The dampening role of large repulsive convection in a chemotaxis system modeling tumor angiogenesis

Youshan Tao*
School of Mathematical Sciences, Shanghai Jiao Tong University,
Shanghai 200240, P.R. China

Michael Winkler#
Institut für Mathematik, Universität Paderborn,
33098 Paderborn, Germany

Abstract

This paper is concerned with a parabolic-parabolic-elliptic system arising as a simplifed model for the initial phase of tumor-related angiogenesis. As essential characteristics, this system contains a cascade-like coupling of two chemotaxis processes involving signal production as in classical Keller-Segel systems, linked to a further repulsive cross-diffusive mechanism. It is shown that in n-dimensional bounded convex domains with $n \leq 3$ and for any given suitably regular initial data, a suitable assumption on largeness of the parameter corresponding to the latter repulsion term ensures global existence of a bounded classical solution, and hence rules out the possibility of finite-time explosions which in the absence of any such chemorepulsion are known to occur at least in a certain borderline case.

Key words: chemotaxis; attraction; repulsion; angiogenesis
MSC (2010): 35A01, 35K57, 35Q92, 92C17

*taoys@sjtu.edu.cn
#michael.winkler@math.uni-paderborn.de

1 Introduction

In the past decade, focal points in the analysis of cross-diffusive migration processes have to a considerable extent been oriented toward the understanding of taxis mechanisms in increasingly complex frameworks. Based on appropriate methodological advances, the recent literature in this direction has been able to address contexts noticeably far beyond those covered by classical Keller-Segel type systems, and the list of examples in this field meanwhile includes not only rather close relatives of the latter concerned with chemotaxis under the influence of logistic sources or indirect types of signal production ([21], [31], [28], [30]), but also couplings to liquid environments ([15], [6], [34]) and even to further taxis mechanisms driven by either diffusible ([25], [17]), or non-diffusible cues ([27], [36], [22]).

In contrast to this, with regard to models simultaneously accounting for taxis processes that are coupled in a cascade-like manner by involving signals which themselves undergo cross-diffusive movement, the knowledge seems yet at a rather rudimentary stage. Indeed, reflecting a substantial increase of mathematical complexity apparently going along with the concatenation of taxis mechanisms, the literature in this direction so far seems to have essentially concentrated on the analysis of systems generalizing a prototypical model for cascade-type coupling of chemotaxis, which in its most crucial part reduces to the two equations

$$\begin{cases} u_t &= \Delta u - \chi_1 \nabla \cdot (u \nabla v), \\ v_t &= \Delta v - \chi_2 \nabla \cdot (v \nabla w), \end{cases} \tag{1.1}$$

coupled to an appropriate evolution problem for the third among the unknown quantities u, v and w. In fact, if in accordance with the modeling background described in [24] this latter variable is governed by sufficiently dissipative processes, such as e.g. those underlying the absorption-diffusion equation

$$w_t = \Delta w - w - (u+v)w, \tag{1.2}$$

then the resulting so-called forager-exploiter system and various close relatives have been found to admit at least some basic global existence theories even when both taxis processes therein are assumed to be attractive by choosing $\chi_1 > 0$ and $\chi_2 > 0$ ([29], [19], [2], [4], [35]).

Compared to this, in cases in which taxis cascades are additionally influenced by potentially aggregation-supporting feedback loops in the style of those in classical Keller-Segel systems, much less seems known despite the fact that such situations apparently arise in quite a natural manner in relevant application contexts. Indeed, such a type of interplay forms a key coupling in a well-known model for the initial phase of tumor-related angiogenesis ([20], [23]), which in its essentially most comprehensive version describes the evolution of endothelial cells at population density u, fibronectin as a certain adhesive chemical at concentration v, and the distribution w of the so-called extracellular matrix according to

$$\begin{cases} u_t &= d_1 \Delta u - \chi_1 \nabla \cdot (u \nabla v) + \xi_1 \nabla \cdot (u \nabla w), \\ v_t &= d_2 \Delta v + \xi_2 \nabla \cdot (v \nabla w) - \mu_2 v + \lambda_2 u, \\ w_t &= d_3 \Delta w - \mu_3 w + \lambda_3 u, \end{cases} \tag{1.3}$$

with positive parameters $\chi_1, \xi_1, \xi_2, \mu_2, \mu_3, \lambda_2, \lambda_3, d_1, d_2$ and d_3. Inter alia, this model accounts for the hypotheses that the endothelial cells secrete both matrix and fibronectin, that the endothelial cells as well as the adhesive sites are carried with the extracellular matrix, and that the endothelial cells move chemotactically toward increasing fibronectin concentrations (cf. also [16] for more details about the background of (1.3)).

In particular, contrary to (1.1)-(1.2) this system contains production terms $+\lambda_2 u$ and $+\lambda_3 u$ in both its second and its third equations, and at least in the borderline case $\lambda_3 = 0$ in which e.g. no-flux boundary conditions imply $w \equiv 0$ whenever $w(\cdot, 0) \equiv 0$, as a subsystem this model includes the classical two-component Keller-Segel system with its well-known ability to enforce finite-time blow-up in two- or higher-dimensional settings ([9], [33]). Describing the considered angiogenesis processes within the realm of singularity-free solutions thus amounts to an ambition of actually rather qualitative character and thus going somewhat beyond mere issues of existence, namely to adequately identifying potentially stabilizing effects of the two repulsive taxis mechanisms in (1.3). Thus henceforth concerned with the question under which circumstances global bounded solutions exist, we will especially focus on the challenge related to determining how far large repulsive chemotaxis may overbalance a simultaneously present chemoattraction when both involved signals are produced by individuals of the considered population. Affirmative results have been achieved for more classical attraction-repulsion systems essentially corresponding to the choice $\xi_2 = 0$ in (1.3) ([25]; cf. also [13] for a simplified parabolic-parabolic-elliptic version), but in the presence of a cascade-type taxis coupling such as obtained on letting $\xi_2 > 0$, available knowledge seems to reduce to a statement on global existence of bounded classical solutions to a one-dimensional Neumann problem ([16]).

Main results. In order to supplement this by a corresponding result applicable to two- and three-dimensional frameworks, in this paper we shall make use of a standard quasi-stationary approximation procedure, quite well-established in the context of chemotaxis systems ([12], [11]) and relying on the circumstance that matrix diffuses much faster than fibronectin and endothelial cells. For convenience in notation moreover keeping $\xi := \xi_1$ as the only free parameter, we shall subsequently be concerned with the initial-boundary value problem

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w), & x \in \Omega,\ t > 0, \\ v_t = \Delta v + \nabla \cdot (v \nabla w) - v + u, & x \in \Omega,\ t > 0, \\ 0 = \Delta w - w + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, where throughout this paper we shall assume that

$$\begin{cases} u_0 \in C^0(\overline{\Omega}) \text{ is nonnegative with } u_0 \not\equiv 0, \quad \text{and that} \\ v_0 \in W^{1,\infty}(\Omega) \text{ is nonnegative.} \end{cases} \tag{1.5}$$

Our main results then confirms that indeed in any physically relevant space dimension, suitably large repulsive convection prevents any blow-up phenomenon:

Theorem 1.1 *Let $n \le 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with smooth boundary, and suppose that u_0 and v_0 satisfy (1.5). Then there exists $\xi_0(u_0, v_0) > 0$ with that property that for any choice of $\xi > \xi_0(u_0, v_0)$, the problem (1.4) admits a global classical solution (u, v, w), uniquely determined* *through the inclusions*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ v \in \bigcap_{q>n} C^0([0,\infty); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \qquad \textit{and} \\ w \in C^{2,0}(\overline{\Omega} \times (0,\infty)), \end{cases}$$

which is bounded in the sense that

$$\sup_{t\to\infty} \Big\{ \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} \Big\} < \infty. \tag{1.6}$$

Two crucial ideas. A cornerstone of our analysis will consist in the derivation of a priori bounds for v in L^∞, which will be achieved in Section 3 by making appropriate use of the repulsive character of cross-diffusion in the second equation from (1.4), and by relying on a Moser-type iteration technique which appears to be novel in the context of taxis-type problems. Thereafter, analyzing the time evolution of the functional

$$\int_\Omega u^2 + a \int_\Omega |\nabla v|^4 \tag{1.7}$$

with suitably chosen $a > 0$ will provide estimates for (u, v) with respect to the norm in $L^2(\Omega)\times W^{1,4}(\Omega)$ (Section 4). Suitable bootstrap arguments thereupon provide L^∞ bounds for u and hence enable us to complete the proof of Theorem 1.1 in Section 5.

2 Local existence and extensibility

The following statement on local existence and extensibility can be verified by adapting arguments well-established in the context of related chemotaxis problems, such as that detailed e.g. in [31], to the present setting in a straightforward manner, so that we may refrain from presenting an elaborate proof here.

Lemma 2.1 *Let $n \ge 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and suppose that $\xi \in \mathbb{R}$, and that u_0 and v_0 comply with (1.5). Then there exist $T_{max} = T_{max}(u_0, v_0, \xi) \in (0,\infty]$ and a uniquely determined triple of functions*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{max})), \\ v \in \bigcap_{q>n} C^0([0,T_{max}); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,T_{max})) \qquad \textit{and} \\ w \in C^{2,0}(\overline{\Omega} \times (0,T_{max})), \end{cases}$$

such that u, v and w are positive in $\overline{\Omega} \times (0, T_{max})$, that (u, v, w) solves (1.4) in the classical sense in $\Omega \times (0, T_{max})$, and that

$$\textit{if } T_{max} < \infty, \quad \textit{then} \quad \limsup_{t\nearrow T_{max}} \Big\{ \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,q}(\Omega)} \Big\} = \infty \textit{ for all } q > \max\{n, 2\}. \tag{2.1}$$

Furthermore, this solution has the property that

$$\int_\Omega u(\cdot,t) = \int_\Omega u_0 \qquad \textit{for all } t \in (0, T_{max}). \tag{2.2}$$

3 An L^∞ estimate for v

Let us first make sure that thanks to our overall assumption on the space dimension, (2.2) entails a basic regularity feature of w.

Lemma 3.1 *Assume (1.5). Then there exists $K(u_0, v_0) > 0$ such that whenever $\xi > 0$, the corresponding solution of (1.4) satisfies*

$$\|w(\cdot,t)\|_{L^2(\Omega)} \le K(u_0,v_0) \qquad \text{for all } t \in (0,T_{max}). \tag{3.1}$$

PROOF. Since $n \le 3$, we have $\frac{2n}{n+2} < \frac{n}{n-1}$, so that we can pick $q \ge 1$ such that

$$\frac{2n}{n+2} \le q < \frac{n}{n-1}. \tag{3.2}$$

Here the second inequality warrants applicability of a well-known result from elliptic regularity theory ([3]), which namely provides $c_1 > 0$ such that whenever $f \in C^0(\overline{\Omega})$ and $\varphi \in C^2(\overline{\Omega})$ are such that

$$\begin{cases} -\Delta\varphi + \varphi = f, & x \in \Omega, \\ \frac{\partial\varphi}{\partial\nu} = 0, & x \in \partial\Omega, \end{cases}$$

we have

$$\|\varphi\|_{W^{1,q}(\Omega)} \le c_1 \|f\|_{L^1(\Omega)}. \tag{3.3}$$

On the other hand, the first inequality in (3.2) guarantees continuity of the embedding $W^{1,q}(\Omega) \hookrightarrow L^2(\Omega)$, which enables us to pick $c_2 > 0$ fulfilling

$$\|\varphi\|_{L^2(\Omega)} \le c_2 \|\varphi\|_{W^{1,q}(\Omega)} \qquad \text{for all } \varphi \in W^{1,q}(\Omega). \tag{3.4}$$

In view of (1.4) and (2.2), combining (3.3) with (3.4) yields the claim if we let $K(u_0,v_0) := c_1 c_2 \int_\Omega u_0$. □

We next make use of the latter in the context of a standard L^p testing procedure, and add a suitable recursive argument of Moser-type, in order to see that thanks to the repulsive character of the cross-diffusion mechanism appearing in the second equation from (1.4), the component v enjoys time-independent bounds not only in $L^p(\Omega)$ for all finite p, but in fact even in $L^\infty(\Omega)$. In view of the presence of said cross-diffusive interaction between v and w, the latter conclusion seems not accessible to straightforward reasonings e.g. involving smoothing estimates for the heat semigroup, nor to standard results on parabolic Moser iterations (as documented in [26], for instance), so that including a detailed argument appears to be in order here.

Lemma 3.2 *If (1.5) holds, then there exists $K(u_0, v_0) > 0$ such that for any choice of $\xi > 0$, the solution of (1.4) has the property that*

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le K(u_0,v_0) \qquad \text{for all } t \in (0,T_{max}). \tag{3.5}$$

Proof. In accordance with Lemma 3.1, given (u_0, v_0) fulfilling (1.5) let us fix $k_1 = k_1(u_0, v_0) > 0$ such that whenever $\xi > 0$, we have

$$\|w(\cdot,t)\|_{L^2(\Omega)} \le k_1 \qquad \text{for all } t \in (0, T_{max}(u_0, v_0, \xi)), \tag{3.6}$$

and apart from that, by means of the Gagliardo-Nirenberg inequality we can find $c_1 > 0$ satisfying

$$\|\varphi\|^2_{L^4(\Omega)} \le c_1 \|\nabla\varphi\|^{2\theta}_{L^2(\Omega)} \|\varphi\|^{2(1-\theta)}_{L^1(\Omega)} + c_1 \|\varphi\|^2_{L^1(\Omega)} \qquad \text{for all } \varphi \in W^{1,2}(\Omega), \tag{3.7}$$

where $\theta := \frac{3n}{2n+4} \in (0, 1)$. Now given $\xi > 0$, we fix $T_{max} = T_{max}(u_0, v_0, \xi)$ and (u, v, w) as correspondingly be provided by Lemma 2.1, and for integers $j \ge 0$ we let $p_j := 2^j$ and

$$M_j(\xi, T) := 1 + \max_{t\in[0,T]} \int_\Omega v^{p_j}(\cdot, t), \qquad \xi > 0,\ T \in (0, T_{max}(u_0, v_0, \xi)). \tag{3.8}$$

To appropriately estimate $M_j(\xi, T)$, for $p = p_j$ with $j \ge 1$ we use (1.4) to compute

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_\Omega v^p + (p-1)\int_\Omega v^{p-2}|\nabla v|^2 \\ &\quad = \frac{p-1}{p}\int_\Omega v^p \Delta w + \int_\Omega u v^{p-1} - \int_\Omega v^p \\ &\quad = \frac{p-1}{p}\int_\Omega v^p w - \frac{p-1}{p}\int_\Omega u v^p + \int_\Omega u v^{p-1} - \int_\Omega v^p \qquad \text{for all } t \in (0, T_{max}), \end{aligned} \tag{3.9}$$

where since $p \ge 2$, we have

$$(p-1)\int_\Omega v^{p-2}|\nabla v|^2 \ge \frac{p}{2}\int_\Omega v^{p-2}|\nabla v|^2 = \frac{2}{p}\int_\Omega |\nabla v^{\frac{p}{2}}|^2 \qquad \text{for all } t \in (0, T_{max}), \tag{3.10}$$

while Young's inequality implies that thanks to (2.2),

$$\begin{aligned} \int_\Omega u v^{p-1} - \frac{p-1}{p}\int_\Omega u v^p &\le \int_\Omega u \cdot \Big(\frac{p-1}{p} v^p + \frac{1}{p}\Big) - \frac{p-1}{p}\int_\Omega u v^p \\ &\le \frac{1}{p}\int_\Omega u = \frac{k_2}{p} \qquad \text{for all } t \in (0, T_{max}) \end{aligned} \tag{3.11}$$

with $k_2 = k_2(u_0, v_0) := \int_\Omega u_0$. In coping with the first summand on the right of (3.9) we use the Cauchy-Schwarz inequality and apply (3.6) and (3.7) to see that

$$\begin{aligned} \frac{p-1}{p}\int_\Omega v^p w &\le \int_\Omega v^p w \\ &\le \Big\{\int_\Omega v^{2p}\Big\}^{\frac12} \cdot \Big\{\int_\Omega w^2\Big\}^{\frac12} \\ &\le k_1 \cdot \Big\{\int_\Omega v^{2p}\Big\}^{\frac12} \\ &= k_1 \|v^{\frac{p}{2}}\|^2_{L^4(\Omega)} \\ &\le c_1 k_1 \|\nabla v^{\frac{p}{2}}\|^{2\theta}_{L^2(\Omega)} \|v^{\frac{p}{2}}\|^{2(1-\theta)}_{L^1(\Omega)} + c_1 k_1 \|v^{\frac{p}{2}}\|^2_{L^1(\Omega)} \qquad \text{for all } t \in (0, T_{max}). \end{aligned}$$

As herein for each $T \in (0, T_{max})$ we have

$$\|v^{\frac{p}{2}}\|_{L^1(\Omega)} = \int_\Omega v^{p_{j-1}} \le M_{j-1}(\xi, T) \qquad \text{for all } t \in (0, T),$$

we may again rely on Young's inequality in estimating

$$\begin{aligned}
\frac{p-1}{p} \int_\Omega v^p w &\le c_1 k_1 M_{j-1}^{2(1-\theta)}(\xi, T) \|\nabla v^{\frac{p}{2}}\|_{L^2(\Omega)}^{2\theta} + c_1 k_1 M_{j-1}^2(\xi, T) \\
&= \left\{ \frac{2}{p} \int_\Omega |\nabla v^{\frac{p}{2}}|^2 \right\}^\theta \cdot \left\{ c_1 k_1 \cdot \left(\frac{p}{2}\right)^\theta \cdot M_{j-1}^{2(1-\theta)}(\xi, T) \right\} + c_1 k_1 M_{j-1}^2(\xi, T) \\
&\le \frac{2}{p} \int_\Omega |\nabla v^{\frac{p}{2}}|^2 + 2^{-\frac{\theta}{1-\theta}} (c_1 k_1)^{\frac{1}{1-\theta}} p^{\frac{\theta}{1-\theta}} M_{j-1}^2(\xi, T) + c_1 k_1 M_{j-1}^2(\xi, T) \\
&\le \frac{2}{p} \int_\Omega |\nabla v^{\frac{p}{2}}|^2 + k_3 p^{\frac{\theta}{1-\theta}} M_{j-1}^2(\xi, T) \qquad \text{for all } t \in (0, T)
\end{aligned}$$

with $k_3 = k_3(u_0, v_0) := 2^{-\frac{\theta}{1-\theta}} (c_1 k_1)^{\frac{1}{1-\theta}} + c_1 k_1$, because $p \ge 1$. Together with (3.10) and (3.11), this shows that for any such T, (3.9) warrants that again since $p \ge 1$, and since $M_{j-1}(\xi, T) \ge 1$,

$$\begin{aligned}
\frac{d}{dt} \int_\Omega v^p + p \int_\Omega v^p &\le k_3 p^{\frac{1}{1-\theta}} M_{j-1}^2(\xi, T) + k_2 \\
&\le (k_2 + k_3) p^{\frac{1}{1-\theta}} M_{j-1}^2(\xi, T) \qquad \text{for all } t \in (0, T),
\end{aligned}$$

which through an ODE comparison argument entails that

$$\int_\Omega v^p \le \max \left\{ \int_\Omega v_0^p \, , \, (k_2 + k_3) p^{\frac{1}{1-\theta}} M_{j-1}^2(\xi, T) \right\} \qquad \text{for all } t \in (0, T).$$

Thus, for any choice of $\xi > 0$ and each $j \ge 1$,

$$M_j(\xi, T) \le 1 + \max \left\{ \int_\Omega v_0^{p_j} \, , \, (k_2 + k_3) p_j^{\frac{1}{1-\theta}} M_{j-1}^2(\xi, T) \right\} \qquad \text{for all } T \in (0, T_{max}(u_0, v_0, \xi)), \quad (3.12)$$

which we combine with the observation that due to (1.4) and (2.2),

$$\frac{d}{dt} \int_\Omega v + \int_\Omega v = \int_\Omega u = k_2 \qquad \text{for all } t \in (0, T_{max}(u_0, v_0, \xi)),$$

and that hence, again by an ODE comparison, $\int_\Omega v \le k_4 = k_4(u_0, v_0) := \max \left\{ \int_\Omega v_0 \, , \, k_2 \right\}$ for all $t \in (0, T_{max}(u_0, v_0, \xi))$. Therefore, $M_0(\xi, T) \le 1 + k_4$ for all $\xi > 0$ and $T \in (0, T_{max}(u_0, v_0, \xi))$, so that from (3.12) we firstly conclude by means of a recursive argument that, in fact, for each integer $j \ge 0$ letting

$$\overline{M}_j := \sup_{\xi > 0} \sup_{T \in (0, T_{max}(u_0, v_0, \xi))} M_j(\xi, T)$$

introduces a sequence $(\overline{M}_j)_{j\ge 0}$ of finite numbers $\overline{M}_j \ge 1$ which satisfy

$$\overline{M}_0 \le 1+k_4 \qquad \text{and} \qquad \overline{M}_j \le 1+\max\left\{\int_\Omega v_0^{p_j}\,,\ (k_2+k_3)p_j^{\frac{1}{1-\theta}}\overline{M}_{j-1}^2\right\} \quad \text{for all } j\ge 1. \tag{3.13}$$

Now the remainder is quite standard: If $\overline{M}_j \le 1+\int_\Omega v_0^{p_j}$ for infinitely many $j\ge 0$, then from (3.8) it directly follows that for any choice of $\xi>0$,

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le \liminf_{j\to\infty}(\overline{M}_j-1)^{\frac{1}{p_j}} \le \liminf_{j\to\infty}\left\{\int_\Omega v_0^{p_j}\right\}^{\frac{1}{p_j}} = \|v_0\|_{L^\infty(\Omega)} \qquad \text{for all } t\in(0,T_{max}(u_0,v_0,\xi)). \tag{3.14}$$

Otherwise, (3.13) warrants the existence of $j_0\in\mathbb{N}$ such that

$$\overline{M}_j \le (k_2+k_3)p_j^{\frac{1}{1-\theta}}\overline{M}_{j-1}^2 \qquad \text{for all } j\ge j_0,$$

meaning that with some $k_5=k_5(u_0,v_0)>1$, clearly independent of ξ by construction of $(\overline{M}_j)_{j\ge 0}$, we actually have

$$\overline{M}_j \le k_5^j\overline{M}_{j-1}^2 \qquad \text{for all } j\ge 1.$$

Hence, by a straightforward induction we infer that in this case,

$$\overline{M}_j \le k_5^{2^{j+1}-j-2}\overline{M}_0^{2^j} \le k_5^{2^{j+1}}\overline{M}_0^{2^j} \qquad \text{for all } j\ge 1$$

and that thus, by (3.13), given any $\xi>0$ we have

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le \liminf_{j\to\infty}\overline{M}_j^{\frac{1}{2^j}} \le k_5^2\overline{M}_0 \le k_5^2\cdot(1+k_4) \qquad \text{for all } t\in(0,T_{max}(u_0,v_0,\xi)).$$

Along with (3.14), this establishes the claim. □

4 $L^2\times W^{1,4}$ bounds for (u,v) for large ξ

The purpose of this section consists in analyzing the functional in (1.7), and in describing its evolution through a linear ODI that contains a suitable absorption term whenever $\xi>0$ is appropriately large. Our first step toward this makes use of the repulsive taxis interplay between u and w in the course of the following quite straightforward testing procedure.

Lemma 4.1 *Assume (1.5). Then there exists $K(u_0,v_0)>0$ such that if $\xi\ge 1$, then*

$$\frac{d}{dt}\int_\Omega u^2+\int_\Omega|\nabla u|^2+\frac{\xi}{2}\int_\Omega u^3 \le K(u_0,v_0)\int_\Omega|\nabla v|^6+K(u_0,v_0)\cdot\xi \qquad \textit{for all } t\in(0,T_{max}). \tag{4.1}$$

Proof. Integrating by parts in the first equation from (1.4) and using that $\Delta w = w - u$, for arbitrary $\xi > 0$ we obtain the identity

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 + \int_\Omega |\nabla u|^2 &= \int_\Omega u\nabla u\cdot\nabla v - \xi\int_\Omega u\nabla u\cdot\nabla w \\ &= \int_\Omega u\nabla u\cdot\nabla v - \frac{\xi}{2}\int_\Omega \nabla u^2\cdot\nabla w \\ &= \int_\Omega u\nabla u\cdot\nabla v + \frac{\xi}{2}\int_\Omega u^2 w - \frac{\xi}{2}\int_\Omega u^3 \qquad \text{for all } t\in(0,T_{max}), \qquad (4.2)\end{aligned}$$

where by Young's inequality we can find $c_1 > 0$ such that whenever $\xi \geq 1$,

$$\begin{aligned}\int_\Omega u\nabla u\cdot\nabla v &\leq \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega u^2|\nabla v|^2 \\ &\leq \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\xi}{8}\int_\Omega u^3 + c_1\int_\Omega |\nabla v|^6 \qquad \text{for all } t\in(0,T_{max}). \qquad (4.3)\end{aligned}$$

Also due to Young's inequality, there exists $c_2 > 0$ such that if $\xi > 0$, then

$$\frac{\xi}{2}\int_\Omega u^2 w \leq \frac{\xi}{16}\int_\Omega u^3 + c_2\xi\int_\Omega w^3 \qquad \text{for all } t\in(0,T_{max}), \qquad (4.4)$$

and to further estimate the rightmost expression herein we invoke the Gagliardo-Nirenberg inequality along with standard elliptic regularity theory ([8]) to fix $c_3 > 0$ and $c_4 > 0$ fulfilling

$$\begin{aligned}\|\varphi\|^3_{L^3(\Omega)} &\leq c_3\|\varphi\|^{3\theta}_{W^{2,3}(\Omega)}\|\varphi\|^{3(1-\theta)}_{L^2(\Omega)} \\ &\leq c_4\|-\Delta\varphi+\varphi\|^{3\theta}_{L^3(\Omega)}\|\varphi\|^{3(1-\theta)}_{L^2(\Omega)} \qquad \text{for all } \varphi\in W^{2,3}(\Omega) \text{ such that } \tfrac{\partial\varphi}{\partial\nu}|_{\partial\Omega}=0,\end{aligned}$$

with $\theta := \frac{n}{n+12} \in (0,1)$. As Lemma 3.1 provides $k_1 = k_1(u_0,v_0) > 0$ such that

$$\|w\|_{L^2(\Omega)} \leq k_1 \qquad \text{for all } t\in(0,T_{max}),$$

namely, this shows that

$$\begin{aligned}c_2\xi\int_\Omega w^3 &\leq c_2c_4k_1^{3(1-\theta)}\xi\|-\Delta w + w\|^{3\theta}_{L^3(\Omega)} \\ &= c_2c_4k_1^{3(1-\theta)}\xi\|u\|^{3\theta}_{L^3(\Omega)} \qquad \text{for all } t\in(0,T_{max}),\end{aligned}$$

so that another application of Young's inequality, relying on the inequality $\theta < 1$, yields $k_2 = k_2(u_0,v_0) > 0$ satisfying

$$c_2\xi\int_\Omega w^3 \leq \frac{\xi}{16}\|u\|^3_{L^3(\Omega)} + k_2\xi \qquad \text{for all } t\in(0,T_{max}).$$

Therefore, (4.4) entails that

$$\frac{\xi}{2}\int_\Omega u^2 w \leq \frac{\xi}{8}\int_\Omega u^3 + k_2\xi \qquad \text{for all } t\in(0,T_{max}),$$

which together with (4.3) implies that as a consequence of (4.2), indeed

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 + \frac{1}{2}\int_\Omega |\nabla u|^2 \le c_1 \int_\Omega |\nabla v|^6 + k_2\xi - \frac{\xi}{4}\int_\Omega u^3$$

for all $t \in (0, T_{max})$. □

To appropriately absorb the first summand on the right-hand side of (4.1) on the basis of the diffusive contribution to the second equation from (1.4), in Lemma 4.3 we shall investigate the evolution of the integral $\int_\Omega |\nabla v|^4$. To prepare our estimation of respectively appearing ill-signed integrals, let us separately state the following interpolation lemma.

Lemma 4.2 *Let $n \ge 1$ and $G \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let $\varphi \in C^2(\overline{G})$ and $\psi \in C^2(\overline{G})$ be such that $\frac{\partial\varphi}{\partial\nu} = 0$ of ∂G. Then*

$$\left| \int_G |\nabla\varphi|^2 \nabla\varphi \cdot \nabla(\nabla\varphi \cdot \nabla\psi) \right| \le \Big(1 + \frac{\sqrt{n}}{4}\Big) \|\nabla\varphi\|_{L^6(G)}^4 \|D^2\psi\|_{L^3(G)} \tag{4.5}$$

and

$$\left| \int_G \varphi \Delta\psi \nabla \cdot (|\nabla\varphi|^2 \nabla\varphi) \right| \le (2 + \sqrt{n}) \|\varphi\|_{L^\infty(G)} \|\nabla\varphi\|_{L^6(G)} \cdot \left\{ \int_G |\nabla\varphi|^2 |D^2\varphi|^2 \right\}^{\frac{1}{2}} \cdot \|\Delta\psi\|_{L^3(G)} \tag{4.6}$$

as well as

$$\int_G |\nabla\varphi|^6 \le (4 + \sqrt{n})^2 \|\varphi\|_{L^\infty(G)}^2 \int_G |\nabla\varphi|^2 |D^2\varphi|^2. \tag{4.7}$$

Proof. Expanding

$$\nabla(\nabla\varphi \cdot \nabla\psi) = D^2\varphi \cdot \nabla\psi + D^2\psi \cdot \nabla\varphi$$

and

$$\nabla|\nabla\varphi|^4 = 2|\nabla\varphi|^2 \nabla|\nabla\varphi|^2 = 4|\nabla\varphi|^2 D^2\varphi \cdot \nabla\varphi, \tag{4.8}$$

we may integrate by parts to see that

$$\begin{aligned} \int_G |\nabla\varphi|^2 \nabla\varphi \cdot \nabla(\nabla\varphi \cdot \nabla\psi) &= \int_G |\nabla\varphi|^2 (D^2\varphi \cdot \nabla\varphi) \cdot \nabla\psi + \int_G |\nabla\varphi|^2 \nabla\varphi \cdot (D^2\psi \cdot \nabla\varphi) \\ &= \frac{1}{4}\int_G \nabla|\nabla\varphi|^4 \cdot \nabla\psi + \int_G |\nabla\varphi|^2 \nabla\varphi \cdot (D^2\psi \cdot \nabla\varphi) \\ &= -\frac{1}{4}\int_G |\nabla\varphi|^4 \Delta\psi + \int_G |\nabla\varphi|^2 \nabla\varphi \cdot (D^2\psi \cdot \nabla\varphi). \end{aligned}$$

Since for all $\rho \in C^2(G)$ we have

$$|\Delta\rho| \le \sqrt{n}|D^2\rho| \qquad \text{in } G, \tag{4.9}$$

due to the Hölder inequality this implies that

$$\begin{aligned} \left| \int_G |\nabla\varphi|^2 \nabla\varphi \cdot \nabla(\nabla\varphi \cdot \nabla\psi) \right| &\le \frac{\sqrt{n}}{4}\int_G |\nabla\varphi|^4 |D^2\psi| + \int_G |\nabla\varphi|^4 |D^2\psi| \\ &\le \Big(\frac{\sqrt{n}}{4} + 1\Big) \cdot \left\{ \int_G |\nabla\varphi|^6 \right\}^{\frac{2}{3}} \cdot \left\{ \int_G |D^2\psi|^3 \right\}^{\frac{1}{3}} \end{aligned}$$

and hence establishes (4.5).

To derive (4.6), we again rely on (4.9) and employ the Hölder inequality to indeed obtain

$$
\begin{aligned}
&\Big|\int_G \varphi\Delta\psi\nabla\cdot(|\nabla\varphi|^2\nabla\varphi)\Big| \\
&\qquad = \Big|2\int_G \varphi\Delta\psi\nabla\varphi\cdot(D^2\varphi\cdot\nabla\varphi)+\int_G\varphi\Delta\psi|\nabla\varphi|^2\Delta\varphi\Big| \\
&\qquad \le (2+\sqrt{n})\int_G|\varphi|\cdot|\Delta\psi|\cdot|\nabla\varphi|^2\cdot|D^2\varphi| \\
&\qquad \le (2+\sqrt{n})\|\varphi\|_{L^\infty(G)}\cdot\Big\{\int_G|\Delta\psi|^3\Big\}^{\frac{1}{3}}\cdot\Big\{\int_G|\nabla\varphi|^6\Big\}^{\frac{1}{6}}\cdot\Big\{\int_G|\nabla\varphi|^2|D^2\varphi|^2\Big\}^{\frac{1}{2}}.
\end{aligned}
$$

Finally, integrating by parts and once more using (4.8) and (4.9) we find that thanks to the Cauchy-Schwarz inequality,

$$
\begin{aligned}
\int_G|\nabla\varphi|^6 &= \int_G|\nabla\varphi|^4\nabla\varphi\cdot\nabla\varphi \\
&= -\int_G\varphi\nabla\varphi\cdot\nabla|\nabla\varphi|^4-\int_G\varphi|\nabla\varphi|^4\Delta\varphi \\
&= -4\int_G\varphi|\nabla\varphi|^2\nabla\varphi\cdot(D^2\varphi\cdot\nabla\varphi)-\int_G\varphi|\nabla\varphi|^4\Delta\varphi \\
&\le (4+\sqrt{n})\int_G|\varphi|\cdot|\nabla\varphi|^4\cdot|D^2\varphi| \\
&\le (4+\sqrt{n})\|\varphi\|_{L^\infty(G)}\cdot\Big\{\int_G|\nabla\varphi|^6\Big\}^{\frac{1}{2}}\cdot\Big\{\int_G|\nabla\varphi|^2D^2\varphi|^2\Big\}^{\frac{1}{2}},
\end{aligned}
$$

from which (4.7) readily follows. □

Relying on our the convexity assumption on Ω, by means of the latter we can indeed control the evolution of $\int_\Omega|\nabla v|^4$ as follows.

Lemma 4.3 *If (1.5) is valid, then there exists $K(u_0,v_0)>0$ with the property that whenever $\xi>0$,*

$$
\frac{d}{dt}\int_\Omega|\nabla v|^4+2\int_\Omega|\nabla v|^2|D^2v|^2+4\int_\Omega|\nabla v|^4\le K(u_0,v_0)\int_\Omega u^3 \qquad \textit{for all } t\in(0,T_{max}). \tag{4.10}
$$

PROOF. We first note that according to Lemma 3.2 we can fix $k_1=k_1(u_0,v_0)>0$ such that if $\xi>0$ is arbitrary, then

$$
\|v(\cdot,t)\|_{L^\infty(\Omega)}\le k_1 \qquad \text{for all } t\in(0,T_{max}), \tag{4.11}
$$

and recall standard elliptic regularity theory ([8]) to find $c_1>0$ fulfilling

$$
\|D^2\varphi\|_{L^3(\Omega)}\le c_1\|-\Delta\varphi+\varphi\|_{L^3(\Omega)} \qquad \text{for all } \varphi\in W^{2,3}(\Omega) \text{ such that } \tfrac{\partial\varphi}{\partial\nu}|_{\partial\Omega}=0. \tag{4.12}
$$

Now assuming that (1.5) holds and that $\xi > 0$ since $\frac{\partial|\nabla v|^2}{\partial\nu} \le 0$ on $\partial\Omega \times (0, T_{max})$ by convexity of Ω ([18]), on the basis of an integration by parts in the second equation from (1.4) we obtain that

$$\begin{aligned}
\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla v|^4 &= \int_\Omega |\nabla v|^2 \nabla v \cdot \nabla \Big\{ \Delta v + \nabla \cdot (v\nabla w) - v + u \Big\} \\
&= \frac{1}{2}\int_\Omega |\nabla v|^2 \Delta |\nabla v|^2 - \int_\Omega |\nabla v|^2 |D^2 v|^2 \\
&\quad + \int_\Omega |\nabla v|^2 \nabla v \cdot \nabla(\nabla v \cdot \nabla w) + \int_\Omega |\nabla v|^2 \nabla v \cdot \nabla (v\Delta w) \\
&\quad - \int_\Omega |\nabla v|^4 + \int_\Omega |\nabla v|^2 \nabla v \cdot \nabla u \\
&= -\frac{1}{2}\int_\Omega \Big|\nabla |\nabla v|^2\Big|^2 + \frac{1}{2}\int_{\partial\Omega} |\nabla v|^2 \frac{\partial |\nabla v|^2}{\partial\nu} - \int_\Omega |\nabla v|^2 |D^2 v|^2 \\
&\quad + \int_\Omega |\nabla v|^2 \nabla v \cdot \nabla(\nabla v \cdot \nabla w) - \int_\Omega v\Delta w \nabla \cdot (|\nabla v|^2 \nabla v) \\
&\quad - \int_\Omega |\nabla v|^4 - \int_\Omega u \cdot \Big\{ 2\nabla v \cdot (D^2 v \cdot \nabla v) + |\nabla v|^2 \Delta v \Big\} \\
&\le -\int_\Omega |\nabla v|^2 |D^2 v|^2 \\
&\quad + \int_\Omega |\nabla v|^2 \nabla v \cdot \nabla(\nabla v \cdot \nabla w) - \int_\Omega v\Delta w \nabla \cdot (|\nabla v|^2 \nabla v) \\
&\quad - \int_\Omega |\nabla v|^4 + (2+\sqrt{n})\int_\Omega u|\nabla v|^2 |D^2 v| \qquad \text{for all } t \in (0, T_{max}),
\end{aligned} \tag{4.13}$$

again because

$$|\Delta \rho| \le \sqrt{n}|D^2\rho| \quad \text{in } \Omega \qquad \text{for all } \rho \in C^2(\Omega). \tag{4.14}$$

Here we combine (4.5) with (4.7), (4.11) and (4.12) to see that due to the third equation in (1.4),

$$\begin{aligned}
\int_\Omega |\nabla v|^2 \nabla v \cdot \nabla(\nabla v \cdot \nabla w) &\le \Big(1 + \frac{\sqrt{n}}{4}\Big) \|\nabla v\|_{L^6(\Omega)}^4 \|D^2 w\|_{L^3(\Omega)} \\
&\le \Big(1 + \frac{\sqrt{n}}{4}\Big)(4+\sqrt{n})^{\frac{4}{3}} \|v\|_{L^\infty(\Omega)}^{\frac{4}{3}} \cdot \Big\{ \int_\Omega |\nabla v|^2 |D^2 v|^2 \Big\}^{\frac{2}{3}} \cdot \|D^2 w\|_{L^3(\Omega)} \\
&\le \Big(1 + \frac{\sqrt{n}}{4}\Big)(4+\sqrt{n})^{\frac{4}{3}} c_1 k_1^{\frac{4}{3}} \cdot \Big\{ \int_\Omega |\nabla v|^2 |D^2 v|^2 \Big\}^{\frac{2}{3}} \cdot \|u\|_{L^3(\Omega)}
\end{aligned}$$

for all $t \in (0, T_{max})$, while (4.6) together with (4.7), (4.11), (4.14) and (4.12) shows that, similarly,

$$\begin{aligned}
-\int_\Omega v\Delta w \nabla \cdot (|\nabla v|^2 \nabla v) &\le (2+\sqrt{n})\|v\|_{L^\infty(\Omega)} \|\nabla v\|_{L^6(\Omega)} \cdot \Big\{ \int_\Omega |\nabla v|^2 |D^2 v|^2 \Big\}^{\frac{1}{2}} \cdot \|\Delta w\|_{L^3(\Omega)} \\
&\le (2+\sqrt{n})(4+\sqrt{n})^{\frac{1}{3}} \|v\|_{L^\infty(\Omega)}^{\frac{4}{3}} \cdot \Big\{ \int_\Omega |\nabla v|^2 |D^2 v|^2 \Big\}^{\frac{2}{3}} \cdot \|\Delta w\|_{L^3(\Omega)} \\
&\le (2+\sqrt{n})(4+\sqrt{n})^{\frac{1}{3}} \sqrt{n} c_1 k_1^{\frac{4}{3}} \cdot \Big\{ \int_\Omega |\nabla v|^2 |D^2 v|^2 \Big\}^{\frac{2}{3}} \cdot \|u\|_{L^3(\Omega)}
\end{aligned}$$

for all $t \in (0, T_{max})$. Apart from that, the Hölder inequality in conjunction with (4.7) and (4.11) enables us to estimate

$$
\begin{aligned}
(2+\sqrt{n}) \int_\Omega u|\nabla v|^2 |D^2 v| &\le (2+\sqrt{n}) \cdot \left\{ \int_\Omega u^2 |\nabla v|^2 \right\}^{\frac{1}{2}} \cdot \left\{ \int_\Omega |\nabla v|^2 |D^2 v|^2 \right\}^{\frac{1}{2}} \\
&\le (2+\sqrt{n}) \|u\|_{L^3(\Omega)} \|\nabla v\|_{L^6(\Omega)} \cdot \left\{ \int_\Omega |\nabla v|^2 |D^2 v|^2 \right\}^{\frac{1}{2}} \\
&\le (2+\sqrt{n})(4+\sqrt{n})^{\frac{1}{3}} \|u\|_{L^3(\Omega)} \|v\|_{L^\infty(\Omega)}^{\frac{1}{3}} \cdot \left\{ \int_\Omega |\nabla v|^2 |D^2 v|^2 \right\}^{\frac{2}{3}} \\
&\le (2+\sqrt{n})(4+\sqrt{n})^{\frac{1}{3}} k_1^{\frac{1}{3}} \|u\|_{L^3(\Omega)} \cdot \left\{ \int_\Omega |\nabla v|^2 |D^2 v|^2 \right\}^{\frac{2}{3}}
\end{aligned}
$$

for all $t \in (0, T_{max})$, so that abbreviating

$$
k_2 = k_2(u_0, v_0) := \Big(1 + \frac{\sqrt{n}}{4}\Big)(4+\sqrt{n})^{\frac{4}{3}} c_1 k_1^{\frac{4}{3}} + (2+\sqrt{n})(4+\sqrt{n})^{\frac{1}{3}} \sqrt{n} c_1 k_1^{\frac{4}{3}} + (2+\sqrt{n})(4+\sqrt{n})^{\frac{1}{3}} k_1^{\frac{1}{3}},
$$

in view of Young's inequality we obtain that

$$
\begin{aligned}
&\int_\Omega |\nabla v|^2 \nabla v \cdot \nabla(\nabla v \cdot \nabla w) - \int_\Omega v \Delta w \nabla \cdot \nabla \cdot (|\nabla v|^2 \nabla v) + (2+\sqrt{n}) \int_\Omega u |\nabla v|^2 |D^2 v| \\
&\qquad \le \left\{ \frac{1}{2} \int_\Omega |\nabla v|^2 |D^2 v|^2 \right\}^{\frac{2}{3}} \cdot \left\{ 2^{\frac{2}{3}} k_2 \|u\|_{L^3(\Omega)} \right\} \\
&\qquad \le \frac{1}{2} \int_\Omega |\nabla v|^2 |D^2 v|^2 + \left\{ 2^{\frac{2}{3}} k_2 \|u\|_{L^3(\Omega)} \right\}^3 \\
&\qquad = \frac{1}{2} \int_\Omega |\nabla v|^2 |D^2 v|^2 + 4 k_2^3 \|u\|_{L^3(\Omega)}^3 \qquad \text{for all } t \in (0, T_{max}).
\end{aligned}
$$

Therefore, (4.14) implies that

$$
\frac{1}{4} \frac{d}{dt} \int_\Omega |\nabla v|^4 \le -\frac{1}{2} \int_\Omega |\nabla v|^2 |D^2 v|^2 + 4 k_2^3 \|u\|_{L^3(\Omega)}^3 - \int_\Omega |\nabla v|^4 \qquad \text{for all } t \in (0, T_{max}),
$$

whereby the proof becomes complete. □

Now an appropriate combination of the inequalities provided by Lemma 4.1 and Lemma 4.3 enables us to derive the following as the main outcome of this section.

Lemma 4.4 *Suppose that (1.5) holds. Then there exists $\xi_0(u_0, v_0) \ge 1$ such that whenever $\xi > \xi_0(u_0, v_0)$, one can find $C = C(u_0, v_0, \xi) > 0$ such that*

$$
\int_\Omega u^2(\cdot, t) \le C \qquad \textit{for all } t \in (0, T_{max}) \tag{4.15}
$$

and

$$
\int_\Omega |\nabla v(\cdot, t)|^4 \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{4.16}
$$

PROOF. According to Lemma 3.2, Lemma 4.1 and Lemma 4.3, given (u_0, v_0) fulfilling (1.5) we fix positive constants $k_i = k_i(u_0, v_0) > 0$, $i \in \{1, 2, 3\}$, such that if $\xi \ge 1$, then

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le k_1 \qquad \text{for all } t \in (0, T_{max}) \tag{4.17}$$

and

$$\frac{d}{dt}\int_\Omega u^2 + \frac{\xi}{2}\int_\Omega u^3 \le k_2 \int_\Omega |\nabla v|^6 + k_2\xi \qquad \text{for all } t \in (0, T_{max}) \tag{4.18}$$

as well as

$$\frac{d}{dt}\int_\Omega |\nabla v|^4 + 2\int_\Omega |\nabla v|^2 |D^2 v|^2 + 4\int_\Omega |\nabla v|^4 \le k_3 \int_\Omega u^3 \qquad \text{for all } t \in (0, T_{max}). \tag{4.19}$$

We thereupon choose $a = a(u_0, v_0) > 0$ such that

$$a \ge \frac{1}{2} \cdot (4 + \sqrt{n})^2 k_1^2 k_2 \tag{4.20}$$

and define

$$\xi_0 = \xi_0(u_0, v_0) := \max\{1\,,\, 4ak_3\}. \tag{4.21}$$

Then for arbitrary $\xi > \xi_0$, once more relying on Lemma 4.2 we make use of (4.17) in estimating

$$\begin{aligned} k_2 \int_\Omega |\nabla v|^6 &\le (4+\sqrt{n})^2 k_2 \|v\|^2_{L^\infty(\Omega)} \int_\Omega |\nabla v|^2 |D^2 v|^2 \\ &\le (4+\sqrt{n})^2 k_1^2 k_2 \int_\Omega |\nabla v|^2 |D^2 v|^2 \qquad \text{for all } t \in (0, T_{max}), \end{aligned}$$

whence combining (4.18) with (4.19) shows that

$$y(t) := \int_\Omega u^2(\cdot,t) + a\int_\Omega |\nabla v(\cdot,t)|^4, \qquad t \in [0, T_{max}),$$

satisfies

$$\begin{aligned} y'(t) + 4y(t) &\le \Big\{ -\frac{\xi}{2}\int_\Omega u^3 + (4+\sqrt{n})^2 k_1^2 k_2 \int_\Omega |\nabla v|^2 |D^2 v|^2 + k_2 \xi \Big\} \\ &\quad + \Big\{ -2a\int_\Omega |\nabla v|^2 |D^2 v|^2 - 4a\int_\Omega |\nabla v|^4 + ak_3 \int_\Omega u^3 \Big\} \\ &\quad + 4 \cdot \Big\{ \int_\Omega u^2 + a\int_\Omega |\nabla v|^4 \Big\} \\ &= \Big\{ -\frac{\xi}{2} + ak_3 \Big\} \cdot \int_\Omega u^3 + 4\int_\Omega u^2 + k_2\xi + \Big\{ (4+\sqrt{n})^2 k_1^2 k_2 - 2a \Big\} \cdot \int_\Omega |\nabla v|^2 |D^2 v|^2 \\ &\le -\frac{\xi}{4}\int_\Omega u^3 + 4\int_\Omega u^2 + k_2 \xi \qquad \text{for all } t \in (0, T_{max}) \end{aligned}$$

thanks to (4.20) and (4.21). Since by Young's inequality,

$$4u^2 = \Big\{\frac{\xi}{4}u^3\Big\}^{\frac{2}{3}} \cdot \Big\{4 \cdot \Big(\frac{4}{\xi}\Big)^{\frac{2}{3}}\Big\} \le \frac{\xi}{4}u^3 + \Big\{4 \cdot \Big(\frac{4}{\xi}\Big)^{\frac{2}{3}}\Big\}^3 = \frac{\xi}{4}u^3 + \frac{1024}{\xi^2}$$

in $\Omega \times (0, T_{max})$ and hence

$$-\frac{\xi}{4}\int_\Omega u^3 + 4\int_\Omega u^2 + k_2\xi \le c_1 = c_1(u_0, v_0, \xi) := \frac{1024|\Omega|}{\xi^2} + k_2\xi \qquad \text{for all } t \in (0, T_{max}),$$

this implies that

$$y'(t) + 4y(t) \le c_1 \qquad \text{for all } t \in (0, T_{max}).$$

As a consequence of an ODE comparison, we thus infer that

$$y(t) \le \max\left\{\int_\Omega u_0^2 + a\int_\Omega |\nabla v_0|^4,\, \frac{c_1}{4}\right\} \qquad \text{for all } t \in (0, T_{max}),$$

and that thus both (4.15) and (4.16) result upon an obvious choice of $C(u_0, v_0, \xi)$. $\square$

5 L^∞ boundedness of u. Proof of Theorem 1.1

Once again based on elliptic regularity theory, we may firstly use the information from (4.15) to considerably improve our knowledge on regularity of w.

Lemma 5.1 *Assume (1.5), and let $\xi > \xi_0(u_0, v_0)$ with $\xi_0(u_0, v_0)$ taken from Lemma 4.4. Then there exists $C = C(u_0, v_0, \xi) > 0$ such that*

$$\|w(\cdot, t)\|_{W^{1,4}(\Omega)} \le C \qquad \textit{for all } t \in (0, T_{max}).$$

PROOF. Since standard elliptic regularity theory ([8]) provides $c_1 > 0$ such that

$$\|\varphi\|_{W^{2,2}(\Omega)} \le c_1\| - \Delta\varphi + \varphi\|_{L^2(\Omega)} \qquad \text{for all } \varphi \in W^{2,2}(\Omega) \text{ such that } \tfrac{\partial\varphi}{\partial\nu}|_{\partial\Omega} = 0,$$

and since our overall assumption $n \le 3$ warrants continuity of the embedding $W^{2,2}(\Omega) \hookrightarrow W^{1,4}(\Omega)$, hence implying the existence of $c_2 > 0$ fulfilling

$$\|\varphi\|_{W^{1,4}(\Omega)} \le c_2\|\varphi\|_{W^{2,2}(\Omega)} \qquad \text{for all } \varphi \in W^{2,2}(\Omega),$$

due to the third equation in (1.4) we can estimate

$$\|w\|_{W^{1,4}(\Omega)} \le c_1c_2\| - \Delta w + w\|_{L^2(\Omega)} \le c_1c_2\|u\|_{L^2(\Omega)} \qquad \text{for all } t \in (0, T_{max}).$$

Therefore, the claim is a consequence of Lemma 4.4. $\square$

Along with Lemma 5.1, a second application of Lemma 4.4 now provides an L^∞ estimate for u through well-known smoothing properties of the Neumann heat semigroup.

Lemma 5.2 *Suppose that (1.5) holds, and that $\xi > \xi_0(u_0, v_0)$ with $\xi_0(u_0, v_0)$ as given by Lemma 4.4. Then one can find $C = C(u_0, v_0, \xi) > 0$ such that*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{5.1}$$

PROOF. Again using that $n \le 3$, we fix any $p \in (n,4)$, and due to well-known smoothing properties of the Neumann heat semigroup $(e^{t\Delta})_{t\ge 0}$ on Ω ([7], [32]) we can then pick $c_1>0$ and $c_2>0$ such that for all $t>0$,

$$\|e^{t\Delta}\nabla\cdot\varphi\|_{L^\infty(\Omega)} \le c_1\cdot(1+t^{-\frac{1}{2}-\frac{n}{2p}})\|\varphi\|_{L^p(\Omega)} \qquad \text{for all } \varphi\in C^1(\overline{\Omega};\mathbb{R}^n) \text{ such that } (\varphi\cdot\nu)|_{\partial\Omega}=0, \tag{5.2}$$

and that

$$\|e^{t\Delta}\varphi\|_{L^\infty(\Omega)} \le c_2\cdot(1+t^{-\frac{n}{4}})\|\varphi\|_{L^2(\Omega)} \qquad \text{for all } \varphi\in C^0(\overline{\Omega}). \tag{5.3}$$

Apart from that, given $\xi > \xi_0(u_0,v_0)$ we may employ Lemma 4.4 along with Lemma 5.1 to find $c_i = c_i(u_0,v_0,\xi)>0$, $i\in\{3,4\}$ such that writing $h:=-\nabla v+\xi\nabla w$ we have

$$\|h(\cdot,t)\|_{L^4(\Omega)} \le c_3 \qquad \text{for all } t\in(0,T_{max}) \tag{5.4}$$

and

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le c_4 \qquad \text{for all } t\in(0,T_{max}). \tag{5.5}$$

Now on the basis of a Duhamel representation associated with the identity $u_t=\Delta u - u + \nabla\cdot(uh)+u$, we can use the maximum principle, (5.2) and (5.3) to see that thanks to (5.4) the Hölder inequality and (5.5),

$$\begin{aligned}
\|u(\cdot,t)\|_{L^\infty(\Omega)} &= \left\| e^{t(\Delta-1)}u_0 + \int_0^t e^{(t-s)(\Delta-1)}\nabla\cdot\Big\{u(\cdot,s)h(\cdot,s)\Big\}ds + \int_0^t e^{(t-s)(\Delta-1)}u(\cdot,s)ds\right\|_{L^\infty(\Omega)} \\
&\le e^{-t}\|u_0\|_{L^\infty(\Omega)} + c_1\int_0^t \Big(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}}\Big)e^{-(t-s)}\|u(\cdot,s)h(\cdot,s)\|_{L^p(\Omega)}ds \\
&\quad + c_2\int_0^t\Big(1+(t-s)^{-\frac{n}{4}}\Big)e^{-(t-s)}\|u(\cdot,s)\|_{L^2(\Omega)}ds \\
&\le \|u_0\|_{L^\infty(\Omega)} + c_1\int_0^t\Big(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}}\Big)e^{-(t-s)}\|u(\cdot,s)\|_{L^{\frac{4p}{4-p}}(\Omega)}\|h(\cdot,s)\|_{L^4(\Omega)}ds \\
&\quad + c_2\int_0^t\Big(1+(t-s)^{-\frac{n}{4}}\Big)e^{-(t-s)}\|u(\cdot,s)\|_{L^2(\Omega)}ds \\
&\le \|u_0\|_{L^\infty(\Omega)} + c_1c_3\int_0^t\Big(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}}\Big)e^{-(t-s)}\|u(\cdot,s)\|_{L^{\frac{4p}{4-p}}(\Omega)}ds \\
&\quad + c_2c_4\int_0^t\Big(1+(t-s)^{-\frac{n}{4}}\Big)e^{-(t-s)}ds \qquad \text{for all } t\in(0,T_{max}).
\end{aligned}$$

As writing $\theta:=\frac{3p-4}{2p}\in(0,1)$ and

$$M(T):=\max_{t\in[0,T]}\|u(\cdot,t)\|_{L^\infty(\Omega)}, \qquad T>0,$$

we can again rely on (5.5) to see that for any $T>0$,

$$\|u(\cdot,s)\|_{L^{\frac{4p}{4-p}}(\Omega)} \le \|u(\cdot,s)\|^{\theta}_{L^\infty(\Omega)}\|u(\cdot,s)\|^{1-\theta}_{L^2(\Omega)} \le c_4^{1-\theta}M^\theta(T) \qquad \text{for all } s\in(0,T),$$

this entails that for each $T > 0$,

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + c_1 c_3 c_4^{1-\theta} c_5 M^\theta(T) + c_1 c_4 c_6 \qquad \text{for all } t \in (0,T),$$

with $c_5 := \int_0^\infty (1+\sigma^{-\frac{1}{2}-\frac{n}{2p}}) e^{-\sigma} d\sigma$ and $c_6 := \int_0^\infty (1+\sigma^{-\frac{n}{4}}) e^{-\sigma} d\sigma$ being finite since $p > n$ and $n < 4$. As a consequence,

$$M(T) \le c_1 c_3 c_4^{1-\theta} c_5 M^\theta(T) + \|u_0\|_{L^\infty(\Omega)} + c_1 c_4 c_6 \qquad \text{for all } T \in (0,T_{max})$$

and thus, since $\theta < 1$,

$$M(T) \le \max\left\{ \left(\frac{\|u_0\|_{L^\infty(\Omega)} + c_1 c_4 c_6}{c_1 c_3 c_4^{1-\theta} c_5} \right)^{\frac{1}{\theta}}, (2c_1 c_3 c_4^{1-\theta} c_5)^{\frac{1}{1-\theta}} \right\} \qquad \text{for all } T \in (0,T_{max}),$$

which on taking $T \nearrow T_{max}$ establishes (5.1). $\square$

Collecting Lemma 4.4, Lemma 5.1 and Lemma 5.2 now finally yields our main result.

PROOF of Theorem 1.1. Taking $\xi_0(u_0,v_0) \ge 1$ as given by Lemma 4.4, from Lemma 5.2, Lemma 3.2, Lemma 4.4 and Lemma 5.1 we know that whenever $\xi > \xi_0(u_0,v_0)$, the solution (u,v,w) obtained in Lemma 2.1 has the property that $((u(\cdot,t),v(\cdot,t),w(\cdot,t))_{t\in(0,T_{max})}$ is bounded in $L^\infty(\Omega) \times W^{1,4}(\Omega) \times W^{1,4}(\Omega)$. According to (2.1), this means that necessarily $T_{max} = \infty$, and that moreover also (1.6) holds, because $W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$ due to our assumption that $n \le 3$. $\square$

Acknowledgement. Youshan Tao was supported by the *National Natural Science Foundation of China (No. 11861131003)*. The second author acknowledges support of the *Deutsche Forschungsgemeinschaft* in the context of the project *Emergence of structures and advantages in cross-diffusion systems* (No. 411007140, GZ: WI 3707/5-1).

References

[1] AMANN, H.: *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems.* In: Schmeisser, H., Triebel, H. eds. Function Spaces, Differential Operators and Nonlinear Analysis, Teubner-Texte Math., Vol.133, 9-126 (1993)

[2] BLACK, T.: *Global generalized solutions to a forager-exploiter model with superlinear degradation and their eventual regularity properties.* arXiv:1911.02321

[3] BREZIS, H., STRAUSS, W.A.: *Semi-linear second-order elliptic equations in L^1.* J. Math. Soc. Japan **25**, 565-590 (1973)

[4] CAO, X.: *Global radial renormalized solution to a producer-scrounger model with singular sensitivities.* Preprint

[5] CIEŚLAK, T., P. LAURENÇOT, P., MORALES-RODRIGO, C.: *Global existence and convergence to steady-states in a chemorepulsion system.* Banach Center Publ. **81**, Polish Acad. Sci., Warsaw, 105-117 (2008)

[6] Duan, R.J., Lorz, A., Markowich, P.A.: *Global solutions to the coupled chemotaxis-fluid equations.* Comm. Part. Differ. Eq. **35**, 1635-1673 (2010)

[7] Fujie, K., Ito, A., Winkler, M., Yokota, T.: *Stabilization in a chemotaxis model for tumor invasion.* Discrete Cont. Dyn. Syst. **36**, 151-169 (2016)

[8] Gilbarg, D., Trudinger, N.S.: *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, Berlin/Heidelberg, 2001

[9] Herrero, M.A., Velázquez, J.J.L.: *A blow-up mechanism for a chemotaxis model.* Ann. Scuola Normale Superiore Pisa **24**, 633-683 (1997)

[10] Hieber, M., Prüss, J.: *Heat kernels and maximal L^p-L^q estimates for parabolic evolution equations.* Comm. Partial Differential Equations **22**, 1647-1669 (1997)

[11] Horstmann, D.: *From 1970 until present: the Keller-Segel model in chemotaxis and its consequences I.* Jahresber. DMV **105**, 103-165 (2003)

[12] Jäger, W., Luckhaus, S.: *On explosions of solutions to a system of partial differential equations modelling chemotaxis.* Trans. Amer. Math. Soc. **329**, 819-824 (1992)

[13] Jin, H.Y., Wang, Z.A.: *Boundedness, blowup and critical mass phenomenon in competing chemotaxis.* J. Differential Equations **260**, 162-196 (2016)

[14] Keller, E.F., Segel, L.A.: *Initiation of slime mold aggregation viewed as an instability.* J. Theor. Biol. **26**, 399-415 (1970)

[15] Kiselev, A., Xu, X.: *Suppression of chemotactic explosion by mixing.* Arch. Ration. Mech. Anal. **222**, 1077-1112 (2016)

[16] Li, G., Tao, Y.: *Analysis of a chemotaxis-convection model of capillary-sprout growth during tumor angiogenesis.* J. Math. Anal. Appl. **481**, 123474 (2020)

[17] Lin, K., Mu, C., Gao, Y.: *Boundedness and blow up in the higher-dimensional attraction-repulsion chemotaxis system with nonlinear diffusion.* J. Differential Equations **261**, 4524-4572 (2016)

[18] Lions, P.L.: *Résolution de problèmes elliptiques quasilinéaires.* Arch. Rat. Mech. Anal. **74**, 335-353 (1980)

[19] Liu, Y.: *Global existence and boundedness of classical solutions to a forager-exploiter model with volume-filling effects.* Nonlin. Anal. Real World Appl. **50**, 519-531 (2019)

[20] Orme, M.E., Chaplain, M.A.J.: *A mathematical model of the first steps of tumour-related angiogenesis: Capillary sprout formation and secondary branching.* IMA J. Math. Appl. Med. Biol. **13**, 73-98 (1996)

[21] Osaki, K., Tsujikawa, T., Yagi, A., Mimura, M.: *Exponential attractor for a chemotaxis-growth system of equations.* Nonlinear Anal. TMA **51**, 119-144 (2002)

[22] Pang, P.Y.H., Wang, Y.: *Global existence of a two-dimensional chemotaxis-haptotaxis model with remodeling of non-diffusible attractant.* J. Differential Eq. **263**, 1269-1292 (2017)

[23] Paweletz, N., Knierim, M.: *Tumor related angiogenesis.* Crit. Rev. Oncol. Hematol. **9**, 197-242 (1989)

[24] Tania, N., Vanderlei, B., Heath, J.P., Edelstein-Keshet, L.: *Role of social interactions in dynamic patterns of resource patches and forager aggregation.* PNAS**109**, 11228-11233 (2012)

[25] Tao, Y., Wang, Z.A.: *Competing effects of attraction vs. repulsion in chemotaxis.* Math. Models Methods Appl. Sci. **23**, 1-36 (2013)

[26] Tao, Y., Winkler, M.: *Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity.* J. Differential Equations **252**, 692-715 (2012)

[27] Tao, Y., Winkler, M.: *Large time behavior in a mutidimensional chemotaxis-haptotaxis model with slow signal diffusion.* SIAM J. Math. Anal.**47**, 4229-4250 (2015)

[28] Tao, Y., Winkler, M.: *Critical mass for infinite-time aggregation in a chemotaxis model with indirect signal production.* J. European Math. Soc. **19**, 3641-3678 (2017)

[29] Tao, Y., Winkler, M: *Large time behavior in a forager-exploiter model with different taxis strategies for two groups in search of food.* Math. Models Methods Appl. Sci. **29**, 2151-2182 (2019)

[30] Wang, W.: *A quasilinear fully parabolic chemotaxis system with indirect signal production and logistic source.* J. Math. Anal. Appl. **477**, 488-522 (2019)

[31] Winkler, M.: *Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source.* Commun. Partial Differential Equations **35**, 1516-1537 (2010)

[32] Winkler, M.: *Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model.* J. Differential Eq. **248**, 2889-2905 (2010)

[33] Winkler, M.: *Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system.* J. Math. Pures Appl. **100**, 748-767 (2013), `arXiv:1112.4156v1`

[34] Winkler, M.: *Global weak solutions in a three-dimensional chemotaxis-Navier-Stokes system.* Ann. Inst. H. Poincaré Anal. Non Linéaire **33**, 1329-1352 (2016)

[35] Winkler, M: *Global generalized solutions to a multi-dimensional doubly tactic resource consumption model accounting for social interactions* Math. Mod. Meth. Appl. Sci. **29**, 373-418 (2019)

[36] Zhigun, A., Surulescu, C., Uatay, A.: *Global existence for a degenerate haptotaxis model of cancer invasion.* Z. Angew. Math. Phys. **67**, Art. 146 (2016)