Global weak solutions and absorbing sets in a chemotaxis-Navier–Stokes system with prescribed signal concentration on the boundary

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Abstract

The chemotaxis-Navier–Stokes system

$$\begin{cases} n_t + u \cdot \nabla n = \nabla \cdot (n^{m-1}\nabla n - n\nabla c), \\ c_t + u \cdot \nabla c = \Delta c - nc, \\ u_t + (u \cdot \nabla)u = \Delta u + \nabla P + n\nabla\phi, \qquad \nabla \cdot u = 0 \end{cases}$$

is considered in a smoothly bounded domain $\Omega \subset \mathbb{R}^3$, along with the boundary conditions

$$\big(n^{m-1}\nabla n - n\nabla c\big) \cdot \nu = 0, \quad c = c_\star, \quad u = 0, \quad x \in \partial\Omega, \ t > 0,$$

with a given nonnegative constant $c_\star$.

Under the standing assumption $m > \frac{7}{6}$, it is firstly shown that for all suitably regular initial data, a corresponding initial-boundary value problem is globally solvable in a natural weak sense. Secondly, some information on the large time behavior of these solutions is provided by asserting the existence of a ball in $L^p(\Omega) \times W^{1,q}(\Omega) \times L^2(\Omega;\mathbb{R}^3)$, with radius depending on $p \in [1, 3m - \frac{7}{3})$, $q \in (1,2)$, $c_\star$ and the conserved total population size $\int_\Omega n$ only, which eventually absorbs each individual among the obtained trajectories.

Key words: chemotaxis-fluid; Navier–Stokes; inhomogeneous boundary condition; global weak solution; absorbing set
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1 Introduction

Although individual cells and bacteria might appear simple and the purposefulness of their behavior quite limited, populations as a whole may act in a rather orderly fashion. Considerably well-organized movement has been discovered even in some very primitive cases, such as that of the ubiquitous aerobic bacterium *Bacillus subtilis*, and the mere ability to perform taxis-type aligned movement, partially directed upward concentration gradients of a signal substance, has been identified as a mechanism of key importance in this regard in various situations ([11, 39, 15, 31]). The ambition to describe key aspects of the observed dynamics by corresponding solution behavior in systems of evolution equations has enthralled the interest of mathematicians for the better part of four decades, and despite considerable analytical efforts the theoretical understanding in this field seems yet far from complete. Of particular interest in this direction seem questions related to the interaction of microbial populations with liquid environments, and indeed some recent results have revealed noticeable effects going along with such types of interplay already in quite simple settings ([21, 22, 23, 16]).

In the context of markedly conspicuous experimental findings on formation of plume-like aggregates in populations of *B. subtilis* at the water-air interface in a sessile drop of water ([11, 39]), coupled chemotaxis–(Navier–)Stokes systems of the form

$$\begin{cases} n_t + u\cdot\nabla n = \nabla\cdot(n^{m-1}\nabla n) - \nabla\cdot(n\nabla c), & x\in\Omega,\quad t>0,\\ c_t + u\cdot\nabla c = \Delta c - nc, & x\in\Omega,\quad t>0,\\ u_t + \kappa(u\cdot\nabla)u = \Delta u + \nabla P + n\nabla\phi,\quad \nabla\cdot u = 0, & x\in\Omega,\quad t>0, \end{cases} \tag{1.1}$$

have been proposed as models appropriately accounting for the key mechanisms of nutrient taxis, convection and buoyancy-driven fluid forcing ([39, 1, 2, 10]); here, $\Omega\subset\mathbb{R}^N$ and ϕ represent the fixed physical domain and the prescribed gravitational potential, $m\ge 1$ and $\kappa\in\mathbb{R}$ are given constants, and the unknown functions n, c, u, P denote the density of the bacteria, the concentration of oxygen, the fluid velocity field and the associated pressure, respectively.

The challenge of controlling effects of boundary conditions on global behavior. In analytical studies concerned with (1.1), initial-boundary value problems are usually considered in the mathematically convenient setting of boundary conditions given by

$$(n^{m-1}\nabla n - n\nabla c)\cdot\nu = 0,\quad \nabla c\cdot\nu = 0\quad\text{and}\quad u = 0,\qquad x\in\partial\Omega,\ t>0. \tag{1.2}$$

Indeed, such a framework allows for some expedient control of the cross-diffusive coupling in (1.1), which is reflected in the option to achieve an essentially exact cancellation of the interaction functional $\int_\Omega \nabla n\cdot\nabla c$ that arises when testing the first equation in (1.1) against $\ln n$. In the presence of homogeneous Neumann boundary conditions for c, namely, with negative sign this expression simultaneously emerges as a contribution to an identity satisfied by $\frac{d}{dt}\int_\Omega \frac{|\nabla c|^2}{c}$, in which the additionally appearing boundary integral

$$\int_{\partial\Omega} \frac{1}{c}\frac{\partial|\nabla c|^2}{\partial\nu} \tag{1.3}$$

is favorably signed when Ω is convex ([48]), or alternatively allows for a convenient one-sided estimate in terms of the corresponding dissipated quantities for general Ω ([32, 20]). A resulting quasi-energy structure can be used as a fundamental piece of regularity information, and accordingly quite comprehensive solution theories could be developed throughout noticeably wide ranges of the key parameter m, both in the full Navier–Stokes case $\kappa = 1$, which meanwhile seems covered quite completely by the literature on two-dimensional versions of (1.1)-(1.2) ([19, 10, 17, 35, 45]), and in the case $\kappa = 0$ of a simplified Stokes-type fluid evolution, in which especially in three-dimensional settings the derivation of global boundedness results has been achieved under mild assumptions on diffusion enhancement at large population densities, to date reducing to the mere hypothesis that $m > \frac{9}{8}$ ([50, 36, 40, 13]; cf. also [12] and [30] for precedents, [53] and [48] for some existence results addressing the case $N = 3$ and $\kappa = 1$, and the recent survey [3] for a broader overview).

With regard to questions concerned with qualitative solution behavior, however, the available literature seems to confirm the intuitive guess that at least on large time scales, the dissipative contributions to (1.1) enforce equilibration, in the context of the above type of boundary conditions implying trivial asymptotics due to fact that then each steady state belongs to the family of spatially homogeneous distributions described by the identity $(n, c, u) \equiv (a, 0, 0)$ with some $a \geq 0$. Specifically, stabilization results on (1.1)-(1.2) exclusively assert convergence towards spatially constant states, as illustrated in [52, 46, 49] and [50], and also in [26] for a related system with additional logistic source terms (cf. also [8] and [28] for corresponding results in associated Cauchy problems and a fluid-free high-dimensional case); rigorous analytical descriptions of pattern formation in any nontrivial flavor, however, seem to remain elusive.

In pursuit of statements concerning convergence towards non-constant states, inhomogeneous boundary data for the chemical signal have received increasing attention in the mathematical literaure on (1.1), either in form of Robin boundary conditions allowing oxygen influx into the domain proportional to the local concentration near the boundary ([5]), or in form of Dirichlet data directly prescribing a certain concentration on the boundary, as already already suggested in [39]. However, the analysis of such potentially more realistic boundary value problems for (1.1) seems yet to be at a rather early stage only, which gives the impression of being mainly due to the loss of a favorable energy-like structure that apparently goes along with such a change in boundary conditions.

After all, addressing the Robin-type setting going back to [5], and concentrating on a simplified fluid-free version of (1.1) with $m = 1$, the study [6] asserts the existence of a unique steady state, in the case of inhomogeneous data necessarily nonconstant, at any prescribed mass level $M := \int_\Omega n > 0$. A corresponding evolutionary variant of parabolic-elliptic type has recently been considered in [14], where results on the existence of global classical solutions for arbitrary initial mass and, under certain smallness conditions, large-time convergence towards the stationary solutions of the doubly-elliptic system can be found. For coupled chemotaxis-fluid systems of the form (1.1) but with nonlinear diffusion the works [38] and [51] obtain global weak solutions for $N = 3, \kappa = 0$ and $m > \frac{7}{6}$, and global weak solutions in the case $N = 2$ for $\kappa = 1$ and any $m > 1$, respectively. Existence of solutions in the linear diffusion case $m = 1$ is addressed in [5] in the presence of additional logistic source terms, and in [7] without; in both these works, global classical solutions are obtained for $N = 2$ and global weak solutions for $N = 3$.

For the scenario with Dirichlet boundary conditions for the signal, physically motivated in [39] by the fact that due to significant differences oxygen-diffusion coefficients between air and water the concentration on the boundary can be assumed to be equal to its saturation value, in line with lacking a priori information on regularity specifically near the boundary, available results especially in three-dimensonal settings seem to date restricted not only to the Stokes case $\kappa = 0$, but moreover, and more drastically, to statements on generalized solvability in classes of possibly quite nonsmooth functions. In this Stokes variant of (1.1), some global generalized solutions have been found for $N = 3$ and $m = 1$ ([42]), whereas in the framework with nonlinear diffusion and Stokes fluid global weak solutions where shown to exist under the assumption that $m \geq 1$ if $N = 2$ and $m > \frac{3N-2}{2N}$ if $N \geq 3$ ([41]). The existence of global weak solutions enjoying more usual regularity features has been achieved for (1.1) with $\kappa = 0$ and additional logistic source terms in [4]. At significantly higher levels of regularity, existence results concerned with boundary conditions alternative to those in (1.2) seem limited to fluid-free model variants ([24, 25, 29]). For some simulation-based approaches to understand possible effects of inhomogeneities in boundary data, we may refer to [9] and [33], for instance.

Main results. Motivated by the above, the present manuscript intends to develop an approach capable of suitably coping with the challenges linked to question how far inhomogeneous Dirichlet boundary conditions for the signal may affect global regularity in the three-dimensional full Navier–Stokes version of (1.1). This will amount to identifying a class of functionals which enjoy certain quasi-dissipative properties despite cross-diffusive interaction, where we note that in the Dirichlet case under consideration, regularity features of the energies encountered in the analysis of (1.1)-(1.2) appear to remain unclear due to apparent obstacles linked to an appropriate estimation of the second order expressions in (1.3).

Thus led to identifying suitable alternative testing procedures in which corresponding boundary integrals can appropriately be controlled in terms of associated dissipation rates, on the basis of an elementary pointwise boundary estimate satisfied by fairly arbitrary functions attaining the value $c_\star$ on $\partial\Omega$ (Lemma 3.4) we shall see that in the presence of appropriately strong enhancement of cell diffusion, expressions of the form

$$\int_\Omega n^p + \int_\Omega (|\nabla c|^2 + 1)^{\frac{q}{2}} + \int_\Omega |u|^2 \tag{1.4}$$

will indeed exhibit some energy-like properties during evolution within some range of $p > 1$ and $q > 1$ (Lemma 3.8). An appropriately arranged combination of the knowledge on regularity, as thereby generated, with certain elementary relaxation properties of (1.1) (Lemma 3.1) will lead not only to a statement on global weak solvability, but moreover provide some qualitative information on large time relaxation, in the flavor of a result on the existence of bounded absorbing sets with conveniently controllable size.

To make this more precise, let us henceforth consider the initial-boundary value problem given by

$$\begin{cases} n_t + u\cdot\nabla n = \nabla\cdot(n^{m-1}\nabla n) - \nabla\cdot(n\nabla c), & x\in\Omega, \quad t>0,\\ c_t + u\cdot\nabla c = \Delta c - nc, & x\in\Omega, \quad t>0,\\ u_t + (u\cdot\nabla)u = \Delta u + \nabla P + n\nabla\phi, \qquad \nabla\cdot u = 0, & x\in\Omega, \quad t>0,\\ \big(\nabla n - n\nabla c\big)\cdot\nu = 0, \qquad c = c_\star, \qquad u = 0, & x\in\partial\Omega, \quad t>0,\\ n(x,0) = n_0(x), \quad c(x,0) = c_0(x), \quad u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1.5}$$

in a smoothly bounded domain $\Omega \subset \mathbb{R}^3$, with $c_\star \in [0,\infty)$, $m > 1$ and $\phi \in W^{2,\infty}(\Omega)$, where throughout the remainder we let $A := -\mathcal{P}\Delta$ denote the Stokes operator with its domain $D(A) := W^{2,2}\left(\Omega;\mathbb{R}^3\right) \cap$

$W_0^{1,2}\left(\Omega;\mathbb{R}^3\right) \cap L^2_\sigma(\Omega)$, with $L^2_\sigma(\Omega) := \left\{\varphi \in L^p(\Omega;\mathbb{R}^3) \,|\, \nabla\cdot\varphi = 0\right\}$ and $\mathcal{P}$ representing the Helmholtz projection of $L^2(\Omega;\mathbb{R}^3)$ onto $L^2_\sigma(\Omega)$. Similarly, for $p \ge 2$ we let $W^{1,p}_{0,\sigma}(\Omega) := W_0^{1,p}(\Omega;\mathbb{R}^3) \cap L^2_\sigma(\Omega)$.

The first of our main results then asserts global solvability under a mild condition on m and for widely arbitrary initial data:

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, and let $\phi \in W^{2,\infty}(\Omega)$,*

$$m > \frac{7}{6}$$

and $c_\star \ge 0$. Then whenever

$$\begin{cases} n_0 \in W^{1,\infty}(\Omega) \quad \text{is such that } n_0 \ge 0 \text{ and } n_0 \not\equiv 0, \\ c_0 \in W^{1,\infty}(\Omega) \quad \text{is such that } c_0 > 0 \text{ in } \overline{\Omega} \text{ with } c_0|_{\partial\Omega} = c_\star, \quad \text{and} \\ u_0 \in W^{2,\infty}(\Omega;\mathbb{R}^3) \quad \text{is such that } \nabla\cdot u_0 \equiv 0 \text{ and } u_0|_{\partial\Omega} = 0, \end{cases} \tag{1.6}$$

there exist functions

$$\begin{cases} n \in \bigcap_{p\in[1,3m-\frac{7}{3})} L^\infty((0,\infty);L^p(\Omega)) \cap \bigcap_{s\in[1,6m-\frac{44}{9})} L^s_{loc}(\overline{\Omega}\times[0,\infty)), \\ c \in L^\infty(\Omega\times(0,\infty)) \quad \text{with} \quad c - c_\star \in L^2_{loc}([0,\infty);W_0^{1,2}(\Omega)) \cap \bigcap_{q\in[1,2)} L^\infty((0,\infty);W_0^{1,q}(\Omega)) \quad \text{and} \\ u \in L^\infty((0,\infty);L^2_\sigma(\Omega)) \cap L^2_{loc}([0,\infty);W^{1,2}_{0,\sigma}(\Omega)) \end{cases} \tag{1.7}$$

such that (n,c,u) forms a global weak solution of (1.5) in the sense of Definition 2.1 below. This solution can be obtained as the limit of solutions to the approximate problems (2.7) in that there exists $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_j \searrow 0$ as $j\to\infty$, and that $(n_\varepsilon,c_\varepsilon,u_\varepsilon) \to (n,c,u)$ a.e. in $\Omega\times(0,\infty)$ as $\varepsilon = \varepsilon_j \searrow 0$.

In line with corresponding limitations concerning the knowledge on regularity already in the three-dimensional Navier–Stokes subsystem of (1.5) ([44]), at a temporally global level we do not expect availability of regularity features significantly beyond those in (1.7). Nevertheless, the organization of our analysis will enable us to identify a qualitative feature reflecting genuine relaxation in (1.5) at least in the long term. In fact, the following second of our main results indicates the existence of a conveniently small absorbing set for (1.5), bounded in size exclusively by the quantities $c_\star$ and $\int_\Omega n_0$ of immediate physical relevance, in the following sense:

Theorem 1.2 *Suppose that $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, that $\phi \in W^{2,\infty}(\Omega)$, and that $m > \frac{7}{6}$. Then given any*

$$p \in [1, 3m - \tfrac{7}{3}) \quad \text{and} \quad q \in (1,2), \tag{1.8}$$

for each $K > 0$ one can find $R(K,p,q) > 0$ with the property that whenever $c_\star \ge 0$ and (1.6) holds with

$$c_\star \le K \qquad \text{and} \qquad \int_\Omega n_0 \le K, \tag{1.9}$$

there exists $t_0(p,q,n_0,c_0,u_0) > 0$ such that the global weak solution of (1.5) constructed in Theorem 1.1 satisfies

$$\|n(\cdot,t)\|_{L^p(\Omega)} + \|\nabla c(\cdot,t)\|_{L^q(\Omega)} + \|u(\cdot,t)\|_{L^2(\Omega)} \le R(K,p,q) \qquad \text{for a.e. } t > t_0(p,q,n_0,c_0,u_0). \tag{1.10}$$

Further steps of our analysis can be outlined as follows. In Section 2 we will, after specifying the concept of weak solvability in question here, introduce a family of appropriately regularized variants of (1.5), which admit global classical solutions. The third section will be dedicated to the derivation of a differential inequality for quantities of the form in (1.4); after first establishing some spatio-temporal information on ∇c_ε by exploiting an ODE comparison argument for the difference $c_\varepsilon - c_\star$ (Lemma 3.1), we proceed here by establishing individual differential inequalities for the solutions components (Lemma 3.2, Lemma 3.3 and Lemma 3.6). The main obstacle therein is the control of unfavorable boundary terms appearing when integrating by parts in the second equation of our regularized systems (Lemma 3.4 and Lemma 3.5). Combination of the quasi energy-like structure with a basic boundedness information on c_ε provided by Lemma 3.1 will then constitute fundamental a priori knowledge, which in Section 4 can be refined (Lemma 4.3 and Lemma 4.4) so as to become a sufficiently strong preparation for the limit procedure undertaken in Lemma 5.1 of Section 5. The final steps of Section 5 will then consist in verifying the claimed weak solution property, and in exploiting some basic exponential decay (Lemma 3.1) to construct absorbing sets in the intended flavor.

2 A concept of weak solvability and global approximating solutions

The following solution concept, to be pursued below, seems fairly natural in the considered context of (1.5), especially by involving standard weak formulations of the respective sub-problems therein, and by thus including requirements somewhat stronger than those introduced in the more generalized framework from [42].

Definition 2.1 *Let $m > 1$, $\phi \in W^{2,\infty}(\Omega)$ and $c_\star \geq 0$, assume (1.6), and let*

$$\begin{cases} n \in L^1_{loc}(\overline{\Omega} \times [0,\infty)), \\ c \in L^1_{loc}(\overline{\Omega} \times [0,\infty)) \quad \textit{with} \quad c - c_\star \in L^1_{loc}([0,\infty); W^{1,1}_0(\Omega)) \qquad \textit{and} \\ u \in L^2_{loc}([0,\infty); W^{1,2}_{0,\sigma}(\Omega)) \end{cases} \tag{2.1}$$

be such that $n \geq 0$ and $c \geq 0$ a.e. in $\Omega \times (0,\infty)$, that

$$nc \in L^1_{loc}(\overline{\Omega} \times [0,\infty)), \tag{2.2}$$

and that

$$\{\nabla n^m \,,\, n\nabla c \,,\, nu \,,\, cu\} \subset L^1_{loc}(\overline{\Omega} \times [0,\infty); \mathbb{R}^3). \tag{2.3}$$

Then (n, c, u) will be called a global weak solution of (1.5) if

$$-\int_0^\infty \int_\Omega n\varphi_t - \int_\Omega n_0 \varphi(\cdot,0) = -\frac{1}{m}\int_0^\infty \int_\Omega \nabla n^m \cdot \nabla\varphi + \int_0^\infty \int_\Omega n\nabla c \cdot \nabla\varphi + \int_0^\infty \int_\Omega nu \cdot \nabla\varphi \tag{2.4}$$

for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$, if

$$-\int_0^\infty \int_\Omega c\varphi_t - \int_\Omega c_0 \varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla c \cdot \nabla\varphi - \int_0^\infty \int_\Omega nc\varphi + \int_0^\infty \int_\Omega cu \cdot \nabla\varphi \tag{2.5}$$

for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$, and if

$$-\int_0^\infty \int_\Omega u \cdot \varphi_t - \int_\Omega u_0 \cdot \varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla u \cdot \nabla\varphi + \int_0^\infty \int_\Omega (u \otimes u) \cdot \nabla\varphi + \int_0^\infty \int_\Omega n\nabla\phi \cdot \varphi \tag{2.6}$$

for each $\varphi \in C_0^\infty(\Omega \times [0,\infty);\mathbb{R}^3)$ fulfilling $\nabla \cdot \varphi = 0$. Here, for vectors $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ and $b = (b_1, b_2, b_3) \in \mathbb{R}^3$ we have defined the matrix $a \otimes b = (M_{ij})_{i,j\in\{1,2,3\}}$ by letting $M_{ij} := a_i b_j$ for $(i,j) \in \{1,2,3\}^2$.

Our approach toward constructing such solutions will be based on an analysis of the regularized variants of (1.5) which, for $\varepsilon \in (0,1)$, are given by

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon = \nabla \cdot \big((n_\varepsilon + \varepsilon)^{m-1} \nabla n_\varepsilon\big) - \nabla \cdot \big(n_\varepsilon F_\varepsilon'(n_\varepsilon) \nabla c_\varepsilon\big), & x \in \Omega, t > 0, \\ c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - F_\varepsilon(n_\varepsilon) c_\varepsilon, & x \in \Omega, t > 0, \\ u_{\varepsilon t} + (Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon = \Delta u_\varepsilon + \nabla P_\varepsilon + F_\varepsilon(n_\varepsilon) \nabla \phi, \quad \nabla \cdot u_\varepsilon = 0, & x \in \Omega, t > 0, \\ \frac{\partial n_\varepsilon}{\partial \nu} - n_\varepsilon F_\varepsilon'(n_\varepsilon) \frac{\partial c_\varepsilon}{\partial \nu} = 0, \qquad c_\varepsilon = c_\star, \qquad u_\varepsilon = 0, & x \in \partial\Omega, t > 0, \\ n_\varepsilon(x,0) = n_0(x), \quad c_\varepsilon(x,0) = c_0(x), \quad u_\varepsilon(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.7}$$

with $Y_\varepsilon := (1 + \varepsilon A)^{-1}$ denoting a Yosida-type approximation, and with

$$F_\varepsilon(\xi) := \frac{\xi}{1 + \varepsilon\xi}, \qquad \xi \geq 0,$$

satisfying

$$0 \leq F_\varepsilon(\xi) \leq \xi \quad \text{and} \quad 0 \leq F_\varepsilon'(\xi) = \frac{1}{(1+\varepsilon\xi)^2} \leq 1 \qquad \text{for all } \xi \geq 0 \text{ and } \varepsilon \in (0,1) \tag{2.8}$$

as well as

$$F_\varepsilon(\xi) \nearrow \xi \quad \text{and} \quad F_\varepsilon'(\xi) \nearrow 1 \qquad \text{for all } \xi \geq 0 \text{ as } \varepsilon \searrow 0. \tag{2.9}$$

Then the saturation effects thereby introduced can readily be seen to exert regularization to an extent sufficient to warrant global smooth solvability in each of these problems. Indeed, the following basic existence statement can be derived by minor modification of the reasoning in [42, Lemma 2.2] (cf. also [4]), augmented by arguments detailed in [27, Lemma 2.2] to cope with the nonlinearity in the considered diffusion mechanism.

Lemma 2.2 *Let $\varepsilon \in (0,1)$. Then there exist functions*

$$\begin{cases} n_\varepsilon \in C^0([\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ c_\varepsilon \in \bigcap_{q\geq 1} C^0([0,\infty); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ u_\varepsilon \in C^0(\overline{\Omega} \times [0,\infty);\mathbb{R}^3) \cap C^{2,1}(\overline{\Omega} \times (0,\infty);\mathbb{R}^3) \qquad \textit{and} \\ P_\varepsilon \in C^{1,0}(\Omega \times (0,\infty)), \end{cases} \tag{2.10}$$

with $n_\varepsilon > 0$ in $\overline{\Omega} \times (0,\infty)$ and $c_\varepsilon > 0$ in $\Omega \times (0,\infty)$, such that $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ solves (2.7) in the classical sense, and that furthermore

$$\int_\Omega n_\varepsilon(\cdot,t) = \int_\Omega n_0 \qquad \textit{for all } t > 0 \tag{2.11}$$

and

$$\|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)} \qquad \textit{for all } t > 0. \tag{2.12}$$

3 The path toward an energy-type inequality

3.1 An exponential relaxation property of c_ε

Making explicit use of the fact that $c_\star$ is constant in time and space, as a first regularity property beyond those in (2.11) and (2.12) let us record the following relaxation feature which yet acts at rather low levels with regard to the topologies involved, but which through its favorably traceable dependence on the data will later on form a cornerstone for our asymptotic analysis related to Theorem 1.2.

Lemma 3.1 *Let $m > 1$. Then there exist $C > 0$ and $(\lambda(k))_{k\in\mathbb{N}} \subset (0,\infty)$ with the following property: Whenever $c_\star \ge 0$ and (n_0, c_0, u_0) satisfies (1.6), one can find $(\Gamma(k, c_0))_{k\in\mathbb{N}} \subset (0,\infty)$ such that*

$$\int_t^{t+1}\int_\Omega |\nabla c_\varepsilon|^2 \le C c_\star^2 \int_\Omega n_0 + \Gamma^2(1, c_0)e^{-2\lambda(1)t} \qquad \text{for all } t \ge 0 \text{ and } \varepsilon \in (0,1), \tag{3.1}$$

and that for each positive integer k,

$$\big\|c_\varepsilon(\cdot,t) - c_\star\big\|_{L^{2k}(\Omega)} \le C c_\star \cdot \Big\{\int_\Omega n_0\Big\}^{\frac{1}{2k}} + \Gamma(k, c_0)e^{-\lambda(k)t} \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1). \tag{3.2}$$

PROOF. We fix $\lambda_1 > 0$ such that in accordance with a Poincaré inequality we have

$$\lambda_1 \int_\Omega \varphi^2 \le \int_\Omega |\nabla\varphi|^2 \qquad \text{for all } \varphi \in W_0^{1,2}(\Omega), \tag{3.3}$$

and use that $(\frac{2k}{\lambda_1})^{\frac{1}{2k}} \to 1$ as $k \to \infty$ in choosing $C_1 > 0$ fulfilling

$$\Big(\frac{2k}{\lambda_1}\Big)^{\frac{1}{2k}} \le C_1 \qquad \text{for all } k \in \{1, 2, 3, ...\}. \tag{3.4}$$

Given any such k and an arbitrary $\varepsilon \in (0,1)$, we may then rely on the solenoidality of u_ε to see that since $\frac{2k-1}{k} \ge 1$,

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (c_\varepsilon - c_\star)^{2k} &= -2k(2k-1)\int_\Omega (c_\varepsilon - c_\star)^{2k-2}|\nabla c_\varepsilon|^2 - 2k\int_\Omega F_\varepsilon(n_\varepsilon)c_\varepsilon(c_\varepsilon - c_\star)^{2k-1} \\
&\quad - \int_\Omega u_\varepsilon \cdot \nabla (c_\varepsilon - c_\star)^{2k} \\
&= -\frac{2(2k-1)}{k}\int_\Omega \Big|\nabla (c_\varepsilon - c_\star)^k\Big|^2 - 2k\int_\Omega F_\varepsilon(n_\varepsilon)c_\varepsilon(c_\varepsilon - c_\star)^{2k-1} \\
&\le -2\int_\Omega \Big|\nabla (c_\varepsilon - c_\star)^k\Big|^2 - 2k\int_\Omega F_\varepsilon(n_\varepsilon)c_\varepsilon(c_\varepsilon - c_\star)^{2k-1} \qquad \text{for all } t > 0,
\end{aligned} \tag{3.5}$$

where since $2k-1$ is odd, recalling that $0 \le F_\varepsilon(n_\varepsilon) \le n_\varepsilon$ we can estimate

$$\begin{aligned}
-2k\int_\Omega F_\varepsilon(n_\varepsilon)c_\varepsilon(c_\varepsilon - c_\star)^{2k-1} &\le 2k\int_{\{c_\varepsilon < c_\star\}} F_\varepsilon(n_\varepsilon)c_\varepsilon(c_\star - c_\varepsilon)^{2k-1} \\
&\le 2kc_\star^{2k}\int_\Omega n_\varepsilon
\end{aligned}$$

$$= \quad 2kc_\star^{2k}\int_\Omega n_0 \qquad \text{for all } t>0$$

by (2.11). In line with (3.3), from (3.5) we thus infer that

$$\frac{d}{dt}\int_\Omega (c_\varepsilon - c_\star)^{2k} + \lambda_1\int_\Omega (c_\varepsilon - c_\star)^{2k} + \int_\Omega \Big|\nabla (c_\varepsilon - c_\star)^k\Big|^2 \le 2kc_\star^{2k}\int_\Omega n_0 \qquad \text{for all } t>0, \tag{3.6}$$

which through an ODE comparison argument firstly entails that

$$\int_\Omega (c_\varepsilon - c_\star)^{2k} \le e^{-\lambda_1 t}\cdot \int_\Omega (c_0 - c_\star)^{2k} + \frac{2k}{\lambda_1}c_\star^{2k}\int_\Omega n_0 \qquad \text{for all } t>0, \tag{3.7}$$

and that thus, since $(\xi+\eta)^\alpha \le \xi^\alpha + \eta^\alpha$ for all $\xi \ge 0, \eta \ge 0$ and $\alpha \in (0,1)$,

$$\begin{aligned}
\|c_\varepsilon - c_\star\|_{L^{2k}(\Omega)} &\le \|c_0 - c_\star\|_{L^{2k}(\Omega)} e^{-\frac{\lambda_1}{2k}t} + \Big(\frac{2k}{\lambda_1}\Big)^{\frac{1}{2k}} c_\star \cdot \Big\{\int_\Omega n_0\Big\}^{\frac{1}{2k}} \\
&\le \|c_0 - c_\star\|_{L^{2k}(\Omega)} e^{-\frac{\lambda_1}{2k}t} + C_1 c_\star \cdot \Big\{\int_\Omega n_0\Big\}^{\frac{1}{2k}} \qquad \text{for all } t>0
\end{aligned} \tag{3.8}$$

thanks to (3.4). Apart from this, by utilizing (3.7) we find that when restricted to $k := 1$ and directly integrated, (3.6) secondly implies that

$$\begin{aligned}
\int_t^{t+1}\int_\Omega |\nabla c_\varepsilon|^2 &= \int_t^{t+1}\int_\Omega \Big|\nabla (c_\varepsilon - c_\star)\Big|^2 \\
&\le \int_\Omega \Big(c_\varepsilon(\cdot,t) - c_\star\Big)^2 + 2c_\star^2\int_\Omega n_0 \\
&\le e^{-\lambda_1 t}\cdot\int_\Omega (c_0 - c_\star)^2 + \Big(\frac{2}{\lambda_1} + 2\Big)c_\star^2\int_\Omega n_0 \qquad \text{for all } t \ge 0.
\end{aligned} \tag{3.9}$$

From (3.9) and (3.8) we therefore obtain (3.1) and (3.2) if we let $C := \max\{\frac{2}{\lambda_1} + 2\,,\, C_1\}$ as well as $\Gamma(k, c_0) := \|c_0 - c_\star\|_{L^{2k}(\Omega)}$ and $\lambda(k) := \frac{\lambda_1}{2k}$ for $k \ge 1$. $\square$

3.2 A basic evolution property of $\int_\Omega (n_\varepsilon + \varepsilon)^p$. The condition $p < 3m - \frac{7}{3}$

Now under the key assumption that $p < 3m - \frac{7}{3}$, in the course of a standard L^p testing procedure applied to the first equation in (2.7) the respective cross-diffusive contribution can be estimated, up to additive constants essentially depending on $\int_\Omega n_0$ only, against an expression containing ∇c_ε which, as our subsequent analysis will show, can suitably be controlled due to the diffusive action in the second equation from (2.7).

Lemma 3.2 *Let $m > \frac{10}{9}$ and $p > \max\big\{1\,,\, m - \frac{1}{2}\,,\, 2m-2\big\}$ be such that*

$$p < 3m - \frac{7}{3}. \tag{3.10}$$

Then there exists $q_0 = q_0(p) \in (1,2)$ with the property that given any $q \in (q_0, 2)$ and $K > 0$ one can find $C(K,p,q) > 0$ such that whenever $c_\star \ge 0$ and (1.6) holds with $\int_\Omega n_0 \le K$, we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega (n_\varepsilon+\varepsilon)^p + \int_\Omega (n_\varepsilon+\varepsilon)^p + \frac{p(p-1)}{2}\int_\Omega (n_\varepsilon+\varepsilon)^{p+m-3}|\nabla n_\varepsilon|^2 \\ \le \int_\Omega |\nabla c_\varepsilon|^{2q} + C(K,p,q) \qquad \textit{for all } t>0 \textit{ and } \varepsilon \in (0,1). \end{aligned} \tag{3.11}$$

PROOF. Since $\frac{2(p-m+1)q}{(p+m-1)(q-1)} \to \frac{4(p-m+1)}{p+m-1}$ as $q \to 2$, with our assumptions $p > m - \frac{1}{2}$ and $m > 1$ ensuring that $\frac{2}{p+m-1} < \frac{4(p-m+1)}{p+m-1} < 6$, since furthermore the hypothesis (3.10) warrants that

$$\lim_{q\to 2} \frac{3(pq-mq+1)}{(3p+3m-4)(q-1)} = \frac{6p-6m+3}{3p+3m-4} = 2 - \frac{12m-11}{3p+3m-4} < 2 - \frac{12m-11}{3\cdot(3m-\frac{7}{3})+3m-4} = 1,$$

and since apart from that we have $\lim_{q\to 2}\frac{(p-m+1)q}{q-1} = 2(p-m+1) > p$ due to the restriction $p > 2m-2$, we can fix $q_0 = q_0(p) \in (1,2)$ suitably close to 2 such that

$$\frac{2}{p+m-1} < \frac{2(p-m+1)q}{(p+m-1)(q-1)} < 6 \qquad \text{for all } q \in (q_0,2) \tag{3.12}$$

and

$$\frac{3(pq-mq+1)}{(3p+3m-4)(q-1)} < 1 \qquad \text{for all } q \in (q_0,2) \tag{3.13}$$

as well as

$$\frac{(p-m+1)q}{q-1} > p \qquad \text{for all } q \in (q_0,2). \tag{3.14}$$

Henceforth fixing $q \in (q_0,2)$ and $K > 0$ and assuming (1.6) with $\int_\Omega n_0 \le K$, we use that $\nabla\cdot u_\varepsilon = 0$ on $\Omega\times(0,\infty)$ and $0 \le F_\varepsilon' \le 1$ for all $\varepsilon \in (0,1)$ to see that due to the first equation in (2.7) and Young's inequality, for all $t > 0$ and $\varepsilon \in (0,1)$ we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega (n_\varepsilon+\varepsilon)^p + \int_\Omega (n_\varepsilon+\varepsilon)^p + p(p-1)\int_\Omega (n_\varepsilon+\varepsilon)^{p+m-3}|\nabla n_\varepsilon|^2 \\ &= p(p-1)\int_\Omega (n_\varepsilon+\varepsilon)^{p-2} n_\varepsilon F_\varepsilon'(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla c_\varepsilon + \int_\Omega (n_\varepsilon+\varepsilon)^p \\ &\le \frac{p(p-1)}{4}\int_\Omega (n_\varepsilon+\varepsilon)^{p+m-3}|\nabla n_\varepsilon|^2 + p(p-1)\int_\Omega (n_\varepsilon+\varepsilon)^{p-m-1} n_\varepsilon^2 F_\varepsilon'^2(n_\varepsilon)|\nabla c_\varepsilon|^2 + \int_\Omega (n_\varepsilon+\varepsilon)^p \\ &\le \frac{p(p-1)}{4}\int_\Omega (n_\varepsilon+\varepsilon)^{p+m-3}|\nabla n_\varepsilon|^2 + p(p-1)\int_\Omega (n_\varepsilon+\varepsilon)^{p-m+1}|\nabla c_\varepsilon|^2 + \int_\Omega (n_\varepsilon+\varepsilon)^p, \end{aligned} \tag{3.15}$$

where since $q > 1$ and $p < \frac{(p-m+1)q}{q-1}$ by (3.14), we may again draw on Young's inequality to infer that with some $C_1 = C_1(p,q) > 0$,

$$p(p-1)\int_\Omega (n_\varepsilon+\varepsilon)^{p-m+1}|\nabla c_\varepsilon|^2 + \int_\Omega (n_\varepsilon+\varepsilon)^p$$

$$\begin{aligned}
&\le \int_\Omega |\nabla c_\varepsilon|^{2q} + C_1 \int_\Omega (n_\varepsilon+\varepsilon)^{\frac{(p-m+1)q}{q-1}} + \int_\Omega (n_\varepsilon+\varepsilon)^p \\
&\le \int_\Omega |\nabla c_\varepsilon|^{2q} + (C_1+1) \int_\Omega (n_\varepsilon+\varepsilon)^{\frac{(p-m+1)q}{q-1}} + |\Omega| \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1). \qquad (3.16)
\end{aligned}$$

Here we rely on (3.12) in employing the Gagliardo–Nirenberg inequality to find $C_2 = C_2(p,q) > 0$ and $C_3 = C_3(K,p,q) > 0$ satisfying

$$\begin{aligned}
&(C_1+1)\int_\Omega (n_\varepsilon+\varepsilon)^{\frac{(p-m+1)q}{q-1}} \\
&\quad = (C_1+1)\Big\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\Big\|^{\frac{2(p-m+1)q}{(p+m-1)(q-1)}}_{L^{\frac{2(p-m+1)q}{(p+m-1)(q-1)}}(\Omega)} \\
&\quad \le C_2\Big\|\nabla(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\Big\|^{\frac{2(p-m+1)q\theta}{(p+m-1)(q-1)}}_{L^2(\Omega)} \Big\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\Big\|^{\frac{2(p-m+1)q(1-\theta)}{(p+m-1)(q-1)}}_{L^{\frac{2}{p+m-1}}(\Omega)} \\
&\quad\quad + \Big\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\Big\|^{\frac{2(p-m+1)q}{(p+m-1)(q-1)}}_{L^{\frac{2}{p+m-1}}(\Omega)} \\
&\quad \le C_3\Big\|\nabla(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\Big\|^{2\cdot\frac{3(pq-mq+1)}{(3p+3m-4)(q-1)}}_{L^2(\Omega)} + C_3 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1), \qquad (3.17)
\end{aligned}$$

because $\Big\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\Big\|^{\frac{2}{p+m-1}}_{L^{\frac{2}{p+m-1}}(\Omega)} = \int_\Omega (n_\varepsilon+\varepsilon) = \int_\Omega (n_0+\varepsilon) \le K + |\Omega|$ for all $t>0$ and $\varepsilon\in(0,1)$ by (2.11), and because the number

$$\theta := \frac{3(p+m-1)(pq-mq+1)}{(3p+3m-4)(p-m+1)q} \in (0,1)$$

appearing herein satisfies $\frac{(p-m+1)q\theta}{(p+m-1)(q-1)} = \frac{3(pq-mq+1)}{(3p+3m-4)(q-1)}$. As (3.13) enables us to once again invoke Young's inequality to obtain $C_4 = C_4(K,p,q) > 0$ fulfilling

$$\begin{aligned}
C_3\Big\|\nabla(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\Big\|^{2\cdot\frac{3(pq-mq+1)}{(3p+3m-4)(q-1)}}_{L^2(\Omega)} &\le \frac{p(p-1)}{(p+m-1)^2}\Big\|\nabla(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\Big\|^2_{L^2(\Omega)} + C_4 \\
&= \frac{p(p-1)}{4}\int_\Omega (n_\varepsilon+\varepsilon)^{p+m-3}|\nabla n_\varepsilon|^2 + C_4
\end{aligned}$$

for all $t>0$ and $\varepsilon\in(0,1)$, it only remains to collect (3.15), (3.16) and (3.17) to derive (3.11) with $C(K,p,q) := |\Omega| + C_3 + C_4$. $\square$

3.3 Tracing the evolution of $\int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q}{2}}$. The condition $p > \frac{7}{3} - m$

The main part of our estimation process will now be launched by the following observation concerned with the evolution of $\frac{d}{dt}\int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q}{2}}$ for $q\in(1,2)$. In fact, whenever $p > \frac{7}{3} - m$, all corresponding nonlinear interaction terms can be controlled by the dissipated quantities from (3.10) and the classical Navier–Stokes energy inequality (cf. Lemma 3.6 below), and the sum of two boundary integrals to be further analyzed in Lemma 3.5 below. Together with the restrictions from Lemma 3.2, this newly arising condition on p forms the main reason behind the limitation of Theorem 1.1 to the case of $m > \frac{7}{6}$.

Lemma 3.3 *Let $m > 1$ and $p > 1$ be such that*

$$p > \frac{7}{3} - m. \tag{3.18}$$

Then one can fix $r_0 = r_0(p) > 2$ such that for each $q \in (1,2)$, $r > r_0$ and $K > 0$ there exists $C(K,p,q,r) > 0$ with the property that if $c_\star \geq 0$ and (1.6) is valid with $\int_\Omega n_0 \leq K$, the solutions of (2.7) satisfy

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q}{2}} &+ \frac{1}{C(K,p,q,r)}\int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q}{2}} + \frac{q(q-1)}{2}\int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}|D^2c_\varepsilon|^2 \\
\leq\ & \frac{p(p-1)}{8}\int_\Omega (n_\varepsilon+\varepsilon)^{p+m-3}|\nabla n_\varepsilon|^2 + \int_\Omega c_\varepsilon^r + 2\int_\Omega |\nabla c_\varepsilon|^{2q} + \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 \\
&+\frac{q}{2}\int_{\partial\Omega}\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}\frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu} - q\int_{\partial\Omega} F_\varepsilon(n_\varepsilon)c_\varepsilon\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}\frac{\partial c_\varepsilon}{\partial\nu} \\
&+C(K,p,q,r) \qquad \textit{for all } t>0 \textit{ and } \varepsilon\in(0,1).
\end{aligned} \tag{3.19}$$

Proof. Since $3p+3m-7>0$ by (3.18), the number

$$r_0 = r_0(p) := 2\cdot\frac{3p+3m-1}{3p+3m-7} \tag{3.20}$$

satisfies $r_0 > 2$, and to see that the claimed conclusion holds with this selection kept fixed, assuming that $q\in(1,2)$, $r>r_0$ and $K>0$, and that $c_\star \geq 0$ and (1.6) holds with $\int_\Omega n_0 \leq K$, for $\varepsilon\in(0,1)$ we use the second equation in (2.7) to compute

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q}{2}} &= q\int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}\nabla c_\varepsilon\cdot\nabla\Big\{\Delta c_\varepsilon - F_\varepsilon(n_\varepsilon)c_\varepsilon - u_\varepsilon\cdot\nabla c_\varepsilon\Big\} \\
&= \frac{q}{2}\int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}\Delta|\nabla c_\varepsilon|^2 - q\int_\Omega\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}|D^2c_\varepsilon|^2 \\
&\quad -q\int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}\nabla c_\varepsilon\cdot\nabla\big(F_\varepsilon(n_\varepsilon)c_\varepsilon\big) \\
&\quad -q\int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}\nabla c_\varepsilon\cdot\nabla(u_\varepsilon\cdot\nabla c_\varepsilon) \qquad \text{for all } t>0,
\end{aligned} \tag{3.21}$$

because $\nabla c_\varepsilon\cdot\nabla\Delta c_\varepsilon = \frac{1}{2}\Delta|\nabla c_\varepsilon|^2 - |D^2c_\varepsilon|^2$. Here an integration by parts shows that since $\big|\nabla|\nabla c_\varepsilon|^2\big|^2 = |2D^2c_\varepsilon\cdot\nabla c_\varepsilon|^2 \leq 4(|\nabla c_\varepsilon|^2+1)|D^2c_\varepsilon|^2$,

$$\begin{aligned}
&\frac{q}{2}\int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}\Delta|\nabla c_\varepsilon|^2 - q\int_\Omega\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}|D^2c_\varepsilon|^2 \\
&\quad= \frac{q(2-q)}{4}\int_\Omega\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-4}{2}}\Big|\nabla|\nabla c_\varepsilon|^2\Big|^2 + \frac{q}{2}\int_{\partial\Omega}\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}\frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu} \\
&\qquad -q\int_\Omega\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}|D^2c_\varepsilon|^2 \\
&\quad\leq -q(q-1)\int_\Omega\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}|D^2c_\varepsilon|^2 + \frac{q}{2}\int_{\partial\Omega}\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}\frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu}
\end{aligned} \tag{3.22}$$

for all $t > 0$, while due to the identity $\nabla \cdot u_\varepsilon = 0$, combining another integration by parts with Young's inequality we find that

$$\begin{aligned}
-q\int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}\nabla c_\varepsilon\cdot\nabla(u_\varepsilon\cdot\nabla c_\varepsilon) &= -q\int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}\nabla c_\varepsilon\cdot(\nabla u_\varepsilon\cdot\nabla c_\varepsilon) \\
&\quad -q\int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}\nabla c_\varepsilon\cdot(D^2c_\varepsilon\cdot u_\varepsilon) \\
&= -q\int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}\nabla c_\varepsilon\cdot(\nabla u_\varepsilon\cdot\nabla c_\varepsilon) \\
&\quad -\int_\Omega u_\varepsilon\cdot\nabla\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q}{2}} \\
&= -q\int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}\nabla c_\varepsilon\cdot(\nabla u_\varepsilon\cdot\nabla c_\varepsilon) \\
&\le 2\int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{q-2}|\nabla c_\varepsilon|^4+\frac{q^2}{8}\int_\Omega|\nabla u_\varepsilon|^2 \\
&\le 2\int_\Omega|\nabla c_\varepsilon|^{2q}+\frac12\int_\Omega|\nabla u_\varepsilon|^2 \qquad \text{for all } t>0, \qquad (3.23)
\end{aligned}$$

because our assumption $q < 2$ warrants that $\big(|\nabla c_\varepsilon|^2+1\big)^{q-2} \le |\nabla c_\varepsilon|^{2q-4}$, and that $\frac{q^2}{8} \le \frac12$.
In estimating the second summand on the right of (3.21), we use the inequality $|\Delta c_\varepsilon| \le \sqrt{3}|D^2c_\varepsilon|$ along with (2.8) and Young's inequality to see that writing $C_1 = C_1(q) := \frac{q(2-q+\sqrt{3})^2}{q(q-1)}$ we have

$$\begin{aligned}
&-q\int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}\nabla c_\varepsilon\cdot\nabla\big(F_\varepsilon(n_\varepsilon)c_\varepsilon\big)+q\int_{\partial\Omega}F_\varepsilon(n_\varepsilon)c_\varepsilon\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}\frac{\partial c_\varepsilon}{\partial\nu} \\
&\quad= q\int_\Omega F_\varepsilon(n_\varepsilon)c_\varepsilon\cdot\Big\{(q-2)\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-4}{2}}\nabla c_\varepsilon\cdot(D^2c_\varepsilon\cdot\nabla c_\varepsilon)+\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}\Delta c_\varepsilon\Big\} \\
&\quad\le q(2-q+\sqrt{3})\int_\Omega n_\varepsilon c_\varepsilon\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}|D^2c_\varepsilon| \\
&\quad\le \frac{q(q-1)}{4}\int_\Omega\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}|D^2c_\varepsilon|^2+C_1\int_\Omega n_\varepsilon^2c_\varepsilon^2\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}} \\
&\quad\le \frac{q(q-1)}{4}\int_\Omega\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}|D^2c_\varepsilon|^2+C_1^{\frac{r}{r-2}}\int_\Omega n_\varepsilon^{\frac{2r}{r-2}}+\int_\Omega c_\varepsilon^r \qquad \text{for all } t>0, \qquad (3.24)
\end{aligned}$$

since $\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}} \le 1$. Here the second last expression can be controlled by means of the Gagliardo–Nirenberg inequality, which in conjunction with (2.11), namely, provides $C_2 = C_2(p,q,r) > 0$ and $C_3 = C_3(K,p,q,r) > 0$ fulfilling

$$\begin{aligned}
C_1^{\frac{r}{r-2}}\int_\Omega n_\varepsilon^{\frac{2r}{r-2}} &\le C_1^{\frac{r}{r-2}}\Big\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\Big\|_{L^{\frac{4r}{(p+m-1)(r-2)}}(\Omega)}^{\frac{4r}{(p+m-1)(r-2)}} \\
&\le C_2\Big\|\nabla(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\Big\|_{L^2(\Omega)}^{\frac{4r\theta}{(p+m-1)(r-2)}}\Big\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\Big\|_{L^{\frac{2}{p+m-1}}(\Omega)}^{\frac{4r(1-\theta)}{(p+m-1)(r-2)}} \\
&\quad+\Big\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\Big\|_{L^{\frac{2}{p+m-1}}(\Omega)}^{\frac{4r}{(p+m-1)(r-2)}}
\end{aligned}$$

$$\leq \quad C_3 \Big\| \nabla (n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}} \Big\|_{L^2(\Omega)}^{\frac{4r\theta}{(p+m-1)(r-2)}} + C_3 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1), \quad (3.25)$$

with $\theta = \theta(p,r) := \frac{3(p+m-1)(r+2)}{(3p+3m-4)\cdot 2r}$ satisfying $0<\theta<1$ due to the inequalities $p+m-1>1+1-1>0$ and $3p+3m-4>3+3-4>0$, and due to the fact that since $r>r_0$,

$$\theta = \frac{3(p+m-1)}{3p+3m-4}\cdot\Big(\frac{1}{2}+\frac{1}{r}\Big) < \frac{3(p+m-1)}{3p+3m-4}\cdot\Big(\frac{1}{2}+\frac{3p+3m-7}{2(3p+3m-1)}\Big) = \frac{3(p+m-1)}{3p+3m-1} < 1,$$

by (3.20). As taking full advantage of the assumption $r>r_0$ we see that moreover

$$\frac{4r\theta}{(p+m-1)(r-2)} = \frac{6}{3p+3m-4}\cdot\frac{1+\frac{2}{r}}{1-\frac{2}{r}} < \frac{6}{3p+3m-4}\cdot\frac{1+\frac{3p+3m-7}{3p+3m-1}}{1-\frac{3p+3m-7}{3p+3m-1}} = 2,$$

we may once again rely on Young's inequality to infer from (3.25) that there exists $C_4 = C_4(K,p,q,r) > 0$ with the property that

$$\begin{aligned} C_1^{\frac{r}{r-2}}\int_\Omega n_\varepsilon^{\frac{2r}{r-2}} &\leq \frac{p(p-1)}{2(p+m-1)^2}\Big\|\nabla(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\Big\|_{L^2(\Omega)}^2 + C_4 \\ &= \frac{p(p-1)}{8}\int_\Omega (n_\varepsilon+\varepsilon)^{p+m-3}|\nabla n_\varepsilon|^2 + C_4 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1). \quad (3.26)\end{aligned}$$

Since, finally, a Poincaré inequality yields $C_5 = C_5(q)>0$ such that

$$\int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q}{2}} \leq C_5\int_\Omega \Big|\nabla\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q}{4}}\Big|^2 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),$$

and that thus

$$\begin{aligned} \frac{q-1}{C_5 q}\int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q}{2}} &\leq \frac{q(q-1)}{4}\int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-4}{2}}\big|D^2c_\varepsilon\cdot\nabla c_\varepsilon\big|^2 \\ &\leq \frac{q(q-1)}{4}\int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}|D^2c_\varepsilon|^2 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),\end{aligned}$$

upon collecting (3.21)-(3.24) and (3.26) we conclude that (3.19) holds if we let $C(K,p,q,r) := \max\{\frac{C_5 q}{q-1}, C_4\}$. $\Box$

3.4 Controlling boundary integrals. A pointwise inequality for normal derivatives

A next step of key importance will now consist in appropriately estimating the boundary integrals appearing in (3.19). This will be prepared by the following observation on a one-sided pointwise inequality for normal derivatives, reminiscent of an estimate previously known for functions additionally satisfying homogeneous Neumann boundary conditions ([32]).

Lemma 3.4 *Let $N\geq 2$ and $G\subset\mathbb{R}^N$ be a bounded domain with boundary of class C^2, and let $\kappa\in\mathbb{R}$ denote the maximum of the curvatures on ∂G. Then whenever $\varphi\in C^2(\overline{G})$ and $\varphi_\star\in\mathbb{R}$ are such that $\varphi=\varphi_\star$ on ∂G,*

$$\frac{\partial|\nabla\varphi|^2}{\partial\nu} \leq 2\frac{\partial\varphi}{\partial\nu}\Delta\varphi + 2\kappa\Big|\frac{\partial\varphi}{\partial\nu}\Big|^2 \qquad \text{on } \partial G. \quad (3.27)$$

PROOF. Since ∂G is of class C^2, for each point $x_\star \in \partial G$ we can find open neighborhoods $U \subset \mathbb{R}^N$ of $x_\star$ and $V \subset \mathbb{R}^{N-1}$ of the origin as well as a function $f = f^{(x_\star)} \in C^2(V)$ such that after an affine coordinate transformation we have $x_\star = 0$, $G \cap U = \{x \in U \mid x_N < f(x_1, ..., x_{N-1})\}$ and $\partial G \cap U = \{x \in U \mid x_N = f(x_1, ..., x_{N-1})\}$, and that $f(0) = 0, \nabla f(0) = 0$ and $\Delta f(0) \le \kappa$. Then differentiating the identity

$$\varphi\big(\widetilde{x}, f(\widetilde{x})\big) = \varphi_\star, \qquad \widetilde{x} = (x_1, ..., x_{N-1}) \in V,$$

we see that

$$\partial_{x_i}\varphi\big(\widetilde{x}, f(\widetilde{x})\big) + \partial_{x_i} f(\widetilde{x}) \cdot \partial_{x_N}\varphi\big(\widetilde{x}, f(\widetilde{x})\big) = 0 \quad \text{for all } \widetilde{x} = (x_1, ..., x_{N-1}) \in V \text{ and each } i \in \{1, ..., N-1\}, \tag{3.28}$$

and that thus, since $\partial_{x_i} f(0) = 0$ for all $i \in \{1, ..., N-1\}$,

$$\begin{aligned} \partial_{x_i x_i}\varphi(0) &= -\partial_{x_i x_i} f(0)\partial_{x_N}\varphi(0) - \partial_{x_i} f(0)\partial_{x_i x_N}\varphi(0) \\ &= -\partial_{x_i x_i} f(0)\partial_{x_N}\varphi(0) \qquad \text{for all } i \in \{1, ..., N-1\}. \end{aligned}$$

As (3.28) furthermore implies that, for the same reason,

$$\partial_{x_i}\varphi(0) = 0 \qquad \text{for all } i \in \{1, ..., N-1\}$$

we therefore obtain that since $\nu = e_N$ at $x = 0$,

$$\begin{aligned} \frac{1}{2}\frac{\partial|\nabla\varphi|^2}{\partial\nu}(0) &= \sum_{i=1}^{N-1} \partial_{x_i}\varphi(0)\partial_{x_i x_N}\varphi(0) + \partial_{x_N}\varphi(0) \cdot \Big\{\Delta\varphi(0) - \sum_{i=1}^{N-1} \partial_{x_i x_i}\varphi(0)\Big\} \\ &= \frac{\partial\varphi}{\partial\nu}(0)\Delta\varphi(0) + \frac{\partial\varphi}{\partial\nu}(0) \cdot \sum_{i=1}^{N-1} \partial_{x_i x_i} f(0)\partial_{x_N}\varphi(0) \\ &\le \frac{\partial\varphi}{\partial\nu}(0)\Delta\varphi(0) + \kappa\Big|\frac{\partial\varphi}{\partial\nu}(0)\Big|^2, \end{aligned} \tag{3.29}$$

as claimed. ☐

By making appropriate use of trace embedding inequalities and our assumption that $c_\star$ be constant in time, we can utilize the above to estimate the boundary terms.

Lemma 3.5 *Let $m > 1$, and $q \in (1, 2)$. Then there exists $C(q) > 0$ such that whenever $c_\star \ge 0$ and (1.6) is valid,*

$$\begin{aligned} &\frac{q}{2}\int_{\partial\Omega} \big(|\nabla c_\varepsilon|^2 + 1\big)^{\frac{q-2}{2}} \frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu} - q\int_{\partial\Omega} F_\varepsilon(n_\varepsilon)c_\varepsilon\big(|\nabla c_\varepsilon|^2 + 1\big)^{\frac{q-2}{2}} \frac{\partial c_\varepsilon}{\partial\nu} \\ &\qquad \le \frac{q(q-1)}{4}\int_\Omega \big(|\nabla c_\varepsilon|^2 + 1\big)^{\frac{q-2}{2}} |D^2 c_\varepsilon|^2 + \int_\Omega |\nabla c_\varepsilon|^{2q} + C(q) \quad \textit{for all } t > 0 \textit{ and } \varepsilon \in (0, 1). \end{aligned} \tag{3.30}$$

PROOF. From Lemma 3.4 we know that if again we let κ denote the maximal curvature on $\partial\Omega$, then since

$$\Delta c_\varepsilon = c_{\varepsilon t} + F_\varepsilon(n_\varepsilon)c_\varepsilon + u_\varepsilon \cdot \nabla c_\varepsilon = F_\varepsilon(n_\varepsilon)c_\star \qquad \text{on } \partial\Omega \times (0, \infty) \text{ for all } \varepsilon \in (0, 1)$$

by (2.7), we have

$$\begin{aligned}
\frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu} &\le 2\frac{\partial c_\varepsilon}{\partial\nu}\Delta c_\varepsilon + 2\kappa\Big|\frac{\partial c_\varepsilon}{\partial\nu}\Big|^2 \\
&= 2c_\star F_\varepsilon(n_\varepsilon)\frac{\partial c_\varepsilon}{\partial\nu} + 2\kappa\Big|\frac{\partial c_\varepsilon}{\partial\nu}\Big|^2 \qquad \text{on } \partial\Omega\times(0,\infty) \text{ for all } \varepsilon\in(0,1)
\end{aligned}$$

and hence

$$\begin{aligned}
&\frac{q}{2}\int_{\partial\Omega}\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}\frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu} - q\int_{\partial\Omega}F_\varepsilon(n_\varepsilon)c_\varepsilon\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}\frac{\partial c_\varepsilon}{\partial\nu} \\
&\qquad\le q\kappa\int_{\partial\Omega}\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q}{2}} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1), \qquad (3.31)
\end{aligned}$$

because $|\nabla c_\varepsilon| = \big|\frac{\partial c_\varepsilon}{\partial\nu}\big|$ on $\partial\Omega\times(0,\infty)$ for all $\varepsilon\in(0,1)$. Here the integral on the right can be controlled by a standard argument: Using the continuity of the trace embedding from $W^{\frac{3}{4},2}(\Omega)$ into $L^2(\partial\Omega)$, and compactness of the inclusion $W^{1,2}(\Omega)\hookrightarrow W^{\frac{3}{4},2}(\Omega)$, we find $C_1=C_1(q)>0$ and $C_2=C_2(q)>0$ such that

$$\begin{aligned}
q\kappa\int_{\partial\Omega}\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q}{2}} &= q\kappa\Big\|\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q}{4}}\Big\|_{L^2(\partial\Omega)}^2 \\
&\le C_1\Big\|\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q}{4}}\Big\|_{W^{\frac{3}{4},2}(\Omega)}^2 \\
&\le \frac{q-1}{q}\Big\|\nabla\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q}{4}}\Big\|_{L^2(\Omega)}^2 + C_2\Big\|\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q}{4}}\Big\|_{L^2(\Omega)}^2 \\
&= \frac{q(q-1)}{4}\int_\Omega\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-4}{2}}|D^2c_\varepsilon\cdot\nabla c_\varepsilon|^2 \\
&\quad +C_2\int_\Omega\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q}{2}} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1). \qquad (3.32)
\end{aligned}$$

Since

$$\int_\Omega\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-4}{2}}|D^2c_\varepsilon\cdot\nabla c_\varepsilon|^2 \le \int_\Omega\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}|D^2c_\varepsilon|^2 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),$$

and since Young's inequality ensures that

$$\begin{aligned}
C_2\int_\Omega\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q}{2}} &\le C_2\int_\Omega|\nabla c_\varepsilon|^q + C_2|\Omega| \\
&\le \int_\Omega|\nabla c_\varepsilon|^{2q} + \frac{C_2^2|\Omega|}{4} + C_2|\Omega| \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),
\end{aligned}$$

a combination of (3.31) with (3.32) establishes (3.30). □

3.5 Controlling forces in the Navier–Stokes energy inequality

Our derivation of basic fluid regularity features, apparently accessible essentially through the standard Navier–Stokes energy inequality only, can now be accomplished in a fairly straightforward manner, and under an assumption on p less restrictive than that from Lemma 3.3.

Lemma 3.6 *Assume that $m > 1$ and that $p > 1$ is such that*

$$p > \frac{5}{3} - m, \tag{3.33}$$

and let $K > 0$. Then there exists $C(K,p) > 0$ such that if $c_\star \geq 0$ and (1.6) holds with $\int_\Omega n_0 \leq K$,

$$\begin{aligned}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \frac{1}{C(K,p)}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 \\ \leq \frac{p(p-1)}{8}\int_\Omega (n_\varepsilon+\varepsilon)^{p+m-3}|\nabla n_\varepsilon|^2 + C(K,p) \qquad \textit{for all } t>0 \textit{ and } \varepsilon \in (0,1).\end{aligned} \tag{3.34}$$

Proof. In a standard manner, we test the approximate Navier–Stokes subsystem of (2.7) against u_ε to see that due to (2.8) and the continuity of the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, there exists $C_1 > 0$ such that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 &= \int_\Omega F_\varepsilon(n_\varepsilon)u_\varepsilon \cdot \nabla\phi \\ &\leq \|\nabla\phi\|_{L^\infty(\Omega)}\|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}\|u_\varepsilon\|_{L^6(\Omega)} \\ &\leq \frac{1}{4}\int_\Omega |\nabla u_\varepsilon|^2 + C_1\|n_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1).\end{aligned} \tag{3.35}$$

Here we use that (3.33) guarantees that $\theta := \frac{p+m-1}{2(3p+3m-4)}$ is well-defined with $\theta \in (0,1)$, and that moreover

$$\frac{4\theta}{p+m-1} = \frac{2}{3p+3m-4} < 2,$$

so that a Gagliardo–Nirenberg inequality in conjunction with (2.11), our hypothesis $\int_\Omega n_0 \leq K$ and Young's inequality shows that with some positive constants $C_2 = C_2(p), C_3 = C_3(K,p)$ and $C_4 = C_4(K,p)$ we have

$$\begin{aligned}C_1\|n_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} &\leq C_1\|n_\varepsilon+\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} \\ &= C_1\Big\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\Big\|^{\frac{4}{p+m-1}}_{L^{\frac{12}{5(p+m-1)}}(\Omega)} \\ &\leq C_2\Big\|\nabla(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\Big\|^{\frac{4\theta}{p+m-1}}_{L^2(\Omega)}\Big\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\Big\|^{\frac{4(1-\theta)}{p+m-1}}_{L^{\frac{2}{p+m-1}}(\Omega)} \\ &\quad + C_2\Big\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\Big\|^{\frac{4}{p+m-1}}_{L^{\frac{2}{p+m-1}}(\Omega)} \\ &\leq C_3\Big\|\nabla(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\Big\|^{\frac{4\theta}{p+m-1}}_{L^2(\Omega)} + C_3 \\ &\leq \frac{p(p-1)}{4(p+m-1)^2}\Big\|\nabla(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\Big\|^2_{L^2(\Omega)} + C_4 \\ &= \frac{p(p-1)}{16}\int_\Omega (n_\varepsilon+\varepsilon)^{p+m-3}|\nabla n_\varepsilon|^2 + C_4 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1).\end{aligned} \tag{3.36}$$

We thus only need to recall that a Poncaré inequality provides $C_5 > 0$ fulfilling

$$\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 \ge C_5 \int_\Omega |u_\varepsilon|^2 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),$$

to conclude (3.34) from (3.35) and (3.36) upon letting $C(K,p) := \max\{\frac{1}{C_5}\,,\, 2C_4\}$. □

3.6 Adding three functionals for (p,q) close to $(3m-\frac{7}{3},2)$. The condition $m>\frac{7}{6}$

Now, due to the circumstance that the second-order functional on the right of (3.19) contains the inverse power $\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}$ of $|\nabla c_\varepsilon|^2+1$ as a potentially strength-diminishing factor, utilizing the dissipative action encoded therein to control the first-order expressions $\int_\Omega |\nabla c_\varepsilon|^{2q}$ seen on the right-hand sides of the inequalities from Lemma 3.2, Lemma 3.3, Lemma 3.5 and Lemma 3.6 seems impossible on the basis of standard interpolation features reported in the literature, such as those formulated in [43] and [47], for instance. We therefore prepare our argument in this direction by deriving a functional inequality precisely designed for such situations.

Lemma 3.7 *Let $q\in(1,2)$. Then there exists $C(q)>0$ such that for each $\varphi\in C^2(\overline{\Omega})$ with $\varphi=0$ on $\partial\Omega$,*

$$\begin{aligned}\int_\Omega |\nabla\varphi|^{2q} \ &\le\ C\cdot\bigg\{\int_\Omega \big(|\nabla\varphi|^2+1\big)^{\frac{q-2}{2}}|D^2\varphi|^2\bigg\}^{\frac{2q}{q+2}}\cdot\bigg\{\int_\Omega|\varphi|^{\frac{4q}{2-q}}\bigg\}^{\frac{2-q}{q+2}} \\ &\quad +C\cdot\bigg\{\int_\Omega \big(|\nabla\varphi|^2+1\big)^{\frac{q-2}{2}}|D^2\varphi|^2\bigg\}^{\frac{q}{2}}\cdot\bigg\{\int_\Omega|\varphi|^{\frac{4q}{2-q}}\bigg\}^{\frac{2-q}{2}}. \qquad (3.37)\end{aligned}$$

Proof. Using that $\varphi|_{\partial\Omega}=0$, we integrate by parts and apply the pointwise inequality $|\Delta\varphi|\le\sqrt{3}|D^2\varphi|$ as well as the Hölder inequality to see that writing $C_1:=2(q-1)+\sqrt{3}$ as well as $I(\varphi):=\int_\Omega|\nabla\varphi|^{2q}$ and $J(\varphi):=\int_\Omega(|\nabla\varphi|^2+1)^{\frac{q-2}{2}}|D^2\varphi|^2$ we have

$$\begin{aligned} I(\varphi) &= -\int_\Omega \varphi\nabla\cdot\big(|\nabla\varphi|^{2q-2}\nabla\varphi\big) \\ &= -2(q-1)\int_\Omega \varphi|\nabla\varphi|^{2q-4}\nabla\varphi\cdot(D^2\varphi\cdot\nabla\varphi) - \int_\Omega\varphi|\nabla\varphi|^{2q-2}\Delta\varphi \\ &\le C_1\int_\Omega|\varphi|\,|\nabla\varphi|^{2q-2}|D^2\varphi| \\ &\le C_1 J^{\frac12}(\varphi)\cdot\bigg\{\int_\Omega\varphi^2\big(|\nabla\varphi|^2+1\big)^{\frac{2-q}{2}}|\nabla\varphi|^{4q-4}\bigg\}^{\frac12}. \end{aligned}$$

As

$$\bigg\{\int_\Omega\varphi^2|\nabla\varphi|^{3q-2}\bigg\}^{\frac12}\le I^{\frac{3q-2}{4q}}(\varphi)\cdot\bigg\{\int_\Omega|\varphi|^{\frac{4q}{2-q}}\bigg\}^{\frac{2-q}{4q}}$$

and

$$\bigg\{\int_\Omega\varphi^2|\nabla\varphi|^{4q-4}\bigg\}^{\frac12}\le I^{\frac{q-1}{q}}(\varphi)\cdot\bigg\{\int_\Omega|\varphi|^{\frac{4q}{2-q}}\bigg\}^{\frac{2-q}{2q}},$$

making use of the fact that $(\xi+\eta)^\alpha \le \xi^\alpha + \eta^\alpha$ holds for $\xi \ge 0$ and $\eta \ge 0$ and with $\alpha \in \{\frac{2-q}{2}, \frac{1}{2}\}$, we may estimate $(|\nabla\varphi|^2+1)^{\frac{2-q}{2}} \le |\nabla\varphi|^{2-q}+1$ and, by moreover utilizing Young's inequality, thus infer that

$$\begin{aligned}
I(\varphi) &\le C_1 J^{\frac{1}{2}}(\varphi) I^{\frac{3q-2}{4q}}(\varphi) \cdot \left\{ \int_\Omega |\varphi|^{\frac{4q}{2-q}} \right\}^{\frac{2-q}{4q}} \\
&\quad + C_1 J^{\frac{1}{2}}(\varphi) I^{\frac{q-1}{q}}(\varphi) \cdot \left\{ \int_\Omega |\varphi|^{\frac{4q}{2-q}} \right\}^{\frac{2-q}{2q}} \\
&= \left\{ \frac{1}{4} I(\varphi) \right\}^{\frac{3q-2}{4q}} \cdot 4^{\frac{3q-2}{4q}} C_1 J^{\frac{1}{2}}(\varphi) \cdot \left\{ \int_\Omega |\varphi|^{\frac{4q}{2-q}} \right\}^{\frac{2-q}{4q}} \\
&\quad + \left\{ \frac{1}{4} I(\varphi) \right\}^{\frac{q-1}{q}} \cdot 4^{\frac{q-1}{q}} C_1 J^{\frac{1}{2}}(\varphi) \cdot \left\{ \int_\Omega |\varphi|^{\frac{4q}{2-q}} \right\}^{\frac{2-q}{2q}} \\
&\le \frac{1}{4} I(\varphi) + \left\{ 4^{\frac{3q-2}{4q}} C_1 J^{\frac{1}{2}}(\varphi) \cdot \left\{ \int_\Omega |\varphi|^{\frac{4q}{2-q}} \right\}^{\frac{2-q}{4q}} \right\}^{\frac{4q}{q+2}} \\
&\quad + \frac{1}{4} I(\varphi) + \left\{ 4^{\frac{q-1}{q}} C_1 J^{\frac{1}{2}}(\varphi) \cdot \left\{ \int_\Omega |\varphi|^{\frac{4q}{2-q}} \right\}^{\frac{2-q}{2q}} \right\}^{q}.
\end{aligned}$$

This implies (3.37) with $C := 2\max\left\{ 4^{\frac{3q-2}{q+2}} C_1^{\frac{4q}{q+2}}, \ 4^{q-1} C_1^q \right\}$. □

We are now in the position to make sure that under the assumption on m from Theorem 1.1, for (p,q) arbitrarily close to $(m-\frac{7}{3}, 2)$ the above inequalities can be combined so as to establish the following which, in light of Lemma 3.1, can in fact be interpreted as revealing an energy-like property.

Lemma 3.8 *Let*

$$m > \frac{7}{6}.$$

Then there exists $p_0 = p_0(m) \in [1, 3m-\frac{7}{3})$ such that for any $p \in (p_0, 3m-\frac{7}{3})$ one can find $q_0 = q_0(p) \in (1,2)$ such that to each $q \in (q_0, 2)$ and any $K > 0$ there correspond some $r = r(p,q) > 2$ and $C(K,p,q) > 0$ with the property that if $c_\star \in [0,K]$ and (1.6) holds with $\int_\Omega n_0 \le K$, then

$$\begin{aligned}
&\frac{d}{dt}\left\{ \int_\Omega (n_\varepsilon+\varepsilon)^p + \int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q}{2}} + \int_\Omega |u_\varepsilon|^2 \right\} \\
&\qquad + \frac{1}{C(K,p,q)} \cdot \left\{ \int_\Omega (n_\varepsilon+\varepsilon)^p + \int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q}{2}} + \int_\Omega |u_\varepsilon|^2 \right\} \\
&\qquad + \frac{p(p-1)}{8} \int_\Omega (n_\varepsilon+\varepsilon)^{p+m-3} |\nabla n_\varepsilon|^2 + \frac{q(q-1)}{8} \int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}} |D^2 c_\varepsilon|^2 + \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 \\
&\le C(K,p,q) \int_\Omega c_\varepsilon^r + C(K,p,q) \qquad \textit{for all } t>0 \textit{ and } \varepsilon \in (0,1). \qquad (3.38)
\end{aligned}$$

PROOF. We fix

$$p_0 := \max\Big\{1\,,\,m-\frac{1}{2}\,,\,2m-2\,,\,\frac{7}{3}-m\Big\}$$

and observe that then, as our assumption $m>\frac{7}{6}$ ensures that $m>\frac{10}{9}, 2m>\frac{11}{6}, m>\frac{1}{3}$ and $4m>\frac{14}{3}$, we have $1\le p_0<3m-\frac{7}{3}$. For $p\in(p_0,3m-\frac{7}{3})$, we may then simultaneously apply Lemma 3.2, Lemma 3.3, Lemma 3.5 and Lemma 3.6 to pick $q_0=q_0(p)\in(1,2)$ and $r_1=r_1(p)>2$ such that whenever $q\in(q_0,2)$ and $K>0$, one can find $C_1=C_1(K,p,q)>0$, $C_2=C_2(K,p,q)>0$ and $C_3=C_3(K,p)>0$ such that if $c_\star\in[0,K]$ and (1.6) holds with $\int_\Omega n_0\le K$,

$$\begin{aligned}\frac{d}{dt}\int_\Omega(n_\varepsilon+\varepsilon)^p+\int_\Omega(n_\varepsilon+\varepsilon)^p+\frac{p(p-1)}{2}\int_\Omega(n_\varepsilon+\varepsilon)^{p+m-3}|\nabla n_\varepsilon|^2\\ \le \int_\Omega|\nabla c_\varepsilon|^{2q}+C_1 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1)\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}\int_\Omega\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q}{2}}+\frac{1}{C_2}\int_\Omega\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q}{2}}+\frac{q(q-1)}{4}\int_\Omega\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}|D^2c_\varepsilon|^2\\ \le \frac{p(p-1)}{4}\int_\Omega(n_\varepsilon+\varepsilon)^{p+m-3}|\nabla n_\varepsilon|^2+3\int_\Omega|\nabla c_\varepsilon|^{2q}\\ +\frac{1}{2}\int_\Omega|\nabla u_\varepsilon|^2+\int_\Omega c_\varepsilon^{r_1}+C_2 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1)\end{aligned}$$

as well as

$$\begin{aligned}\frac{d}{dt}\int_\Omega|u_\varepsilon|^2+\frac{1}{C_3}\int_\Omega|u_\varepsilon|^2+\int_\Omega|\nabla u_\varepsilon|^2\\ \le \frac{p(p-1)}{8}\int_\Omega(n_\varepsilon+\varepsilon)^{p+m-3}|\nabla n_\varepsilon|^2+C_3 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),\end{aligned}$$

so that

$$\begin{aligned}\frac{d}{dt}\Big\{\int_\Omega(n_\varepsilon+\varepsilon)^p+\int_\Omega\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q}{2}}+\int_\Omega|u_\varepsilon|^2\Big\}\\ +C_4\cdot\Big\{\int_\Omega(n_\varepsilon+\varepsilon)^p+\int_\Omega\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q}{2}}+\int_\Omega|u_\varepsilon|^2\Big\}\\ +\frac{p(p-1)}{8}\int_\Omega(n_\varepsilon+\varepsilon)^{p+m-3}|\nabla n_\varepsilon|^2+\frac{q(q-1)}{4}\int_\Omega\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}|D^2c_\varepsilon|^2+\frac{1}{2}\int_\Omega|\nabla u_\varepsilon|^2\\ \le 4\int_\Omega|\nabla c_\varepsilon|^{2q}+\int_\Omega c_\varepsilon^{r_1}+C_1+C_2+C_3 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1)\end{aligned}\tag{3.39}$$

with $C_4:=\min\{1,\frac{1}{C_2},\frac{1}{C_3}\}$. Here the first summand on the right can be estimated by means of Lemma 3.7, which in combination with Young's inequality, namely, shows that with some $C_5=C_5(q)>0$ and $C_6=C_6(q)>0$ we have

$$4\int_\Omega|\nabla c_\varepsilon|^{2q} \;=\; 4\int_\Omega\Big|\nabla(c_\varepsilon-c_\star)\Big|^{2q}$$

$$
\begin{aligned}
&\le C_5 \cdot \Big\{ \int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}} |D^2 c_\varepsilon|^2 \Big\}^{\frac{2q}{q+2}} \cdot \Big\{ \int_\Omega |c_\varepsilon - c_\star|^{\frac{4q}{2-q}} \Big\}^{\frac{2-q}{q+2}} \\
&\quad + C_5 \cdot \Big\{ \int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}} |D^2 c_\varepsilon|^2 \Big\}^{\frac{q}{2}} \cdot \Big\{ \int_\Omega |c_\varepsilon - c_\star|^{\frac{4q}{2-q}} \Big\}^{\frac{2-q}{2}} \\
&\le \frac{q(q-1)}{8} \int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}} |D^2 c_\varepsilon|^2 + 2C_6 \int_\Omega |c_\varepsilon - c_\star|^{\frac{4q}{2-q}}
\end{aligned}
$$

for all $t>0$ and $\varepsilon\in(0,1)$. Since writing $r = r(p,q) := \max\{r_1,\ \frac{4q}{2-q}\}$ we have

$$
\begin{aligned}
2C_6 \int_\Omega |c_\varepsilon - c_\star|^{\frac{4q}{2-q}} &\le 2^{\frac{4q}{2-q}} C_6 \int_\Omega c_\varepsilon^{\frac{4q}{2-q}} + (2c_\star)^{\frac{4q}{2-q}} C_6 |\Omega| \\
&\le 2^{\frac{4q}{2-q}} C_6 \int_\Omega c_\varepsilon^r + 2^{\frac{4q}{2-q}} C_6|\Omega| + (2K)^{\frac{4q}{2-q}} C_6 |\Omega| \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1)
\end{aligned}
$$

as well as

$$
\int_\Omega c_\varepsilon^{r_1} \le \int_\Omega c_\varepsilon^r + |\Omega| \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1)
$$

by Young's inequality and our assumption that $c_\star\in[0,K]$, the claim thus follows from (3.39) if we let

$$
C(K,p,q) := \max\Big\{ \frac{1}{C_4}\,,\ 2^{\frac{4q}{2-q}} C_6 + 1\,,\ 2^{\frac{4q}{2-q}} C_6|\Omega| + (2K)^{\frac{4q}{2-q}} C_6|\Omega| + |\Omega| + C_1 + C_2 + C_3 \Big\},
$$

for instance. □

As the constant appearing on the right of (3.38) depends on $c_\star$ and the initial data only through an upper bound K for $c_\star$ and $\int_\Omega n_0$, in line with Lemma 3.1 our conclusion from Lemma 3.8 can be formulated in such a way that not only regularity features appropriate for our construction of global solutions are documented for $((n_\varepsilon, c_\varepsilon, u_\varepsilon))_{\varepsilon\in(0,1)}$ with each fixed $c_\star$ and (n_0,c_0,u_0), but that moreover already a major step toward the relaxation statement from Theorem 1.2 is accomplished.

Lemma 3.9 *Let $m>\frac{7}{6}$, and with $p_0(m)\in[1,3m-\frac{7}{3})$ and $(q_0(p))_{p\in(p_0(m),3m-\frac{7}{3})}$ taken from Lemma 3.8, suppose that $p\in(p_0(m),3m-\frac{7}{3})$, $q\in(q_0(p),2)$ and $K>0$. Then there exist $C(K,p,q)>0$ and $\lambda(K,p,q)>0$ such that whenever $c_\star\in[0,K]$ and (1.6) holds with $\int_\Omega n_0\le K$, one can fix $\Gamma(p,q,n_0,c_0,u_0)>0$ such that*

$$
\begin{aligned}
&\int_\Omega \big(n_\varepsilon(\cdot,t)+\varepsilon\big)^p + \int_\Omega \Big(|\nabla c_\varepsilon(\cdot,t)|^2+1\Big)^{\frac{q}{2}} + \int_\Omega \big|u_\varepsilon(\cdot,t)\big|^2 \\
&\qquad \le C(K,p,q) + \Gamma(p,q,n_0,c_0,u_0) e^{-\lambda(K,p,q)t} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1)
\end{aligned}
\tag{3.40}
$$

and

$$
\begin{aligned}
&\int_t^{t+1}\int_\Omega \Big\{ (n_\varepsilon+\varepsilon)^{p+m-3}|\nabla n_\varepsilon|^2 + \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q-2}{2}}|D^2c_\varepsilon|^2 + |\nabla u_\varepsilon|^2 \Big\} \\
&\qquad \le C(K,p,q) + \Gamma(p,q,n_0,c_0,u_0) e^{-\lambda(K,p,q)t} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1).
\end{aligned}
\tag{3.41}
$$

PROOF. For fixed $p \in (p_0(m), 3m - \frac{7}{3})$, $q \in (q_0(p), 2)$ and $K > 0$, we invoke Lemma 3.8 to pick $C_i = C_i(K,p,q) > 0$, $i \in \{1,2,3\}$, and $r = r(p,q) > 2$ with the property that if $c_\star \in [0,K]$ and (1.6) holds with $\int_\Omega n_0 \le K$, then for each $\varepsilon \in (0,1)$, the functions y_ε and g_ε defined by letting, for $t \ge 0$,

$$y_\varepsilon(t) := \int_\Omega \big(n_\varepsilon(\cdot,t)+\varepsilon\big)^p + \int_\Omega \Big(|\nabla c_\varepsilon(\cdot,t)|^2+1\Big)^{\frac{q}{2}} + \int_\Omega \big|u_\varepsilon(\cdot,t)\big|^2$$

and, for $t > 0$,

$$g_\varepsilon(t) := \int_\Omega \bigg\{ \big(n_\varepsilon(\cdot,t)+\varepsilon\big)^{p+m-3}\big|\nabla n_\varepsilon(\cdot,t)\big|^2 + \Big(|\nabla c_\varepsilon(\cdot,t)|^2+1\Big)^{\frac{q-2}{2}}\big|D^2 c_\varepsilon(\cdot,t)\big|^2 + \big|\nabla u_\varepsilon(\cdot,t)\big|^2\bigg\},$$

satisfy

$$y_\varepsilon'(t) + C_1 y_\varepsilon(t) + C_2 g_\varepsilon(t) \le C_3 \int_\Omega c_\varepsilon^r + C_3 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1). \tag{3.42}$$

We thereupon choose an integer $k = k(p,q) \ge \frac{r}{2}$ and employ Lemma 3.1 to find $C_4 > 0$ and $\lambda_1 = \lambda_1(p,q) > 0$ such that if $c_\star \ge 0$ and that (1.6) is valid, then there exists $\Gamma_1 = \Gamma_1(p,q,c_0) > 0$ such that

$$\big\|c_\varepsilon(\cdot,t) - c_\star\big\|_{L^{2k}(\Omega)} \le C_4 c_\star \cdot \bigg\{\int_\Omega n_0\bigg\}^{\frac{1}{2k}} + \Gamma_1 e^{-\lambda_1 t} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1). \tag{3.43}$$

Henceforth assuming that $c_\star \in [0,K]$ and that (1.6) is satisfied with $\int_\Omega n_0 \le K$, on the right-hand side of (3.42) we utilize Young's inequality together with (3.43) to see that since $r \le 2k$, with $\Gamma_1 = \Gamma_1(p,q,c_0)$ as above we have

$$\begin{aligned}
C_3 \int_\Omega c_\varepsilon^r + C_3 &\le C_3 \int_\Omega c_\varepsilon^{2k} + C_3|\Omega| + C_3 \\
&\le C_3 \cdot \Big\{\|c_\varepsilon - c_\star\|_{L^{2k}(\Omega)} + c_\star|\Omega|^{\frac{1}{2k}}\Big\}^{2k} + C_3|\Omega| + C_3 \\
&\le C_3 \cdot \bigg\{ C_4 c_\star \cdot \bigg\{\int_\Omega n_0\bigg\}^{\frac{1}{2k}} + \Gamma_1 e^{-\lambda_1 t} + c_\star|\Omega|^{\frac{1}{2k}} \bigg\}^{2k} + C_3|\Omega| + C_3 \\
&\le C_3 \cdot \Big\{ C_4 K^{1+\frac{1}{2k}} + |\Omega|^{\frac{1}{2k}} K + \Gamma_1 e^{-\lambda_1 t}\Big\}^{2k} + C_3|\Omega| + C_3 \\
&\le C_5 + \Gamma_2 e^{-\lambda_2 t} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),
\end{aligned} \tag{3.44}$$

with $C_5 = C_5(K,p,q) := 2^{2k-1}C_3 \cdot \Big\{C_4 K^{1+\frac{1}{2k}} + |\Omega|^{\frac{1}{2k}} K\Big\}^{2k} + C_3|\Omega| + C_3$, $\Gamma_2 = \Gamma_2(p,q,c_0) := 2^{2k-1} C_3 \Gamma_1^{2k}$ and $\lambda_2 = \lambda_2(p,q) := 2k\lambda_1$. An ODE comparison argument applied to (3.42) thus shows that since g_ε is nonnegative,

$$\begin{aligned}
y_\varepsilon(t) &\le y_\varepsilon(0)e^{-C_1 t} + \int_0^t e^{-C_1(t-s)} \cdot \big(C_5 + \Gamma_2 e^{-\lambda_2 s}\big) ds \\
&= y_\varepsilon(0)e^{-C_1 t} + \frac{C_5}{C_1}(1 - e^{-C_1 t}) + \Gamma_2 e^{-C_1 t}\int_0^t e^{(C_1-\lambda_2)s} ds \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),
\end{aligned}$$

where abbreviating $\lambda_3 = \lambda_3(K,p,q) := \min\{\lambda_2, \frac{C_1}{2}\}$ we see that for all $t > 0$,

$$e^{-C_1 t}\int_0^t e^{(C_1-\lambda_2)s}ds \le e^{-C_1 t}\int_0^t e^{(C_1-\lambda_3)s}ds \quad = \quad \frac{1}{C_1-\lambda_3}(e^{-\lambda_3 t} - e^{-C_1 t}) \le \frac{1}{C_1-\lambda_3}e^{-\lambda_3 t},$$

so that

$$y_\varepsilon(t) \le \frac{C_5}{C_1} + \Gamma_3 e^{-\lambda_3 t} \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1)$$

with $\Gamma_3 = \Gamma_3(p,q,n_0,c_0,u_0) := \int_\Omega (n_0+1)^p + \int_\Omega \big(|\nabla c_0|^2+1\big)^{\frac{q}{2}} + \int_\Omega |u_0|^2 + \frac{\Gamma_2}{C_1-\lambda_3}$. Thereafter, a direct integration of (3.42) reveals that once more due to (3.44),

$$\begin{aligned} C_2\int_t^{t+1} g_\varepsilon(s)ds &\le y_\varepsilon(t) + \int_t^{t+1}\big(C_5 + \Gamma_2 e^{-\lambda_2 s}\big)ds \\ &\le \frac{C_5}{C_1} + \Gamma_3 e^{-\lambda_3 t} + C_5 + \Gamma_2 e^{-\lambda_2 t} \\ &\le \frac{C_5}{C_1} + C_5 + (\Gamma_2+\Gamma_3)e^{-\lambda_3 t} \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1), \end{aligned}$$

whence both (3.40) and (3.41) follow upon obvious choices of the constants $C(K,p,q)$, $\Gamma(p,q,n_0,c_0,u_0)$ and $\lambda(K,p,q)$. $\square$

4 Further compactness features. Time regularity

In order to prepare an Aubin–Lions type argument guaranteeing approximation properties also at a level of pointwise convergence a.e. in $\Omega\times(0,\infty)$, the next lemma once again returns to the first equation in (2.7) to provide bounds on first-order derivatives, both in space and in time, for an appropriate power of $n_\varepsilon + \varepsilon$ within fixed time intervals of finite length.

Lemma 4.1 *Let $m > \frac{7}{6}$, and suppose that $c_\star \ge 0$ and that (1.6) is valid. Then for all $T > 0$ there exists $C(T,n_0,c_0,u_0) > 0$ such that*

$$\int_0^T\int_\Omega \Big|\nabla(n_\varepsilon+\varepsilon)^{m-1}\Big|^2 \le C(T,n_0,c_0,u_0) \qquad \textit{for all } \varepsilon \in (0,1) \tag{4.1}$$

and

$$\int_0^T \Big\|\partial_t\big(n_\varepsilon(\cdot,t)+\varepsilon\big)^{m-1}\Big\|_{(W^{3,2}(\Omega))^\star} dt \le C(T,n_0,c_0,u_0) \qquad \textit{for all } \varepsilon \in (0,1). \tag{4.2}$$

Proof. For $\varepsilon \in (0,1)$ and $\xi \ge 0$, we let

$$\Phi_\varepsilon(\xi) := \begin{cases} \frac{1}{(m-1)(m-2)}(\xi+\varepsilon)^{m-1} & \text{if } m \ne 2, \\ (\xi+\varepsilon)\ln(\xi+\varepsilon) & \text{if } m = 2, \end{cases}$$

and observe that in both these cases we have $\Phi_\varepsilon'' = (\xi)(\xi+\varepsilon)^{m-3}$ for all $\xi \ge 0$ and $\varepsilon \in (0,1)$, whence using (2.7) we find that by Young's inequality and (2.8),

$$\frac{d}{dt}\int_\Omega \Phi_\varepsilon(n_\varepsilon) \quad = \quad -\int_\Omega (n_\varepsilon+\varepsilon)^{2m-4}|\nabla n_\varepsilon|^2 + \int_\Omega n_\varepsilon F_\varepsilon'(n_\varepsilon)(n_\varepsilon+\varepsilon)^{m-3}\nabla n_\varepsilon\cdot\nabla c_\varepsilon$$

$$
\begin{aligned}
&\le -\frac{1}{2}\int_\Omega (n_\varepsilon+\varepsilon)^{2m-4}|\nabla n_\varepsilon|^2 + \frac{1}{2}\int_\Omega \frac{n_\varepsilon^2 F_\varepsilon'^2(n_\varepsilon)}{(n_\varepsilon+\varepsilon)^2}|\nabla c_\varepsilon|^2 \\
&\le -\frac{1}{2}\int_\Omega (n_\varepsilon+\varepsilon)^{2m-4}|\nabla n_\varepsilon|^2 + \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^2 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1).
\end{aligned}
$$

Therefore,

$$
\frac{1}{2}\int_0^T\int_\Omega (n_\varepsilon+\varepsilon)^{2m-4}|\nabla n_\varepsilon|^2 \le \int_\Omega \Phi_\varepsilon(n_0) - \int_\Omega \Phi_\varepsilon\big(n_\varepsilon(\cdot,T)\big) + \frac{1}{2}\int_0^T\int_\Omega |\nabla c_\varepsilon|^2
$$

for all $T>0$ and $\varepsilon\in(0,1)$, whence (4.1) results upon recalling Lemma 3.1 and observing that if $m<2$ then

$$
\begin{aligned}
&\int_\Omega \Phi_\varepsilon(n_0) - \int_\Omega \Phi_\varepsilon\big(n_\varepsilon(\cdot,T)\big) \\
&\qquad = -\frac{1}{(m-1)(2-m)}\int_\Omega (n_0+\varepsilon)^{m-1} + \frac{1}{(m-1)(2-m)}\int_\Omega \big(n_\varepsilon(\cdot,T)+\varepsilon\big)^{m-1} \\
&\qquad \le \frac{|\Omega|^{2-m}}{(m-1)(2-m)}\cdot\Big\{\int_\Omega \big(n_\varepsilon(\cdot,T)+1\big)\Big\}^{m-1} \\
&\qquad = \frac{|\Omega|^{2-m}}{(m-1)(2-m)}\cdot\Big\{\int_\Omega (n_0+1)\Big\}^{m-1} \qquad \text{for all } T>0 \text{ and } \varepsilon\in(0,1),
\end{aligned}
$$

by the Hölder inequality and (2.11), that if $m>2$ then

$$
\int_\Omega \Phi_\varepsilon(n_0) - \int_\Omega \Phi_\varepsilon\big(n_\varepsilon(\cdot,T)\big) \le \frac{1}{(m-1)(m-2})\int_\Omega (n_0+1)^{m-1} \qquad \text{for all } T>0 \text{ and } \varepsilon\in(0,1),
$$

amd that in the case when $m=2$,

$$
\int_\Omega \Phi_\varepsilon(n_0) - \int_\Omega \Phi_\varepsilon\big(n_\varepsilon(\cdot,T)\big) \le (n_0+1)\ln(n_0+1) + \frac{|\Omega|}{e} \qquad \text{for all } T>0 \text{ and } \varepsilon\in(0,1),
$$

due to the fact that $\xi\ln\xi \ge -\frac{1}{e}$ for all $\xi>0$.

To derive (4.2), we fix $\psi\in C^\infty(\overline{\Omega})$ and use (2.7) and again (2.8) in estimating

$$
\begin{aligned}
&\left|\frac{1}{m-1}\int_\Omega \partial_t (n_\varepsilon+\varepsilon)^{m-1}\psi\right| \\
&\quad = \bigg|\int_\Omega (n_\varepsilon+\varepsilon)^{m-2}\nabla\cdot\Big\{(n_\varepsilon+\varepsilon)^{m-1}\nabla n_\varepsilon - n_\varepsilon F_\varepsilon'(n_\varepsilon)\nabla c_\varepsilon\Big\}\psi \\
&\qquad\qquad + \int_\Omega (n_\varepsilon+\varepsilon)^{m-2}(u_\varepsilon\cdot\nabla n_\varepsilon)\psi\bigg| \\
&\quad = \bigg| -(m-2)\int_\Omega (n_\varepsilon+\varepsilon)^{2m-4}|\nabla n_\varepsilon|^2\psi + (m-2)\int_\Omega n_\varepsilon F_\varepsilon'(n_\varepsilon)(n_\varepsilon+\varepsilon)^{m-3}(\nabla n_\varepsilon\cdot\nabla c_\varepsilon)\psi \\
&\qquad - \int_\Omega (n_\varepsilon+\varepsilon)^{2m-3}\nabla n_\varepsilon\cdot\nabla\psi + \int_\Omega n_\varepsilon F_\varepsilon'(n_\varepsilon)(n_\varepsilon+\varepsilon)^{m-2}\nabla c_\varepsilon\cdot\nabla\psi
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{m-1}\int_\Omega (n_\varepsilon+\varepsilon)^{m-1}u_\varepsilon\cdot\nabla\psi\Bigg| \\
\le\ & |m-2|\cdot\Big\{\int_\Omega (n_\varepsilon+\varepsilon)^{2m-4}|\nabla n_\varepsilon|^2\Big\}\cdot\|\psi\|_{L^\infty(\Omega)} \\
&+|m-2|\cdot\Big\{\int_\Omega (n_\varepsilon+\varepsilon)^{2m-4}|\nabla n_\varepsilon|^2\Big\}^{\frac12}\cdot\Big\{\int_\Omega |\nabla c_\varepsilon|^2\Big\}^{\frac12}\cdot\|\psi\|_{L^\infty(\Omega)} \\
&+\Big\{\int_\Omega (n_\varepsilon+\varepsilon)^{2m-4}|\nabla n_\varepsilon|^2\Big\}^{\frac12}\cdot\Big\{\int_\Omega (n_\varepsilon+\varepsilon)^{2m-2}\Big\}^{\frac12}\cdot\|\nabla\psi\|_{L^\infty(\Omega)} \\
&+\Big\{\int_\Omega (n_\varepsilon+\varepsilon)^{2m-2}\Big\}^{\frac12}\cdot\Big\{\int_\Omega |\nabla c_\varepsilon|^2\Big\}^{\frac12}\cdot\|\nabla\psi\|_{L^\infty(\Omega)} \\
&+\frac{1}{m-1}\cdot\Big\{\int_\Omega (n_\varepsilon+\varepsilon)^{2m-2}\Big\}^{\frac12}\cdot\Big\{\int_\Omega |u_\varepsilon|^2\Big\}^{\frac12}\cdot\|\nabla\psi\|_{L^\infty(\Omega)} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1).
\end{aligned}
$$

As $W^{3,2}(\Omega)\hookrightarrow W^{1,\infty}(\Omega)$, in line with Young's inequality this ensures the existence of $C_1>0$ such that

$$
\begin{aligned}
\Big\|\partial_t\big(n_\varepsilon(\cdot,t)+\varepsilon\big)^{m-1}\Big\|_{(W^{3,2}(\Omega))^\star} \le\ & C_1\cdot\Big\{\int_\Omega (n_\varepsilon+\varepsilon)^{2m-4}|\nabla n_\varepsilon|^2+\int_\Omega|\nabla c_\varepsilon|^2 \\
&+\int_\Omega (n_\varepsilon+\varepsilon)^{2m-2}+\int_\Omega |u_\varepsilon|^2\Big\}
\end{aligned}
$$

for all $t>0$ and $\varepsilon\in(0,1)$, so that since a Poincaré inequality provides $C_2>0$ fulfilling

$$
\begin{aligned}
\int_\Omega (n_\varepsilon+\varepsilon)^{2m-2} =\ & \big\|(n_\varepsilon+\varepsilon)^{m-1}\big\|_{L^2(\Omega)}^2 \\
\le\ & C_2\Big\|\nabla(n_\varepsilon+\varepsilon)^{m-1}\Big\|_{L^2(\Omega)}^2+C_2\big\|(n_\varepsilon+\varepsilon)^{m-1}\big\|_{L^{\frac{1}{m-1}}(\Omega)}^2 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),
\end{aligned}
$$

the estimate in (4.2) follows from that in (4.1) when combined with (2.11), (3.1) and (3.40). $\square$

For the second and third solution components, similar estimates will be needed only for the respective time derivatives:

Lemma 4.2 *Let $m>\frac76$ and $s>3$, and assume that $c_\star\ge0$ and that (1.6) holds. Then for all $T>0$ there exists $C(T,n_0,c_0,u_0)>0$ such that*

$$
\int_0^T \big\|c_{\varepsilon t}(\cdot,t)\big\|_{(W_0^{1,s}(\Omega))^\star}^2 dt \le C(T,n_0,c_0,u_0) \qquad \textit{for all } \varepsilon\in(0,1) \tag{4.3}
$$

and

$$
\int_0^T \big\|u_{\varepsilon t}(\cdot,t)\big\|_{(W_{0,\sigma}^{1,s}(\Omega))^\star}^2 dt \le C(T,n_0,c_0,u_0) \qquad \textit{for all } \varepsilon\in(0,1). \tag{4.4}
$$

Proof. For fixed $\psi\in C_0^\infty(\Omega)$ and any $t>0$ and $\varepsilon\in(0,1)$,

$$
\Big|\int_\Omega c_{\varepsilon t}\psi\Big| = \Big|-\int_\Omega\nabla c_\varepsilon\cdot\nabla\psi-\int_\Omega F_\varepsilon(n_\varepsilon)c_\varepsilon\psi+\int_\Omega c_\varepsilon u_\varepsilon\cdot\nabla\psi\Big|
$$

$$\leq \|\nabla c_\varepsilon\|_{L^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega)} + \|n_\varepsilon\|_{L^1(\Omega)}\|c_\varepsilon\|_{L^\infty(\Omega)}\|\psi\|_{L^\infty(\Omega)} + \|c_\varepsilon\|_{L^\infty(\Omega)}\|u_\varepsilon\|_{L^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega)},$$

again according to (2.8). As $W^{1,s}(\Omega) \hookrightarrow L^\infty(\Omega)$, this entails the existence of $C_1 > 0$ such that for all $t > 0$ and $\varepsilon \in (0,1)$,

$$\big\|c_{\varepsilon t}(\cdot,t)\big\|^2_{(W_0^{1,s}(\Omega))^\star} \leq C_1 \cdot \Big\{\|\nabla c_\varepsilon\|^2_{L^2(\Omega)} + \|n_\varepsilon\|^2_{L^1(\Omega)}\|c_\varepsilon\|^2_{L^\infty(\Omega)} + \|c_\varepsilon\|^2_{L^\infty(\Omega)}\|u_\varepsilon\|^2_{L^2(\Omega)}\Big\},$$

whence (4.3) results from Lemma 3.1, (2.11), (2.12) and (3.40).

Similarly, for $\psi \in C_0^\infty(\Omega;\mathbb{R}^3)$ with $\nabla\cdot\psi = 0$ we have

$$\begin{aligned}
\Big|\int_\Omega u_\varepsilon(\cdot,t)\cdot\psi\Big| &= \Big|-\int_\Omega \nabla u_\varepsilon\cdot\nabla\psi + \int_\Omega F_\varepsilon(n_\varepsilon)\nabla\phi\cdot\psi - \int_\Omega \big\{(Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon\cdot\psi\Big| \\
&\leq \|\nabla u_\varepsilon\|_{L^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega)} + \|\nabla\phi\|_{L^\infty(\Omega)}\|n_\varepsilon\|_{L^1(\Omega)}\|\psi\|_{L^\infty(\Omega)} \\
&\quad + \|Y_\varepsilon u_\varepsilon\|_{L^2(\Omega)}\|\nabla u_\varepsilon\|_{L^2(\Omega)}\|\psi\|_{L^\infty(\Omega)} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),
\end{aligned}$$

so that thanks to the boundedness of $\nabla\phi$, once more using that $s > 3$ we find $C_2 > 0$ such that for all $t > 0$ and $\varepsilon \in (0,1)$,

$$\big\|u_{\varepsilon t}(\cdot,t)\big\|^2_{(W_{0,\sigma}^{1,s}(\Omega))^\star} \leq C_2 \cdot \Big\{\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + \|n_\varepsilon\|^2_{L^1(\Omega)} + \|Y_\varepsilon u_\varepsilon\|^2_{L^2(\Omega)}\|\nabla u_\varepsilon\|^2_{L^2(\Omega)}\Big\}.$$

Since Y_ε is nonexpansive on $L^2_\sigma(\Omega)$, we thus obtain (4.4) as a consequence of (3.41), (2.11) and (3.40). □

Even if pointwise convergence toward a limit object can already be attained from an Aubin–Lions type argument, the precompactness features entailed by the bounds from Lemma 3.9 seem yet insufficient for the requirements in Definition 2.1. By a straightforward interpolation, the following slightly enriches our knowledge on n_ε in this direction, unlike the previous lemmata referring to the quantities $(n_\varepsilon+\varepsilon)^m$ of explicit relevance in (2.3) and (2.4).

Lemma 4.3 *Let $m > \frac{7}{6}$. Then there exists $\theta = \theta(m) > 1$ such that whenever $c_\star \geq 0$ and (1.6) holds, for all $T > 0$ one can find $C = C(T, n_0, c_0, u_0) > 0$ such that*

$$\int_0^T \int_\Omega \Big|\nabla(n_\varepsilon+\varepsilon)^m\Big|^\theta \leq C(T,n_0,c_0,u_0) \qquad \textit{for all } \varepsilon\in(0,1). \tag{4.5}$$

Proof. As the inequality $m > \frac{7}{6}$ ensures that $3m - \frac{7}{3} > m$, we can pick $p = p(m) \in (m, m+1)$ such that $p < 3m - \frac{7}{3}$, and then choose any $p_\star = p_\star(m) \in (p, 3m-\frac{7}{3})$ such that with $p_0(m)$ taken from Lemma 3.8 we have $p_\star > p_0$. Then since $m < p < m+1$, the number $\theta = \theta(m) := \frac{2}{m+2-p}$ satisfies $1 < \theta < 2$, whence we may employ Young's inequality to see that if $c_\star \geq 0$ and (1.6) holds, then according to (2.11), for all $T > 0$ and $\varepsilon \in (0,1)$ we have

$$\begin{aligned}
\int_0^T\int_\Omega \Big|\nabla(n_\varepsilon+\varepsilon)^m\Big|^\theta &= m^\theta \int_0^T\int_\Omega \Big\{(n_\varepsilon+\varepsilon)^{p+m-3}|\nabla n_\varepsilon|^2\Big\}^{\frac{\theta}{2}} \cdot (n_\varepsilon+\varepsilon)^{\frac{(-p+m+1)\theta}{2}} \\
&\leq m^\theta \int_0^T\int_\Omega (n_\varepsilon+\varepsilon)^{p+m-3}|\nabla n_\varepsilon|^2 + m^\theta \int_0^T\int_\Omega (n_\varepsilon+\varepsilon)^{\frac{(-p+m+1)\theta}{2-\theta}}
\end{aligned}$$

$$= m^\theta \int_0^T \int_\Omega (n_\varepsilon+\varepsilon)^{p+m-3}|\nabla n_\varepsilon|^2 + m^\theta T \int_\Omega (n_0+\varepsilon), \tag{4.6}$$

because $\frac{(-p+m+1)\theta}{2-\theta}=1$. Using that $m-1<p<p_\star$, we may here again invoke Young's inequality to estimate

$$\int_0^T \int_\Omega (n_\varepsilon+\varepsilon)^{p+m-3}|\nabla n_\varepsilon|^2 \le \int_0^T \int_\Omega (n_\varepsilon+\varepsilon)^{2m-4}|\nabla n_\varepsilon|^2 + \int_0^T \int_\Omega (n_\varepsilon+\varepsilon)^{p_\star+m-3}|\nabla n_\varepsilon|^2 \tag{4.7}$$

for all $T>0$ and $\varepsilon\in(0,1)$, so that (4.5) results from (4.6) upon a combination of Lemma 4.1 with Lemma 3.9, the latter being applicable to the rightmost integral in (4.7) due to the inclusion $p_\star \in (p_0(m), 3m-\frac{7}{3})$. $\Box$

Once more by interpolation, from Lemma 3.9 we can moreover derive bounds for n_ε itself in space-time Lebesgue norms at suitably high integrability level.

Lemma 4.4 *Let $m>\frac{7}{6}$, $c_\star\ge 0$ and $s\in[1,6m-\frac{44}{9})$, and assume (1.6). Then for any $T>0$ there exists $C(T,s,n_0,c_0,u_0)>0$ such that*

$$\int_0^T \int_\Omega (n_\varepsilon+\varepsilon)^s \le C(T,s,n_0,c_0,u_0) \qquad \textit{for all } \varepsilon\in(0,1). \tag{4.8}$$

Proof. As $3m-\frac{7}{3}<6m-\frac{44}{9}$, without loss of generality we may assume that $s>3m-\frac{7}{3}$, and noting that our hypothesis ensures that

$$\frac{3s-3m+3}{5} < \frac{3\cdot(6m-\frac{44}{9})-3m+3}{5} = 3m-\frac{7}{3},$$

with $p_0(m)\in(1,3m-\frac{7}{3})$ taken from Lemma 3.8 we can choose $p=p(s)\in(p_0(m),3m-\frac{7}{3})$ in such a way that $p\ge\frac{3s-3m+3}{5}$. Therefore,

$$p<3m-\frac{7}{3}<s\le\frac{5p+3m-3}{3}<3(p+m-1),$$

so that $\theta := \frac{3(p+m-1)(s-p)}{(2p+3m-3)s}$ satisfies $0<\theta<1$ as well as

$$\frac{2s\theta}{p+m-1} = 2\cdot\frac{3(s-p)}{2p+3m-3} \le 2\cdot\frac{3\cdot\left(\frac{5p+3m-3}{3}-p\right)}{2p+3m-3} = 2.$$

We may hence rely on the Gagliardo–Nirenberg inequality and thereafter apply (3.40) and Young's inequality to see that with some $C_1=C_1(s)>0$ and $C_2=C_2(s,n_0,c_0,u_0)>0$,

$$\begin{aligned}
\int_\Omega (n_\varepsilon+\varepsilon)^s &= \left\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\right\|_{L^{\frac{2s}{p+m-1}}(\Omega)}^{\frac{2s}{p+m-1}} \\
&\le C_1\left\|\nabla(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\right\|_{L^2(\Omega)}^{\frac{2s\theta}{p+m-1}} \left\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\right\|_{L^{\frac{2p}{p+m-1}}(\Omega)}^{\frac{2s(1-\theta)}{p+m-1}} \\
&\quad +C_1\left\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\right\|_{L^{\frac{2p}{p+m-1}}(\Omega)}^{\frac{2s}{p+m-1}}
\end{aligned}$$

$$\leq C_2 \Big\| \nabla (n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}} \Big\|_{L^2(\Omega)}^{\frac{2s\theta}{p+m-1}} + C_2$$
$$\leq C_2 \Big\| \nabla (n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}} \Big\|_{L^2(\Omega)}^{2} + 2C_2 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1).$$

In light of (3.41), the claim thus results upon an integration in time. □

5 Construction of weak solutions. Proof of Theorem 1.1

Having the estimates from the previous two sections at hand, we can now rely on an Aubin–Lions type lemma to obtain limit functions which do not only comply with the regularity requirements imposed on a weak solution by Definition 2.1, but which also satisfy the integral identities (2.4)–(2.6).

Lemma 5.1 *Let $m>\frac{7}{6}$. Then there exist $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ and functions n,c and u on $\Omega\times(0,\infty)$ such that $\varepsilon_j\searrow 0$ as $j\to\infty$, that the inclusions in (1.7) hold with $n\geq 0$ and $c\geq 0$ a.e. in $\Omega\times(0,\infty)$, and that as $\varepsilon=\varepsilon_j\searrow 0$ we have*

$$(n_\varepsilon,c_\varepsilon,u_\varepsilon)\to(n,c,u) \qquad a.e.\ in\ \Omega\times(0,\infty), \tag{5.1}$$
$$n_\varepsilon\to n \qquad in\ L^s_{loc}(\overline{\Omega}\times[0,\infty)) \quad for\ all\ s\in[1,6m-\tfrac{44}{9}), \tag{5.2}$$
$$\nabla(n_\varepsilon+\varepsilon)^m \rightharpoonup \nabla n^m \qquad in\ L^1_{loc}(\overline{\Omega}\times[0,\infty)), \tag{5.3}$$
$$c_\varepsilon \overset{\star}{\rightharpoonup} c \qquad in\ L^\infty(\Omega\times(0,\infty)), \tag{5.4}$$
$$\nabla c_\varepsilon \rightharpoonup \nabla c \qquad in\ L^2_{loc}(\overline{\Omega}\times[0,\infty)), \tag{5.5}$$
$$u_\varepsilon\to u \quad in\ L^2_{loc}(\overline{\Omega}\times[0,\infty)) \quad and \tag{5.6}$$
$$u_\varepsilon(\cdot,t)\to u(\cdot,t)\ in\ L^2(\Omega)\ for\ a.e.\ t>0, \qquad and \tag{5.7}$$
$$\nabla u_\varepsilon \rightharpoonup \nabla u \qquad in\ L^2_{loc}(\overline{\Omega}\times[0,\infty)). \tag{5.8}$$

Moreover, (n,c,u) is a global weak solution of (1.5) in the sense of Definition 2.1.

Proof. Let $T>0$. Then according to Lemma 4.1, (2.11) and a Poincaré inequality,

$$\Big((n_\varepsilon+\varepsilon)^{m-1}\Big)_{\varepsilon\in(0,1)} \text{ is bounded in } L^2\big((0,T);W^{1,2}(\Omega)\big)$$

and

$$\Big(\partial_t(n_\varepsilon+\varepsilon)^{m-1}\Big)_{\varepsilon\in(0,1)} \text{ is bounded in } L^1\Big((0,T);\big(W^{3,2}(\Omega)\big)^\star\Big),$$

while combining Lemma 3.1, (2.12) and Lemma 3.9 with Lemma 4.2 shows that

$$(c_\varepsilon-c_\star)_{\varepsilon\in(0,1)} \text{ is bounded in } L^2\big((0,T);W^{1,2}_0(\Omega)\big)$$

and

$$\Big(\partial_t(c_\varepsilon-c_\star)\Big)_{\varepsilon\in(0,1)} \text{ is bounded in } L^2\Big((0,T);\big(W^{1,4}_0(\Omega)\big)^\star\Big),$$

that

$$(u_\varepsilon)_{\varepsilon\in(0,1)} \text{ is bounded in } L^2\big((0,T);W^{1,2}_{0,\sigma}(\Omega)\big)$$

and that

$$(u_{\varepsilon t})_{\varepsilon\in(0,1)} \text{ is bounded in } L^2\Big((0,T);\big(W^{1,4}_{0,\sigma}(\Omega)\big)^\star\Big).$$

Three applications of Aubin–Lions lemmata ([37]) thus yield $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ as well as functions n,c and u on $\Omega\times(0,\infty)$ such that $\varepsilon_j\searrow 0$ as $j\to\infty$, that $n\ge 0$ and $c\ge 0$ a.e. in $\Omega\times(0,\infty)$, and that (5.1), (5.5), (5.7) and (5.8) hold as $\varepsilon=\varepsilon_j\searrow 0$, where in view of Lemma 4.3, (2.12), Lemma 4.4 and the Vitali convergence theorem we can achieve upon passing to a suitable subsequence if necessary that also (5.3), (5.4) and (5.2) hold. Since the inequality $m>\frac{7}{6}$ particularly ensures that $6m-\frac{44}{9}>7-\frac{44}{9}>2$, (5.2) together with (2.9), and again the Vitali convergence theorem, implies that

$$n_\varepsilon\to n,\quad n_\varepsilon F'_\varepsilon(n_\varepsilon)\to n \quad\text{and}\quad F_\varepsilon(n_\varepsilon)\to n \qquad \text{in } L^2_{loc}(\overline{\Omega}\times[0,\infty)) \tag{5.9}$$

as $\varepsilon=\varepsilon_j\searrow 0$, so that for each fixed $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$, in the identity

$$\begin{aligned} -\int_0^\infty\int_\Omega n_\varepsilon\varphi_t-\int_\Omega n_0\varphi(\cdot,0) &= -\frac{1}{m}\int_0^\infty\int_\Omega\nabla(n_\varepsilon+\varepsilon)^m\cdot\nabla\varphi+\int_0^\infty\int_\Omega n_\varepsilon F'_\varepsilon(n_\varepsilon)\nabla c_\varepsilon\cdot\nabla\varphi \\ &\quad +\int_0^\infty\int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\varphi, \qquad \varepsilon\in(0,1), \end{aligned}$$

as implied by (2.7), we may take $\varepsilon=\varepsilon_j\searrow 0$ in each of the summands separately to obtain (2.4) from (5.2), (5.3), (5.9), (5.5) and (5.7).

The derivation of (2.5) and (2.6) can similarly be based on (5.2)-(5.9), so that we may refrain from giving details on this here, and rather refer to [48, Lemma 4.1] for a fully documented reasoning in a closely related setting. □

PROOF of Theorem 1.1. The claim has completely been covered by Lemma 5.1. □

6 Absorbing sets in $L^p\times W^{1,q}\times L^2$. Proof of Theorem 1.2

Thanks to the above arrangement of estimates, without any substantial further efforts we can finally derive our main result on large time relaxation in (1.5) by drawing on the information about data dependence contained in Lemma 3.9, and especially on the exponential decay of those contributions to (3.40) which involve (n_0,c_0,u_0) through quantities beyond the mere mass functional $\int_\Omega n_0$.

PROOF of Theorem 1.2. Since $p<3m-\frac{7}{3}$ and $q<2$, with $p_0(m)$ and $(q_0(p))_{p\in(p_0(m),3m-\frac{7}{3})}$ as in Lemma 3.8 we can first pick $p_\star=p_\star(p)\in(p_0(m),3m-\frac{7}{3})$ such that $p_\star\ge p$, and then choose $q_\star=q_\star(p,q)\in(q_0(p_\star),2)$ such that $q_\star\ge q$. Then for fixed $K>0$, an application of Lemma 3.9 provides $C_1(K,p,q)>0$ and $\lambda=\lambda(K,p,q)>0$ such that if $c_\star\in[0,K]$, then for any (n_0,c_0,u_0) fulfilling (1.6) with $\int_\Omega n_0\le K$ we can find $C_2(p,q,n_0,c_0,u_0)>0$ such that

$$\int_\Omega(n_\varepsilon+\varepsilon)^{p_\star}+\int_\Omega\big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q_\star}{2}}+\int_\Omega|u_\varepsilon|^2$$

$$\leq \quad C_1(K,p,q) + C_2(p,q,n_0,c_0,u_0)e^{-\lambda(K,p,q)t} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1).$$

As $p_\star \geq p$ and $q_\star \geq q$, thanks to Young's inequality this implies that

$$\begin{aligned}
&\int_\Omega n_\varepsilon^p + \int_\Omega |\nabla c_\varepsilon|^q + \int_\Omega |u_\varepsilon|^2 \\
&\qquad \leq \quad \int_\Omega (n_\varepsilon+\varepsilon)^{p_\star} + \int_\Omega \big(|\nabla c_\varepsilon|^2+1\big)^{\frac{q_\star}{2}} + \int_\Omega |u_\varepsilon|^2 + 2|\Omega| \\
&\qquad \leq \quad C_1(K,p,q) + 2|\Omega| + C_2(p,q,n_0,c_0,u_0)e^{-\lambda(K,p,q)t} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),
\end{aligned}$$

whence letting $t_0(p,q,n_0,c_0,u_0) := \frac{1}{\lambda(K,p,q)} \ln C_2(p,q,n_0,c_0,u_0)$ we obtain that

$$\begin{aligned}
&\int_\Omega n_\varepsilon^p + \int_\Omega |\nabla c_\varepsilon|^q + \int_\Omega |u_\varepsilon|^2 \\
&\qquad \leq \quad C_3(K,p,q) := C_1(K,p,q) + 2|\Omega| + 1 \qquad \text{for all } t>t_0(p,q,n_0,c_0,u_0) \text{ and } \varepsilon\in(0,1). \qquad (6.1)
\end{aligned}$$

Since with $(\varepsilon_j)_{j\in\mathbb{N}}$ taken from Lemma 5.1 we clearly have

$$\begin{aligned}
&\|n\|_{L^\infty((t_\star,\infty);L^p(\Omega))} + \|\nabla c\|_{L^\infty((t_\star,\infty);L^q(\Omega))} + \|u\|_{L^\infty((t_\star,\infty);L^2(\Omega))} \\
&\quad \leq \quad \liminf_{\varepsilon=\varepsilon_j\searrow 0} \Big\{ \|n_\varepsilon\|_{L^\infty((t_\star,\infty);L^p(\Omega))} + \|\nabla c_\varepsilon\|_{L^\infty((t_\star,\infty);L^q(\Omega))} + \|u_\varepsilon\|_{L^\infty((t_\star,\infty);L^2(\Omega))} \Big\} \qquad \text{for all } t_\star \geq t_0
\end{aligned}$$

according to (5.1) and (5.5), we readily arrive at (1.10) if we let

$$R(K,p,q) := C_3^{\frac{1}{p}}(K,p,q) + C_3^{\frac{1}{q}}(K,p,q) + C_3^{\frac{1}{2}}(K,p,q),$$

for example. $\square$

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References

[1] N. Bellomo, A. Bellouquid, Y. Tao, and M. Winkler. Toward a mathematical theory of Keller–Segel models of pattern formation in biological tissues. *Math. Models Methods Appl. Sci.*, 25(9):1663–1763, 2015.

[2] N. Bellomo, A. Bellouquid and N. Chouhad. From a Multiscale Derivation of Nonlinear Cross Diffusion Models to Keller-Segel Models in a Navier-Stokes Fluid. *Math. Models Methods Appl. Sci.*, 26: 2041–2069, 2016.

[3] N. Bellomo, N. Outada, J. Soler, Y. Tao and M. Winkler. Chemotaxis and Cross Diffusion Models in Complex Environments: Modeling Towards a Multiscale Vision. In preparation.

[4] T. Black and C. Wu. Prescribed signal concentration on the boundary: Weak solvability in a chemotaxis-Stokes system with proliferation. 2020. Preprint.

[5] M. Braukhoff. Global (weak) solution of the chemotaxis-Navier-Stokes equations with non-homogeneous boundary conditions and logistic growth. *Ann. Inst. H. Poincar Anal. Non Linaire*, 34(4):1013–1039, 2017.

[6] M. Braukhoff and J. Lankeit. Stationary solutions to a chemotaxis-consumption model with realistic boundary conditions for the oxygen. *Math. Models Methods Appl. Sci.*, 29(11):2033–2062, 2019.

[7] M. Braukhoff and B. Q. Tang. Global solutions for chemotaxis-Navier–Stokes system with Robin boundary conditions. *J. Differential Eq.*, 269(12):10630–10669, 2020.

[8] M. Chae, K. Kang and J. Lee. Global existence and temporal decay in Keller-Segel models coupled to fluid equations. *Commun. Partial Differential Eq.*, 39: 1205–1235, 2014.

[9] A. Chertock, K. Fellner, A. Kurganov, A. Lorz, and P. A. Markowich. Sinking, merging and stationary plumes in a coupled chemotaxis-fluid model: a high-resolution numerical approach. *J. Fluid Mech.*, 694:155–190, 2012.

[10] M. Di Francesco, A. Lorz, and P. Markowich. Chemotaxis-fluid coupled model for swimming bacteria with nonlinear diffusion: Global existence and asymptotic behavior. *Discrete Contin. Dyn. Syst.*, 28(4):1437–1453, 2010.

[11] C. Dombrowski, L. Cisneros, S. Chatkaew, R. E. Goldstein, and J. O. Kessler. Self-Concentration and Large-Scale Coherence in Bacterial Dynamics. *Physical Review Letters*, 93(9), 2004.

[12] R. Duan, A. Lorz, and P. A. Markowich. Global solutions to the coupled chemotaxis-fluid equations. *Comm. Partial Differential Eq.*, 35(9):1635–1673, 2010.

[13] R. Duan and Z. Xiang. A note on global existence for the chemotaxis-Stokes model with nonlinear diffusion. *Int. Math. Res. Notices*, 2014(7):1833–1852, 2014.

[14] M. Fuest, J. Lankeit, and M. Mizukami. Long-term behaviour in a parabolicelliptic chemotaxisconsumption model. *J. Differential Eq.*, 271:254–279, 2021.

[15] H. Fujikawa and M. Matsushita. Fractal Growth of *Bacillus subtilis* on Agar Plates. *J. Phys. Soc. Japan*, 58(11): 3875–3878, 1989.

[16] S. He and E. Tadmor. Suppressing Chemotactic Blow-Up Through a Fast Splitting Scenario on the Plane. *Arch. Ration. Mech. Anal.*, 232: 951–986, 2019.

[17] F. Heihoff. Global mass-preserving solutions for a two-dimensional chemotaxis system with rotational flux components coupled with a full Navier–Stokes equation. *Discrete Contin. Dyn. Syst. Ser. B*, 25(12):4703–4719, 2020.

[18] D. Horstmann. From 1970 until present: the Keller–Segel model in chemotaxis and its consequences. I. *Jahresber. Deutsch. Math.-Verein.*, 105(3):103–165, 2003.

[19] S. Ishida. Global existence and boundedness for chemotaxis-Navier–Stokes systems with position-dependent sensitivity in 2D bounded domains. *Discrete Contin. Dyn. Syst.*, 35(8):3463–3482, 2015.

[20] S. Ishida, K. Seki, and T. Yokota. Boundedness in quasilinear Keller–Segel systems of parabolic-parabolic type on non-convex bounded domains. *J. Differential Eq.*, 256(8) :2993–3010, 2014.

[21] A. Kiselev and L. Ryzhik. Biomixing by chemotaxis and enhancement of biological reactions. *Comm. Partial Differ. Eq.*, 37: 298–318, 2012.

[22] A. Kiselev and L. Ryzhik. Biomixing by chemotaxis and efficiency of biological reactions: the critical reaction case. *J. Math. Phys.*, 53: 115609, 2012.

[23] A. Kiselev and X. Xu. Suppression of Chemotactic Explosion by Mixing. *Arch. Ration. Mech. Anal.*, 222: 1077–1112, 2016.

[24] P. Knosalla. Global solutions of aerotaxis equations. *Applicationes Mathematicae*, 44(1):135–148, 2017.

[25] P. Knosalla and T. Nadzieja. Stationary solutions of aerotaxis equations. *Appl. Math. (Warsaw)*, 42(2-3):125–135, 2015.

[26] J. Lankeit. Long-term behaviour in a chemotaxis-fluid system with logistic source. *Math. Models Methods Appl. Sci.*, 26(11):2071–2109, 2016.

[27] J. Lankeit. Locally bounded global solutions to a chemotaxis consumption model with singular sensitivity and nonlinear diffusion. *J. Differential Eq.*, 262(7):4052–4084, 2017.

[28] J. Lankeit and Y. Wang. Global existence, boundedness and stabilization in a high-dimensional chemotaxis system with consumption. *Discrete Contin. Dyn. Syst.*, 37(12):6099–6121, 2017.

[29] J. Lankeit and M. Winkler. Radial solutions to a chemotaxis-consumption model involving prescribed signal concentration on the boundary. 2021. Preprint.

[30] J. Liu and A. Lorz. A coupled chemotaxis-fluid model: Global existence. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28(5):643–652, 2011.

[31] M. Matsushita and H. Fujikawa. Diffusion-limited growth in bacterial colony formation. *Physica A: Statistical Mechanics and its Applications*, 168(1):498–506, 1990.

[32] N. Mizoguchi and Ph. Souplet. Nondegeneracy of blow-up points for the parabolic Keller–Segel system. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 31:(4):851–875, 2014.

[33] Y. Peng and Z. Xiang. Global existence and convergence rates to a chemotaxis-fluids system with mixed boundary conditions. *J. Differential Eq.*, 267(2):1277–1321, 2019.

[34] Y. Tao and M. Winkler. Eventual smoothness and stabilization of large-data solutions in a three-dimensional chemotaxis system with consumption of chemoattractant. *J. Differential Eq.*, 252(3):2520 – 2543, 2012.

[35] Y. Tao and M. Winkler. Global existence and boundedness in a Keller–Segel–Stokes model with arbitrary porous medium diffusion. *Discrete Contin. Dyn. Syst.*, 32(5):1901–1914, 2012.

[36] Y. Tao and M. Winkler. Locally bounded global solutions in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion. *Ann. Inst. H. Poincar Anal. Non Linaire*, 30(1):157–178, 2013.

[37] R. Temam. *Navier-Stokes equations. Theory and numerical analysis.* North-Holland, 1977. Studies in Mathematics and its Applications, Vol. 2.

[38] Y. Tian and Z. Xiang. Global solutions to a 3D chemotaxis-Stokes system with nonlinear cell diffusion and Robin signal boundary condition. *J. Differential Eq.*, 269(3):2012–2056, 2020.

[39] I. Tuval, L. Cisneros, C. Dombrowski, C. W. Wolgemuth, J. O. Kessler, and R. E. Goldstein. Bacterial swimming and oxygen transport near contact lines. *Proc. Natl. Acad. Sci. U.S.A.*, 102(7):2277–2282, 2005.

[40] Y. Wang and X. Li. Boundedness for a 3D chemotaxis-Stokes system with porous medium diffusion and tensor-valued chemotactic sensitivity. *Z. Angew. Math. Phys.*, 68(2):Paper No. 29, 23, 2017.

[41] Y. Wang, M. Winkler, and Z. Xiang. Global mass-preserving solutions to a chemotaxis-fluid model involving Dirichlet boundary conditions for the signal. 2020. Preprint.

[42] Y. Wang, M. Winkler, and Z. Xiang. Local energy estimates and global solvability in a three-dimensional chemotaxis-fluid system with prescribed signal on the boundary. 2020. Preprint.

[43] Y. Wang, Y and Z. Xiang. Global existence and boundedness in a Keller–Segel–Stokes system involving a tensor-valued sensitivity with saturation. *J. Differential Eq.*, 259(12):7578–7609, 2015.

[44] M. Wiegner. The Navier–Stokes equations — a neverending challenge? *Jahresber. Deutsch. Math.-Verein.*, 101: 1–25, 1999.

[45] M. Winkler. Global large-data solutions in a chemotaxis-(Navier–)Stokes system modeling cellular swimming in fluid drops. *Comm. Partial Differential Eq.*, 37(2):319–351, 2012.

[46] M. Winkler. Stabilization in a two-dimensional chemotaxis-Navier–Stokes system. *Arch. Ration. Mech. Anal.*, 211 (2):455–487, 2014.

[47] M. Winkler: Boundedness and large time behavior in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion and general sensitivity. *Calc. Var. Part. Differential Eq.*, 54(4):3789–3828, 2016.

[48] M. Winkler. Global weak solutions in a three-dimensional chemotaxis-Navier–Stokes system. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33(5):1329–1352, 2016.

[49] M. Winkler. How far do chemotaxis-driven forces influence regularity in the Navier–Stokes system? *Trans. Amer. Math. Soc.*, 369(5):3067–3125, 2017.

[50] M. Winkler. Global existence and stabilization in a degenerate chemotaxis-Stokes system with mildly strong diffusion enhancement. *J. Differential Eq.*, 264(10):6109–6151, 2018.

[51] C. Wu and Z. Xiang. Asymptotic dynamics on a chemotaxis-Navier-Stokes system with nonlinear diffusion and inhomogeneous boundary conditions. *Math. Models Methods Appl. Sci.*, 30(07):1325–1374, 2020.

[52] Q. Zhang and Y. Li. Convergence rates of solutions for a two-dimensional chemotaxis-Navier–Stokes system. *Discrete Contin. Dyn. Syst.*, 20(8):2751–2759, 2015.

[53] Q. Zhang and Y. Li. Global weak solutions for the three-dimensional chemotaxis-Navier–Stokes system with nonlinear diffusion. *J. Differential Eq.*, 259(8):3730–3754, 2015.