Global weak solutions in a three-dimensional Keller-Segel-Navier-Stokes system with gradient-dependent flux limitation

Michael Winkler*
Institut für Mathematik, Universität Paderborn,
Warburger Str. 100, 33098 Paderborn, Germany

Abstract

In a bounded domain $\Omega \subset \mathbb{R}^3$, we are concerned with the evolution system

$$\begin{cases} n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot \Big(n f(|\nabla c|^2) \nabla c \Big), \\ c_t + u \cdot \nabla c &= \Delta c - c + n, \\ u_t + (u \cdot \nabla) u &= \Delta u + \nabla P + n \nabla \Phi, \qquad \nabla \cdot u = 0, \end{cases} \qquad (\star)$$

coupling the incompressible Navier-Stokes equations to a class of flux-limited Keller-Segel systems which has received noticeable attention in the recent biomathematical literature. When considered without such fluid interaction, no-flux boundary value problems for chemotaxis systems of the latter type are known to admit global bounded solutions for widely arbitrary initial data whenever f is a suitably smooth function fulfilling

$$|f(\xi)| \le K_f \cdot (\xi + 1)^{-\frac{\alpha}{2}} \qquad \text{for all } \xi \ge 0 \qquad (\star\star)$$

with some $K_f > 0$ and $\alpha > \frac{1}{2}$, while if here the converse inequality holds with some $K_f > 0$ and $\alpha < \frac{1}{2}$, then blow-up occurs at least in some simplified parabolic-elliptic counterpart.

The present work now asserts that the former condition remains sufficient to ensure global solvability in a corresponding initial-boundary value problem for the fully coupled system $(\star)$, within a natural weak solution concept consistent with those underlying well-established theories for the Navier-Stokes equations. This indicates that the saturation exponent $\alpha = \frac{1}{2}$ in $(\star\star)$ continues to play the role of a critical flux limitation parameter also in the presence of fluid interaction.

Key words: chemotaxis; Navier-Stokes; flux limitation
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*michael.winkler@math.uni-paderborn.de

1 Introduction

The coaction of chemotaxis systems with liquid environments has attracted considerable attention in the recent literature. Motivated by experimental findings which attest a significant relevance of chemotaxis-fluid couplings in various application contexts ([12], [13], [32], [37], [39]), several branches of mathematical research have been devoted to the understanding of possible influences generated by such interaction mechanisms, and beyond numerical evidence ([39], [31]), even some rigurous analytical studies have revealed noticeable effects which at least some suitably designed given fluid fields may exert on the qualitative behavior in Keller-Segel type cross-diffusion systems ([26], [27], [28], [22], [17]).

In comparison to the latter class of situations involving prescribed and hence essentially passive liquid surroundings, the knowledge seems much less comprehensive in frameworks when the fluid itself forms a system variable, e.g. due to buoyancy-induced influences of microbial movement on fluid motion ([39]). Indeed, already in cases of fairly dissipation-dominated chemotaxis systems in which the respective attractant is consumed by bacteria, the additional coupling to equations from fluid mechanics, as thus necessary for an adequate description in such situations ([39], [30]), seems to go along with substantial challenges due to which even at basic levels of existence and qualitative analysis, the development of satisfactory theories apparently required a considerable history in the literature ([15], [14], [29], [16], [10], [30], [46], [49], [48], [50]).

This issue seems to gain yet further cruciality in models linking fluid dynamics to chemotaxis systems in which, similar to situations addressed by classical Keller-Segel systems, already the cross-diffusive interaction itself bears significant destabilizing potential by accounting for signal evolution that is determined by production, rather than consumption, of the considered directing chemical through cells ([24], [47]). Especially in parameter settings near criticality of such systems, the nearby question how far their explosion-enforcing properties may be affected by fluid interaction quite naturally becomes increasingly demanding. Accordingly, when interacting with the (Navier-)Stokes system, the classical quasilinear Keller-Segel model involving density-dependent diffusion and cross-diffusion rates, as in this fully coupled form given by

$$\begin{cases} n_t + u\cdot\nabla n = \nabla\cdot\big(D(n)\nabla n\big) - \nabla\cdot\big(S(n)\nabla c\big), \\ c_t + u\cdot\nabla c = \Delta c - c + n, \\ u_t + \kappa(u\cdot\nabla)u = \Delta u + \nabla P + n\nabla\Phi, \qquad \nabla\cdot u = 0, \end{cases} \tag{1.1}$$

has yet not completely been understood with regard to the occurrence of blow-up phenomena: While in two-dimensional settings, parameter conditions essentially optimal in this respect could be identified even for the case $\kappa = 1$ involving the full Navier-Stokes equations ([55], [41], [42], [25], [45]), comparably exhaustive results in three-dimensional frameworks seem available only for the simplified chemotaxis-Stokes versions of (1.1) in which $\kappa = 0$ ([9], [43], [40], [52], [51], [45]). In the corresponding fully coupled three-dimensional Keller-Segel-Navier-Stokes obtained on letting $\kappa = 1$, at least within standard weak solution concepts, naturally extending those that underlie classical solution theories for the corresponding taxis-free counterpart, global existence results so far seem available only in parameter regimes significantly smaller than the complement of ranges inside which blow-up is known to occur already in the fluid-free case when $u \equiv 0$ ([7]); statements on global solvability for parameter constellations close to optimality, the construction of global solutions has so far been possible only upon resorting to quite drastically relaxed notions of solvability ([7]).

Approaching optimality in flux-limited Keller-Segel-fluid systems. Main results. The present manuscript now addresses the issue of coupling the three-dimensional Navier-Stokes system to a Keller-Segel type chemotaxis model which itself apparently has been much less thoroughly studied than the corresponding fluid-free counterpart of (1.1) that with regard to several aspects can be regarded as essentially well-understood. Specifically, we shall below consider the problem

$$\begin{cases} n_t + u\cdot\nabla n &=& \Delta n - \nabla\cdot\Big(nf(|\nabla c|^2)\nabla c\Big), & & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c &=& \Delta c - c + n, & & x\in\Omega,\ t>0,\\ u_t + (u\cdot\nabla)u &=& \Delta u + \nabla P + n\nabla\Phi, & \nabla\cdot u = 0, & x\in\Omega,\ t>0,\\ \frac{\partial n}{\partial\nu} = \frac{\partial c}{\partial\nu} = 0, \quad u=0, & & & & x\in\partial\Omega,\ t>0,\\ n(x,0)=n_0(x), \quad c(x,0)=c_0(x), \quad u(x,0)=u_0(x), & & & & x\in\Omega, \end{cases} \tag{1.2}$$

in a bounded domain $\Omega\subset\mathbb{R}^3$ with smooth boundary, where we assume that Φ represents a given gravitational potenatial, and that f is a suitably regular function quantifying saturation effects through which cross-diffusion is inhibited near points at which the gradient of the signal concentration c is large. Such flux-limited migration mechanisms have been at the core of various developments in the recent literature concerned with refinements of classical Keller-Segel models ([2], [35], [5], [6]), but rigorous analytical findings addressing existence theories or even qualitative properties yet seem to reduce to some scattered studies concerned with quite particular among accordingly resulting chemotaxis systems ([1], [8], [35], [11], [33], [3], [4]). After all, it is known that if f generalizes an algebraically decaying prototype by satisfying

$$|f(\xi)| \le K_f\cdot(\xi+1)^{-\frac{\alpha}{2}} \qquad \text{for all } \xi\ge 0 \tag{1.3}$$

with some $K_f>0$ and $\alpha>0$, then the corresponding fluid-free version of (1.2), when posed in N-dimensional domains Ω with $N\ge 2$, admits global bounded classical solutions for all suitably regular but arbitrarily large initial data whenever $\alpha>\alpha_c(N):=\frac{N-2}{N-1}$ ([54]; cf. also [34]), while if in (1.3) the converse inequality holds with some $K_f>0$ and $\alpha<\alpha_c(N)$ when $N\ge 3$, then a related parabolic-elliptic simplification of (1.2) possesses classical solutions which blow up in finite time with respect to the spatial L^∞ norm of the population density n ([53]).

Our goal now consists in establishing a theory of global weak solvability for the full flux-limited Keller-Segel-Navier-Stokes system (1.2)-(1.3) throughout the parameter range which the above findings suggest to be essentially maximal in this regard. Thus pursuing the ambition of constructing global solutions within the entire range determined by the inequality $\alpha>\alpha_c(3)=\frac{1}{2}$ in (1.3), in the centerpiece of our analysis we shall need to adequately cope with the circumstance that the only meaningful source of information on fluid regularity appears to be the natural energy identity associated with the Navier-Stokes subsystem of (1.2) (cf. (4.3). At its first stage, our approach will accordingly focus on the derivation of some basic integrability features of the forcing term $n\nabla\Phi$ therein, which will be achieved through the detection of certain entropy-like properties that functionals of the form $-\int_\Omega n^p + \int_\Omega c^2$ enjoy along trajectories of suitably regularized problems (see (2.6), provided that the mapping $0\le\xi\mapsto\xi^p$ exhibits sufficiently slow *sublinear* growth near $\xi=\infty$. As the essential part of our analysis in this respect will reveal in Lemma 3.6, a corresponding sufficient condition on smallness of p can be satisfied by exponents p which are yet suitably large so as to allow for expedient conclusions, asserting sufficient regularity of the fluid force $n\nabla\Phi$ through bounds for associated dissipation rates, exactly under the assumption that (1.3) be valid with some $\alpha > \frac{1}{2}$.

In order to precisely formulate the main results which we thereby plan to accomplish, given a smoothly bounded domain $\Omega \subset \mathbb{R}^3$ and $p > 1$, as usual we let $W^{1,p}_{0,\sigma}(\Omega) := W^{1,p}_0(\Omega;\mathbb{R}^3) \cap L^p_\sigma(\Omega) \equiv W^{1,p}_{0,\sigma}(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{W^{1,p}(\Omega)}}$ with $L^p_\sigma(\Omega) := \{\varphi \in L^p(\Omega;\mathbb{R}^3) \mid \nabla\cdot\varphi = 0\}$ and $C^\infty_{0,\sigma}(\Omega) := C^\infty_0(\Omega;\mathbb{R}^3) \cap L^2_\sigma(\Omega)$, and we let $\mathcal{P} : L^p(\Omega;\mathbb{R}^3) \to L^p_\sigma(\Omega)$ represent the realization of the Helmholtz projection in $L^p(\Omega;\mathbb{R}^3)$.

The following main outcome of this study now indeed asserts global solvability in (1.2), in the framework of a fairly natural solution concept, under a condition on the strength of flux limitation which the literature on corresponding fluid-free counterparts suggests to be essentially optimal:

Theorem 1.1 *Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, that $\Phi \in W^{2,\infty}(\Omega)$, and that $f \in C^2([0,\infty))$ satisfies (1.3) with some*

$$\alpha > \frac{1}{2}.$$

Then writing

$$q_\alpha := \min\left\{\frac{5}{3}, \frac{2}{3(1-\alpha)_+}\right\} > 1 \qquad \textit{and} \qquad r_\alpha := \min\left\{\frac{5}{4}, \frac{1}{2(1-\alpha)_+}\right\} > 1, \tag{1.4}$$

given any initial data n_0, c_0 and u_0 which are such that

$$\begin{cases} n_0 \in C^0(\overline{\Omega}) \text{ is nonnegative with } n_0 \not\equiv 0, \\ c_0 \in W^{1,\infty}(\Omega) \text{ is nonnegative, } \quad \text{and} \\ u_0 \in W^{1,2}_{0,\sigma}(\Omega) \cap W^{2,2}(\Omega;\mathbb{R}^3), \end{cases} \tag{1.5}$$

one can find functions

$$\begin{cases} n \in L^\infty((0,\infty);L^1(\Omega)) \cap \bigcap_{q\in(1,q_\alpha)} L^q_{loc}(\overline{\Omega}\times[0,\infty)) \cap \bigcap_{r\in(1,r_\alpha)} L^r_{loc}([0,\infty);W^{1,r}(\Omega)), \\ c \in L^\infty_{loc}([0,\infty);L^2(\Omega)) \cap L^{\frac{10}{3}}_{loc}(\overline{\Omega}\times[0,\infty)) \cap L^2_{loc}([0,\infty);W^{1,2}(\Omega)) \qquad \text{and} \\ u \in L^\infty_{loc}([0,\infty);L^2_\sigma(\Omega)) \cap L^{\frac{10}{3}}_{loc}(\overline{\Omega}\times[0,\infty);\mathbb{R}^3) \cap L^2_{loc}([0,\infty);W^{1,2}_{0,\sigma}(\Omega)) \end{cases} \tag{1.6}$$

with the properties that $n \ge 0$ and $c \ge 0$ a.e. in $\Omega\times(0,\infty)$, and that (n,c,u) forms a global weak solution of (1.2) in the sense of Definition 2.1 below.

2 Approximation by smooth solutions to regularized problems

To make our objective more precise, let us first substantiate what is to be understood as a global weak solution of (1.2) in the sequel. Here, for vectors $v = (v_1,v_2,v_3) \in \mathbb{R}^3$ and $w = (w_1,w_2,w_3) \in \mathbb{R}^3$ we let $v \otimes w$ denote the matrix given by $(v_iw_j)_{i,j\in\{1,2,3\}} \in \mathbb{R}^{3\times3}$.

Definition 2.1 *Let $f \in C^0([0,\infty)$ and $\Phi \in W^{1,\infty}(\Omega)$, and suppose that $n_0 \in L^1(\Omega), c_0 \in L^1(\Omega)$ and $u_0 \in L^1(\Omega;\mathbb{R}^3)$. Then a triple (n,c,u) of functions*

$$\begin{cases} n \in L^1_{loc}([0,\infty);W^{1,1}(\Omega)), \\ c \in L^1_{loc}([0,\infty);W^{1,1}(\Omega)) \qquad \text{and} \\ u \in L^1_{loc}([0,\infty);W^{1,1}_0(\Omega;\mathbb{R}^3)) \end{cases} \tag{2.1}$$

will be called a global weak solution of (1.2) *if* $n \geq 0, c \geq 0$ *and* $\nabla \cdot u = 0$ *a.e. in* $\Omega \times (0,\infty)$, *if*

$$nf(|\nabla c|^2)\nabla c,\ nu,\ cu \text{ and } u \otimes u \quad \text{belong to } L^1_{loc}(\overline{\Omega} \times [0,\infty); \mathbb{R}^3), \tag{2.2}$$

if for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$ *we have*

$$-\int_0^\infty \int_\Omega n\varphi_t - \int_\Omega n_0\varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla n \cdot \nabla\varphi + \int_0^\infty \int_\Omega nf(|\nabla c|^2)\nabla c \cdot \nabla\varphi + \int_0^\infty \int_\Omega nu \cdot \nabla\varphi \tag{2.3}$$

and

$$-\int_0^\infty \int_\Omega c\varphi_t - \int_\Omega c_0\varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla c \cdot \nabla\varphi - \int_0^\infty \int_\Omega c\varphi + \int_0^\infty \int_\Omega cu \cdot \nabla\varphi, \tag{2.4}$$

and if

$$-\int_0^\infty \int_\Omega u \cdot \varphi_t - \int_\Omega u_0 \cdot \varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla u \cdot \nabla\varphi + \int_0^\infty \int_\Omega (u \otimes u) \cdot \nabla\varphi + \int_0^\infty \int_\Omega n\nabla\Phi \cdot \varphi \tag{2.5}$$

for all $\varphi \in C_{0,\sigma}^\infty(\Omega \times [0,\infty))$.

Following precedents concerned with various related three-dimensional chemotaxis-fluid systems (see, e.g., [7], [49]), our construction of a function (n,c,u) solving (1.2) in the above sense will be based on approximation by global smooth solutions to suitably regularized counterparts of (1.2). Specifically, for $\varepsilon \in (0,1)$ we shall henceforth consider

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon &= \Delta n_\varepsilon - \nabla \cdot \Big(\frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} f(|\nabla c_\varepsilon|^2)\nabla c_\varepsilon\Big), & x \in \Omega,\ t > 0, \\ c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon &= \Delta c_\varepsilon - c_\varepsilon + \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon}, & x \in \Omega,\ t > 0, \\ u_{\varepsilon t} + (Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon &= \Delta u_\varepsilon + \nabla P_\varepsilon + \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon}\nabla\Phi, \qquad \nabla \cdot u_\varepsilon = 0, & x \in \Omega,\ t > 0, \\ \frac{\partial n_\varepsilon}{\partial\nu} = \frac{\partial c_\varepsilon}{\partial\nu} = 0, \quad u_\varepsilon = 0, & & x \in \partial\Omega,\ t > 0, \\ n_\varepsilon(x,0) = n_0(x), \quad c_\varepsilon(x,0) = c_0(x), \quad u_\varepsilon(x,0) = u_0(x), & & x \in \Omega, \end{cases} \tag{2.6}$$

with the Yosida-type approximation defined by setting

$$Y_\varepsilon v := (1+\varepsilon A)^{-1}v, \qquad v \in L^2_\sigma(\Omega),\ \varepsilon \in (0,1),$$

where here and below, we let $A = -\mathcal{P}\Delta$ and $(A^\vartheta)_{\vartheta>0}$ denote the Stokes operator in $L^2(\Omega;\mathbb{R}^3)$, with domain given by $D(A) := W^{2,2}(\Omega;\mathbb{R}^3) \cap W^{1,3}_{0,\sigma}(\Omega)$, and the family of its corresponding fractional powers, respectively.

In fact, all these problems admit global mass-preserving classical solutions:

Lemma 2.2 *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with smooth boundary, and suppose that* $f \in C^2([0,\infty)) \cap L^\infty((0,\infty))$ *and that* (n_0, c_0, u_0) *satisfy (1.5). Then for each* $\varepsilon \in (0,1)$, *there exist uniquely determined functions*

$$\begin{cases} n_\varepsilon \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ c_\varepsilon \in \bigcap_{p>3} C^0([0,\infty); W^{1,p}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \qquad \text{and} \\ u_\varepsilon \in \bigcap_{\vartheta \in (\frac{3}{4},1)} C^0([0,\infty); D(A^\vartheta)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty); \mathbb{R}^3) \end{cases}$$

such that $n_\varepsilon > 0$ and $c_\varepsilon \ge 0$ in $\overline{\Omega} \times (0,\infty)$, and that $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ solves (2.6) classically in $\Omega \times (0,\infty)$ with some $P_\varepsilon \in C^{1,0}(\Omega \times (0,\infty))$. Moreover,

$$\int_\Omega n_\varepsilon(\cdot,t) = \int_\Omega n_0 \qquad \textit{for all } t > 0. \tag{2.7}$$

Proof. Arguments well-known from the literature on local existence and extensibility in related chemotaxis-fluid systems, as detailed, e.g., in [46], yield $T_{max,\varepsilon} \in (0,\infty]$ and a unique triple $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ of functions

$$\begin{cases} n_\varepsilon \in C^0(\overline{\Omega} \times [0,T_{max,\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{max,\varepsilon})), \\ c_\varepsilon \in \bigcap_{p>3} C^0([0,T_{max,\varepsilon}); W^{1,p}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,T_{max,\varepsilon})) \qquad \text{and} \\ u_\varepsilon \in \bigcap_{\vartheta \in (\frac{3}{4},1)} C^0([0,T_{max,\varepsilon}); D(A^\vartheta)) \cap C^{2,1}(\overline{\Omega} \times (0,T_{max,\varepsilon}); \mathbb{R}^3) \end{cases}$$

for which $n_\varepsilon > 0$ and $c_\varepsilon \ge 0$ in $\overline{\Omega} \times (0,T_{max,\varepsilon})$ as well as

$$\int_\Omega n_\varepsilon(\cdot,t) = \int_\Omega n_0 \qquad \text{for all } t \in (0,T_{max,\varepsilon}), \tag{2.8}$$

for which one can find $P_\varepsilon \in C^{1,0}(\Omega \times (0,T_{max,\varepsilon}))$ such that $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ forms a classical solution of (2.6) in $\Omega \times (0,T_{max,\varepsilon})$, and for which

$$\begin{aligned} &\text{either } T_{max,\varepsilon} = \infty, \quad \text{or for all } p > 3 \text{ and } \vartheta \in (\tfrac{3}{4},1) \text{ we have} \\ &\qquad \limsup_{t \nearrow T_{max,\varepsilon}} \Big\{ \|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot,t)\|_{W^{1,p}(\Omega)} + \|A^\vartheta u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \Big\} = \infty. \end{aligned} \tag{2.9}$$

Now if $T_{max,\varepsilon}$ was finite for any such ε, we could use the rough estimate $\frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} \le \frac{1}{\varepsilon}$ to see upon testing the third equation in (2.6) by u_ε that due to Young's inequality,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} u_\varepsilon \cdot \nabla\Phi \le \int_\Omega |u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_\Omega |\nabla\Phi|^2 \qquad \text{for all } t \in (0,T_{max,\varepsilon}),$$

and that therefore

$$\sup_{t\in(0,T_{max,\varepsilon})} \int_\Omega |u_\varepsilon(\cdot,t)|^2 < \infty \qquad \text{and} \qquad \int_0^{T_{max,\varepsilon}} \int_\Omega |\nabla u_\varepsilon|^2 < \infty.$$

Since Y_ε acts as a bounded operator from $L^2_\sigma(\Omega)$ to $W^{2,2}(\Omega;\mathbb{R}^3) \cap W^{1,2}_{0,\sigma}(\Omega) \hookrightarrow L^\infty(\Omega;\mathbb{R}^3)$, this would imply that $h_\varepsilon := \mathcal{P}[-(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon + \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon}\nabla\Phi]$ would belong to $L^2(\Omega \times (0,T_{max,\varepsilon});\mathbb{R}^3)$, and that thus

$$\sup_{t\in(0,T_{max,\varepsilon})} \int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 < \infty, \tag{2.10}$$

because again by (2.6) and Young's inequality,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u_\varepsilon|^2 = -\int_\Omega |Au_\varepsilon|^2 + \int_\Omega Au_\varepsilon \cdot h_\varepsilon \le \frac{1}{4}\int_\Omega |h_\varepsilon|^2 \qquad \text{for all } t \in (0,T_{max,\varepsilon}).$$

From (2.10), however, we now obtain that actually $\sup_{t\in(0,T_{max,\varepsilon})}\int_\Omega |h_\varepsilon(\cdot,t)|^2 < \infty$, so that if we fix any $\vartheta \in (\frac{3}{4},1)$ and rely on well-known regularization features of the Dirichlet Stokes semigroup $(e^{-tA})_{t\ge 0}$ ([18]) in choosing $C_1 > 0$ such that

$$\begin{aligned}
\|A^\vartheta u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} &= \Big\|A^\vartheta e^{-tA}u_0 + \int_0^t A^\vartheta e^{-(t-s)A}h_\varepsilon(\cdot,s)ds\Big\|_{L^2(\Omega)} \\
&\le C_1 + C_1\int_0^t (t-s)^{-\vartheta}\|h_\varepsilon(\cdot,s)\|_{L^2(\Omega)}ds \qquad \text{for all } t\in(0,T_{max,\varepsilon}),
\end{aligned}$$

we infer that

$$\sup_{t\in(0,T_{max,\varepsilon})}\|A^\vartheta u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} < \infty. \tag{2.11}$$

As $\vartheta > \frac{3}{4}$, according to a known embedding property ([23], [20]) this would especially warrant that

$$\sup_{t\in(0,T_{max,\varepsilon})}\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} < \infty, \tag{2.12}$$

so that picking any $p > 3$ and $\theta \in (\frac{1}{2},1)$, abbreviating $\tau := \frac{1}{2}T_{max,\varepsilon}$ and letting B denote the realization of $-\Delta + 1$ under homogeneous Neumann boundary conditions in $L^p(\Omega)$, by means of corresponding smoothing estimates for the associated semigroup $(e^{-tB})_{t\ge 0}$ ([18]) we could infer that with some $C_2 = C_2(\varepsilon) > 0$ and $C_3 = C_3(\varepsilon) > 0$, whenever $T \in (\tau, T_{max,\varepsilon})$ we would have

$$\begin{aligned}
\|B^\theta c_\varepsilon(\cdot,t)\|_{L^p(\Omega)} &= \Big\|B^\theta e^{-(t-\tau)B}c_\varepsilon(\cdot,\tau) + \int_\tau^t B^\theta e^{-(t-s)B}\Big\{\frac{n_\varepsilon(\cdot,s)}{1+\varepsilon n_\varepsilon(\cdot,s)} - u_\varepsilon(\cdot,s)\cdot\nabla c_\varepsilon(\cdot,s)\Big\}ds\Big\|_{L^p(\Omega)} \\
&\le C_2 + C_2\int_\tau^t (t-s)^{-\theta}\cdot\Big\{\Big\|\frac{n_\varepsilon(\cdot,s)}{1+\varepsilon n_\varepsilon(\cdot,s)}\Big\|_{L^p(\Omega)} + \|u_\varepsilon(\cdot,s)\cdot\nabla c_\varepsilon(\cdot,s)\|_{L^p(\Omega)}\Big\}ds \\
&\le C_3 + C_3\sup_{s\in(\tau,T)}\|\nabla c_\varepsilon(\cdot,s)\|_{L^p(\Omega)} \qquad \text{for all } t\in(\tau,T).
\end{aligned}$$

Since a direct integration in (2.6) shows that $\sup_{t\in(0,T_{max,\varepsilon})}\int_\Omega c_\varepsilon(\cdot,t)$ is finite, and since $D(B^\theta)$ is compactly embedded into $W^{1,p}(\Omega)$ ([23]), through an associated Ehrling-type inequality this readily implies that $\sup_{T\in(\tau,T_{max,\varepsilon})}\sup_{t\in(\tau,T)}\|B^\theta c_\varepsilon(\cdot,t)\|_{L^p(\Omega)}$ is finite, and that thus, in particular,

$$\sup_{t\in(\tau,T_{max,\varepsilon})}\|\nabla c_\varepsilon(\cdot,t)\|_{L^p(\Omega)} < \infty. \tag{2.13}$$

As combining standard regularization properties of the Neumann heat semigroup $(e^{t\Delta})_{t\ge 0}$ on Ω ([19]) provides $C_4 = C_4(\varepsilon) > 0$ and $C_5 = C_5(\varepsilon) > 0$ such that

$$\begin{aligned}
&\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \\
&= \Big\|e^{(t-\tau)\Delta}n_\varepsilon(\cdot,\tau) - \int_\tau^t e^{(t-s)\Delta}\nabla\cdot\Big\{\frac{n_\varepsilon(\cdot,s)}{1+\varepsilon n_\varepsilon(\cdot,s)}f(|\nabla c_\varepsilon(\cdot,s)|^2)\nabla c_\varepsilon(\cdot,s) + n_\varepsilon(\cdot,s)u_\varepsilon(\cdot,s)\Big\}ds\Big\|_{L^\infty(\Omega)} \\
&\le C_4 + C_4\int_\tau^t (t-s)^{-\frac{1}{2}-\frac{3}{2p}}\cdot\Big\{\Big\|\frac{n_\varepsilon(\cdot,s)}{1+\varepsilon n_\varepsilon(\cdot,s)}f(|\nabla c_\varepsilon(\cdot,s)|^2)\nabla c_\varepsilon(\cdot,s)\Big\|_{L^p(\Omega)}
\end{aligned}$$

$$+\|n_\varepsilon(\cdot,s)u_\varepsilon(\cdot,s)\|_{L^p(\Omega)}\Big\}ds$$

$$\le \quad C_4 + C_5 \sup_{s\in(\tau,T_{max,\varepsilon})} \|\nabla c_\varepsilon(\cdot,s)\|_{L^p(\Omega)} + C_5 \cdot \Big\{ \sup_{s\in(\tau,T_{max,\varepsilon})} \|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)} \Big\} \cdot \sup_{s\in(\tau,t)} \|n_\varepsilon(\cdot,s)\|_{L^p(\Omega)}$$

for all $t \in (\tau, T_{max,\varepsilon})$, using (2.13) along with (2.12) and (2.8) shows that with some $C_6 = C_6(\varepsilon) > 0$ and $C_7 = C_7(\varepsilon) > 0$ we have

$$\begin{aligned}
\sup_{t\in(\tau,T)} \|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} &\le C_6 + C_6 \sup_{t\in(\tau,T)} \|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \\
&\le \frac{1}{2} \sup_{t\in(\tau,T)} \|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + C_7 \qquad \text{for all } T \in (\tau, T_{max,\varepsilon})
\end{aligned}$$

thanks to Young's inequality. This, however, guarantees boundedness of n_ε in $\Omega \times (\tau, T_{max,\varepsilon})$ and hence, together with (2.11) and (2.13), contradicts our hypothesis that $T_{max,\varepsilon}$ be finite. The proof is thus complete. $\Box$

3 Estimates implied by a quasi-entropy property of $-\int_\Omega n_\varepsilon^p + \int_\Omega c_\varepsilon^2$

The goal of this section is to derive some basic regularity features of n_ε and c_ε which do not require any knowledge on integrability properties of u_ε that go beyond the mere fact that $\nabla \cdot u_\varepsilon = 0$. This will be set about by analyzing the time evolution of the coupled quantity $\mathcal{F} := -\int_\Omega n_\varepsilon^p + \int_\Omega c_\varepsilon^2$, where Lemma 3.6 will reveal that under the subcriticality assumption on f and α in Theorem 1.1, the free parameter $p \in (0,1)$ herein can be chosen in such a manner that said expression plays the role of a quasi-entropy functional.

Our considerations in this direction will be prepared by three interpolation-based observations about zero-oder quantities which can suitably be controlled in terms of a dissipation rate functional appearing in the course of a corresponding testing procedure in Lemma 3.4, provided that p does not fall below a certain threshold value. Our first statement in this regard is a consequence of the Gagliardo-Nirenberg inequality and the mass conservation feature in (2.7):

Lemma 3.1 *Let $p > \frac{1}{3}$ and $q \in (1, 3p]$. Then there exists $C(p,q) > 0$ such that*

$$\|n_\varepsilon\|_{L^q(\Omega)}^{\frac{(3p-1)q}{3(q-1)}} \le C(p,q) \int_\Omega n_\varepsilon^{p-2} |\nabla n_\varepsilon|^2 + C(p,q) \qquad \textit{for all } t > 0 \textit{ and } \varepsilon \in (0,1). \tag{3.1}$$

Proof. Since $q \le 3p$ and thus $\frac{2q}{p} \le 6$, we may invoke the Gagliardo-Nirenberg inequality to fix $C_1 = C_1(p,q) > 0$ such that

$$\|\varphi\|_{L^{\frac{2q}{p}}(\Omega)}^{\frac{2(3p-1)q}{3p(q-1)}} \le C_1 \|\nabla\varphi\|_{L^2(\Omega)}^2 \|\varphi\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(3p-q)}{3p(q-1)}} + C_1 \|\varphi\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(3p-1)q}{3p(q-1)}} \qquad \text{for all } \varphi \in W^{1,2}(\Omega).$$

As $\|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{p}} = C_2 := \int_\Omega n_0$ for all $t>0$ and $\varepsilon\in(0,1)$ by (2.7), an application thereof to $\varphi := n_\varepsilon^{\frac{2}{p}}$ shows that

$$
\begin{aligned}
\|n_\varepsilon\|_{L^q(\Omega)}^{\frac{(3p-1)q}{3(q-1)}} &= \|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2q}{p}}(\Omega)}^{\frac{2(3p-1)q}{3p(q-1)}} \\
&\le C_1 C_2^{\frac{3p-q}{3(q-1)}} \|\nabla n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^2 + C_1 C_2^{\frac{(3p-1)q}{3(q-1)}} \\
&= \frac{p^2}{4} C_1 C_2^{\frac{3p-q}{3(q-1)}} \int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + C_1 C_2^{\frac{(3p-1)q}{3(q-1)}}
\end{aligned}
$$

for all $t>0$ and $\varepsilon\in(0,1)$. $\square$

Combined with a straightforward second interpolation, now at the level of zero-order expressions, the latter implies the following.

Corollary 3.2 *Let $p>\frac{1}{3}$ and $q\in(1,p+\frac{2}{3})$. Then for all $\eta>0$ one can find $C(\eta,p,q)>0$ with the property that*

$$
\int_\Omega n_\varepsilon^q \le \eta \int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + C(\eta,p,q) \qquad \textit{for all } t>0 \textit{ and } \varepsilon\in(0,1). \tag{3.2}
$$

PROOF. Since $p+\frac{2}{3}\le 3p$ due to the inequality $p\ge\frac{1}{3}$, our assumption on q particularly ensures that $q\le 3p$, so that we may apply Lemma 3.1 to find $C_1=C_1(p,q)>0$ such that writing $\theta := \frac{(3p-1)q}{3(q-1)}$ we have

$$
\|n_\varepsilon\|_{L^q(\Omega)}^\theta \le C_1 \int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + C_1 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1). \tag{3.3}
$$

As the hypothesis $q<p+\frac{2}{3}$ furthermore warrants that

$$
\frac{\theta}{q} = \frac{3p-1}{3q-3} > \frac{3p-1}{3\cdot(p+\frac{2}{3})-3} = 1,
$$

and that hence $ab\le a^{\frac{\theta}{q}} + b^{\frac{\theta}{\theta-q}}$ for all $a\ge0$ and $b\ge0$ by Young's inequality, given $\eta>0$ we can use (3.3) to see that

$$
\begin{aligned}
\int_\Omega n_\varepsilon^q &= \Big\{\frac{\eta}{C_1}\|n_\varepsilon\|_{L^q(\Omega)}^\theta\Big\}^{\frac{q}{\theta}} \cdot \Big(\frac{C_1}{\eta}\Big)^{\frac{q}{\theta}} \\
&\le \frac{\eta}{C_1}\|n_\varepsilon\|_{L^q(\Omega)}^\theta + \Big(\frac{C_1}{\eta}\Big)^{\frac{q}{\theta-q}} \\
&\le \eta\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + \eta + \Big(\frac{C_1}{\eta}\Big)^{\frac{q}{\theta-q}} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1)
\end{aligned}
$$

and conclude as intended. $\square$

Quite a similar argument shows that whenever $p\in(0,1)$ is not too small, the weighted Dirichlet integral under consideration moreover conveniently dominates an expression which will turn out to be of immediate relevance for our analysis of the Navier-Stokes energy (cf. Lemma 4.1), but which prior to that will also be made use of in Lemma 3.5.

Corollary 3.3 *Let $p \ge \frac{2}{5}$ and $s \in (1, 6p-2)$. Then given any $\eta > 0$, one can fix $C(\eta,p,s) > 0$ in such a way that*

$$\|n_\varepsilon\|^s_{L^{\frac{6}{5}}(\Omega)} \le \eta \int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + C(\eta,p,s) \qquad \textit{for all } t>0 \textit{ and } \varepsilon \in (0,1). \tag{3.4}$$

Proof. Using that our assumption $p \ge \frac{2}{5}$ entails that both $p > \frac{1}{3}$ and $\frac{6}{5} \le 3p$, from Lemma 3.1 we obtain $C_1 = C_1(p) > 0$ such that

$$\|n_\varepsilon\|^\theta_{L^{\frac{6}{5}}(\Omega)} \le C_1 \int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + C_1 \qquad \text{for all } t>0 \text{ and } \varepsilon \in (0,1), \tag{3.5}$$

where $\theta := \frac{(3p-1)\cdot\frac{6}{5}}{3\cdot(\frac{6}{5}-1)} \equiv 2\cdot(3p-1)$ satisfies $\theta > s$ by hypothesis. Therefore, Young's inequality applies so as to guaranteee that for any choice of $\eta > 0$,

$$\begin{aligned}
\|n_\varepsilon\|^s_{L^{\frac{6}{5}}(\Omega)} &= \Big\{\frac{\eta}{C_1}\|n_\varepsilon\|^\theta_{L^{\frac{6}{5}}(\Omega)}\Big\}^{\frac{s}{\theta}} \cdot \Big(\frac{C_1}{\eta}\Big)^{\frac{s}{\theta}} \\
&\le \frac{\eta}{C_1}\|n_\varepsilon\|^\theta_{L^{\frac{6}{5}}(\Omega)} + \Big(\frac{C_1}{\eta}\Big)^{\frac{s}{\theta-s}} \qquad \text{for all } t>0 \text{ and } \varepsilon \in (0,1),
\end{aligned}$$

whence (3.4) results from (3.5) if we let $C(\eta,p,s) := \eta + (\frac{C_1(p)}{\eta})^{\frac{s}{\theta-s}}$. □

Having at hand these preparations, and especially Corollary 3.2, we can now perform a standard testing procedure to describe the evolution of the first summand in $\mathcal{F}$, provided that p, in addition to the above, satisfies the crucial condition (3.6) which will actually form the core of our overall restriction on α from Theorem 1.1 (see Lemma 3.6).

Lemma 3.4 *Assume (1.3) with some $\alpha > 0$, and let $p \in (\frac{1}{3}, 1)$ satisfy*

$$p < \frac{2\alpha}{3(1-\alpha)_+}. \tag{3.6}$$

Then there exists $C(p) > 0$ such that

$$-\frac{d}{dt}\int_\Omega n_\varepsilon^p + \frac{p(1-p)}{4}\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 \le \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^2 + C(p) \qquad \textit{for all } t>0 \textit{ and } \varepsilon \in (0,1). \tag{3.7}$$

Proof. Relying on the positivity of n_ε in $\overline{\Omega}\times(0,\infty)$, we may test the first equation in (2.6) against n_ε^{p-1} to see that since $\nabla\cdot u_\varepsilon = 0$, Young's inequality and (1.3) imply that

$$\begin{aligned}
-\frac{1}{p}\frac{d}{dt}\int_\Omega n_\varepsilon^p &+ (1-p)\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 \\
&= (1-p)\int_\Omega \frac{n_\varepsilon^{p-1}}{1+\varepsilon n_\varepsilon} f(|\nabla c_\varepsilon|^2)\nabla n_\varepsilon \cdot \nabla c_\varepsilon \\
&\le \frac{1-p}{2}\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + \frac{1-p}{2}\int_\Omega \frac{n_\varepsilon^p}{(1+\varepsilon n_\varepsilon)^2} f^2(|\nabla c_\varepsilon|^2)|\nabla c_\varepsilon|^2
\end{aligned}$$

$$\leq \frac{1-p}{2}\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + \frac{1-p}{2}K_f^2\int_\Omega n_\varepsilon^p(|\nabla c_\varepsilon|^2+1)^{-\alpha}|\nabla c_\varepsilon|^2 \tag{3.8}$$

for all $t>0$ and $\varepsilon\in(0,1)$. Here when $\alpha\geq 1$, by means of the Hölder inequality and (2.7) we can simply estimate

$$\begin{aligned}
&\frac{1-p}{2}K_f^2\int_\Omega n_\varepsilon^p(|\nabla c_\varepsilon|^2+1)^{-\alpha}|\nabla c_\varepsilon|^2 \\
&\quad\leq \frac{1-p}{2}K_f^2\int_\Omega n_\varepsilon^p \\
&\quad\leq \frac{1-p}{2}K_f^2|\Omega|^{1-p}\cdot\Big\{\int_\Omega n_\varepsilon\Big\}^p \\
&\quad= C_1=C_1(p):=\frac{1-p}{2}K_f^2|\Omega|^{1-p}\cdot\Big\{\int_\Omega n_0\Big\}^p \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),
\end{aligned}$$

so that in this case we obtain from (3.8) that

$$-\frac{1}{p}\frac{d}{dt}\int_\Omega n_\varepsilon^p + \frac{1-p}{2}\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 \leq C_1 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),$$

which trivially entails (3.7).
If $\alpha<1$, however, then thanks to Young's inequality,

$$\begin{aligned}
&\frac{1-p}{2}K_f^2\int_\Omega n_\varepsilon^p(|\nabla c_\varepsilon|^2+1)^{-\alpha}|\nabla c_\varepsilon|^2 \\
&\quad\leq \frac{1-p}{2}K_f^2\int_\Omega n_\varepsilon^p|\nabla c_\varepsilon|^{2-2\alpha} \\
&\quad= \int_\Omega\Big\{\frac{1-p}{2}K_f^2\cdot(2p)^{1-\alpha}n_\varepsilon^p\Big\}\cdot\Big\{\frac{1}{2p}|\nabla c_\varepsilon|^2\Big\}^{1-\alpha} \\
&\quad\leq \frac{1}{2p}\int_\Omega|\nabla c_\varepsilon|^2 + C_2\int_\Omega n_\varepsilon^{\frac{p}{\alpha}} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1)
\end{aligned} \tag{3.9}$$

with $C_2=C_2(p):=\Big\{\frac{1-p}{2}K_f^2\cdot(2p)^{1-\alpha}\Big\}^{\frac{1}{\alpha}}$, where we note that as a consequence of our assumption (3.6), the exponent in the rightmost integral satisfies

$$\frac{p}{\alpha}-\Big(p+\frac{2}{3}\Big)=\frac{1-\alpha}{\alpha}\cdot p-\frac{2}{3}<\frac{1-\alpha}{\alpha}\cdot\frac{2\alpha}{3(1-\alpha)}-\frac{2}{3}=0.$$

Therefore, Corollary 3.2 becomes applicable so as to provide $C_3=C_3(p)>0$ fulfilling

$$C_2\int_\Omega n_\varepsilon^{\frac{p}{\alpha}} \leq \frac{1-p}{4}\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + C_3 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1).$$

Together with (3.9) and (3.8), this implies that

$$-\frac{1}{p}\frac{d}{dt}\int_\Omega n_\varepsilon^p + \frac{1-p}{4}\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 \leq \frac{1}{2p}\int_\Omega|\nabla c_\varepsilon|^2 + C_3 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1)$$

and hence shows that (3.7) can be achieved also in this case. $\square$

Upon an application to suitably large $p \in (0,1)$, Corollary 3.3 now enables us to appropriately estimate the first summand on the right of (3.7) in terms of the dissipation rate that appears in the course of a standard testing procedure performed to the second sub-problem contained in (2.6):

Lemma 3.5 *Let $p \in (\frac{2}{3}, 1)$. Then there exists $C(p) > 0$ satisfying*

$$\frac{d}{dt}\int_\Omega c_\varepsilon^2 + \int_\Omega |\nabla c_\varepsilon|^2 \le \frac{p(1-p)}{8}\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + C(p) \qquad \textit{for all } t>0 \textit{ and } \varepsilon \in (0,1). \tag{3.10}$$

Proof. According to the second equation in (2.6), again by solenoidality of u_ε we obtain that due to the Hölder inequality,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega c_\varepsilon^2 + \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega c_\varepsilon^2 &= \int_\Omega \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} c_\varepsilon \\ &\le \|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} \|c_\varepsilon\|_{L^6(\Omega)} \qquad \text{for all } t>0 \text{ and } \varepsilon \in (0,1), \end{aligned} \tag{3.11}$$

where by continuity of the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, with some $C_1 > 0$ we have

$$\begin{aligned} \|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} \|c_\varepsilon\|_{L^6(\Omega)} &\le C_1 \|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} \cdot \Big\{ \|\nabla c_\varepsilon\|_{L^2(\Omega)}^2 + \|c_\varepsilon\|_{L^2(\Omega)}^2 \Big\}^{\frac{1}{2}} \\ &\le \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^2 + \frac{1}{2}\int_\Omega c_\varepsilon^2 + \frac{C_1^2}{2}\|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}^2 \qquad \text{for all } t>0 \text{ and } \varepsilon \in (0,1) \end{aligned} \tag{3.12}$$

due to Young's inequality.
We now make use of our hypothesis $p > \frac{2}{3}$, which namely means that $6p-2 > 2$ and that hence Corollary 3.3 applies to yield $C_2 = C_2(p) > 0$ fulfilling

$$\frac{C_1^2}{2}\|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}^2 \le \frac{p(1-p)}{16}\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + C_2 \qquad \text{for all } t>0 \text{ and } \varepsilon \in (0,1).$$

Consequently, (3.11) along with (3.12) shows that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega c_\varepsilon^2 + \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^2 + \frac{1}{2}\int_\Omega c_\varepsilon^2 \le \frac{p(1-p)}{16}\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + C_2 \qquad \text{for all } t>0 \text{ and } \varepsilon \in (0,1),$$

which clearly entails (3.10) if we let $C(p) := 2C_2(p)$, for instance. $\square$

By suitably selecting p in such a way that the above hypotheses are simultaneously fulfilled, under the assumptions from Theorem 1.1 we can now derive the fundamental regularity properties of n_ε and c_ε that form the main result of this section:

Lemma 3.6 *Assume (1.3) with some $\alpha > \frac{1}{2}$, and let $q>1, r>1$ and $s>1$ be such that with q_α and r_α taken from (1.4), and with*

$$s_\alpha := \min\Big\{4, \frac{6\alpha-2}{(1-\alpha)_+}\Big\} > 1 \tag{3.13}$$

we have

$$q < q_\alpha, \quad r < r_\alpha \quad \textit{and} \quad s < s_\alpha. \tag{3.14}$$

Then for all $T>0$ there exists $C(T)=C(T;q,r,s)>0$ such that

$$\int_\Omega c_\varepsilon^2(\cdot,t) \le C(T) \qquad \textit{for all } t\in(0,T) \textit{ and } \varepsilon\in(0,1) \tag{3.15}$$

and

$$\int_0^T\int_\Omega |\nabla c_\varepsilon|^2 \le C(T) \qquad \textit{for all } \varepsilon\in(0,1) \tag{3.16}$$

as well as

$$\int_0^T\int_\Omega n_\varepsilon^q \le C(T) \qquad \textit{for all } \varepsilon\in(0,1) \tag{3.17}$$

and

$$\int_0^T\int_\Omega |\nabla n_\varepsilon|^r \le C(T) \qquad \textit{for all } \varepsilon\in(0,1) \tag{3.18}$$

and

$$\int_0^T \|n_\varepsilon(\cdot,t)\|^s_{L^{\frac{6}{5}}(\Omega)} dt \le C(T) \qquad \textit{for all } \varepsilon\in(0,1). \tag{3.19}$$

PROOF. In order to appropriately prepare our applications of Lemma 3.4 and Lemma 3.5, we let $\alpha>\frac{1}{2}$, $q\in(1,q_\alpha), r\in(1,r_\alpha)$ and $s\in(1,s_\alpha)$ be given and then claim that it is possible to choose $p\in(\frac{2}{3},1)$ in such a way that

$$p<\frac{2\alpha}{3(1-\alpha)_+}, \tag{3.20}$$

and that

$$q<p+\frac{2}{3} \tag{3.21}$$

and

$$r<\frac{3p+2}{4} \tag{3.22}$$

as well as

$$s<6p-2. \tag{3.23}$$

Indeed, if $\alpha\ge\frac{3}{5}$ then it follows from (3.14) that

$$\max\Big\{q-\frac{2}{3}\,,\,\frac{4r-2}{3}\,,\,\frac{s+2}{6}\Big\}<\max\Big\{q_\alpha-\frac{2}{3}\,,\,\frac{4r_\alpha-2}{3}\,,\,\frac{s_\alpha+2}{6}\Big\}=1,$$

so that (3.21), (3.22) and (3.23) can simultaneously be achieved for some $p\in(\frac{2}{3},1)$ suitably close to 1, whereupon observing that for such α we have $\frac{2\alpha}{3(1-\alpha)_+}\ge 1$, we furthermore see that then (3.20) is trivially satisfied.

In the case when $\alpha\in(\frac{1}{2},\frac{3}{5})$, however, we note that then (3.14) says that

$$q-\frac{2}{3}<q_\alpha-\frac{2}{3}=\frac{2\alpha}{3(1-\alpha)}, \quad \frac{4r-2}{3}<\frac{4r_\alpha-2}{3}=\frac{2\alpha}{3(1-\alpha)} \quad \text{and} \quad \frac{s+2}{6}<\frac{s_\alpha+2}{6}=\frac{2\alpha}{3(1-\alpha)}.$$

Since for such α we have

$$1=\frac{2}{3\cdot(\frac{5}{3}-1)}>\frac{2}{3\cdot(\frac{1}{\alpha}-1)}=\frac{2\alpha}{3(1-\alpha)}>\frac{2}{3\cdot(2-1)}=\frac{2}{3},$$

we thereby see that it is possible to pick $p \in \mathbb{R}$ such that

$$\max\Big\{\frac{2}{3}\,,\,q-\frac{2}{3}\,,\,\frac{4r-2}{3}\,,\,\frac{s+2}{6}\Big\} < p < \min\Big\{1\,,\,\frac{2\alpha}{3(1-\alpha)}\Big\} = \frac{2\alpha}{3(1-\alpha)},$$

and that this selection warrants that, evidently, p in fact belongs to the interval $(\frac{2}{3},1)$ and satisfies (3.20)-(3.23).

Keeping this value of p fixed henceforth, we may rely on (3.20) to infer from Lemma 3.4 that there exists $C_1 > 0$ such that

$$-\frac{d}{dt}\int_\Omega n_\varepsilon^p + \frac{p(1-p)}{4}\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 \le \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^2 + C_1 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),$$

while Lemma 3.5 provides $C_2 > 0$ fulfilling

$$\frac{d}{dt}\int_\Omega c_\varepsilon^2 + \int_\Omega |\nabla c_\varepsilon|^2 \le \frac{p(1-p)}{8}\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + C_2 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1).$$

Therefore,

$$\frac{d}{dt}\Big\{-\int_\Omega n_\varepsilon^p + \int_\Omega c_\varepsilon^2\Big\} + \frac{p(1-p)}{8}\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^2 \le C_1 + C_2 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),$$

which upon an integration in time shows that due to the Hölder inequality and (2.7),

$$\begin{aligned}
\int_\Omega c_\varepsilon^2(\cdot,t) &+ \frac{p(1-p)}{8}\int_0^t\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + \frac{1}{2}\int_0^t\int_\Omega |\nabla c_\varepsilon|^2 \\
&\le \int_\Omega n_\varepsilon^p(\cdot,t) + \int_\Omega c_0^2 + (C_1+C_2)t \\
&\le |\Omega|^{1-p}\cdot\Big\{\int_\Omega n_\varepsilon(\cdot,t)\Big\}^p + \int_\Omega c_0^2 + (C_1+C_2)t \\
&= C_3(t) := |\Omega|^{1-p}\cdot\Big\{\int_\Omega n_0\Big\}^p + \int_\Omega c_0^2 + (C_1+C_2)t \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1). \qquad (3.24)
\end{aligned}$$

This immediately implies the properties claimed both in (3.15) and in (3.16), and moreover entails that

$$\int_0^T\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 \le \frac{8C_3(T)}{p(1-p)} \qquad \text{for all } T>0 \text{ and } \varepsilon\in(0,1), \qquad (3.25)$$

so that we may draw on (3.21) and (3.23) to conclude the boundedness features in (3.17) and (3.19) from Corollary 3.2 and Corollary 3.3, respectively. Finally, using that $r < 2$ we may employ Young's inequality to estimate

$$\begin{aligned}
\int_\Omega |\nabla n_\varepsilon|^r &= \int_\Omega \Big\{n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2\Big\}^{\frac{r}{2}}\cdot n_\varepsilon^{\frac{(2-p)r}{2}} \\
&\le \int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + \int_\Omega n_\varepsilon^{\frac{(2-p)r}{2-r}} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1), \qquad (3.26)
\end{aligned}$$

and noting that here

$$\frac{\frac{(2-p)r}{2-r}}{p+\frac{2}{3}} = \frac{3(2-p)}{(3p+2)\cdot(\frac{2}{r}-1)} < \frac{3(2-p)}{(3p+2)\cdot\left(\frac{2}{\frac{3p+2}{4}}-1\right)} = 1$$

and hence $\frac{(2-p)r}{2-r} < p+\frac{2}{3}$, we may once again invoke Corollary 3.2 to obtain $C_4 > 0$ such that

$$\int_\Omega n_\varepsilon^{\frac{(2-p)r}{2-r}} \le \int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + C_4 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1).$$

Consequently, (3.26) together with (3.25) guarantees that

$$\begin{aligned}\int_0^T\int_\Omega |\nabla n_\varepsilon|^r &\le 2\int_0^T\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + C_4T \\ &\le \frac{16C_3(T)}{p(1-p)} + C_4T \qquad \text{for all } T>0 \text{ and } \varepsilon\in(0,1),\end{aligned}$$

and thus establishes the bound in (3.18). □

For later use, let us already here record the following by-product of (3.15) and (3.16).

Corollary 3.7 *Let f be such that (1.3) holds with some $\alpha > \frac{1}{2}$, and let $T>0$. Then one can pick $C(T)>0$ in such a manner that*

$$\int_0^T\int_\Omega c_\varepsilon^{\frac{10}{3}} \le C(T) \qquad \textit{for all } \varepsilon\in(0,1). \tag{3.27}$$

Proof. By means of a Gagliardo-Nirenberg interpolation, we obtain $C_1>0$ such that

$$\int_\Omega |\varphi|^{\frac{10}{3}} \le C_1\cdot\left\{\int_\Omega|\nabla\varphi|^2\right\}\cdot\left\{\int_\Omega|\varphi|^2\right\}^{\frac{2}{3}} + C_1\cdot\left\{\int_\Omega|\varphi|^2\right\}^{\frac{5}{3}} \qquad \text{for all } \varphi\in W^{1,2}(\Omega).$$

Applying this to c_ε directly shows that (3.27) is a consequence of (3.15) and (3.16). □

4 Regularity features of u_ε. The Navier-Stokes energy

Since within the considered range of α we may apply (3.19) to $s:=2$, we can now suitably estimate the forcing term in the approximate Navier-Stokes subsystem of (2.6), and thereby derive the following regularity features of the corresponding fluid field as natural consequences.

Lemma 4.1 *Suppose that (1.3) is satisfied with some $\alpha>\frac{1}{2}$. Then for all $T>0$ there exists $C(T) => 0$ with the property that*

$$\int_\Omega |u_\varepsilon(\cdot,t)|^2 \le C(T) \qquad \textit{for all } t\in(0,T) \textit{ and } \varepsilon\in(0,1) \tag{4.1}$$

and

$$\int_0^T\int_\Omega |\nabla u_\varepsilon|^2 \le C(T) \qquad \textit{for all } \varepsilon\in(0,1). \tag{4.2}$$

PROOF. According to a standard reasoning on the basis of the approximate Navier-Stokes subsystem of (2.6) ([49, Lemma 3.5]), we can obtain the identity

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} u_\varepsilon \cdot \nabla\Phi \qquad \text{for all } t>0 \text{ and } \varepsilon \in (0,1), \tag{4.3}$$

in which we again we use the Hölder inequality along with the continuity of the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ to obtain from an associated Poincaré-Sobolev inequality that with some $C_1 > 0$ we have

$$\begin{aligned}
\int_\Omega \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} u_\varepsilon \cdot \nabla\Phi &\le \|\nabla\Phi\|_{L^\infty(\Omega)} \|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} \|u_\varepsilon\|_{L^6(\Omega)} \\
&\le C_1 \|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} \|\nabla u_\varepsilon\|_{L^2(\Omega)} \\
&\le \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 + \frac{C_1^2}{2}\|n_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1)
\end{aligned}$$

thanks to Young's inequality. Therefore,

$$\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 \le C_1^2 \|n_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1)$$

and hence

$$\int_\Omega |u_\varepsilon(\cdot,t)|^2 + \int_0^t\int_\Omega |\nabla u_\varepsilon|^2 \le \int_\Omega |u_0|^2 + C_1^2 \int_0^t \int_\Omega \|n_\varepsilon(\cdot,\tau)\|^2_{L^{\frac{6}{5}}(\Omega)} d\tau \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),$$

so that both (4.1) and (4.2) become consequences of (3.19) when applied to $s := 2$, because the number s_α from (3.13) satisfies $s_\alpha > 2$ due to the fact that since $\alpha > \frac{1}{2}$ we have $\frac{6\alpha-2}{(1-\alpha)_+} > 2$. ☐

Again by interpolation in the style of that performed in Corollary 3.7, the latter implies the following zero-order spatio-temporal integral bound.

Corollary 4.2 *If (1.3) holds with some $\alpha > \frac{1}{2}$, then for all $T>0$ one can find $C(T)>0$ such that*

$$\int_0^T\int_\Omega |u_\varepsilon|^{\frac{10}{3}} \le C(T) \qquad \textit{for all } \varepsilon\in(0,1). \tag{4.4}$$

PROOF. This can be derived from (4.1) and (4.2) through an essentially verbatim copy of the argument from Corollary 3.7. ☐

In order to prepare a convenient limit passage in the fluid-related part of the weak formulation associated with the first equation in (2.6), let us make sure that when combined with (3.19), Lemma 4.1 secondly ensures a uniform integrability property of $(n_\varepsilon u_\varepsilon)_{\varepsilon\in(0,1)}$ through the following boundedness feature.

Lemma 4.3 *Assume that (1.3) holds with some $\alpha > \frac{1}{2}$. Then there exists $\kappa > 1$ with the property that given any $T>0$ one can find $C(T)>0$ fulfilling*

$$\int_0^T\int_\Omega |n_\varepsilon u_\varepsilon|^\kappa \le C(T) \qquad \textit{for all } \varepsilon\in(0,1). \tag{4.5}$$

PROOF. We first note that our assumption $\alpha > \frac{1}{2}$ especially ensures that $\frac{2}{3(1-\alpha)_+} > \frac{4}{3} > \frac{6}{5}$ and $\frac{6\alpha-2}{(1-\alpha)_+} \ge \frac{1}{(1-\alpha)_+} > 2$, and that according to (1.4) and (3.13) we thus have

$$q_\alpha > \frac{6}{5} \qquad \text{and} \qquad s_\alpha > 2. \tag{4.6}$$

Here the left inequality enables us to pick $q > \frac{6}{5}$ such that

$$q < q_\alpha, \tag{4.7}$$

whence observing that $\frac{10q}{5q+6} > 1$, we see that

$$s(\kappa) := \frac{2(6q-6\kappa-q\kappa)}{10q-6\kappa-5q\kappa}, \qquad \kappa \in \Big(1, \frac{10q}{5q+6}\Big),$$

is well-defined with

$$s(\kappa) \to \frac{2\cdot(6q-6-q)}{10q-6-5q} = 2 \qquad \text{as } \kappa \searrow 1.$$

Thanks to the right inequality in (4.6), we can therefore fix $\kappa > 1$ in such a way that

$$\kappa < \frac{10q}{5q+6} \tag{4.8}$$

and

$$s(\kappa) < s_\alpha, \tag{4.9}$$

where again relying on the inequality $q > \frac{6}{5}$ we can clearly achieve that also

$$\frac{6\kappa}{6-\kappa} < q.$$

As thus $\frac{6}{5} < \frac{6\kappa}{6-\kappa} < q$, we may thereupon employ the Hölder inequality and Young's inequality to estimate

$$\begin{aligned}
\int_0^T \int_\Omega |n_\varepsilon u_\varepsilon|^\kappa &\le \int_0^T \|n_\varepsilon(\cdot,t)\|^\kappa_{L^{\frac{6\kappa}{6-\kappa}}(\Omega)} \|u_\varepsilon(\cdot,t)\|^\kappa_{L^6(\Omega)} dt \\
&\le \int_0^T \|n_\varepsilon(\cdot,t)\|^{\frac{6q(\kappa-1)}{5q-6}}_{L^q(\Omega)} \|n_\varepsilon(\cdot,t)\|^{\frac{6q-6\kappa-q\kappa}{5q-6}}_{L^{\frac{6}{5}}(\Omega)} \|u_\varepsilon(\cdot,t)\|^\kappa_{L^6(\Omega)} dt \\
&\le \int_0^T \|u_\varepsilon(\cdot,t)\|^2_{L^6(\Omega)} dt + \int_0^T \|n_\varepsilon(\cdot,t)\|^{\frac{12q(\kappa-1)}{(5q-6)(2-\kappa)}}_{L^q(\Omega)} \|n_\varepsilon(\cdot,t)\|^{\frac{2(6q-6\kappa-q\kappa)}{(5q-6)(2-\kappa)}}_{L^{\frac{6}{5}}(\Omega)} dt
\end{aligned}$$

for all $T > 0$ and $\varepsilon \in (0,1)$, where we note that according to (4.8) we know that $\gamma := \frac{(5q-6)(2-\kappa)}{12(\kappa-1)}$ satisfies $\gamma > 1$, and that hence a second application of Young's inequality shows that

$$\int_0^T \|n_\varepsilon(\cdot,t)\|^{\frac{12q(\kappa-1)}{(5q-6)(2-\kappa)}}_{L^q(\Omega)} \|n_\varepsilon(\cdot,t)\|^{\frac{2(6q-6\kappa-q\kappa)}{(5q-6)(2-\kappa)}}_{L^{\frac{6}{5}}(\Omega)} dt \le \int_0^T \|n_\varepsilon(\cdot,t)\|^q_{L^q(\Omega)} dt + \int_0^T \|n_\varepsilon(\cdot,t)\|^{\frac{2(6q-6\kappa-q\kappa)}{(5q-6)(2-\kappa)}\cdot\frac{\gamma}{\gamma-1}}_{L^{\frac{6}{5}}(\Omega)} dt$$

for all $T > 0$ and $\varepsilon \in (0,1)$. Computing

$$\frac{2(6q-6\kappa-q\kappa)}{(5q-6)(2-\kappa)} \cdot \frac{\gamma}{\gamma-1} = \frac{2(6q-6\kappa-q\kappa)}{(5q-6)(2-\kappa)} \cdot \frac{1}{1-\frac{12(\kappa-1)}{(5q-6)(2-\kappa)}} = \frac{2(6q-6\kappa-q\kappa)}{10q-6\kappa-5q\kappa} = s(\kappa)$$

and using the continuity of the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, we thus infer that with some $C_1 > 0$ we have

$$\int_0^T \int_\Omega |n_\varepsilon u_\varepsilon|^\kappa \le C_1 \int_0^T \int_\Omega |\nabla u_\varepsilon|^2 + \int_0^T \int_\Omega n_\varepsilon^q + \int_0^T \|n_\varepsilon(\cdot,t)\|_{L^{\frac{6}{5}}(\Omega)}^{s(\kappa)} dt \qquad \text{for all } T > 0 \text{ and } \varepsilon \in (0,1),$$

and that the claim threfeore results from (4.2), (3.17) and (3.19) because of (4.7) and (4.9). □

5 Estimates for $D^2 c_\varepsilon$ and $c_{\varepsilon t}$

Beyond those documented in Lemma 3.6 and Corollary 3.7, due to an argument based on maximal Sobolev regularity the solution components c_ε can be seen to actually satisfy certain higher-order estimates.

Lemma 5.1 *Suppose that (1.3) is valid with some $\alpha > \frac{1}{2}$, and let $\lambda > 1$ be such that*

$$\lambda \le \frac{5}{4} \qquad \textit{and} \qquad \lambda < q_\alpha, \tag{5.1}$$

with q_α taken from (1.4). Then for any $\tau > 0$ and each $T > \tau$ there exists $C(\tau, T) = C(\tau, T; \lambda) > 0$ fulfilling

$$\int_\tau^T \|c_\varepsilon(\cdot,t)\|_{W^{2,\lambda}(\Omega)}^\lambda dt \le C(\tau,T) \qquad \textit{for all } \varepsilon \in (0,1) \tag{5.2}$$

and

$$\int_\tau^T \int_\Omega |c_{\varepsilon t}|^\lambda \le C(\tau,T) \qquad \textit{for all } \varepsilon \in (0,1). \tag{5.3}$$

Proof. Given $\tau > 0$, we fix a noncecreasing cut-off function $\zeta \in C^\infty([0,\infty))$ such that $\zeta \equiv 0$ in $[0, \frac{\tau}{2}]$ and $\zeta \equiv 1$ in $[\tau, \infty)$, and let

$$z_\varepsilon(x,t) := \zeta(t) \cdot c_\varepsilon(x,t), \qquad x \in \overline{\Omega},\ t \ge 0,\ \varepsilon \in (0,1).$$

Then by (2.6),

$$\begin{cases} z_{\varepsilon t} = \Delta z_\varepsilon - z_\varepsilon + h_\varepsilon(x,t), & x \in \Omega,\ t > 0, \\ \frac{\partial z_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ z_\varepsilon(x,0) = 0, & x \in \Omega, \end{cases} \tag{5.4}$$

where

$$h_\varepsilon(x,t) := \zeta(t) \cdot \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} - \zeta(t) u_\varepsilon \cdot \nabla c_\varepsilon + \zeta'(t) c_\varepsilon, \qquad x \in \Omega,\ t > 0,\ \varepsilon \in (0,1).$$

Here since $0 \le \zeta \le 1$, and since $\lambda \le \frac{5}{4}$ and $\lambda < \frac{10}{3}$, we may use Young's inequality to estimate

$$\begin{aligned}
\int_0^T \int_\Omega |h_\varepsilon|^\lambda &\le 3^\lambda \int_0^T \int_\Omega n_\varepsilon^\lambda + 3^\lambda \int_0^T \int_\Omega |u_\varepsilon \cdot \nabla c_\varepsilon|^\lambda + 3^\lambda C_1(\tau) \int_0^T \int_\Omega c_\varepsilon^\lambda \\
&\le 3^\lambda \int_0^T \int_\Omega n_\varepsilon^\lambda + 3^\lambda \int_0^T \int_\Omega |u_\varepsilon \cdot \nabla c_\varepsilon|^{\frac{5}{4}} + 3^\lambda C_1(\tau) \int_0^T \int_\Omega c_\varepsilon^{\frac{10}{3}} \\
&\quad + 3^\lambda |\Omega| T + 3^\lambda C_1(\tau) |\Omega| T \qquad \text{for all } T > 0 \text{ and } \varepsilon \in (0,1),
\end{aligned} \tag{5.5}$$

with $C_1(\tau) := \|\zeta'\|_{L^\infty((0,\infty))}^\lambda$. As another application of Young's inequality shows that

$$3^\lambda \int_0^T \int_\Omega |u_\varepsilon \cdot \nabla c_\varepsilon|^{\frac{5}{4}} \le 3^\lambda \int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}} + 3^\lambda \int_0^T \int_\Omega |\nabla c_\varepsilon|^2 \qquad \text{for all } T > 0 \text{ and } \varepsilon \in (0,1),$$

relying on the fact that $\lambda < q_\alpha$ by (5.1) we infer on combining (5.5) with (3.17), (4.4), (3.16) and (3.27) that for any $T > \tau$ we can find $C_2(\tau, T) > 0$ fulfilling

$$\int_0^T \int_\Omega |h_\varepsilon|^\lambda \le C_2(\tau, T) \qquad \text{for all } \varepsilon \in (0,1).$$

Now in view of (5.4), a standard result on maximal Sobolev regularity in the Neumann problem for the inhomigeneous linear heat equation ([21]) provides $C_3(T) > 0$ such that

$$\int_0^T \Big\{ \|z_\varepsilon(\cdot,t)\|_{W^{2,\lambda}(\Omega)}^\lambda + \|z_{\varepsilon t}(\cdot,t)\|_{L^\lambda(\Omega)}^\lambda \Big\} dt \le C_3(T) \int_0^T \int_\Omega |h_\varepsilon|^\lambda \qquad \text{for all } \varepsilon \in (0,1).$$

Since $z_\varepsilon \equiv c_\varepsilon$ in $\Omega \times (\tau, \infty)$ for all $\varepsilon \in (0,1)$, in conjunction with (5.5) this establishes both (5.2) and (5.3). $\square$

6 Time regularity of n_ε and u_ε

In preparation for an appropriate application of Aubin-Lions type statements on strong precompactness, let us furthermore note some straightforward implications of the estimates from Lemma 3.6 and Lemma 4.1 on regularity of the time derivatives $n_{\varepsilon t}$ and $u_{\varepsilon t}$.

Lemma 6.1 *Assume (1.3) with some $\alpha > \frac{1}{2}$. Then for all $T > 0$ there exists $C(T) > 0$ such that*

$$\int_\tau^T \|n_{\varepsilon t}(\cdot,t)\|_{(W^{3,2}(\Omega))^\star} dt \le C(T) \qquad \textit{for all } \varepsilon \in (0,1) \tag{6.1}$$

and

$$\int_\tau^T \|u_{\varepsilon t}(\cdot,t)\|_{(W_{0,\sigma}^{1,3}(\Omega))^\star}^2 dt \le C(T) \qquad \textit{for all } \varepsilon \in (0,1). \tag{6.2}$$

Proof. Since $W^{3,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, there exists $C_1 > 0$ such that $\|\nabla\psi\|_{L^\infty(\Omega)} \le C_1\|\psi\|_{W^{3,2}(\Omega)}$ for all $\psi \in C^3(\overline{\Omega})$. Thus, for fixed $t > 0$ and $\psi \in C^3(\overline{\Omega})$ with $\|\psi\|_{W^{3,2}(\Omega)} \le 1$ we can integrate by parts in (2.6) and estimate

$$\begin{aligned}
\left|\int_\Omega n_{\varepsilon t}(\cdot,t)\psi\right| &= \left|-\int_\Omega \nabla n_\varepsilon\cdot\nabla\psi + \int_\Omega \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} f(|\nabla c_\varepsilon|^2)\nabla c_\varepsilon\cdot\nabla\psi + \int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\psi\right| \\
&\le C_1\int_\Omega |\nabla n_\varepsilon| + C_1\int_\Omega \left|\frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} f(|\nabla c_\varepsilon|^2)\nabla c_\varepsilon\right| + C_1\int_\Omega |n_\varepsilon u_\varepsilon| \\
&\le C_1\int_\Omega |\nabla n_\varepsilon| + C_1K_f\int_\Omega n_\varepsilon(1+|\nabla c_\varepsilon|^2)^{-\frac{\alpha}{2}}|\nabla c_\varepsilon| + C_1\int_\Omega n_\varepsilon|u_\varepsilon| \\
&\le C_1\int_\Omega |\nabla n_\varepsilon| + C_1K_f\int_\Omega n_\varepsilon|\nabla c_\varepsilon|^{(1-\alpha)_+} + C_1\int_\Omega n_\varepsilon|u_\varepsilon| \qquad \text{for all } \varepsilon\in(0,1)
\end{aligned}$$

due to (1.3). Since Young's inequality implies that in both cases $\alpha < 1$ and $\alpha \ge 1$ we have

$$\int_\Omega n_\varepsilon|\nabla c_\varepsilon|^{(1-\alpha)_+} \le \int_\Omega n_\varepsilon^q + \int_\Omega |\nabla c_\varepsilon|^2 \qquad \text{for all } \varepsilon\in(0,1)$$

with $q := \max\{\frac{2}{1+\alpha},1\}$, and that

$$\int_\Omega n_\varepsilon|u_\varepsilon| \le \|n_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} + \|u_\varepsilon\|^2_{L^6(\Omega)} \qquad \text{for all } \varepsilon\in(0,1),$$

again relying on the continuity of the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ we thus obtain $C_2 > 0$ such that

$$\begin{aligned}
\|n_{\varepsilon t}(\cdot,t)\|_{(W^{3,2}(\Omega))^\star} &\le C_1\int_\Omega |\nabla n_\varepsilon| + C_1K_f\int_\Omega n_\varepsilon^q + C_1K_f\int_\Omega |\nabla c_\varepsilon|^2 \\
&\qquad +C_1\|n_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} + C_2\int_\Omega |\nabla u_\varepsilon|^2 \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1). \qquad (6.3)
\end{aligned}$$

Here we only need to observe that since $\alpha > \frac{1}{2}$, with q_α and s_α taken from (1.4) and (3.13) we have $q < q_\alpha$ and $2 < s_\alpha$, to see that (6.1) results from (6.3) when combined with (3.18), (3.17), (3.16), (3.19) and (4.2).

Likewise, we may use that $W^{1,3}(\Omega) \hookrightarrow L^6(\Omega)$ in choosing $C_3 > 0$ with the property that $\|\psi\|_{L^6(\Omega)} + \|\nabla\psi\|_{L^2(\Omega)} + \|\nabla\psi\|_{L^3(\Omega)} \le C_3$ for all $\psi \in C^\infty_{0,\sigma}(\Omega)$ with $\|\psi\|_{W^{1,3}(\Omega)} \le 1$, to see that given any such ψ, due to (2.6) and the Hölder inequality we have

$$\begin{aligned}
&\left|\int_\Omega u_{\varepsilon t}(\cdot,t)\cdot\psi\right| \\
&\quad = \left|-\int_\Omega \nabla u_\varepsilon\cdot\nabla\psi + \int_\Omega (Y_\varepsilon u_\varepsilon\otimes u_\varepsilon)\cdot\nabla\psi + \int_\Omega \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon}\nabla\Phi\cdot\psi\right| \\
&\quad \le C_3\|\nabla u_\varepsilon\|_{L^2(\Omega)} + C_3\|Y_\varepsilon u_\varepsilon\otimes u_\varepsilon\|_{L^{\frac{3}{2}}(\Omega)} + C_4\|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1) \quad (6.4)
\end{aligned}$$

with $C_4 := C_3 \|\nabla\Phi\|_{L^\infty(\Omega)}$. Here once more due to the continuity of $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, we may rely on the fact that Y_ε commutes with $A^{\frac{1}{2}}$ on $D(A^{\frac{1}{2}})$ to see that with some $C_5 > 0$ we have

$$\begin{aligned}
C_3 \|Y_\varepsilon u_\varepsilon \otimes u_\varepsilon\|_{L^{\frac{3}{2}}(\Omega)} &\le C_1 \|Y_\varepsilon u_\varepsilon\|_{L^6(\Omega)} \|u_\varepsilon\|_{L^2(\Omega)} \\
&\le C_5 \|\nabla Y_\varepsilon u_\varepsilon\|_{L^2(\Omega)} \|u_\varepsilon\|_{L^2(\Omega)} \\
&= C_5 \|A^{\frac{1}{2}} Y_\varepsilon u_\varepsilon\|_{L^2(\Omega)} \|u_\varepsilon\|_{L^2(\Omega)} \\
&= C_5 \|Y_\varepsilon A^{\frac{1}{2}} u_\varepsilon\|_{L^2(\Omega)} \|u_\varepsilon\|_{L^2(\Omega)} \\
&\le C_5 \|A^{\frac{1}{2}} u_\varepsilon\|_{L^2(\Omega)} \|u_\varepsilon\|_{L^2(\Omega)} \\
&= C_5 \|\nabla u_\varepsilon\|_{L^2(\Omega)} \|u_\varepsilon\|_{L^2(\Omega)} \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1),
\end{aligned}$$

because Y_ε is nonexpansive on $L^2(\Omega;\mathbb{R}^3)$. From (6.4) we therefore obtain that for all $t > 0$ and $\varepsilon \in (0,1)$,

$$\|u_{\varepsilon t}(\cdot,t)\|^2_{(W^{1,3}_{0,\sigma}(\Omega))^\star} \le 3C_3^2 \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + 3C_5^2 \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \|u_\varepsilon\|^2_{L^2(\Omega)} + 3C_4^2 \|n_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)},$$

which in conjunction with (4.2), (4.1) and (3.19) yields (6.2) after a time integration. $\Box$

7 Passing to the limit. Proof of Theorem 1.1

It remains to suitably exploit the weak and strong compactness features, as implied by the estimates gathered above, to construct a global weak solution with the claimed additional regularity properties through a straightforward extraction process.

Lemma 7.1 *Suppose that (1.3) is fulfilled with some $\alpha > \frac{1}{2}$, and let q_α and r_α be as defined in (1.4). Then there exist $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ and functions n, c and u, defined a.e. on $\Omega \times (0,\infty)$ and fulfilling (1.6), such that $\varepsilon_j \searrow 0$ as $j \to \infty$, that $n \ge 0$ and $c \ge 0$ a.e. in $\Omega \times (0,\infty)$, and that as $\varepsilon = \varepsilon_j \searrow 0$ we have*

$$n_\varepsilon \to n \quad \text{a.e. in } \Omega \times (0,\infty) \text{ and in } L^q_{loc}(\overline{\Omega} \times [0,\infty)) \text{ for all } q \in (1,q_\alpha), \tag{7.1}$$
$$\nabla n_\varepsilon \rightharpoonup \nabla n \quad \text{in } L^r_{loc}(\overline{\Omega} \times [0,\infty)) \text{ for all } r \in (1,r_\alpha), \tag{7.2}$$
$$c_\varepsilon \to c \quad \text{a.e. in } \Omega \times (0,\infty) \text{ and in } L^2_{loc}(\overline{\Omega} \times [0,\infty)), \tag{7.3}$$
$$\nabla c_\varepsilon \to \nabla c \quad \text{a.e. in } \Omega \times (0,\infty) \text{ and } L^\mu_{loc}(\overline{\Omega} \times [0,\infty)) \text{ for all } \mu \in (1,2), \tag{7.4}$$
$$u_\varepsilon \to u \quad \text{a.e. in } \Omega \times (0,\infty) \text{ and in } L^2_{loc}(\overline{\Omega} \times [0,\infty)), \tag{7.5}$$
$$u_\varepsilon(\cdot,t) \to u(\cdot,t) \quad \text{in } L^2(\Omega) \text{ for a.e. } t > 0 \qquad \text{and} \tag{7.6}$$
$$\nabla u_\varepsilon \rightharpoonup \nabla u \quad \text{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)). \tag{7.7}$$

Moreover, (n,c,u) is a global weak solution of (1.2) in the sense of Definition 2.1.

PROOF. We fix $q \in (1,q_\alpha)$ and $r \in (1,r_\alpha)$ to see that for arbitrary $T > 0$, Lemma 3.6 ensures that

$$(n_\varepsilon)_{\varepsilon\in(0,1)} \text{ is bounded in } L^q(\Omega \times (0,T)) \text{ and in } L^r((0,T); W^{1,r}(\Omega)), \tag{7.8}$$

whereas due to Lemma 6.1,

$$(n_{\varepsilon t})_{\varepsilon\in(0,1)} \text{ is bounded in } L^1\big((0,T);(W^{3,2}(\Omega))^\star\big).$$

Apart from that, Lemma 3.6 together with Corollary 3.7 shows that for each $T>0$,

$$(c_\varepsilon)_{\varepsilon\in(0,1)} \text{ is bounded in } L^\infty((0,T);L^2(\Omega)), \text{ in } L^{\frac{10}{3}}(\Omega\times(0,T)) \text{ and in } L^2((0,T);W^{1,2}(\Omega)), \tag{7.9}$$

and Lemma 5.1 provides $\lambda>1$ with the property that whenever $\tau>0$ and $T>\tau$,

$$(c_\varepsilon)_{\varepsilon\in(0,1)} \text{ is bounded in } L^\lambda((\tau,T);W^{2,\lambda}(\Omega)) \tag{7.10}$$

and

$$(c_{\varepsilon t})_{\varepsilon\in(0,1)} \text{ is bounded in } L^\lambda(\Omega\times(\tau,T)).$$

Finally, Lemma 4.1 and Corollary 4.2 warrant that for any $T>0$,

$$(u_\varepsilon)_{\varepsilon\in(0,1)} \text{ is bounded in } L^\infty((0,T);L^2_\sigma(\Omega)) \text{ and in } L^2((0,T);W^{1,2}_{0,\sigma}(\Omega)), \tag{7.11}$$

while Lemma 6.1 asserts that

$$(u_{\varepsilon t})_{\varepsilon\in(0,1)} \text{ is bounded in } L^2\big((0,T);(W^{1,3}_{0,\sigma}(\Omega))^\star\big).$$

Performing a straighforward extraction procedure based on appropriate Aubin-Lions lemmata ([38]), through compactness of the embeddings $W^{1,r}(\Omega)\hookrightarrow L^1(\Omega)$, $W^{2,\lambda}(\Omega)\hookrightarrow W^{1,1}(\Omega)$ and $W^{1,2}(\Omega)\hookrightarrow L^2(\Omega)$ we thus readily obtain $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ and functions n,c and u on $\Omega\times(0,\infty)$ such that $\varepsilon_j\searrow 0$ as $j\to\infty$, that n,c and u enjoy the regularity properties in (1.6), and that as $\varepsilon=\varepsilon_j\searrow 0$, besides (7.2), (7.3), (7.5), (7.6) and (7.7) we have

$$n_\varepsilon\to n \quad\text{and}\quad \nabla c_\varepsilon\to\nabla c \qquad \text{a.e. in } \Omega\times(0,\infty).$$

We therefore may once again use (7.8) and (7.9) to see by means of the Vitali convergence theorem that also (7.1) and (7.4) are valid as $\varepsilon=\varepsilon_j\searrow 0$.

Now to verify (2.3) for each $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$, given any such φ we go back to (2.6) to see that

$$\begin{aligned} -\int_0^\infty\int_\Omega n_\varepsilon\varphi_t-\int_\Omega n_0\varphi(\cdot,0) &= -\int_0^\infty\int_\Omega\nabla n_\varepsilon\cdot\nabla\varphi+\int_0^\infty\int_\Omega\frac{n_\varepsilon}{1+\varepsilon n_\varepsilon}f(|\nabla c_\varepsilon|^2)\nabla c_\varepsilon\cdot\nabla\varphi \\ &\quad +\int_0^\infty\int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\varphi \qquad \text{for all } \varepsilon\in(0,1), \end{aligned} \tag{7.12}$$

where clearly

$$-\int_0^\infty\int_\Omega n_\varepsilon\varphi_t\to-\int_0^\infty\int_\Omega n\varphi_t \quad\text{and}\quad -\int_0^\infty\int_\Omega\nabla n_\varepsilon\cdot\nabla\varphi\to-\int_0^\infty\int_\Omega\nabla n\cdot\nabla\varphi \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0 \tag{7.13}$$

by (7.1) and (7.2). Moreover, since Lemma 4.3 provides $\kappa > 1$ and $C_1 > 0$ such that taking $T > 0$ large fulfilling $\varphi \equiv 0$ in $\Omega \times (T, \infty)$ we have

$$\int_0^T \int_\Omega |n_\varepsilon u_\varepsilon|^\kappa \le C_1 \qquad \text{for all } \varepsilon \in (0,1),$$

the uniform integrability property of $(n_\varepsilon u_\varepsilon)_{\varepsilon\in(0,1)}$ thereby expressed can be combined with the fact that $n_\varepsilon u_\varepsilon \to nu$ a.e. in $\Omega \times (0,T)$ as $\varepsilon = \varepsilon_j \searrow 0$, as asserted by (7.1) and (7.5), to confirm that due to the Vitali convergence theorem we have $n_\varepsilon u_\varepsilon \to nu$ in $L^1(\Omega \times (0,T))$ and hence

$$\int_0^\infty \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla\varphi \to \int_0^\infty \int_\Omega nu \cdot \nabla\varphi \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0. \tag{7.14}$$

In order to proceed similarly in the second last summand in (7.12), we first note that the number

$$\mu_\alpha := \frac{2q_\alpha}{2 + (1-\alpha)_+ q_\alpha}$$

satisfies $\mu_\alpha > 1$, which actually is obvious if $\alpha \ge 1$, which if $\alpha \in [\frac{3}{5}, 1)$ followd from the fact that then

$$\frac{2q_\alpha}{2 + (1-\alpha)_+ q_\alpha} = \frac{\frac{10}{3}}{2 + \frac{5}{3}\cdot(1-\alpha)} = \frac{10}{11 - 5\alpha} \ge \frac{10}{11 - 5\cdot\frac{3}{5}} = \frac{5}{4},$$

and which for $\alpha \in (\frac{1}{2}, \frac{3}{5})$ can be seen by estimating

$$\frac{2q_\alpha}{2 + (1-\alpha)_+ q_\alpha} = \frac{\frac{4}{3(1-\alpha)}}{2 + (1-\alpha)\cdot\frac{2}{3(1-\alpha)}} = \frac{1}{2(1-\alpha)} > \frac{1}{2\cdot(1-\frac{1}{2})} = 1.$$

Based on this observation, we may pick any $\mu \in (1, \mu_\alpha)$ and rely on (1.3) and Young's inequality to see that with $T > 0$ as fixed above,

$$\begin{aligned}
\int_0^T \int_\Omega \Big| \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} f(|\nabla c_\varepsilon|^2)\nabla c_\varepsilon \Big|^\mu &\le K_f^\mu \int_0^T \int_\Omega n_\varepsilon^\mu |\nabla c_\varepsilon|^{\mu(1-\alpha)_+} \\
&\le K_f^\mu \int_0^T \int_\Omega n_\varepsilon^{\frac{2\mu}{2-\mu(1-\alpha)_+}} + K_f^\mu \int_0^T \int_\Omega |\nabla c_\varepsilon|^2 \qquad \text{for all } \varepsilon \in (0,1),
\end{aligned}$$

where due to the restriction $\mu < \mu_\alpha$,

$$\frac{2\mu}{2 - \mu(1-\alpha)_+} = \frac{2}{\frac{2}{\mu} - (1-\alpha)_+} < \frac{2}{\frac{2}{\mu_\alpha} - (1-\alpha)_+} = \frac{2}{\frac{2+(1-\alpha)_+ q_\alpha}{q_\alpha} - (1-\alpha)_+} = q_\alpha.$$

We may therefore once again resort to (7.8) and (7.9) to obtain $C_2 > 0$ such that

$$\int_0^T \int_\Omega \Big| \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} f(|\nabla c_\varepsilon|^2)\nabla c_\varepsilon \Big|^\mu \le C_2 \qquad \text{for all } \varepsilon \in (0,1),$$

and to thus infer from the pointwise convergence statements in (7.1) and (7.4) that again thanks to the Vitali convergence theorem,

$$\frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} f(|\nabla c_\varepsilon|^2)\nabla c_\varepsilon \to n f(|\nabla c|^2)\nabla c \quad \text{in } L^1(\Omega\times(0,T)) \qquad \text{as } \varepsilon=\varepsilon_j \searrow 0$$

and that hence, in particular,

$$\int_0^\infty \int_\Omega \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} f(|\nabla c_\varepsilon|^2)\nabla c_\varepsilon \cdot \nabla\varphi \to \int_0^\infty \int_\Omega n f(|\nabla c|^2)\nabla c\cdot\nabla\varphi \qquad \text{as } \varepsilon=\varepsilon_j \searrow 0,$$

because f is continuous on $[0,\infty)$. Together with (7.13) and (7.14), this shows that (7.12) entails the claimed identity in (2.3) upon letting $\varepsilon=\varepsilon_j \searrow 0$.

The derivation of (2.4) for $\varphi \in C_0^\infty(\overline{\Omega}\times[0,\infty))$ can be accomplished in quite a straightforward manner on the basis of (7.3), (7.4), (7.1) and (7.5), while (2.5) for any fixed $\varphi \in C_{0,\sigma}^\infty(\Omega\times[0,\infty))$ can be verified by combining (7.5) and (7.6) with (7.7) and (7.1) through a standard argument in treating the corresponding nonlinear convective contributions (see, e.g., [49, Lemma 4.1] for a detailed reasoning in this regard). As the regularity requirements in (2.1) and (2.2) are clearly implied by the features stated in (1.6) when combined with (1.3), we thereby conclude that indeed (n,c,u) is a global weak solution of (1.2) in the claimed sense. $\Box$

Proof of Theorem 1.1. The statement is actually part of what has been derived in Lemma 7.1. $\Box$

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