Local energy estimates and global solvability in a three-dimensional chemotaxis-fluid system with prescribed signal on the boundary

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Abstract

The chemotaxis-Stokes system

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla c), \\ c_t + u \cdot \nabla c = \Delta c - nc, \\ u_t = \Delta u + \nabla P + n \nabla \phi, \qquad \nabla \cdot u = 0, \end{cases} \tag{$\star$}$$

is considered in a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary. The corresponding solution theory is quite well-developed in the case when $(\star)$ is accompanied by homogeneous boundary conditions of no-flux type for n and c, and of Dirichlet type for u. In such situations, namely, a quasi-Lyapunov structure provides regularity features sufficient to facilitate not only a basic existence theory, but also a comprehensive qualitative analysis.

However, if in line with what is suggested by the modeling literature the boundary condition for the signal is changed so as to become

$$c(x,t) = c_\star, \quad x \in \partial\Omega, \ t > 0,$$

with some constant $c_\star \geq 0$, then such structures apparently cease to be present at spatially global levels. The present work reveals that such properties persist at least in a weakened form of suitably localized variants, and on the basis of accordingly obtained *a priori* estimates it is shown that for widely arbitrary initial data an associated initial-boundary value problem for $(\star)$ admits a globally defined generalized solution.

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1 Introduction

Quasi-Lyapunov structures in chemotaxis-fluid systems. This work deals with a mathematical model for the interaction of bacterial populations with a surrounding fluid in which their nutrient is dissolved. In fact, experimental studies have revealed partially rather complex facets of the spatio-temporal behavior in colonies of the aerobic species *Bacillus subtilis* when suspended in sessile water drops, and in order to achieve an appropriate description thereof at macroscopic levels, the authors in [32] proposed the chemotaxis-fluid system

$$\begin{cases} n_t + u\cdot\nabla n &= \Delta n - \nabla\cdot(n\nabla c), & & x\in\Omega,\ t>0, \\ c_t + u\cdot\nabla c &= \Delta c - nc, & & x\in\Omega,\ t>0, \\ u_t + \kappa(u\cdot\nabla)u &= \Delta u + \nabla P + n\nabla\phi, & \nabla\cdot u = 0, & x\in\Omega,\ t>0, \end{cases} \tag{1.1}$$

as a model for for the unknown (n,c,u,P) in the physical domain $\Omega\subset\mathbb{R}^N$, with given gravitational potential ϕ, and with $\kappa\in\mathbb{R}$. When the fluid motion is slow, the Stokes flow ($\kappa=0$) is used rather than the Navier-Stokes one. Here the unknown $n=n(x,t)$ denotes the bacteria density, and $u=u(x,t)$ and $P=P(x,t)$ represent the velocity field of the incompressible fluid and an associated pressure, respectively. This model is based on the hypothesis that besides all of these components, the only further component relevant for such phenomena is the oxygen with concentration denoted by $c=c(x,t)$.

Since their initial introduction in 2005, chemotaxis-fluid systems of this and related types have inspired considerable activity in the analytical literature (see [2], [8], [10], [12], [22], [24], [36], [46] for instance). Particularly, when Ω is a bounded domain, questions concerning the global well-posedness of (1.1) have meanwhile been answered to quite a comprehensive extent in the case when the homogeneous boundary conditions

$$\frac{\partial n}{\partial\nu} = \frac{\partial c}{\partial\nu} = 0, \quad u=0, \quad x\in\partial\Omega \tag{1.2}$$

are imposed: Indeed, in the two-dimensional case the corresponding full chemotaxis-Navier-Stokes system is known to admit uniquely determined global smooth solutions for widely arbitrary initial data ([37]), whereas for the three-dimensional version of (1.1) it is at least possible to construct global weak solutions ([40]) which eventually become smooth and classical ([41]). Extensions in various directions are concerned with global classical solvability for small data ([7], [17]), or address modified systems involving nonlinear cell diffusion of porous medium type ([9], [11], [30], [42], [45]), or also accounting for saturation effects in chemotactic migration ([23], [33], [34]).

A core characteristic of (1.1) when accompanied by (1.2) seems to consist in a quasi-Lyapunov structure formally becoming manifest in the inequality

$$\frac{d}{dt}\Big\{\int_\Omega n\ln n + \frac12\int_\Omega\frac{|\nabla c|^2}{c} + \alpha\int_\Omega|u|^2\Big\} + \frac1C\Big\{\int_\Omega\frac{|\nabla n|^2}{n} + \int_\Omega\frac{|D^2c|^2}{c} + \int_\Omega|\nabla u|^2\Big\} \le C, \qquad t>0, \tag{1.3}$$

valid for suitably regular solutions to (1.1)-(1.2) in bounded convex domains with certain positive constants α and C. In fact, *a priori* estimates implied by (1.3) and appropriately adapted counterparts have, firstly, constituted crucial ingredients in the development of corresponding existence theories for (1.1) and several of its close relatives ([3], [19], [35], [30], [40], [45]). A second essential role played by inequalities of the form in (1.3), apparently yet more important with regard to aspects related to model validation, is related to the description of qualitative solution properties: In the particular context of (1.1)-(1.2), for instance, regularity information gained from (1.3) has proved sufficient to turn the fundamental relaxation features expressed in the two basic inequalities

$$\int_0^\infty \int_\Omega nc \le \int_\Omega c(\cdot,0) \qquad \text{and} \qquad \int_0^\infty \int_\Omega |\nabla c|^2 \le \frac{1}{2} \int_\Omega c^2(\cdot,0), \tag{1.4}$$

into statements on asymptotic behavior, indeed asserting that both in two- and three-dimensional frameworks all solutions approach spatially homogeneous equilibria in the large time limit ([38], [41], [44]; cf. also [39] and [43] for similar conclusions relying on (1.4) also in more general situations).

Results of the latter flavor, however, seem to reflect experimentally observed behavior, inter alia involving the emergence of large scale coherence patterns and structure formation especially near interfaces, only to a limited extent. Accordingly, in the recent literature an increasing part attempts to go beyond the first analytical steps concerned with the mathematically convenient boundary conditions in (1.2) by concentrating in a refined manner on more realistic types of assumptions on interface behavior. For instance, a family of Robin boundary conditions was first discussed in the context of a chemotaxis-Navier-Stokes system in [4], and more recently a corresponding global existence statement quite precisely addressing (1.1) has been derived in [6]; a steady-state version of a corresponding fluid-free variant has been analyzed [5].

Dirichlet conditions for the signal: local persistence of energy structure. Main results. The purpose of the present work is to consider (1.1) along with a set of boundary conditions which in [32] were proposed as appropriate especially near liquid-air interfaces. In line with the circumstance that oxygen diffuses substantially faster in air than in water, namely, both the formal and the numerical study in [32] presupposes a fixed given oxygen concentration on such boundary parts. Despite their correspondingly considerable potential relevance for the comprehension of the phenomena under consideration, such Dirichlet-type boundary conditions have so far been understood analytically only to a rudimentary extent, with the available knowledge apparently being restricted to a single result on global existence of small-data solutions to a problem related to (1.1) in a particular domain bounded by two parallel planes, involving given data for c on one of these planes ([25]).

In contrast to this intending to include arbitrary bounded domains and initial data of arbitrary size, in order to limit complexity we shall here assume for simplicity that the entire fluid is surrounded by air. Accordingly, we shall be concerned with (1.1) under boundary conditions which differ from those in (1.2) by requiring c to attain a prescribed value $c_\star$ throughout $\partial\Omega$, and hence we will subsequently consider the no-flux-Dirichlet-Dirichlet initial-boundary value problem

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(n\nabla c), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - nc, & x\in\Omega,\ t>0,\\ u_t = \Delta u + \nabla P + n\nabla\phi, \qquad \nabla\cdot u = 0, & x\in\Omega,\ t>0,\\ \frac{\partial n}{\partial\nu} - n\frac{\partial c}{\partial\nu} = 0, \quad c = c_\star, \quad u = 0, & x\in\partial\Omega,\ t>0,\\ n(x,0)=n_0(x), \quad c(x,0)=c_0(x), \quad u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.5}$$

in a smoothly bounded domain $\Omega\subset\mathbb{R}^3$, with $c_\star \ge 0$ and

$$\phi \in W^{2,\infty}(\Omega), \tag{1.6}$$

and with initial data such that

$$\begin{cases} n_0 \in W^{1,\infty}(\Omega) \quad \text{is nonnegative with } n_0 \not\equiv 0, \\ c_0 \in W^{1,\infty}(\Omega) \quad \text{is positive in } \Omega \text{ with } c_0|_{\partial\Omega} = c_\star, \quad \text{and that} \\ u_0 \in W^{2,\infty}(\Omega;\mathbb{R}^3) \quad \text{satisfies } \nabla\cdot u_0 \equiv 0 \text{ and } u_0|_{\partial\Omega} = 0. \end{cases} \tag{1.7}$$

A particular objective, potentially of relevance in wider frameworks of chemotaxis-fluid interaction under more general boundary conditions, consists in the question how far the apparent loss of global structural properties of the form in (1.3) can be compensated by the persistence of at least certain spatially localized variants thereof. In this direction, our analysis will reveal that indeed the natural candidate

$$\int_\Omega \zeta^4 n \ln n + \frac{1}{2}\int_\Omega \zeta^4 \frac{|\nabla c|^2}{c} + \alpha\int_\Omega \zeta^4 |u|^2 \tag{1.8}$$

can be shown to play a corresponding role of a functional enjoying certain energy-like features whenever $\zeta = \zeta(x)$ is suitably smooth and compactly supported in Ω. This observation, forming the core of our analysis and to be detailed in Section 6, will enable us to supplement some basic regularity information, to be documented in Section 3, by a crucial strong L^1 compactness feature for the first solution component (Corollary 6.10).

In consequence, we shall see that within an appropriately mild solution concept the problem (1.5) indeed allows for a fairly comprehensive statement on global existence for widely arbitrary initial data, as contained in the following main result of this study:

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, let $\phi \in W^{2,\infty}(\Omega)$ and $c_\star \ge 0$, and assume that (n_0, c_0, u_0) is such that (1.7) holds. Then there exist functions*

$$\begin{cases} n \in L^\infty((0,\infty);L^1(\Omega)), \\ c \in L^\infty_{loc}(\overline{\Omega}\times[0,\infty)) \ \textit{with}\ c - c_\star \in L^2_{loc}([0,\infty);W_0^{1,2}(\Omega)) \qquad \textit{and} \\ u \in \bigcap_{p\in[1,\frac{3}{2})} L^\infty((0,\infty);W_0^{1,p}(\Omega)) \end{cases} \tag{1.9}$$

such that $n \ge 0$ and $c \ge 0$ a.e. in $\Omega\times(0,\infty)$, that

$$\ln(n+1) \in L^2_{loc}([0,\infty);W^{1,2}(\Omega)), \tag{1.10}$$

and that (n,c,u) forms a global generalized solution of (1.5) in the sense of Definition 2.1 below.

2 Approximation and basic estimates

Let us first specify the solution concept which we pursue throughout the sequel, and which by referring to products of functions simultaneously involving the first two solution components partially parallels a similar approach designed in [20] for a fluid-free chemotaxis system.

Definition 2.1 *Assume that $c_\star \ge 0$, that $n_0 \in L^1(\Omega)$ and $c_0 \in L^1(\Omega)$ are nonnegative, and that $u_0 \in L^1(\Omega;\mathbb{R}^3)$. Then a triple of functions*

$$\begin{cases} n \in L^\infty((0,\infty);L^1(\Omega)), \\ c \in L^\infty_{loc}(\bar{\Omega}\times[0,\infty)) \ \textit{with}\ c - c_\star \in L^2_{loc}([0,\infty);W_0^{1,2}(\Omega)) \qquad \textit{and} \\ u \in L^1_{loc}([0,\infty);(W_0^{1,1}(\Omega));R^3) \end{cases} \tag{2.1}$$

will be called a global generalized solution of (1.5) *if* $n \ge 0$ *and* $c \ge 0$ *a.e. in* $\Omega \times (0,\infty)$ *and* $\nabla \cdot u = 0$ *a.e. in* $\Omega \times (0,\infty)$*, if for any* $\varphi \in C_0^\infty(\Omega \times [0,\infty))$ *the equality*

$$\int_0^\infty \int_\Omega c\varphi_t + \int_\Omega c_0 \varphi(\cdot,0) = \int_0^\infty \int_\Omega \nabla c \cdot \nabla\varphi + \int_0^\infty \int_\Omega nc\varphi - \int_0^\infty \int_\Omega c(u\cdot\nabla\varphi) \tag{2.2}$$

holds, if for each $\varphi \in C_0^\infty(\Omega \times [0,\infty);\mathbb{R}^3)$ *with* $\nabla\cdot\varphi = 0$ *the identity*

$$\int_0^\infty \int_\Omega u\cdot\varphi_t + \int_\Omega u_0\cdot\varphi(\cdot,0) = \int_0^\infty \int_\Omega \nabla u\cdot\nabla\varphi - \int_0^\infty \int_\Omega n\nabla\phi\cdot\varphi \tag{2.3}$$

is satisfied, if there exist positive functions $\psi \in C^2([0,\infty))$ *and* $\rho \in C^2([0,\infty))$ *fulfilling* $\psi' < 0$ *on* $[0,\infty)$ *and* $\rho'(c_\star) = 0$ *such that*

$$\sqrt{|\psi''(n)|}\nabla n,\ \psi'(n)\nabla n,\ n\psi''(n)\nabla n \ \text{ and } \ n\psi'(n)\nabla c \ \text{ belong to } \ L^2_{loc}(\overline{\Omega}\times[0,\infty);\mathbb{R}^3), \tag{2.4}$$

and that the inequality

$$\begin{aligned}
-\int_0^\infty &\int_\Omega \psi(n)\rho(c)\varphi_t - \int_\Omega \psi(n_0)\rho(c_0)\varphi(\cdot,0)\\
&\le -\int_0^\infty \int_\Omega \psi''(n)\rho(c)|\nabla n|^2\varphi\\
&\quad + \int_0^\infty \int_\Omega \Big\{ -2\psi'(n)\rho'(c) + n\psi''(n)\rho(c)\Big\}(\nabla n\cdot\nabla c)\varphi\\
&\quad + \int_0^\infty \int_\Omega \Big\{ -\psi(n)\rho''(c) + n\psi'(n)\rho'(c)\Big\}|\nabla c|^2\varphi\\
&\quad - \int_0^\infty \int_\Omega \psi'(n)\rho(c)\nabla n\cdot\nabla\varphi + \int_0^\infty \int_\Omega \Big\{n\psi'(n)\rho(c) - \psi(n)\rho'(c)\Big\}\nabla c\cdot\nabla\varphi\\
&\quad + \int_0^\infty \int_\Omega \psi(n)\rho(c)(u\cdot\nabla\varphi) - \int_0^\infty \int_\Omega n\psi(n)c\rho'(c)\varphi
\end{aligned} \tag{2.5}$$

is valid for all nonnegative $\varphi \in C_0^\infty(\overline{\Omega}\times[0,\infty))$*, and if moreover*

$$\int_\Omega n(\cdot,t) \le \int_\Omega n_0 \qquad \textit{for a.e. } t>0. \tag{2.6}$$

Remark. i) It can readily be verified that the hypotheses in (2.1) and (2.4) are sufficient to warrant that all expressions in (2.2), (2.3) and (2.5) are indeed well-defined.

ii) By straightforward adaptation of the reasoning in e.g. [20, Lemma 2.5], it can be seen that whenever (n,c,u) is a global generalized solution in the above sense which has the additional properties that n and c belong to $C^0(\bar\Omega\times[0,\infty))\cap C^{2,1}(\bar\Omega\times(0,\infty))$, and that $u \in C^0(\bar\Omega\times[0,\infty);\mathbb{R}^3)\cap C^{2,1}(\Omega\times(0,\infty);\mathbb{R}^3)$, then (n,c,u), along with some associated pressure $P \in C^{1,0}(\Omega\times(0,\infty))$ constructed in a standard manner ([28]), actually forms a global classical solution of (1.5).

In order to construct such solutions via suitable approximation, for $\varepsilon \in (0,1)$ we consider

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon &= \Delta n_\varepsilon - \nabla \cdot (n_\varepsilon F'_\varepsilon(n_\varepsilon) \nabla c_\varepsilon), & x \in \Omega,\ t > 0, \\ c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon &= \Delta c_\varepsilon - F_\varepsilon(n_\varepsilon) c_\varepsilon, & x \in \Omega,\ t > 0, \\ u_{\varepsilon t} &= \Delta u_\varepsilon + \nabla P_\varepsilon + F_\varepsilon(n_\varepsilon)\nabla\phi, \quad \nabla \cdot u_\varepsilon = 0, & x \in \Omega,\ t > 0, \\ \frac{\partial n_\varepsilon}{\partial \nu} - n_\varepsilon F'_\varepsilon(n_\varepsilon) \frac{\partial c_\varepsilon}{\partial \nu} = 0, \quad c_\varepsilon = c_\star, \quad u_\varepsilon = 0, & & x \in \partial\Omega,\ t > 0, \\ n_\varepsilon(x,0) = n_0(x), \quad c_\varepsilon(x,0) = c_0(x), \quad u_\varepsilon(x,0) = u_0(x), & & x \in \Omega, \end{cases} \tag{2.7}$$

where

$$F_\varepsilon(s) := \frac{s}{1+\varepsilon s}, \qquad s \ge 0, \tag{2.8}$$

satisfies

$$0 \le F_\varepsilon(s) \le s \quad \text{and} \quad 0 \le F'_\varepsilon(s) = \frac{1}{(1+\varepsilon s)^2} \le 1 \qquad \text{for all } s \ge 0 \text{ and } \varepsilon \in (0,1) \tag{2.9}$$

and

$$F_\varepsilon(s) \nearrow s \quad \text{and} \quad F'_\varepsilon(s) \nearrow 1 \qquad \text{for all } s \ge 0 \text{ as } \varepsilon \searrow 0. \tag{2.10}$$

Similar regularizations have been underlying several previous works on problems of type (1.1)-(1.2) (cf. [37] or [40], for instance); a slight difference to most precedents consists in the artificial ε-dependent dampening expressed in the source term $F_\varepsilon(n_\varepsilon)\nabla\phi$ entering the Stokes subsystem of (2.7).

Now by means of a suitably arranged additional approximation procedure, it can be seen that indeed each of these problems admits a global classical solution enjoying some basic regularity and boundedness features. In comparison to well-established reasonings in the literature e.g. on (1.1)-(1.2), some nontrivial adaptations seem in order here in order to appropriately cope with the no-flux boundary condition for the first solution component, which in the framework of (2.7) apparently cannot be reduced to a requirement solely referring to n_ε in a trivial manner.

Lemma 2.2 *Let $\varepsilon \in (0,1)$. Then there exist functions*

$$\begin{cases} n_\varepsilon \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ c_\varepsilon \in \bigcap_{q \ge 1} C^0([0,\infty); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ u_\varepsilon \in C^0(\overline{\Omega} \times [0,\infty); \mathbb{R}^3) \cap C^{2,1}(\Omega \times (0,\infty); \mathbb{R}^3) \qquad \text{and} \\ P_\varepsilon \in C^{1,0}(\Omega \times (0,\infty)) \end{cases} \tag{2.11}$$

such that $n_\varepsilon > 0$ in $\overline{\Omega} \times (0,\infty)$ and $c_\varepsilon > 0$ in $\Omega \times (0,\infty)$, and that $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ solves (2.7) in the classical sense. Moreover,

$$\int_\Omega n_\varepsilon(\cdot,t) = \int_\Omega n_0 \qquad \text{for all } t > 0 \tag{2.12}$$

and

$$\|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le M := \max\Big\{ \|c_0\|_{L^\infty(\Omega)}\,,\, c_\star \Big\} \qquad \text{for all } t > 0. \tag{2.13}$$

Proof. Following [39], we fix $(\chi_\delta)_{\delta\in(0,1)} \subset C_0^\infty(\Omega)$ such that $0 \le \chi_\delta \le 1$ in Ω and $\chi_\delta \nearrow 1$ in Ω as $\delta \searrow 0$, and rewriting (2.7) in the equivalent form

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon &= \Delta n_\varepsilon - \nabla \cdot (n_\varepsilon F_\varepsilon'(n_\varepsilon)\nabla \widehat{c}_\varepsilon), & x \in \Omega,\ t > 0,\\ \widehat{c}_{\varepsilon t} + u_\varepsilon \cdot \nabla \widehat{c}_\varepsilon &= \Delta \widehat{c}_\varepsilon - F_\varepsilon(n_\varepsilon)\widehat{c}_\varepsilon - F_\varepsilon(n_\varepsilon)c_\star, & x \in \Omega,\ t > 0,\\ u_{\varepsilon t} &= \Delta u_\varepsilon + \nabla P_\varepsilon + F_\varepsilon(n_\varepsilon)\nabla\phi, \quad \nabla \cdot u_\varepsilon = 0, & x \in \Omega,\ t > 0,\\ \frac{\partial n_\varepsilon}{\partial \nu} - n_\varepsilon F_\varepsilon'(n_\varepsilon)\frac{\partial \widehat{c}_\varepsilon}{\partial\nu} = 0, \quad \widehat{c}_\varepsilon = 0, \quad u_\varepsilon = 0, & & x \in \partial\Omega,\ t > 0,\\ n_\varepsilon(x,0) = n_0(x), \quad \widehat{c}_\varepsilon(x,0) = c_0(x) - c_\star, \quad u_\varepsilon(x,0) = u_0(x), & & x \in \Omega \end{cases} \tag{2.14}$$

by substituting $c_\varepsilon = \widehat{c}_\varepsilon + c_\star$, we consider the approximation thereof given by

$$\begin{cases} n_{\varepsilon\delta t} + u_{\varepsilon\delta} \cdot \nabla n_{\varepsilon\delta} &= \Delta n_{\varepsilon\delta} - \nabla \cdot (\chi_\delta(x) n_{\varepsilon\delta} F_\varepsilon'(n_{\varepsilon\delta})\nabla \widehat{c}_{\varepsilon\delta}), & x \in \Omega,\ t > 0,\\ \widehat{c}_{\varepsilon\delta t} + u_{\varepsilon\delta} \cdot \nabla \widehat{c}_{\varepsilon\delta} &= \Delta \widehat{c}_{\varepsilon\delta} - F_\varepsilon(n_{\varepsilon\delta})\widehat{c}_{\varepsilon\delta} - F_\varepsilon(n_{\varepsilon\delta})c_\star, & x \in \Omega,\ t > 0,\\ u_{\varepsilon\delta t} &= \Delta u_{\varepsilon\delta} + \nabla P_{\varepsilon\delta} + F_\varepsilon(n_{\varepsilon\delta})\nabla\phi, \quad \nabla \cdot u_{\varepsilon\delta} = 0, & x \in \Omega,\ t > 0,\\ \frac{\partial n_{\varepsilon\delta}}{\partial \nu} = 0, \quad \widehat{c}_{\varepsilon\delta} = 0, \quad u_{\varepsilon\delta} = 0, & & x \in \partial\Omega,\ t > 0,\\ n_{\varepsilon\delta}(x,0) = n_0(x), \quad \widehat{c}_{\varepsilon\delta}(x,0) = c_0(x) - c_\star, \quad u_{\varepsilon\delta}(x,0) = u_0(x), & & x \in \Omega \end{cases} \tag{2.15}$$

for $\delta \in (0,1)$. Here thanks to the homogeneity of all boundary conditions, (2.15) becomes accessible to standard parabolic and Stokes semigroup estimates, which enable us to perform standard arguments in suitable fixed point frameworks (e.g. in the style of the reasonings detailed in [37]) to infer the existence of $T_{\varepsilon\delta} \in (0,\infty]$ and a classical solution $(n_{\varepsilon\delta}, c_{\varepsilon\delta}, u_{\varepsilon\delta}, P_{\varepsilon\delta})$ of (2.15) in $\Omega \times (0, T_{\varepsilon\delta})$, with components belonging to the corresponding analogues of the spaces in (2.11), such that

$$\text{if } T_{\varepsilon\delta} < \infty, \quad \text{then} \quad \limsup_{t\nearrow T_{\varepsilon\delta}} \Big\{ \|n_{\varepsilon\delta}(\cdot,t)\|_{L^\infty(\Omega)} + \|\widehat{c}_{\varepsilon\delta}(\cdot,t)\|_{W^{1,q}(\Omega)} + \|A^\beta u_{\varepsilon\delta}(\cdot,t)\|_{L^2(\Omega)} \Big\} = \infty$$
$$\text{for all } q > 3 \text{ and any } \beta > \frac{3}{4}, \tag{2.16}$$

where A denotes the L^2 realization of the Stokes operator under homogeneous Dirichlet boundary conditions. Moreover, three applications of the maximum principle show that $n_{\varepsilon\delta} \ge 0$ in $\Omega \times (0, T_{\varepsilon\delta})$ and that

$$0 \le \widehat{c}_{\varepsilon\delta} + c_\star \le M \qquad \text{in } \Omega \times (0, T_{\varepsilon\delta}) \tag{2.17}$$

for all $\varepsilon \in (0,1)$ and $\delta \in (0,1)$, and using that $\chi_\delta = 0$ on $\partial\Omega$, upon integrating the first equation in (2.15) we see that

$$\int_\Omega n_{\varepsilon\delta}(\cdot,t) = \int_\Omega n_0 \qquad \text{for all } t \in (0, T_{\varepsilon\delta}), \text{ each } \varepsilon \in (0,1) \text{ and any } \delta \in (0,1). \tag{2.18}$$

Now since

$$0 \le F_\varepsilon(s) \le \frac{1}{\varepsilon} \qquad \text{for all } s \ge 0 \text{ and each fixed } \varepsilon \in (0,1) \tag{2.19}$$

by (2.8), invoking well-known smoothing estimates for the Stokes semigroup ([14]) we see that for all $\beta \in (0,1)$ and any $T > 0$ we can find $C_1(\varepsilon,\beta,T) > 0$ such that

$$\|A^\beta u_{\varepsilon\delta}(\cdot,t)\|_{L^2(\Omega)} \le C_1(\varepsilon,\beta,T) \qquad \text{for all } t \in (0, \widehat{T}_{\varepsilon\delta}) \text{ and any } \delta \in (0,1), \tag{2.20}$$

where here and below we abbreviate $\widehat{T}_{\varepsilon\delta} := \min\{T_{\varepsilon\delta}, T\}$ for $T > 0$. Thereupon, known $L^p - L^q$ estimates for the Dirichlet heat semigroup on Ω ([27]) can be applied to the second equation from (2.15) to show that again thanks to (2.19), for all $q \geq 1$ and each $T > 0$ there exists $C_2(\varepsilon, q, T) > 0$ fulfilling

$$\|\widehat{c}_{\varepsilon\delta}(\cdot,t)\|_{W^{1,q}(\Omega)} \leq C_2(\varepsilon,q,T) \qquad \text{for all } t \in (0,\widehat{T}_{\varepsilon\delta}) \text{ and } \delta \in (0,1). \tag{2.21}$$

In view of (2.15) and corresponding smoothing properties of the Neumann heat semigroup over Ω ([13]), together with the fact that

$$0 \leq sF'_\varepsilon(s) \leq \frac{1}{2\varepsilon} \qquad \text{for all } s \geq 0 \tag{2.22}$$

by (2.8), this in turn implies that for all $T > 0$ one can choose $C_3(\varepsilon, T) > 0$ such that

$$\|n_{\varepsilon\delta}(\cdot,t)\|_{L^\infty(\Omega)} \leq C_3(\varepsilon,T) \qquad \text{for all } t \in (0,\widehat{T}_{\varepsilon\delta}) \text{ and } \delta \in (0,1). \tag{2.23}$$

In light of (2.16), the estimates in (2.20), (2.21) and (2.23) particularly warrant that actually $T_{\varepsilon\delta} = \infty$ for all $\varepsilon \in (0,1)$ and $\delta \in (0,1)$, and apart from that they can be used in suitably passing to the limit $\delta \searrow 0$ in (2.15): Namely, combining (2.21) with (2.20) and the fact that $0 \leq \chi_\delta \leq 1$, we readily obtain boundedness of $(a_{\varepsilon\delta})_{\delta\in(0,1)}$ in $L^\infty((0,T);L^q(\Omega))$ for all $T > 0$ and $q > 1$, where $a_{\varepsilon\delta}(x,t) := \chi_\delta(x)F'_\varepsilon(n_{\varepsilon\delta})\nabla\widehat{c}_{\varepsilon\delta} + u_{\varepsilon\delta}$, $\varepsilon \in (0,1)$, $\delta \in (0,1)$. As $n_{\varepsilon\delta t} = \Delta n_{\varepsilon\delta} - \nabla\cdot(a_{\varepsilon\delta}(x,t)n_{\varepsilon\delta})$ in $\Omega\times(0,\infty)$ due to (2.15), by means of a standard testing procedure this firstly implies the uniform bound

$$\begin{aligned}\int_0^T\int_\Omega |\nabla n_{\varepsilon\delta}|^2 &\leq \int_\Omega n_0^2 + \int_0^T\int_\Omega |a_{\varepsilon\delta}n_{\varepsilon\delta}|^2 \\ &\leq \int_\Omega n_0^2 + C_3^2(\varepsilon,T)T \sup_{\delta\in(0,1)} \|a_{\varepsilon\delta}\|^2_{L^\infty((0,T);L^2(\Omega))} \qquad \text{for all } \delta \in (0,1),\end{aligned} \tag{2.24}$$

and secondly, through known results on Hölder regularity in scalar parabolic equations ([26]), this boundedness property of $a_{\varepsilon\delta}$ ensures that for all $T > 0$ there exist $\theta_1 = \theta_1(\varepsilon,T) \in (0,1)$ and $C_4(\varepsilon,T) > 0$ such that

$$\|n_{\varepsilon\delta}\|_{C^{\theta_1,\frac{\theta_1}{2}}(\overline{\Omega}\times[0,T])} \leq C_4(\varepsilon,T) \qquad \text{for all } \delta \in (0,1). \tag{2.25}$$

By the same token, (2.20), (2.21) and (2.23) imply that for all $T > 0$ one can find $\theta_2 = \theta_2(\varepsilon,T) \in (0,1)$ and $C_5(\varepsilon,T) > 0$ satisfying

$$\|\widehat{c}_{\varepsilon\delta}\|_{C^{\theta_2,\frac{\theta_2}{2}}(\overline{\Omega}\times[0,T])} \leq C_5(\varepsilon,T) \qquad \text{for all } \delta \in (0,1), \tag{2.26}$$

while merely relying on (2.23), (1.7) and maximal Sobolev regularity theory for the Stokes evolution system ([15]) we see that to each $T > 0$ there corresponds some $C_6(\varepsilon,p,T) > 0$ fulfilling

$$\int_0^T \Big\{\|u_{\varepsilon\delta}(\cdot,t)\|^p_{W^{2,p}(\Omega)} + \|u_{\varepsilon\delta t}(\cdot,t)\|^p_{L^p(\Omega)}\Big\}dt \leq C_6(\varepsilon,p,T) \qquad \text{for all } \delta \in (0,1). \tag{2.27}$$

Apart from that, estimates local in time and excluding the temporal origin can be achieved by successively employing Schauder theories for the Stokes equations ([29]) and for scalar parabolic equations

([18]) to see that thanks to (2.25), for all $\tau > 0$ and $T > \tau$ there exist $\theta_3 = \theta_3(\varepsilon, \tau, T) \in (0,1)$, $\theta_4 = \theta_4(\varepsilon, \tau, T) \in (0,1)$, $C_7(\varepsilon, \tau, T) > 0$ and $C_8(\varepsilon, \tau, T) > 0$ such that

$$\|u_{\varepsilon\delta}\|_{C^{2+\theta_3, 1+\frac{\theta_3}{2}}(\overline{\Omega}\times[\tau,T])} \le C_7(\varepsilon, \tau, T) \qquad \text{for all } \delta \in (0,1) \tag{2.28}$$

and

$$\|\widehat{c}_{\varepsilon\delta}\|_{C^{2+\theta_4, 1+\frac{\theta_4}{2}}(\overline{\Omega}\times[\tau,T])} \le C_8(\varepsilon, \tau, T) \qquad \text{for all } \delta \in (0,1). \tag{2.29}$$

Now using that due to a well-known embedding result ([1]) the validity of (2.27) for suitably large $p > 1$ in particular warrants that for each $T > 0$, $(u_{\varepsilon\delta})_{\delta\in(0,1)}$ is bounded in $C^{\theta_5, \frac{\theta_5}{2}}(\overline{\Omega} \times [0,T])$ for some $\theta_5 = \theta_5(\varepsilon, T) \in (0,1)$, in view of (2.24), (2.25), (2.26), (2.28) and (2.29) we may extract a subsequence $(\delta_k)_{k\in\mathbb{N}} \subset (0,1)$ such that $\delta_k \searrow 0$ as $k \to \infty$, and that

$$\begin{array}{l}
n_{\varepsilon\delta} \to n_\varepsilon \text{ in } L^\infty_{loc}(\overline{\Omega} \times [0,\infty)), \\
\nabla n_{\varepsilon\delta} \rightharpoonup \nabla n_\varepsilon \text{ in } L^2_{loc}(\overline{\Omega} \times [0,\infty)), \\
\widehat{c}_{\varepsilon\delta} \to \widehat{c}_\varepsilon \text{ in } L^\infty_{loc}(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}_{loc}(\overline{\Omega} \times (0,\infty)) \qquad \text{and} \\
u_{\varepsilon\delta} \to u_\varepsilon \text{ in } L^\infty_{loc}(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}_{loc}(\overline{\Omega} \times (0,\infty))
\end{array} \tag{2.30}$$

as $\delta = \delta_k \searrow 0$, with some limit triple $(n_\varepsilon, \widehat{c}_\varepsilon, u_\varepsilon)$ which, along with a corresponding associated pressure $P_\varepsilon \in C^{1,0}(\Omega \times (0,\infty))$ constructed via a standard argument ([28]), satisfies the second and third subproblem of (2.14) in the classical sense in $\Omega \times (0,\infty)$, and which solves the first evolution problem in (2.14) in the natural weak sense specified e.g. in [18]. According to the smoothness properties of $n_\varepsilon, u_\varepsilon$ and $\widehat{c}_\varepsilon$ asserted by (2.30), (2.25), (2.28) and (2.29), however, upon application of well-known regularity theory for generalized solutions of scalar parabolic equations ([21], [18]) it can, finally, a posteriori be shown that actually n_ε also belongs to $C^{2,1}(\overline{\Omega} \times (0,\infty))$ and satisfies its respective initial-boundary value problem in (2.14) in the classical sense (cf. also [7] for a detailed reasoning in a closely related situation).
Transforming back via defining $c_\varepsilon := \widehat{c}_\varepsilon + c_\star$, we readily infer that indeed $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ enjoys the regularity features in (2.11) and solves (2.7) classically. According to (2.17) and (2.30), this solution moreover satisfies (2.13), while (2.12) is a consequence of (2.18) and (2.30), and the claimed strict positivity features result from (1.7) and the strong maximum principle. $\square$

3 Basic regularity features of ∇c_ε and ∇n_ε

A basic but crucial regularity information on the taxis gradient can be obtained through quite a standard testing procedure:

Lemma 3.1 *Let $T > 0$. Then there exists $C(T) > 0$ such that*

$$\int_0^T \int_\Omega |\nabla c_\varepsilon|^2 \le C(T) \qquad \textit{for all } \varepsilon \in (0,1). \tag{3.1}$$

Proof. Using that $c_\varepsilon - c_\star = 0$ on $\partial\Omega \times (0,\infty)$ and that $\nabla \cdot u_\varepsilon = 0$, from the second equation in (2.7), (2.9), (2.12) and (2.13) we obtain that

$$\frac{d}{dt}\left\{\frac{1}{2}\int_\Omega c_\varepsilon^2 - c_\star \int_\Omega c_\varepsilon\right\} + \int_\Omega |\nabla c_\varepsilon|^2 \;\; = \;\; -\int_\Omega F_\varepsilon(n_\varepsilon) c_\varepsilon^2 + c_\star \int_\Omega F_\varepsilon(n_\varepsilon) c_\varepsilon$$

$$\leq \quad c_\star M \int_\Omega n_0 \qquad \text{for all } t>0,$$

from which (3.1) directly results upon a time integration. □

As a consequence, we obtain an estimate for ∇n_ε if a suitably strong n_ε-dependent weight is included:

Lemma 3.2 *For all $T>0$ there exists $C(T)>0$ such that*

$$\int_0^T \int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} \leq C(T) \qquad \textit{for all } \varepsilon \in (0,1). \tag{3.2}$$

PROOF. According to the first equation in (2.7) and Young's inequality,

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \ln(n_\varepsilon+1) &= \int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} - \int_\Omega \frac{n_\varepsilon F'_\varepsilon(n_\varepsilon)}{(n_\varepsilon+1)^2}\nabla n_\varepsilon \cdot \nabla c_\varepsilon \\
&\geq \frac{1}{2}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} - \frac{1}{2}\int_\Omega \frac{n_\varepsilon^2 F'^2_\varepsilon(n_\varepsilon)}{(n_\varepsilon+1)^2}|\nabla c_\varepsilon|^2 \\
&\geq \frac{1}{2}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} - \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^2 \qquad \text{for all } t>0,
\end{aligned}$$

again because $\nabla\cdot u_\varepsilon = 0$ and $0 \leq F'_\varepsilon \leq 1$. Since $0 \leq \int_\Omega \ln(n_\varepsilon+1) \leq \int_\Omega n_\varepsilon = \int_\Omega n_0$ for all $t>0$ by (2.12), an integration thereof shows that (3.2) is a consequence of Lemma 3.1. □

4 Regularity of velocity and pressure in the Stokes system with sources in L^1

This purpose of this preparatory section is to provide some regularity information on solutions to the incompressible Stokes evolution equations driven by forcing terms for which bounds are known only with respect to spatial L^1 norms. Whereas regularity features of the corresponding velocity field can be obtained in quite a straightforward manner, deriving appropriate estimates for the pressure, as playing a crucial role in the course of our subsequent localization procedure (see Lemma 6.9), seems to require some additional efforts.

Our first step toward this consists in an essentially well-known observation on regularity of velocities.

Lemma 4.1 *Let $p \in (1,\frac{3}{2})$. Then there exists $C(p)>0$ with the property that whenever $T>0$, $v_0 \in C^0(\overline{\Omega};\mathbb{R}^3) \cap L^2_\sigma(\Omega)$, $v \in C^0(\overline{\Omega}\times[0,T];\mathbb{R}^3) \cap L^2((0,T);W_0^{1,2}(\Omega;\mathbb{R}^3)) \cap C^{2,1}(\Omega\times(0,T);\mathbb{R}^3)$ and $f \in C^0(\overline{\Omega}\times[0,T];\mathbb{R}^3)$ are such that $\nabla\cdot v = 0$ in $\Omega\times(0,T)$ and*

$$\begin{cases} v_t + Av = \mathcal{P}f & \textit{in } \Omega\times(0,T), \\ v(\cdot,0) = v_0 & \textit{in } \Omega, \end{cases} \tag{4.1}$$

we have

$$\|v(\cdot,t)\|_{L^{2p}(\Omega)} \leq C(p)\cdot\Big\{\|v_0\|_{L^{2p}(\Omega)} + \sup_{s\in(0,T)}\|f(\cdot,s)\|_{L^1(\Omega)}\Big\} \qquad \textit{for all } t\in(0,T) \tag{4.2}$$

and

$$\|v(\cdot,t)\|_{W^{1,p}(\Omega)} \le C(p)\cdot\Big\{\|v_0\|_{W^{1,2}(\Omega)} + \sup_{s\in(0,T)}\|f(\cdot,s)\|_{L^1(\Omega)}\Big\} \qquad \textit{for all } t\in(0,T). \tag{4.3}$$

PROOF. As solutions in the indicated class admit a representation in terms of an associated Duhamel formula ([28, Theorem IV 2.4.1]), this can be seen in a standard manner using well-known smoothing properties of the Stokes semigroup (cf. e.g. [39, Corollary 3.4]). □

By suitably combining this with the comprehensive theory on pressure regularity contained in [28], we obtain the following consequence which in Section 6 will turn out to be an indispensible cornerstone for our approach.

Lemma 4.2 *Let $T>0$, and let $r>1$ and $p\in(1,\frac{3}{2})$. Then there exists $C(p,r,T)>0$ such that if $v_0\in C^0(\overline{\Omega};\mathbb{R}^3)\cap L^2_\sigma(\Omega)$, $v\in C^0(\overline{\Omega}\times[0,T];\mathbb{R}^3)\cap L^2((0,T);W^{1,2}_0(\Omega;\mathbb{R}^3))\cap C^{2,1}(\Omega\times(0,T);\mathbb{R}^3)$ and $f\in C^0(\overline{\Omega}\times[0,T];\mathbb{R}^3)$ satisfy $\nabla\cdot v=0$ in $\Omega\times(0,T)$ and are such that (4.1) holds, then there exists a unique*

$$P\in L^2((0,T);L^p(\Omega)) \quad \textit{such that} \quad \int_\Omega P(\cdot,t)=0 \qquad \textit{for a.e. } t\in(0,T), \tag{4.4}$$

that

$$v_t=\Delta v+\nabla P+f \qquad \textit{in } \mathcal{D}'(\Omega\times(0,T)), \tag{4.5}$$

and that

$$\int_0^T \|P(\cdot,t)\|^r_{L^p(\Omega)}dt \le C(p,r,T)\cdot\Big\{\|v_0\|_{W^{2,\frac{3}{2}}(\Omega)} + \sup_{t\in(0,T)}\|f(\cdot,t)\|_{L^1(\Omega)}\Big\}. \tag{4.6}$$

PROOF. In view of the assumed regularity properties of v_0, v and f, the statement on existence and uniqeness of a function P fulfilling (4.4) and (4.5) follows from standard theory on the Stokes system ([28, Lemma II 2.1.1, Lemma II 2.2.2, Theorem IV 2.6.3]), so that it remains to verify (4.6). To this end, given $p\in(1,\frac{3}{2})$ we fix $q\in(p,\frac{3}{2})$ and use that then $\frac{3(q-1)}{2q}<\frac{1}{2}$, to choose some number α such that both $\alpha>\frac{3(q-1)}{2q}$ and $\alpha<\frac{1}{2}$, and then observe that the former inequality warrants the existence of $C_1>0$ fulfilling

$$\|A^{-\alpha}\mathcal{P}\varphi\|_{L^q(\Omega)} \le C_1\|\varphi\|_{L^1(\Omega)} \qquad \text{for all } \varphi\in L^1(\Omega;\mathbb{R}^3) \tag{4.7}$$

(see e.g. [39, Lemma 3.3]), and that the second restriction on α ensures that $W^{1,\frac{p}{p-1}}(\Omega;\mathbb{R}^3)\hookrightarrow D(A^\alpha_{\frac{q}{q-1}})$, and that thus

$$\|A^\alpha\varphi\|_{L^{\frac{q}{q-1}}(\Omega)} \le C_2\|\varphi\|_{W^{1,\frac{p}{p-1}}(\Omega)} \qquad \text{for all } \varphi\in D(A^\alpha_{\frac{q}{q-1}}) \tag{4.8}$$

with some $C_2>0$ ([16]). We furthermore note that our hypothesis $p<\frac{3}{2}$ guarantees that $\frac{p}{p-1}>3$ and hence $W^{1,\frac{p}{p-1}}(\Omega)\hookrightarrow L^\infty(\Omega)$, so that we can pick $C_3>0$ such that

$$\|\varphi\|_{L^\infty(\Omega)} \le C_3\|\varphi\|_{W^{1,\frac{p}{p-1}}(\Omega)} \qquad \text{for all } \varphi\in W^{1,\frac{p}{p-1}}(\Omega;\mathbb{R}^3). \tag{4.9}$$

Now on the basis of (4.5), for arbitrary $\varphi \in C_0^\infty(\Omega \times (0,T);\mathbb{R}^3)$ fulfilling $\nabla \cdot \varphi = 0$ we can integrate by parts and use (4.8) and (4.9) to estimate

$$\begin{aligned}
\Big|\int_0^T \int_\Omega \nabla P \cdot \varphi\Big| &= \Big|\int_0^T \int_\Omega v_t \cdot \varphi + \int_0^T \int_\Omega \nabla v \cdot \nabla \varphi - \int_0^T \int_\Omega f \cdot \varphi\Big| \\
&= \Big|\int_0^T \int_\Omega A^{-\alpha} v_t \cdot A^\alpha \varphi + \int_0^T \int_\Omega \nabla v \cdot \nabla \varphi - \int_0^T \int_\Omega f \cdot \varphi\Big| \\
&\le \int_0^T \|A^{-\alpha} v_t(\cdot,t)\|_{L^q(\Omega)} \|A^\alpha \varphi(\cdot,t)\|_{L^{\frac{q}{q-1}}(\Omega)} dt \\
&\quad + \int_0^T \|\nabla v(\cdot,t)\|_{L^p(\Omega)} \|\nabla \varphi(\cdot,t)\|_{L^{\frac{p}{p-1}}(\Omega)} dt \\
&\quad + \int_0^T \|f(\cdot,t)\|_{L^1(\Omega)} \|\varphi(\cdot,t)\|_{L^\infty(\Omega)} dt \\
&\le C_2 \int_0^T \|A^{-\alpha} v_t(\cdot,t)\|_{L^q(\Omega)} \|\varphi(\cdot,t)\|_{W^{1,\frac{p}{p-1}}(\Omega)} dt \\
&\quad + \int_0^T \|v(\cdot,t)\|_{W^{1,p}(\Omega)} \|\varphi\|_{W^{1,\frac{p}{p-1}}(\Omega)} dt \\
&\quad + C_3 \int_0^T \|f(\cdot,t)\|_{L^1(\Omega)} \|\varphi(\cdot,t)\|_{W^{1,\frac{p}{p-1}}(\Omega)} dt \\
&\le \Big\{ C_2 \|A^{-\alpha} v_t\|_{L^r((0,T);L^q(\Omega))} + T^{\frac{1}{r}} \|v\|_{L^\infty((0,T);W^{1,p}(\Omega))} \\
&\qquad\qquad + C_3 T^{\frac{1}{r}} \|f\|_{L^\infty((0,T);L^1(\Omega))} \Big\} \cdot \|\varphi\|_{L^{\frac{r}{r-1}}((0,T);W^{1,\frac{p}{p-1}}(\Omega))}.
\end{aligned}$$

Therefore, a well-known result on pressure regularity in the Stokes system ([28, Lemma IV 1.4.1], cf. also [28, Lemma II 2.1.1 and Lemma II 2.2.2]) says that with some $C_4 = C_4(p,r,T) > 0$ we have

$$\|P\|_{L^r((0,T);L^p(\Omega))} \le C_4 \cdot \Big\{ \|A^{-\alpha} v_t\|_{L^r((0,T);L^q(\Omega))} + \|v\|_{L^\infty((0,T);W^{1,p}(\Omega))} + \|f\|_{L^\infty((0,T);L^1(\Omega))} \Big\}. \tag{4.10}$$

Here recalling Lemma 4.1, we obtain $C_5 = C_5(p) > 0$ such that

$$\|v\|_{L^\infty((0,T);W^{1,p}(\Omega))} \le C_5 \cdot \Big\{ \|v_0\|_{W^{1,2}(\Omega)} + \|f\|_{L^\infty((0,T);L^1(\Omega))} \Big\}, \tag{4.11}$$

whereas observing that (4.1) implies that $\partial_t A^{-\alpha} v + A(A^{-\alpha} v) = A^{-\alpha} \mathcal{P} f$ in $\Omega \times (0,T)$ allows us to invoke a standard result on maximal Sobolev regularity in the Stokes evolution equation ([15]) to fix $C_6 = C_6(p,r,T) > 0$ satisfying

$$\|A^{-\alpha} v_t\|_{L^r((0,T);L^q(\Omega))} \le C_6 \cdot \Big\{ \|A^{-\alpha} v_0\|_{W^{2,q}(\Omega)} + \|A^{-\alpha} \mathcal{P} f\|_{L^r((0,T);L^q(\Omega))} \Big\}. \tag{4.12}$$

Since clearly $\|A^{-\alpha} v_0\|_{W^{2,q}(\Omega)} \le C_7 \|v_0\|_{W^{2,\frac{3}{2}}(\Omega)}$ with some $C_7 = C_7(p) > 0$, and since

$$\begin{aligned}
\|A^{-\alpha} \mathcal{P} f\|_{L^r((0,T);L^q(\Omega))} &\le T^{\frac{1}{r}} \|A^{-\alpha} \mathcal{P} f\|_{L^\infty((0,T);L^q(\Omega))} \\
&\le C_1 T^{\frac{1}{r}} \|f\|_{L^\infty((0,T);L^1(\Omega))}
\end{aligned}$$

according to (4.7), inserting (4.11) and (4.12) into (4.10) yields (4.6). $\Box$

Let us explicitly extract two immediate conclusions from Lemma 4.1 and Lemma 4.2 in a form in which they will be referred to below.

Lemma 4.3 *Let $p \in (1, \frac{3}{2})$. Then there exists $C(p) > 0$ such that for each $\varepsilon \in (0,1)$,*

$$\|u_\varepsilon(\cdot,t)\|_{L^{2p}(\Omega)} + \|\nabla u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le C(p) \qquad \text{for all } t>0. \tag{4.13}$$

Proof. In view of (2.12), (1.7), this is actually a by-product of Lemma 4.1. $\Box$

Lemma 4.4 *Let $r > 1$ and $p \in (1, \frac{3}{2})$. Then for all $T > 0$ there exists $C(p,r,T) > 0$ such that*

$$\int_0^T \|P_\varepsilon(\cdot,t)\|_{L^p(\Omega)}^r dt \le C(p,r,T) \qquad \text{for all } \varepsilon \in (0,1). \tag{4.14}$$

Proof. This follows directly from Lemma 4.2, (1.7) and (2.12). $\Box$

5 Estimates for time derivatives. Construction of a limit (n, c, u)

Before proceeding to the main part of our analysis, let us utilize the estimates for $n_\varepsilon, c_\varepsilon$ and u_ε from Lemma 2.2, Lemma 3.1, Lemma 3.2 and Lemma 4.3 to construct a candidate (n, c, u) for a solution to (1.5) via an appropriate subsequence extraction. A last preparation therefor is provided by the following statement on regularity with respect to the time variable.

Lemma 5.1 *For all $T > 0$ there exists $C(T) > 0$ such that*

$$\int_0^T \Big\|\partial_t \ln\Big(n_\varepsilon(\cdot,t)+1\Big)\Big\|_{(W_0^{2,2}(\Omega))^\star} dt \le C(T) \qquad \text{for all } \varepsilon \in (0,1) \tag{5.1}$$

and

$$\int_0^T \|c_{\varepsilon t}(\cdot,t)\|^2_{(W_0^{1,4}(\Omega))^\star} dt \le C(T) \qquad \text{for all } \varepsilon \in (0,1). \tag{5.2}$$

Proof. Proceeding in a standard manner, for fixed $\varphi \in C_0^\infty(\Omega)$ and $t > 0$ we integrate by parts in (2.7) and use the Cauchy-Schwarz inequality and again the identity $\nabla \cdot u_\varepsilon = 0$ to obtain that

$$\begin{aligned}
\Big|\int_\Omega \partial_t \ln(n_\varepsilon+1)\cdot\varphi\Big| &= \Big| \int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2}\varphi - \int_\Omega \frac{\nabla n_\varepsilon}{n_\varepsilon+1}\cdot\nabla\varphi \\
&\quad - \int_\Omega \frac{n_\varepsilon F_\varepsilon'(n_\varepsilon)}{(n_\varepsilon+1)^2}(\nabla n_\varepsilon\cdot\nabla c_\varepsilon)\varphi + \int_\Omega \frac{n_\varepsilon F_\varepsilon'(n_\varepsilon)}{n_\varepsilon+1}\nabla c_\varepsilon\cdot\nabla\varphi \\
&\quad + \int_\Omega \ln(n_\varepsilon+1)(u_\varepsilon\cdot\nabla\varphi)\Big| \\
&\le \Big\{\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2}\Big\}\cdot\|\varphi\|_{L^\infty(\Omega)} + \Big\{\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2}\Big\}^{\frac12}\cdot\|\nabla\varphi\|_{L^2(\Omega)} \\
&\quad + \Big\{\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2}\Big\}^{\frac12}\cdot\Big\{\int_\Omega |\nabla c_\varepsilon|^2\Big\}^{\frac12}\cdot\|\varphi\|_{L^\infty(\Omega)} + \Big\{\int_\Omega |\nabla c_\varepsilon|^2\Big\}^{\frac12}\cdot\|\nabla\varphi\|_{L^2(\Omega)} \\
&\quad + \Big\{\int_\Omega \ln^3(n_\varepsilon+1)\Big\}^{\frac13}\cdot\|u_\varepsilon\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^6(\Omega)} \qquad \text{for all } \varepsilon\in(0,1).
\end{aligned}$$

As $W^{2,2}(\Omega) \hookrightarrow W^{1,6}(\Omega) \hookrightarrow L^\infty(\Omega)$, this entails the existence of $C_1 > 0$ such that due to Young's inequality, for all $\varepsilon \in (0,1)$ and $t > 0$ we have

$$
\begin{aligned}
\Big\|\partial_t \ln(n_\varepsilon+1)\Big\|_{(W_0^{2,2}(\Omega))^\star} \le\ & C_1 \cdot \Bigg\{ \int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} + \Bigg\{ \int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} \Bigg\}^{\frac{1}{2}} \\
& + \Bigg\{ \int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} \Bigg\}^{\frac{1}{2}} \cdot \Bigg\{ \int_\Omega |\nabla c_\varepsilon|^2 \Bigg\}^{\frac{1}{2}} + \Bigg\{ \int_\Omega |\nabla c_\varepsilon|^2 \Bigg\}^{\frac{1}{2}} \\
& + \Bigg\{ \int_\Omega \ln^3(n_\varepsilon+1) \Bigg\}^{\frac{1}{3}} \|u_\varepsilon\|_{L^2(\Omega)} \Bigg\} \\
\le\ & C_1 \cdot \Bigg\{ 2\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2} + \int_\Omega |\nabla c_\varepsilon|^2 + \Bigg\{ \int_\Omega \ln^3(n_\varepsilon+1) \Bigg\}^{\frac{1}{3}} \|u_\varepsilon\|_{L^2(\Omega)} + 1 \Bigg\},
\end{aligned}
$$

which in view of Lemma 3.2, Lemma 3.1, Lemma 4.3 and (2.12) entails (5.1) upon a time integration, because $\ln^3(s+1) \le \frac{54}{e^2}(s+1)$ for all $s \ge 0$.

Likewise, for $\varphi \in C_0^\infty(\Omega)$, $t > 0$ and $\varepsilon \in (0,1)$ we can estimate

$$
\begin{aligned}
\Bigg| \int_\Omega c_{\varepsilon t}\varphi \Bigg| =\ & \Bigg| - \int_\Omega \nabla c_\varepsilon \cdot \nabla\varphi - \int_\Omega F_\varepsilon(n_\varepsilon) c_\varepsilon \varphi - \int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon)\varphi \Bigg| \\
\le\ & \Bigg\{ \int_\Omega |\nabla c_\varepsilon|^2 \Bigg\}^{\frac{1}{2}} \|\nabla\varphi\|_{L^2(\Omega)} + \|n_\varepsilon\|_{L^1(\Omega)} \|c_\varepsilon\|_{L^\infty(\Omega)} \|\varphi\|_{L^\infty(\Omega)} \\
& + \|u_\varepsilon\|_{L^2(\Omega)} \cdot \Bigg\{ \int_\Omega |\nabla c_\varepsilon|^2 \Bigg\}^{\frac{1}{2}} \cdot \|\varphi\|_{L^\infty(\Omega)}
\end{aligned}
$$

and use that $W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$ and $W^{1,4}(\Omega) \hookrightarrow W^{1,2}(\Omega)$ to find $C_2 > 0$ such that consequently

$$
\|c_{\varepsilon t}\|^2_{(W_0^{1,4}(\Omega))^\star} \le C_2 \cdot \Bigg\{ \int_\Omega |\nabla c_\varepsilon|^2 + \|n_\varepsilon\|^2_{L^1(\Omega)} \|c_\varepsilon\|^2_{L^\infty(\Omega)} + \|u_\varepsilon\|^2_{L^2(\Omega)} \int_\Omega |\nabla c_\varepsilon|^2 \Bigg\}
$$

for all $\varepsilon \in (0,1)$ and $t > 0$, hence implying (5.2) through Lemma 3.1, (2.12), (2.13) and Lemma 4.3. $\square$

The following conclusion thereby becomes rather straightforward:

Lemma 5.2 *There exist functions n, c and u such that (1.9) holds, that $n \ge 0$, $c \ge 0$ and $\nabla \cdot u = 0$ a.e. in $\Omega \times (0,\infty)$, that (1.10) and (2.6) are satisfied, and such that with some $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ fulfilling $\varepsilon_j \searrow 0$ as $j \to \infty$ we have*

$$n_\varepsilon \to n \qquad a.e.\ in\ \Omega \times (0,\infty), \tag{5.3}$$
$$\ln(n_\varepsilon+1) \to \ln(n+1) \qquad in\ L^2_{loc}(\overline{\Omega} \times [0,\infty)), \tag{5.4}$$
$$\nabla \ln(n_\varepsilon+1) \rightharpoonup \nabla \ln(n+1) \qquad in\ L^2_{loc}(\overline{\Omega} \times [0,\infty)), \tag{5.5}$$
$$c_\varepsilon \to c \qquad in\ L^2_{loc}(\overline{\Omega} \times [0,\infty))\ and\ a.e.\ in\ \Omega \times (0,\infty), \tag{5.6}$$
$$c_\varepsilon \stackrel{\star}{\rightharpoonup} c \qquad in\ L^\infty_{loc}(\overline{\Omega} \times [0,\infty)), \tag{5.7}$$

$$\nabla c_\varepsilon \rightharpoonup \nabla c \qquad \text{in } L^2_{loc}(\overline{\Omega}\times[0,\infty)), \tag{5.8}$$
$$u_\varepsilon \stackrel{\star}{\rightharpoonup} u \qquad \text{in } L^\infty((0,\infty);L^p(\Omega)) \quad \text{for all } p\in(1,3) \qquad \text{as well as} \tag{5.9}$$
$$\nabla u_\varepsilon \stackrel{\star}{\rightharpoonup} \nabla u \qquad \text{in } L^\infty((0,\infty);L^p(\Omega)) \quad \text{for all } p\in(1,\frac{3}{2}). \tag{5.10}$$

PROOF. All statements can readily be seen by means of a straightforward extraction procedure based on the estimates provided by (2.12), (2.13), Lemma 3.1, Lemma 3.2, Lemma 4.3 and the relative compactness of $((\ln(n_\varepsilon+1))_{\varepsilon\in(0,1)}$ and $(c_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^2_{loc}(\overline{\Omega}\times[0,\infty))$, as thereby implied through an Aubin-Lions type lemma ([31]). $\Box$

6 Local energy. Precompactness of $(n_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^1_{loc}(\Omega\times[0,\infty))$

6.1 Collecting prospective constituents for a local energy inequality

Next approaching the core of our analysis, we prepare our study of a suitably localized variant of (1.3) by collecting some basic evolution properties of respectively localized versions of the integrals making up the quasi-energy functional therein. Our first observation in this regard concerns the corresponding logarithmic entropy.

Lemma 6.1 *Let $\chi\in C_0^\infty(\Omega)$ and $\varepsilon\in(0,1)$. Then*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \chi n_\varepsilon \ln n_\varepsilon + \int_\Omega \chi\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} \\
&\quad= \int_\Omega \chi F_\varepsilon'(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla c_\varepsilon - \int_\Omega (\ln n_\varepsilon+1)\nabla n_\varepsilon\cdot\nabla\chi \\
&\qquad -\int_\Omega c_\varepsilon\cdot\Big\{(\ln n_\varepsilon+2)F_\varepsilon'(n_\varepsilon)+n_\varepsilon(\ln n_\varepsilon+1)F_\varepsilon''(n_\varepsilon)\Big\}\nabla n_\varepsilon\cdot\nabla\chi - \int_\Omega n_\varepsilon(\ln n_\varepsilon+1)F_\varepsilon'(n_\varepsilon)c_\varepsilon\Delta\chi \\
&\qquad +\int_\Omega n_\varepsilon\ln n_\varepsilon(u_\varepsilon\cdot\nabla\chi) \qquad \text{for all } t>0.
\end{aligned} \tag{6.1}$$

PROOF. We integrate by parts in (2.7) to see that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \chi n_\varepsilon\ln n_\varepsilon &= \int_\Omega \chi(\ln n_\varepsilon+1)\nabla\cdot\Big\{\nabla n_\varepsilon - n_\varepsilon F_\varepsilon'(n_\varepsilon)\nabla c_\varepsilon\Big\} - \int_\Omega \chi(\ln n_\varepsilon+1)(u_\varepsilon\cdot\nabla n_\varepsilon) \\
&= -\int_\Omega \Big\{\chi\frac{\nabla n_\varepsilon}{n_\varepsilon}+(\ln n_\varepsilon+1)\nabla\chi\Big\}\cdot\Big\{\nabla n_\varepsilon - n_\varepsilon F_\varepsilon'(n_\varepsilon)\nabla c_\varepsilon\Big\} \\
&\quad -\int_\Omega \chi\Big\{u_\varepsilon\cdot\nabla(n_\varepsilon\ln n_\varepsilon)\Big\} \\
&= -\int_\Omega \chi\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \int_\Omega \chi F_\varepsilon'(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla c_\varepsilon \\
&\quad -\int_\Omega (\ln n_\varepsilon+1)\nabla n_\varepsilon\cdot\nabla\chi + \int_\Omega n_\varepsilon(\ln n_\varepsilon+1)F_\varepsilon'(n_\varepsilon)\nabla c_\varepsilon\cdot\nabla\chi \\
&\quad +\int_\Omega n_\varepsilon\ln n_\varepsilon(u_\varepsilon\cdot\nabla\chi) \qquad \text{for all } t>0,
\end{aligned}$$

because $\nabla \cdot u_\varepsilon = 0$. As herein another integration by parts shows that

$$\int_\Omega n_\varepsilon(\ln n_\varepsilon + 1)F'_\varepsilon(n_\varepsilon)\nabla c_\varepsilon \cdot \nabla\chi = -\int_\Omega c_\varepsilon \nabla\Big\{n_\varepsilon(\ln n_\varepsilon+1)F'_\varepsilon(n_\varepsilon)\Big\}\cdot\nabla\chi - \int_\Omega n_\varepsilon(\ln n_\varepsilon+1)F'_\varepsilon(n_\varepsilon)c_\varepsilon\Delta\chi$$

for all $t > 0$, computing

$$\frac{d}{ds}\Big\{s(\ln s+1)F'_\varepsilon(s)\Big\} = (\ln s+2)F'_\varepsilon(s) + s(\ln s+1)F''_\varepsilon(s), \qquad s>0,$$

we immediately obtain (6.1) from this. □

Next, the evolution of a localized version of the Fisher information functional in (1.3) can be described as follows.

Lemma 6.2 *If $\chi \in C_0^\infty(\Omega)$, then for any $\varepsilon \in (0,1)$,*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \chi\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \int_\Omega \chi c_\varepsilon |D^2\ln c_\varepsilon|^2 \\
&\quad= -\int_\Omega \chi F'_\varepsilon(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla c_\varepsilon - \frac12\int_\Omega \chi F_\varepsilon(n_\varepsilon)\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} - \int_\Omega \chi\frac{1}{c_\varepsilon}\nabla c_\varepsilon\cdot(\nabla u_\varepsilon\cdot\nabla c_\varepsilon) \\
&\qquad -\int_\Omega \frac{1}{c_\varepsilon}(D^2c_\varepsilon\cdot\nabla c_\varepsilon)\cdot\nabla\chi + \frac12\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(\nabla c_\varepsilon\cdot\nabla\chi) \\
&\qquad +\frac12\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}(u_\varepsilon\cdot\nabla\chi) \qquad \textit{for all } t>0.
\end{aligned} \tag{6.2}$$

Proof. Using the pointwise identities $\nabla c_\varepsilon\cdot\nabla\Delta c_\varepsilon = \frac12\Delta|\nabla c_\varepsilon|^2 - |D^2c_\varepsilon|^2$ and $\nabla|\nabla c_\varepsilon|^2 = 2D^2c_\varepsilon\cdot\nabla c_\varepsilon$, on the basis of (2.7) and several integrations by parts we compute

$$\begin{aligned}
\frac12\frac{d}{dt}\int_\Omega\chi\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} &= \int_\Omega \chi\frac{\nabla c_\varepsilon}{c_\varepsilon}\cdot\nabla\Big\{\Delta c_\varepsilon - F_\varepsilon(n_\varepsilon)c_\varepsilon - u_\varepsilon\cdot\nabla c_\varepsilon\Big\} - \frac12\int_\Omega\chi\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}\Big\{\Delta c_\varepsilon - F_\varepsilon(n_\varepsilon)c_\varepsilon - u_\varepsilon\cdot\nabla c_\varepsilon\Big\} \\
&= \frac12\int_\Omega\chi\frac{1}{c_\varepsilon}\Delta|\nabla c_\varepsilon|^2 - \int_\Omega\chi\frac{|D^2c_\varepsilon|^2}{c_\varepsilon} - \int_\Omega\chi F'_\varepsilon(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla c_\varepsilon - \int_\Omega\chi F_\varepsilon(n_\varepsilon)\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} \\
&\quad -\int_\Omega\chi\frac{1}{c_\varepsilon}\nabla c_\varepsilon\cdot(\nabla u_\varepsilon\cdot\nabla c_\varepsilon) - \frac12\int_\Omega\chi\frac{1}{c_\varepsilon}(u_\varepsilon\cdot\nabla|\nabla c_\varepsilon|^2) \\
&\quad +\int_\Omega\chi\frac{1}{c_\varepsilon^2}\nabla c_\varepsilon\cdot(D^2c_\varepsilon\cdot\nabla c_\varepsilon) - \int_\Omega\chi\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac12\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}\nabla c_\varepsilon\cdot\nabla\chi \\
&\quad +\frac12\int_\Omega\chi F_\varepsilon(n_\varepsilon)\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \frac12\int_\Omega\chi\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon\cdot\nabla c_\varepsilon) \\
&= \int_\Omega\chi\frac{1}{c_\varepsilon^2}\nabla c_\varepsilon\cdot(D^2c_\varepsilon\cdot\nabla c_\varepsilon) - \int_\Omega\frac{1}{c_\varepsilon}(D^2c_\varepsilon\cdot\nabla c_\varepsilon)\cdot\nabla\chi - \int_\Omega\chi\frac{|D^2c_\varepsilon|^2}{c_\varepsilon} \\
&\quad -\int_\Omega\chi F'_\varepsilon(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla c_\varepsilon - \frac12\int_\Omega\chi F_\varepsilon(n_\varepsilon)\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} - \int_\Omega\chi\frac{1}{c_\varepsilon}\nabla c_\varepsilon\cdot(\nabla u_\varepsilon\cdot\nabla c_\varepsilon) \\
&\quad -\frac12\int_\Omega\chi\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon\cdot\nabla c_\varepsilon) + \frac12\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}(u_\varepsilon\cdot\nabla\chi)
\end{aligned}$$

$$
\begin{aligned}
& +\int_\Omega \chi \frac{1}{c_\varepsilon^2} \nabla c_\varepsilon \cdot (D^2 c_\varepsilon \cdot \nabla c_\varepsilon) - \int_\Omega \chi \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{1}{2} \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2} \nabla c_\varepsilon \cdot \nabla \chi \\
& + \frac{1}{2} \int_\Omega \chi \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2} (u_\varepsilon \cdot \nabla c_\varepsilon) \\
= \quad & - \int_\Omega \chi \frac{|D^2 c_\varepsilon|^2}{c_\varepsilon} + 2 \int_\Omega \chi \frac{1}{c_\varepsilon^2} \nabla c_\varepsilon \cdot (D^2 c_\varepsilon \cdot \nabla c_\varepsilon) - \int_\Omega \chi \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \\
& - \int_\Omega \chi F_\varepsilon'(n_\varepsilon) \nabla n_\varepsilon \cdot \nabla c_\varepsilon - \frac{1}{2} \int_\Omega \chi F_\varepsilon(n_\varepsilon) \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} \\
& - \int_\Omega \chi \frac{1}{c_\varepsilon} \nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon) \\
& - \int_\Omega \frac{1}{c_\varepsilon} (D^2 c_\varepsilon \cdot \nabla c_\varepsilon) \cdot \nabla \chi + \frac{1}{2} \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2} \nabla c_\varepsilon \cdot \nabla \chi + \frac{1}{2} \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} (u_\varepsilon \cdot \nabla \chi)
\end{aligned}
$$

for all $t > 0$. As

$$
c_\varepsilon |D^2 \ln c_\varepsilon|^2 = \frac{|D^2 c_\varepsilon|^2}{c_\varepsilon} - 2 \frac{\nabla c_\varepsilon \cdot (D^2 c_\varepsilon \cdot \nabla c_\varepsilon)}{{c_\varepsilon}^2} + \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \qquad \text{in } \Omega \times (0, \infty),
$$

this is equivalent to (6.2). $\square$

Finally, localizing in the corresponding fluid part does involve the pressure in a natural manner:

Lemma 6.3 *Let $\chi \in C_0^\infty(\Omega)$. Then*

$$
\frac{1}{2} \frac{d}{dt} \int_\Omega \chi |u_\varepsilon|^2 + \int_\Omega \chi |\nabla u_\varepsilon|^2 = \int_\Omega \chi F_\varepsilon(n_\varepsilon)(u_\varepsilon \cdot \nabla \phi) - \int_\Omega u_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla \chi) - \int_\Omega P_\varepsilon (u_\varepsilon \cdot \nabla \chi) \qquad (6.3)
$$

for all $t > 0$ and $\varepsilon \in (0,1)$.

Proof. On integrating by parts, from the third equation in (2.7) we directly obtain that again since $\nabla \cdot u_\varepsilon = 0$,

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_\Omega \chi |u_\varepsilon|^2 &= \int_\Omega \chi u_\varepsilon \cdot \Big\{ \Delta u_\varepsilon + \nabla P_\varepsilon + F_\varepsilon(n_\varepsilon) \nabla \phi \Big\} \\
&= - \int_\Omega \chi |\nabla u_\varepsilon|^2 - \int_\Omega u_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla \chi) - \int_\Omega P_\varepsilon (u_\varepsilon \cdot \nabla \chi) + \int_\Omega \chi F_\varepsilon(n_\varepsilon)(u_\varepsilon \cdot \nabla \phi)
\end{aligned}
$$

for all $t > 0$, as claimed. $\square$

6.2 Estimating the right-hand sides of (6.1), (6.2) and (6.3)

In order to prepare an appropriate estimation of the ill-signed contributions to (6.1), (6.2) and (6.3), let us state the result of a straightforward application of the Hölder and the Sobolev inequality that will explicitly be referred to in Lemma 6.5 and Lemma 6.7.

Lemma 6.4 *Let $\zeta \in C^\infty(\overline{\Omega})$ be nonnegative, and suppose that $m \in (0, 12)$ and $q \geq \frac{m}{4}$ are such that $q \leq 1 + \frac{m}{6}$. Then there exists $C(m,q) > 0$ such that*

$$
\int_\Omega \zeta^m n_\varepsilon^q \leq C(m,q) \cdot \bigg\{ \int_\Omega \zeta^4 \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} \bigg\}^{\frac{m}{4}} + C(m,q) \qquad (6.4)
$$

for all $t>0$ and $\varepsilon\in(0,1)$.

PROOF. We first use the Hölder inequality to see that thanks to (2.12),

$$\begin{aligned}\int_\Omega \zeta^m n_\varepsilon^q &= \int_\Omega (\zeta^2\sqrt{n_\varepsilon})^{\frac{m}{2}}\cdot n_\varepsilon^{\frac{4q-m}{4}}\\ &\le |\Omega|^{\frac{m-6q+6}{6}}\|n_\varepsilon\|_{L^1(\Omega)}^{\frac{4q-m}{4}}\|\zeta^2\sqrt{n_\varepsilon}\|_{L^6(\Omega)}^{\frac{m}{2}} \qquad \text{for all } t>0, \end{aligned} \tag{6.5}$$

because $4q-m\ge 0$ and $\frac{\frac{m}{2}}{6}+\frac{4q-m}{4}\le 1$ according to our restrictions that $q\ge\frac{m}{4}$ and $q\le 1+\frac{m}{6}$. Here a Sobolev inequality provides $C_1>0$ such that for all $\varepsilon\in(0,1)$,

$$\begin{aligned}\|\zeta^2\sqrt{n_\varepsilon}\|_{L^6(\Omega)}^2 &\le C_1\int_\Omega\Big|\nabla(\zeta^2\sqrt{n_\varepsilon})\Big|^2+C_1\int_\Omega(\zeta^2\sqrt{n_\varepsilon})^2\\ &= C_1\int_\Omega\Big|\frac{1}{2}\zeta^2\frac{\nabla n_\varepsilon}{\sqrt{n_\varepsilon}}+2\zeta\sqrt{n_\varepsilon}\nabla\zeta\Big|^2+C_1\int_\Omega\zeta^4 n_\varepsilon\\ &\le \frac{1}{2}C_1\int_\Omega\zeta^4\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon}+8C_1\|\zeta\nabla\zeta\|_{L^\infty(\Omega)}^2\int_\Omega n_\varepsilon+C_1\|\zeta\|_{L^\infty(\Omega)}^4\int_\Omega n_\varepsilon \qquad \text{for all } t>0.\end{aligned}$$

Once more by means of (2.12), combining this with (6.5) yields (6.4) □

A first application thereof indeed enables us to appropriately relate the right-hand side in (6.1) to the dissipated quantity on the left, provided that the weight function therein can be written as the fourth power of a suitably smooth function.

Lemma 6.5 *Let $\zeta\in C^\infty(\overline{\Omega})$ be nonnegative. Then for each $\eta>0$ one can find $C(\eta)>0$ such that*

$$\begin{aligned}&-\int_\Omega(\ln n_\varepsilon+1)\nabla n_\varepsilon\cdot\nabla\zeta^4-\int_\Omega c_\varepsilon\cdot\Big\{(\ln n_\varepsilon+2)F_\varepsilon'(n_\varepsilon)+n_\varepsilon(\ln n_\varepsilon+1)F_\varepsilon''(n_\varepsilon)\Big\}\nabla n_\varepsilon\cdot\nabla\zeta^4\\ &\qquad -\int_\Omega n_\varepsilon(\ln n_\varepsilon+1)F_\varepsilon'(n_\varepsilon)c_\varepsilon\Delta\zeta^4+\int_\Omega n_\varepsilon\ln n_\varepsilon(u_\varepsilon\cdot\nabla\zeta^4)\\ &\quad\le \eta\int_\Omega\zeta^4\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon}+C(\eta)\qquad \textit{for all } t>0 \textit{ and } \varepsilon\in(0,1).\end{aligned}\tag{6.6}$$

PROOF. By means of Young's inequality and (2.12), for all $t>0$ we can estimate

$$\begin{aligned}-\int_\Omega(\ln n_\varepsilon+1)\nabla n_\varepsilon\cdot\nabla\zeta^4 &= -4\int_\Omega\zeta^3(\ln n_\varepsilon+1)\nabla n_\varepsilon\cdot\nabla\zeta\\ &\le \frac{\eta}{4}\int_\Omega\zeta^4\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon}+\frac{16}{\eta}\int_\Omega\zeta^2|\nabla\zeta|^2 n_\varepsilon\Big(|\ln n_\varepsilon|+1\Big)^2\\ &\le \frac{\eta}{4}\int_\Omega\zeta^4\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon}\\ &\quad+\frac{32}{\eta}\|\nabla\zeta\|_{L^\infty(\Omega)}^2\int_\Omega\zeta^2 n_\varepsilon\ln^2 n_\varepsilon+\frac{32}{\eta}\|\zeta\nabla\zeta\|_{L^\infty(\Omega)}^2\int_\Omega n_\varepsilon,\end{aligned}\tag{6.7}$$

and using (2.13) and the inequalities $|F_\varepsilon'(s)| \le 1$ and $|sF_\varepsilon''(s)| = \frac{\varepsilon s}{(1+\varepsilon s)^2} \le \frac{1}{4}$, $s \ge 0$, we similarly obtain that

$$\begin{aligned}
&-\int_\Omega c_\varepsilon \cdot \Big\{(\ln n_\varepsilon + 2)F_\varepsilon'(n_\varepsilon) + n_\varepsilon(\ln n_\varepsilon + 1)F_\varepsilon''(n_\varepsilon)\Big\}\nabla n_\varepsilon \cdot \nabla\zeta^4 \\
&\quad\le M\int_\Omega \Big\{|\ln n_\varepsilon| + 2 + \frac{1}{4}(|\ln n_\varepsilon| + 1)\Big\}|\nabla n_\varepsilon| \cdot |\nabla\zeta^4| \\
&\quad= M\int_\Omega \zeta^3\Big(5|\ln n_\varepsilon| + 9\Big)|\nabla n_\varepsilon| \cdot |\nabla\zeta| \\
&\quad\le \frac{\eta}{4}\int_\Omega \zeta^4\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \frac{M^2}{\eta}\int_\Omega \zeta^2|\nabla\zeta|^2 n_\varepsilon\Big(5|\ln n_\varepsilon| + 9\Big)^2 \\
&\quad\le \frac{\eta}{4}\int_\Omega \zeta^4\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \frac{50M^2}{\eta}\|\nabla\zeta\|_{L^\infty(\Omega)}^2\int_\Omega \zeta^2 n_\varepsilon \ln^2 n_\varepsilon + \frac{162M^2}{\eta}\|\zeta\nabla\zeta\|_{L^\infty(\Omega)}^2\int_\Omega n_\varepsilon \qquad (6.8)
\end{aligned}$$

for all $t > 0$. Since $\Delta\zeta^4 = 4\zeta^3\Delta\zeta + 12\zeta^2|\nabla\zeta|^2$ in Ω, we furthermore see that

$$\begin{aligned}
&-\int_\Omega n_\varepsilon(\ln n_\varepsilon + 1)F_\varepsilon'(n_\varepsilon)c_\varepsilon\Delta\zeta^4 \\
&\qquad\le M \cdot \Big\{4\|\zeta\Delta\zeta\|_{L^\infty(\Omega)} + 12\|\nabla\zeta\|_{L^\infty(\Omega)}^2\Big\}\int_\Omega \zeta^2 n_\varepsilon(|\ln n_\varepsilon| + 1) \qquad \text{for all } t > 0, \qquad (6.9)
\end{aligned}$$

and in order to finally estimate the rightmost summand on the left of (6.6), we fix any $p \in (\frac{3}{2}, 2)$ and use the Hölder inequality to obtain that

$$\begin{aligned}
-\int_\Omega n_\varepsilon \ln n_\varepsilon(u_\varepsilon \cdot \nabla\zeta^4) &= -4\int_\Omega \zeta^3 n_\varepsilon \ln n_\varepsilon(u_\varepsilon \cdot \nabla\zeta) \\
&\le 4\|\nabla\zeta\|_{L^\infty(\Omega)}\|u_\varepsilon\|_{L^{\frac{p}{p-1}}(\Omega)} \cdot \Big\{\int_\Omega \zeta^{3p} n_\varepsilon^p |\ln n_\varepsilon|^p\Big\}^{\frac{1}{p}} \qquad \text{for all } t > 0. \quad (6.10)
\end{aligned}$$

Collecting (6.7)-(6.10), we thus infer the existence of $C_1(\eta) > 0$ such that for any choice of $\varepsilon \in (0,1)$,

$$\begin{aligned}
&-\int_\Omega (\ln n_\varepsilon + 1)\nabla n_\varepsilon \cdot \nabla\zeta^4 - \int_\Omega \cdot\Big\{(\ln n_\varepsilon + 2)F_\varepsilon'(n_\varepsilon) + n_\varepsilon(\ln n_\varepsilon + 1)F_\varepsilon''(n_\varepsilon)\Big\}\nabla n_\varepsilon \cdot \nabla\zeta^4 \\
&\qquad - \int_\Omega n_\varepsilon(\ln n_\varepsilon + 1)F_\varepsilon'(n_\varepsilon)c_\varepsilon\Delta\zeta^4 + \int_\Omega n_\varepsilon \ln n_\varepsilon(u_\varepsilon \cdot \nabla\zeta^4) \\
&\quad\le \frac{3\eta}{4}\int_\Omega \zeta^4\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + C_1(\eta)\int_\Omega \zeta^2\Big(n_\varepsilon \ln^2 n_\varepsilon + n_\varepsilon|\ln n_\varepsilon| + n_\varepsilon\Big) \\
&\qquad + C_1(\eta)\|u_\varepsilon\|_{L^{\frac{p}{p-1}}(\Omega)} \cdot \Big\{\int_\Omega \zeta^{3p} n_\varepsilon^p |\ln n_\varepsilon|^p\Big\}^{\frac{1}{p}} \qquad \text{for all } t > 0. \qquad (6.11)
\end{aligned}$$

Here to control the second summand on the right, we take $C_2 > 0$ large enough such that $s\ln^2 s + s|\ln s| + s \le C_2 s^{\frac{4}{3}} + 1$ for all $s > 0$, and then invoke Lemma 6.4 and Young's inequality to find $C_3(\eta) > 0$ fulfilling

$$C_1(\eta)\int_\Omega \zeta^2\Big(n_\varepsilon \ln^2 n_\varepsilon + n_\varepsilon|\ln n_\varepsilon| + n_\varepsilon\Big)$$

$$\begin{aligned}
&\le C_1(\eta)C_2\int_\Omega \zeta^2 n_\varepsilon^{\frac{4}{3}} + C_1(\eta)\|\zeta\|_{L^2(\Omega)}^2 \\
&\le C_3(\eta)\cdot\Big\{\int_\Omega \zeta^4\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon}\Big\}^{\frac12} + C_3(\eta) \\
&\le \frac{\eta}{8}\int_\Omega \zeta^4\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \frac{2C_3^2(\eta)}{\eta} + C_3(\eta) \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1).
\end{aligned} \tag{6.12}$$

In the last summand of (6.11), we use that the restriction $p>\frac32$ warrants that $\frac{p}{p-1}<3$, which through Lemma 4.3 entails the existence of $C_4>0$ satisfying

$$\|u_\varepsilon\|_{L^{\frac{p}{p-1}}(\Omega)} \le C_4 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1). \tag{6.13}$$

Furthermore, due to the inequality $p<2$ we have $p<1+\frac{p}{2}$, whence it is possible to fix $q>p$ such that $q\le 1+\frac{p}{2}$. Then since $q>p$, there exists $C_5>0$ such that $s^p|\ln s|^p\le C_5 s^q+1$ for all $s>0$, so that using that $m:=3p$ satisfies $m<6<12$ as well as $\frac{m}{4}=\frac{3p}{4}<p<q\le 1+\frac{p}{2}=1+\frac{m}{6}$, once more relying on Lemma 6.4 we see that with some $C_6>0$,

$$\begin{aligned}
\Big\{\int_\Omega \zeta^{3p} n_\varepsilon^p|\ln n_\varepsilon|^p\Big\}^{\frac1p} &\le \Big\{C_5\int_\Omega \zeta^{3p}n_\varepsilon^q + \int_\Omega \zeta^{3p}\Big\}^{\frac1p} \\
&\le \Big\{C_6\cdot\Big\{\int_\Omega \zeta^4\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon}\Big\}^{\frac{3p}{4}} + C_6\Big\}^{\frac1p} \\
&\le (2C_6)^{\frac1p}\cdot\Big\{\zeta^4\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon}\Big\}^{\frac34} + (2C_6)^{\frac1p} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1).
\end{aligned}$$

In conjunction with (6.13) and again Young's inequality, this shows that there exists $C_7(\eta)>0$ with the property that whenever $\varepsilon\in(0,1)$,

$$\begin{aligned}
&C_1(\eta)\|u_\varepsilon\|_{L^{\frac{p}{p-1}}(\Omega)}\cdot\Big\{\int_\Omega \zeta^{3p}n_\varepsilon^p|\ln n_\varepsilon|^p\Big\}^{\frac1p} \\
&\qquad\le \frac{\eta}{8}\int_\Omega \zeta^4\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + C_7(\eta) \qquad \text{for all } t>0,
\end{aligned}$$

and thereby establishes (6.6) when combined with (6.12) and (6.11). □

In estimating the right-hand side of (6.2), we basically only need to suitably combine Young's inequality with (2.13) and Lemma 4.3.

Lemma 6.6 *Let $\zeta\in C^\infty(\overline{\Omega})$ be such that $\zeta\ge 0$ in Ω. Then for all $\eta>0$ one can find $C(\eta)>0$ such that for all $t>0$ and $\varepsilon\in(0,1)$,*

$$\begin{aligned}
&-\int_\Omega \zeta^4\frac{1}{c_\varepsilon}\nabla c_\varepsilon\cdot(\nabla u_\varepsilon\cdot\nabla c_\varepsilon) - \int_\Omega \frac{1}{c_\varepsilon}(D^2c_\varepsilon\cdot\nabla c_\varepsilon)\cdot\nabla\zeta^4 + \frac12\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}\nabla c_\varepsilon\cdot\nabla\zeta^4 \\
&\qquad + \frac12\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}(u_\varepsilon\cdot\nabla\zeta^4) \\
&\quad\le \eta\int_\Omega \zeta^4\frac{|D^2c_\varepsilon|^2}{c_\varepsilon} + \eta\int_\Omega \zeta^4\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + C(\eta)\int_\Omega \zeta^4|\nabla u_\varepsilon|^2 + C(\eta).
\end{aligned} \tag{6.14}$$

PROOF. By Young's inequality and (2.13),

$$
\begin{aligned}
-\int_\Omega \zeta^4 \frac{1}{c_\varepsilon} \nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon) &\le \frac{\eta}{4} \int_\Omega \zeta^4 \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{1}{\eta} \int_\Omega \zeta^4 c_\varepsilon |\nabla u_\varepsilon|^2 \\
&\le \frac{\eta}{4} \int_\Omega \zeta^4 \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{M}{\eta} \int_\Omega \zeta^4 |\nabla u_\varepsilon|^2 \qquad \text{for all } t>0 \qquad (6.15)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{2} \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} (u_\varepsilon \cdot \nabla \zeta^4) &= 2 \int_\Omega \zeta^3 \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} (u_\varepsilon \cdot \nabla \zeta) \\
&\le \frac{\eta}{4} \int_\Omega \zeta^4 \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{4}{\eta} \int_\Omega \zeta^2 |\nabla \zeta|^2 c_\varepsilon |u_\varepsilon|^2 \\
&\le \frac{\eta}{4} \int_\Omega \zeta^4 \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{4M}{\eta} \|\zeta \nabla \zeta\|_{L^\infty(\Omega)}^2 \|u_\varepsilon\|_{L^2(\Omega)}^2 \qquad \text{for all } t>0. \quad (6.16)
\end{aligned}
$$

Moreover, by the same tokens,

$$
\begin{aligned}
-\int_\Omega \frac{1}{c_\varepsilon} (D^2 c_\varepsilon \cdot \nabla c_\varepsilon) \cdot \nabla \zeta^4 &= -4 \int_\Omega \zeta^3 \frac{1}{c_\varepsilon} (D^2 c_\varepsilon \cdot \nabla c_\varepsilon) \cdot \nabla \zeta \\
&\le \eta \int_\Omega \zeta^4 \frac{|D^2 c_\varepsilon|^2}{c_\varepsilon} + \frac{4}{\eta} \int_\Omega \zeta^2 |\nabla \zeta|^2 \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} \\
&\le \eta \int_\Omega \zeta^4 \frac{|D^2 c_\varepsilon|^2}{c_\varepsilon} + \frac{\eta}{4} \int_\Omega \zeta^4 \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{16}{\eta^3} \int_\Omega |\nabla \zeta|^4 c_\varepsilon \\
&\le \eta \int_\Omega \zeta^4 \frac{|D^2 c_\varepsilon|^2}{c_\varepsilon} + \frac{\eta}{4} \int_\Omega \zeta^4 \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{16M}{\eta^3} \|\nabla \zeta\|_{L^4(\Omega)}^4 \quad \text{for all } t>0 (6.17)
\end{aligned}
$$

as well as

$$
\begin{aligned}
\frac{1}{2} \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2} \nabla c_\varepsilon \cdot \nabla \zeta^4 &= 2 \int_\Omega \zeta^3 \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2} \nabla c_\varepsilon \cdot \nabla \zeta \\
&\le \frac{\eta}{8} \int_\Omega \zeta^4 \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{8}{\eta} \int_\Omega \zeta^2 |\nabla \zeta|^2 \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} \\
&\le \frac{\eta}{4} \int_\Omega \zeta^4 \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{128}{\eta^3} \int_\Omega |\nabla \zeta|^4 c_\varepsilon \\
&\le \frac{\eta}{4} \int_\Omega \zeta^4 \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{128M}{\eta^3} \|\nabla \zeta\|_{L^4(\Omega)}^4 \qquad \text{for all } t>0. \qquad (6.18)
\end{aligned}
$$

Since $\sup_{\varepsilon\in(0,1)} \sup_{t>0} \|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}$ is finite according to Lemma 4.3, (6.15)-(6.18) directly lead to (6.14). □

The following estimate concerned with (6.3) now again relies on Lemma 6.4.

Lemma 6.7 *For any nonnegative $\zeta \in C^\infty(\overline{\Omega})$ and each $\eta > 0$, there exists $C(\eta) > 0$ such that for all $\varepsilon \in (0,1)$,*

$$\begin{aligned}
&\int_\Omega \zeta^4 F_\varepsilon(n_\varepsilon)(u_\varepsilon \cdot \nabla\phi) - \int_\Omega u_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla\zeta^4) - \int_\Omega P_\varepsilon(u_\varepsilon \cdot \nabla\zeta^4) \\
&\qquad \le \eta \int_\Omega \zeta^4 |\nabla u_\varepsilon|^2 + \eta \int_\Omega \zeta^4 \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + C(\eta)\|P_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} + C(\eta) \qquad \textit{for all } t > 0. \qquad (6.19)
\end{aligned}$$

PROOF. Once more thanks to Young's inequality,

$$\begin{aligned}
-\int_\Omega u_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla\zeta^4) &= -4\int_\Omega \zeta^3 u_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla\zeta) \\
&\le \frac{\eta}{2}\int_\Omega \zeta^4|\nabla u_\varepsilon|^2 + \frac{8}{\eta}\int_\Omega \zeta^2|\nabla\zeta|^2|u_\varepsilon|^2 \\
&\le \frac{\eta}{2}\int_\Omega \zeta^4|\nabla u_\varepsilon|^2 + C_1(\eta) \qquad \text{for all } t > 0, \qquad (6.20)
\end{aligned}$$

with $C_1(\eta) := \frac{8}{\eta}\|\zeta\nabla\zeta\|^2_{L^\infty(\Omega)} \sup_{\varepsilon\in(0,1)} \sup_{t>0} \|u_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)}$ being finite by Lemma 4.3. Furthermore, choosing any $p \in (\frac{3}{2}, 3)$ and letting $m := 6(p-1)$, we use that then $4p - m = 6 - 2p > 0$ in employing the Hölder inequality to estimate

$$\int_\Omega \zeta^4 F_\varepsilon(n_\varepsilon)(u_\varepsilon \cdot \nabla\phi) \le \|\nabla\phi\|_{L^\infty(\Omega)}\|u_\varepsilon\|_{L^{\frac{p}{p-1}}(\Omega)}\|\zeta\|^{\frac{4p-m}{p}}_{L^\infty(\Omega)} \cdot \left\{\int_\Omega \zeta^m n_\varepsilon^p\right\}^{\frac{1}{p}} \qquad \text{for all } t > 0,$$

where $\sup_{\varepsilon\in(0,1)}\sup_{t>0}\|u_\varepsilon(\cdot,t)\|_{L^{\frac{p}{p-1}}(\Omega)} < \infty$ due to Lemma 4.3 and the fact that $\frac{p}{p-1} < 3$. As the restriction $p < 3$ ensures that $\frac{m}{4p} < 1$, noting that $p = 1 + \frac{m}{6}$ we may combine this with Lemma 6.4 and Young's inequality to find $C_2 > 0$ and $C_3(\eta) > 0$ satisfying

$$\begin{aligned}
\int_\Omega \zeta^4 F_\varepsilon(n_\varepsilon)(u_\varepsilon \cdot \nabla\phi) &\le C_2 \cdot \left\{\left\{\int_\Omega \zeta^4 \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon}\right\}^{\frac{m}{4}} + 1\right\}^{\frac{1}{p}} \\
&\le \eta\int_\Omega \zeta^4\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + C_3(\eta) \qquad \text{for all } t > 0. \qquad (6.21)
\end{aligned}$$

In the last term on the left-hand side of (6.19), we first use the Hölder inequality to see that writing $C_4 := 4\|\zeta\nabla\zeta\|_{L^\infty(\Omega)}$ we have

$$\begin{aligned}
-\int_\Omega P_\varepsilon(u_\varepsilon \cdot \nabla\zeta^4) &= -4\int_\Omega \zeta^3 P_\varepsilon(u_\varepsilon\cdot\nabla\zeta) \\
&\le C_4\|P_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}\|\zeta^2 u_\varepsilon\|_{L^6(\Omega)} \qquad \text{for all } t > 0,
\end{aligned}$$

where a Poincaré-Sobolev inequality yields $C_5 > 0$ such that for all $\varepsilon \in (0,1)$,

$$\begin{aligned}
\|\zeta^2 u_\varepsilon\|^2_{L^6(\Omega)} &\le C_5\int_\Omega \Big|\nabla(\zeta^2 u_\varepsilon)\Big|^2 \\
&= C_5\int_\Omega \Big|\zeta^2\nabla u_\varepsilon + 2\zeta(u_\varepsilon\cdot\nabla\zeta)\Big|^2 \\
&\le 2C_5\int_\Omega \zeta^4|\nabla u_\varepsilon|^2 + C_6 \qquad \text{for all } t > 0
\end{aligned}$$

due to Young's inequality, with $C_6 := 8C_5 \|\zeta\nabla\zeta\|^2_{L^\infty(\Omega)} \sup_{\varepsilon\in(0,1)} \sup_{t>0} \|u_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)} < \infty$ again by Lemma 4.3. Therefore, a final application of Young's inequality shows that

$$
\begin{aligned}
-\int_\Omega P_\varepsilon(u_\varepsilon\cdot\nabla\zeta^4) &\le \frac{\eta}{4C_5}\|\zeta^2 u_\varepsilon\|^2_{L^6(\Omega)} + \frac{C_4^2 C_5}{\eta}\|P_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} \\
&\le \frac{\eta}{2}\int_\Omega \zeta^4|\nabla u_\varepsilon|^2 + \frac{C_6\eta}{4C_5} + \frac{C_4^2C_5}{\eta}\|P_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} \qquad \text{for all } t>0,
\end{aligned}
$$

which together with (6.20) and (6.21) establishes (6.19). □

In order to adequately make use of the dissipation mechanism expressed in (6.2), let us note two functional inequalities which in quite a natural manner extend their corresponding spatially global counterparts that have already been relied on in previous literature on (1.1) (see, e.g., [37] and [40]).

Lemma 6.8 *Let $\zeta \in C_0^\infty(\Omega)$. Then for all $\xi\in C^2(\overline{\Omega})$ such that $\xi>0$ in Ω, we have*

$$
\int_\Omega \zeta^4\frac{|\nabla\xi|^4}{\xi^3} \le 28\int_\Omega \zeta^4\xi|D^2\ln\xi|^2 + 4096\int_\Omega |\nabla\zeta|^4\xi \tag{6.22}
$$

and

$$
\int_\Omega \zeta^4\frac{|D^2\xi|^2}{\xi} \le 58\int_\Omega \zeta^4\xi|D^2\ln\xi|^2 + 8192\int_\Omega |\nabla\zeta|^4\xi. \tag{6.23}
$$

PROOF. We first adapt an argument from [37, Lemma 3.3] to see integrating by parts and using Young's inequality that since $|\Delta\ln\xi|^2 \le 3|D^2\ln\xi|^2$ in Ω,

$$
\begin{aligned}
\int_\Omega \zeta^4\frac{|\nabla\xi|^4}{\xi^3} &= \int_\Omega \zeta^4|\nabla\ln\xi|^2\nabla\ln\xi\cdot\nabla\xi \\
&= -\int_\Omega \zeta^4\xi|\nabla\ln\xi|^2\Delta\ln\xi - 2\int_\Omega \zeta^4\xi\nabla\ln\xi\cdot(D^2\ln\xi\cdot\nabla\ln\xi) - 4\int_\Omega \zeta^3\xi|\nabla\ln\xi|^2\nabla\ln\xi\cdot\nabla\zeta \\
&= -\int_\Omega \zeta^4\frac{|\nabla\xi|^2}{\xi}\Delta\ln\xi - 2\int_\Omega \zeta^4\frac{\nabla\xi}{\xi}\cdot(D^2\ln\xi\cdot\nabla\xi) - 4\int_\Omega \zeta^3\frac{|\nabla\xi|^2}{\xi^2}\nabla\xi\cdot\nabla\zeta \\
&\le \frac{1}{8}\int_\Omega \zeta^4\frac{|\nabla\xi|^4}{\xi^3} + 2\int_\Omega \zeta^4\xi|\Delta\ln\xi|^2 \\
&\quad +\frac{1}{8}\int_\Omega \zeta^4\frac{|\nabla\xi|^4}{\xi^3} + 8\int_\Omega \zeta^4\xi|D^2\ln\xi|^2 \\
&\quad +\frac{1}{8}\int_\Omega \zeta^4\frac{|\nabla\xi|^4}{\xi^3} + 32\int_\Omega \zeta^2|\nabla\zeta|^2\frac{|\nabla\xi|^2}{\xi} \\
&\le \frac{3}{8}\int_\Omega \zeta^4\frac{|\nabla\xi|^4}{\xi^3} + 14\int_\Omega \zeta^4\xi|D^2\ln\xi|^2 + 32\Big\{\frac{1}{256}\int_\Omega \zeta^4\frac{|\nabla\xi|^4}{\xi^3} + \frac{256}{4}\int_\Omega |\nabla\zeta|^4\xi\Big\} \\
&\le \frac{1}{2}\int_\Omega \zeta^4\frac{|\nabla\xi|^4}{\xi^3} + 14\int_\Omega \zeta^4\xi|D^2\ln\xi|^2 + 2048\int_\Omega |\nabla\zeta|^4\xi,
\end{aligned}
$$

from which (6.22) follows.

As, again by Young's inequality,

$$
\xi|D^2\ln\xi|^2 = \frac{|D^2\xi|^2}{\xi} - 2\frac{\nabla\xi\cdot(D^2\xi\cdot\nabla\xi)}{\xi^2} + \frac{|\nabla\xi|^4}{\xi^3}
$$

$$\geq \frac{1}{2}\frac{|D^2\xi|^2}{\xi} - \frac{|\nabla\xi|^4}{\xi^3} \qquad \text{in } \Omega$$

and thus

$$\int_\Omega \zeta^4 \frac{|D^2\xi|^2}{\xi} \leq 2\int_\Omega \zeta^4 \xi |D^2 \ln\xi|^2 + 2\int_\Omega \zeta^4 \frac{|\nabla\xi|^4}{\xi^3},$$

from (6.22) we moreover readily obtain (6.23). $\square$

By suitably combining the above pieces of information, we can now proceed to derive the following on the basis of an argument based on correspondingly localized versions of (1.3).

Lemma 6.9 *Let $K \subset \Omega$ be compact and $T > 0$. Then there exists $C(K,T) > 0$ such that whenever $\varepsilon \in (0,1)$,*

$$\int_K n_\varepsilon(\cdot,t)\Big|\ln n_\varepsilon(\cdot,t)\Big| \leq C(K,T) \qquad \textit{for all } t \in (0,T). \tag{6.24}$$

PROOF. We fix a nonnegative $\zeta \in C_0^\infty(\Omega)$ such that $\zeta \equiv 1$ in K, and then obtain by combining Lemma 6.1 and Lemma 6.2 with Lemma 6.5, Lemma 6.6, Lemma 6.8 and (2.13) that there exist $C_1(K) > 0$ and $C_2(K) > 0$ such that for all $\varepsilon \in (0,1)$,

$$\begin{aligned}\frac{d}{dt}\Big\{\int_\Omega \zeta^4 n_\varepsilon \ln n_\varepsilon + \frac{1}{2}\int_\Omega \zeta^4 \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}\Big\} + \frac{1}{2}\int_\Omega \zeta^4 \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} \\ \leq \ C_1(K)\int_\Omega \zeta^4 |\nabla u_\varepsilon|^2 + C_2(K) \qquad \text{for all } t > 0.\end{aligned}$$

Since Lemma 6.3 in conjunction with Lemma 6.7 yields $C_3(K) > 0$ such that

$$C_1(K)\frac{d}{dt}\int_\Omega \zeta^4|u_\varepsilon|^2 + C_1(K)\int_\Omega \zeta^4|\nabla u_\varepsilon|^2 \leq \frac{1}{2}\int_\Omega \zeta^4\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + C_3(K)\|P_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} + C_3(K)$$

for all $t > 0$ and $\varepsilon \in (0,1)$, we thus infer that

$$\mathcal{F}_\varepsilon(t) := \int_\Omega \zeta^4 n_\varepsilon(\cdot,t)\ln n_\varepsilon(\cdot,t) + \frac{1}{2}\int_\Omega \zeta^4\frac{|\nabla c_\varepsilon(\cdot,t)|^2}{c_\varepsilon(\cdot,t)} + C_1(K)\int_\Omega \zeta^4 |u_\varepsilon(\cdot,t)|^2, \qquad t \geq 0,\ \varepsilon \in (0,1) \tag{6.25}$$

satisfies

$$\mathcal{F}'_\varepsilon(t) \leq C_2(K) + C_3(K) + C_3(K)\|P_\varepsilon(\cdot,t)\|^2_{L^{\frac{6}{5}}(\Omega)} \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1). \tag{6.26}$$

As Lemma 4.4 provides $C_4(T) > 0$ such that

$$\int_0^T \|P_\varepsilon(\cdot,t)\|^2_{L^{\frac{6}{5}}(\Omega)}\,dt \leq C_4(T) \qquad \text{for all } \varepsilon \in (0,1)$$

due to the fortunate circumstance that $\frac{6}{5} < \frac{3}{2}$, an integration of (6.26) shows that

$$\begin{aligned}\mathcal{F}_\varepsilon(t) \ \leq\ & \int_\Omega \zeta^4 n_0 \ln n_0 + \frac{1}{2}\int_\Omega \zeta^4\frac{|\nabla c_0|^2}{c_0} + C_1(K)\int_\Omega \zeta^4|u_0|^2 \\ & +(C_2(K) + C_3(K))\cdot T + C_3(K)C_4(T)\end{aligned}$$

for all $t \in (0,T)$ and each $\varepsilon \in (0,1)$. Since $s \ln s \ge -\frac{1}{e}$ for all $s > 0$ and hence

$$\mathcal{F}_\varepsilon(t) \ge \int_K \ln n_\varepsilon = \int_K n_\varepsilon |\ln n_\varepsilon| + 2\int_{\{n_\varepsilon < 1\}} n_\varepsilon \ln n_\varepsilon \ge \int_K n_\varepsilon |\ln n_\varepsilon| - \frac{2|\Omega|}{e}$$

for all $t > 0$ and $\varepsilon \in (0,1)$, this entails (6.24). $\square$

As a by-product, the latter strengthens the convergence statements concerning the first solution component in (5.3) as follows:

Corollary 6.10 *Let $(\varepsilon_j)_{j\in\mathbb{N}}$ be as in Lemma 5.2. Then*

$$n_\varepsilon \to n \quad \textit{and} \quad F_\varepsilon(n_\varepsilon) \to n \quad \textit{in } L^1_{loc}(\Omega \times [0,\infty)) \qquad \textit{as } \varepsilon = \varepsilon_j \searrow 0. \tag{6.27}$$

PROOF. As Lemma 6.9 guarantees uniform integrability of $(n_\varepsilon)_{\varepsilon\in(0,1)}$ and thus, through (2.9) and (2.10), also of $(F_\varepsilon(n_\varepsilon))_{\varepsilon\in(0,1)}$ over $K \times (0,T)$ for all compact $K \subset \Omega$ and any $T > 0$, in view of the Vitali convergence theorem this is a direct consequence of the pointwise convergence property in (5.3). $\square$

7 Solution properties of c, u and n. Proof of Theorem 1.1

Having at hand the key information gained in Corollary 6.10 now, in this section we can proceed to make sure that the limit triple gained in Lemma 5.2 indeed solves (1.5) in the sense specified in Definition 2.1. Already our first statement in this direction essentially relies on one of the strong convergence features in (6.27).

Lemma 7.1 *The identity (2.3) is valid for each $\varphi \in C_0^\infty(\Omega \times [0,\infty);\mathbb{R}^3)$ fulfilling $\nabla\cdot\varphi = 0$.*

PROOF. Given any such φ, from (2.7) we know that

$$-\int_0^\infty \int_\Omega u_\varepsilon \cdot \varphi_t - \int_\Omega u_0 \cdot \varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla u_\varepsilon \cdot \nabla\varphi + \int_0^\infty \int_\Omega F_\varepsilon(n_\varepsilon)(\varphi\cdot\nabla\phi) \tag{7.1}$$

for all $\varepsilon \in (0,1)$. Here the convergence properties (5.9) and (5.10) ensure that with $(\varepsilon_j)_{j\in\mathbb{N}}$ as provided by Lemma 5.2 we have

$$-\int_0^\infty \int_\Omega u_\varepsilon \cdot \varphi_t \to -\int_0^\infty \int_\Omega u \cdot \varphi_t \quad \text{and} \quad -\int_0^\infty \int_\Omega \nabla u_\varepsilon \cdot \nabla\varphi \to -\int_0^\infty \int_\Omega \nabla u \cdot \nabla\varphi$$

as $\varepsilon = \varepsilon_j \searrow 0$, and since $\operatorname{supp}\varphi$ is a compact subset of $\Omega \times [0,\infty)$, the local convergence feature asserted by Corollary 6.10 is sufficient to guarantee that furthermore

$$\int_0^\infty \int_\Omega F_\varepsilon(n_\varepsilon)(\varphi\cdot\nabla\phi) \to \int_0^\infty \int_\Omega n(\varphi\cdot\nabla\phi)$$

as $\varepsilon = \varepsilon_j \searrow 0$. The claim therefore results from (7.1). $\square$

Also our verification of (2.2) makes substantial use of Corollary 6.10.

Lemma 7.2 *Let $\varphi \in C_0^\infty(\Omega \times [0,\infty))$. Then (2.2) holds.*

PROOF. According to (2.7), for all $\varepsilon \in (0,1)$ we have

$$-\int_0^\infty \int_\Omega c_\varepsilon \varphi_t - \int_\Omega c_0 \varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla c_\varepsilon \cdot \nabla\varphi - \int_0^\infty \int_\Omega F_\varepsilon(n_\varepsilon) c_\varepsilon \varphi + \int_0^\infty \int_\Omega c_\varepsilon (u_\varepsilon \cdot \nabla\varphi) \qquad (7.2)$$

where due to (5.6), (5.8) and (5.9),

$$\begin{aligned}
&-\int_0^\infty \int_\Omega c_\varepsilon \varphi_t \to -\int_0^\infty \int_\Omega c\varphi_t, \\
&-\int_0^\infty \int_\Omega \nabla c_\varepsilon \cdot \nabla\varphi \to -\int_0^\infty \int_\Omega \nabla c \cdot \nabla\varphi \qquad \text{and} \\
&\int_0^\infty \int_\Omega c_\varepsilon (u_\varepsilon \cdot \nabla\varphi) \to \int_0^\infty \int_\Omega c(u \cdot \nabla\varphi)
\end{aligned}$$

as $\varepsilon = \varepsilon_j \searrow 0$, with $(\varepsilon_j)_{j\in\mathbb{N}}$ as in Lemma 5.2. Since moreover, by compactness of $\operatorname{supp}\varphi \subset \Omega \times [0,\infty)$, Corollary 6.10 in conjunction with (5.7) warrants that also

$$-\int_0^\infty \int_\Omega F_\varepsilon(n_\varepsilon) c_\varepsilon \varphi \to -\int_0^\infty \int_\Omega nc\varphi$$

as $\varepsilon = \varepsilon_j \searrow 0$, from (7.2) we directly obtain (2.2). □

Our derivation of the corresponding solution properties of n in Definition 2.1, though now independent of Corollary 6.10, is significantly more involved and requires a careful choice of the renormalization functions ψ and ρ appearing therein:

Lemma 7.3 *Let*

$$\psi(s) := \frac{1}{s+1} \quad \textit{and} \quad \rho(s) := e^{s^2 - 2c_\star s} \qquad \textit{for } s \geq 0. \qquad (7.3)$$

Then (2.4) is satisfied, and (2.5) holds for each nonnegative $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$.

PROOF. Using that

$$\psi'(s) = -\frac{1}{(s+1)^2} \quad \text{and} \quad \psi''(s) = \frac{2}{(s+1)^3} \qquad \text{for all } s \geq 0, \qquad (7.4)$$

for each $T > 0$ we can estimate

$$\begin{aligned}
\int_0^T \int_\Omega \Big\{ |\psi''(n)| + \psi'^2(n) + n^2 \psi''^2(n) \Big\} |\nabla n|^2 &= \int_0^T \int_\Omega \Big\{ \frac{2}{(n+1)^3} + \frac{1}{(n+1)^4} + \frac{4n^2}{(n+1)^6} \Big\} |\nabla n|^2 \\
&\leq 7 \int_0^T \int_\Omega \frac{|\nabla n|^2}{(n+1)^2}
\end{aligned}$$

and

$$\int_0^T \int_\Omega n^2 \psi'^2(n) |\nabla c|^2 \leq \int_0^T \int_\Omega |\nabla c|^2,$$

so that (2.4) is a consequence of Lemma 3.2, Lemma 3.1 and Lemma 5.2.

To verify (2.5) for fixed nonnegative $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$, we go back to (2.7) and integrate by parts several times to see that due to (7.4),

$$
\begin{aligned}
-\int_0^\infty \int_\Omega \psi(n_\varepsilon)\rho(c_\varepsilon)\varphi_t - \int_\Omega \psi(n_0)\rho(c_0)\varphi(\cdot,0) \\
= \quad & \int_0^\infty \int_\Omega \partial_t\Big(\psi(n_\varepsilon)\rho(c_\varepsilon)\Big)\varphi \\
= \quad & \int_0^\infty \int_\Omega \psi'(n_\varepsilon)\rho(c_\varepsilon)\varphi \cdot \Big\{\Delta n_\varepsilon - \nabla\cdot(n_\varepsilon F_\varepsilon'(n_\varepsilon)\nabla c_\varepsilon) - u_\varepsilon\cdot\nabla n_\varepsilon\Big\} \\
& + \int_0^\infty \int_\Omega \psi(n_\varepsilon)\rho'(c_\varepsilon)\varphi \cdot \Big\{\Delta c_\varepsilon - F_\varepsilon(n_\varepsilon)c_\varepsilon - u_\varepsilon\cdot\nabla c_\varepsilon\Big\} \\
= \quad & -\int_0^\infty \int_\Omega \psi''(n_\varepsilon)\rho(c_\varepsilon)|\nabla n_\varepsilon|^2\varphi - \int_0^\infty \int_\Omega \psi'(n_\varepsilon)\rho'(c_\varepsilon)(\nabla n_\varepsilon\cdot\nabla c_\varepsilon)\varphi \\
& + \int_0^\infty \int_\Omega n_\varepsilon F_\varepsilon'(n_\varepsilon)\psi''(n_\varepsilon)\rho(c_\varepsilon)(\nabla n_\varepsilon\cdot\nabla c_\varepsilon)\varphi + \int_0^\infty \int_\Omega n_\varepsilon F_\varepsilon'(n_\varepsilon)\psi'(n_\varepsilon)\rho'(c_\varepsilon)|\nabla c_\varepsilon|^2\varphi \\
& - \int_0^\infty \int_\Omega \psi'(n_\varepsilon)\rho'(c_\varepsilon)(\nabla n_\varepsilon\cdot\nabla c_\varepsilon)\varphi - \int_0^\infty \int_\Omega \psi(n_\varepsilon)\rho''(c_\varepsilon)|\nabla c_\varepsilon|^2\varphi \\
& - \int_0^\infty \int_\Omega F_\varepsilon(n_\varepsilon)\psi(n_\varepsilon)c_\varepsilon\rho'(c_\varepsilon)\varphi \\
& - \int_0^\infty \int_\Omega \psi'(n_\varepsilon)\rho(c_\varepsilon)\nabla n_\varepsilon\cdot\nabla\varphi + \int_0^\infty \int_\Omega n_\varepsilon F_\varepsilon'(n_\varepsilon)\psi'(n_\varepsilon)\rho(c_\varepsilon)\nabla c_\varepsilon\cdot\nabla\varphi \\
& - \int_0^\infty \int_\Omega \psi(n_\varepsilon)\rho'(c_\varepsilon)\nabla c_\varepsilon\cdot\nabla\varphi + \int_0^\infty \int_\Omega \psi(n_\varepsilon)\rho(c_\varepsilon)(u_\varepsilon\cdot\nabla\varphi) \\
= \quad & -2\int_0^\infty \int_\Omega \rho(c_\varepsilon)\frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^3}\varphi \\
& +2\int_0^\infty \int_\Omega \Big\{\rho'(c_\varepsilon) + \frac{n_\varepsilon F_\varepsilon'(n_\varepsilon)}{n_\varepsilon+1}\rho(c_\varepsilon)\Big\}\frac{\nabla n_\varepsilon\cdot\nabla c_\varepsilon}{(n_\varepsilon+1)^2}\varphi \\
& - \int_0^\infty \int_\Omega \Big\{\rho''(c_\varepsilon) + \frac{n_\varepsilon F_\varepsilon'(n_\varepsilon)}{n_\varepsilon+1}\rho'(c_\varepsilon)\Big\}\frac{|\nabla c_\varepsilon|^2}{n_\varepsilon+1}\varphi \\
& - \int_0^\infty \int_\Omega \frac{F_\varepsilon(n_\varepsilon)}{n_\varepsilon+1}c_\varepsilon\rho'(c_\varepsilon)\varphi \\
& + \int_0^\infty \int_\Omega \frac{\rho(c_\varepsilon)}{(n_\varepsilon+1)^2}\nabla n_\varepsilon\cdot\nabla\varphi \\
& - \int_0^\infty \int_\Omega \Big\{\frac{n_\varepsilon F_\varepsilon'(n_\varepsilon)}{(n_\varepsilon+1)^2}\rho(c_\varepsilon) + \frac{\rho'(c_\varepsilon)}{n_\varepsilon+1}\Big\}\nabla c_\varepsilon\cdot\nabla\varphi \\
& + \int_0^\infty \int_\Omega \frac{\rho(c_\varepsilon)}{n_\varepsilon+1}(u_\varepsilon\cdot\nabla\varphi) \qquad \text{for all } \varepsilon\in(0,1),
\end{aligned}
\tag{7.5}
$$

because $\nabla\cdot u_\varepsilon = 0$, and because $\rho'(c_\varepsilon) = 0$ on $\partial\Omega\times(0,\infty)$ due to (2.7) and the fact that by (7.3), $\rho'(s) = 2(s-c_\star)e^{s^2-2c_\star s}$ for $s \ge 0$ and hence $\rho'(c_\star) = 0$. Furthermore computing $\rho''(s) = \{2+4(s-c_\star)^2\}e^{s^2-2c_\star s}$

for $s \ge 0$ and observing that thus

$$\begin{aligned}
h_{1,\varepsilon}(x,t) &:= \rho''(c_\varepsilon) - \frac{1}{2}\frac{\rho'^2(c_\varepsilon)}{\rho(c_\varepsilon)} - \frac{1}{2}\frac{n_\varepsilon^2 F_\varepsilon'^2(n_\varepsilon)}{(n_\varepsilon+1)^2}\rho(c_\varepsilon) \\
&= \Big\{2 + 4(c_\varepsilon - c_\star)^2 - \frac{1}{2}\cdot 4(c_\varepsilon - c_\star)^2 - \frac{1}{2}\frac{n_\varepsilon^2 F_\varepsilon'^2(n_\varepsilon)}{(n_\varepsilon+1)^2}\Big\} e^{c_\varepsilon^2 - 2c_\star c_\varepsilon} \\
&= \Big\{2 + 2(c_\varepsilon - c_\star)^2 - \frac{1}{2}\frac{n_\varepsilon^2 F_\varepsilon'^2(n_\varepsilon)}{(n_\varepsilon+1)^2}\Big\} e^{c_\varepsilon^2 - 2c_\star c_\varepsilon}, \qquad (x,t) \in \Omega \times (0,\infty), \qquad (7.6)
\end{aligned}$$

we rewrite

$$\begin{aligned}
&-2\rho(c_\varepsilon)\frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^3} + 2\Big\{\rho'(c_\varepsilon) + \frac{n_\varepsilon F_\varepsilon'(n_\varepsilon)}{n_\varepsilon+1}\rho(c_\varepsilon)\Big\}\frac{\nabla n_\varepsilon \cdot \nabla c_\varepsilon}{(n_\varepsilon+1)^2} - \Big\{\rho''(c_\varepsilon) + \frac{n_\varepsilon F_\varepsilon'(n_\varepsilon)}{n_\varepsilon+1}\rho'(c_\varepsilon)\Big\}\frac{|\nabla c_\varepsilon|^2}{n_\varepsilon+1} \\
&\quad = -2\left|\sqrt{\rho(c_\varepsilon)}\frac{\nabla n_\varepsilon}{(n_\varepsilon+1)^{\frac{3}{2}}} - \Big\{\frac{\rho'(c_\varepsilon)}{2\sqrt{\rho(c_\varepsilon)}} + \frac{1}{2}\frac{n_\varepsilon F_\varepsilon'(n_\varepsilon)}{n_\varepsilon+1}\sqrt{\rho(c_\varepsilon)}\Big\}\frac{\nabla c_\varepsilon}{(n_\varepsilon+1)^{\frac{1}{2}}}\right|^2 - \frac{h_{1,\varepsilon}}{n_\varepsilon+1}|\nabla c_\varepsilon|^2
\end{aligned}$$

in $\Omega \times (0,\infty)$ and obtain from (7.5) and again (7.3) that if we moreover define

$$h_{2,\varepsilon}(x,t) := \frac{\rho'(c_\varepsilon)}{2\sqrt{\rho(c_\varepsilon)}} + \frac{1}{2}\frac{n_\varepsilon F_\varepsilon'(n_\varepsilon)}{n_\varepsilon+1}\sqrt{\rho(c_\varepsilon)}, \qquad (x,t) \in \Omega \times (0,\infty), \qquad (7.7)$$

and

$$h_{3,\varepsilon}(x,t) := \frac{n_\varepsilon F_\varepsilon'(n_\varepsilon)}{(n_\varepsilon+1)^2}\rho(c_\varepsilon) + \frac{\rho'(c_\varepsilon)}{n_\varepsilon+1}, \qquad (x,t) \in \Omega \times (0,\infty), \qquad (7.8)$$

then

$$\begin{aligned}
2\int_0^\infty \int_\Omega &\left|\sqrt{\rho(c_\varepsilon)}\frac{\nabla n_\varepsilon}{(n_\varepsilon+1)^{\frac{3}{2}}} - \frac{h_{2,\varepsilon}}{(n_\varepsilon+1)^{\frac{1}{2}}}\nabla c_\varepsilon\right|^2 \varphi + \int_0^\infty \int_\Omega \frac{h_{1,\varepsilon}}{n_\varepsilon+1}|\nabla c_\varepsilon|^2 \varphi \\
= &\int_0^\infty \int_\Omega \frac{\rho(c_\varepsilon)}{n_\varepsilon+1}\varphi_t + \int_\Omega \frac{\rho(c_0)}{n_0+1}\varphi(\cdot,0) \\
&- \int_0^\infty \int_\Omega \frac{F_\varepsilon(n_\varepsilon)}{n_\varepsilon+1} c_\varepsilon \rho'(c_\varepsilon)\varphi \\
&+ \int_0^\infty \int_\Omega \frac{\rho(c_\varepsilon)}{(n_\varepsilon+1)^2}\nabla n_\varepsilon \cdot \nabla\varphi \\
&- \int_0^\infty \int_\Omega h_{3,\varepsilon}\nabla c_\varepsilon \cdot \nabla\varphi \\
&+ \int_0^\infty \int_\Omega \frac{\rho(c_\varepsilon)}{n_\varepsilon+1}(u_\varepsilon \cdot \nabla\varphi) \qquad \text{for all } \varepsilon \in (0,1). \qquad (7.9)
\end{aligned}$$

Now taking $(\varepsilon_j)_{j\in\mathbb{N}}$ as provided by Lemma 5.2, from (5.3), (5.6) and (2.13) we readily infer that

$$\int_0^\infty \int_\Omega \frac{\rho(c_\varepsilon)}{n_\varepsilon+1}\varphi_t \to \int_0^\infty \int_\Omega \frac{\rho(c)}{n+1}\varphi_t = \int_0^\infty \int_\Omega \psi(n)\rho(c)\varphi_t \qquad (7.10)$$

and

$$-\int_0^\infty \int_\Omega \frac{F_\varepsilon(n_\varepsilon)}{n_\varepsilon+1} c_\varepsilon \rho'(c_\varepsilon)\varphi \to -\int_0^\infty \int_\Omega \frac{n}{n+1} c\rho'(c)\varphi = -\int_0^\infty \int_\Omega n\psi(n) c\rho'(c)\varphi \qquad (7.11)$$

as $\varepsilon = \varepsilon_j \searrow 0$, while additionally invoking (5.5) and (5.9) we see that

$$\int_0^\infty \int_\Omega \frac{\rho(c_\varepsilon)}{n_\varepsilon + 1}(u_\varepsilon \cdot \nabla\varphi) \to \int_0^\infty \int_\Omega \frac{\rho(c)}{n+1}(u \cdot \nabla\varphi) = \int_0^\infty \int_\Omega \psi(n)\rho(c)(u \cdot \nabla\varphi) \tag{7.12}$$

and

$$\begin{aligned} \int_0^\infty \int_\Omega \frac{\rho(c_\varepsilon)}{(n_\varepsilon+1)^2}\nabla n_\varepsilon \cdot \nabla\varphi &= \int_0^\infty \int_\Omega \frac{\rho(c_\varepsilon)}{n_\varepsilon+1}\nabla \ln(n_\varepsilon+1) \cdot \nabla\varphi \\ &\to \int_0^\infty \int_\Omega \frac{\rho(c)}{n+1}\nabla \ln(n+1) \cdot \nabla\varphi \\ &= -\int_0^\infty \int_\Omega \psi'(n)\rho(c)\nabla n \cdot \nabla\varphi \end{aligned} \tag{7.13}$$

as $\varepsilon = \varepsilon_j \searrow 0$. Since (5.3), (5.6) and (2.13) apart from that assert that $\Big(\frac{\sqrt{\rho(c_\varepsilon)}}{(n_\varepsilon+1)^{\frac{1}{2}}}\Big)_{\varepsilon\in(0,1)}$, $\Big(\frac{h_{1,\varepsilon}}{n_\varepsilon+1}\Big)_{\varepsilon\in(0,1)}$, $\Big(\frac{h_{2,\varepsilon}}{(n_\varepsilon+1)^{\frac{1}{2}}}\Big)_{\varepsilon\in(0,1)}$ and $(h_{3,\varepsilon})_{\varepsilon\in(0,1)}$ are bounded in $L^\infty(\Omega \times (0,\infty))$ and that a.e. in $\Omega \times (0,\infty)$ we have

$$\frac{\sqrt{\rho(c_\varepsilon)}}{(n_\varepsilon+1)^{\frac{1}{2}}} \to \frac{\sqrt{\rho(c)}}{(n+1)^{\frac{1}{2}}}$$

and

$$\frac{h_{1,\varepsilon}}{n_\varepsilon+1} \to \frac{h_1}{n+1} \qquad \text{with} \qquad h_1(x,t) := \rho''(c) - \frac{\rho'^2(c)}{2\rho(c)} - \frac{1}{2}\frac{n^2}{(n+1)^2}\rho(c), \quad (x,t) \in \Omega \times (0,\infty),$$

as well as

$$\frac{h_{2,\varepsilon}}{(n_\varepsilon+1)^{\frac{1}{2}}} \to \frac{h_2}{(n+1)^{\frac{1}{2}}} \qquad \text{with} \qquad h_2(x,t) := \frac{\rho'(c)}{2\sqrt{\rho(c)}} + \frac{1}{2}\frac{n}{n+1}\sqrt{\rho(c)}, \quad (x,t) \in \Omega \times (0,\infty),$$

and

$$h_{3,\varepsilon} \to \frac{n}{(n+1)^2}\rho(c) + \frac{\rho'(c)}{n+1}$$

as $\varepsilon = \varepsilon_j \searrow 0$, using (5.5), (5.8) and the fact that $h_{1,\varepsilon}$ is nonnegative due to (7.6) and (2.9), we see that furthermore

$$\begin{aligned} \sqrt{\rho(c_\varepsilon)}\frac{\nabla n_\varepsilon}{(n_\varepsilon+1)^{\frac{3}{2}}} &= \frac{\sqrt{\rho(c_\varepsilon)}}{(n_\varepsilon+1)^{\frac{1}{2}}}\nabla \ln(n_\varepsilon+1) \\ &\rightharpoonup \frac{\sqrt{\rho(c)}}{(n+1)^{\frac{1}{2}}}\nabla \ln(n+1) \\ &= \sqrt{\rho(c)}\frac{\nabla n}{(n+1)^{\frac{3}{2}}} \qquad \text{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)) \end{aligned} \tag{7.14}$$

and

$$\frac{h_{2,\varepsilon}}{(n_\varepsilon+1)^{\frac{1}{2}}}\nabla c_\varepsilon \rightharpoonup \frac{h_2}{(n+1)^{\frac{1}{2}}}\nabla c \qquad \text{in } L^2_{loc}(\overline{\Omega}\times[0,\infty)) \tag{7.15}$$

as well as

$$\sqrt{\frac{h_{1,\varepsilon}}{n_\varepsilon+1}}\nabla c_\varepsilon \rightharpoonup \sqrt{\frac{h_1}{n+1}}\nabla c \qquad \text{in } L^2_{loc}(\overline{\Omega}\times[0,\infty)) \tag{7.16}$$

and

$$h_{3,\varepsilon}\nabla c_\varepsilon \rightharpoonup \Big\{\frac{n}{(n+1)^2}\rho(c)+\frac{\rho'(c)}{n+1}\Big\}\nabla c \qquad \text{in } L^2_{loc}(\overline{\Omega}\times[0,\infty)) \tag{7.17}$$

as $\varepsilon=\varepsilon_j\searrow 0$. While (7.17) enables us to pass to the limit in the second last summand in (7.9), thereby deducing that

$$\begin{aligned}
-\int_0^\infty\int_\Omega h_{3,\varepsilon}\nabla c_\varepsilon\cdot\nabla\varphi &\to -\int_0^\infty\int_\Omega \Big\{\frac{n}{(n+1)^2}\rho(c)+\frac{\rho'(c)}{n+1}\Big\}\nabla c\cdot\nabla\varphi \\
&= \int_0^\infty\int_\Omega \Big\{n\psi'(n)\rho(c)-\psi(n)\rho'(c)\Big\}\nabla c\cdot\nabla\varphi
\end{aligned} \tag{7.18}$$

as $\varepsilon=\varepsilon_j\searrow 0$, on combining (7.14) with (7.15) and an argument based on lower semicontinuity of the norm in $L^2(\operatorname{supp}\varphi)$ with respect to weak convergence we infer that thanks to the nonnegativity of φ,

$$2\int_0^\infty\int_\Omega \Big|\sqrt{\rho(c)}\frac{\nabla n}{(n+1)^{\frac{3}{2}}}-\frac{h_2}{(n+1)^{\frac{1}{2}}}\nabla c\Big|^2\varphi \le \liminf_{\varepsilon=\varepsilon_j\searrow 0}\Bigg\{2\int_0^\infty\int_\Omega \Big|\sqrt{\rho(c_\varepsilon)}\frac{\nabla n_\varepsilon}{(n_\varepsilon+1)^{\frac{3}{2}}}-\frac{h_{2,\varepsilon}}{(n_\varepsilon+1)^{\frac{1}{2}}}\nabla c_\varepsilon\Big|^2\varphi\Bigg\}, \tag{7.19}$$

and similarly we obtain that moreover

$$\int_0^\infty\int_\Omega \frac{h_1}{n+1}|\nabla c|^2\varphi \le \liminf_{\varepsilon=\varepsilon_j\searrow 0}\int_0^\infty\int_\Omega \frac{h_{1,\varepsilon}}{n_\varepsilon+1}|\nabla c_\varepsilon|^2\varphi. \tag{7.20}$$

It remains to collect (7.10), (7.11), (7.12), (7.13), (7.18), (7.19) and (7.20) to conclude from (7.9) upon a straightforward rearrangement that indeed

$$\begin{aligned}
&\int_0^\infty\int_\Omega \psi''(n)\rho(c)|\nabla n|^2\varphi - \int_0^\infty\int_\Omega\Big\{-2\psi'(n)\rho'(c)+n\psi''(n)\rho(c)\Big\}(\nabla n\cdot\nabla c)\varphi \\
&\qquad -\int_0^\infty\int_\Omega\Big\{-\psi(n)\rho''(c)+n\psi'(n)\rho'(c)\Big\}|\nabla c|^2\varphi \\
&= 2\int_0^\infty\int_\Omega\Big|\sqrt{\rho(c)}\frac{\nabla n}{(n+1)^{\frac{3}{2}}}-\frac{h_2}{(n+1)^{\frac{1}{2}}}\nabla c\Big|^2\varphi + \int_0^\infty\int_\Omega\frac{h_1}{n+1}|\nabla c|^2\varphi \\
&\le \liminf_{\varepsilon=\varepsilon_j\searrow 0}\Bigg\{2\int_0^\infty\int_\Omega\Big|\sqrt{\rho(c_\varepsilon)}\frac{\nabla n_\varepsilon}{(n_\varepsilon+1)^{\frac{3}{2}}}-\frac{h_{2,\varepsilon}}{(n_\varepsilon+1)^{\frac{1}{2}}}\nabla c_\varepsilon\Big|^2\varphi + \int_0^\infty\int_\Omega\frac{h_{1,\varepsilon}}{n_\varepsilon+1}|\nabla c_\varepsilon|^2\varphi\Bigg\} \\
&= \liminf_{\varepsilon=\varepsilon_j\searrow 0}\Bigg\{\int_0^\infty\int_\Omega\frac{\rho(c_\varepsilon)}{n_\varepsilon+1}\varphi_t + \int_\Omega\frac{\rho(c_0)}{n_0+1}\varphi(\cdot,0) \\
&\qquad -\int_0^\infty\int_\Omega\frac{F_\varepsilon(n_\varepsilon)}{n_\varepsilon+1}c_\varepsilon\rho'(c_\varepsilon)\varphi
\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_0^\infty \int_\Omega \frac{\rho(c_\varepsilon)}{(n_\varepsilon+1)^2} \nabla n_\varepsilon \cdot \nabla\varphi - \int_0^\infty \int_\Omega h_{3,\varepsilon} \nabla c_\varepsilon \cdot \nabla\varphi \\
&\quad + \int_0^\infty \int_\Omega \frac{\rho(c_\varepsilon)}{n_\varepsilon+1} (u_\varepsilon \cdot \nabla\varphi) \Big\} \\
= &\int_0^\infty \int_\Omega \psi(n)\rho(c)\varphi_t + \int_\Omega \psi(n_0)\rho(c_0)\varphi(\cdot,0) \\
&- \int_0^\infty \int_\Omega n\psi(n) c\rho'(c)\varphi \\
&- \int_0^\infty \int_\Omega \psi'(n)\rho(c)\nabla n \cdot \nabla\varphi + \int_0^\infty \int_\Omega \Big\{ n\psi'(n)\rho(c) - \psi(n)\rho'(c) \Big\} \nabla c \cdot \nabla\varphi \\
&+ \int_0^\infty \int_\Omega \psi(n)\rho(c)(u\cdot\nabla\varphi),
\end{aligned}
$$

as intended. □

Our main result on global solvability in (1.5) has thereby actually been completed already:

Proof of Theorem 1.1. We take (n, c, u) as constructed in Lemma 5.2, and then only need to combine the statements from Lemma 5.2, Lemma 7.1, Lemma 7.2 and Lemma 7.3. □

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