{2}$ and $\frac{2(p+1)(p+2)}{p+4} \ge 2$. Furthermore, Lemma 2.2 allows us to estimate

$$\int_0^\infty \int_\Omega u_\varepsilon v_\varepsilon \le \|v_0\|_{L^1(\Omega)} \qquad \text{for all } \varepsilon \in (0,1),$$

and Lemma 4.8 yields $c_8 = c_8(p,K) > 0$ such that

$$\varepsilon \int_0^T \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}^{2p+m-1} dt \le c_8 \cdot (1+\varepsilon T) \qquad \text{for all } \varepsilon \in (0,1) \text{ and each } T > 0,$$

so that by Young's inequality,

$$\begin{aligned} c_3\varepsilon \int_0^T \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}^m dt + c_3\varepsilon T &\le c_3\varepsilon \int_0^T \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}^{2p+m-1} dt + 2c_3\varepsilon T \\ &\le c_3c_8 \cdot (1+\varepsilon T) + 2c_3\varepsilon T \\ &\le 2c_3c_8 + 2c_3 \qquad \text{for all } \varepsilon \in (0,1) \text{ and } T \in \Big(0, \frac{1}{\varepsilon}\Big), \end{aligned}$$

as clearly $2p+m-1 \ge m$. From (11.8) we therefore conclude that for all $\varepsilon \in (0,1)$ and any $t \in (0, \frac{1}{\varepsilon})$,

$$\begin{aligned} \|u_\varepsilon(\cdot,t) - u_0\|_{(W_0^{2,q}(\Omega))^\star} &\le \int_0^t \|u_{\varepsilon t}(\cdot,t)\|_{(W_0^{2,q}(\Omega))^\star} dt \\ &\le \int_0^\infty \|u_{\varepsilon t}(\cdot,t)\|_{(W_0^{2,q}(\Omega))^\star} dt \\ &\le c_3 c_4^{\frac{3(1-\gamma_1)}{4}} c_5^{\frac{1}{4}} \|v_0\|_{L^1(\Omega)}^{\frac{3\gamma_1}{4}} + c_3 c_4^{1-\gamma_2} \|v_0\|_{L^1(\Omega)}^{\gamma_2} + c_3\|v_0\|_{L^1(\Omega)} + 2c_3c_8 + 2c_3, \end{aligned}$$

which already establishes (11.1) with $\gamma(p,q) := \min\{\frac{3\gamma_1}{4}, \gamma_2\} \in (0,1)$, because according to (11.5) we have $\|v_0\|_{L^1(\Omega)} \le \|v_0\|_{L^1(\Omega)}^\gamma \cdot |\Omega|^{\beta-\gamma} c_2^{\beta-\gamma}$ for arbitrary $\beta \ge \gamma$. □

This firstly implies that indeed arbitrary equilibria of (1.6) have the stability property described in Theorem 1.4:

Proof of Theorem 1.4. Given $p > \frac{8}{3}, q > \frac{p}{p-2}$ and $K > 0$, we apply Lemma 11.1 to find $\gamma \in (0,1)$ and $c_1 > 0$ such that whenever (1.7), (1.18) and (1.19) hold, for all $\varepsilon \in (0,1)$ we have

$$\|u_\varepsilon(\cdot,t) - u_0\|_{(W_0^{2,q}(\Omega))^\star} \le c_1 \|v_0\|_{L^1(\Omega)}^\gamma \qquad \text{for all } t \in \Big(0, \frac{1}{\varepsilon}\Big). \tag{11.9}$$

Now for fixed $\eta > 0$ we choose $\delta > 0$ small enough fulfilling

$$\delta \le \frac{\eta}{2} \qquad \text{and} \qquad c_1\delta^\gamma \le \frac{\eta}{2} \tag{11.10}$$

and suppose that $u_\star \in (W_0^{q,2}(\Omega))^\star$, u_0 and v_0 are such that (1.7) as well as (1.18)-(1.20) are satisfied. Then using that (8.4) clearly implies that $u_\varepsilon(\cdot,t) \to u(\cdot,t)$ in $(W_0^{q,2}(\Omega))^\star$ as $\varepsilon = \varepsilon_j \searrow 0$ for all $t > 0$, from

(11.9), (1.20) and (11.10) we infer that

$$\begin{aligned}
\|u(\cdot,t)-u_\star\|_{(W_0^{2,q}(\Omega))^\star} &\le \|u(\cdot,t)-u_0\|_{(W_0^{2,q}(\Omega))^\star} + \|u_0-u_\star\|_{(W_0^{2,q}(\Omega))^\star} \\
&\le c_1\|v_0\|_{L^1(\Omega)}^\gamma + \|u_0-u_\star\|_{(W_0^{2,q}(\Omega))^\star} \\
&\le c_1\delta^\gamma + \delta \\
&\le \frac{\eta}{2}+\frac{\eta}{2} = \eta \qquad \text{for all } t>0,
\end{aligned}$$

whereas (1.20) and (11.10) in view of Lemma 2.2 and (8.6) trivially entail that

$$\|v(\cdot,t)\|_{L^1(\Omega)} \le \|v_0\|_{L^1(\Omega)} \le \delta < \eta \qquad \text{for all } t>0,$$

as claimed. □

Beyond the latter, Lemma 11.1 secondly entails that stabilization toward spatially heterogeneous steady states in fact occurs within considerably large sets of initial data.

Proof of Corollary 1.5. Since u_0 is nonconstant and continuous, we can pick $c_1>0$ and an open subinterval Ω_0 of Ω such that

$$u_0(x)+c_1 \le \overline{u}_0 := \frac{1}{|\Omega|}\int_\Omega u_0 \qquad \text{for all } x\in\Omega_0, \tag{11.11}$$

and thereafter fix some nonnegative $\psi\in C_0^\infty(\Omega)$ such that $\operatorname{supp}\psi\subset\Omega_0$ and $\int_\Omega\psi=1$.
Moreover, given $K>0$ we apply Theorem 1.4 to any conveniently chosen $p>\frac{8}{3}$ and $q>\frac{p}{p-2}$ and to $\eta:=\frac{c_1}{2\|\psi\|_{W^{2,q}(\Omega)}}$ and $u_\star:=u_0$ and thereby obtain $\delta>0$ with the property that whenever $v_0\in W^{1,\infty}(\Omega)$ is positive in $\overline{\Omega}$ and such that both (1.22) and (1.23) are valid, for the corresponding solution (u,v) of (1.6) from Theorem 1.1 we have

$$\|u(\cdot,t)-u_0\|_{(W_0^{2,q}(\Omega))^\star} \le \frac{c_1}{2\|\psi\|_{W^{2,q}(\Omega))^\star}} \qquad \text{for all } t>0. \tag{11.12}$$

Now fixing any such v_0, since we already know from Theorem 1.2 and Theorem 1.3 that (1.24) holds with some $u_\infty\in C^0(\overline{\Omega})$, verifying the claimed inhomogeneity property amounts to showing that the hypothesis that

$$u_\infty\equiv c_2 \quad \text{in } \Omega \qquad \text{for some } c_2\ge 0 \tag{11.13}$$

is absurd. To achieve this, assuming (11.13) to be valid we firstly use (1.24) to see that

$$\int_\Omega u(\cdot,t)\psi \to \int_\Omega u_\infty\psi = c_2\int_\Omega\psi = c_2 \qquad \text{as } t\to\infty, \tag{11.14}$$

whereas combining (1.24) with (11.12) shows that

$$\|u_\infty-u_0\|_{(W_0^{2,q}(\Omega))^\star} \le \frac{c_1}{2}$$

and hence, in particular,

$$\int_\Omega u_\infty\psi - \int_\Omega u_0\psi \le \|u_\infty-u_0\|_{(W_0^{2,q}(\Omega))^\star}\cdot\|\psi\|_{W^{2,q}(\Omega)} \le \frac{c_1}{2}. \tag{11.15}$$

Here we note that according to our choice of ψ and (11.11),

$$\int_\Omega u_0\psi = \int_{\Omega_0} u_0\psi \le (\overline{u}_0-c_1)\int_\Omega\psi = \overline{u}_0-c_1,$$

so that (11.14) and (11.15) yield

$$c_2 \le \int_\Omega u_0\psi + \frac{c_1}{2} \le (\overline{u}_0 - c_1) + \frac{c_1}{2} = \overline{u}_0 - \frac{c_1}{2}. \tag{11.16}$$

But from Lemma 2.2 we know that

$$\int_\Omega u_\varepsilon(\cdot,t) \ge \int_\Omega u_0 \qquad \text{for all } t>0 \text{ and any } \varepsilon \in (0,1),$$

which clearly implies that

$$\int_\Omega u_\infty \ge \int_\Omega u_0$$

and that thus, again by (11.14),

$$c_2 \ge \overline{u}_0.$$

This contradiction falsifies (11.13), so that in consequence indeed (1.25) must be true. □

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