A fully cross-diffusive two-component evolution system: Existence and qualitative analysis via entropy-consistent thin-film-type approximation

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Abstract

This work is concerned with a two-component parabolic system accounting for a doubly cross-diffusive interaction mechanism which was was predicted in *Physical Review Letters* **91**, *218102 (2003)* as responsible for the occurrence of certain solitary propagating waves in so-called pursuit-evasion systems. This system formally possesses two basic entropy-like structures, but especially in the presence of large data the regularity features thereby implied seem insufficient to ensure global extensibility of local-in-time classical solutions provided by known results on classical solvability in general parabolic systems of not necessarily tridiagonal type.

Attempting to nevertheless develop a basic theory of existence and qualitative behavior, the manuscript firstly constructs global solutions within a natural concept of weak solvability and for arbitrarily large data, and secondly derives a result on large-time stabilization toward homogeneous equilibria. A major challenge connected with this appears to consist in designing a suitable regularization which complies with the two requirements of asserting global solvability in the corresponding approximate systems on the one hand, and of retaining consistency with essential structural properties on the other. To adequately cope with this, a fourth-order regularization is pursued which, besides essentially respecting said entropy features, conforms to the fundamental *sine qua non* of positivity preservation by involving thin-film type degeneracies in the associated artificial diffusion operators.

Here the use of embeddings enforces a restriction to spatially one-dimensional settings, in which an apparently novel refinement of Gagliardo-Nirenberg interpolation reveals a crucial L^1 compactness feature of corresponding cross-diffusive fluxes.

Key words: cross-diffusion, thin-film equation, global existence, asymptotic behavior
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1 Introduction

Beyond classical reaction-diffusion processes: Cross-diffusion and taxis in science. Taxis-type cross-diffusion mechanisms are known to play a fundamental role at several levels of complexity in various branches of science. Prominent examples range from the dynamics of aggregation phenomena in contexts of self-gravitating systems or biological self-organization ([45], [11], [41], [33]), electro-diffusion of ions in electrolytes ([16]), and transport of electrons and holes in semiconductors ([50]), to intelligent migration of macroscopic individuals in living systems ([31]). Beyond this, related types of cross-diffusive motion have been identified as a non-negligible constituent at more subtle stages of relevance in numerous further models in, e.g., mechanical, electrochemical, magnetohydrodynamical or also multi-physical frameworks (see [28], [21], [10], [24], [51] and [29] for some examples).

As indicated by a correspondingly considerable mathematical literature, the evident additional challenges originating from taxis-type interaction can be coped with to a relatively comprehensive extent if the collection of all migration mechanisms results in a *triangular* diffusion matrix. Indeed, due to their accessibility to arguments from scalar parabolic theories in essential parts, several important subclasses of such triangular taxis systems have allowed for the development of quite thorough understanding. Accordingly, beyond quite far-reaching statements concerned with questions from basic solution theories, occasionally yet providing essentially optimal criteria for global existence ([39], [30], [5], [40], [27], [9], [7], [8], [52]), or including irregular initial data ([38], [6]), in various relevant cases the literature moreover contains noticeably detailed information on qualitative aspects of solution behavior ([46], [22], [23], [32]), partially even in contexts of couplings to further delicate processes such as fluid interaction ([20], [35], [53], [54]).

Doubly tactic interaction vs. Shigesada-Kawasaki-Teramoto cross-diffusion. As contrasted to the latter class of situations, the present study will be concerned with doubly taxis-driven dynamics by considering constellations in which not only the motion of particles or individuals within one group is biased in a taxis-type manner by members of a second group, but that moreover the former similarly influence movement of the latter. A context of paradigmatic character in this regard was originally addressed in [48], where as a description of pursuit-evasion dynamics in two-component predator-prey systems the authors proposed the doubly cross-diffusive parabolic system

$$\begin{cases} u_t = D_1\Delta u - \chi_1\nabla\cdot(u\nabla v) + f(u,v), \\ v_t = D_2\Delta v + \chi_2\nabla\cdot(v\nabla u) + g(u,v), \end{cases} \tag{1.1}$$

with positive parameters D_1, D_2, χ_1, χ_2 and local kinetics functions f and g, and with $u = u(x,t)$ and $v = v(x,t)$ denoting the population densities of predators and preys, respectively. As indicated by numerical evidence and formal linearized analysis, the simultaneous presence of two taxis terms can significantly affect the solution behavior, e.g. by facilitating a novel type of solitary waves, both in (1.1) ([48]) and in some close relatives ([49], [26] and [55]).

Through its thus fully cross-diffusive character, (1.1) can be viewed as a far relative of the renowned Shigesada-Kawasaki-Teramoto ([43]) class of reaction-diffusion systems given by

$$\begin{cases} u_t = \Delta[(d_1 + a_{11}u + a_{12}v)u] + f(u,v), \\ v_t = \Delta[(d_2 + a_{21}u + a_{22}v)v] + g(u,v). \end{cases} \tag{1.2}$$

In fact, both in (1.1) and in (1.2) some quite fundamental obstructions for the analysis stem from the circumstance that unlike in triangular relatives, the use of scalar parabolic techniques seems rather limited. In particular, for general and especially large data an application of classical theory on abstract parabolic problems seems to at most provide results on local existence of classical solutions, with the option of their global extensibility typically relying on time-independent $W^{1,p}$ boundedness properties of both solution components, for some p exceeding the spatial dimension ([1]); only in exceptionally favorable situations, such as those generated by appropriate smallness assumptions on f and g and either the initial data or the cross-diffusion coefficients, it can be expected – and in fact has partially been confirmed for (1.2) in [17] – that straightforward perturbation arguments can efficiently make use of the essentially quadratic character of the cross-diffusion terms in (1.1) to construct global smooth solutions. For wider ranges of parameters and initial data, an apparent need to resort to solution theories in adequately generalized frameworks, as successfully concretized for (1.2) in [12] and [13] (cf. also [14] and [18]), seems to form one crucial methodology-related feature that (1.1) shares with (1.2).

Essential differences to (1.2), however, originate from the particular taxis-type interaction mechanisms in (1.1): Firstly, unlike in (1.2) the second-order terms in (1.1) can apparently not be interpreted as resulting from the action of a single Laplacian on some appropriate function of (u, v); in fact, by providing access to duality-based arguments in the style of those developed in [42] this latter structural property has been forming an essential ingredient in the apparently only result on global classical solvability available for a multi-dimensional version of (1.2) in the presence of arbitrary positive cross-diffusion rates a_{12} and a_{21} ([37]). Secondly, through their mere nature, pursuit-evasion systems of the form in (1.1) need to account for one cross-diffusion process which in contrast to both of those in (1.2) is attractive, rather than repulsive; in light of the rich knowledge collected for corresponding Keller-Segel type systems concentrating on single taxis mechanisms, actually even reporting on some blow-up phenomena in attractive but exclusively providing global existence results in repulsive cases ([27], [52], [15]), it may be expected that choosing χ_1 to be positive will go along with a significant tendency toward destabilization. This is further indicated by a result on nonexistence, even of local-in-time solutions and even in one-dimensional contexts, for (1.1) in the case when both taxis processes are assumed to be attractive in the sense that $\chi_1 > 0$ and $\chi_2 < 0$ (cf. Proposition 1.3 below).

The challenge of designing structure-consistent approximations. The purpose of the present work now consists in creating an analytical approach which despite these obstacles is capable of establishing a basic theory not only of global solvability, but also of some essential qualitative features. To address this in a framework which captures the apparently most essential features of (1.1) but beyond this remains as simple as possible, we concentrate on the source-free case when $f = g = 0$ in a spatially one-dimensional setting, and hence subsequently consider the initial-boundary value problem

$$\begin{cases} u_t = D_1 u_{xx} - \chi_1 (u v_x)_x, & x \in \Omega,\ t > 0, \\ v_t = D_2 v_{xx} + \chi_2 (v u_x)_x, & x \in \Omega,\ t > 0, \\ u_x = v_x = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \qquad (1.3)$$

in an open bounded interval $\Omega \subset \mathbb{R}$, where D_1, D_1, χ_1 and χ_2 are positive parameters.

A major challenge to be adequately dealt with in this regard will be connected to the design of a suitable approximation procedure on the basis of which appropriate generalized solutions can be obtained through a limit process. Here especially in view of our ambition to include a qualitative analysis, the approximation we are seeking for should simultaneously comply with several system-inherent stuctural properties: Besides evident basic features of nonnegativity and mass conservation, namely, (1.3) formally enjoys two entropy-like features potentially blazing a trail not only toward the construction of global weak solutions, but also toward a description of their large time behavior in quite a strong topological setting. Going in search of a parabolic regularization in order to adequately respect this, we firstly observe that second-order parabolic smoothing procedures, as frequently performed in previous literature on taxis-type cross-diffusion systems ([19], [36], [47]), seem to go along with substantial difficulties already at the level of asserting global solvability in the respective approximate systems.

Therefore preferring to pursue an essentially fourth-order regularization, as newly arising problems we then naturally encounter the requirements of positivity and mass conservation. Fortunately, it will turn out that the introduction of carefully chosen degeneracies of the respective artificial diffusion mechanisms, quite in the style of the well-studied thin film equation ([4], [3]; cf. also [25] and [34])), does not only solve these problems but also provides convenient consistency with both of said entropy properties. The obtained fourth-order parabolic approximation thereby seems to become more efficient here than, for instance, discretization-based approaches such as those which have been underlying the analysis for (1.2) in [12] and [13], but for which it seems unclear how far they can retain applicability also in the present setting, mainly because of the characteristic differences due to a deviating general structure of the migration operators in (1.3) and (1.2), and due to the simultaneous presence of a repulsive and an attractive cross-diffusive mechanism in (1.3). In particular, it appears unsure whether such discretization strategies can be hoped for to adequately cooperate with higher-order nonlinear testing procedures, as already at a formal level required, e.g., for the derivation of (1.8) below.

Main results I: Global existence. To describe this essential part of our approach in more detail, and to formulate our main results obtained on the basis thereof, with parameters $n > 0, m \in (0,n), \alpha > 0$ and $\beta > 0$ to be specified below and for $\varepsilon \in (0,1)$, let us consider the regularized versions of (1.3) given by

$$\begin{cases} u_{\varepsilon t} = -\varepsilon\Big(\frac{u_\varepsilon^n}{u_\varepsilon^{n-m}+\varepsilon}u_{\varepsilon xxx}\Big)_x + \varepsilon^\beta(u_\varepsilon^{-\alpha}u_{\varepsilon x})_x + D_1 u_{\varepsilon xx} - \chi_1\Big(\frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m}+\varepsilon}v_{\varepsilon x}\Big)_x, & x\in\Omega,\ t>0,\\ v_{\varepsilon t} = -\varepsilon\Big(\frac{v_\varepsilon^n}{v_\varepsilon^{n-m}+\varepsilon}v_{\varepsilon xxx}\Big)_x + \varepsilon^\beta(v_\varepsilon^{-\alpha}v_{\varepsilon x})_x + D_2 v_{\varepsilon xx} + \chi_2\Big(\frac{v_\varepsilon^{n-m+1}}{v_\varepsilon^{n-m}+\varepsilon}u_{\varepsilon x}\Big)_x, & x\in\Omega,\ t>0,\\ u_{\varepsilon x} = u_{\varepsilon xxx} = v_{\varepsilon x} = v_{\varepsilon xxx} = 0, & x\in\partial\Omega,\ t>0,\\ u_\varepsilon(x,0) = u_{0\varepsilon}(x), \quad v_\varepsilon(x,0) = v_{0\varepsilon}(x). & x\in\Omega, \end{cases} \tag{1.4}$$

Here the particular choice of the degenerate fourth-order operators is inspired by quite well-established approaches to adequately approximate thin-film evolution ([4], [3]), whereupon the particular design of the regularization in the respective cross-diffusive contributions has been motivated by the ambition to maintain entropy consistency at approximate levels within suitable ranges of the free parameters n and m (see, e.g., Lemma 3.1). The artifical second-order diffusion operators of fast-diffusion type, finally, can be viewed as instruments to provide convenient control over some ill-signed contributions which due to the latter modification appear in the justification of positivity preservation (Lemma 2.3).

Apart from that, to appropriately regularize the initial data we shall here and throughout the sequel, given $\{u_0,v_0\}\subset W^{1,2}(\Omega)$ fulfilling $u_0>0$ and $v_0>0$ in $\overline{\Omega}$, fix families $(u_{0\varepsilon})_{\varepsilon\in(0,1)}$ and $(v_{0\varepsilon})_{\varepsilon\in(0,1)}$ such that

$$\begin{cases} u_{0\varepsilon}\in C^5(\overline{\Omega}) \text{ and } v_{0\varepsilon}\in C^5(\overline{\Omega}) \text{ satisfy } u_{0\varepsilon}>0 \text{ and } v_{0\varepsilon}>0 \text{ in } \overline{\Omega} \text{ for all } \varepsilon\in(0,1) \qquad \text{and} \\ u_{0\varepsilon x}=u_{0\varepsilon xxx}=v_{0\varepsilon x}=v_{0\varepsilon xxx}=0 \text{ on } \partial\Omega \qquad \text{for all } \varepsilon\in(0,1), \qquad \text{that} \\ \int_\Omega u_{0\varepsilon}=\int_\Omega u_0 \quad \text{and} \quad \int_\Omega v_{0\varepsilon}=\int_\Omega v_0 \qquad \text{for all } \varepsilon\in(0,1), \qquad \text{and that} \\ u_{0\varepsilon}\to u_0 \ \text{ and } \ v_{0\varepsilon}\to v_0 \quad \text{in } W^{1,2}(\Omega) \quad \text{as } \varepsilon\searrow 0. \end{cases} \tag{1.5}$$

A first and constitutive observation will reveal that in accordance with well-known results on positivity of solutions to scalar thin film equations in one-dimensional settings ([4], [3]), suitably strong degeneracies in the considered fourth-order diffusion mechanisms warrant preservation of positivity also in the coupled system (1.4). More precisely, Lemma 2.6 will reveal that whenever $n>\frac{7}{2}$, under the additional assumptions that $m\in(0,n-1]$, $\alpha\in(0,\frac{1}{2})$ and $\beta>0$ are such that $m\le\frac{n+2}{2}$ and $\alpha\ge 4-n$, for each suitably small $\varepsilon\in(0,1)$ the problem (1.4) admits a globally defined positive classical solution.

Now a key question in the course of our existence analysis for the original problem will be how far this approximation is consistent with a first and fundamental entropy structure inherent to (1.3), formally becoming manifest in the identity

$$\frac{d}{dt}\Big\{\chi_2\int_\Omega u\ln u+\chi_1\int_\Omega v\ln v\Big\}+\chi_2 D_1\int_\Omega\frac{u_x^2}{u}+\chi_1 D_2\int_\Omega\frac{v_x^2}{v}=0 \tag{1.6}$$

satisfied by smooth positive solutions to the boundary value problem in (1.3). Indeed, we shall see that under the above assumptions on n,α and β, this structure will be adequately respected by (1.4) if m satisfies the stronger restriction that $m\in(\frac{1}{2},2]$. Accordingly implied a priori estimates, inter alia relying on an apparently novel Gagliardo-Nirenberg-type interpolation inequality involving certain Orlicz space norms (Lemma 7.5), will thereby lead us to our following main result on global solvability in (1.3). Here and below, as usual we shall let $C^0_w(J;L^1(\Omega))$ denote the space of $L^1(\Omega)$-valued functions on the interval $J\subset\mathbb{R}$ which are continuous with respect to the weak topology in $L^1(\Omega)$, and let the Orlicz space $L\log L(\Omega)$ consist of all measurable functions φ on Ω which are such that $\int_\Omega|\varphi|\ln(|\varphi|+1)<\infty$.

Theorem 1.1 *Let $\Omega\subset\mathbb{R}$ be a bounded open interval, and let $D_1>0,D_2>0$, $\chi_1>0$ and $\chi_2>0$. Then for any choice of $u_0\in W^{1,2}(\Omega)$ and $v_0\in W^{1,2}(\Omega)$ satisfying $u_0>0$ and $v_0>0$ in $\overline{\Omega}$, in the sense of Definition 4.1 the problem (1.3) possesses a global weak solution which has the additional properties that*

$$\{u,v\}\subset C^0_w([0,\infty);L^1(\Omega)) \ \cap\ L^3_{loc}(\overline{\Omega}\times[0,\infty)) \ \cap\ L^{\frac{3}{2}}_{loc}([0,\infty);W^{1,\frac{3}{2}}(\Omega)) \ \cap\ L^\infty((0,\infty);L\log L(\Omega)). \tag{1.7}$$

Moreover, given families $(u_{0\varepsilon})_{\varepsilon\in(0,1)}$ and $(v_{0\varepsilon})_{\varepsilon\in(0,1)}$ fulfilling (1.5), and parameters $n>\frac{7}{2}$, $m\in(\frac{1}{2},2]$, $\alpha\in(0,\frac{1}{2})$ and $\beta>0$ such that $\alpha\ge 4-n$, one can find $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ such that $\varepsilon_j\searrow 0$ as $j\to\infty$, and such that for the solutions $(u_\varepsilon,v_\varepsilon)$ of (1.4) we have $u_\varepsilon\to u$ as well as $v_\varepsilon\to v$ a.e. in $\Omega\times(0,\infty)$ as $\varepsilon=\varepsilon_j\searrow 0$.

Even in the considered one-dimensional setting, the regularity information implied by entropy dissipation, as here expressed through (1.7) seems rather moderate only; accordingly, we have to leave open here the interesting question whether solutions are unique.

Main results II: Eventual boundedness and uniform stabilization. Addressing qualitative aspects of the solution behavior in (1.3) requires to adequately respect the evident circumstance that due to their essentially quadratic growth with respect to the unknown (u,v), the cross-diffusive contributions to (1.3) can in general apparently not be viewed as reasonably small perturbations to a predominantly diffusion-driven system; accordingly, we do not expect the solutions gained above to be smooth near the initial time, especially in cases when χ_1 and χ_2 and the initial data are inconveniently large. On the other hand, the basic dissipation process implicitly expressed through (1.6) indicates a certain global relaxation property at least in a suitable weak sense, inter alia excluding any collapse into persistent singular profiles unbounded in space. At a formal level, a second fundamental gradient structure, corresponding to a now *conditional* entropy inequality of the form

$$\frac{d}{dt}\mathcal{F}(t)+\Big\{\frac{1}{K}-K\mathcal{F}(t)\Big\}\cdot\Big\{\int_\Omega\frac{u_{xx}^2}{u}+\int_\Omega\frac{v_{xx}^2}{v}\Big\}\le 0, \tag{1.8}$$

along smooth and positive trajectories satisfied by

$$\mathcal{F}(t):=\chi_2\int_\Omega\frac{u_x^2}{u}+\chi_1\int_\Omega\frac{v_x^2}{v}$$

with some $K>0$, suggests that the weak decay information on the dissipation rate in (1.6) can actually be turned into genuine decay.

The second of our objectives will consist in revealing that this heuristic argument can be transferred to a rigorous stage for arbitrary ingredients to (1.3), provided that the approximation parameters in (1.4) are chosen appropriately. In particular, we shall see that (1.4) is essentially consistent with the structural property (1.8) if beyond further requirements, mainly on the parameters α and β referring to the artificial second-order fast diffusion therein, the crucial restrictions $m\ge 2$ and $m>n-2$ are satisfied (Lemma 5.1). Fortunately, these further assumptions are all compatible with Theorem 1.1, thus enabling us to achieve the following second of our main results, in which we adopt the commonly used notational convention to write $\overline{\varphi}:=\frac{1}{|\Omega|}\int_\Omega\varphi$ for $\varphi\in L^1(\Omega)$:

Theorem 1.2 *Let $n\in(\frac{7}{2},4)$ and $m=2$, and let $\alpha\in(0,\frac{1}{2})$ and $\beta>0$ be such that $\alpha\ge 4-n,\alpha>\frac{n-3}{2}$ and $\beta<\frac{\alpha}{n-2}$. Then if $\Omega\subset\mathbb{R}$ is a bounded open interval, D_1,D_2,χ_1 and χ_2 are positive and $u_0\in W^{1,2}(\Omega)$ and $v_0\in W^{1,2}(\Omega)$ are such that $u_0>0$ and $v_0>0$ in $\overline{\Omega}$, and if $(u_{0\varepsilon})_{\varepsilon\in(0,1)}$ and $(v_{0\varepsilon})_{\varepsilon\in(0,1)}$ satisfy (1.5), then the global weak solution (u,v) of (1.3) obtained in Theorem 1.1 has the additional properties that there exist $T>0$ and $C>0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}+\|v(\cdot,t)\|_{L^\infty(\Omega)}\le C \qquad \textit{for all } t>T, \tag{1.9}$$

and that (u,v) stabilizes toward $(\overline{u}_0,\overline{v}_0)$ in the sense that

$$u(\cdot,t)\to\overline{u}_0 \textit{ in } L^\infty(\Omega) \quad \textit{and} \quad v(\cdot,t)\to\overline{v}_0 \textit{ in } L^\infty(\Omega) \qquad \textit{as } t\to\infty. \tag{1.10}$$

In light of the above, a natural question seems to consist in deciding how far the approximation properties (1.10), and especially the topological framework therein, may imply that solutions even become smooth eventually. Appropriately addressing this challenging topic in the context of the approximation procedure pursued here, however, would go beyond the scope of the present work and will thus be left for future researach.

Let us finally add a simple observation indicating a crucial importance, also beyond technical issues, of our overall assumption on the cross-diffusive interplay in (1.3), namely that χ_1 and χ_2 both be positive. In fact, in order to briefly address a prototypical situation in which unlike in (1.3) two taxis-type cross-diffusive mechanisms *both* act attractively, let us consider the variant of (1.3) given by

$$\begin{cases} u_t = u_{xx} - (uv_x)_x, & x \in \Omega,\ t>0, \\ v_t = v_{xx} - (vu_x)_x, & x \in \Omega,\ t>0, \\ u_x = v_x = 0, & x \in \partial\Omega,\ t>0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{1.11}$$

In sharp contrast to large classes of related triangular chemotaxis models or more general cross-diffusion systems to which Amann's theory ([1]) applies, problems of this form cannot be expected to admit local-in-time regular solutions, not even for initial data in $C^\infty(\overline{\Omega})$. This is an evident consequence of the following.

Proposition 1.3 *Let $\Omega \subset \mathbb{R}$ be a bounded open interval, and suppose that $\phi \in C^0(\overline{\Omega})$ is such that $\phi > 1$ in $\overline{\Omega}$, that $T>0$, and that u and v are nonnegative functions on $\overline{\Omega} \times [0,T)$ fulfilling*

$$\{u,v\} \subset C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T)) \cap L^\infty_{loc}([0,T); W^{1,2}(\Omega)), \tag{1.12}$$

which are such that (u,v) solves (1.11) classically in $\Omega \times (0,T)$ with $u_0 \equiv v_0 \equiv \phi$. Then necessarily ϕ is analytic in Ω.

2 Global existence in the approximate problems

To begin with, let us first employ standard abstract parabolic theory to obtain local existence of smooth positive solutions to (1.4), as well as a handy criterion for their extensibility.

Lemma 2.1 *Let $n>0, m \in (0,n), \alpha > 0$, $\beta > 0$ and $s \in (\frac{3}{2}, 2)$, and suppose that (1.5) holds. Then for all $\varepsilon \in (0,1)$ there exist $T_{max,\varepsilon} \in (0,\infty]$ and a pair $(u_\varepsilon, v_\varepsilon)$ of functions*

$$\begin{cases} u_\varepsilon \in C^0([0,T_{max,\varepsilon}); W^{s,2}(\Omega)) \cap C^{4,1}(\overline{\Omega} \times (0,T_{max,\varepsilon})) \qquad \text{and} \\ v_\varepsilon \in C^0([0,T_{max,\varepsilon}); W^{s,2}(\Omega)) \cap C^{4,1}(\overline{\Omega} \times (0,T_{max,\varepsilon})), \end{cases}$$

satisfying $u_\varepsilon > 0$ and $v_\varepsilon > 0$ in $\overline{\Omega} \times [0,T_{max,\varepsilon})$, which are such that $(u_\varepsilon, v_\varepsilon)$ solves (1.4) classically in $\Omega \times (0,T_{max,\varepsilon})$, and that

either $T_{max,\varepsilon} = \infty$, or

$$\limsup_{t \nearrow T_{max,\varepsilon}} \left\{ \|u_\varepsilon(\cdot,t)\|_{W^{s,2}(\Omega)} + \left\| \frac{1}{u_\varepsilon(\cdot,t)} \right\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot,t)\|_{W^{s,2}(\Omega)} + \left\| \frac{1}{v_\varepsilon(\cdot,t)} \right\|_{L^\infty(\Omega)} \right\} = \infty. \tag{2.1}$$

Moreover,

$$\int_\Omega u_\varepsilon(x,t)dx = \int_\Omega u_0 \quad \text{and} \quad \int_\Omega v_\varepsilon(x,t)dx = \int_\Omega v_0 \qquad \text{for all } t \in (0,T_{max,\varepsilon}). \tag{2.2}$$

PROOF. To apply the theory for abstract quasilinear parabolic problems developed in [1], for $\varepsilon \in (0,1)$ we let

$$g_\varepsilon(s) := \varepsilon \frac{s^n}{s^{n-m}+\varepsilon}, \quad g_{1\varepsilon}(s) := \varepsilon^\beta s^{-\alpha} \quad \text{and} \quad g_{2\varepsilon}(s) := \frac{s^{n-m+1}}{s^{n-m}+\varepsilon} \qquad \text{for } s>0$$

and introduce matrix-valued functions $A_\varepsilon, A_{1\varepsilon}$ and $A_{2\varepsilon}$ by defining

$$A_\varepsilon \begin{pmatrix} \xi \\ \eta \end{pmatrix} := \begin{pmatrix} g_\varepsilon(\xi) & 0 \\ 0 & g_\varepsilon(\eta) \end{pmatrix}$$

as well as

$$A_{1\varepsilon}\left(\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \kappa \\ \lambda \end{pmatrix} \right) := \begin{pmatrix} g'_\varepsilon(\xi)\kappa & 0 \\ 0 & g'_\varepsilon(\eta)\lambda \end{pmatrix} \quad \text{and} \quad A_{2\varepsilon}\begin{pmatrix} \xi \\ \eta \end{pmatrix} := \begin{pmatrix} g_{1\varepsilon}(\xi)+D_1 & -\chi_1 g_{2\varepsilon}(\xi) \\ \chi_2 g_{2\varepsilon}(\eta) & g_{1\varepsilon}(\eta)+D_2 \end{pmatrix}$$

for positive numbers ξ, η, κ and λ. When rewritten in the new variable $U_\varepsilon := \begin{pmatrix} u_\varepsilon \\ v_\varepsilon \end{pmatrix}$, (1.4) then takes the divergence form

$$\begin{cases} U_{\varepsilon t} = -(A_\varepsilon(U_\varepsilon)U_{\varepsilon xx})_{xx} + (A_{1\varepsilon}(U_\varepsilon, U_{\varepsilon x})U_{\varepsilon xx} + A_{2\varepsilon}(U_\varepsilon)U_{\varepsilon x})_x, & x \in \Omega,\ t>0, \\ U_{\varepsilon x} = U_{\varepsilon xxx} = 0, & x \in \partial\Omega,\ t>0, \\ U_\varepsilon(x,0) = \begin{pmatrix} u_{0\varepsilon}(x) \\ v_{0\varepsilon}(x) \end{pmatrix}, & x \in \Omega. \end{cases} \tag{2.3}$$

Given $s \in (\frac{3}{2}, 2)$, we now let $\delta := \frac{s}{4}$ and can then pick some $\vartheta \in (\frac{3}{8}, \frac{1}{2})$ fulfilling $\vartheta < \delta$, and define

$$E_0 := L^2(\Omega), \qquad E_1 := W^{4,2}(\Omega), \qquad E_\vartheta := W^{4\vartheta,2}(\Omega) \qquad \text{and} \qquad E_\delta := W^{4\delta,2}(\Omega).$$

As long as u_ε and v_ε have a positive lower bound, the matrix $A_\varepsilon(U_\varepsilon)$ is positive definite due to the fact that $g'_\varepsilon(s) > 0$ for all $s>0$, and thereby an application of Amann's theory (cf. [1, Theorem 12.1 and Theorem 12.5]) asserts the existence of $T_{max,\varepsilon} \in (0,\infty]$ such that U_ε possesses the claimed positivity and regularity properties and solves (2.3) classically in $\Omega \times (0, T_{max,\varepsilon})$, and that moreover

$$\begin{aligned} &\text{either } T_{max,\varepsilon} = \infty, \quad \text{or} \\ &\qquad U_\varepsilon(\cdot,t) \to \partial\, \mathrm{dom} A_\varepsilon \qquad \text{or} \qquad \|U_\varepsilon(\cdot,t)\|_{W^{4\delta,2}(\Omega)} \to \infty \qquad \text{as } t \nearrow T_{max,\varepsilon}, \end{aligned}$$

where the latter entails (2.1) due to the fact that $\frac{3}{2} < 4\vartheta < 4\delta < 2$. Finally, (2.2) follows from straightforward integration in (1.4). $\square$

2.1 ε-dependent a priori estimates: H^1 regularity and positivity

In view of (2.1), in order to assert global extensibility of the above solutions it will be sufficient to establish bounds, possibly depending on ε, for $\frac{1}{u_\varepsilon}$ and $\frac{1}{v_\varepsilon}$ in L^∞ and for u_ε and v_ε with respect to the spatial H^2 norm. In a first step toward the latter, we shall extend a standard H^1 testing procedure, well-known as a fundamental constituent in the analysis of scalar thin film equations ([4]), to the present context:

Lemma 2.2 *Let $n > 0$ and $m \in (0, n)$ be such that $m \le \frac{n+2}{2}$, and let $\alpha \in (0, \frac{1}{2})$ and $\beta > 0$. Then for all $\varepsilon \in (0, 1)$ and each $T > 0$ one can find $C(\varepsilon, T) > 0$ such that*

$$\int_\Omega u_{\varepsilon x}^2(x,t)dx + \int_\Omega v_{\varepsilon x}^2(x,t)dx \le C(\varepsilon, T) \qquad \text{for all } t \in (0, \widehat{T}_\varepsilon) \tag{2.4}$$

and

$$\int_0^t \int_\Omega \frac{u_\varepsilon^n}{u_\varepsilon^{n-m} + \varepsilon} u_{\varepsilon xxx}^2 + \int_0^t \int_\Omega \frac{v_\varepsilon^n}{v_\varepsilon^{n-m} + \varepsilon} v_{\varepsilon xxx}^2 \le C(\varepsilon, T) \qquad \text{for all } t \in (0, \widehat{T}_\varepsilon) \tag{2.5}$$

as well as

$$\int_0^t \int_\Omega u_\varepsilon^{-\alpha} u_{\varepsilon xx}^2 + \int_0^t \int_\Omega v_\varepsilon^{-\alpha} v_{\varepsilon xx}^2 \le C(\varepsilon, T) \qquad \text{for all } t \in (0, \widehat{T}_\varepsilon) \tag{2.6}$$

and

$$\int_0^t \int_\Omega u_\varepsilon^{-\alpha-2} u_{\varepsilon x}^4 + \int_0^t \int_\Omega v_\varepsilon^{-\alpha-2} v_{\varepsilon x}^4 \le C(\varepsilon, T) \qquad \text{for all } t \in (0, \widehat{T}_\varepsilon), \tag{2.7}$$

where $\widehat{T}_\varepsilon := \min\{T, T_{max,\varepsilon}\}$.

Proof. We multiply the first equation in (1.4) by $-u_{\varepsilon xx}$ to see upon integrating by parts that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega u_{\varepsilon x}^2 &= -\varepsilon \int_\Omega \frac{u_\varepsilon^n}{u_\varepsilon^{n-m} + \varepsilon} u_{\varepsilon xxx}^2 - \varepsilon^\beta \int_\Omega u_\varepsilon^{-\alpha} u_{\varepsilon xx}^2 + \alpha\varepsilon^\beta \int_\Omega u_\varepsilon^{-\alpha-1} u_{\varepsilon x}^2 u_{\varepsilon xx} \\ &\quad - D_1 \int_\Omega u_{\varepsilon xx}^2 - \chi_1 \int_\Omega \frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m} + \varepsilon} v_{\varepsilon x} u_{\varepsilon xxx} \qquad \text{for all } t \in (0, T_{max,\varepsilon}), \end{aligned} \tag{2.8}$$

where another integration by parts together with an application of Lemma 7.1 shows that

$$\begin{aligned} &-\varepsilon^\beta \int_\Omega u_\varepsilon^{-\alpha} u_{\varepsilon xx}^2 + \alpha\varepsilon^\beta \int_\Omega u_\varepsilon^{-\alpha-1} u_{\varepsilon x}^2 u_{\varepsilon xx} \\ &\qquad = -\varepsilon^\beta \int_\Omega u_\varepsilon^{-\alpha} u_{\varepsilon xx}^2 - \frac{\alpha(\alpha+1)}{3}\varepsilon^\beta \int_\Omega u_\varepsilon^{-\alpha-2} u_{\varepsilon x}^4 \\ &\qquad \le -\varepsilon^\beta \int_\Omega u_\varepsilon^{-\alpha} u_{\varepsilon xx}^2 + \frac{3\alpha}{\alpha+1}\varepsilon^\beta \int_\Omega u_\varepsilon^{-\alpha} u_{\varepsilon xx}^2 \\ &\qquad = -\frac{1-2\alpha}{\alpha+1}\varepsilon^\beta \int_\Omega u_\varepsilon^{-\alpha} u_{\varepsilon xx}^2 \qquad \text{for all } t \in (0, T_{max,\varepsilon}), \end{aligned} \tag{2.9}$$

and where by Young's inequality,

$$-\chi_1 \int_\Omega \frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m} + \varepsilon} v_{\varepsilon x} u_{\varepsilon xxx} \le \frac{\varepsilon}{2} \int_\Omega \frac{u_\varepsilon^n}{u_\varepsilon^{n-m} + \varepsilon} u_{\varepsilon xxx}^2 + \frac{\chi_1^2}{2\varepsilon} \int_\Omega \frac{u_\varepsilon^{n-2m+2}}{u_\varepsilon^{n-m} + \varepsilon} v_{\varepsilon x}^2 \qquad \text{for all } t \in (0, T_{max,\varepsilon}).$$

Here we note that if $u_\varepsilon^{n-m} \le \varepsilon$, then since $m < n$ and $m \le \frac{n+2}{2}$ we can estimate

$$\frac{u_\varepsilon^{n-2m+2}}{u_\varepsilon^{n-m} + \varepsilon} \le \frac{1}{\varepsilon} u_\varepsilon^{n-2m+2} \le \frac{1}{\varepsilon} \cdot \varepsilon^{\frac{n-2m+2}{n-m}},$$

while at points where $u_\varepsilon^{n-m} > \varepsilon$,

$$\frac{u_\varepsilon^{n-2m+2}}{u_\varepsilon^{n-m} + \varepsilon} \le u_\varepsilon^{2-m},$$

so that regardless of the sign of $2-m$, for each $\varepsilon \in (0,1)$ we can find $c_1(\varepsilon)>0$ such that

$$\frac{u_\varepsilon^{n-2m+2}}{u_\varepsilon^{n-m}+\varepsilon} \le c_1(\varepsilon)\cdot (1+u_\varepsilon^p) \qquad \text{in } \Omega\times(0,T_{max,\varepsilon})$$

with $p:=\max\{1,2-m\}\in[1,2)$. We may therefore use the Gagliardo-Nirenberg inequality along with (2.2) to see that with some $c_2(\varepsilon)>0$ and $c_3(\varepsilon)>0$ we have

$$\begin{aligned}
&\frac{\chi_1^2}{2\varepsilon}\int_\Omega \frac{u_\varepsilon^{n-2m+2}}{u_\varepsilon^{n-m}+\varepsilon} v_{\varepsilon x}^2 \\
&\quad\le \frac{\chi_1^2}{2\varepsilon}\|v_{\varepsilon x}\|_{L^\infty(\Omega)}^2 \cdot c_1(\varepsilon)\cdot\left\{|\Omega|+\int_\Omega u_\varepsilon^p\right\} \\
&\quad\le c_2(\varepsilon)\cdot\left\{\|v_{\varepsilon xx}\|_{L^2(\Omega)}^{\frac{8}{5}}\|v_\varepsilon\|_{L^1(\Omega)}^{\frac{2}{5}}+\|v_\varepsilon\|_{L^1(\Omega)}^2\right\}\cdot\left\{1+\|u_{\varepsilon xx}\|_{L^2(\Omega)}^{\frac{2(p-1)}{5}}\|u_\varepsilon\|_{L^1(\Omega)}^{\frac{3p+2}{5}}+\|u_\varepsilon\|_{L^1(\Omega)}^p\right\} \\
&\quad\le c_3(\varepsilon)\cdot\left\{\|v_{\varepsilon xx}\|_{L^2(\Omega)}^{\frac{8}{5}}+1\right\}\cdot\left\{\|u_{\varepsilon xx}\|_{L^2(\Omega)}^{\frac{2(p-1)}{5}}+1\right\} \qquad \text{for all } t\in(0,T_{max,\varepsilon}).
\end{aligned}$$

As $\frac{8}{5}+\frac{2(p-1)}{5}<2$ due to the fact that $m>0$, two applications of Young's inequality thus readily reveal the existence of $c_4(\varepsilon)>0$ such that

$$\frac{\chi_1^2}{2\varepsilon}\int_\Omega \frac{u_\varepsilon^{n-2m+2}}{u_\varepsilon^{n-m}+\varepsilon} v_{\varepsilon x}^2 \le \frac{D_1}{2}\int_\Omega u_{\varepsilon xx}^2+\frac{D_2}{2}\int_\Omega v_{\varepsilon xx}^2 + c_4(\varepsilon) \qquad \text{for all } t\in(0,T_{max,\varepsilon}), \tag{2.10}$$

whence combining (2.8) with (2.9) and (2.10) we infer that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega u_{\varepsilon x}^2+\frac{\varepsilon}{2}\int_\Omega\frac{u_\varepsilon^n}{u_\varepsilon^{n-m}+\varepsilon}u_{\varepsilon xxx}^2+\frac{1-2\alpha}{\alpha+1}\varepsilon^\beta\int_\Omega u_\varepsilon^{-\alpha}u_{\varepsilon xx}^2+\frac{D_1}{2}\int_\Omega u_{\varepsilon xx}^2 \\
&\qquad \le \frac{D_2}{2}\int_\Omega v_{\varepsilon xx}^2+c_4(\varepsilon) \qquad \text{for all } t\in(0,T_{max,\varepsilon}).
\end{aligned}$$

Since in quite a similar manner we obtain $c_5(\varepsilon)>0$ satisfying

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega v_{\varepsilon x}^2+\frac{\varepsilon}{2}\int_\Omega\frac{v_\varepsilon^n}{v_\varepsilon^{n-m}+\varepsilon}v_{\varepsilon xxx}^2+\frac{1-2\alpha}{\alpha+1}\varepsilon^\beta\int_\Omega v_\varepsilon^{-\alpha}v_{\varepsilon xx}^2+\frac{D_2}{2}\int_\Omega v_{\varepsilon xx}^2 \\
&\qquad \le \frac{D_1}{2}\int_\Omega u_{\varepsilon xx}^2+c_5(\varepsilon) \qquad \text{for all } t\in(0,T_{max,\varepsilon}),
\end{aligned}$$

upon an addition followed by an integration we conclude that

$$\begin{aligned}
&\frac{1}{2}\cdot\left\{\int_\Omega u_{\varepsilon x}^2(\cdot,t)+\int_\Omega v_{\varepsilon x}^2(\cdot,t)\right\}+\frac{\varepsilon}{2}\cdot\left\{\int_0^t\int_\Omega\frac{u_\varepsilon^n}{u_\varepsilon^{n-m}+\varepsilon}u_{\varepsilon xxx}^2+\int_0^t\int_\Omega\frac{v_\varepsilon^n}{v_\varepsilon^{n-m}+\varepsilon}v_{\varepsilon xxx}^2\right\} \\
&\qquad +\frac{1-2\alpha}{\alpha+1}\varepsilon^\beta\cdot\left\{\int_0^t\int_\Omega u_\varepsilon^{-\alpha}u_{\varepsilon xx}^2+\int_0^t\int_\Omega v_\varepsilon^{-\alpha}v_{\varepsilon xx}^2\right\} \\
&\qquad \le \frac{1}{2}\cdot\left\{\int_\Omega u_{0\varepsilon x}^2+\int_\Omega v_{0\varepsilon x}^2\right\}+\Big(c_4(\varepsilon)+c_5(\varepsilon)\Big)\cdot t \qquad \text{for all } t\in(0,T_{max,\varepsilon}).
\end{aligned}$$

As $\alpha < \frac{1}{2}$, this directly yields (2.4), (2.6) and (2.5), whereafter (2.7) results from (2.6) by once more employing Lemma 7.1. □

Now if the degeneracy in the fourth-order operators in (1.4) is suitably strong, then a second standard testing procedure can be imported from the analysis of thin film problems ([4], [3]) so as to warrant L^2 bounds for $\frac{1}{u_\varepsilon}$ and $\frac{1}{v_\varepsilon}$. Here we emphasize that somewhat in contrast to corresponding arguments in the thin film analysis, an appropriate treatment of contributions stemming from the second-order diffusion and cross-diffusion mechanisms in (1.4) seems to require utilization of the estimates provided by Lemma 2.2, and thereby particularly rely on the presence of the fast diffusion operators in (1.4).

Lemma 2.3 *Let $n > \frac{7}{2}$, $m \in (0, n-1]$ and $\alpha \in (0, \frac{1}{2})$ be such that $m \le \frac{n+2}{2}$ and $\alpha \ge 4 - n$, and let $\beta > 0$. Then for all $\varepsilon \in (0,1)$ and $T > 0$ there exists $C(\varepsilon, T) > 0$ such that*

$$\int_\Omega \frac{1}{u_\varepsilon^2(x,t)} dx + \int_\Omega \frac{1}{v_\varepsilon^2(x,t)} dx \le C(\varepsilon, T) \qquad \textit{for all } t \in (0, \widehat{T}_\varepsilon), \tag{2.11}$$

where again $\widehat{T}_\varepsilon := \min\{T, T_{max,\varepsilon}\}$.

Proof. Using (1.4) we see that

$$\begin{aligned} \frac{1}{6}\frac{d}{dt}\int_\Omega \frac{1}{u_\varepsilon^2} &= -\frac{1}{3}\int_\Omega \frac{1}{u_\varepsilon^3} \cdot \Big\{ -\varepsilon \frac{u_\varepsilon^n}{u_\varepsilon^{n-m}+\varepsilon} u_{\varepsilon xxx} + \varepsilon^\beta u_\varepsilon^{-\alpha} u_{\varepsilon x} + D_1 u_{\varepsilon x} - \chi_1 \frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m}+\varepsilon} v_{\varepsilon x} \Big\}_x \\ &= \varepsilon \int_\Omega \frac{u_\varepsilon^{n-4}}{u_\varepsilon^{n-m}+\varepsilon} u_{\varepsilon x} u_{\varepsilon xxx} - \varepsilon^\beta \int_\Omega u_\varepsilon^{-\alpha-4} u_{\varepsilon x}^2 \\ &\quad - D_1 \int_\Omega u_\varepsilon^{-4} u_{\varepsilon x}^2 + \chi_1 \int_\Omega \frac{u_\varepsilon^{n-m-3}}{u_\varepsilon^{n-m}+\varepsilon} u_{\varepsilon x} v_{\varepsilon x} \qquad \text{for all } t \in (0, T_{max,\varepsilon}), \end{aligned} \tag{2.12}$$

where due to Young's inequality, our assumption that $m \le n-1$ entails that

$$\begin{aligned} \chi_1 \int_\Omega \frac{u_\varepsilon^{n-m-3}}{u_\varepsilon^{n-m}+\varepsilon} u_{\varepsilon x} v_{\varepsilon x} &\le \frac{\chi_1}{\varepsilon} \int_\Omega u_\varepsilon^{n-m-3} |u_{\varepsilon x} v_{\varepsilon x}| \\ &\le D_1 \int_\Omega u_\varepsilon^{-4} u_{\varepsilon x}^2 + \frac{\chi_1^2}{4\varepsilon^2 D_1} \int_\Omega u_\varepsilon^{2n-2m-2} v_{\varepsilon x}^2 \\ &\le D_1 \int_\Omega u_\varepsilon^{-4} u_{\varepsilon x}^2 + \frac{\chi_1^2}{4\varepsilon^2 D_1} \|u_\varepsilon\|_{L^\infty(\Omega)}^{2n-2m-2} \int_\Omega v_{\varepsilon x}^2 \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \end{aligned} \tag{2.13}$$

Moreover, using that with $c_1 := \max\{|m-4|, |n-4|\}$ we have

$$\begin{aligned} \Big| \frac{d}{ds} \Big(\frac{s^{n-4}}{s^{n-m}+\varepsilon} \Big) \Big| &= \Big| \frac{(m-4)s^{2n-m-5} + (n-4)\varepsilon s^{n-5}}{(s^{n-m}+\varepsilon)^2} \Big| \\ &\le \frac{c_1 s^{2n-m-5} + c_1 \varepsilon s^{n-5}}{(s^{n-m}+\varepsilon)^2} \\ &= \frac{c_1 s^{n-5}}{s^{n-m}+\varepsilon} \le \frac{c_1}{\varepsilon} s^{n-5} \qquad \text{for all } s > 0, \end{aligned}$$

once again integrating by parts we obtain

$$
\begin{aligned}
\varepsilon \int_\Omega \frac{u_\varepsilon^{n-4}}{u_\varepsilon^{n-m}+\varepsilon} u_{\varepsilon x} u_{\varepsilon xxx} &= -\varepsilon \int_\Omega \frac{u_\varepsilon^{n-4}}{u_\varepsilon^{n-m}+\varepsilon} u_{\varepsilon xx}^2 - \varepsilon \int_\Omega \frac{d}{ds}\Big(\frac{s^{n-4}}{s^{n-m}+\varepsilon}\Big)\Big|_{s=u_\varepsilon} u_{\varepsilon x}^2 u_{\varepsilon xx} \\
&\le c_1 \int_\Omega u_\varepsilon^{n-5} u_{\varepsilon x}^2 |u_{\varepsilon xx}| \qquad \text{for all } t\in(0,T_{max,\varepsilon}), \qquad (2.14)
\end{aligned}
$$

and here thanks to Young's inequality,

$$
\begin{aligned}
\int_\Omega u_\varepsilon^{n-5} u_{\varepsilon x}^2 |u_{\varepsilon xx}| &\le \frac{1}{2}\int_\Omega u_\varepsilon^{-\alpha} u_{\varepsilon xx}^2 + \frac{1}{2}\int_\Omega u_\varepsilon^{2n+\alpha-10} u_{\varepsilon x}^4 \\
&\le \frac{1}{2}\int_\Omega u_\varepsilon^{-\alpha} u_{\varepsilon xx}^2 + \frac{1}{2}\|u_\varepsilon\|_{L^\infty(\Omega)}^{2n+2\alpha-8} \int_\Omega u_\varepsilon^{-\alpha-2} u_{\varepsilon x}^4 \qquad \text{for all } t\in(0,T_{max,\varepsilon}), \quad (2.15)
\end{aligned}
$$

because $\alpha \ge 4-n$ by hypothesis. As the constant numbers $c_2(\varepsilon,T) := \sup_{t\in(0,\widehat{T}_\varepsilon)} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$, $c_3(\varepsilon,T) := \sup_{t\in(0,\widehat{T}_\varepsilon)} \int_\Omega v_{\varepsilon x}^2(\cdot,t)$, $c_4(\varepsilon,T) := \int_0^{\widehat{T}_\varepsilon}\int_\Omega u_\varepsilon^{-\alpha} u_{\varepsilon xx}^2$ and $c_5(\varepsilon,T) := \int_0^{T_{max,\varepsilon}}\int_\Omega u_\varepsilon^{-\alpha-2} u_{\varepsilon x}^4$ are all finite according to Lemma 2.2 and the continuity of $W^{1,2}(\Omega) \hookrightarrow L^\infty(\Omega)$, collecting (2.12)-(2.15) we thus infer upon an integration that

$$
\begin{aligned}
\frac{1}{6}\int_\Omega \frac{1}{u_\varepsilon^2(\cdot,t)} &\le \frac{1}{6}\int_\Omega \frac{1}{u_{0\varepsilon}^2} + \frac{\chi_1^2}{4\varepsilon^2 D_1} c_2^{2n-2m-2}(\varepsilon,T)\cdot c_3(\varepsilon,T)\cdot T \\
&\quad + \frac{c_1}{2}\cdot c_4(\varepsilon,T) + \frac{c_1}{2}\cdot c_2^{2n+2\alpha-8}(\varepsilon,T)\cdot c_5(\varepsilon,T) \qquad \text{for all } t\in(0,\widehat{T}_\varepsilon),
\end{aligned}
$$

which along with an analogous argument for the second solution component establishes (2.11). $\Box$

Thanks to the H^1 estimates from Lemma 2.2, the latter extends so as to warrant boundedness of $\frac{1}{u_\varepsilon}$ and $\frac{1}{v_\varepsilon}$ actually in L^∞ for each suitably small ε.

Lemma 2.4 *Let $n > \frac{7}{2}$, $m \in (0,n-1]$ and $\alpha \in (0,\frac{1}{2})$ be such that $m \le \frac{n+2}{2}$ and $\alpha \ge 4-n$, let $\beta > 0$, and let $\varepsilon_\star \in (0,1)$ be as in Lemma 2.3. Then for all $\varepsilon \in (0,\varepsilon_\star)$ and $T>0$ there exists $C(\varepsilon,T)>0$ such that again writing $\widehat{T}_\varepsilon := \min\{T,T_{max,\varepsilon}\}$ we have*

$$
u_\varepsilon(x,t) \ge C(\varepsilon,T) \quad \textit{and} \quad v_\varepsilon(x,t) \ge C(\varepsilon,T) \qquad \textit{for all } x\in\Omega \textit{ and } t\in(0,\widehat{T}_\varepsilon). \qquad (2.16)
$$

PROOF. For fixed $\varepsilon \in (0,\varepsilon_\star)$ and $T>0$, Lemma 2.2 and Lemma 2.3 provide $c_1(\varepsilon,T)>0$ and $c_2(\varepsilon,T)>0$ fulfilling

$$
\int_\Omega u_{\varepsilon x}^2 \le c_1(\varepsilon,T) \qquad \text{for all } t\in(0,\widehat{T}_\varepsilon) \qquad (2.17)
$$

and

$$
\int_\Omega \frac{1}{u_\varepsilon^2} \le c_2(\varepsilon,T) \qquad \text{for all } t\in(0,\widehat{T}_\varepsilon). \qquad (2.18)
$$

Now for $\varepsilon \in (0,\varepsilon_\star)$ and $t \in (0,\widehat{T}_\varepsilon)$ we let $\underline{u}_\varepsilon(t) := \min_{x\in\overline{\Omega}} u_\varepsilon(x,t)$ and pick $x_0(\varepsilon,t) \in \overline{\Omega}$ in such a way that $u_\varepsilon(x_0(\varepsilon,t),t) = \underline{u}_\varepsilon(t)$, where without loss of generality, for definiteness we may assume that $\Omega = (-\frac{|\Omega|}{2},\frac{|\Omega|}{2})$ and $x_0(\varepsilon,t) \le 0$. Then using (2.17) we can estimate

$$
u_\varepsilon(x,t) = u_\varepsilon(x_0(\varepsilon,t),t) + \int_{x_0(\varepsilon,t)}^x u_{\varepsilon x}(y,t)dy \le \underline{u}_\varepsilon(t) + \sqrt{c_1(\varepsilon,T)}|x-x_0(\varepsilon,t)|^{\frac{1}{2}} \qquad \text{for all } x\in\Omega
$$

and hence particularly obtain that

$$u_\varepsilon(x,t) \le 2\sqrt{c_1(\varepsilon,T)}(x - x_0(\varepsilon,t))^{\frac{1}{2}} \qquad \text{whenever } x \in \Omega \text{ is such that } x \ge x_0 + \frac{4\underline{u}_\varepsilon^2(t)}{c_1(\varepsilon,T)}. \tag{2.19}$$

Under the hypothesis that

$$\frac{\underline{u}_\varepsilon^2(t)}{c_1(\varepsilon,T)} \le \frac{|\Omega|}{4}, \tag{2.20}$$

however, the latter region is conveniently large and enables us to infer from (2.18) and (2.19) that

$$\begin{aligned} c_2(\varepsilon,T) &\ge \int_\Omega \frac{1}{u_\varepsilon^2(x,t)} dx \\ &\ge \frac{1}{4c_1(\varepsilon,T)} \int_{x_0(\varepsilon,t)+\frac{\underline{u}_\varepsilon^2(t)}{c_1(\varepsilon,T)}}^{\frac{|\Omega|}{2}} \frac{dx}{x - x_0(\varepsilon,t)} \\ &= \frac{1}{4c_1(\varepsilon,T)} \ln \left(\frac{\frac{|\Omega|}{2} - x_0(\varepsilon,t)}{\frac{\underline{u}_\varepsilon^2(t)}{c_1(\varepsilon,T)}} \right) \\ &\ge \frac{1}{4c_1(\varepsilon,T)} \ln \frac{c_1(\varepsilon,T)|\Omega|}{2\underline{u}_\varepsilon^2(t)}, \end{aligned}$$

because $x_0(\varepsilon,t) \le 0$. In this case, we thus conclude that

$$\underline{u}_\varepsilon^2(t) \ge c_3(\varepsilon,T) := \frac{c_1(\varepsilon,T)|\Omega|}{2} \cdot e^{-4c_1(\varepsilon,T)c_2(\varepsilon,T)},$$

which in conjunction with (2.20) implies that in any event,

$$u_\varepsilon(x,t) \ge \min\left\{ \sqrt{c_3(\varepsilon,T)}\, , \frac{\sqrt{c_1(\varepsilon,T)|\Omega|}}{2} \right\} \qquad \text{for all } x \in \Omega \text{ and } t \in (0, \widehat{T}_\varepsilon).$$

A corresponding lower bound for v_ε can be found similarly. □

2.2 ε-dependent a priori estimates: H^2 bounds

We now slightly exceed the realm of classical thin film analysis by proceeding toward the derivation of H^2 estimates. In the course of a corresponding third testing-based argument, we shall make substantial use not only of the first-order estimates from Lemma 2.2, but also of the two-sided pointwise bounds for u_ε and v_ε implied by the latter in conjunction with Lemma 2.4.

Lemma 2.5 *Let $n > \frac{7}{2}$, $m \in (0, n-1]$ and $\alpha \in (0, \frac{1}{2})$ be such that $m \le \frac{n+2}{2}$ and $\alpha \ge 4 - n$, and let $\beta > 0$. Then with $\varepsilon_\star \in (0,1)$ taken from Lemma 2.3, for all $\varepsilon \in (0, \varepsilon_\star)$ and $T > 0$ one can find $C(\varepsilon,T) > 0$ such that*

$$\int_\Omega u_{\varepsilon xx}^2(x,t)dx + \int_\Omega v_{\varepsilon xx}^2(x,t)dx \le C(\varepsilon,T) \qquad \textit{for all } t \in (0, \widehat{T}_\varepsilon), \tag{2.21}$$

where again $\widehat{T}_\varepsilon := \min\{T, T_{max,\varepsilon}\}$.

PROOF. We first make use of Lemma 2.4, Lemma 2.2 and the fact that $W^{1,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ to find positive constants $c_1(\varepsilon,T), c_2(\varepsilon,T)$ and $c_3(\varepsilon,T)$ such that

$$c_1(\varepsilon,T) \le u_\varepsilon(x,t) \le c_2(\varepsilon,T) \quad \text{and} \quad c_1(\varepsilon,T) \le v_\varepsilon(x,t) \le c_2(\varepsilon,T) \qquad \text{for all } x \in \Omega \text{ and } t \in (0,\widehat{T}_\varepsilon), \tag{2.22}$$

and that

$$\int_\Omega u_{\varepsilon x}^2 + \int_\Omega v_{\varepsilon x}^2 \le c_3(\varepsilon,T) \qquad \text{for all } t \in (0,\widehat{T}_\varepsilon). \tag{2.23}$$

In particular, (2.22) entails that e.g. with $c_4(\varepsilon,T) := \frac{\varepsilon c_1^n(\varepsilon,T)}{c_2^{n-m}(\varepsilon,T)+\varepsilon}$ we have

$$\varepsilon \cdot \frac{u_\varepsilon^n}{u_\varepsilon^{n-m}+\varepsilon} \ge c_4(\varepsilon,T) \qquad \text{in } \Omega \times (0,\widehat{T}_\varepsilon),$$

and that moreover

$$\varepsilon \cdot \frac{m u_\varepsilon^{2n-m-1} + n\varepsilon u_\varepsilon^{n-1}}{(u_\varepsilon^{n-m}+\varepsilon)^2} \le c_5(\varepsilon,T) := \varepsilon \cdot \frac{m c_2^{2n-m-1}(\varepsilon,T) + n\varepsilon c_2^{n-1}(\varepsilon,T)}{\varepsilon^2} \qquad \text{in } \Omega \times (0,\widehat{T}_\varepsilon)$$

as well as

$$\begin{aligned} \chi_1 \frac{u_\varepsilon^{2n-2m} + (n-m+1)\varepsilon u_\varepsilon^{n-m}}{(u_\varepsilon^{n-m}+\varepsilon)^2} &\le c_6(\varepsilon,T) \\ &:= \chi_1 \cdot \frac{c_2^{2n-2m}(\varepsilon,T) + (n-m+1)\varepsilon c_2^{n-m}(\varepsilon,T)}{\varepsilon^2} \qquad \text{in } \Omega \times (0,\widehat{T}_\varepsilon). \end{aligned}$$

Therefore, testing the first equation in (1.4) against $u_{\varepsilon xxxx}$ and using Young's inequality shows that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega u_{\varepsilon xx}^2 &= -\varepsilon \int_\Omega \frac{u_\varepsilon^n}{u_\varepsilon^{n-m}+\varepsilon} u_{\varepsilon xxxx}^2 - \varepsilon \int_\Omega \frac{m u_\varepsilon^{2n-m-1} + n\varepsilon u_\varepsilon^{n-1}}{(u_\varepsilon^{n-m}+\varepsilon)^2} u_{\varepsilon x} u_{\varepsilon xxx} u_{\varepsilon xxxx} \\ &\quad + \varepsilon^\beta \int_\Omega (u_\varepsilon^{-\alpha} u_{\varepsilon x})_x u_{\varepsilon xxxx} - D_1 \int_\Omega u_{\varepsilon xxx}^2 \\ &\quad - \chi_1 \int_\Omega \frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m}+\varepsilon} v_{\varepsilon xx} u_{\varepsilon xxxx} - \chi_1 \int_\Omega \frac{u_\varepsilon^{2n-2m} + (n-m+1)\varepsilon u_\varepsilon^{n-m}}{(u_\varepsilon^{n-m}+\varepsilon)^2} u_{\varepsilon x} v_{\varepsilon x} u_{\varepsilon xxxx} \\ &\le -c_4(\varepsilon,T) \int_\Omega u_{\varepsilon xxxx}^2 + c_5(\varepsilon,T) \int_\Omega |u_{\varepsilon x} u_{\varepsilon xxx} u_{\varepsilon xxxx}| + \varepsilon^\beta \int_\Omega (u_\varepsilon^{-\alpha} u_{\varepsilon x})_x u_{\varepsilon xxxx} \\ &\quad + c_2(\varepsilon,T)\chi_1 \int_\Omega |v_{\varepsilon xx} u_{\varepsilon xxxx}| + c_6(\varepsilon,T) \int_\Omega |u_{\varepsilon x} v_{\varepsilon x} u_{\varepsilon xxxx}| \\ &\le -\frac{c_4(\varepsilon,T)}{2} \int_\Omega u_{\varepsilon xxxx}^2 + \frac{2c_5^2(\varepsilon,T)}{c_4(\varepsilon,T)} \int_\Omega u_{\varepsilon x}^2 u_{\varepsilon xxx}^2 \\ &\quad + \frac{2\varepsilon^{2\beta}}{c_4(\varepsilon,T)} \int_\Omega \Big|(u_\varepsilon^{-\alpha} u_{\varepsilon x})_x\Big|^2 + \frac{2c_2^2(\varepsilon,T)}{c_4(\varepsilon,T)}\chi_1^2 \int_\Omega v_{\varepsilon xx}^2 + \frac{2c_6^2(\varepsilon,T)}{c_4(\varepsilon,T)} \int_\Omega u_{\varepsilon x}^2 v_{\varepsilon x}^2 \end{aligned}$$

for all $t \in (0,\widehat{T}_\varepsilon)$. As herein, once more by (2.22) and Young's inequality,

$$\int_\Omega \Big|(u_\varepsilon^{-\alpha} u_{\varepsilon x})_x\Big|^2 = \int_\Omega |u_\varepsilon^{-\alpha} u_{\varepsilon xx} - \alpha u_\varepsilon^{-\alpha-1} u_{\varepsilon x}^2|^2$$

$$\begin{aligned}
&\leq 2\int_\Omega u_\varepsilon^{-2\alpha} u_{\varepsilon xx}^2 + 2\alpha^2 \int_\Omega u_\varepsilon^{-2\alpha-2} u_{\varepsilon x}^4 \\
&\leq 2c_1^{-2\alpha}(\varepsilon,T)\int_\Omega u_{\varepsilon xx}^2 + 2\alpha^2 c_1^{-2\alpha-2}(\varepsilon,T)\int_\Omega u_{\varepsilon x}^4 \qquad \text{for all } t\in(0,\widehat{T}_\varepsilon)
\end{aligned}$$

and $\int_\Omega u_{\varepsilon x}^2 v_{\varepsilon x}^2 \leq \frac{1}{2}\int_\Omega u_{\varepsilon x}^4 + \frac{1}{2}\int_\Omega v_{\varepsilon x}^4$ for all $t\in(0,T_{max,\varepsilon})$, on performing a similar procedure to the second solution component we infer the existence of $c_7(\varepsilon,T)>0$ and $c_8(\varepsilon,T)>0$ such that for all $t\in(0,\widehat{T}_\varepsilon)$,

$$\begin{aligned}
&\frac{d}{dt}\Big\{\int_\Omega u_{\varepsilon xx}^2 + \int_\Omega v_{\varepsilon xx}^2\Big\} + c_7(\varepsilon,T)\cdot\Big\{\int_\Omega u_{\varepsilon xxxx}^2 + \int_\Omega v_{\varepsilon xxxx}^2\Big\} \\
&\qquad \leq c_8(\varepsilon,T)\int_\Omega u_{\varepsilon x}^2 u_{\varepsilon xxx}^2 + c_8(\varepsilon,T)\int_\Omega v_{\varepsilon x}^2 v_{\varepsilon xxx}^2 \\
&\qquad\quad + c_8(\varepsilon,T)\int_\Omega u_{\varepsilon xx}^2 + c_8(\varepsilon,T)\int_\Omega v_{\varepsilon xx}^2 + c_8(\varepsilon,T)\int_\Omega u_{\varepsilon x}^4 + c_8(\varepsilon,T)\int_\Omega v_{\varepsilon x}^4. \qquad (2.24)
\end{aligned}$$

We now invoke the Gagliardo-Nirenberg inequality along with (2.23) and again Young's inequality to see that with some $c_9(\varepsilon,T)>0$ and $c_{10}(\varepsilon,T)>0$,

$$\begin{aligned}
c_8(\varepsilon,T)\int_\Omega u_{\varepsilon x}^2 u_{\varepsilon xxx}^2 &\leq c_8(\varepsilon,T)\|u_{\varepsilon x}\|_{L^2(\Omega)}^2 \|u_{\varepsilon xxx}\|_{L^\infty(\Omega)}^2 \\
&\leq c_9(\varepsilon,T)\|u_{\varepsilon x}\|_{L^2(\Omega)}^2 \cdot \Big\{\|u_{\varepsilon xxxx}\|_{L^2(\Omega)}^{\frac{5}{3}} \|u_{\varepsilon x}\|_{L^2(\Omega)}^{\frac{1}{3}} + \|u_{\varepsilon x}\|_{L^2(\Omega)}^2\Big\} \\
&\leq c_3^{\frac{7}{6}}(\varepsilon,T)c_9(\varepsilon,T)\|u_{\varepsilon xxxx}\|_{L^2(\Omega)}^{\frac{5}{3}} + c_3^2(\varepsilon,T)c_9(\varepsilon,T) \\
&\leq \frac{c_7(\varepsilon,T)}{2}\int_\Omega u_{\varepsilon xxxx}^2 + c_{10}(\varepsilon,T) \qquad \text{for all } t\in(0,\widehat{T}_\varepsilon),
\end{aligned}$$

while similarly

$$\begin{aligned}
c_8(\varepsilon,T)\int_\Omega u_{\varepsilon x}^4 &\leq c_{11}(\varepsilon,T)\|u_{\varepsilon xxxx}\|_{L^2(\Omega)}^{\frac{1}{3}} \|u_{\varepsilon x}\|_{L^2(\Omega)}^{\frac{11}{3}} + c_{11}(\varepsilon,T)\|u_{\varepsilon x}\|_{L^2(\Omega)}^4 \\
&\leq c_3^{\frac{11}{6}}(\varepsilon,T)c_{11}(\varepsilon,T)\|u_{\varepsilon xxxx}\|_{L^2(\Omega)}^{\frac{1}{3}} + c_3^2(\varepsilon,T)c_{11}(\varepsilon,T) \\
&\leq \frac{c_7(\varepsilon,T)}{2}\int_\Omega u_{\varepsilon xxxx}^2 + c_{12}(\varepsilon,T) \qquad \text{for all } t\in(0,\widehat{T}_\varepsilon)
\end{aligned}$$

with appropriately large constants $c_{11}(\varepsilon,T)$ and $c_{12}(\varepsilon,T)$. Along with essentially identical arguments for v_ε, from (2.24) we thus conclude that with some $c_{13}(\varepsilon,T)>0$ we have

$$\frac{d}{dt}\Big\{\int_\Omega u_{\varepsilon xx}^2 + \int_\Omega v_{\varepsilon xx}^2\Big\} \leq c_{13}(\varepsilon,T)\cdot\Big\{\int_\Omega u_{\varepsilon xx}^2 + \int_\Omega v_{\varepsilon xx}^2\Big\} + c_{13}(\varepsilon,T) \qquad \text{for all } t\in(0,\widehat{T}_\varepsilon),$$

which upon an integration directly leads to (2.21) in view of the assumed inclusions $u_{0\varepsilon}\in W^{2,2}(\Omega)$ and $v_{0\varepsilon}\in W^{2,2}(\Omega)$ asserted by (1.5). $\square$

Within the parameter setting created above, we can thereby complete our reasoning concerning global solvability in the approximate problems:

Lemma 2.6 *Let $n > \frac{7}{2}$, $m \in (0, n-1]$, $\alpha \in (0, \frac{1}{2})$ and $\beta > 0$ be such that $m \le \frac{n+2}{2}$ and $\alpha \ge 4-n$, and let $\varepsilon_\star \in (0,1)$ ba as given by Lemma 2.3. Then for each $\varepsilon \in (0, \varepsilon_\star)$, the solution $(u_\varepsilon, v_\varepsilon)$ of (1.4) from Lemma 2.1 is global in time.*

Proof. By means of the extensibility criterion (2.1), in view of the continuity of the embedding $W^{2,2}(\Omega) \hookrightarrow W^{s,2}(\Omega)$ for any $s \in (\frac{3}{2}, 2)$ we immediately obtain this as a consequence of Lemma 2.5, Lemma 2.4 and (2.2). $\square$

3 Consistency with a global entropy structure: The condition $m \le 2$

We shall next focus on the question how far the fundamental identity (1.6) is respected by the chosen approximation scheme. Our first observation in this regard reveals that indeed a suitably modified variant of (1.6) can rigorously be derived for solutions to (1.4) whenever the requirements on the parameter m therein are suitably sharepened in comparison to the above. Let us underline already here that of particular importance for our qualitative analysis in Section 5 will be the fortunate circumstance that the range of admissible m includes some conveniently large number by containing the value $m = 2$.

Lemma 3.1 *Let $n > \frac{7}{2}$, $m \in (\frac{1}{2}, 2]$ and $\alpha \in (0, \frac{1}{2})$ be such that $\alpha \ge 4-n$, and let $\beta > 0$. For $\varepsilon \in (0,1)$, define*

$$\begin{aligned}\mathcal{E}_\varepsilon(t) := \; & \chi_2 \int_\Omega u_\varepsilon(\cdot,t)\ln u_\varepsilon(\cdot,t) + \chi_1 \int_\Omega v_\varepsilon(\cdot,t)\ln v_\varepsilon(\cdot,t) \\ & + \frac{\chi_2 \varepsilon}{(n-m)(n-m-1)} \int_\Omega \frac{1}{u_\varepsilon^{n-m-1}(\cdot,t)} + \frac{\chi_1 \varepsilon}{(n-m)(n-m-1)} \int_\Omega \frac{1}{v_\varepsilon^{n-m-1}(\cdot,t)}, \quad t \ge 0, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\mathcal{D}_\varepsilon(t) := \; & \chi_2 D_1 \int_\Omega \frac{u_{\varepsilon x}^2(\cdot,t)}{u_\varepsilon(\cdot,t)} + \chi_1 D_2 \int_\Omega \frac{v_{\varepsilon x}^2(\cdot,t)}{v_\varepsilon(\cdot,t)} \\ & + \chi_2 D_1 \varepsilon \int_\Omega u_\varepsilon^{-n+m-1}(\cdot,t) u_{\varepsilon x}^2(\cdot,t) + \chi_1 D_2 \varepsilon \int_\Omega v_\varepsilon^{-n+m-1}(\cdot,t) v_{\varepsilon x}^2(\cdot,t) \\ & + \min\Big\{1\,,\, \frac{2m-1}{2-m}\Big\} \cdot \Big\{ \chi_2 \varepsilon \int_\Omega u_\varepsilon^{m-1}(\cdot,t) u_{\varepsilon xx}^2(\cdot,t) + \chi_1 \varepsilon \int_\Omega v_\varepsilon^{m-1}(\cdot,t) v_{\varepsilon xx}^2(\cdot,t) \Big\}, \quad t > 0. \end{aligned} \tag{3.2}$$

Then

$$\mathcal{E}_\varepsilon'(t) \le -\mathcal{D}_\varepsilon(t) \qquad \textit{for all } t > 0 \textit{ and any } \varepsilon \in (0,1). \tag{3.3}$$

Proof. Using (1.4) and (2.2), we compute

$$\begin{aligned} & \frac{d}{dt}\int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{\varepsilon}{(n-m)(n-m-1)} \frac{d}{dt}\int_\Omega \frac{1}{u_\varepsilon^{n-m-1}} \\ & = \int_\Omega \Big\{ \ln u_\varepsilon + 1 - \frac{\varepsilon}{(n-m)u_\varepsilon^{n-m}} \Big\} \cdot u_{\varepsilon t} \\ & = \int_\Omega \Big\{ \ln u_\varepsilon + 1 - \frac{\varepsilon}{(n-m)u_\varepsilon^{n-m}} \Big\} \cdot \Big\{ -\varepsilon \frac{u_\varepsilon^n}{u_\varepsilon^{n-m}+\varepsilon} u_{\varepsilon xxx} + \varepsilon^\beta u_\varepsilon^{-\alpha} u_{\varepsilon x} + D_1 u_{\varepsilon x} - \chi_1 \frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m}+\varepsilon} v_{\varepsilon x} \Big\}_x \end{aligned}$$

$$
\begin{aligned}
&= -\int_\Omega \Big\{\frac{1}{u_\varepsilon} + \frac{\varepsilon}{u_\varepsilon^{n-m+1}}\Big\} u_{\varepsilon x} \cdot \Big\{ -\varepsilon \frac{u_\varepsilon^n}{u_\varepsilon^{n-m}+\varepsilon} u_{\varepsilon xxx} + \varepsilon^\beta u_\varepsilon^{-\alpha} u_{\varepsilon x} + D_1 u_{\varepsilon x} - \chi_1 \frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m}+\varepsilon} v_{\varepsilon x}\Big\} \\
&= \varepsilon \int_\Omega u_\varepsilon^{m-1} u_{\varepsilon x} u_{\varepsilon xxx} \\
&\quad -\varepsilon^\beta \int_\Omega u_\varepsilon^{-\alpha-1} u_{\varepsilon x}^2 - \varepsilon^{\beta+1} \int_\Omega u_\varepsilon^{-n+m-\alpha-1} u_{\varepsilon x}^2 \\
&\quad -D_1 \int_\Omega \frac{u_{\varepsilon x}^2}{u_\varepsilon} - D_1 \varepsilon \int_\Omega u_\varepsilon^{-n+m-1} u_{\varepsilon x}^2 \\
&\quad +\chi_1 \int_\Omega u_{\varepsilon x} v_{\varepsilon x} \qquad \text{for all } t>0.
\end{aligned}
\tag{3.4}
$$

Here two more integrations by parts show that

$$
\begin{aligned}
\varepsilon \int_\Omega u_\varepsilon^{m-1} u_{\varepsilon x} u_{\varepsilon xxx} &= -\varepsilon \int_\Omega u_\varepsilon^{m-1} u_{\varepsilon xx}^2 - (m-1)\varepsilon \int_\Omega u_\varepsilon^{m-2} u_{\varepsilon x}^2 u_{\varepsilon xx} \\
&= -\varepsilon \int_\Omega u_\varepsilon^{m-1} u_{\varepsilon xx}^2 + \frac{(m-1)(m-2)\varepsilon}{3} \int_\Omega u_\varepsilon^{m-3} u_{\varepsilon x}^4 \qquad \text{for all } t>0,
\end{aligned}
$$

where the last summand is nonpositive if $m \in [1,2]$, while in the case when $m<1$ we invoke Lemma 7.1 to see that then

$$
-\varepsilon \int_\Omega u_\varepsilon^{m-1} u_{\varepsilon xx}^2 + \frac{(m-1)(m-2)\varepsilon}{3} \int_\Omega u_\varepsilon^{m-3} u_{\varepsilon x}^4 \le -\Big\{1 - \frac{3(1-m)}{2-m}\Big\} \cdot \varepsilon \int_\Omega u_\varepsilon^{m-1} u_{\varepsilon xx}^2
$$

for all $t>0$, with the factor $1-\frac{3(1-m)}{2-m} = \frac{2m-1}{2-m}$ being positive thanks to the assumption that $m>\frac{1}{2}$. On dropping nonpositive summands, from (3.4) we thus infer that for arbitrary $m \in (\frac{1}{2},2]$ and any $t>0$,

$$
\begin{aligned}
&\frac{d}{dt} \int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{\varepsilon}{(n-m)(n-m-1)} \frac{d}{dt} \int_\Omega \frac{1}{u_\varepsilon^{n-m-1}} \\
&\quad \le -\min\Big\{1\,,\,\frac{2m-1}{2-m}\Big\} \cdot \varepsilon \int_\Omega u_\varepsilon^{m-1} u_{\varepsilon xx}^2 - D_1 \int_\Omega \frac{u_{\varepsilon x}^2}{u_\varepsilon} - D_1\varepsilon \int_\Omega u_\varepsilon^{-n+m-1} u_{\varepsilon x}^2 + \chi_1 \int_\Omega u_{\varepsilon x} v_{\varepsilon x},
\end{aligned}
$$

so that since similarly

$$
\begin{aligned}
&\frac{d}{dt} \int_\Omega v_\varepsilon \ln v_\varepsilon + \frac{\varepsilon}{(n-m)(n-m-1)} \frac{d}{dt} \int_\Omega \frac{1}{v_\varepsilon^{n-m-1}} \\
&\quad \le -\min\Big\{1\,,\,\frac{2m-1}{2-m}\Big\} \cdot \varepsilon \int_\Omega v_\varepsilon^{m-1} v_{\varepsilon xx}^2 - D_2 \int_\Omega \frac{v_{\varepsilon x}^2}{v_\varepsilon} - D_2\varepsilon \int_\Omega v_\varepsilon^{-n+m-1} v_{\varepsilon x}^2 - \chi_2 \int_\Omega u_{\varepsilon x} v_{\varepsilon x}
\end{aligned}
$$

for all $t>0$, by taking a suitable linear combination of these two inequalities we arrive at (3.3). $\square$

An integration of the latter immediately implies some first ε-independent regularity features beyond those from (2.2):

Corollary 3.2 *Let $n > \frac{7}{2}$, $m \in (\frac{1}{2}, 2]$ and $\alpha \in (0, \frac{1}{2})$ be such that $\alpha \geq 4 - n$, and let $\beta > 0$. Then there exist $C > 0$ and $\varepsilon_\star \in (0,1)$ such that if $\varepsilon \in (0, \varepsilon_\star)$, then*

$$\int_\Omega u_\varepsilon(x,t) \ln\Big(u_\varepsilon(x,t) + e\Big) + \int_\Omega v_\varepsilon(x,t) \ln\Big(v_\varepsilon(x,t) + e\Big) \leq C \qquad \textit{for all } t > 0 \tag{3.5}$$

and

$$\int_0^t \int_\Omega \frac{u_{\varepsilon x}^2}{u_\varepsilon} + \int_0^t \int_\Omega \frac{v_{\varepsilon x}^2}{v_\varepsilon} \leq C \qquad \textit{for all } t > 0 \tag{3.6}$$

as well as

$$\varepsilon \int_0^t \int_\Omega u_\varepsilon^{-n+m-1} u_{\varepsilon x}^2 + \varepsilon \int_0^t \int_\Omega v_\varepsilon^{-n+m-1} v_{\varepsilon x}^2 \leq C \qquad \textit{for all } t > 0 \tag{3.7}$$

and

$$\varepsilon \int_0^t \int_\Omega u_\varepsilon^{m-1} u_{\varepsilon xx}^2 + \varepsilon \int_0^t \int_\Omega v_\varepsilon^{m-1} v_{\varepsilon xx}^2 \leq C \qquad \textit{for all } t > 0. \tag{3.8}$$

PROOF. Integrating (3.3), we obtain that for all $\varepsilon \in (0,1)$,

$$\mathcal{E}_\varepsilon(t) + \int_0^t \mathcal{D}_\varepsilon(s) ds \leq \mathcal{E}_\varepsilon(0) \qquad \text{for all } t > 0, \tag{3.9}$$

where since (1.5) ensures that $(u_{0\varepsilon}, v_{0\varepsilon}) \to (u_0, v_0)$ in $(L^\infty(\Omega))^2$ as $\varepsilon \searrow 0$, by positivity of both u_0 and v_0 in $\overline{\Omega}$ it follows that

$$\mathcal{E}_\varepsilon(0) \to c_1 := \chi_2 \int_\Omega u_0 \ln u_0 + \chi_1 \int_\Omega v_0 \ln v_0 \qquad \text{as } \varepsilon \searrow 0,$$

and that hence there exists $\varepsilon_\star \in (0,1)$ such that

$$\mathcal{E}_\varepsilon(0) \leq c_1 + 1 \qquad \text{for all } \varepsilon \in (0, \varepsilon_\star). \tag{3.10}$$

Moreover, using that $\xi \ln \xi \geq -\frac{1}{e}$ for all $\xi > 0$, by recalling (2.2) we can estimate

$$\begin{aligned}
\int_\Omega u_\varepsilon \ln(u_\varepsilon + e) &= \int_{\{u_\varepsilon > e\}} u_\varepsilon \ln(u_\varepsilon + e) + \int_{\{u_\varepsilon \leq e\}} u_\varepsilon \ln(u_\varepsilon + e) \\
&\leq \int_{\{u_\varepsilon > e\}} u_\varepsilon \ln(2u_\varepsilon) + \ln(2e) \int_{\{u_\varepsilon \leq e\}} u_\varepsilon \\
&= \int_\Omega u_\varepsilon \ln u_\varepsilon - \int_{\{u_\varepsilon \leq e\}} u_\varepsilon \ln u_\varepsilon + \ln 2 \cdot \int_\Omega u_\varepsilon + \int_{\{u_\varepsilon \leq e\}} u_\varepsilon \\
&\leq \int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{|\Omega|}{e} + (\ln 2 + 1) \int_\Omega u_0 \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1),
\end{aligned}$$

which combined with a similar inequality for v_ε shows that

$$\begin{aligned}
\mathcal{E}_\varepsilon(t) \geq\ & \chi_2 \int_\Omega u_\varepsilon \ln(u_\varepsilon + e) + \chi_1 \int_\Omega v_\varepsilon \ln(v_\varepsilon + e) \\
& -(\chi_1 + \chi_2) \cdot \frac{|\Omega|}{e} - (\ln 2 + 1) \cdot \Big\{ \chi_2 \int_\Omega u_0 + \chi_1 \int_\Omega v_0 \Big\} \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1).
\end{aligned}$$

In view of our definition (3.2) of $\mathcal{D}_\varepsilon$, both (3.5) as well as (3.6), (3.7) and (3.8) thus result from (3.9) and (3.10). $\Box$

Our derivation of further implications thereof will, besides utilizing a straightforward Hölder interpolation, make essential use of a Gagliardo-Nirenberg type inequality containing certain logarithmic corrections to standard Lebesgue norms. In view of its potential independent interest, we formulate and verify this apparently novel type of interpolation inequality in a separate appendix below, within a context slightly more general than needed here (cf. Corollary 7.6).

Corollary 3.3 *Suppose that $n > \frac{7}{2}$, $m \in (\frac{1}{2}, 2]$, $\alpha \in (0, \frac{1}{2})$ and $\beta > 0$ are such that $\alpha \geq 4 - n$, and let $\varepsilon_\star \in (0,1)$ be as given by Corollary 3.2. Then for all $T > 0$ there exist $C(T) > 0$ such that*

$$\int_0^T \int_\Omega u_\varepsilon^3 \ln(u_\varepsilon + e) + \int_0^T \int_\Omega v_\varepsilon^3 \ln(v_\varepsilon + e) \leq C(T) \qquad \text{for all } \varepsilon \in (0, \varepsilon_\star) \tag{3.11}$$

and

$$\int_0^T \int_\Omega |u_{\varepsilon x}|^{\frac{3}{2}} + \int_0^T \int_\Omega |v_{\varepsilon x}|^{\frac{3}{2}} \leq C(T) \qquad \text{for all } \varepsilon \in (0, \varepsilon_\star). \tag{3.12}$$

Proof. According to the refined Gagliardo-Nirenberg type inequality from Corollary 7.6 below, (3.11) readily follows from (3.5) and (3.6). As a Hölder interpolation ensures that

$$\begin{aligned} \int_0^T \int_\Omega |u_{\varepsilon x}|^{\frac{3}{2}} &= \int_0^T \int_\Omega \Big| \frac{u_{\varepsilon x}}{\sqrt{u_\varepsilon}} \Big|^{\frac{3}{2}} \cdot |\sqrt{u_\varepsilon}|^{\frac{3}{2}} \\ &\leq \bigg\{ \int_0^T \int_\Omega \frac{u_{\varepsilon x}^2}{u_\varepsilon} \bigg\}^{\frac{3}{4}} \cdot \bigg\{ \int_0^T \int_\Omega u_\varepsilon^3 \bigg\}^{\frac{1}{4}} \qquad \text{for all } \varepsilon \in (0, \varepsilon_\star), \end{aligned}$$

combining an evident consequence of the inequality in (3.11) with (3.6) thereafter yields (3.12). $\Box$

In preparation of a subsequence extraction procedure based on Aubin-Lions-type compactness statements, let us draw a further and conclusion of Corollary 3.2 and Corollary 3.3 on regularity properties of time derivatives in (1.4). Due to the structure of the fourth-order diffusion terms in (1.4), our argument in this regard needs to slightly deviate from fully straightforward reasonings by involving two further interpolation inequalities, now essentially based on the Hölder inequality and again swapped out to an appendix (Lemma 7.4), which allow for a convenient control of corresponding highest-order contributions in terms of the quantities addressed in (3.11) and (3.8).

Lemma 3.4 *Suppose that $n > \frac{7}{2}$, $m \in (\frac{1}{2}, 2]$, $\alpha \in (0, \frac{1}{2})$ and $\beta > 0$ are such that $\alpha \geq 4 - n$, and let $\varepsilon_\star \in (0,1)$ be as given by Corollary 3.2. Then for all $T > 0$ there exist $C(T) > 0$ such that*

$$\int_0^T \|u_{\varepsilon t}(\cdot, t)\|_{(W_0^{3,2}(\Omega))^\star} dt + \int_0^T \|v_{\varepsilon t}(\cdot, t)\|_{(W_0^{3,2}(\Omega))^\star} dt \leq C(T) \qquad \text{for all } \varepsilon \in (0, \varepsilon_\star). \tag{3.13}$$

Proof. Given $t > 0$ and $\psi \in C_0^\infty(\Omega)$, we use (1.4) and integrate by parts to see that writing $F_\varepsilon(s) := \frac{s^n}{s^{n-m}+\varepsilon}$ for $s \geq 0$ and $\varepsilon \in (0,1)$, we have

$$\Big| \int_\Omega u_{\varepsilon t}(\cdot, t) \psi \Big| \;=\; \bigg| - \varepsilon \int_\Omega F_\varepsilon'(u_\varepsilon) u_{\varepsilon x} u_{\varepsilon xxx} \psi_x - \varepsilon \int_\Omega F_\varepsilon(u_\varepsilon) u_{\varepsilon xxx} \psi_{xx} + \frac{\varepsilon^\beta}{1-\alpha} \int_\Omega u_\varepsilon^{1-\alpha} \psi_{xx}$$

$$
\begin{aligned}
& -D_1 \int_\Omega u_{\varepsilon x}\psi_x + \chi_1 \int_\Omega \frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m}+\varepsilon} v_{\varepsilon x}\psi_x \Bigg| \\
\le \quad & \varepsilon \cdot \left\{ \int_\Omega |F_\varepsilon'(u_\varepsilon) u_{\varepsilon x} u_{\varepsilon xx}| \right\} \cdot \|\psi_x\|_{L^\infty(\Omega)} + \varepsilon \cdot \left\{ \int_\Omega |F_\varepsilon(u_\varepsilon) u_{\varepsilon xx}| \right\} \cdot \|\psi_{xx}\|_{L^\infty(\Omega)} \\
& + \frac{\varepsilon^\beta}{1-\alpha} \cdot \left\{ \int_\Omega u_\varepsilon^{1-\alpha} \right\} \cdot \|\psi_{xx}\|_{L^\infty(\Omega)} \\
& + D_1 \cdot \left\{ \int_\Omega |u_{\varepsilon x}| \right\} \cdot \|\psi_x\|_{L^\infty(\Omega)} + \chi_1 \cdot \left\{ \int_\Omega |u_\varepsilon v_{\varepsilon x}| \right\} \cdot \|\psi_x\|_{L^\infty(\Omega)}, \qquad (3.14)
\end{aligned}
$$

because $\frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m}+\varepsilon} \le u_\varepsilon$. Here employing the two interpolation properties asserted by Lemma 7.4, followed by applying Young's inequality, shows that with $c_1 := \frac{n|\Omega|^{\frac{2-m}{12}}}{\sqrt{m}}$ and $c_2 := |\Omega|^{\frac{2-m}{6}}$,

$$
\begin{aligned}
\varepsilon \int_\Omega |F_\varepsilon'(u_\varepsilon) u_{\varepsilon x} u_{\varepsilon xx}| \quad & \le \quad c_1 \varepsilon \cdot \left\{ \int_\Omega u_\varepsilon^{m-1} u_{\varepsilon xx}^2 \right\}^{\frac{3}{4}} \cdot \left\{ \int_\Omega u_\varepsilon^3 \right\}^{\frac{m+1}{12}} \\
& \le \quad c_1 \varepsilon \int_\Omega u_\varepsilon^{m-1} u_{\varepsilon xx}^2 + c_1 \int_\Omega u_\varepsilon^3 + c_1 |\Omega| \qquad (3.15)
\end{aligned}
$$

and that

$$
\begin{aligned}
\varepsilon \int_\Omega |F_\varepsilon(u_\varepsilon) u_{\varepsilon xx}| \quad & \le \quad c_2 \varepsilon \cdot \left\{ \int_\Omega u_\varepsilon^{m-1} u_{\varepsilon xx}^2 \right\}^{\frac{1}{2}} \cdot \left\{ \int_\Omega u_\varepsilon^3 \right\}^{\frac{m+1}{6}} \\
& \le \quad c_2 \varepsilon \int_\Omega u_\varepsilon^{m-1} u_{\varepsilon xx}^2 + c_2 \int_\Omega u_\varepsilon^3 + c_2 |\Omega| \qquad (3.16)
\end{aligned}
$$

for all $\varepsilon \in (0,1)$, because $\frac{3}{4} + \frac{m+1}{12} = \frac{m+10}{12} \le 1$ and $\frac{1}{2} + \frac{m+1}{6} = \frac{m+4}{6} \le 1$. Moreover, several further applications of Young's inequality imply that

$$
\frac{\varepsilon^\beta}{1-\alpha} \int_\Omega u_\varepsilon^{1-\alpha} \le \frac{1}{1-\alpha} \int_\Omega u_\varepsilon^3 + \frac{|\Omega|}{1-\alpha} \qquad (3.17)
$$

and

$$
D_1 \int_\Omega |u_{\varepsilon x}| \le D_1 \int_\Omega |u_{\varepsilon x}|^{\frac{3}{2}} + D_1 |\Omega| \qquad (3.18)
$$

as well as

$$
\chi_1 \int_\Omega |u_\varepsilon v_{\varepsilon x}| \le \chi_1 \int_\Omega u_\varepsilon^3 + \chi_1 \int_\Omega |v_{\varepsilon x}|^{\frac{3}{2}}. \qquad (3.19)
$$

As $W_0^{3,2}(\Omega) \hookrightarrow W^{2,\infty}(\Omega)$, inserting (3.15)-(3.19) into (3.14) thus entails that with some $c_3 > 0$, for all $\varepsilon \in (0,1)$ and any $t > 0$ we have

$$
\|u_{\varepsilon t}(\cdot,t)\|_{(W_0^{3,2}(\Omega))^\star} \le c_3 \cdot \left\{ \varepsilon \int_\Omega u_\varepsilon^{m-1} u_{\varepsilon xx}^2 + \int_\Omega |u_{\varepsilon x}|^{\frac{3}{2}} + \int_\Omega |v_{\varepsilon x}|^{\frac{3}{2}} + \int_\Omega u_\varepsilon^3 + 1 \right\},
$$

so that recalling Corollary 3.2 and Corollary 3.3 we infer that indeed for each $T > 0$, $(u_{\varepsilon t})_{\varepsilon \in (0,\varepsilon_\star)}$ has the boundedness property claimed in (3.13). Along with quite a similar reasoning for $v_{\varepsilon t}$, this proves the lemma. $\square$

4 Global existence. Proof of Theorem 1.1

We can now proceed to appropriately pass to the limit in (1.4) so as to construct a pair of functions solving (1.3) in the following natural weak sense.

Definition 4.1 *Let $D_1 > 0, D_2 > 0$, $\chi_1 > 0$ and $\chi_2 > 0$, and suppose that $u_0 \in L^1(\Omega)$ and $v_0 \in L^1(\Omega)$ are nonnegative. Then if u and v are nonnegative functions defined a.e. in $\Omega \times (0,\infty)$ which are such that*

$$u, v, u_x, v_x, uv_x \text{ and } vu_x \quad \text{belong to } L^1_{loc}(\overline{\Omega} \times [0,\infty)), \tag{4.1}$$

then (u,v) will be called a global weak solution *of (1.3) if for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$, the identities*

$$-\int_0^\infty \int_\Omega u\varphi_t - \int_\Omega u_0\varphi(\cdot,0) = -D_1 \int_0^\infty \int_\Omega u_x\varphi_x + \chi_1 \int_0^\infty \int_\Omega uv_x\varphi_x \tag{4.2}$$

and

$$-\int_0^\infty \int_\Omega v\varphi_t - \int_\Omega v_0\varphi(\cdot,0) = -D_2 \int_0^\infty \int_\Omega v_x\varphi_x - \chi_2 \int_0^\infty \int_\Omega vu_x\varphi_x \tag{4.3}$$

hold.

Indeed, the ε-independent estimates collected above imply compactness features sufficient to ensure the following.

Lemma 4.1 *Suppose that $n > \frac{7}{2}$, $m \in (\frac{1}{2}, 2]$, $\alpha \in (0, \frac{1}{2})$ and $\beta > 0$ are such that $\alpha \geq 4 - n$. Then there exist $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$, a null set $N \subset (0,\infty)$ and nonnegative functions u and v defined a.e. in $\Omega \times (0,\infty)$ such that*

$$\{u,v\} \subset L^3_{loc}(\overline{\Omega} \times [0,\infty)) \cap L^{\frac{3}{2}}_{loc}([0,\infty); W^{1,\frac{3}{2}}(\Omega)) \cap L^\infty((0,\infty); L\log L(\Omega)), \tag{4.4}$$

that $\varepsilon_j \searrow 0$ as $j \to \infty$ and

$$u_\varepsilon \to u \quad \text{and} \quad v_\varepsilon \to v \qquad \text{a.e. in } \Omega \times (0,\infty), \tag{4.5}$$

$$u_\varepsilon(\cdot,t) \to u(\cdot,t) \quad \text{and} \quad v_\varepsilon(\cdot,t) \to v(\cdot,t) \qquad \text{a.e. in } \Omega \text{ for all } t \in (0,\infty) \setminus N, \tag{4.6}$$

$$u_\varepsilon \to u \quad \text{and} \quad v_\varepsilon \to v \qquad \text{in } L^3_{loc}(\overline{\Omega} \times [0,\infty)) \qquad \text{and} \tag{4.7}$$

$$u_{\varepsilon x} \rightharpoonup u_x \quad \text{and} \quad v_{\varepsilon x} \rightharpoonup v_x \qquad \text{in } L^{\frac{3}{2}}_{loc}(\overline{\Omega} \times [0,\infty)) \tag{4.8}$$

as $\varepsilon = \varepsilon_j \searrow 0$, and such that (u,v) is a global weak solution of (1.3) in the sense of Definition 4.1. Furthermore, both u and v belong to $C^0_w([0,\infty); L^1(\Omega))$.

Proof. If $\varepsilon_\star \in (0,1)$ is as in Corollary 3.2, then according to Corollary 3.3,

$$(u_\varepsilon)_{\varepsilon\in(0,\varepsilon_\star)} \text{ and } (v_\varepsilon)_{\varepsilon\in(0,\varepsilon_\star)} \quad \text{are bounded in } L^{\frac{3}{2}}_{loc}([0,\infty); W^{1,\frac{3}{2}}(\Omega)), \tag{4.9}$$

whereas Lemma 3.4 says that

$$(u_{\varepsilon t})_{\varepsilon\in(0,\varepsilon_\star)} \text{ and } (v_{\varepsilon t})_{\varepsilon\in(0,\varepsilon_\star)} \quad \text{are bounded in } L^1_{loc}([0,\infty); (W_0^{3,2}(\Omega))^\star).$$

Therefore, an Aubin-Lions lemma ([44]) applies so as to ensure that $(u_\varepsilon)_{\varepsilon\in(0,\varepsilon_\star)}$ and $(v_\varepsilon)_{\varepsilon\in(0,\varepsilon_\star)}$ are relatively compact in $L^1_{loc}(\overline{\Omega}\times[0,\infty))$, whence we can find $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,\varepsilon_\star)$ such that $\varepsilon_j\searrow 0$ as $j\to\infty$ and that both (4.5) and (4.6) are valid with some null set $N\subset(0,\infty)$ and nonnegative functions u and v belonging to $L^1_{loc}(\overline{\Omega}\times[0,\infty))$. In view of (4.9), it is clear that on passing to a subsequence if necessary we may also assume that (4.8) holds, and since again from Corollary 3.3 we know that furthermore

$$\Big(u_\varepsilon^3\ln(u_\varepsilon+e)\Big)_{\varepsilon\in(0,\varepsilon_\star)} \text{ and } \Big(v_\varepsilon^3\ln(v_\varepsilon+e)\Big)_{\varepsilon\in(0,\varepsilon_\star)} \quad \text{are bounded in } L^1_{loc}(\overline{\Omega}\times[0,\infty)) \tag{4.10}$$

and that hence

$$(u_\varepsilon^3)_{\varepsilon\in(0,\varepsilon_\star)} \text{ and } (v_\varepsilon^3)_{\varepsilon\in(0,\varepsilon_\star)} \quad \text{are equi-integrable over } \Omega\times(0,T) \text{ for all } T>0,$$

it follows from (4.5) and the Vitali convergence theorem that upon a final extraction we can also achieve (4.7). It is therefore evident that both u and v belong to $L^3_{loc}(\overline{\Omega}\times[0,\infty))\cap L^{\frac{3}{2}}_{loc}([0,\infty);W^{1,\frac{3}{2}}(\Omega))$, and recalling that $(u_\varepsilon)_{\varepsilon\in(0,\varepsilon_\star)}$ and $(v_\varepsilon)_{\varepsilon\in(0,\varepsilon_\star)}$ are bounded in $L^\infty((0,\infty);L\log L(\Omega))$ by (3.5), an application of Fatou's lemma in conjunction with (4.6) shows that u and v also lie in $L^\infty((0,\infty);L\log L(\Omega))$. This establishes (4.4), which in turn immediately implies the regularity requirements in (4.1), because in view of the Hölder inequality the inclusions $\{u,v\}\subset L^3_{loc}(\overline{\Omega}\times[0,\infty))$ and $\{u_x,v_x\}\subset L^{\frac{3}{2}}_{loc}(\overline{\Omega}\times[0,\infty))$ ensure that uv_x and vu_x are locally integrable in $\overline{\Omega}\times[0,\infty)$.

Now in order to verify (4.2), we fix $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$ and let $(\varphi_k)_{k\in\mathbb{N}}\subset C_0^\infty(\overline{\Omega}\times[0,\infty))$ be such that

$$\varphi_k\to\varphi, \quad \varphi_{kt}\overset{\star}{\rightharpoonup}\varphi_t \quad \text{and} \quad \varphi_{kx}\rightharpoonup\varphi_x \quad \text{in } L^\infty(\Omega\times(0,\infty)) \qquad \text{as } k\to\infty, \tag{4.11}$$

and that in addition

$$\varphi_{kx}=0 \quad \text{on } \partial\Omega\times(0,\infty) \qquad \text{for all } k\in\mathbb{N}, \tag{4.12}$$

which can easily be seen to be possible by means of an essentially elementary construction.

Then for each $k\in\mathbb{N}$, thanks to (4.12) we may integrate by parts in (1.4) without encountering nonzero lateral boundary terms to find that for all $\varepsilon\in(0,1)$,

$$\begin{aligned} -\int_0^\infty\int_\Omega u_\varepsilon\varphi_{kt}-\int_\Omega u_{0\varepsilon}\varphi_k(\cdot,0) &= \varepsilon\int_0^\infty\int_\Omega F_\varepsilon(u_\varepsilon)u_{\varepsilon xxx}\varphi_{kx}+\frac{\varepsilon^\beta}{1-\alpha}\int_\Omega u_\varepsilon^{1-\alpha}\varphi_{kxx} \\ &\quad -D_1\int_0^\infty\int_\Omega u_{\varepsilon x}\varphi_{kx}+\chi_1\int_0^\infty\int_\Omega\frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m}+\varepsilon}v_{\varepsilon x}\varphi_{kx}, \end{aligned} \tag{4.13}$$

where again we have abbreviated $F_\varepsilon(s)=\frac{s^n}{s^{n-m}+\varepsilon}$ for $s\ge 0$ and $\varepsilon\in(0,1)$.

Here for a second time relying on (4.12), we may once more apply Lemma 7.4 to see that if we let $T_k>0$ be large enough such that $\operatorname{supp}\varphi_{kx}\subset\overline{\Omega}\times[0,T_k]$, then due to the Hölder inequality,

$$\begin{aligned} &\left|\varepsilon\int_0^\infty\int_\Omega F_\varepsilon(u_\varepsilon)u_{\varepsilon xxx}\varphi_{kx}\right| \\ &\quad = \left|-\varepsilon\int_0^\infty\int_\Omega F_\varepsilon(u_\varepsilon)u_{\varepsilon xx}\varphi_{kxx}-\varepsilon\int_0^\infty\int_\Omega F'_\varepsilon(u_\varepsilon)u_{\varepsilon x}u_{\varepsilon xx}\varphi_{kx}\right| \end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{n|\Omega|^{\frac{2-m}{12}}}{\sqrt{m}}\|\varphi_{kx}\|_{L^\infty(\Omega\times(0,T_k))}\cdot\varepsilon\int_0^{T_k}\left\{\int_\Omega u_\varepsilon^{m-1}u_{\varepsilon xx}^2\right\}^{\frac{3}{4}}\cdot\left\{\int_\Omega u_\varepsilon^3\right\}^{\frac{m+1}{12}} \\
& +|\Omega|^{\frac{2-m}{6}}\|\varphi_{kxx}\|_{L^\infty(\Omega\times(0,T_k))}\cdot\varepsilon\int_0^{T_k}\left\{\int_\Omega u_\varepsilon^{m-1}u_{\varepsilon xx}^2\right\}^{\frac{1}{2}}\cdot\left\{\int_\Omega u_\varepsilon^3\right\}^{\frac{m+1}{6}} \\
\leq\ & \frac{n(|\Omega|T_k)^{\frac{2-m}{12}}}{\sqrt{m}}\|\varphi_{kx}\|_{L^\infty(\Omega\times(0,T_k))}\cdot\varepsilon\cdot\left\{\int_0^{T_k}\int_\Omega u_\varepsilon^{m-1}u_{\varepsilon xx}^2\right\}^{\frac{3}{4}}\cdot\left\{\int_0^{T_k}\int_\Omega u_\varepsilon^3\right\}^{\frac{m+1}{12}} \\
& +(|\Omega|T_k)^{\frac{2-m}{6}}\|\varphi_{kxx}\|_{L^\infty(\Omega\times(0,T_k))}\cdot\varepsilon\cdot\left\{\int_0^{T_k}\int_\Omega u_\varepsilon^{m-1}u_{\varepsilon xx}^2\right\}^{\frac{1}{2}}\cdot\left\{\int_0^{T_k}\int_\Omega u_\varepsilon^3\right\}^{\frac{m+1}{6}}
\end{aligned}
$$

for all $\varepsilon\in(0,1)$. Recalling that Corollary 3.2 provides $c_1(k)>0$ fulfilling

$$
\int_0^{T_k}\int_\Omega u_\varepsilon^{m-1}u_{\varepsilon xx}^2\leq\frac{c_1(k)}{\varepsilon}\qquad\text{for all }\varepsilon\in(0,\varepsilon_\star),
$$

again making use of (4.10) we thereby infer that for each fixed $k\in\mathbb{N}$,

$$
\varepsilon\int_0^\infty\int_\Omega F_\varepsilon(u_\varepsilon)u_{\varepsilon xxx}\varphi_{kx}\to 0\qquad\text{as }\varepsilon\searrow 0. \tag{4.14}
$$

Now the limit procedure in the remaining expressions in (4.13) is rather straightforward: From (4.5) and (4.7) we particularly obtain that $u_\varepsilon\to u$, $u_\varepsilon^{1-\alpha}\to u^{1-\alpha}$ and $u_{\varepsilon x}\rightharpoonup u_x$ in $L^1(\Omega\times(0,T_k))$, implying that

$$
-\int_0^\infty\int_\Omega u_\varepsilon\varphi_{kt}\to-\int_0^\infty\int_\Omega u\varphi_{kt},\quad\text{and}\quad\frac{\varepsilon^\beta}{1-\alpha}\int_0^\infty\int_\Omega u_\varepsilon^{1-\alpha}\varphi_{kxx}\to 0 \tag{4.15}
$$

and that

$$
-D_1\int_0^\infty\int_\Omega u_{\varepsilon x}\varphi_{kx}\to-D_1\int_0^\infty\int_\Omega u_x\varphi_{kx} \tag{4.16}
$$

as $\varepsilon=\varepsilon_j\searrow 0$. Moreover, using that evidently $|\frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m}+\varepsilon}|\leq u_\varepsilon$ in $\Omega\times(0,\infty)$ and that hence by (4.10)

$$
\left(\left(\frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m}+\varepsilon}\right)^3\right)_{\varepsilon\in(0,\varepsilon_\star)}\qquad\text{is equi-integrable in }\Omega\times(0,T_k),
$$

again invoking the Vitali convergence theorem and (4.5) we see that $\frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m}+\varepsilon}\to u$ in $L^3(\Omega\times(0,T_k))$ and that thus, due to (4.8),

$$
\chi_1\int_0^\infty\int_\Omega\frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m}+\varepsilon}v_{\varepsilon x}\varphi_{kx}\to\chi_1\int_0^\infty\int_\Omega uv_x\varphi_{kx} \tag{4.17}
$$

as $\varepsilon=\varepsilon_j\searrow 0$. Since clearly $\int_\Omega u_{0\varepsilon}\varphi_k(\cdot,0)\to\int_\Omega u_0\varphi_k(\cdot,0)$ and $\varepsilon\searrow 0$ by (1.5), collecting (4.14)-(4.17) we conclude from (4.13) that

$$
-\int_0^\infty\int_\Omega u\varphi_{kt}-\int_\Omega u_0\varphi_k(\cdot,0)=-D_1\int_0^\infty\int_\Omega u_x\varphi_{kx}+\chi_1\int_0^\infty\int_\Omega uv_x\varphi_{kxx}\qquad\text{for all }k\in\mathbb{N}.
$$

In light of (4.11), we may now take $k \to \infty$ here to readily derive (4.2) from this, while (4.3) can be verified by exactly the same arguments.

To finally make sure that possibly after modification of N, and of u and v outside a null set of times, it can be achieved that

$$u \text{ and } v \text{ belong to } C^0_w([0,T]; L^1(\Omega)) \tag{4.18}$$

for all $T > 0$, given any such T we use the last boundedness statement contained in (4.4) to see that

$$(u(\cdot,t))_{t\in(0,T)\setminus N} \text{ and } (v(\cdot,t))_{t\in(0,T)\setminus N} \qquad \begin{array}{l} \text{are relatively compact in } L^1(\Omega) \\ \text{with respect to the weak topology} \end{array} \tag{4.19}$$

according to the Dunford-Pettis theorem. Since now knowing (4.2) to be valid, we may infer from the inclusion $\{u, uv_x\} \subset L^1(\Omega \times (0,T))$ that the distribution u_t lies in e.g. $L^1((0,T); (W^{2,2}(\Omega))^\star)$, and that thus after redefinition of u on a null set of times we have $u \in C^0([0,T]; (W^{2,2}(\Omega))^\star)$. Along with a similar argument for v, a standard approximation argument now readily shows that (4.18) therefore is a consequence of (4.19). $\Box$

Our main result on solvability in (1.3) has thereby actually been fully settled already.

Proof of Theorem 1.1. All statements immediately result as by-products of Lemma 4.1. $\Box$

5 Large time behavior. Proof of Theorem 1.2

5.1 Consistency with a conditional energy functional: The requirement $m \ge 2$

Next concerned with the large time behavior of the solutions constructed above, in order to follow the strategy outlined in the introduction we shall launch our analysis in this direction by examining how far (1.8) can be further developed so as to become rigorously justifiable for solutions to (1.4).

It will turn out that this will in fact be possible upon some suitable modification whenever the auxiliary parameter m in (1.4), beyond fulfilling $m \in (n-2, n-1)$, is chosen large enough by satisfying the condition $m \ge 2$ which, fortunately, can be fulfilled simultaneously with the requirements from the pervious sections. Indeed, we shall see that when tracking the time evolution of a modified variant of $\mathcal{F}$ from (1.8), besides making use of another favorable exact cancellation of corresponding taxis-induced contributions, for such m one can appropriately relate certain $O(\varepsilon)$-sized error terms to the considered entropy functional itself (cf. the argument near (5.26) and (5.27) below), and thereby finally derive a conditional entropy inequality in the following flavor.

Lemma 5.1 *Let $n > \frac{7}{2}$, $m \in (n-2, n-1)$, $\alpha \in (0, \frac{1}{2})$ and $\beta > 0$ be such that*

$$m \ge 2, \tag{5.1}$$

that

$$\alpha \ge 4 - n \quad \textit{and} \quad \alpha > \frac{n-m-1}{2}, \tag{5.2}$$

and that

$$\beta < \frac{\alpha}{n-m}. \tag{5.3}$$

Then there exist $K>0$ and $\varepsilon_{\star\star}\in(0,1)$ such that for any choice of $\varepsilon\in(0,\varepsilon_{\star\star})$, writing

$$\mathcal{F}_\varepsilon(t):=\chi_2\int_\Omega \psi_\varepsilon(u_\varepsilon(\cdot,t))u_{\varepsilon x}^2(\cdot,t)+\chi_1\int_\Omega \psi_\varepsilon(v_\varepsilon(\cdot,t))v_{\varepsilon x}^2(\cdot,t), \qquad t\ge 0, \tag{5.4}$$

with

$$\psi_\varepsilon(s):=\frac{1}{s}+\frac{\varepsilon}{s^{n-m+1}}, \qquad s>0, \tag{5.5}$$

we have

$$\mathcal{F}_\varepsilon'(t)+\Big\{\frac{1}{K}-K\mathcal{F}_\varepsilon^{\frac{m+2}{2}}(t)-K\mathcal{F}_\varepsilon(t)\Big\}\cdot\Big\{\int_\Omega\frac{u_{\varepsilon xx}^2}{u_\varepsilon}+\int_\Omega\frac{v_{\varepsilon xx}^2}{v_\varepsilon}\Big\}\le 0 \qquad \textit{for all } t>0. \tag{5.6}$$

Proof. By means of several integrations by parts, on the basis of (1.4) we compute

$$\begin{aligned}
\frac{d}{dt}\int_\Omega\psi_\varepsilon(u_\varepsilon)u_{\varepsilon x}^2 &= 2\int_\Omega\psi_\varepsilon(u_\varepsilon)u_{\varepsilon x}u_{\varepsilon xt}+\int_\Omega\psi_\varepsilon'(u_\varepsilon)u_{\varepsilon x}^2u_{\varepsilon t}\\
&= -2\int_\Omega\psi_\varepsilon(u_\varepsilon)u_{\varepsilon xx}u_{\varepsilon t}-\int_\Omega\psi_\varepsilon'(u_\varepsilon)u_{\varepsilon x}^2u_{\varepsilon t}\\
&= 2\varepsilon\int_\Omega\psi_\varepsilon(u_\varepsilon)u_{\varepsilon xx}\cdot\Big(\frac{u_\varepsilon^n}{u_\varepsilon^{n-m}+\varepsilon}u_{\varepsilon xxx}\Big)_x+\varepsilon\int_\Omega\psi_\varepsilon'(u_\varepsilon)u_{\varepsilon x}^2\cdot\Big(\frac{u_\varepsilon^n}{u_\varepsilon^{n-m}+\varepsilon}u_{\varepsilon xxx}\Big)_x\\
&\quad -2\varepsilon^\beta\int_\Omega\psi_\varepsilon(u_\varepsilon)u_{\varepsilon xx}\cdot(u_\varepsilon^{-\alpha}u_{\varepsilon x})_x-\varepsilon^\beta\int_\Omega\psi_\varepsilon'(u_\varepsilon)u_{\varepsilon x}^2\cdot(u_\varepsilon^{-\alpha}u_{\varepsilon x})_x\\
&\quad -2D_1\int_\Omega\psi_\varepsilon(u_\varepsilon)u_{\varepsilon xx}^2-D_1\int_\Omega\psi_\varepsilon'(u_\varepsilon)u_{\varepsilon x}^2u_{\varepsilon xx}\\
&\quad +2\chi_1\int_\Omega\psi_\varepsilon(u_\varepsilon)u_{\varepsilon xx}\cdot\Big(\frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m}+\varepsilon}v_{\varepsilon x}\Big)_x+\chi_1\int_\Omega\psi_\varepsilon'(u_\varepsilon)u_{\varepsilon x}^2\cdot\Big(\frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m}+\varepsilon}v_{\varepsilon x}\Big)_x
\end{aligned}\tag{5.7}$$

for all $t>0$. Here we first address the third and fourth last summands, in which we perform another integration by parts to see that since

$$\psi_\varepsilon'(s)=-\frac{1}{s^2}-(n-m+1)\varepsilon s^{-n+m-2} \quad\text{and}\quad \psi_\varepsilon''(s)=\frac{2}{s^3}+(n-m+1)(n-m+2)\varepsilon s^{-n+m-3} \quad \text{for all } s>0 \tag{5.8}$$

by (5.5), we have

$$\begin{aligned}
&-2D_1\int_\Omega\psi_\varepsilon(u_\varepsilon)u_{\varepsilon xx}^2-D_1\int_\Omega\psi_\varepsilon'(u_\varepsilon)u_{\varepsilon x}^2u_{\varepsilon xx}\\
&\quad= -2D_1\int_\Omega\psi_\varepsilon(u_\varepsilon)u_{\varepsilon xx}^2+\frac{D_1}{3}\int_\Omega\psi_\varepsilon''(u_\varepsilon)u_{\varepsilon x}^4\\
&\quad= -2D_1\int_\Omega\frac{u_{\varepsilon xx}^2}{u_\varepsilon}-2D_1\varepsilon\int_\Omega u_\varepsilon^{-n+m-1}u_{\varepsilon xx}^2\\
&\qquad +\frac{2D_1}{3}\int_\Omega\frac{u_{\varepsilon x}^4}{u_\varepsilon^3}+\frac{(n-m+1)(n-m+2)D_1\varepsilon}{3}\int_\Omega u_\varepsilon^{-n+m-3}u_{\varepsilon x}^4\\
&\quad\le -\frac{D_1}{2}\int_\Omega\frac{u_{\varepsilon xx}^2}{u_\varepsilon}+\frac{(n-m+1)(n-m+2)D_1\varepsilon}{3}\int_\Omega u_\varepsilon^{-n+m-3}u_{\varepsilon x}^4 \qquad \text{for all } t>0,
\end{aligned}\tag{5.9}$$

because

$$\int_\Omega \frac{u_{\varepsilon x}^4}{u_\varepsilon^3} \le \frac{9}{4} \int_\Omega \frac{u_{\varepsilon xx}^2}{u_\varepsilon} \qquad \text{for all } t>0 \tag{5.10}$$

as a consequence of the elementary functional inequality stated in Lemma 7.1 below.

In order to suitably compensate the ill-signed rightmost summand in (5.9), we now consider the expressions in (5.7) which originate from the artificial second-order fast diffusion introduced in (1.4): In fact, by definition of ψ_ε and two further integrations by parts we can rewrite

$$\begin{aligned}
-2\int_\Omega & \psi_\varepsilon(u_\varepsilon)u_{\varepsilon xx}\cdot(u_\varepsilon^{-\alpha}u_{\varepsilon x})_x - \int_\Omega \psi_\varepsilon'(u_\varepsilon)u_{\varepsilon x}^2\cdot(u_\varepsilon^{-\alpha}u_{\varepsilon x})_x \\
&= -2\int_\Omega u_\varepsilon^{-\alpha-1}u_{\varepsilon xx}^2 - 2\varepsilon\int_\Omega u_\varepsilon^{-n+m-\alpha-1}u_{\varepsilon xx}^2 \\
&\quad +(2\alpha+1)\int_\Omega u_\varepsilon^{-\alpha-2}u_{\varepsilon x}^2 u_{\varepsilon xx} + (n-m+2\alpha+1)\varepsilon\int_\Omega u_\varepsilon^{-n+m-\alpha-2}u_{\varepsilon x}^2 u_{\varepsilon xx} \\
&\quad -\alpha\int_\Omega u_\varepsilon^{-\alpha-3}u_{\varepsilon x}^4 - (n-m+1)\alpha\varepsilon\int_\Omega u_\varepsilon^{-n+m-\alpha-3}u_{\varepsilon x}^4 \\
&= -2\int_\Omega u_\varepsilon^{-\alpha-1}u_{\varepsilon xx}^2 - 2\varepsilon\int_\Omega u_\varepsilon^{-n+m-\alpha-1}u_{\varepsilon xx}^2 \\
&\quad +\Big\{\frac{(2\alpha+1)(\alpha+2)}{3}-\alpha\Big\}\cdot\int_\Omega u_\varepsilon^{-\alpha-3}u_{\varepsilon x}^4 \\
&\quad +\Big\{\frac{(n-m+2\alpha+1)(n-m+\alpha+2)}{3}-(n-m+1)\alpha\Big\}\cdot\varepsilon\int_\Omega u_\varepsilon^{-n+m-\alpha-3}u_{\varepsilon x}^4
\end{aligned} \tag{5.11}$$

for $t>0$, where two more applications of Lemma 7.1 show that

$$\begin{aligned}
\Big\{\frac{(2\alpha+1)(\alpha+2)}{3}-\alpha\Big\}\cdot\int_\Omega u_\varepsilon^{-\alpha-3}u_{\varepsilon x}^4 &= \frac{2(\alpha^2+\alpha+1)}{3}\int_\Omega u_\varepsilon^{-\alpha-3}u_{\varepsilon x}^4 \\
&\le \frac{2(\alpha^2+\alpha+1)}{3}\cdot\frac{9}{(\alpha+2)^2}\int_\Omega u_\varepsilon^{-\alpha-1}u_{\varepsilon xx}^2
\end{aligned} \tag{5.12}$$

and

$$\begin{aligned}
\Big\{&\frac{(n-m+2\alpha+1)(n-m+\alpha+2)}{3}-(n-m+1)\alpha\Big\}\cdot\varepsilon\int_\Omega u_\varepsilon^{-n+m-\alpha-3}u_{\varepsilon x}^4 \\
&= \frac{2\alpha^2+2\alpha+(n-m+1)(n-m+2)}{3}\cdot\varepsilon\int_\Omega u_\varepsilon^{-n+m-\alpha-3}u_{\varepsilon x}^4 \\
&\le \frac{2\alpha^2+2\alpha+(n-m+1)(n-m+2)}{3}\cdot\frac{9}{(n-m+\alpha+2)^2}\cdot\varepsilon\int_\Omega u_\varepsilon^{-n+m-\alpha-1}u_{\varepsilon xx}^2
\end{aligned} \tag{5.13}$$

for all $t>0$. Now since

$$\frac{2(\alpha^2+\alpha+1)}{3}\cdot\frac{9}{(\alpha+2)^2}-2 = 2\cdot\frac{(2\alpha+1)(\alpha-1)}{\alpha^2+4\alpha+4}\le 0$$

due to the fact that $\alpha \in [-\frac{1}{2}, 1]$, and since

$$\begin{aligned}
\frac{2\alpha^2 + 2\alpha + (n-m+1)(n-m+2)}{3} \cdot \frac{9}{(n-m+\alpha+2)^2} - 2 \\
&= \frac{4}{(n-m+\alpha+2)^2} \cdot \Big(\alpha - \frac{n-m+2}{2}\Big)\Big(\alpha - \frac{n-m-1}{2}\Big) \\
&< 0
\end{aligned}$$

thanks to (5.2) and the evident inequality $\alpha < 1 < \frac{n-m+2}{2}$, from (5.11)-(5.13) we all in all infer the existence of $c_1 > 0$ such that for all $t > 0$ and any $\varepsilon \in (0,1)$,

$$-2\varepsilon^\beta \int_\Omega \psi_\varepsilon(u_\varepsilon) u_{\varepsilon xxx} \cdot (u_\varepsilon^{-\alpha} u_{\varepsilon x})_x - \varepsilon^\beta \int_\Omega \psi'_\varepsilon(u_\varepsilon) u_{\varepsilon x}^2 \cdot (u_\varepsilon^{-\alpha} u_{\varepsilon x})_x \le -c_1 \varepsilon^{\beta+1} \int_\Omega u_\varepsilon^{-n+m-\alpha-3} u_{\varepsilon x}^4. \tag{5.14}$$

In order to take appropriate advantage of this, e.g. by means of Young's inequality we fix $c_2 > 0$ such that

$$\frac{(n-m+1)(n-m+2)D_1}{3} \cdot s \le \frac{D_1}{9} + c_2 s^{\frac{n-m+\alpha}{n-m}} \qquad \text{for all } s > 0,$$

and let $\varepsilon_{\star\star} \in (0,1)$ be small enough fulfilling

$$c_2 \varepsilon^{\frac{n-m+\alpha}{n-m}} \le c_1 \varepsilon^{\beta+1} \qquad \text{for all } \varepsilon \in (0, \varepsilon_{\star\star}),$$

which is possible in view of the fact that $\beta + 1 < \frac{\alpha}{n-m} + 1 = \frac{n-m+\alpha}{n-m}$ by (5.3). Then resorting to correspondingly small values of ε, we may again rely on (5.10) in estimating

$$\begin{aligned}
\frac{(n-m+1)(n-m+2)D_1\varepsilon}{3} \int_\Omega u_\varepsilon^{-n+m-3} u_{\varepsilon x}^4 &= \int_\Omega u_\varepsilon^{-3} \cdot \Big\{ \frac{(n-m+1)(n-m+2)D_1}{3} \cdot \varepsilon u_\varepsilon^{-n+m} \Big\} \cdot u_{\varepsilon x}^4 \\
&\le \int_\Omega u_\varepsilon^{-3} \cdot \Big\{ \frac{D_1}{9} + c_2 (\varepsilon u_\varepsilon^{-n+m})^{\frac{n-m+\alpha}{n-m}} \Big\} \cdot u_{\varepsilon x}^4 \\
&= \frac{D_1}{9} \int_\Omega \frac{u_{\varepsilon x}^4}{u_\varepsilon^3} + c_2 \varepsilon^{\frac{n-m+\alpha}{n-m}} \int_\Omega u_\varepsilon^{-n+m-\alpha-3} u_{\varepsilon x}^4 \\
&\le \frac{D_1}{4} \int_\Omega \frac{u_{\varepsilon xx}^2}{u_\varepsilon} + c_1 \varepsilon^{\beta+1} \int_\Omega u_\varepsilon^{-n+m-\alpha-3} u_{\varepsilon x}^4
\end{aligned} \tag{5.15}$$

for all $t > 0$ and any $\varepsilon \in (0, \varepsilon_{\star\star})$.

Next, with regard to the cross-diffusive contributions to (5.7) we note that our particular choice of ψ_ε, especially thanks to the ε-dependent correction therein, warrants that the crucial second last summand in (5.7) can be rewritten according to

$$\begin{aligned}
2\chi_1 \int_\Omega \psi_\varepsilon(u_\varepsilon) u_{\varepsilon xxx} \cdot \Big(\frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m} + \varepsilon} v_{\varepsilon x} \Big)_x &= 2\chi_1 \int_\Omega \psi_\varepsilon(u_\varepsilon) u_{\varepsilon xxx} \cdot \frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m} + \varepsilon} v_{\varepsilon xx} \\
&\quad + 2\chi_1 \int_\Omega \psi_\varepsilon(u_\varepsilon) u_{\varepsilon xxx} \cdot \frac{d}{ds}\Big(\frac{s^{n-m+1}}{s^{n-m} + \varepsilon} \Big)\Big|_{s=u_\varepsilon} \cdot u_{\varepsilon x} v_{\varepsilon x}
\end{aligned}$$

$$
\begin{aligned}
&= 2\chi_1 \int_\Omega u_{\varepsilon xx} v_{\varepsilon xx} \\
&\quad +2\chi_1 \int_\Omega \psi_\varepsilon(u_\varepsilon) u_{\varepsilon xx} \cdot \frac{d}{ds}\Big(\frac{s^{n-m+1}}{s^{n-m}+\varepsilon}\Big)\bigg|_{s=u_\varepsilon} \cdot u_{\varepsilon x} v_{\varepsilon x} \qquad (5.16)
\end{aligned}
$$

for $t>0$. Since here the fact that $m \le n$ warrants that

$$
0 \le \frac{d}{ds}\Big(\frac{s^{n-m+1}}{s^{n-m}+\varepsilon}\Big) = \frac{s^{2n-2m} + (n-m+1)\varepsilon s^{n-m}}{(s^{n-m}+\varepsilon)^2} \le \frac{(n-m+1)s^{n-m}}{s^{n-m}+\varepsilon} \qquad \text{for all } s>0,
$$

once more recalling (5.5) we may use Young's inequality to estimate

$$
\begin{aligned}
2\chi_1 \int_\Omega \psi_\varepsilon(u_\varepsilon) u_{\varepsilon xx} \cdot \frac{d}{ds}\Big(\frac{s^{n-m+1}}{s^{n-m}+\varepsilon}\Big)\bigg|_{s=u_\varepsilon} \cdot u_{\varepsilon x} v_{\varepsilon x} &\le 2(n-m+1)\chi_1 \int_\Omega \frac{u_\varepsilon^{n-m-1} + \varepsilon u_\varepsilon^{-1}}{u_\varepsilon^{n-m}+\varepsilon} |u_{\varepsilon x} v_{\varepsilon x} u_{\varepsilon xx}| \\
&= 2(n-m+1)\chi_1 \int_\Omega \frac{|u_{\varepsilon x} v_{\varepsilon x} u_{\varepsilon xx}|}{u_\varepsilon} \\
&\le \frac{D_1}{8} \int_\Omega \frac{u_{\varepsilon xx}^2}{u_\varepsilon} + \frac{8(n-m+1)^2\chi_1^2}{D_1} \int_\Omega \frac{u_{\varepsilon x}^2 v_{\varepsilon x}^2}{u_\varepsilon} \qquad (5.17)
\end{aligned}
$$

for all $t>0$. As Lemma 7.3 along with (2.2) asserts that

$$
\begin{aligned}
\int_\Omega \frac{u_{\varepsilon x}^2 v_{\varepsilon x}^2}{u_\varepsilon} &\le \|v_{\varepsilon x}\|_{L^\infty(\Omega)}^2 \int_\Omega \frac{u_{\varepsilon x}^2}{u_\varepsilon} \\
&\le \Big\{\int_\Omega v_\varepsilon\Big\} \cdot \Big\{\int_\Omega \frac{v_{\varepsilon xx}^2}{v_\varepsilon}\Big\} \cdot \int_\Omega \frac{u_{\varepsilon x}^2}{u_\varepsilon} \\
&\le \Big\{\int_\Omega v_0\Big\} \cdot \frac{1}{\chi_2}\mathcal{F}_\varepsilon(t) \cdot \int_\Omega \frac{v_{\varepsilon xx}^2}{v_\varepsilon} \qquad \text{for all } t>0, \qquad (5.18)
\end{aligned}
$$

from (5.16) and (5.17) it thus follows that whenever $\varepsilon \in (0,1)$,

$$
\begin{aligned}
&2\chi_1 \int_\Omega \psi_\varepsilon(u_\varepsilon) u_{\varepsilon xx} \cdot \Big(\frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m}+\varepsilon} v_{\varepsilon x}\Big)_x \\
&\quad \le 2\chi_1 \int_\Omega u_{\varepsilon xx} v_{\varepsilon xx} + \frac{D_1}{8} \int_\Omega \frac{u_{\varepsilon xx}^2}{u_\varepsilon} + c_3 \mathcal{F}_\varepsilon(t) \cdot \int_\Omega \frac{v_{\varepsilon xx}^2}{v_\varepsilon} \qquad \text{for all } t>0 \qquad (5.19)
\end{aligned}
$$

if we let $c_3 := \frac{8(n-m+1)^2\chi_1^2}{D_1\chi_2} \cdot \int_\Omega v_0$.

The last integral in (5.7) can similarly be estimated after another integration by parts, observing that in

$$
\begin{aligned}
\chi_1 \int_\Omega \psi_\varepsilon'(u_\varepsilon) u_{\varepsilon x}^2 \cdot \Big(\frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m}+\varepsilon} v_{\varepsilon x}\Big)_x &= -2\chi_1 \int_\Omega \psi_\varepsilon'(u_\varepsilon) \cdot \frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m}+\varepsilon} \cdot u_{\varepsilon x} v_{\varepsilon x} u_{\varepsilon xx} \\
&\quad -\chi_1 \int_\Omega \psi_\varepsilon''(u_\varepsilon) \cdot \frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m}+\varepsilon} \cdot u_{\varepsilon x}^3 v_{\varepsilon x}, \qquad t>0, \qquad (5.20)
\end{aligned}
$$

by (5.8) and again since $m \le n$ we have

$$0 \le -\psi_\varepsilon'(s) \le (n-m+1)s^{-n+m-2} \cdot (s^{n-m}+\varepsilon) \qquad \text{for all } s>0 \tag{5.21}$$

and

$$0 \le \psi_\varepsilon''(s) \le (n-m+1)(n-m+2)s^{-n+m-3} \cdot (s^{n-m}+\varepsilon) \qquad \text{for all } s>0. \tag{5.22}$$

Therefore, namely, by the same argument as in (5.17) and (5.18),

$$\begin{aligned} -2\chi_1 \int_\Omega \psi_\varepsilon'(u_\varepsilon) \cdot \frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m}+\varepsilon} \cdot u_{\varepsilon x} v_{\varepsilon x} u_{\varepsilon xx} &\le 2(n-m+1)\chi_1 \int_\Omega \frac{|u_{\varepsilon x} v_{\varepsilon x} u_{\varepsilon xx}|}{u_\varepsilon} \\ &\le \frac{D_1}{16} \int_\Omega \frac{u_{\varepsilon xx}^2}{u_\varepsilon} + c_4 \mathcal{F}_\varepsilon(t) \cdot \int_\Omega \frac{v_{\varepsilon xx}^2}{v_\varepsilon} \qquad \text{for all } t>0 \end{aligned} \tag{5.23}$$

with $c_4 := \frac{16(n-m+1)^2\chi_1^2}{D_1\chi_2} \cdot \int_\Omega v_0$, whereas once more due to Young's inequality, (5.10) and (5.18),

$$\begin{aligned} -\chi_1 \int_\Omega \psi_\varepsilon''(u_\varepsilon) \cdot \frac{u_\varepsilon^{n-m+1}}{u_\varepsilon^{n-m}+\varepsilon} \cdot u_{\varepsilon x}^3 v_{\varepsilon x} &\le (n-m+1)(n-m+2)\chi_1 \int_\Omega \frac{|u_{\varepsilon x}^3 v_{\varepsilon x}|}{u_\varepsilon^2} \\ &\le \frac{D_1}{72} \int_\Omega \frac{u_{\varepsilon x}^4}{u_\varepsilon^3} + \frac{18(n-m+1)^2(n-m+2)^2\chi_1^2}{D_1} \int_\Omega \frac{u_{\varepsilon x}^2 v_{\varepsilon x}^2}{u_\varepsilon} \\ &\le \frac{D_1}{32} \int_\Omega \frac{u_{\varepsilon xx}^2}{u_\varepsilon} + c_5 \mathcal{F}_\varepsilon(t) \cdot \int_\Omega \frac{v_{\varepsilon xx}^2}{v_\varepsilon} \qquad \text{for all } t>0 \end{aligned} \tag{5.24}$$

if we abbreviate $c_5 := \frac{18(n-m+1)^2(n-m+2)^2\chi_1^2}{D_1\chi_2} \cdot \int_\Omega v_0$.

It finally remains to adequately cope with the first two summands on the right of (5.7), in which we firstly again integrate by parts and recall (5.5), (5.21) and (5.8) to see that

$$\begin{aligned} & 2\varepsilon \int_\Omega \psi_\varepsilon(u_\varepsilon) u_{\varepsilon xx} \cdot \Big(\frac{u_\varepsilon^n}{u_\varepsilon^{n-m}+\varepsilon} u_{\varepsilon xxx} \Big)_x + \varepsilon \int_\Omega \psi_\varepsilon'(u_\varepsilon) u_{\varepsilon x}^2 \cdot \Big(\frac{u_\varepsilon^n}{u_\varepsilon^{n-m}+\varepsilon} u_{\varepsilon xxx} \Big)_x \\ &\quad = -2\varepsilon \int_\Omega \psi_\varepsilon(u_\varepsilon) \cdot \frac{u_\varepsilon^n}{u_\varepsilon^{n-m}+\varepsilon} u_{\varepsilon xxx}^2 - 4\varepsilon \int_\Omega \psi_\varepsilon'(u_\varepsilon) \cdot \frac{u_\varepsilon^n}{u_\varepsilon^{n-m}+\varepsilon} u_{\varepsilon x} u_{\varepsilon xx} u_{\varepsilon xxx} \\ &\qquad - \varepsilon \int_\Omega \psi_\varepsilon''(u_\varepsilon) \cdot \frac{u_\varepsilon^n}{u_\varepsilon^{n-m}+\varepsilon} u_{\varepsilon x}^3 u_{\varepsilon xxx} \\ &\quad \le -2\varepsilon \int_\Omega u_\varepsilon^{m-1} u_{\varepsilon xxx}^2 + 4(n-m+1)\varepsilon \int_\Omega u_\varepsilon^{m-2} |u_{\varepsilon x} u_{\varepsilon xx} u_{\varepsilon xxx}| \\ &\qquad - \varepsilon \int_\Omega \psi_\varepsilon''(u_\varepsilon) \cdot \frac{u_\varepsilon^n}{u_\varepsilon^{n-m}+\varepsilon} u_{\varepsilon x}^3 u_{\varepsilon xxx} \end{aligned} \tag{5.25}$$

for $t>0$. Here Young's inequality together with Lemma 7.2 guarantees that for all $\varepsilon \in (0,1)$ and any $t>0$,

$$\begin{aligned} 4(n-m+1)\varepsilon \int_\Omega u_\varepsilon^{m-2} |u_{\varepsilon x} u_{\varepsilon xx} u_{\varepsilon xxx}| &\le \frac{\varepsilon}{2} \int_\Omega u_\varepsilon^{m-1} u_{\varepsilon xxx}^2 + 8(n-m+1)^2 \varepsilon \int_\Omega u_\varepsilon^{m-3} u_{\varepsilon x}^2 u_{\varepsilon xx}^2 \\ &\le \varepsilon \int_\Omega u_\varepsilon^{m-1} u_{\varepsilon xxx}^2 + c_6 \int_\Omega u_\varepsilon^{m-5} u_{\varepsilon x}^6 \end{aligned} \tag{5.26}$$

with $c_6 := 8(n-m+1)^2 \cdot \Big\{ \frac{|m-3|\cdot|m-4|}{15} + \frac{4(n-m+1)^2}{9} \Big\}$, while combining (5.22) with Young's inequality shows that for all $\varepsilon \in (0,1)$ and $t>0$,

$$\begin{aligned} -\varepsilon \int_\Omega \psi_\varepsilon''(u_\varepsilon) \cdot \frac{u_\varepsilon^n}{u_\varepsilon^{n-m}+\varepsilon} u_{\varepsilon x}^3 u_{\varepsilon xxx} &\le (n-m+1)(n-m+2)\varepsilon \int_\Omega u_\varepsilon^{m-3} |u_{\varepsilon x}^3 u_{\varepsilon xxx}| \\ &\le \varepsilon \int_\Omega u_\varepsilon^{m-1} u_{\varepsilon xxx}^2 + c_7 \int_\Omega u_\varepsilon^{m-5} u_{\varepsilon x}^6 \end{aligned} \tag{5.27}$$

if we set $c_7 := \frac{(n-m+1)^2(n-m+2)^2}{4}$. Now since we are assuming that $m \ge 2$, the rightmost summands in (5.26) and (5.27) can be related to $\mathcal{F}_\varepsilon$ through an interpolation argument: Indeed, thanks to this restriction on m we may invoke the Gagliardo-Nirenberg inequality to find $c_8 > 0$ such that for all $\varepsilon \in (0,1)$ and $t>0$,

$$\begin{aligned} &(c_6+c_7) \int_\Omega u_\varepsilon^{m-5} u_{\varepsilon x}^6 \\ &\quad \le (c_6+c_7) \|u_\varepsilon\|_{L^\infty(\Omega)}^{m-2} \int_\Omega \frac{u_{\varepsilon x}^6}{u_\varepsilon^3} \\ &\quad = 64(c_6+c_7) \|\sqrt{u_\varepsilon}\|_{L^\infty(\Omega)}^{2(m-2)} \|(\sqrt{u_\varepsilon})_x\|_{L^6(\Omega)}^6 \\ &\quad \le c_8 \cdot \Big\{ \|(\sqrt{u_\varepsilon})_x\|_{L^2(\Omega)}^{m-2} \|\sqrt{u_\varepsilon}\|_{L^2(\Omega)}^{m-2} + \|\sqrt{u_\varepsilon}\|_{L^2(\Omega)}^{2(m-2)} \Big\} \cdot \|(\sqrt{u_\varepsilon})_{xx}\|_{L^2(\Omega)}^2 \|(\sqrt{u_\varepsilon})_x\|_{L^2(\Omega)}^4 \\ &\quad = \frac{c_8}{16} \cdot \Bigg\{ \Big\{ \frac{1}{4} \int_\Omega \frac{u_{\varepsilon x}^2}{u_\varepsilon} \Big\}^{\frac{m-2}{2}} \cdot \Big\{ \int_\Omega u_0 \Big\}^{\frac{m-2}{2}} + \Big\{ \int_\Omega u_0 \Big\}^{m-2} \Bigg\} \cdot \Big\{ \int_\Omega \frac{u_{\varepsilon x}^2}{u_\varepsilon} \Big\}^2 \cdot \int_\Omega (\sqrt{u_\varepsilon})_{xx}^2 \end{aligned}$$

according to (2.2). As herein by Young's inequality and again (5.10),

$$\begin{aligned} \int_\Omega (\sqrt{u_\varepsilon})_{xx}^2 &= \int_\Omega \Big\{ \frac{u_{\varepsilon xx}}{2\sqrt{u_\varepsilon}} - \frac{u_{\varepsilon x}^2}{4\sqrt{u_\varepsilon}^3} \Big\}^2 \\ &\le \frac{1}{2} \int_\Omega \frac{u_{\varepsilon xx}^2}{u_\varepsilon} + \frac{1}{8} \int_\Omega \frac{u_{\varepsilon x}^4}{u_\varepsilon^3} \\ &\le \frac{25}{32} \int_\Omega \frac{u_{\varepsilon xx}^2}{u_\varepsilon} \qquad \text{for all } t>0, \end{aligned}$$

once more using that $\int_\Omega \frac{u_{\varepsilon x}^2}{u_\varepsilon} \le \frac{1}{\chi_2} \mathcal{F}_\varepsilon(t)$ and estimating $\mathcal{F}_\varepsilon^2(t) \le \mathcal{F}_\varepsilon^{\frac{m+2}{2}}(t) + \mathcal{F}_\varepsilon(t)$ for $t>0$ we thus obtain that with some $c_9 > 0$ we have

$$(c_6+c_7) \int_\Omega u_\varepsilon^{m-5} u_{\varepsilon x}^6 \le c_9 \cdot \Big\{ \mathcal{F}_\varepsilon^{\frac{m+2}{2}}(t) + \mathcal{F}_\varepsilon(t) \Big\} \cdot \int_\Omega \frac{u_{\varepsilon xx}^2}{u_\varepsilon} \qquad \text{for all } t>0 \text{ and } \varepsilon \in (0,1).$$

In conjunction with (5.9), (5.14)-(5.27) and (5.19)-(5.24), this shows that (5.7) implies the inequality

$$\begin{aligned} \frac{d}{dt} \int_\Omega \psi_\varepsilon(u_\varepsilon) u_{\varepsilon x}^2 &\le 2\chi_1 \int_\Omega u_{\varepsilon xx} v_{\varepsilon xx} - \frac{D_1}{32} \int_\Omega \frac{u_{\varepsilon xx}^2}{u_\varepsilon} \\ &\quad + (c_3+c_4+c_5)\mathcal{F}_\varepsilon(t) \cdot \int_\Omega \frac{v_{\varepsilon xx}^2}{v_\varepsilon} + c_9 \cdot \Big\{ \mathcal{F}_\varepsilon^{\frac{m+2}{2}}(t) + \mathcal{F}_\varepsilon(t) \Big\} \cdot \int_\Omega \frac{u_{\varepsilon xx}^2}{u_\varepsilon} \end{aligned} \tag{5.28}$$

for all $t > 0$ whenever $\varepsilon \in (0, \varepsilon_{\star\star})$. Since a similar reasoning reveals that upon diminishing $\varepsilon_{\star\star}$ if necessary we can also achieve that with some $c_{10} > 0$, for all $t > 0$ and any $\varepsilon \in (0, \varepsilon_{\star\star})$ we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \psi_\varepsilon(v_\varepsilon)v_{\varepsilon x}^2 \ &\le\ -2\chi_2\int_\Omega u_{\varepsilon xx}v_{\varepsilon xx} - \frac{D_2}{32}\int_\Omega \frac{v_{\varepsilon xx}^2}{v_\varepsilon} \\
&\quad + c_{10}\mathcal{F}_\varepsilon(t)\cdot\int_\Omega \frac{u_{\varepsilon xx}^2}{u_\varepsilon} + c_{10}\cdot\Big\{\mathcal{F}_\varepsilon^{\frac{m+2}{2}}(t) + \mathcal{F}_\varepsilon(t)\Big\}\cdot\int_\Omega \frac{v_{\varepsilon xx}^2}{v_\varepsilon},
\end{aligned} \tag{5.29}$$

on combining (5.28) with (5.29) we readily end up with (5.6). □

5.2 Proof of Theorem 1.2

In view of the fact that Corollary 3.2 warrants a certain eventual smallness property of $\mathcal{F}_\varepsilon$, possibly beyond some suitably large waiting time the inequality in (5.6) can be turned into a genuine monotonicity feature, in particular implying ultimate boundedness and stabilization even in L^∞ topologies:

Lemma 5.2 *Let $n \in (\frac{7}{2}, 4)$ and $m = 2$, and let $\alpha \in (0, \frac{1}{2})$ and $\beta > 0$ be such that $\alpha \ge 4 - n, \alpha > \frac{n-3}{2}$ and $\beta < \frac{\alpha}{n-2}$. Then there exists $\varepsilon_0 \in (0,1)$ such that for some $T_0 > 0$ we have*

$$\sup_{t>T_0}\ \sup_{\varepsilon\in(0,\varepsilon_0)} \Big\{\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\Big\} < \infty, \tag{5.30}$$

and such that moreover

$$\sup_{t>T}\ \sup_{\varepsilon\in(0,\varepsilon_0)} \Big\{\|u_\varepsilon(\cdot,t) - \overline{u}_0\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot,t) - \overline{v}_0\|_{L^\infty(\Omega)}\Big\} \to 0 \qquad \text{as } T \to \infty. \tag{5.31}$$

Proof. Since our parameter restrictions warrant applicability of Lemma 5.1, we may fix $K > 0$ and $\varepsilon_{\star\star} \in (0,1)$ as provided by the latter, and thereupon pick $\eta_0 > 0$ small enough fulfilling

$$K\cdot(\eta_0 + \eta_0^2) \le \frac{1}{2K}. \tag{5.32}$$

Next, since we have chosen m so as to satisfy $m \le 2$, recalling the definition (5.4) of $\mathcal{F}_\varepsilon$ we may invoke Corollary 3.2 to find $\varepsilon_\star \in (0,1)$ and $c_1 > 0$ such that whenever $\varepsilon \in (0, \varepsilon_\star)$,

$$\int_0^T \mathcal{F}_\varepsilon(t) \le c_1 \qquad \text{for all } T > 0.$$

In particular, this implies that if for $\eta \in (0, \eta_0]$ we let $T_\eta := \frac{2c_1}{\eta}$, then for arbitrary $\varepsilon \in (0, \varepsilon_\star)$ we can fix $t_0(\eta,\varepsilon) \in (0, T_\eta)$ such that

$$\mathcal{F}_\varepsilon(t_0(\eta,\varepsilon)) \le \frac{\eta}{2}, \tag{5.33}$$

and we claim that actually

$$\mathcal{F}_\varepsilon(t) < \eta \qquad \text{for all } t \ge t_0(\eta,\varepsilon) \tag{5.34}$$

whenever $\eta \in (0, \eta_0]$ and $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0 := \min\{\varepsilon_\star, \varepsilon_{\star\star}\}$.

Indeed, this follows from a straightforward comparison argument on the basis of Lemma 5.1: According to (5.33) and the continuity of $\mathcal{F}_\varepsilon$, namely, for any such η and ε the set $S := \{T > t_0(\eta,\varepsilon) \mid \mathcal{F}_\varepsilon(t) < \eta$ for all $t \in [t_0(\eta,\varepsilon),T]\}$ is not empty and hence $T := \sup S$ a well-defined element of $(t_0(\eta,\varepsilon),\infty]$, and if T was finite, then $\mathcal{F}_\varepsilon < \eta$ on $(t_0(\eta,\varepsilon),T)$ and $\mathcal{F}_\varepsilon(T) = \eta$. By (5.6), however, due to (5.32) this would imply that

$$\begin{aligned}
\mathcal{F}_\varepsilon'(t) &\le -\Big\{\frac{1}{K} - K\mathcal{F}_\varepsilon(t) - K\mathcal{F}_\varepsilon^2(t)\Big\}\cdot\Big\{\int_\Omega \frac{u_{\varepsilon xx}^2}{u_\varepsilon} + \int_\Omega \frac{v_{\varepsilon xx}^2}{v_\varepsilon}\Big\} \\
&\le -\Big\{\frac{1}{K} - K\eta - K\eta^2\Big\}\cdot\Big\{\int_\Omega \frac{u_{\varepsilon xx}^2}{u_\varepsilon} + \int_\Omega \frac{v_{\varepsilon xx}^2}{v_\varepsilon}\Big\} \\
&\le -\frac{1}{2K}\cdot\Big\{\int_\Omega \frac{u_{\varepsilon xx}^2}{u_\varepsilon} + \int_\Omega \frac{v_{\varepsilon xx}^2}{v_\varepsilon}\Big\} \\
&\le 0 \qquad \text{for all } t \in (t_0(\eta,\varepsilon),T)
\end{aligned}$$

and hence lead to the absurd conclusion that $\mathcal{F}_\varepsilon(T) \le \mathcal{F}_\varepsilon(t_0(\eta,\varepsilon)) \le \frac{\eta}{2} < \eta$ according to (5.33).

Having thus asserted (5.34), by a first application thereof to e.g. $\eta := \eta_0$ we particularly infer that

$$\chi_2\int_\Omega \frac{u_{\varepsilon x}^2}{u_\varepsilon} + \chi_1\int_\Omega \frac{v_{\varepsilon x}^2}{v_\varepsilon} \le \eta_0 \qquad \text{for all } t > T_0 := T_{\eta_0} \text{ and any } \varepsilon \in (0,\varepsilon_0). \tag{5.35}$$

Since the Gagliardo-Nirenberg inequality says that with some $c_2 > 0$ we have

$$\|\sqrt{\varphi}\|_{L^\infty(\Omega)}^4 \le c_2\|(\sqrt{\varphi})_x\|_{L^2(\Omega)}^2\|\sqrt{\varphi}\|_{L^2(\Omega)}^2 + c_2\|\sqrt{\varphi}\|_{L^2(\Omega)}^4 \qquad \text{for all } \varphi \in W^{1,2}(\Omega) \text{ such that } \varphi > 0 \text{ in } \overline{\Omega},$$

in view of (2.2) this entails that for all $t > T_0$ and $\varepsilon \in (0,\varepsilon_0)$,

$$\begin{aligned}
\chi_2\|u_\varepsilon\|_{L^\infty(\Omega)}^2 + \chi_1\|v_\varepsilon\|_{L^\infty(\Omega)}^2 &\le c_2\chi_2\|(\sqrt{u_\varepsilon})_x\|_{L^2(\Omega)}^2\cdot\int_\Omega u_0 + c_2\chi_2\cdot\Big\{\int_\Omega u_0\Big\}^2 \\
&\quad + c_2\chi_1\|(\sqrt{v_\varepsilon})_x\|_{L^2(\Omega)}^2\cdot\int_\Omega v_0 + c_2\chi_1\cdot\Big\{\int_\Omega v_0\Big\}^2 \\
&\le c_3 := \frac{c_2}{4}\cdot\eta_0\cdot\int_\Omega u_0 + c_2\chi_2\cdot\Big\{\int_\Omega u_0\Big\}^2 \\
&\quad + \frac{c_2}{4}\cdot\eta_0\cdot\int_\Omega v_0 + c_2\chi_1\cdot\Big\{\int_\Omega v_0\Big\}^2,
\end{aligned} \tag{5.36}$$

and that hence (5.30) holds.

In order to secondly derive (5.31) from (5.33), given $\delta > 0$ we fix $\eta \in (0,\eta_0]$ small enough such that $\sqrt{\frac{c_3}{\chi_i^3}}|\Omega|\eta \le \frac{\delta^2}{4}$ for $i \in \{1,2\}$, and noting that

$$\|\varphi - \overline{\varphi}\|_{L^\infty(\Omega)}^2 \le |\Omega|\cdot\int_\Omega \varphi_x^2 \qquad \text{for all } \varphi \in W^{1,2}(\Omega),$$

we conclude from (5.36), (5.33) and again (2.2) that

$$\begin{aligned}
\|u_\varepsilon - \overline{u}_0\|_{L^\infty(\Omega)}^2 &\le |\Omega|\cdot\int_\Omega u_{\varepsilon x}^2 \le |\Omega|\cdot\|u_\varepsilon\|_{L^\infty(\Omega)}\int_\Omega \frac{u_{\varepsilon x}^2}{u_\varepsilon} \le \sqrt{\frac{c_3}{\chi_2}}|\Omega|\cdot\int_\Omega \frac{u_{\varepsilon x}^2}{u_\varepsilon} \\
&\le \sqrt{\frac{c_3}{\chi_2^3}}|\Omega|\cdot\mathcal{F}_\varepsilon(t) \le \sqrt{\frac{c_3}{\chi_2^3}}|\Omega|\cdot\eta \qquad \text{for all } t \ge T_\eta \text{ and each } \varepsilon \in (0,\varepsilon_0).
\end{aligned}$$

Combined with a similar inequality for the second solution component, according to our restriction on η this shows that indeed

$$\|u_\varepsilon - \overline{u}_0\|_{L^\infty(\Omega)} + \|v_\varepsilon - \overline{v}_0\|_{L^\infty(\Omega)} \le \frac{\delta}{2} + \frac{\delta}{2} = \delta \qquad \text{for all } t \ge T_\eta \text{ and } \varepsilon \in (0, \varepsilon_0)$$

and hence verifies (5.31). $\square$

A simple limit passage leads to our main result on eventual boundedness and uniform stabilization in (1.3):

Proof of Theorem 1.2. Thanks to the approximation property asserted by Lemma 4.1, both claims immediately result from Lemma 5.2. $\square$

6 Local nonexistence for doubly attractive systems. Proof of Proposition 1.3

Proof of Proposition 1.3. Given $T > 0$ and any such solution, we first claim that

$$u \equiv v \qquad \text{in } \Omega \times (0, T). \tag{6.1}$$

To verify this, we use (1.11) to see that $w := u - v$ satisfies

$$w_t = w_{xx} - (uv_x)_x + (vu_x)_x = w_{xx} - uv_{xx} + vu_{xx} = (1 + u)w_{xx} - u_{xx}w$$

in $\Omega \times (0, T)$, which when tested against w implies that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2 + \int_\Omega (1 + u)w_x^2 = -\int_\Omega wu_xw_x - \int_\Omega u_{xx}w^2 = \int_\Omega wu_xw_x \tag{6.2}$$

for all $t \in (0, T)$. Now for arbitrary $T_0 \in (0, T)$, relying on (1.12) we can pick $c_1 = c_1(T_0) > 0$ such that $\int_\Omega u_x^2 \le c_1$ for all $t \in (0, T_0)$, and therefore we may use the compactness of the embedding $W^{1,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ to conclude from an associated Ehrling-type lemma that with some $c_2 > 0$ we have

$$\|\varphi\|^2_{L^\infty(\Omega)} \le \frac{1}{c_1}\int_\Omega \varphi_x^2 + c_2\int_\Omega \varphi^2 \qquad \text{for all } \varphi \in W^{1,2}(\Omega).$$

By Young' inequality, we can therefore estimate

$$\begin{aligned}
\int_\Omega wu_xw_x &\le \frac{1}{2}\int_\Omega w_x^2 + \frac{1}{2}\int_\Omega w^2u_x^2 \\
&\le \frac{1}{2}\int_\Omega w_x^2 + \frac{c_1}{2}\|w\|^2_{L^\infty(\Omega)} \\
&\le \int_\Omega w_x^2 + \frac{c_1c_2}{2}\int_\Omega w^2 \qquad \text{for all } t \in (0, T_0),
\end{aligned}$$

so that (6.2) entails that

$$\frac{d}{dt}\int_\Omega w^2 \le c_1c_2\int_\Omega w^2 \qquad \text{for all } t \in (0, T_0),$$

because u is nonnegative. As our assumptions in particular warrant continuity of w on $\overline{\Omega} \times \{0\}$ with $w|_{t=0} \equiv 0$, by integration this implies that $w \equiv 0$ in $\Omega \times (0, T_0)$ and thereby yields (6.1) on taking $T_0 \nearrow T$.
Next, by continuity of u our hypothesis that $\phi > 1$ in $\overline{\Omega}$ ensures the existence of $\delta > 0$ and $t_0 \in (0,T)$ such that $u \geq 1 + \delta$ in $\overline{\Omega} \times [0, t_0]$, and that accordingly,

$$z(x,s) := u(x, t_0 - s) - 1, \qquad x \in \overline{\Omega}, \ s \in [0, t_0],$$

satisfies $z \geq \delta$ in $\overline{\Omega} \times [0, t_0]$. Since furthermore z belongs to $C^0(\overline{\Omega} \times [0,t_0]) \cap C^{2,1}(\overline{\Omega} \times [0, t_0))$ by (1.12), with

$$z_s = -u_t = -u_{xx} + (uv_x)_x = (zz_x)_x \qquad \text{in } \Omega \times (0, t_0)$$

and $z_x|_{\partial\Omega \times (0,t_0)} = 0$ due to (1.11), well-known arguments asserting analyticity of solutions to one-dimensional porous medium equations inside their positivity set ([2]) become applicable so as to assert that $z(\cdot, t_0)$ and hence also $\phi = z(\cdot, t_0) + 1$ must be analytic throughout Ω. $\square$

7 Appendix: Some functional inequalities

In this appendix we collect some classes of functional inequalities needed for our analysis.

7.1 Bernis-type weighted embedding and interpolation inequalities

In a first group of inequalities, each member is in essence accessible to methods of Hölder-type interpolation when suitably combined with integration by parts. Numerous precedents of a similar flavor have been forming core ingredients in the study of scalar thin film equations (cf. e.g. [4], [3]).

The first inequality from this context which is needed here is quite elementary:

Lemma 7.1 *Let $\lambda \in \mathbb{R}$ be such that $\lambda \neq 1$. Then*

$$\int_\Omega \varphi^{\lambda-2} \varphi_x^4 \leq \frac{9}{(\lambda-1)^2} \int_\Omega \varphi^\lambda \varphi_{xx}^2 \tag{7.1}$$

for all $\varphi \in C^2(\overline{\Omega})$ which are such that $\varphi > 0$ in $\overline{\Omega}$ and $\varphi_x = 0$ on $\partial\Omega$.

Proof. We integrate by parts and use the Cauchy-Schwarz inequality to see that

$$\int_\Omega \varphi^{\lambda-2}\varphi_x^4 = -\frac{3}{\lambda-1} \int_\Omega \varphi^{\lambda-1} \varphi_x^2 \varphi_{xx} \leq \frac{3}{|\lambda-1|} \bigg\{ \int_\Omega \varphi^{\lambda-2}\varphi_x^4 \bigg\}^{\frac{1}{2}} \cdot \bigg\{ \int_\Omega \varphi^\lambda \varphi_{xx}^2 \bigg\}^{\frac{1}{2}},$$

from which (7.1) immediately follows. $\square$

The derivation of the following interpolation inequality is comparably simple:

Lemma 7.2 *Let $\lambda \in \mathbb{R}$ and $\varphi \in C^3(\overline{\Omega})$ be such that $\varphi > 0$ in $\overline{\Omega}$ and $\varphi_x = 0$ on $\partial\Omega$. Then*

$$\int_\Omega \varphi^{\lambda-2}\varphi_x^2\varphi_{xx}^2 \leq \eta \int_\Omega \varphi^\lambda \varphi_{xxx}^2 + \bigg\{ \frac{(\lambda-2)(\lambda-3)}{15} + \frac{1}{36\eta} \bigg\} \cdot \int_\Omega \varphi^{\lambda-4}\varphi_x^6 \qquad \textit{for all } \eta > 0. \tag{7.2}$$

PROOF. Using that $\varphi_x|_{\partial\Omega} = 0$, we twice integrate by parts to obtain

$$\begin{aligned}\int_\Omega \varphi^{\lambda-2}\varphi_x^2\varphi_{xx}^2 &= \frac{1}{3}\int_\Omega \varphi^{\lambda-2}(\varphi_x^3)_x\varphi_{xx} \\ &= -\frac{1}{3}\int_\Omega \varphi^{\lambda-2}\varphi_x^3\varphi_{xxx} - \frac{\lambda-2}{3}\int_\Omega \varphi^{\lambda-3}\varphi_x^4\varphi_{xx} \\ &= -\frac{1}{3}\int_\Omega \varphi^{\lambda-2}\varphi_x^3\varphi_{xxx} + \frac{(\lambda-2)(\lambda-3)}{15}\int_\Omega \varphi^{\lambda-4}\varphi_x^6.\end{aligned}$$

As by Young's inequality,

$$-\frac{1}{3}\int_\Omega \varphi^{\lambda-2}\varphi_x^3\varphi_{xxx} \le \eta\int_\Omega \varphi^\lambda\varphi_{xxx}^2 + \frac{1}{36\eta}\int_\Omega \varphi^{\lambda-4}\varphi_x^6 \qquad \text{for all } \eta > 0,$$

this implies (7.2). $\Box$

Likewise, the following one-dimensional version of a weighted Gagliardo-Nirenberg inequality can be obtained in quite a simple manner:

Lemma 7.3 *Let $\varphi \in C^2(\overline{\Omega})$ be such that $\varphi > 0$ in $\overline{\Omega}$ and $\varphi_x = 0$ on $\partial\Omega$. Then*

$$\|\varphi_x\|_{L^\infty(\Omega)} \le \left\{\int_\Omega \frac{\varphi_{xx}^2}{\varphi}\right\}^{\frac{1}{2}} \cdot \left\{\int_\Omega \varphi\right\}^{\frac{1}{2}}. \tag{7.3}$$

PROOF. Without loss of generality assuming that $\Omega = (0, |\Omega|)$ and that hence $\varphi_x(0) = 0$, by means of the Cauchy-Schwarz inequality we can estimate

$$|\varphi_x(x)| = \left|\int_0^x \varphi_{xx}(y)dy\right| \le \left\{\int_0^x \frac{\varphi_{xx}^2(y)}{\varphi(y)}dy\right\}^{\frac{1}{2}} \cdot \left\{\int_0^x \varphi(y)dy\right\}^{\frac{1}{2}} \qquad \text{for all } x \in \Omega,$$

from which (7.3) directly follows. $\Box$

The following two inequalities of quite a similar flavor are more specifically designed so as to serve in the particular frameworks of Lemma 3.4 and Lemma 4.1.

Lemma 7.4 *Let $n > 0$ and $m \in (0, n)$ be such that $m \le 2$, and for $\varepsilon \in (0,1)$ let*

$$F_\varepsilon(s) := \frac{s^n}{s^{n-m}+\varepsilon}, \qquad s \ge 0. \tag{7.4}$$

Then for any $\varphi \in C^2(\overline{\Omega})$ such that $\varphi > 0$ in $\overline{\Omega}$ and $\varphi_x = 0$ on $\partial\Omega$,

$$\int_\Omega |F'_\varepsilon(\varphi)\varphi_x\varphi_{xx}| \le \frac{n|\Omega|^{\frac{2-m}{12}}}{\sqrt{m}} \cdot \left\{\int_\Omega \varphi^{m-1}\varphi_{xx}^2\right\}^{\frac{3}{4}} \cdot \left\{\int_\Omega \varphi^3\right\}^{\frac{m+1}{12}} \tag{7.5}$$

and

$$\int_\Omega |F_\varepsilon(\varphi)\varphi_{xx}| \le |\Omega|^{\frac{2-m}{6}} \cdot \left\{\int_\Omega \varphi^{m-1}\varphi_{xx}^2\right\}^{\frac{1}{2}} \cdot \left\{\int_\Omega \varphi^3\right\}^{\frac{m+1}{6}}. \tag{7.6}$$

Proof. Since

$$0 \le F_\varepsilon'(s) = \frac{ms^{2n-m-1} + n\varepsilon s^{n-1}}{(s^{n-m}+\varepsilon)^2} \le \frac{ns^{n-1}}{s^{n-m}+\varepsilon} \le ns^{m-1}$$

for all $s > 0$ and $\varepsilon \in (0,1)$, given any positive $\varphi \in C^2(\overline{\Omega})$ with $\varphi_x|_{\partial\Omega} = 0$ we can use the Cauchy-Schwarz inequality to see that for arbitrary $m > 0$,

$$\int_\Omega |F_\varepsilon'(\varphi)\varphi_x\varphi_{xx}| \le n \cdot \left\{ \int_\Omega \varphi^{m-1}\varphi_{xx}^2 \right\}^{\frac{1}{2}} \cdot \left\{ \int_\Omega \varphi^{m-1}\varphi_x^2 \right\}^{\frac{1}{2}}. \tag{7.7}$$

Here, integrating by parts and once more relying on the Cauchy-Schwarz inequality we find that since $\varphi_x = 0$ on $\partial\Omega$,

$$\int_\Omega \varphi^{m-1}\varphi_x^2 = -\frac{1}{m}\int_\Omega \varphi^m \varphi_{xx} \le \frac{1}{m} \cdot \left\{ \int_\Omega \varphi^{m-1}\varphi_{xx}^2 \right\}^{\frac{1}{2}} \cdot \left\{ \int_\Omega \varphi^{m+1} \right\}^{\frac{1}{2}}, \tag{7.8}$$

so that since our assumption $m \le 2$ warrants that

$$\int_\Omega \varphi^{m+1} \le |\Omega|^{\frac{2-m}{3}} \cdot \left\{ \int_\Omega \varphi^3 \right\}^{\frac{m+1}{3}} \tag{7.9}$$

by the Hölder inequality, combining (7.7) with (7.8) yields (7.6).

As for (7.5), we only need to observe that $0 \le F_\varepsilon(s) \le s^m$ for $s \ge 0$ and $\varepsilon \in (0,1)$, and once more use the Cauchy-Schwarz inequality to see that

$$\int_\Omega |F_\varepsilon(\varphi)\varphi_{xx}| \le \int_\Omega \varphi^m|\varphi_{xx}| \le \left\{ \int_\Omega \varphi^{m-1}\varphi_{xx}^2 \right\}^{\frac{1}{2}} \cdot \left\{ \int_\Omega \varphi^{m+1} \right\}^{\frac{1}{2}},$$

which in view of (7.9) establishes the claimed estimate. □

7.2 A class of Gagliardo-Nirenberg type inequalities in Orlicz spaces

Although its proof is quite simple, the following class of inequalities seems to provide an efficient means to suitably make use of certain Orlicz space bounds in the course of Gagliardo-Nirenberg type interpolation.

Lemma 7.5 *Let $N \ge 1$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, and suppose that $p > 0$ and $q > 0$ are such that $p < \frac{2N}{(N-2)_+}$ and $q < p$. Assume furthermore that $\Lambda \in C^0(\mathbb{R})$ is such that $\Lambda \ge 1$ on $\mathbb{R}$, and that $\theta \in (0,1]$ is such that*

$$\begin{cases} \theta \le 1 & \text{if } N = 1, \\ \theta < 1 & \text{if } N = 2, \\ \theta \le \frac{2N-(N-2)p}{2N-(N-2)q} & \text{if } N \ge 3. \end{cases} \tag{7.10}$$

Then there exists $C>0$ such that for all $\varphi \in W^{1,2}(\Omega)$,

$$\int_\Omega |\varphi|^p \Lambda^\theta(\varphi) \le C \cdot \Big\{ \int_\Omega |\nabla\varphi|^2 \Big\}^{\frac{pa}{2}} \cdot \Big\{ \int_\Omega |\varphi|^q \Lambda(\varphi) \Big\}^{\frac{p(1-a)}{q}} + C \cdot \Big\{ \int_\Omega |\varphi|^q \Lambda(\varphi) \Big\}^{\frac{p}{q}} \tag{7.11}$$

where

$$a := \frac{\frac{N}{q} - \frac{N}{p}}{1 - \frac{N}{2} + \frac{N}{q}} \in (0,1). \tag{7.12}$$

PROOF. Writing $r := \theta q \in (0,q]$, from (7.10) we infer that if we let $s \in (0,\infty]$ be defined by $s := \frac{(p-r)q}{q-r}$, then $s \le \infty$ when $N=1$, that $s<\infty$ if $N=2$ and that $s \le \frac{2N}{N-2}$ in the case $N \ge 3$. As moreover $s>q$ due to our assumption that $p>q$, the Gagliardo-Nirenberg inequality provides $c_1>0$ such that

$$\|\varphi\|_{L^s(\Omega)}^{p-r} \le c_1 \|\nabla\varphi\|_{L^2(\Omega)}^{(p-r)b} \|\varphi\|_{L^q(\Omega)}^{(p-r)(1-b)} + c_1 \|\varphi\|_{L^q(\Omega)}^{p-r} \qquad \text{for all } \varphi \in W^{1,2}(\Omega)$$

with

$$b := \frac{\frac{N}{q} - \frac{N}{s}}{1 - \frac{N}{2} + \frac{N}{q}}. \tag{7.13}$$

Therefore, an application of the Hölder inequality shows that since $\frac{q\theta}{r} = 1$,

$$\begin{aligned}
\int_\Omega |\varphi|^p \Lambda^\theta(\varphi) &= \int_\Omega |\varphi|^{p-r} \cdot \Big(|\varphi|^r \Lambda^\theta(\varphi) \Big) \\
&\le \|\varphi\|_{L^s(\Omega)}^{p-r} \cdot \Big\{ \int_\Omega |\varphi|^q \Lambda(\varphi) \Big\}^{\frac{r}{q}} \\
&\le c_1 \|\nabla\varphi\|_{L^2(\Omega)}^{(p-r)b} \|\varphi\|_{L^q(\Omega)}^{(p-r)(1-b)} \cdot \Big\{ \int_\Omega |\varphi|^q \Lambda(\varphi) \Big\}^{\frac{r}{q}} \\
&\quad + c_1 \|\varphi\|_{L^q(\Omega)}^{p-r} \cdot \Big\{ \int_\Omega |\varphi|^q \Lambda(\varphi) \Big\}^{\frac{r}{q}} \qquad \text{for all } \varphi \in W^{1,2}(\Omega),
\end{aligned} \tag{7.14}$$

where we note that by (7.13), our definition of s and (7.12),

$$(p-r)b = (p-r) \cdot \frac{\frac{N}{q} \cdot \frac{p-q}{p-r}}{1 - \frac{N}{2} + \frac{N}{q}} = \frac{\frac{N}{q} \cdot (p-q)}{1 - \frac{N}{2} + \frac{N}{q}} = pa. \tag{7.15}$$

Since the hypothesis $\Lambda \ge 1$ ensures that $\|\varphi\|_{L^q(\Omega)}^q \le \int_\Omega |\varphi|^q \Lambda(\varphi)$ and thus, by (7.15),

$$\|\varphi\|_{L^q(\Omega)}^{(p-r)(1-b)} \cdot \Big\{ \int_\Omega |\varphi|^q \Lambda(\varphi) \Big\}^{\frac{r}{q}} \le \Big\{ \int_\Omega |\varphi|^q \Lambda(\varphi) \Big\}^{\frac{p(1-a)}{q}}$$

and

$$\|\varphi\|_{L^q(\Omega)}^{p-r} \cdot \Big\{ \int_\Omega |\varphi|^q \Lambda(\varphi) \Big\}^{\frac{r}{q}} \le \Big\{ \int_\Omega |\varphi|^q \Lambda(\varphi) \Big\}^{\frac{p}{q}}$$

for any such φ, from (7.14) we immediately obtain (7.11) with $C := c_1$. $\square$

Among numerous extractable special cases thereof, let us concentrate on the one explicitly referred to in the core part of our analysis (cf. Corollary 3.3).

Corollary 7.6 *Let $\Omega \subset \mathbb{R}$ be a bounded interval. Then there exists $C > 0$ such that for all $\varphi \in W^{1,2}(\Omega)$ satisfying $\varphi > 0$ in $\overline{\Omega}$,*

$$\int_\Omega \varphi^3 \ln(\varphi + e) \le C \cdot \bigg\{ \int_\Omega \frac{\varphi_x^2}{\varphi} \bigg\} \cdot \bigg\{ \int_\Omega \varphi \ln(\varphi + e) \bigg\}^2 + C \cdot \bigg\{ \int_\Omega \varphi \ln(\varphi + e) \bigg\}^3. \tag{7.16}$$

PROOF. We apply Lemma 7.5 to $p := 6, q := 2$ and $\theta := 1$ and $\Lambda(s) := \ln(s^2 + e)$, $s \in \mathbb{R}$, to find $c_1 > 0$ such that whenever $\psi \in W^{1,2}(\Omega)$,

$$\int_\Omega \psi^6 \ln(\psi^2 + e) \le c_1 \cdot \bigg\{ \int_\Omega \psi_x^2 \bigg\} \cdot \bigg\{ \int_\Omega \psi^2 \ln(\psi^2 + e) \bigg\}^2 + c_1 \cdot \bigg\{ \int_\Omega \psi^2 \ln(\psi^2 + e) \bigg\}^3.$$

For fixed $\varphi \in W^{1,2}(\Omega)$ being positive throughout $\overline{\Omega}$, on setting $\psi := \sqrt{\varphi}$ this directly implies that (7.16) holds with $C := c_1$, because then $\int_\Omega \psi_x^2 = \frac{1}{4} \int_\Omega \frac{\varphi_x^2}{\varphi} \le \int_\Omega \frac{\varphi_x^2}{\varphi}$. $\square$

In order to indicate the potential strength of Lemma 7.5 beyond the latter particular context, let us finally demonstrate how it can be used in an almost trivial manner to deduce a well-known and frequently used variant of a two-dimensional Gagliardo-Nirenberg inequality due to Biler, Hebisch and Nadzieja ([5]).

Corollary 7.7 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Then there exists $C > 0$ such that for each $\varepsilon \in (0, 1)$ one can pick $C_\varepsilon > 0$ with property that*

$$\|\varphi\|_{L^3(\Omega)}^3 \le \varepsilon \|\nabla\varphi\|_{L^2(\Omega)}^2 \|\varphi \ln(\varphi + e)\|_{L^1(\Omega)} + C\|\varphi \ln(\varphi + e)\|_{L^1(\Omega)}^3 + C_\varepsilon \tag{7.17}$$

holds for all nonnegative $\varphi \in W^{1,2}(\Omega)$.

PROOF. Applying Lemma 7.5 to $N := 2, p := 3, q := 1$ and $\theta := \frac{1}{2}$ and $\Lambda(s) := \ln(|s| + e)$, $s \in \mathbb{R}$, we obtain $c_1 > 0$ such that

$$\int_\Omega \varphi^3 \ln^{\frac{1}{2}}(\varphi + e) \le c_1 \cdot \bigg\{ \int_\Omega |\nabla\varphi|^2 \bigg\} \cdot \bigg\{ \int_\Omega \varphi \ln(\varphi + e) \bigg\} + c_1 \cdot \bigg\{ \int_\Omega \varphi \ln(\varphi + e) \bigg\}^3$$

for all $\varphi \in W^{1,2}(\Omega)$. Since for each $\varepsilon > 0$ one can find $c_2(\varepsilon) > 0$ such that

$$\xi^3 \le \frac{\varepsilon}{c_1} \cdot \xi^3 \ln^{\frac{1}{2}}(\xi + e) + c_2(\varepsilon) \qquad \text{for all } \xi > 0,$$

this readily yields (7.17) with $C := 1$ and $C_\varepsilon := c_2(\varepsilon) \cdot |\Omega|$. $\square$

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