Small-density solutions in Keller-Segel systems involving rapidly decaying diffusivities

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Abstract

In a smoothly bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, the quasilinear Keller-Segel system

$$\begin{cases} u_t = \nabla \cdot \big(D(u)\nabla u\big) - \nabla \cdot \big(S(u)\nabla v\big), \\ v_t = \Delta v - v + u, \end{cases} \qquad (\star)$$

is considered under homogeneous no-flux boundary conditions.

It is firstly shown that if D and S, besides belonging to $C^2([0,\infty))$ with $S(0) = 0$, merely satisfy

$$D > 0 \quad \text{in } [0,R] \qquad \text{with some } R > 0,$$

then for all $K > 0$ there exists $\varepsilon_\star(K) \in (0, \frac{R}{2})$ such that whenever $0 \leq u_0 \in W^{1,\infty}(\Omega)$ and $0 \leq v_0 \in W^{1,\infty}(\Omega)$ satisfy

$$\|u_0\|_{L^\infty(\Omega)} \leq \varepsilon_\star(K) \qquad \text{and} \qquad \|v_0\|_{W^{1,\infty}(\Omega)} \leq K,$$

a corresponding initial value problem for $(\star)$ admits a global bounded classical solution with $(u,v)|_{t=0} = (u_0, v_0)$.

Secondly, a more restrictive condition on the initial data, inter alia requiring appropriate smallness of both $\|u_0\|_{L^\infty(\Omega)}$ and $\|v_0\|_{W^{1,\infty}(\Omega)}$, is identified as sufficient to ensure exponential stabilization of the correspondingly obtained solution toward the equilibrium $(\frac{1}{|\Omega|}\int_\Omega u_0, \frac{1}{|\Omega|}\int_\Omega u_0)$.

As a technical ingredient of crucial importance for the derivation of explicit pointwise bounds for the respective first solution components, the analysis relies on a refinement of a Moser-type iterative argument which, formulated here in a general context of parabolic inequalities, provides some quantitative information about the dependence of L^∞ estimates on bounds on the initial data and L^1 bounds.

Key words: chemotaxis; Moser iteration
MSC (2020): 35K65 (primary); 35B45, 92C17 (secondary)

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1 Introduction

This article studies the well-posedness for the following parabolic-parabolic Keller-Segel system

$$\begin{cases} u_t = \nabla \cdot \big(D(u)\nabla u\big) - \nabla \cdot \big(S(u)\nabla v\big), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a smoothly bounded domain in $\mathbb{R}^n$ with $n \geq 1$, and where $\frac{\partial}{\partial \nu}$ denotes the derivative with respect to the outer normal of $\partial\Omega$. Problems of this form arise in the description of reinforced chemotactic migration, that is, of processes in which cell, in (1.1) represented through the population density u, migrate toward locations with a higher concentration of a chemical signal that is produced by themselves, and which is denoted by v. By allowing deviations from the particular choices $D \equiv 1$ and $S(u) = u$, $u \geq 0$, that determine the classical and thoroughly studied Keller-Segel model, the system (1.1) appears to be well-adapted to situations in which both random diffusive and directed cross-diffusive movement are subject to density-dependent influences, among which especially so-called volume-filling effects seem to be of predominant relevance in applications ([15], [8]). In such contexts, a natural focus is on choices of D and S which in comparison to the above appropriately reflect saturation effects at large population densities, in particular leading to decreasing diffusion rates which may give rise to considerable degeneracies of parabolicity near points where u is large.

In order to further illustrate the motivation for this study, especially with regard to mathematical aspects, let first recall some known results that demonstrate effects exerted by the competing mechanisms of diffusion and taxis in (1.1). In fact, a certain setting-dependence in the solution behavior can already be observed in the classical fully parabolic Keller-Segel system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

as forming the most prominent representative thereof: In the dimension $n = 1$, the dissipative effect of diffusion is strong enough to ensure the global and classical solvability for the model (1.2) ([14]). If $n = 2$, a critical mass phenomenon occurs in the radially symmetric setting: The model possesses a global and bounded classical solution whenever $\int_\Omega u_0 < 8\pi$ [13] and there exist finite-time blow-up solutions emanating from some initial data (u_0, v_0) with the property $\int_\Omega u_0 > 8\pi$ ([7]). For the case when $\Omega \subseteq \mathbb{R}^n$ is a ball with $n \geq 3$, it has been shown in [23] that at arbitrarily small levels of $\int_\Omega u_0$, some solutions blow up in finite time, which inter alia indicates that smallness of the total population mass is not sufficient to rule out chemotactic collapse in such higher-dimensional settings. After all, global well-posedness can be inferred from alternative conditions on smallness of initial data which involve norms more selective than L^1: In fact, the work [1] revealed the existence of $\varepsilon_0 > 0$ such that if the initial data (u_0, v_0) satisfy $\|u_0\|_{L^{\frac{n}{2}}(\Omega)} \leq \varepsilon_0$ and $\|\nabla v_0\|_{L^n(\Omega)} \leq \varepsilon_0$, then the system (1.2) admits a classical and globally bounded solution (u, v) which approaches the steady state $(\frac{1}{|\Omega|}\int_\Omega u_0, \frac{1}{|\Omega|}\int_\Omega u_0)$

in the large time limit.

In contexts of more general ingredients for (1.1), existing results concerned with initial data of arbitrary size have indicated that it should essentially be the asymptotics of $\frac{S(u)}{D(u)}$ at large values of u which determines whether diffusion can overbalance the aggregation-enhancing effects of taxis. Indeed, when $\frac{S(u)}{D(u)} \ge cu^{\frac{2}{n}+\varepsilon}$ with $\varepsilon > 0$ for all $u > 1$, by utilizing the presence of a certain Lyapunov functional that may become unbounded along trajectories, it was found in [22] that the problem (1.1) with Ω being a ball possesses smooth solutions that blow up either in finite or infinite time; within some parts of this parameter regime it could even be clarified whether these explosions occur within finite ([2], [3]) or infinite time ([24], [25]). On the other hand, a result on global existence and boundedness of solutions emanating from widely arbitrary, and especially large, initial data was obtained in [18] and [10] provided that $\frac{S(u)}{D(u)} \le cu^{\frac{2}{n}-\varepsilon}$ with $\varepsilon > 0$ for all $u > 1$ (cf. also [17] and [9] for some earlier partial findings concerned with specific choices of D and S); however, as an assumption of apparently crucial technical relevance all these latter approaches toward blow-up exclusion rely on the additional hypothesis that D decays at rates which are at most algebraic, and only in very few exceptional cases some boundedness results could be derived in the presence of exponentially decreasing diffusion rates ([4], [5]).

The methodological core of these shortcomings seems to be linked to the circumstance that in sharp contrast to the situation in associated purely dissipative nonlinear diffusion problems of the form $u_t = \nabla \cdot (D(u)\nabla u)$, an apparent absence of any expedient maximum-principle based strategy toward boundedness proofs for the full system (1.1) enforces resorting to alternative derivations of L^∞ estimates. In fact, approaches based on Moser-type iterations have turned out to provide powerful tools in this direction, and their use in contexts of chemotaxis systems, also within considerably more complex frameworks, has meanwhile become fairly common; however, in their original form such Moser-type methods seem a priori limited in applicability, due to their mere nature, to situations in which the action of diffusion can be controlled from below in an appropriate power-type manner (cf. e.g. [18, Lemma A.1]).

Quite in line with this, also the analysis of small-data solutions to versions of (1.1) in supercritical regimes of $\frac{S}{D}$ has so far been restricted to cases of at most algebraically decaying D; for an associated result on the existence of global and bounded solutions within such aggregation-dominated parameter ranges, involving a smallness condition on (u_0, v_0) in $L^p \times W^{1,q}$ with suitably large $p > 1$ and $q > 1$, we refer to [6]. In the presence of general diffusion rates functions D possibly exhibiting very rapid decay as $u \to \infty$, the question whether any nontrivial global bounded solutions to (1.1) exist at all appears to be completely open, both for subcritical and for supercritical S.

Main results. Now the purpose of this paper is to present an approach capable of answering the latter question in the affirmative under apparently minimal requirements on the model constituents. To achieve this, as a tool of possibly independent interest we shall attempt to further develop Moser's method so as to provide, in contexts of fairly general parabolic inequalities, quite precise quantitative information about the dependence of spatio-temporal L^∞ estimates on uniform bounds for the initial data on the one hand, and L^1 bounds for the solution itself on the other (cf. Section 2). With this tool at hand, we shall in fact be able to assert global classical solvability and boundedness in (1.1) under model assumptions which essentially require nothing further than positivity of D *near* $u = 0$, and for initial data which *only in their first component need to be adequately small.*

More precisely, throughout the sequel we shall only need to assume that

$$\begin{cases} D \in C^2([0,R]) \text{ and } S \in C^2([0,R]) \text{ are such that} \\ D > 0 \text{ in } [0,R] \quad \text{and} \quad S(0) = 0 \end{cases} \tag{1.3}$$

with some $R > 0$, and hence not only include positive $D \in C^2([0,\infty))$ which decay at arbitrarily fast rates as $u \to \infty$, but beyond this allow for diffusion degeneracies even at finite positive densities such as those proposed in [27] and [28], and analyzed in the presence of correspondingly large initial data in [19] and [20], for instance.

Within any such setting, the first of our main results makes sure that even in the presence of taxis gradients of arbitrary size at the initial instant, an appropriate smallness assumption on the first component of the initial data, in view of (1.3) quite naturally formulated in terms of L^∞ norms, ensures global existence and boundedness:

Theorem 1.1 *Let $n \ge 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and suppose that D and S satisfy (1.3) with some $R > 0$. Then given any $K > 0$ one can find $\varepsilon_\star = \varepsilon_\star(K) \in (0,R)$ with the property that whenever $u_0 \in W^{1,\infty}(\Omega)$ and $v_0 \in W^{1,\infty}(\Omega)$ are nonnegative with*

$$\|u_0\|_{L^\infty(\Omega)} \le \varepsilon_\star \qquad \textit{and} \qquad \|v_0\|_{W^{1,\infty}(\Omega)} \le K, \tag{1.4}$$

there exist functions

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \qquad \textit{and} \\ v \in \bigcap_{q>n} C^0([0,\infty); W^{1,q}(\Omega)) \cap L^\infty((0,\infty); W^{1,\infty}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \end{cases} \tag{1.5}$$

such that $0 \le u < R$ and $v \ge 0$ in $\overline{\Omega} \times [0,\infty)$, and that (u,v) solves (1.1) in the classical sense.

If, beyond the above, suitable smallness conditions are satisfied by both components of the initial data, then these solutions can furthermore be seen to stabilize toward homogeneous steady states asymptotically. Indeed, on the basis of an auxiliary result ensuring a certain preservation of smallness in the second solution component (Lemma 3.3), an expression of the form

$$\int_\Omega \Big(u - \frac{1}{|\Omega|} \int_\Omega u_0 \Big)^2 + \int_\Omega \Big(v - \frac{1}{|\Omega|} \int_\Omega u_0 \Big)^2, \qquad t > 0,$$

can be shown to play the role of a genuine Lyapunov functional along small-data trajectories. In Section 4, this will reveal the second of our main results:

Theorem 1.2 *Suppose that $n \ge 1$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, and (1.3) holds with some $R > 0$. Then there exists $\lambda > 0$ such that with $\varepsilon_\star = \varepsilon_\star(1) \in (0,R)$ taken from Theorem 1.1, one can choose $\varepsilon_{\star\star} \in (0, \min\{\varepsilon_\star, 1\}]$ in such a way that if $0 \le u_0 \in W^{1,\infty}(\Omega)$ and $0 \le v_0 \in W^{1,\infty}(\Omega)$ satisfy*

$$\|u_0\|_{L^\infty(\Omega)} \le \varepsilon_{\star\star} \qquad \textit{and} \qquad \|v_0\|_{W^{1,\infty}(\Omega)} \le \varepsilon_{\star\star}, \tag{1.6}$$

the solution (u,v) of (1.1) obtained in Theorem 1.1 has the additional property that with some $C = C(u_0,v_0) > 0$ we have

$$\|u(\cdot,t) - \overline{u}_0\|_{L^\infty(\Omega)} \le Ce^{-\lambda t} \qquad \textit{for all } t > 0 \tag{1.7}$$

and

$$\|v(\cdot,t)-\overline{u}_0\|_{L^\infty(\Omega)} \le Ce^{-\lambda t} \qquad \textit{for all } t>0, \tag{1.8}$$

where $\overline{u}_0 := \frac{1}{|\Omega|}\int_\Omega u_0$.

2 Quantitative dependencies in a parabolic Moser-type iteration

The goal of this preliminary section is to derive some quantitative information on how L^∞ norms of solutions to certain linear parabolic inequalities depend on the respective initial data and, yet more importantly, on a supposedly present temporally uniform spatial L^1 bound. This will be achieved in Lemma 2.2 on the basis of a Moser-type iterative argument which makes use of the following elementary observation.

Lemma 2.1 *Let* $A \ge 0$ *and* $B \ge 1$*, and suppose that* $(M_k)_{k\in\mathbb{N}_0} \subset [0,\infty)$ *is such that*

$$M_k \le \max\Big\{A^{2^k}\,,\,B^k M_{k-1}^2\Big\} \qquad \textit{for all } k\ge 1. \tag{2.1}$$

Then

$$M_k^{\frac{1}{2^k}} \le B^2\cdot\max\big\{A\,,\,M_0\big\} \qquad \textit{for all } k\ge 0. \tag{2.2}$$

Proof. We let

$$\mu_k := \max\Big\{M_k\,,\,A^{2^k}B^{-\frac{k+1}{2}}\Big\}, \qquad k\in\mathbb{N}_0, \tag{2.3}$$

and first claim that

$$\mu_k \le B^k\mu_{k-1}^2 \qquad \text{for all } k\ge 1. \tag{2.4}$$

To verify this, we note that from (2.3) and (2.1) it follows since $B^{-\frac{k+1}{2}} \le 1$ for all $k\ge 1$,

$$\begin{aligned}\mu_k &\le \max\Bigg\{\max\Big\{A^{2^k}\,,\,B^kM_{k-1}^2\Big\}\,,\,A^{2^k}B^{-\frac{k+1}{2}}\Bigg\}\\ &= \max\Big\{A^{2^k}\,,\,B^kM_{k-1}^2\,,\,A^{2^k}B^{-\frac{k+1}{2}}\Big\}\\ &= \max\Big\{A^{2^k}\,,\,B^kM_{k-1}^2\Big\} \qquad \text{for all } k\ge 1.\end{aligned} \tag{2.5}$$

Here, again due to (2.3),

$$A^{2^k} = B^k\cdot\big(A^{2^{k-1}}B^{-\frac{k}{2}}\big)^2 \le B^k\mu_{k-1}^2 \qquad \text{for all } k\ge 1$$

and

$$B^kM_{k-1}^2 \le B^k\mu_{k-1}^2 \qquad \text{for all } k\ge 1.$$

Therefore, (2.5) indeed implies (2.4), which in turn, through a straightforward induction, entails that

$$\mu_k \le B^{2^{k+1}-k-2}\mu_0^{2^k} \qquad \text{for all } k\ge 1.$$

Once more thanks to (2.3) and the inequality $B \ge 1$, we thus infer that

$$M_k^{\frac{1}{2^k}} \le \mu_k^{\frac{1}{2^k}} \le B^{\frac{2^{k+1}-k-2}{2^k}} \mu_0 \le B^2\mu_0 = B^2 \cdot \max\left\{M_0\,,\, AB^{-\frac{1}{2}}\right\} \le B^2 \cdot \max\left\{M_0\,,\, A\right\} \qquad \text{for all } k \ge 1,$$

and conclude as intended. □

In fact, the latter will form the iterative core in our derivation of the following boundedness result which we formulate in a setting slightly more general than actually needed in the sequel. Indeed, in Lemma 3.4 below our application of Lemma 2.2 will exclusively refer to the particular choices $q = \infty$ and $f \equiv 0$ therein.

Lemma 2.2 *Let $n \ge 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let $q \in [2,\infty]$ be such that $q > n$. Then for all $\eta > 0$ and $K > 0$ one can find $\Lambda(q,\eta,K) > 0$ with the property that whenever $T \in (0,\infty]$, $a \in C^1(\overline{\Omega} \times (0,T)), b \in C^1(\overline{\Omega} \times (0,T);\mathbb{R}^n)$, $f \in C^0(\overline{\Omega} \times (0,T))$ and $u \in C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ are such that*

$$a(x,t) \ge \eta \qquad \textit{for all } (x,t) \in \Omega \times (0,T), \tag{2.6}$$

that

$$\|b(\cdot,t)\|_{L^q(\Omega)} \le K \qquad \textit{for all } t \in (0,T), \tag{2.7}$$

that

$$\|f(\cdot,t)\|_{L^{\frac{q}{2}}(\Omega)} \le K \qquad \textit{for all } t \in (0,T), \tag{2.8}$$

and that u is nonnegative with

$$\begin{cases} u_t \le \nabla \cdot \big(a(x,t)\nabla u\big) + \nabla \cdot \big(b(x,t)u\big) + f(x,t)u & x \in \Omega,\ t \in (0,T), \\ \big(a(x,t)\nabla u + b(x,t)u\big) \cdot \nu \le 0, & x \in \partial\Omega,\ t \in (0,T), \end{cases} \tag{2.9}$$

we have

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \Lambda(q,\eta,K) \cdot \max\left\{\|u(\cdot,0)\|_{L^\infty(\Omega)}\,,\ \sup_{s\in(0,T)} \|u(\cdot,s)\|_{L^1(\Omega)}\right\} \qquad \textit{for all } t \in (0,T). \tag{2.10}$$

PROOF. Due to the boundedness of Ω, it is evidently sufficient to restrict our considerations to the case when q is finite. In order to then prepare our definition of $\Lambda(q,\eta,K)$, we abbreviate

$$c_1 = c_1(q,\eta,K) := \frac{K^2}{2\eta} + K + |\Omega|^{\frac{2}{q}} \tag{2.11}$$

and note that since $q > n$ and $q \ge 2$, writing

$$r = r(q) := \begin{cases} +\infty & \text{if } q = 2, \\ \frac{2q}{q-2} & \text{if } q > 2, \end{cases} \tag{2.12}$$

we have $r > 2$ and $(n-2)r < 2n$, because if $q = 2$ then necessarily $n = 1$. Therefore, the Gagliardo-Nirenberg inequality applies so as to yield $c_2 = c_2(q) > 0$ such that

$$\|\varphi\|_{L^r(\Omega)}^2 \le c_2\|\nabla\varphi\|_{L^2(\Omega)}^{2\kappa}\|\varphi\|_{L^1(\Omega)}^{2(1-\kappa)} + c_2\|\varphi\|_{L^1(\Omega)}^2 \qquad \text{for all } \varphi \in W^{1,2}(\Omega) \tag{2.13}$$

with

$$\kappa = \kappa(q) := \begin{cases} \frac{2n}{n+2} = \frac{2}{3} & \text{if } q = 2, \\ \frac{n(q+2)}{(n+2)q} & \text{if } q > 2, \end{cases}$$

satisfying $\kappa \in (0,1)$. We thereupon define

$$c_3 = c_3(q,\eta,K) := (c_1c_2)^{\frac{1}{1-\kappa}}\eta^{-\frac{\kappa}{1-\kappa}}, \qquad c_4 = c_4(q,\eta,K) := c_1c_2 + c_3 \qquad \text{and} \qquad c_5 := \max\big\{|\Omega|\,,\,1\big\} \tag{2.14}$$

as well as

$$B = B(q,\eta,K) := 2^{\frac{2}{1-\kappa}} \cdot \max\big\{c_4\,,\,1\big\}, \tag{2.15}$$

and to verify that the claimed conclusion holds if we let

$$\Lambda(q,\eta,K) := c_5B^2, \tag{2.16}$$

we suppose that $T \in (0,\infty]$, and that a, b, f and u have the listed regularoty properties and satisfy (2.6), (2.7), (2.8) and (2.9). For nonnegative integers k, we then take $p_k := 2^k$ and

$$M_k(T_0) := \sup_{t\in(0,T_0)} \int_\Omega u^{p_k}(\cdot,t), \qquad T_0 \in (0,T), \tag{2.17}$$

and note that to derive (2.10) we only need to consider the case when both $\|u(\cdot,0)\|_{L^\infty(\Omega)}$ and $\overline{M}_0 := \sup_{T_0\in(0,T)} M_0(T_0) \equiv \sup_{s\in(0,T)} \|u(\cdot,s)\|_{L^1(\Omega)}$ are finite.

In such constellations, for $k \ge 1$ we may multiply the first inequality in (2.9) by u^{p_k-1} and integrate by parts to obtain from (2.6) and the nonpositivity of $(a\nabla u + bu)\cdot\nu$ on $\partial\Omega\times(0,T)$ that due to Young's inequality,

$$\begin{aligned} \frac{1}{p_k}\frac{d}{dt}\int_\Omega u^{p_k} + (p_k-1)\eta\int_\Omega u^{p_k-2}|\nabla u|^2 &\le \frac{1}{p_k}\frac{d}{dt}\int_\Omega u^{p_k} + (p_k-1)\int_\Omega a u^{p_k-2}|\nabla u|^2 \\ &\le -(p_k-1)\int_\Omega u^{p_k-1} b\cdot\nabla u + \int_\Omega f u^{p_k} \\ &\le \frac{(p_k-1)\eta}{2}\int_\Omega u^{p_k-2}|\nabla u|^2 \\ &\quad + \frac{p_k-1}{2\eta}\int_\Omega |b|^2 u^{p_k} + \int_\Omega |f| u^{p_k} \qquad \text{for all } t\in(0,T), \end{aligned}$$

so that

$$\frac{d}{dt}\int_\Omega u^{p_k} + \eta\int_\Omega |\nabla u^{\frac{p_k}{2}}|^2 + \int_\Omega u^{p_k} \le \frac{p_k^2}{2\eta}\int_\Omega |b|^2u^{p_k} + p_k^2\int_\Omega |f|u^{p_k} + p_k^2\int_\Omega u^{p_k} \qquad \text{for all } t\in(0,T), \tag{2.18}$$

because $|\nabla u^{\frac{p_k}{2}}|^2 = \frac{p_k^2}{4}u^{p_k-2}|\nabla u|^2$, and because $\frac{2(p_k-1)\eta}{p_k} \ge \eta$ and $\frac{p_k(p_k-1)}{2\eta} \le \frac{p_k^2}{2\eta}$ as well as $p_k \le p_k^2$ and $1 \le p_k^2$ for any such k. Now thanks to the Hölder inequality and (2.12), we may utilize (2.7) and (2.8) to see that since $\frac{r-2}{r} = \frac{2}{q}$,

$$\begin{aligned} \frac{1}{2\eta}\int_\Omega |b|^2u^{p_k} + \int_\Omega |f|u^{p_k} + \int_\Omega u^{p_k} &\le \frac{1}{2\eta}\|b\|^2_{L^q(\Omega)}\|u^{\frac{p_k}{2}}\|^2_{L^r(\Omega)} + \|f\|_{L^{\frac{q}{2}}(\Omega)}\|u^{\frac{p_k}{2}}\|^2_{L^r(\Omega)} + |\Omega|^{\frac{2}{q}}\|u^{\frac{p_k}{2}}\|^2_{L^r(\Omega)} \\ &\le c_1\|u^{\frac{p_k}{2}}\|^2_{L^r(\Omega)} \qquad \text{for all } t\in(0,T) \end{aligned}$$

according to our definition (2.11) of c_1. We can therefore draw on (2.13) to infer that by (2.17), and once more by Young's inequality, given any $T_0 \in (0,T)$ we have

$$\begin{aligned}
\frac{p_k^2}{2\eta}\int_\Omega |b|^2 u^{p_k} + p_k^2 \int_\Omega |f| u^{p_k} + p_k^2 \int_\Omega u^{p_k} \\
&\le c_1 c_2 p_k^2 \|\nabla u^{\frac{p_k}{2}}\|_{L^2(\Omega)}^{2\kappa} \|u^{\frac{p_k}{2}}\|_{L^1(\Omega)}^{2(1-\kappa)} + c_1 c_2 p_k^2 \|u^{\frac{p_k}{2}}\|_{L^1(\Omega)}^2 \\
&\le c_1 c_2 p_k^2 M_{k-1}^{2(1-\kappa)}(T_0) \|\nabla u^{\frac{p_k}{2}}\|_{L^2(\Omega)}^{2\kappa} + c_1 c_2 p_k^2 M_{k-1}^2(T_0) \\
&= \Big\{ \eta \int_\Omega |\nabla u^{\frac{p_k}{2}}|^2 \Big\}^\kappa \cdot \Big\{ c_1 c_2 \eta^{-\kappa} p_k^2 M_{k-1}^{2(1-\kappa)}(T_0) \Big\} + c_1 c_2 p_k^2 M_{k-1}^2(T_0) \\
&\le \eta \int_\Omega |\nabla u^{\frac{p_k}{2}}|^2 + c_3 p_k^{\frac{2}{1-\kappa}} M_{k-1}^2(T_0) + c_1 c_2 p_k^2 M_{k-1}^2(T_0) \\
&\le \eta \int_\Omega |\nabla u^{\frac{p_k}{2}}|^2 + c_4 p_k^{\frac{2}{1-\kappa}} M_{k-1}^2(T_0) \qquad \text{for all } t \in (0,T_0),
\end{aligned}$$

in line with our selections of c_3 and c_4 in (2.14), and with the obvious fact that $p_k^2 \le p_k^{\frac{2}{1-\kappa}}$. In consequence, from (2.18) we obtain the autonomous ODI

$$\frac{d}{dt}\int_\Omega u^{p_k} + \int_\Omega u^{p_k} \le c_4 p_k^{\frac{2}{1-\kappa}} M_{k-1}^2(T_0) \qquad \text{for all } t \in (0,T_0),$$

which upon a comparison argument implies that for any such T_0 and k,

$$\int_\Omega u^{p_k}(\cdot,t) \le \max\Big\{ \int_\Omega u^{p_k}(\cdot,0)\,,\, c_4 p_k^{\frac{2}{1-\kappa}} M_{k-1}^2(T_0) \Big\} \qquad \text{for all } t \in (0,T_0),$$

and thus, since $\int_\Omega u^{p_k}(\cdot,0) \le |\Omega| \cdot \|u(\cdot,0)\|_{L^\infty(\Omega)}^{p_k}$,

$$M_k(T_0) \le \max\Big\{ |\Omega| \cdot \|u(\cdot,0)\|_{L^\infty(\Omega)}^{p_k}\,,\, c_4 p_k^{\frac{2}{1-\kappa}} M_{k-1}^2(T_0) \Big\} \qquad \text{for all } T_0 \in (0,T) \text{ and } k \ge 1. \tag{2.19}$$

By an induction relying on the presupposed finiteness of $\|u(\cdot,0)\|_{L^\infty(\Omega)}$ and of $\overline{M}_0$, this firstly ensures that for each $k \in \mathbb{N}$, $\overline{M}_k := \sup_{T_0 \in (0,T)} M_k(T_0)$ introduces a finite number, whereupon (2.19), secondly, entails that these numbers satisfy

$$\overline{M}_k \le \max\Big\{ |\Omega| \cdot \|u(\cdot,0)\|_{L^\infty(\Omega)}^{p_k}\,,\, c_4 p_k^{\frac{2}{1-\kappa}} \overline{M}_{k-1}^2 \Big\} \qquad \text{for all } k \ge 1. \tag{2.20}$$

To make this inequality accessible to the outcome of Lemma 2.1, we observe that by definition of c_5 in (2.14),

$$|\Omega|^{\frac{1}{p_k}} \cdot \|u(\cdot,0)\|_{L^\infty(\Omega)} \le A := c_5 \|u(\cdot,0)\|_{L^\infty(\Omega)} \qquad \text{for all } k \ge 1,$$

and that our choice of B in (2.15) guarantees that

$$\big(c_4 p_k^{\frac{2}{1-\kappa}}\big)^{\frac{1}{k}} = c_4^{\frac{1}{k}} \cdot 2^{\frac{2}{1-\kappa}} \le B \qquad \text{for all } k \ge 1.$$

Accordingly, from (2.20) we obtain that

$$\overline{M}_k \le \max\Big\{A^{2^k}\,,\,B^k\overline{M}_{k-1}^2\Big\} \qquad \text{for all } k \ge 1,$$

which upon an application of Lemma 2.1 reveals that

$$\begin{aligned}
\overline{M}_k^{\frac{1}{2^k}} &\le B^2 \cdot \max\big\{A\,,\,\overline{M}_0\big\} \\
&= B^2 \cdot \max\Big\{c_5\|u(\cdot,0)\|_{L^\infty(\Omega)}\,,\,\sup_{s\in(0,T)}\|u(\cdot,s)\|_{L^1(\Omega)}\Big\} \\
&\le c_5B^2 \cdot \max\Big\{\|u(\cdot,0)\|_{L^\infty(\Omega)}\,,\,\sup_{s\in(0,T)}\|u(\cdot,s)\|_{L^1(\Omega)}\Big\} \qquad \text{for all } k \ge 1
\end{aligned}$$

due to the fact that $c_5 \ge 1$. Taking $k \to \infty$ here readily establishes (2.10) with $\Lambda(q,\eta,K)$ as in (2.16). □

3 Global existence. Proof of Theorem 1.1

In order to next construct global small-data solutions to (1.1) in the intended flavor, let us first draw on approaches from standard theory to obtain the following basic statement on local existence, extensibility and mass conservation:

Lemma 3.1 *If $n \ge 1$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, if D and S satisfy (1.3) with some $R > 0$, and if*

$$\begin{cases} u_0 \in W^{1,\infty}(\Omega) \text{ satisfies } 0 \le u_0 < R \text{ in } \overline{\Omega} \qquad \text{and} \\ v_0 \in W^{1,\infty}(\Omega) \text{ is nonnegative,} \end{cases} \tag{3.1}$$

there exist $T_m \in (0,\infty]$ and nonnegative functions

$$\begin{cases} u \in C^0(\overline{\Omega}\times[0,T_m)) \cap C^{2,1}(\overline{\Omega}\times(0,T_m)) \qquad \text{and} \\ v \in \bigcap_{q>n} C^0([0,T_m);W^{1,q}(\Omega)) \cap L^\infty_{loc}([0,T_m);W^{1,\infty}(\Omega)) \cap C^{2,1}(\overline{\Omega}\times(0,T_m)) \end{cases}$$

such that $u < R$ in $\overline{\Omega}\times[0,T_m)$, that (u,v) solves (1.1) classically in $\Omega\times(0,T_m)$, and that

$$\text{if } T_m = \infty, \qquad \text{then} \qquad \limsup_{t\nearrow T_m}\|u(\cdot,t)\|_{L^\infty(\Omega)} = R \quad \text{or} \quad \limsup_{t\nearrow T_m}\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} = \infty. \tag{3.2}$$

Moreover, this solution satisfies

$$\int_\Omega u(\cdot,t) = \int_\Omega u_0 \qquad \text{for all } t \in (0,T_m). \tag{3.3}$$

Proof. As this can be verified by straightforward adaptation of standard reasonings, instead of repeating essentially well-known arguments here we may refer to the derivation detailed [11, Section 2] for a closely related framework. □

Now to provide a convenient tool for an appropriate control of the second solution component, and particularly its gradient, we extract the following as a special case of a more general regularity property documented in [26, Proposition 1.1].

Lemma 3.2 *Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let $p > n$. Then writing $\theta(p) := \frac{(n-1)p}{n(p-1)}$, for each $\delta \in (0, 1 - \theta(p))$ one can find $C(p, \delta) > 0$ such that whenever $T \in (0, \infty]$, any $\varphi \in C^0(\overline{\Omega} \times [0, T)) \cap C^{2,1}(\overline{\Omega} \times (0, T))$ with $\varphi_t - \Delta\varphi \in L^1_{loc}(\overline{\Omega} \times [0, T))$ and $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega \times (0, T)$ has the property that*

$$\begin{aligned} &\|\varphi(\cdot, t)\|_{W^{1,\infty}(\Omega)} \\ &\quad \leq C(p, \delta)\|\varphi(\cdot, 0)\|_{W^{1,\infty}(\Omega)} \\ &\qquad + C(p, \delta) \cdot \left\{ \sup_{s\in(0,t)} \Big\|(\varphi_t - \Delta\varphi + \varphi)(\cdot, s)\Big\|_{L^p(\Omega)} \right\}^{\theta(p)+\delta} \times \\ &\qquad\qquad \times \left\{ \sup_{s\in(0,t)} \Big\|(\varphi_t - \Delta\varphi + \varphi)(\cdot, s)\Big\|_{L^1(\Omega)} \right\}^{1-\theta(p)-\delta} \qquad \textit{for all } t \in (0, T). \end{aligned}$$

In fact, this quantitative a priori information enables us to derive the following from the second equation in (1.1) in quite a straightforward manner.

Lemma 3.3 *There exists $\Gamma > 0$ with the property that given any $K > 0$ one can find $\varepsilon_0 = \varepsilon_0(K) \in (0, \frac{R}{4})$ such that whenever u_0 and v_0 satisfy (3.1) with*

$$\|u_0\|_{L^\infty(\Omega)} \leq \varepsilon_0 \qquad \textit{and} \qquad \|v_0\|_{W^{1,\infty}(\Omega)} \leq K, \tag{3.4}$$

then for the solution (u, v) of (1.1) from Lemma 3.1 we have

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} + \|\nabla v(\cdot, t)\|_{L^\infty(\Omega)} \leq \Gamma K \qquad \textit{for all } t \in (0, T_m). \tag{3.5}$$

Proof. Fixing an arbitrary $p > n$, we let $\theta := \frac{(n-1)p}{n(p-1)}$ and pick any $\delta \in (0, 1 - \theta)$, and then infer from Lemma 3.2 that there exists $c_1 > 0$ with the property that for arbitrary $T \in (0, \infty]$, each $\varphi \in C^0(\overline{\Omega} \times [0, T)) \cap C^{2,1}(\overline{\Omega} \times (0, T))$ fulfilling $\varphi_t - \Delta\varphi \in L^1_{loc}(\overline{\Omega} \times [0, T))$ and $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega \times (0, T)$ satisfies

$$\begin{aligned} &\|\varphi(\cdot, t)\|_{L^\infty(\Omega)} + \|\nabla\varphi(\cdot, t)\|_{L^\infty(\Omega)} \\ &\quad \leq c_1\|\varphi(\cdot, 0)\|_{W^{1,\infty}(\Omega)} \\ &\qquad + c_1 \cdot \left\{ \sup_{s\in(0,t)} \Big\|(\varphi_t - \Delta\varphi + \varphi)(\cdot, s)\Big\|_{L^p(\Omega)} \right\}^{\theta+\delta} \times \\ &\qquad\qquad \times \left\{ \sup_{s\in(0,t)} \Big\|(\varphi_t - \Delta\varphi + \varphi)(\cdot, s)\Big\|_{L^1(\Omega)} \right\}^{1-\theta-\delta} \qquad \text{for all } t \in (0, T). \end{aligned} \tag{3.6}$$

To see that this implies the claim if we let

$$\Gamma := 2c_1, \tag{3.7}$$

given $K > 0$ we choose $\varepsilon_0 = \varepsilon_0(K) \in (0, \frac{R}{4})$ small enough such that

$$\big(R|\Omega|^{\frac{1}{p}}\big)^{\theta+\delta} \cdot \big(\varepsilon_0|\Omega|\big)^{1-\theta-\delta} \leq K, \tag{3.8}$$

and henceforth assume that u_0 and v_0 satisfy (3.1) and (3.4), and that $T_m \in (0,\infty]$ and (u,v) are as accordingly provided by Lemma 3.1. Then (3.3) together with (3.4) asserts that

$$\|u(\cdot,t)\|_{L^1(\Omega)} = \int_\Omega u_0 \le \varepsilon_0|\Omega| \qquad \text{for all } t \in (0,T_m), \tag{3.9}$$

whereas from the characterization of T_m contained in Lemma 3.1 we know that $u \le R$ in $\Omega \times (0,T_m)$ and hence

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le R|\Omega|^{\frac{1}{p}} \qquad \text{for all } t \in (0,T_m)$$

thanks to the Hölder inequality. When combined with (3.9) and (3.6), and again with (3.4), in view of (3.6) and (3.7) this ensures that indeed

$$\begin{aligned}
\|v(\cdot,t)\|_{L^\infty(\Omega)} + \|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} &\le c_1\|v_0\|_{W^{1,\infty}(\Omega)} \\
&\quad + c_1 \cdot \Big\{ \sup_{s\in(0,t)} \|u(\cdot,s)\|_{L^p(\Omega)} \Big\}^{\theta+\delta} \cdot \Big\{ \sup_{s\in(0,t)} \|u(\cdot,s)\|_{L^1(\Omega)} \Big\}^{1-\theta-\delta} \\
&\le c_1K + c_1 \cdot \big(R|\Omega|^{\frac{1}{p}}\big)^{\theta+\delta} \cdot \big(\varepsilon_0|\Omega|\big)^{1-\theta-\delta} \\
&\le c_1K + c_1K = \Gamma K \qquad \text{for all } t \in (0,T_m),
\end{aligned}$$

as intended. $\square$

Thanks to our preparatory efforts undertaken to derive Lemma 2.2, the control of ∇v thereby particularly achieved can now quite unpretentiously be seen to imply the following statement on preservation of spatially uniform smallness enjoyed by u, provided that a second and possibly more restrictive smallness hypothesis is met.

Lemma 3.4 *For each $K > 0$ there exists $\varepsilon_\star = \varepsilon_\star(K) \in (0, \frac{R}{4})$ such that if (3.1) holds with $u_0 \not\equiv 0$ as well as*

$$\|u_0\|_{L^\infty(\Omega)} \le \varepsilon_\star \qquad \text{and} \qquad \|v_0\|_{W^{1,\infty}(\Omega)} \le K, \tag{3.10}$$

then with T_m and u taken from Lemma 3.1 we have

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{R}{2} \qquad \text{for all } t \in (0,T_m). \tag{3.11}$$

Proof. Given $K > 0$, we fix $\varepsilon_0 = \varepsilon_0(K)$ as provided by Lemma 3.3, and choosing $c_1 > 0$ and $c_2 > 0$ such that in accordance with (1.3) we have $D(\xi) \ge c_1$ and $|S(\xi)| \le c_2\xi$ for all $\xi \in [0,R]$, we let $\Lambda := \Lambda(\infty, c_1, c_2\Gamma K) > 0$ be as correspondingly provided by Lemma 2.2, with $\Gamma > 0$ taken from Lemma 3.3. We now choose $\varepsilon_\star = \varepsilon_\star(K) \in (0, \frac{R}{4})$ small enough such that both $\varepsilon_\star \le \varepsilon_0$ and

$$\Lambda \cdot \max\big\{|\Omega|\,,\,1\big\} \cdot \varepsilon_\star \le \frac{R}{2}, \tag{3.12}$$

and assuming $0 \not\equiv u_0$ and v_0 to satisfy (3.1) and (3.10) we then firstly observe that since $\varepsilon_\star \le \varepsilon_0$, Lemma 3.3 applies so as to ensure that $|\nabla v| \le \Gamma K$ in $\overline{\Omega} \times (0,T_m)$. Therefore, the function $b(x,t) := \frac{S(u(x,t))}{u(x,t)}\nabla v(x,t)$, $(x,t) \in \overline{\Omega} \times (0,T_m)$, well-defined as an element of $C^1(\overline{\Omega} \times (0,T_m);\mathbb{R}^n)$ due to the fact that $u > 0$ in $\overline{\Omega} \times (0,T_m)$ by the strong maximum principle, satisfies $|b(x,t)| \le c_2 \Gamma K$ for all $(x,t) \in \Omega \times (0,T_m)$, so that since furthermore for $a(x,t) := D(u(x,t))$, $(x,t) \in \overline{\Omega} \times (0,T_m)$, we have $a \in C^1(\overline{\Omega} \times (0,T_m))$ with $a(x,t) \ge c_1$ for all $(x,t) \in \Omega \times (0,T_m)$, in view of (1.1) we may rely on Lemma 2.2 to infer that

$$\begin{aligned} \|u(\cdot,t)\|_{L^\infty(\Omega)} &\le \Lambda \cdot \max\Big\{ \|u_0\|_{L^\infty(\Omega)} \, , \sup_{s\in(0,T_m)} \|u(\cdot,s)\|_{L^1(\Omega)} \Big\} \\ &\le \Lambda \cdot \max\big\{ \varepsilon_\star \, , \, |\Omega| \varepsilon_\star \big\} \qquad \text{for all } t \in (0,T_m), \end{aligned}$$

because $\|u\|_{L^1(\Omega)} = \int_\Omega u_0 \le |\Omega| \varepsilon_\star$ for all $t \in (0,T_m)$ by (3.3) and (3.10). The claim therefore results from (3.12). □

Our main result on global smooth solvability in (1.1) thereby becomes fairly evident:

Proof of Theorem 1.1. For fixed $K > 0$, we take $\varepsilon_\star(K)$ as given by Lemma 3.4 and then obtain the claimed conclusion as a direct consequence of Lemma 3.4 and Lemma 3.3 when combined with Lemma 3.1, because the corresponding statement concerning global existence is trivial in the case $u_0 \equiv 0$ not explicitly covered by Lemma 3.4. □

4 Large time behavior. Proof of Theorem 1.2

When next concerned with the large time behavior of the obtained solutions under a possibly yet stronger restriction on their initial size, in view of the bounds already known from Theorem 1.1, along with interpolation options thereby provided, we may reduce the essence of our argument to a convenient L^2 setting in which two quite basic testing procedures can easily be combined with a standard Poincaré inequality:

Proof of Theorem 1.2. We again use (1.3) to pick $c_1 > 0$ and $c_2 > 0$ such that $D(\xi) \ge c_1$ and $|S(\xi)| \le c_2 \xi$ for all $\xi \in [0,R]$, and employ a Poincaré inequality to fix $c_3 > 0$ fulfilling

$$\int_\Omega |\nabla \varphi|^2 \ge c_3 \int_\Omega \left| \varphi - \frac{1}{|\Omega|} \int_\Omega \varphi \right|^2 \qquad \text{for all } \varphi \in W^{1,2}(\Omega). \tag{4.1}$$

With $\Gamma > 0$ taken from Lemma 3.3, we then abbreviate

$$K := \frac{c_1 \sqrt{c_3}}{2 c_2 \Gamma} \tag{4.2}$$

and take $\varepsilon_0(K) \in (0, \frac{R}{4})$ and $\varepsilon_\star(K) \in (0,R)$ as accordingly provided by Lemma 3.3 and Theorem 1.1, respectively, and writing

$$\kappa := \frac{c_1 c_3}{4}, \tag{4.3}$$

we fix $\varepsilon_{\star\star} \in (0, \min\{\varepsilon_\star(1), 1\}]$ small enough such that $\varepsilon_{\star\star} \le \min\{K \, , \, \varepsilon_0(K) \, , \, \varepsilon_\star(K)\}$ and

$$\frac{c_2^2}{c_1} \varepsilon_{\star\star}^2 \le \kappa. \tag{4.4}$$

Now given any $0 \le u_0 \in W^{1,\infty}(\Omega)$ and $0 \le v_0 \in W^{1,\infty}(\Omega)$ such that (1.6) holds, relying on the inequality $\varepsilon_{\star\star} \le \min\{K, \varepsilon_\star(K)\}$ we take the global classical solution (u,v) of (1.1) from Theorem 1.1

and first integrate by parts in (1.1) to obtain, using Young's inequality along with our selections of c_1 and c_2, that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (u-\overline{u}_0)^2 &= -\int_\Omega D(u)|\nabla u|^2 + \int_\Omega S(u)\nabla u\cdot\nabla v \\
&\le -\frac{1}{2}\int_\Omega D(u)|\nabla u|^2 + \frac{1}{2}\int_\Omega \frac{S^2(u)}{D(u)}|\nabla v|^2 \\
&\le -\frac{c_1}{2}\int_\Omega |\nabla u|^2 + \frac{c_2^2}{2c_1}\int_\Omega u^2|\nabla v|^2 \qquad \text{for all } t>0,
\end{aligned}$$

so that in line with (4.1),

$$\frac{d}{dt}\int_\Omega (u-\overline{u}_0)^2 + c_1c_3\int_\Omega (u-\overline{u}_0)^2 \le \frac{c_2^2}{c_1}\int_\Omega u^2|\nabla v|^2 \qquad \text{for all } t>0. \tag{4.5}$$

Here on the right-hand side we utilize that $\varepsilon_{\star\star} \le \varepsilon_0(K)$ in employing Lemma 3.3 to see that since thus

$$\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \le \Gamma K = \frac{c_1\sqrt{c_3}}{2c_2} \qquad \text{for all } t>0$$

due to (4.2), we have

$$\begin{aligned}
\frac{c_2^2}{c_1}\int_\Omega u^2|\nabla v|^2 &= \frac{c_2^2}{c_1}\int_\Omega (u-\overline{u}_0+\overline{u}_0)^2|\nabla v|^2 \\
&\le \frac{2c_2^2}{c_1}\int_\Omega (u-\overline{u}_0)^2|\nabla v|^2 + \frac{2c_2^2\overline{u}_0^2}{c_1}\int_\Omega |\nabla v|^2 \\
&\le \frac{2c_2^2}{c_1}\|\nabla v\|_{L^\infty(\Omega)}^2\int_\Omega (u-\overline{u}_0)^2 + \frac{2c_2^2\overline{u}_0^2}{c_1}\int_\Omega |\nabla v|^2 \\
&\le \frac{2c_2^2}{c_1}\cdot\Big(\frac{c_1\sqrt{c_3}}{2c_2}\Big)^2\int_\Omega (u-\overline{u}_0)^2 + \frac{2c_2^2\overline{u}_0^2}{c_1}\int_\Omega |\nabla v|^2 \\
&= \frac{c_1c_3}{2}\int_\Omega (u-\overline{u}_0)^2 + \frac{2c_2^2\overline{u}_0^2}{c_1}\int_\Omega |\nabla v|^2 \qquad \text{for all } t>0,
\end{aligned}$$

so that recalling that $\overline{u}_0 \le \|u_0\|_{L^\infty(\Omega)} \le \varepsilon_{\star\star}$ by (1.6), from (4.5) it follows that

$$\frac{d}{dt}\int_\Omega (u-\overline{u}_0)^2 + \frac{c_1c_3}{2}\int_\Omega (u-\overline{u}_0)^2 \le \frac{2c_2^2\varepsilon_{\star\star}^2}{c_1}\int_\Omega |\nabla v|^2 \qquad \text{for all } t>0. \tag{4.6}$$

Next, on the basis of the second equation in (1.1), and again Young's inequality, we find that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (v-\overline{u}_0)^2 &= -\int_\Omega |\nabla v|^2 + \int_\Omega (v-\overline{u}_0)\cdot(-v+u) \\
&= -\int_\Omega |\nabla v|^2 - \int_\Omega (v-\overline{u}_0)^2 + \int_\Omega (u-\overline{u}_0)\cdot(v-\overline{u}_0) \\
&\le -\int_\Omega |\nabla v|^2 - \frac{1}{2}\int_\Omega (v-\overline{u}_0)^2 + \frac{1}{2}\int_\Omega (u-\overline{u}_0)^2 \qquad \text{for all } t>0,
\end{aligned}$$

and that hence

$$\frac{d}{dt}\int_\Omega (v-\overline{u}_0)^2 + 2\int_\Omega |\nabla v|^2 + \int_\Omega (v-\overline{u}_0)^2 \le \int_\Omega (u-\overline{u}_0)^2 \qquad \text{for all } t>0,$$

which combined with (4.6) implies that thanks to our restriction on $\varepsilon_{\star\star}$ in (4.4), in view of our definition of κ in (4.3) we have

$$\begin{aligned}
\frac{d}{dt}\bigg\{ \int_\Omega (u-\overline{u}_0)^2 + \kappa\int_\Omega (v-\overline{u}_0)^2 \bigg\} &+ \frac{c_1c_3}{2}\int_\Omega (u-\overline{u}_0)^2 + 2\kappa\int_\Omega |\nabla v|^2 + \kappa\int_\Omega (v-\overline{u}_0)^2 \\
&\le \frac{2c_2^2\varepsilon_{\star\star}^2}{c_1}\int_\Omega |\nabla v|^2 + \kappa\int_\Omega (u-\overline{u}_0)^2 \\
&\le 2\kappa\int_\Omega |\nabla v|^2 + \frac{c_1c_3}{4}\int_\Omega (u-\overline{u}_0)^2 \qquad \text{for all } t>0.
\end{aligned}$$

Accordingly, writing $y(t) := \int_\Omega \big(u(\cdot,t)-\overline{u}_0\big)^2 + \kappa\int_\Omega \big(v(\cdot,t)-\overline{u}_0\big)^2$, $t\ge 0$, and $\lambda_1 := \min\big\{1\,,\,\frac{c_1c_3}{4}\big\}$, we infer that

$$y'(t) + \lambda_1 y(t) \le 0 \qquad \text{for all } t>0,$$

and that thus with $c_4 = c_4(u_0,v_0) := y(0)$ we have

$$y(t) \le c_4 e^{-\lambda_1 t} \qquad \text{for all } t>0,$$

meaning that

$$\|u(\cdot,t)-\overline{u}_0\|_{L^2(\Omega)} \le \sqrt{c_4}\,e^{-\frac{\lambda_1}{2}t} \qquad \text{for all } t>0 \tag{4.7}$$

as well as

$$\|v(\cdot,t)-\overline{u}_0\|_{L^2(\Omega)} \le \sqrt{\frac{c_4}{\kappa}}\,e^{-\frac{\lambda_1}{2}t} \qquad \text{for all } t>0. \tag{4.8}$$

Apart from that, in view of the bounds asserted by Theorem 1.1 we may invoke standard parabolic regularity theory ([16], [12]) along with Lemma 3.3 to fix $c_5 = c_5(u_0,v_0)>0$ such that

$$\|u(\cdot,t)-\overline{u}_0\|_{W^{1,\infty}(\Omega)} + \|v(\cdot,t)-\overline{u}_0\|_{W^{1,\infty}(\Omega)} \le c_5 \qquad \text{for all } t\ge 1,$$

so that a straightforward Gagliardo-Nirenberg type interpolation shows that with some $c_6>0$,

$$\begin{aligned}
\|u(\cdot,t)-\overline{u}_0\|_{L^\infty(\Omega)} &\le c_6\|u(\cdot,t)-\overline{u}_0\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}}\|u(\cdot,t)-\overline{u}_0\|_{L^2(\Omega)}^{\frac{2}{n+2}} \\
&\le c_4^{\frac{1}{n+2}}c_5^{\frac{n}{n+2}}c_6 e^{-\frac{\lambda_1}{n+2}t} \qquad \text{for all } t\ge 1,
\end{aligned}$$

and that, similarly,

$$\|v(\cdot,t)-\overline{u}_0\|_{L^\infty(\Omega)} \le \Big(\frac{c_4}{\kappa}\Big)^{\frac{1}{n+2}} c_5^{\frac{n}{n+2}}c_6 e^{-\frac{\lambda_1}{n+2}t} \qquad \text{for all } t\ge 1.$$

As u and v are bounded in $\Omega\times(0,1)$, this implies (1.7) and (1.8) with some suitably large $C>0$ if we let $\lambda := \frac{\lambda_1}{n+2}$. □

Acknowledgement. The second author acknowledges support of the *Deutsche Forschungsgemeinschaft* in the context of the project *Emergence of structures and advantages in cross-diffusion systems* (Project No. 411007140, GZ: WI 3707/5-1).

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