Suppressing blow-up by gradient-dependent flux limitation in a planar Keller-Segel-Navier-Stokes system

Michael Winkler[*]
Institut für Mathematik, Universität Paderborn,
Warburger Str. 100, 33098 Paderborn, Germany

Abstract

The flux-limited Keller-Segel-Navier-Stokes system

$$\begin{cases} n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot \Big(n f(|\nabla c|^2) \nabla c \Big), \\ c_t + u \cdot \nabla c &= \Delta c - c + n, \\ u_t + (u \cdot \nabla) u &= \Delta u + \nabla P + n \nabla \Phi, \qquad \nabla \cdot u = 0, \end{cases} \qquad (\star)$$

is considered in a smoothly bounded domain $\Omega \subset \mathbb{R}^2$. It is shown that whenever the suitably smooth function f models any asymptotically algebraic-type saturation of cross-diffusive fluxes in the sense that

$$|f(\xi)| \le K_f \cdot (\xi+1)^{-\frac{\alpha}{2}}$$

holds for all $\xi \ge 0$ with some $K_f > 0$ and $\alpha > 0$, for any all reasonably regular initial data a corresponding no-flux/no-flux/Dirichlet problem admits a globally defined classical solution which is bounded, inter alia, in $L^\infty(\Omega \times (0,\infty))$ with respect to all its components. By extending a corresponding result known for a fluid-free counterpart of $(\star)$, this confirms that with regard to the possible emergence of blow-up phenomena, the choice $f \equiv const.$ retains some criticality also in the presence of fluid interaction.

Key words: chemotaxis; Navier-Stokes; flux limitation
MSC (2010): 35K65 (primary); 35Q55, 92C17 (secondary)

[*]michael.winkler@math.uni-paderborn.de

1 Introduction

The choice $f \equiv K_f = const.$ in the two-dimensional version of the chemotaxis model

$$\begin{cases} n_t = \Delta n - \nabla \cdot \Big(n f(|\nabla c|^2) \nabla c \Big), \\ \tau c_t = \Delta c - c + n, \end{cases} \tag{1.1}$$

is critical in several respects: Firstly, the classical Keller-Segel systems thus obtained are themselves well-known to be mass-critical in the sense that with some $m_c = m_c(K_f) > 0$, global bounded solutions to associated no-flux problems in smoothly bounded planar domains Ω can be found whenever the suitably regular initial data satisfy $\int_\Omega n(\cdot,0) < m_c$ whereas some unbounded solutions emanating from appropriately chosen initial data with $\int_\Omega n(\cdot,0) > m_c$ exist (see [25], [4] and [26] for the simplified parabolic-elliptic case with $\tau = 0$, and [27], [18] as well as [19] for the fully parabolic variant involving $\tau = 1$). Secondly, already small perturbations of such constant f, asymptotically damping the corresponding cross-diffusion mechanism in the spirit of gradient-dependent flux limitations recently receiving considerable attention in the biomathematical literature ([1], [29], [2], [3]), seem to entirely inhibit any such blow-up phenomenon: If f is suitably smooth and such that

$$|f(\xi)| \le K_f \cdot (\xi + 1)^{-\frac{\alpha}{2}} \qquad \text{for all } \xi \ge 0 \tag{1.2}$$

with some $K_f > 0$ and $\alpha > 0$, namely, then all reasonably regular initial data of arbitrary size lead to global bounded solutions both when $\tau = 0$ and when $\tau = 1$ ([28], [45], [46]).

A natural next task now seems to consist in determining how far such criticality features persist when single two-component systems of the form (1.1) are embedded into more complex models. Here in accordance with considerable developments in both the modeling and the analysis-focused literature, of particular interest appear couplings of chemotactically migrating populations to liquid environments ([8], [9], [24], [33], [34]), and indeed some significant effects of fluid interaction have either been indicated by numerical evidence ([34], [23]), or even been verified rigorously ([21], [22], [16], [10]).

In order to address this topic for possibly flux-limited Keller-Segel systems of the form in a framework consistent with the modeling approach in [34], concentrating on the prototypical situation of bacterial populations coupled to a surrounding liquid through transport and buocancy we shall subsequently consider the problem

$$\begin{cases} n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot \Big(n f(|\nabla c|^2) \nabla c \Big), & x \in \Omega,\ t > 0, \\ c_t + u \cdot \nabla c &= \Delta c - c + n, & x \in \Omega,\ t > 0, \\ u_t + (u \cdot \nabla) u &= \Delta u + \nabla P + n \nabla \Phi, \qquad \nabla \cdot u = 0, & x \in \Omega,\ t > 0, \\ \frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0, \quad u = 0, & & x \in \partial\Omega,\ t > 0, \\ n(x,0) = n_0(x), \quad c(x,0) = c_0(x), \quad u(x,0) = u_0(x), & & x \in \Omega, \end{cases} \tag{1.3}$$

in a smoothly bounded domain $\Omega \subset \mathbb{R}^2$, with a suitably regular chemotactic sensitivity coefficient f and a given gravitational potential Φ.

Within this setting and that of a corresponding parabolic-elliptic simplification, the first of the above two criticality features of $f \equiv const.$ has already been investigated numericaly and analytically: While simulations predict that fluid coupling of this type may entail some blow-up delay by enforcing boundedness in the presence of large-mass initial data for which solutions to (1.1) blow up ([23]), a result on global solvability for small-mass initial data indicates that the corresponding critical mass phenomenon for (1.1), albeit with a possibly changed particular threshold value, persists also under the influence of the considered fluid interplay ([43]).

Main results. The purpose of the present study now consists in examining corresponding stability properties of the second among the mentioned aspects of criticality related to the choice of constant f in (1.1). Specifically concerned with the question whether in the fully coupled model (1.3) still an arbitrarily small algebraic flux limitation of the form in (1.2) is sufficient to suppress any unboundedness phenomenon, by means of an analytical approach quite completely different from those pursued in the literature on (1.1) we shall find the following essentially affirmative answer. In its formulation, as well as throughout the sequel, we let $W^{1,2}_{0,\sigma}(\Omega) := W^{1,2}_0(\Omega;\mathbb{R}^2) \cap L^2_\sigma(\Omega)$ with $L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega;\mathbb{R}^2) \mid \nabla\cdot\varphi = 0\}$, and letting $\mathcal{P}$ represent the Helmholtz projection on $\bigcap_{r>1} L^r(\Omega;\mathbb{R}^2)$ we let $A = -\mathcal{P}\Delta$ and $(A^\vartheta)_{\vartheta>0}$ denote the Stokes operator in $L^2(\Omega;\mathbb{R}^2)$, with domain given by $D(A) := W^{2,2}(\Omega;\mathbb{R}^2) \cap W^{1,2}_{0,\sigma}(\Omega)$, and the family of its corresponding fractional powers, respectively.

Theorem 1.1 *Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded convex domain with smooth boundary, and assume that $\Phi \in W^{2,\infty}(\Omega)$, and that $f \in C^2([0,\infty))$ is such that $f \in C^2([0,\infty))$ satisfies (1.2) with some $K_f > 0$ and some*

$$\alpha > 0.$$

Moreover, suppose that

$$\begin{cases} n_0 \in C^0(\overline{\Omega}) \text{ is nonnegative with } n_0 \not\equiv 0, \quad \text{that} \\ c_0 \in W^{1,\infty}(\Omega) \text{ is nonnegative,} \quad \text{and that} \\ u_0 \in W^{1,2}_{0,\sigma}(\Omega) \cap W^{2,2}(\Omega;\mathbb{R}^2). \end{cases} \tag{1.4}$$

Then there exist uniquely determined functions

$$\begin{cases} n \in C^0(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)), \\ c \in \bigcap_{q>1} C^0([0,\infty);W^{1,2q}(\Omega)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)) \qquad \text{and} \\ u \in \bigcap_{\vartheta\in(\frac12,1)} C^0([0,\infty);D(A^\vartheta)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty);\mathbb{R}^2) \end{cases} \tag{1.5}$$

which are such that $n > 0$ and $c \ge 0$ in $\overline{\Omega}\times(0,\infty)$, and that with some $P \in C^{1,0}(\Omega\times(0,\infty))$, the quadruple (n,c,u,P) is a classical solution of (1.3) in $\Omega\times(0,\infty)$. Moreover, this solution is bounded in the sense that one can find $q > 1$ and $\vartheta \in (\frac12,1)$ satisfying

$$\sup_{t>0}\Big\{ \|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,2q}(\Omega)} + \|A^\vartheta u(\cdot,t)\|_{L^2(\Omega)} \Big\} < \infty. \tag{1.6}$$

By thus essentially asserting persistence of a boundedness feature, as known to be enjoyed by a chemotaxis system, under the additional inclusion of fluid interaction, this result appears to be quite in line with some precedent studies concerned with variants of (1.1) involving migration rates which depend on population densities, rather than signal gradients. In fact, the substantially more favorable mathematical structure of such systems has allowed for quite a complete knowledge concerning blow-up dichotomies, and for a widely comprehensive transfer of results from corresponding taxis-only systems to associated Keller-Segel fluid variants, both in cases governed by linear cell diffusion ([36], [37], [35], [41]) and in models involving certain nonlinear diffusion mechanisms ([6], [5], [44], [7], [30], [32], [38]).

Main ideas. In contrast to the simple situation of (1.1), any derivation of boundedness features for solutions to (1.3) evidently needs to adequately cope with the circumstance that regularity properties of the crucial quantity ∇c may potentially be affected by the fluid field which is a priori unknown. Accordingly, this seems to limit accessibility to well-established direct strategies toward establishing L^p bounds for n, e.g. by controlling the time evolution in expressions of the form

$$\int_\Omega n^p + a\int_\Omega |\nabla c|^{2q}, \tag{1.7}$$

as having played key roles in studies on chemotaxis systems for various choices of $a > 0$ and $q \ge 1$ ([32], [47], [20]). In fact, at a first stage the only source for regularity information on u appears to be the standard energy identity for the Navier-Stokes subsystem of (1.3), any meaningful application of which, however, seems to require integrability features of the forcing term $n\nabla\Phi$ therein that go beyond those trivially implied by mass conservation in the first solution component. A first and crucial step in our analysis will accordingly consist in the derivation of some additional basic estimates for n which do not rely on any features of the fluid field beyond its mere solenoidality, and which provide integrability properties sufficient to allow for a successful initial regularity analysis of u (Lemma 3.3). Fortunately, in Section 3 we shall see that precisely when $\alpha > 0$, the assumption (1.2) warrants that this can indeed be achieved by tracing the evolution of

$$-\int_\Omega n^p + \int_\Omega (c+M)^r \tag{1.8}$$

for some $p \in (0,1)$, arbitrarily large r and suitably chosen $M > 0$ (cf. Lemma 3.6); although in its principal design related to the strategy from [35] where similar sublinear powers have been essential, this procedure deviates from the latter by simultaneously providing bounds for c which will later on come in handy when an apparently novel interpolation inequality will be applied (Lemma 4.1, Lemma 4.5 and Lemma 4.6). Only after this preparation, Section 4 will reveal that an analysis of higher regularity features can be based on the examination of functionals in the style of those from (1.7) with arbitrarily large $p > 1$ and $q > 1$, when augmented by an added Dirichlet integral of the fluid velocity. In fact, appropriately estimating respectively appearing ill-signed contributions will enable us to derive temporally uniform bounds for the coupled functionals under consideration (Lemma 4.8), and to thereupon obtain our main result by means of a final bootstrap argument to be performed in Section 5.

2 Local existence and extensibility

Let us first note that according to approaches meanwhile quite standard, unique local-in-time classical solutions enjoying a fairly handy entensibility feature always exist:

Lemma 2.1 *Suppose that $f \in C^2([0,\infty))$ and that (n_0, c_0, u_0) satisfies (1.4). Then there exist $T_{max} \in (0,\infty]$ and unique functions*

$$\begin{cases} n \in C^0(\overline{\Omega} \times [0,T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{max})), \\ c \in \bigcap_{q>1} C^0([0,T_{max}); W^{1,2q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,T_{max})) \qquad \text{and} \\ u \in \bigcap_{\vartheta \in (\frac{1}{2},1)} C^0([0,T_{max}); D(A^\vartheta)) \cap C^{2,1}(\overline{\Omega} \times (0,T_{max,\varepsilon}); \mathbb{R}^2) \end{cases}$$

such that $n > 0$ and $c \geq 0$ in $\overline{\Omega} \times [0,T_{max})$, that

$$\text{if } T_{max} = \infty, \qquad \text{or for all } q > 1 \text{ and } \vartheta \in (\tfrac{1}{2},1) \text{ we have}$$
$$\limsup_{t \nearrow T_{max}} \Big\{ \|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,2q}(\Omega)} + \|A^\vartheta u(\cdot,t)\|_{L^2(\Omega)} \Big\} = \infty, \tag{2.1}$$

and that (n,c,u,P) forms a classical solution of (1.3) in $\Omega \times (0,T_{max})$ with some appropriate $P \in C^{1,0}(\Omega \times (0,T_{max}))$. This solution has the additional property that

$$\int_\Omega n(\cdot,t) = \int_\Omega n_0 \qquad \text{for all } t \in (0,T_{max}). \tag{2.2}$$

PROOF. This can be seen by straightforward adaptation of well-known arguments from the literature on local existence, extensibility, positivity and mass conservation in related chemotaxis-fluid systems, as detailed in [39], for instance. □

3 Basic bounds for n and c. Analyzing the Navier-Stokes energy

The first objective of this section is to make sure that functionals of the form in (1.8) enjoy certain entropy-like features when the integrability exponent $p \in (0,1)$ appearing therein is suitably small. Besides implying bounds for c in L^r spaces with arbitrarily large r, this will reveal some additional regularity features of n that will turn out to be sufficient for a successful analysis of the standard energy identity associated with the Navier-Stokes subsystem of (1.3).

Our analysis in this direction is launched by the following observation which already relies in an essentiall manner on the flux limitation mechanism expressed in our overall assumption (1.2).

Lemma 3.1 *Assume (1.2) with some $K_f > 0$ and $\alpha > 0$, and let $p \in (0,1)$ be such that*

$$p \leq \frac{\alpha}{1-\alpha}. \tag{3.1}$$

Then there exists $C = C(p) > 0$ such that

$$-\frac{d}{dt}\int_\Omega n^p + \frac{p(1-p)}{2}\int_\Omega n^{p-2}|\nabla n|^2 \leq C\int_\Omega |\nabla c|^2 + C \qquad \text{for all } t \in (0,T_{max}). \tag{3.2}$$

Proof. We make use of the positivity of n in $\overline{\Omega}\times(0,T_{max})$ to see upon integrating by parts in the first equation from (1.3) and using Young's inequality along with (1.2) that

$$
\begin{aligned}
&-\frac{1}{p}\frac{d}{dt}\int_\Omega n^p + (1-p)\int_\Omega n^{p-2}|\nabla n|^2 \\
&\quad= (1-p)\int_\Omega n^{p-1} f(|\nabla c|^2)\nabla n\cdot\nabla c \\
&\quad\le \frac{1-p}{4}\int_\Omega n^{p-2}|\nabla n|^2 + (1-p)\int_\Omega n^p f^2(|\nabla c|^2)|\nabla c|^2 \\
&\quad\le \frac{1-p}{4}\int_\Omega n^{p-2}|\nabla n|^2 + (1-p)K_f^2\int_\Omega n^p(|\nabla c|^2+1)^{-\alpha}|\nabla c|^2 \qquad \text{for all } t\in(0,T_{max}).
\end{aligned}
\tag{3.3}
$$

Here if $\alpha\ge 1$, then by the Hölder inequality and (2.2), abbreviating $C_1 := \int_\Omega n_0$ we have

$$
\begin{aligned}
(1-p)K_f^2\int_\Omega n^p(|\nabla c|^2+1)^{-\alpha}|\nabla c|^2 &\le (1-p)K_f^2\int_\Omega n^p \\
&\le (1-p)K_f^2|\Omega|^{1-p}\cdot\Big\{\int_\Omega n\Big\}^p \\
&= (1-p)K_f^2|\Omega|^{1-p}C_1^p \qquad \text{for all } t\in(0,T_{max}),
\end{aligned}
$$

while if $\alpha\in(0,1)$ and $p\le\alpha$, then after employing Young's inequality we may proceed similarly to estimate

$$
\begin{aligned}
(1-p)K_f^2\int_\Omega n^p(|\nabla c|^2+1)^{-\alpha}|\nabla c|^2 &\le (1-p)K_f^2\int_\Omega n^p|\nabla c|^{2-2\alpha} \\
&\le (1-p)K_f^2\int_\Omega n^{\frac{p}{\alpha}} + (1-p)^2K_f^2\int_\Omega|\nabla c|^2 \\
&\le (1-p)K_f^2\cdot|\Omega|^{1-\frac{p}{\alpha}}C_1^{\frac{p}{\alpha}} + (1-p)^2K_f^2\int_\Omega|\nabla c|^2
\end{aligned}
$$

for all $t\in(0,T_{max})$, so that in both these cases, (3.2) is a consequence of (3.3).

If $\alpha<1$ and $\alpha<p\le\frac{\alpha}{1-\alpha}$, finally, then in view of the Gagliardo-Nirenberg inequality we can pick $C_2=C_2(p)>0$ such that

$$
\|\varphi\|_{L^{2\alpha}(\Omega)}^{\frac{2p}{p-\alpha}} \le C_2\|\nabla\varphi\|_{L^2(\Omega)}^2\|\varphi\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2\alpha}{p-\alpha}} + C_2\|\varphi\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2p}{p-\alpha}} \qquad \text{for all } \varphi\in W^{1,2}(\Omega),
$$

to see that again by Young's inequality and (2.2),

$$
\begin{aligned}
&(1-p)K_f^2\int_\Omega n^p(|\nabla c|^2+1)^{-\alpha}|\nabla c|^2 \\
&\quad\le (1-p)K_f^2\cdot\Big\{\int_\Omega n^{\frac{p}{\alpha}}\Big\}^\alpha\cdot\Big\{\int_\Omega|\nabla c|^2\Big\}^{1-\alpha} \\
&\quad\le (1-p)K_f^2C_2^{\frac{p-\alpha}{p}}\cdot\Big\{\|\nabla n^{\frac{p}{2}}\|_{L^2(\Omega)}^2\|n^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2\alpha}{p-\alpha}} + \|n^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2p}{p-\alpha}}\Big\}^{\frac{p-\alpha}{p}}\cdot\Big\{\int_\Omega|\nabla c|^2\Big\}^{1-\alpha} \\
&\quad= (1-p)K_f^2C_2^{\frac{p-\alpha}{p}}\cdot\Big\{\frac{p^2C_1^{\frac{p\alpha}{p-\alpha}}}{4}\int_\Omega n^{p-2}|\nabla n|^2 + C_1^{\frac{4}{p-\alpha}}\Big\}^{\frac{p-\alpha}{p}}\cdot\Big\{\int_\Omega|\nabla c|^2\Big\}^{1-\alpha}
\end{aligned}
$$

for all $t \in (0, T_{max})$. Taking $C_3 = C_3(p) > 0$ such that in accordance with Young's inequality we have

$$ab \le \frac{1-p}{p^2 C_1^{\frac{p\alpha}{p-\alpha}}} a^{\frac{p}{p-\alpha}} + C_3 b^{\frac{p}{\alpha}} \qquad \text{for all } a \ge 0 \text{ and } b \ge 0,$$

we therefore obtain that for all $t \in (0, T_{max})$,

$$\begin{aligned}
&(1-p)K_f^2 \int_\Omega n^p (|\nabla c|^2 + 1)^{-\alpha} |\nabla c|^2 \\
&\le \frac{1-p}{p^2 C_1^{\frac{p\alpha}{p-\alpha}}} \cdot \left\{ \frac{p^2 C_1^{\frac{p\alpha}{p-\alpha}}}{4} \int_\Omega n^{p-2} |\nabla n|^2 + C_1^{\frac{4}{p-\alpha}} \right\} + C_3 \cdot \left\{ (1-p) K_f^2 C_2^{\frac{p-\alpha}{p}} \cdot \left\{ \int_\Omega |\nabla c|^2 \right\}^{1-\alpha} \right\}^{\frac{p}{\alpha}} \\
&= \frac{1-p}{4} \int_\Omega n^{p-2} |\nabla n|^2 + \frac{(1-p) C_1^{\frac{4-p\alpha}{p-\alpha}}}{p^2} + (1-p)^{\frac{p}{\alpha}} K_f^{\frac{2p}{\alpha}} C_2^{\frac{p-\alpha}{\alpha}} C_3 \cdot \left\{ \int_\Omega |\nabla c|^2 \right\}^{\frac{p(1-\alpha)}{\alpha}},
\end{aligned}$$

so that since, clearly,

$$\left\{ \int_\Omega |\nabla c|^2 \right\}^{\frac{p(1-\alpha)}{\alpha}} \le \int_\Omega |\nabla c|^2 + 1 \qquad \text{for all } t \in (0, T_{max})$$

due to the fact that $\frac{p(1-\alpha)}{\alpha} \le 1$, we readily infer (3.2) from (3.3) also for such p. $\square$

Fortunately, the ill-signed contribution to the right-hand side of (3.2) is dominated by the dissipation rate appearing in a zero-order testing procedure which operates on the second equation in (1.3), but which unlike that to be pursued later when analyzing functionals of the form in (1.7) (see Lemma 4.3) does not presuppose any quantitative regularity feature of the fluid field.

Lemma 3.2 *Suppose that (1.2) is satisfied with some $K_f > 0$ and $\alpha > 0$, and let $p \in (0,1)$, $r \ge 3$ and $M \ge 1$. Then there exists $C(M) = C(M; p, r) > 0$ with the property that*

$$\frac{d}{dt} \int_\Omega (c+M)^r + M \int_\Omega |\nabla c|^2 + \int_\Omega (c+M)^r \le \frac{p(1-p)}{4} \int_\Omega n^{p-2} |\nabla n|^2 + C(M) \qquad \textit{for all } t \in (0, T_{max}). \tag{3.4}$$

Proof. We fix $q = q(p,r) > 1$ such that

$$q < \frac{r}{r-p}, \tag{3.5}$$

and observe that then the inequalities $r > p+1 > 1$ ensure that $q < \frac{1}{1-p}$ and $q < r$, and that hence

$$\frac{pq}{q-1} > 1 \qquad \text{and} \qquad \frac{2q(r-1)}{(q-1)r} > 2. \tag{3.6}$$

As furthermore clearly $\frac{2q}{p} > 2$, we may interpolate using the Gagliardo-Nirenberg inequality to find $C_1 = C_1(p,r) > 0$ and $C_2 = C_2(p,r) > 0$ such that

$$\|\varphi\|_{L^{\frac{2q}{p}}(\Omega)}^{\frac{2q}{q-1}} \le C_1 \|\nabla\varphi\|_{L^2(\Omega)}^2 \|\varphi\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{q-1}} + C_1 \|\varphi\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2q}{q-1}} \qquad \text{for all } \varphi \in W^{1,2}(\Omega) \tag{3.7}$$

and

$$\|\varphi\|^2_{L^{\frac{2q(r-1)}{(q-1)r}}(\Omega)} \le C_2\|\nabla\varphi\|^2_{L^2(\Omega)} + C_2\|\varphi\|^2_{L^2(\Omega)} \qquad \text{for all } \varphi\in W^{1,2}(\Omega). \tag{3.8}$$

Moreover, relying on the left inequality in (3.6) we may invoke Young's inequality to see that if we abbreviate $C_3 := \int_\Omega n_0$, then with some $C_4 = C_4(p,r) > 0$ we have

$$ab \le \frac{1-p}{pC_3^{\frac{p}{q-1}}} a^{\frac{pq}{q-1}} + C_4 b^{\frac{pq}{pq-q+1}} \qquad \text{for all } a\ge 0 \text{ and } b\ge 0, \tag{3.9}$$

and to appropriately make use of these selections, we multiply the second equation in (1.3) by $(c+M)^{r-1}$ and integrate by parts to find that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (c+M)^r + \frac{4(r-1)}{r}\int_\Omega \big|\nabla(c+M)^{\frac{r}{2}}\big|^2 \\
&\quad = -r\int_\Omega c(c+M)^{r-1} + r\int_\Omega n(c+M)^{r-1} \\
&\quad = -r\int_\Omega (c+M)^r + rM\int_\Omega (c+M)^{r-1} + r\int_\Omega n(c+M)^{r-1} \qquad \text{for all } t\in(0,T_{max}).
\end{aligned} \tag{3.10}$$

Here by Young's inequality,

$$\begin{aligned}
rM\int_\Omega (c+M)^{r-1} &= \int_\Omega \Big\{\frac{r(r-2)}{r-1}(c+1)^r\Big\}^{\frac{r-1}{r}} \cdot \Big\{r^{\frac{1}{r}}M\cdot\Big(\frac{r-1}{r-2}\Big)^{\frac{r-1}{r}}\Big\} \\
&\le \int_\Omega \Big\{\frac{r-1}{r}\cdot\Big\{\frac{r(r-2)}{r-1}(c+M)^r\Big\} + \frac{1}{r}\cdot\Big\{r^{\frac{1}{r}}M\cdot\Big(\frac{r-1}{r-2}\Big)^{\frac{r-1}{r}}\Big\}^r\Big\} \\
&= (r-2)\int_\Omega (c+M)^r + \Big(\frac{r-1}{r-2}\Big)^{r-1} M^r|\Omega| \qquad \text{for all } t\in(0,T_{max}),
\end{aligned} \tag{3.11}$$

while combining the Hölder inequality with (3.7) and (3.9), we infer that according to (2.2) and our definition of C_3,

$$\begin{aligned}
r\int_\Omega n(c+M)^{r-1} &\le r\cdot\Big\{\int_\Omega n^q\Big\}^{\frac{1}{q}}\cdot\Big\{\int_\Omega (c+M)^{\frac{q(r-1)}{q-1}}\Big\}^{\frac{q-1}{q}} \\
&= r\|n^{\frac{p}{2}}\|^{\frac{2}{p}}_{L^{\frac{2q}{p}}(\Omega)} \|(c+M)^{\frac{r}{2}}\|^{\frac{2(r-1)}{r}}_{L^{\frac{2q(r-1)}{(q-1)r}}(\Omega)} \\
&\le C_1^{\frac{q-1}{pq}} r\cdot\Big\{\|\nabla n^{\frac{p}{2}}\|^2_{L^2(\Omega)}\|n^{\frac{p}{2}}\|^{\frac{2}{q-1}}_{L^{\frac{2}{p}}(\Omega)} + \|n^{\frac{p}{2}}\|^{\frac{2q}{q-1}}_{L^{\frac{2}{p}}(\Omega)}\Big\}^{\frac{q-1}{q}} \cdot \|(c+M)^{\frac{r}{2}}\|^{\frac{2(r-1)}{r}}_{L^{\frac{2q(r-1)}{(q-1)r}}(\Omega)} \\
&= C_1^{\frac{q-1}{pq}} r\cdot\Big\{\frac{p^2C_3^{\frac{p}{q-1}}}{4}\int_\Omega n^{p-2}|\nabla n|^2 + C_3^{\frac{pq}{q-1}}\Big\}^{\frac{q-1}{pq}} \cdot \|(c+M)^{\frac{r}{2}}\|^{\frac{2(r-1)}{r}}_{L^{\frac{2q(r-1)}{(q-1)r}}(\Omega)} \\
&\le \frac{1-p}{pC_3^{\frac{p}{q-1}}}\cdot\Big\{\frac{p^2C_3^{\frac{p}{q-1}}}{4}\int_\Omega n^{p-2}|\nabla n|^2 + C_3^{\frac{pq}{q-1}}\Big\}
\end{aligned}$$

$$+C_5\|(c+M)^{\frac{r}{2}}\|_{L^{\frac{2q(r-1)}{(q-1)r}}(\Omega)}^{\frac{2pq(r-1)}{(pq-q+1)r}}$$

$$= \frac{p(1-p)}{4}\int_\Omega n^{p-2}|\nabla n|^2 + \frac{(1-p)C_3^p}{p} + C_5\|(c+M)^{\frac{r}{2}}\|_{L^{\frac{2q(r-1)}{(q-1)r}}(\Omega)}^{\frac{2pq(r-1)}{(pq-q+1)r}} \qquad (3.12)$$

for all $t\in(0,T_{max})$, with $C_5 = C_5(p,r) := C_4\cdot(C_1^{\frac{q-1}{pq}} r)^{\frac{pq}{pq-q+1}}$. We now employ (3.8) and then use that according to (3.5) the positive number $\theta = \theta(p,r) := \frac{pq(r-1)}{(pq-q+1)r}$ satisfies

$$\theta - 1 = -\frac{r-(r-p)q}{(pq-q+1)r} < 0,$$

and hence $\theta > 1$, to see once more by means of Young's inequality that

$$\begin{aligned} C_5\|(c+M)^{\frac{r}{2}}\|_{L^{\frac{2q(r-1)}{(q-1)r}}(\Omega)}^{\frac{2pq(r-1)}{(pq-q+1)r}} &\le C_2^\theta C_5\cdot\Big\{\int_\Omega|\nabla(c+M)^{\frac{r}{2}}|^2 + \int_\Omega(c+M)^r\Big\}^\theta \\ &\le \int_\Omega|\nabla(c+M)^{\frac{r}{2}}|^2 + \int_\Omega(c+M)^r + (C_2^\theta C_5)^{\frac{1}{1-\theta}} \qquad \text{for all } t\in(0,T_{max}). \end{aligned}$$

Together with (3.12) and (3.11) inserted into (3.10), this shows that

$$\begin{aligned} &\frac{d}{dt}\int_\Omega(c+M)^r + \Big\{\frac{4(r-1)}{r}-1\Big\}\cdot\int_\Omega|\nabla(c+M)^{\frac{r}{2}}|^2 + \Big\{r-(r-2)-1\Big\}\cdot\int_\Omega(c+M)^r \\ &\quad\le \frac{1-p}{4}\int_\Omega n^{p-2}|\nabla n|^2 + \Big(\frac{r-1}{r-2}\Big)^{r-1}M^r|\Omega| + \frac{(1-p)C_3^p}{p} + (C_2^\theta C_5)^{\frac{1}{1-\theta}} \qquad \text{for all } t\in(0,T_{max}) \end{aligned}$$

and thereby establishes (3.4), because $r-(r-2)-1=1$, and because thanks to our restrictions $r\ge 3$ and $M\ge 1$,

$$\Big\{\frac{4(r-1)}{r}-1\Big\}\cdot\int_\Omega|\nabla(c+M)^{\frac{r}{2}}|^2 = \frac{r(3r-4)}{4}\int_\Omega(c+M)^{r-2}|\nabla c|^2 \ge \frac{15}{4}M^{r-2}\int_\Omega|\nabla c|^2 \ge M\int_\Omega|\nabla c|^2$$

for all $t\in(0,T_{max})$. □

Among the consequences implied by an adequate linear combination of the inequalities provided by the latter two lemmata, the following will be referred to below.

Lemma 3.3 *Suppose that (1.2) holds with some $K_f > 0$ and $\alpha > 0$. Then whenever $p\in(0,1)$ satisfies (3.1) and $r\ge 3$, one can find $C = C(p,r) > 0$ such that*

$$\int_\Omega c^r(\cdot,t) \le C \qquad \textit{for all } t\in(0,T_{max}) \qquad (3.13)$$

and

$$\int_t^{t+\tau}\int_\Omega n^{p-2}|\nabla n|^2 \le C \qquad \textit{for all } t\in(0,T_{max}-\tau), \qquad (3.14)$$

where $\tau := \min\{1, \frac{1}{2}T_{max}\}$.

Proof. We first apply Lemma 3.1 to fix $C_1 = C_1(p) > 0$ such that

$$-\frac{d}{dt}\int_\Omega n^p + \frac{p(1-p)}{2}\int_\Omega n^{p-2}|\nabla n|^2 \le C_1 \int_\Omega |\nabla c|^2 + C_1 \qquad \text{for all } t \in (0, T_{max}),$$

and then rely on Lemma 3.2 to find $C_2 = C_2(p, r) > 0$ fulfilling

$$\frac{d}{dt}\int_\Omega (c + C_1)^r + C_1 \int_\Omega |\nabla c|^2 + \int_\Omega (c + C_1)^r \le \frac{p(1-p)}{4}\int_\Omega n^{p-2}|\nabla n|^2 + C_2 \qquad \text{for all } t \in (0, T_{max}).$$

Therefore,

$$y(t) := -\int_\Omega n^p(\cdot, t) + \int_\Omega \Big(c(\cdot, t) + C_1\Big)^r, \qquad t \in [0, T_{max}),$$

and

$$h(t) := \frac{p(1-p)}{4}\int_\Omega n^{p-2}(\cdot, t)|\nabla n(\cdot, t)|^2, \qquad t \in (0, T_{max}),$$

satisfy

$$y'(t) + y(t) + h(t) \le C_1 + C_2 \qquad \text{for all } t \in (0, T_{max}), \tag{3.15}$$

so that by nonnegativity of h,

$$\begin{aligned} y(t) &\le y(0)e^{-t} + (C_1 + C_2)\int_0^t e^{-(t-s)}h(s)ds \\ &= y(0)e^{-t} + (C_1 + C_2)\cdot(1 - e^{-t}) \\ &\le C_3 = C_3(p, r) := \int_\Omega (c_0 + C_1)^r + C_1 + C_2 \qquad \text{for all } t \in (0, T_{max}). \end{aligned} \tag{3.16}$$

As $-y(t) \le \int_\Omega n^p(\cdot, t)$ and

$$\int_\Omega n^p \le |\Omega|^{1-p}\cdot\Big\{\int_\Omega n\Big\}^p = C_4 = C_4(p) := |\Omega|^{1-p}\cdot\Big\{\int_\Omega n_0\Big\}^p \qquad \text{for all } t \in (0, T_{max}) \tag{3.17}$$

by the Hölder inequality and (2.2), combining (3.15) with (3.16) we thereafter obtain that

$$\begin{aligned} \int_t^{t+\tau} h(s)ds &\le y(t) - y(t+\tau) - \int_t^{t+\tau} y(s)ds + (C_1 + C_2)\tau \\ &\le C_3 + C_4 + C_4\tau + (C_1 + C_2)\tau \\ &\le C_1 + C_2 + C_3 + 2C_4 \qquad \text{for all } t \in (0, T_{max} - \tau) \end{aligned}$$

due to the restriction $\tau \le 1$. This implies (3.14), whereas (3.13) results from (3.16) together with (3.17). $\Box$

Through an interpolation along with (2.2), the weighted first-order estimate in (3.14) provides a bound for n with respect to some spatio-temporal Lebesgue norm that combines square integrability in time with some superlinear summability feature in space:

Corollary 3.4 *If (1.2) holds with some $K_f > 0$ and $\alpha > 0$, and if $p \in (0,1)$ satisfies (3.1), then there exists $C = C(p) > 0$ such that*

$$\int_t^{t+\tau} \|n(\cdot,s)\|^2_{L^{\frac{2}{2-p}}(\Omega)} ds \le C \qquad \textit{for all } t \in (0, T_{max} - \tau), \tag{3.18}$$

where again $\tau := \min\{1, \frac{1}{2}T_{max}\}$.

Proof. Using that $\frac{4}{p(2-p)} > \frac{2}{p}$, we employ the Gagliardo-Nirenberg inequality to fix $C_1 = C_1(p) > 0$ such that

$$\|\varphi\|^{\frac{4}{p}}_{L^{\frac{4}{p(2-p)}}(\Omega)} \le C_1 \|\nabla\varphi\|^2_{L^2(\Omega)} \|\varphi\|^{\frac{4-2p}{p}}_{L^{\frac{2}{p}}(\Omega)} + C_1 \|\varphi\|^{\frac{4}{p}}_{L^{\frac{2}{p}}(\Omega)} \qquad \text{for all } \varphi \in W^{1,2}(\Omega),$$

which when applied to $\varphi := n^{\frac{p}{2}}$ shows that

$$\begin{aligned}
\int_t^{t+\tau} \|n(\cdot,s)\|^2_{L^{\frac{2}{2-p}}(\Omega)} ds &= \int_t^{t+\tau} \|n^{\frac{p}{2}}(\cdot,s)\|^{\frac{4}{p}}_{L^{\frac{4}{p(2-p)}}(\Omega)} ds \\
&\le C_1 \int_t^{t+\tau} \|\nabla n^{\frac{p}{2}}(\cdot,s)\|^2_{L^2(\Omega)} \|n^{\frac{p}{2}}(\cdot,s)\|^{\frac{4-2p}{\frac{2}{p}}}_{L^{\frac{2}{p}}(\Omega)} ds \\
&\quad + C_1 \int_t^{t+\tau} \|n^{\frac{p}{2}}(\cdot,s)\|^{\frac{4}{p}}_{L^{\frac{2}{p}}(\Omega)} ds \\
&= \frac{p^2 C_1}{4} \cdot \left\{ \int_\Omega n_0 \right\}^{2-p} \cdot \int_t^{t+\tau} \int_\Omega n^{p-2} |\nabla n|^2 \\
&\quad + C_1 \tau \cdot \left\{ \int_\Omega n_0 \right\}^2 \qquad \text{for all } t \in (0, T_{max})
\end{aligned}$$

according to (2.2). The claim therefore results from the estimate in (3.14). $\square$

In order to make appropriate use of the latter in the context of the announced analysis of the Navier-Stokes system in (1.3), let us recall from [42, Lemma 3.4] the following elementary result concerned with elementary calculus.

Lemma 3.5 *Let $T \in (0, \infty]$ and $\tau \in (0, T)$, and suppose that $h \in L^1_{loc}((0,T))$ is nonnegative and such that*

$$\int_t^{t+\tau} h(s) ds \le b \qquad \textit{for all } t \in (0, T - \tau)$$

with some $b > 0$. Then whenever $\lambda > 0$,

$$\int_0^t e^{-\lambda(t-s)} h(s) ds \le \frac{b}{1 - e^{-\lambda\tau}} \qquad \textit{for all } t \in (0, T).$$

By means of the latter, we can now utilize the outcome of Corollary 3.4 to derive the following basic regularity information on the fluid component.

Lemma 3.6 *Assume that there exist $K_f > 0$ and $\alpha > 0$ such that (1.2) holds. Then one can fix $C > 0$ with the property that*

$$\int_\Omega |u(\cdot,t)|^2 \le C \qquad \textit{for all } t \in (0, T_{max}) \tag{3.19}$$

and

$$\int_t^{t+\tau} \int_\Omega |\nabla u|^2 \le C \qquad \textit{for all } t \in (0, T_{max} - \tau), \tag{3.20}$$

again with $\tau := \min\{1, \frac{1}{2}T_{max}\}$.

Proof. We pick any $p \in (0,1)$ fulfilling (3.1), and then infer from Corollary 3.4 that there exists $C_1 > 0$ satisfying

$$\int_t^{t+\tau} \|n(\cdot,s)\|^2_{L^{\frac{2}{2-p}}(\Omega)} ds \le C_1 \qquad \text{for all } t \in (0, T_{max} - \tau). \tag{3.21}$$

Moreover, by continuity of the embedding $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2}{p}}(\Omega)$ we can employ an associated Poincaré-Sobolev inequality to fix $C_2 > 0$ with the property that

$$\|\varphi\|_{L^{\frac{2}{p}}(\Omega)} + \|\varphi\|_{L^2(\Omega)} \le C_2 \|\nabla\varphi\|_{L^2(\Omega)} \qquad \text{for all } \varphi \in W_0^{1,2}(\Omega;\mathbb{R}^2), \tag{3.22}$$

and abbreviating $C_3 := \|\nabla\Phi\|_{L^\infty(\Omega)}$, we then test the third equation in (1.3) against u and invoke the Hölder inequality along with (3.22) and Young's inequality to see that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 &= \int_\Omega nu\cdot\nabla\Phi \\ &\le C_3 \|n\|_{L^{\frac{2}{2-p}}(\Omega)} \|u\|_{L^{\frac{2}{p}}(\Omega)} \\ &\le C_2C_3 \|n\|_{L^{\frac{2}{2-p}}(\Omega)} \|\nabla u\|_{L^2(\Omega)} \\ &\le \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{C_2^2C_3^2}{2}\|n\|^2_{L^{\frac{2}{2-p}}(\Omega)} \qquad \text{for all } t \in (0, T_{max}). \end{aligned}$$

Again using (3.22), we thus obtain that with $\lambda := \frac{1}{2C_2^2}$ we have

$$\frac{d}{dt}\int_\Omega |u|^2 + \lambda\int_\Omega |u|^2 + \frac{1}{2}\int_\Omega |\nabla u|^2 \le h(t) := C_2^2C_3^2\|n\|^2_{L^{\frac{2}{2-p}}(\Omega)} \qquad \text{for all } t \in (0, T_{max}), \tag{3.23}$$

so that since

$$\int_t^{t+\tau} h(s)ds \le C_4 := C_1C_2^2C_3^2 \qquad \text{for all } t \in (0, T_{max} - \tau) \tag{3.24}$$

by (3.21), we firstly infer upon applying Lemma 3.5 that

$$\begin{aligned} \int_\Omega |u|^2 &\le e^{-\lambda t}\cdot\int_\Omega |u_0|^2 + \int_0^t e^{-\lambda(t-s)}h(s)ds \\ &\le C_5 := \int_\Omega |u_0|^2 + \frac{C_4}{1-e^{-\lambda\tau}} \qquad \text{for all } t \in (0, T_{max}). \end{aligned}$$

Besides directly establishing (3.19), due to (3.23) and (3.24) this furthermore shows that

$$\frac{1}{2}\int_t^{t+\tau}\int_\Omega |\nabla u|^2 \le \int_\Omega |u(\cdot,t)|^2 + \int_t^{t+\tau} h(s)ds \le C_5 + C_4 \qquad \text{for all } t \in (0, T_{max}-\tau)$$

and hence also implies (3.20). $\square$

4 An advanced quasi-entropy property. Estimates in $L^p \times W^{1,2q} \times W^{1,2}$ for some $p > 2$ and $q > 1$

This section will be devoted to the derivation of some time-independent integrability properties for n, and espcially also for the signal gradient ∇c, in some superquadratic Lebesgue spaces. This will be achieved in Lemma 4.8 by analyzing a functional which extends that from (1.7) through an additional summand $\int_\Omega |\nabla u|^2$, and in the course of an appropriate estimation of the respective contributions stemming from the interaction mechanisms in (1.3) we shall make essential use the regularity features of u asserted by Lemma 3.6 together with the L^r bounds for c from Lemma 3.3.

4.1 Interpolating between zero-order and weighted second-order expressions

In order to prepare a suitable exploitation of this latter information currently available for c, let us briefly derive a functional inequality which interpolates gradients between weighted L^2 norms of Hessians and some zero-order Lebesgue norms. Though being quite in the spirit of precedents that involve L^∞ norms of the considered functions ([40, Lemma 3.8]), their particular preciousness in the context of our present purposes originates from their meaningful applicability already in situations when L^r bounds are a priori known for some finite r; in contrast to a previously obtained variant of a similar flavor ([36, Lemma 2.7]), our result does not involve any additional zero-order summands on its right-hand side, and moreover is fully homogeneous with respect to multiplication of the estimated function by constant factors.

Lemma 4.1 *Let $q > 1$ and $\lambda > 2$ be such that $\lambda < 2q+2$. Then there exists $C = C(q,\lambda) > 0$ such that for all $\varphi \in C^2(\overline{\Omega})$ fulfilling $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$,*

$$\int_\Omega |\nabla\varphi|^\lambda \le C \cdot \left\{ \int_\Omega |\nabla\varphi|^{2q-2}|D^2\varphi|^2 \right\}^{\frac{\lambda}{2q+2}} \cdot \left\{ \int_\Omega |\varphi|^{\frac{2\lambda}{2q+2-\lambda}} \right\}^{\frac{2q+2-\lambda}{2q+2}}. \tag{4.1}$$

Proof. Since $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$, an integration by parts followed by an application of the Cauchy-Schwarz inequality shows that since $|\Delta\varphi| \le \sqrt{2}|D^2\varphi|$ in Ω,

$$\begin{aligned}
\int_\Omega |\nabla\varphi|^\lambda &= \int_\Omega |\nabla\varphi|^{\lambda-2}\nabla\varphi\cdot\nabla\varphi \\
&= -\int_\Omega \varphi\nabla\varphi\cdot\nabla|\nabla\varphi|^{\lambda-2} - \int_\Omega \varphi|\nabla\varphi|^{\lambda-2}\Delta\varphi \\
&= -(\lambda-2)\int_\Omega \varphi|\nabla\varphi|^{\lambda-4}\nabla\varphi\cdot(D^2\varphi\cdot\nabla\varphi) - \int_\Omega \varphi|\nabla\varphi|^{\lambda-2}\Delta\varphi
\end{aligned}$$

$$\leq (\lambda-2+\sqrt{2})\int_\Omega |\varphi|\cdot|\nabla\varphi|^{\lambda-2}\cdot|D^2\varphi|$$
$$\leq (\lambda-2+\sqrt{2})\cdot\Big\{\int_\Omega |\nabla\varphi|^{2q-2}|D^2\varphi|^2\Big\}^{\frac{1}{2}}\cdot\Big\{\int_\Omega \varphi^2|\nabla\varphi|^{2\lambda-2q-2}\Big\}^{\frac{1}{2}}. \qquad (4.2)$$

Here we may rely on our assumption $\lambda < 2q+2$ to infer that by the Hölder inequality,

$$\Big\{\int_\Omega \varphi^2|\nabla\varphi|^{2\lambda-2q-2}\Big\}^{\frac{1}{2}} \leq \Big\{\int_\Omega |\varphi|^{\frac{2\lambda}{2q+2-\lambda}}\Big\}^{\frac{2q+2-\lambda}{2\lambda}}\cdot\Big\{\int_\Omega |\nabla\varphi|^\lambda\Big\}^{\frac{\lambda-q-1}{\lambda}},$$

whence (4.2) implies that

$$\begin{aligned}\Big\{\int_\Omega |\nabla\varphi|^\lambda\Big\}^{\frac{q+1}{\lambda}} &= \Big\{\int_\Omega |\nabla\varphi|^\lambda\Big\}^{1-\frac{\lambda-q-1}{\lambda}}\\ &\leq (\lambda-2+\sqrt{2})\cdot\Big\{\int_\Omega |\nabla\varphi|^{2q-2}|D^2\varphi|^2\Big\}^{\frac{1}{2}}\cdot\Big\{\int_\Omega |\varphi|^{\frac{2\lambda}{2q+2-\lambda}}\Big\}^{\frac{2q+2-\lambda}{2\lambda}},\end{aligned}$$

and that thus (4.1) holds with $C(q,\lambda) := (\lambda-2+\sqrt{2})^{\frac{\lambda}{q+1}}$. □

As a particular implication specifically adapted to our context, let us note the following.

Corollary 4.2 *Assume (1.2) with some $K_f > 0$ and $\alpha > 0$, and let $q \geq 1$ and $\lambda > 0$ be such that $\lambda < 2q+2$. Then for all $\eta > 0$ one can find $C(\eta) = C(\eta; q, \lambda) > 0$ such that*

$$\int_\Omega |\nabla c|^\lambda \leq \eta\int_\Omega |\nabla c|^{2q-2}|D^2c|^2 + C(\eta) \qquad \textit{for all } t\in(0,T_{max}). \qquad (4.3)$$

Proof. Due to the Hölder inequality it is sufficient to consider the case when $\lambda > 2$, in which since $\frac{\lambda}{2q+2} < 1$, an application of Young's inequality to (4.1) shows that given any $\eta > 0$ one can pick $C_1 = C_1(\eta, q, \lambda) > 0$ fulfilling

$$\int_\Omega |\nabla c|^\lambda \leq \eta\int_\Omega |\nabla c|^{2q-2}|D^2c|^2 + C_1\int_\Omega c^{\frac{2\lambda}{2q+2-\lambda}} \qquad \text{for all } t\in(0,T_{max}).$$

As from Lemma 3.3 we know that $\sup_{t\in(0,T_{max})}\int_\Omega c^{\frac{2\lambda}{2q+2-\lambda}}(\cdot,t)$ is finite, this implies (4.3). □

4.2 Analyzing $\int_\Omega n^p + \int_\Omega |\nabla c|^{2q} + \int_\Omega |\nabla u|^2$ for some $p > 2$ and $q > 1$

Let us now turn our attention to the major part of this section by noticing the preliminary outcomes of three testing procedures applied to (1.3). For convenience in notation, we may and will assume here that the exponent α in (1.2) be such that $\alpha < 1$.

Lemma 4.3 *Assume (1.2) with some $K_f > 0$ and $\alpha \in (0,1)$, and let $p > 2$ and $q > 2-\alpha$. Then there exists $C = C(p,q) > 0$ such that for all $t \in (0,T_{max})$ we have*

$$\frac{d}{dt}\int_\Omega n^p + \int_\Omega n^{p-2}|\nabla n|^2 \leq C\int_\Omega n^{\frac{p(q-1)-2(1-\alpha)}{q+\alpha-2}} + C\int_\Omega |\nabla c|^{2\cdot\frac{p(q-1)-2(1-\alpha)}{p-2}} + C \qquad (4.4)$$

and

$$\frac{d}{dt}\int_\Omega |\nabla c|^{2q}+\int_\Omega |\nabla c|^{2q-2}|D^2c|^2 \le C\int_\Omega n^{\frac{p(q-1)-2(1-\alpha)}{q+\alpha-2}}+C\int_\Omega |\nabla c|^{2\cdot\frac{p(q-1)-2(1-\alpha)}{p-2}}+C\int_\Omega |\nabla c|^{2q}|\nabla u| \tag{4.5}$$

as well as

$$\frac{d}{dt}\int_\Omega |\nabla u|^2+\int_\Omega |Au|^2 \le C\cdot\Big\{\int_\Omega |\nabla u|^2\Big\}^2+C\int_\Omega n^2+C \tag{4.6}$$

Proof. In view of the first equation from (1.3), using (1.2) along with our assumption that $\alpha<1$ we see that due to Young's inequality,

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega n^p+(p-1)\int_\Omega n^{p-2}|\nabla n|^2 &= (p-1)\int_\Omega n^{p-1}f(|\nabla c|^2)\nabla n\cdot\nabla c\\
&\le \frac{p-1}{2}\int_\Omega n^{p-2}|\nabla n|^2+\frac{p-1}{2}\int_\Omega n^pf^2(|\nabla c|^2)|\nabla c|^2\\
&\le \frac{p-1}{2}\int_\Omega n^{p-2}|\nabla n|^2\\
&\quad+\frac{(p-1)K_f^2}{2}\int_\Omega n^p|\nabla c|^{2-2\alpha} \qquad \text{for all } t\in(0,T_{max}),
\end{aligned}$$

so that since $p\ge 2$ implies that $\frac{p(p-1}{2}\ge 1$, we obtain that

$$\frac{d}{dt}\int_\Omega n^p+\int_\Omega n^{p-2}|\nabla n|^2 \le \frac{p(p-1)K_f^2}{2}\int_\Omega n^p|\nabla c|^{2-2\alpha} \qquad \text{for all } t\in(0,T_{max}). \tag{4.7}$$

Here we note that our hypothesis $p>2$ ensures that $\theta:=\frac{p(q-1)-2(1-\alpha)}{p(q+\alpha-2)}$ satisfies

$$\theta-1=\frac{(p-2)(1-\alpha)}{p(q+\alpha-2)}>0,$$

whence a further application of Young's inequality shows that

$$\int_\Omega n^p|\nabla c|^{2-2\alpha}\le\int_\Omega n^{p\theta}+\int_\Omega|\nabla c|^{\frac{(2-2\alpha)\theta}{\theta-1}} \qquad \text{for all } t\in(0,T_{max}). \tag{4.8}$$

Observing that $p\theta=\frac{p(q-1)-2(1-\alpha)}{q+\alpha-2}$ and that

$$\frac{(2-2\alpha)\theta}{\theta-1}=2\cdot\frac{1-\alpha}{1-\frac{p(q+\alpha-2)}{p(q-1)-2(1-\alpha)}}=2\cdot\frac{p(q-1)-2(1-\alpha)}{p-2},$$

we thus infer (4.4) from (4.8) when inserted into (4.7).

To derive (4.5), we use that according to an argument due to [20, Lemma 3.2] we can find $C_1=C_1(q)>0$ such that

$$\begin{aligned}
-\frac{1}{2}\int_\Omega |\nabla c|^{2q-2}\frac{\partial|\nabla c|^2}{\partial\nu} &\le \frac{2(q-1)}{q^2}\int_\Omega\Big|\nabla|\nabla c|^q\Big|^2+C_1\\
&= \frac{q-1}{2}\int_\Omega|\nabla c|^{2q-4}\Big|\nabla|\nabla c|^2\Big|^2+C_1 \qquad \text{for all } t\in(0,T_{max}),
\end{aligned}$$

to see intergating by parts in the second equation from (1.3) that since $\nabla c \cdot \nabla \Delta c = \frac{1}{2}\Delta|\nabla c|^2 - |D^2 c|^2$ and $|\nabla c| \le \sqrt{2}|D^2 c|$,

$$\begin{aligned}
\frac{1}{2q}\frac{d}{dt}\int_\Omega |\nabla c|^{2q} &= \int_\Omega |\nabla c|^{2q-2}\nabla c \cdot \nabla\Big\{\Delta c - c + n - u\cdot\nabla c\Big\} \\
&= \frac{1}{2}\int_\Omega |\nabla c|^{2q-2}\Delta|\nabla c|^2 - \int_\Omega |\nabla c|^{2q-2}|D^2 c|^2 - \int_\Omega |\nabla c|^{2q} \\
&\quad - \int_\Omega n\nabla\cdot(|\nabla c|^{2q-2}\nabla c) - \int_\Omega |\nabla c|^{2q-2}\nabla c\cdot\nabla(u\cdot\nabla c) \\
&= -\frac{q-1}{2}\int_\Omega |\nabla c|^{2q-4}\Big|\nabla|\nabla c|^2\Big|^2 - \frac{1}{2}\int_{\partial\Omega}|\nabla c|^{2q-2}\frac{\partial|\nabla c|^2}{\partial\nu} \\
&\quad - \int_\Omega |\nabla c|^{2q-2}|D^2 c|^2 - \int_\Omega |\nabla c|^{2q} \\
&\quad - \int_\Omega n\cdot\Big\{2(q-1)|\nabla c|^{2q-4}\nabla c\cdot(D^2 c\cdot\nabla c) - |\nabla c|^{2q-2}\Delta c\Big\} \\
&\quad - \int_\Omega |\nabla c|^{2q-2}\nabla c\cdot(\nabla u\cdot\nabla c) \\
&\le C_1 - \int_\Omega |\nabla c|^{2q-2}|D^2 c|^2 \\
&\quad +(2q-2+\sqrt{2})\int_\Omega n|\nabla c|^{2q-2}|D^2 c| + \int_\Omega |\nabla c|^{2q}|\nabla u| \qquad \text{for all } t\in(0,T_{max}), \quad (4.9)
\end{aligned}$$

because for all $t\in(0,T_{max})$,

$$\begin{aligned}
\int_\Omega |\nabla c|^{2q-2}\nabla c\cdot\nabla(u\cdot\nabla c) - \int_\Omega |\nabla c|^{2q-2}\nabla c\cdot(\nabla u\cdot\nabla c) &= \int_\Omega |\nabla c|^{2q}\nabla c\cdot(D^2 c\cdot u) \\
&= \frac{1}{2q}\int_\Omega u\cdot\nabla|\nabla c|^{2q} = 0
\end{aligned}$$

by solenoidality of u. Since Young's inequality asserts that

$$\begin{aligned}
(2q-2+\sqrt{2})\int_\Omega n|\nabla c|^{2q-2}|D^2 c| &\le \frac{1}{2q}\int_\Omega |\nabla c|^{2q-2}|D^2 c|^2 \\
&\quad +\frac{q(2q-2+\sqrt{2})^2}{2}\int_\Omega n^2|\nabla c|^{2q-2} \qquad \text{for all } t\in(0,T_{max}),
\end{aligned}$$

and that here due to the fact that $\lambda := \frac{p(q-1)-2(1-\alpha)}{2(q+\alpha-2)}$ satisfies $\lambda > \theta > 1$ we have

$$\int_\Omega n^2|\nabla c|^{2q-2} \le \int_\Omega n^{2\lambda} + \int_\Omega |\nabla c|^{(2q-2)\lambda}\lambda - 1 \qquad \text{for all } t\in(0,T_{max}),$$

from (4.9) we readily obtain (4.5) upon noticing that $2q \ge 1$ and computing

$$2\lambda = \frac{p(q-1)-2(1-\alpha)}{q+\alpha-2} \quad \text{as well as} \quad \frac{(2q_2)\lambda}{\lambda-1} = 2\cdot\frac{q-1}{1-\frac{2(q+\alpha-2)}{p(q-1)-2(1-\alpha)}} = 2\cdot\frac{p(q-1)-2(1-\alpha)}{p-2}. \tag{4.10}$$

Finally, testing the Navier-Stokes subsystem of (1.3) by Au we find that thanks to Young's inequality,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 + \int_\Omega |Au|^2 &= -\int_\Omega Au\cdot\mathcal{P}[(u\cdot\nabla)u] + \int_\Omega Au\cdot\mathcal{P}[n\nabla\Phi] \\
&\le \frac{1}{4}\int_\Omega |Au|^2 + 2\int_\Omega |\mathcal{P}[(u\cdot\nabla)u]|^2 + 2\int_\Omega |\mathcal{P}[n\nabla\Phi]|^2 \\
&\le \frac{1}{4}\int_\Omega |Au|^2 + 2\int_\Omega |(u\cdot\nabla)u|^2 + 2\int_\Omega |n\nabla\Phi|^2 \\
&\le \frac{1}{4}\int_\Omega |Au|^2 + 2\|u\|_{L^\infty(\Omega)}^2\int_\Omega |\nabla u|^2 + 2\|\nabla\Phi\|_{L^\infty(\Omega)}^2\int_\Omega n^2 \qquad (4.11)
\end{aligned}$$

for all $t\in(0,T_{max})$, where on the basis of our assumption on f we may clearly invoke Lemma 3.6 to infer finiteness of $\sup_{t\in(0,T_{max})}\int_\Omega |u(\cdot,t)|^2$, and to thus see by means of a Gagliardo-Nirenberg interpolation and Young's inequality that with some $C_2>0$ and $C_3>0$ we have

$$\begin{aligned}
2\|u\|_{L^\infty(\Omega)}^2\int_\Omega |\nabla u|^2 &\le C_2\|Au\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}\int_\Omega |\nabla u|^2 \\
&\le \frac{1}{4}\int_\Omega |Au|^2 + C_3\cdot\Big\{\int_\Omega |\nabla u|^2\Big\}^2 \qquad \text{for all } t\in(0,T_{max}).
\end{aligned}$$

We therefore obtain (4.6) as a consequence of (4.11). □

Next intending to estimate the expressions on the right-hand sides in (4.4), (4.5) and (4.6) through appropriate interpolation, we shall first focus on the integrals in (4.4) and (4.5) that involve n. By merely resorting to mass conservation as the only time-independent information on n currently at hand, we see that these can favorably be controlled by means of the corresponding dissipation rate in (4.4) when the exponent q is suitably large relative to p:

Lemma 4.4 *Let $\alpha\in(0,1)$, $p>2$ and*

$$q > p(1-\alpha)+\alpha. \qquad (4.12)$$

Then for all $\eta>0$ there exists $C(\eta)=C(\eta;p,q)>0$ such that

$$\int_\Omega n^{\frac{p(q-1)-2(1-\alpha)}{q+\alpha-2}} \le \eta\int_\Omega n^{p-2}|\nabla n|^2 + C(\eta) \qquad \textit{for all } t\in(0,T_{max}). \qquad (4.13)$$

Proof. We first note that (4.12) together with our assumption $p>2$ implies that $q>(1-\alpha)\cdot 2+\alpha = 2-\alpha$, and that thus also

$$p(q-1)-2(1-\alpha)-(q+\alpha-2) > 2(q-1)-2(1-\alpha)-(q+\alpha-2) = q+\alpha-2>0,$$

whence $\theta := \frac{p(q-1)-2(1-\alpha)}{q+\alpha-2}$ is well-defined with $\theta>1$. Therefore, the Gagliardo-Nirenberg inequality in conjunction with (2.2) yields $C_1=C_1(p,q)>0$ and $C_2=C_2(p,q)>0$ such that

$$\begin{aligned}
\int_\Omega n^\theta &= \|n^{\frac{p}{2}}\|_{L^{\frac{2\theta}{p}}(\Omega)}^{\frac{2\theta}{p}} \\
&\le C_1\|\nabla n^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{2(\theta-1)}{p}}\|n^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{p}} + C_1\|n^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2\theta}{p}} \\
&\le C_2\|\nabla n^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{2(\theta-1)}{p}} + C_2 \qquad \text{for all } t\in(0,T_{max}),
\end{aligned}$$

so that since (4.12) furthermore ensures that

$$\frac{2(\theta-1)}{p} - 2 = \frac{2}{p} \cdot \frac{p(1-\alpha)+\alpha-q}{q+\alpha-2} < 0,$$

and that thus $\frac{2(\theta-1)}{p} < 2$, an application of Young's inequality readily yields the claim. $\Box$

On the other hand, requiring q not to be too large as compared to p enables us to directly apply Corollary 4.2 in order to similarly estimate the integrals merely involving the taxis gradient against the dissipated quantity in (4.5).

Lemma 4.5 *Assume (1.2) with some $K_f > 0$ and $\alpha \in (0,1)$, and let $p > 2$ and $q > 2-\alpha$ be such that*

$$q < p - \alpha. \tag{4.14}$$

Then given any $\eta > 0$ one can pick $C(\eta) = C(\eta; p, q) > 0$ such that

$$\int_\Omega |\nabla c|^{2\cdot\frac{p(q-1)-2(1-\alpha)}{p-2}} \le \eta \int_\Omega |\nabla c|^{2q-2}|D^2 c|^2 + C(\eta) \qquad \textit{for all } t \in (0, T_{max}). \tag{4.15}$$

Proof. We only need to observe that since $p > 2$, the inequalities $q > 2-\alpha$ and $q < p-\alpha$ warrant that

$$p(q-1) - 2(1-\alpha) > p(1-\alpha) - 2(1-\alpha) > 0$$

and

$$2 \cdot \frac{p(q-1)-2(1-\alpha)}{p-2} - (2q+2) = \frac{4(q+\alpha-p)}{p-2} > 0,$$

respectively, so that $\lambda := 2 \cdot \frac{p(q-1)-2(1-\alpha)}{p-2}$ is positive with $\lambda < 2q+2$. Therefore, namely, the claim results from Corollary 4.2. $\Box$

Apart from that, let us prepare our subsequent estimation of the rightmost summand in (4.5) by the following two lemmata, the first of which again quite trivially results from Corollary 4.2 when q is suitably small.

Lemma 4.6 *Assume (1.2) with some $K_f > 0$ and $\alpha > 0$, and let $q \ge 1$ be such that*

$$q < 2. \tag{4.16}$$

Then for all $\eta > 0$ there exists $C(\eta) = C(\eta; q) > 0$ with the property that

$$\int_\Omega |\nabla c|^{3q} \le \eta \int_\Omega |\nabla c|^{2q-2}|D^2 c|^2 + C(\eta) \qquad \textit{for all } t \in (0, T_{max}). \tag{4.17}$$

Proof. This immediately follows from Corollary 4.2, because (4.16) warrants that $2q - (2q+2) = q - 2 < 0$. $\Box$

Secondly, the corresponding fluid velocity gradient will be controlled by making use of the following result that relies on the L^2 bound for u provided by Lemma 3.6.

Lemma 4.7 *Assume that (1.2) be satisfied with some $K_f > 0$ and $\alpha > 0$. Then there exists $C > 0$ such that*

$$\int_\Omega |\nabla u|^3 \le C \int_\Omega |Au|^2 \qquad \textit{for all } t \in (0, T_{max}). \tag{4.18}$$

PROOF. Since well-known regularity features of the Stokes operator guarantee that $\|A(\cdot)\|_{L^2(\Omega)}$ defines a norm equivalent to $\|\cdot\|_{W^{2,2}(\Omega)}$ on $D(A)$ ([31]), by means of the Gagliardo-Nirenberg inequality we see that with some $C_1 > 0$ and $C_2 > 0$ we have

$$\int_\Omega |\nabla u|^3 \le C_1 \|u\|^2_{W^{2,2}(\Omega)} \|u\|_{L^2(\Omega)} \le C_2 \|Au\|^2_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \qquad \text{for all } t \in (0, T_{max}).$$

Due to the boundedness property in (3.19) this already establishes (4.18). □

It now turns out that if one last time we adequately make explicit use of our overall assumption on validity of (1.2) with some positive α, then we can find $p > 2$ and $q > 1$ in such a way that the assumptions (4.12), (4.14) and (4.16) are simultaneously satisfied. We may therefore derive the following from a combination of Lemma 4.3 with Lemma 4.4–Lemma 4.7.

Lemma 4.8 *Suppose that (1.2) holds with some $K_f > 0$ and $\alpha > 0$. Then there exist $p > 2, q > 1$ and $C > 0$ such that*

$$\int_\Omega n^p(\cdot, t) \le C \qquad \textit{for all } t \in (0, T_{max}) \tag{4.19}$$

and

$$\int_\Omega |\nabla c(\cdot, t)|^{2q} \le C \qquad \textit{for all } t \in (0, T_{max}) \tag{4.20}$$

as well as

$$\int_\Omega |\nabla u(\cdot, t)|^2 \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{4.21}$$

PROOF. Assuming without loss of generality that $\alpha < 1$, we fix any $q \in (2 - \alpha, 2)$ and use that then

$$\frac{q-\alpha}{1-\alpha} - (q + \alpha) = \frac{\alpha \cdot (q + \alpha - 2)}{1 - \alpha} > 0$$

in choosing $p \in (q + \alpha, \frac{q-\alpha}{1-\alpha})$. Then necessarily $p > (2 - \alpha) + \alpha = 2$, so that Lemma 4.3 applies so as to yield $C_1 > 0$ such that

$$\begin{aligned}
&\frac{d}{dt}\Big\{ \int_\Omega n^p + \int_\Omega |\nabla c|^{2q} + \int_\Omega |\nabla u|^2 \Big\} + \int_\Omega n^{p-2}|\nabla n|^2 + \int_\Omega |\nabla c|^{2q-2}|D^2 c|^2 + \int_\Omega |Au|^2 \\
&\quad \le \; C_1 + C_1 \int_\Omega n^{\frac{p(q-1)-2(1-\alpha)}{q+\alpha-2}} + C_1 \int_\Omega n^2 \\
&\qquad + C_1 \int_\Omega |\nabla c|^{2 \cdot \frac{p(q-1)-2(1-\alpha)}{p-2}} \\
&\qquad + C_1 \int_\Omega |\nabla c|^{2q} |\nabla u| + C_1 \cdot \Big\{ \int_\Omega |\nabla u|^2 \Big\}^2 \qquad \text{for all } t \in (0, T_{max}).
\end{aligned} \tag{4.22}$$

To derive (4.19)-(4.21) from this, we employ Lemma 3.6 to fix $C_2 > 0$ such that

$$h(t) := \begin{cases} C_1 \int_\Omega |\nabla u(\cdot,t)|^2, & t \in (0, T_{max}), \\ 0, & t \in \mathbb{R} \setminus (0, T_{max}), \end{cases}$$

has the property that again with $\tau := \min\{1, \frac{1}{2}T_{max}\}$ we have

$$\int_t^{t+\tau} h(s)ds \le C_2 \qquad \text{for all } t \in \mathbb{R}, \tag{4.23}$$

and thereupon let

$$B := 2C_2. \tag{4.24}$$

Then observing that $\frac{p(q-1)-2(1-\alpha)}{q+\alpha-2} - p = \frac{(p-2)(1-\alpha)}{q+\alpha-2} > 0$ and hence $2 < p < \frac{p(q-1)-2(1-\alpha)}{q+\alpha-2}$, by using Young's inequality along with Lemma 4.4 we can find $C_3 > 0$ fulfilling

$$\begin{aligned} B\int_\Omega n^p + C_1 \int_\Omega n^{\frac{p(q-1)-2(1-\alpha)}{q+\alpha-2}} + C_1 \int_\Omega n^2 &\le (B + 2C_1) \int_\Omega n^{\frac{p(q-1)-2(1-\alpha)}{q+\alpha-2}} + (B + C_1)|\Omega| &(4.25) \\ &\le \int_\Omega n^{p-2}|\nabla n|^2 + C_3 \qquad \text{for all } t \in (0, T_{max}). &(4.26) \end{aligned}$$

Moreover, taking $C_4 > 0$ such that in accordance with Lemma 4.7 we have

$$\int_\Omega |\nabla u|^3 \le C_4 \int_\Omega |Au|^2 \qquad \text{for all } t \in (0, T_{max}),$$

in view of Young's inequality we obtain that

$$\begin{aligned} B\int_\Omega |\nabla u|^2 + \frac{1}{2C_4}\int_\Omega |\nabla u|^3 &= \Big(\frac{4C_4}{3}\Big)^{\frac{2}{3}} B \int_\Omega \Big\{\frac{3}{4C_4}|\nabla u|^3\Big\}^{\frac{2}{3}} + \frac{1}{2C_4}\int_\Omega |\nabla u|^3 \\ &\le \frac{2}{3}\cdot \int_\Omega \Big\{\frac{3}{4C_4}|\nabla u|^3\Big\} + \frac{1}{3}\cdot\Big\{\Big(\frac{4C_4}{3}\Big)^{\frac{2}{3}} M\Big\}^3 |\Omega| + \frac{1}{2C_4}\int_\Omega |\nabla u|^3 \\ &= \frac{1}{C_4}\int_\Omega |\nabla u|^3 + C_5 \\ &\le \int_\Omega |Au|^2 + C_5 \qquad \text{for all } t \in (0, T_{max}) \end{aligned} \tag{4.27}$$

with $C_5 := \frac{1}{3}\cdot\Big(\frac{4C_4}{3}\Big)^2 M^3|\Omega|$. Writing $C_6 := \frac{2}{3}\cdot\Big\{\Big(\frac{2C_4}{3}\Big)^{\frac{1}{3}} C_1\Big\}^{\frac{3}{2}}$, we thereafter combine Young's inequality with Lemma 4.6 and Lemma 4.5 to infer the existence of $C_7 > 0$ satisfying

$$\begin{aligned} &B\int_\Omega |\nabla c|^{2q} + C_6 \int_\Omega |\nabla c|^{3q} + C_1 \int_\Omega |\nabla c|^{2\cdot\frac{p(q-1)-2(1-\alpha)}{p-2}} \\ &\quad \le (B + C_6)\int_\Omega |\nabla c|^{3q} + C_1 \int_\Omega |\nabla c|^{2\cdot\frac{p(q-1)-2(1-\alpha)}{p-2}} \\ &\quad \le \int_\Omega |\nabla c|^{2q-2}|D^2 c| + C_7 \qquad \text{for all } t \in (0, T_{max}), \end{aligned} \tag{4.28}$$

so that since, again by Young's inequality,

$$\begin{aligned}
C_1\int_\Omega |\nabla c|^{2q}|\nabla u| &= \int_\Omega \Big\{\frac{3}{2C_4}|\nabla u|^3\Big\}^{\frac13}\cdot\Big\{\Big(\frac{2C_4}{3}\Big)^{\frac13}C_1|\nabla c|^{2q}\Big\} \\
&\le \frac13\cdot\int_\Omega\Big\{\frac{3}{2C_4}|\nabla u|^3\Big\}+\frac23\cdot\int_\Omega\Big\{\Big(\frac{2C_4}{3}\Big)^{\frac13}C_1|\nabla c|^{2q}\Big\}^{\frac32} \\
&= \frac{1}{2C_4}\int_\Omega|\nabla u|^3+C_6\int_\Omega|\nabla c|^{3q} \qquad \text{for all } t\in(0,T_{max}),
\end{aligned}$$

upon collecting (4.25), (4.27) and (4.28) we conclude from (4.22) that

$$y(t):=\int_\Omega n^p(\cdot,t)+\int_\Omega|\nabla c(\cdot,t)|^{2q}+\int_\Omega|\nabla u(\cdot,t)|^2, \qquad t\in[0,T_{max}),$$

has the property that

$$y'(t)+By(t)\le h(t)y(t)+C_8 \qquad \text{for all } t\in(0,T_{max}) \tag{4.29}$$

with $C_8:=C_1+C_3+C_5+C_7$.

To proceed from this, we note that if $T_{max}\le 2$ and hence $\tau=\frac12 T_{max}$, then (4.23) implies that

$$\int_0^{T_{max}}h(t)dt=\int_0^\tau h(t)dt+\int_\tau^{2\tau}h(t)dt\le 2C_2,$$

and that in the case when $T_{max}>2$ and thus $\tau=1$, for each positive integer N and any $k\in\{0,...,N-1\}$ we similarly obtain from (4.23) that

$$\int_k^N h(t)dt\le C_2\cdot(N-k),$$

meaning that for any such N and k,

$$\begin{aligned}
\int_s^t h(\sigma)d\sigma &\le C_2\cdot\Big\{N-(k-1)\Big\} \\
&\le C_2\cdot(t-s)+2C_2 \qquad \text{for all } t\in(N-1,N] \text{ and } s\in(k-1,k].
\end{aligned}$$

Therefore, in both these cases we find that

$$\int_s^t h(\sigma)d\sigma\le C_2\cdot(t-s)+2C_2 \qquad \text{for all } t\in(0,T_{max}) \text{ and each } s\in[0,t),$$

so that an integration of (4.29) shows that due to (4.24),

$$\begin{aligned}
y(t) &\le y(0)e^{\int_0^t(h(\sigma-B)d\sigma}+C_8\int_0^t e^{\int_s^t(h(\sigma)-B)d\sigma}ds \\
&\le y(0)e^{C_2t+2C_2-Bt}+C_8\int_0^t e^{C_2\cdot(t-s)+2C_2-B\cdot(t-s)}ds
\end{aligned}$$

$$
\begin{aligned}
&= y(0)e^{2C_2} \cdot e^{-\frac{B}{2}t} + C_8 e^{2C_2} \int_0^t e^{-\frac{B}{2}(t-s)} ds \\
&= y(0)e^{2C_2} \cdot e^{-\frac{B}{2}t} + \frac{2C_8 e^{2C_2}}{B} \cdot (1 - e^{-\frac{B}{2}t}) \\
&\le y(0)e^{2C_2} + \frac{2C_8 e^{2C_2}}{B} \qquad \text{for all } t \in (0, T_{max}).
\end{aligned}
$$

By definition of y, this yields (4.19)-(4.21). □

5 Bounds for $A^\vartheta u$ in L^2 and for n in L^∞. Proof of Theorem 1.1

The following consequence of the integrability features in (4.19) and (4.21) on further fluid regularity is quite straightforward.

Lemma 5.1 *Assume (1.2) with some $K_f > 0$ and $\alpha > 0$. Then for all $\vartheta \in (\frac{1}{2}, 1)$ there exists $C = C(\theta) > 0$ such that*

$$
\|A^\vartheta u(\cdot, t)\|_{L^2(\Omega)} \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{5.1}
$$

Proof. Since $\vartheta < 1$ and thus $\frac{3}{2} - \vartheta > \frac{1}{2}$, we can pick $r = r(\vartheta) \in (1, 2)$ suitably close to 2 such that $\frac{1}{r} < \frac{3}{2} - \vartheta$. Then by continuity of the embedding $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2r}{2-r}}(\Omega)$, utilizing (4.21) we see that with some $C_1 = C_1(\vartheta) > 0$ and $C_2 = C_2(\vartheta) > 0$ we have

$$
\|(u \cdot \nabla)u\|_{L^r(\Omega)} \le C_1 \|u\|_{L^{\frac{2r}{2-r}}(\Omega)} \|\nabla u\|_{L^2(\Omega)} \le C_2 \qquad \text{for all } t \in (0, T_{max}),
$$

which in conjunction with (4.19) and the boundedness of the Helmholtz projection on $L^r(\Omega; \mathbb{R}^2)$ ([13]) ensures the existence of $C_3 = C_3(\vartheta) > 0$ such that $h := -\mathcal{P}[(u \cdot \nabla)u] + \mathcal{P}[n \nabla \Phi]$ satisfies

$$
\|h(\cdot, t)\|_{L^r(\Omega)} \le C_3 \qquad \text{for all } t \in (0, T_{max}).
$$

Therefore, recalling known smoothing properties of the Stokes semigroup ([11], [15, p.201]) we obtain that with some $C_4 = C_4(\vartheta) > 0$ and $\mu > 0$,

$$
\begin{aligned}
\|A^\vartheta u(\cdot, t)\|_{L^2(\Omega)} &= \left\| A^\vartheta e^{-tA} u_0 + \int_0^t e^{-(t-s)A} h(\cdot, s) ds \right\|_{L^2(\Omega)} \\
&\le \|A^\vartheta u_0\|_{L^2(\Omega)} + C_4 \int_0^t (t-s)^{-\vartheta - \frac{1}{r} + \frac{1}{2}} e^{-\mu(t-s)} \|h(\cdot, s)\|_{L^r(\Omega)} ds \\
&\le \|A^\vartheta u_0\|_{L^2(\Omega)} + C_3 C_4 \int_0^\infty \sigma^{-\vartheta - \frac{1}{r} + \frac{1}{2}} e^{-\mu\sigma} d\sigma
\end{aligned}
$$

for all $t \in (0, T_{max})$. As the latter integral is finite due to the inequality $-\vartheta - \frac{1}{r} + \frac{1}{2} > -1$, in view of the inclusion $u_0 \in D(A^\vartheta)$ asserted by (1.4) this entails (5.1). □

Along with the fact that in (4.19) and (4.20) we have $p > 2$ and $2q > 2$, the latter now provides sufficient information on regularity in the cross-diffusive and transport-related contributions to the first equation from (1.3) to ensure boundedness of n actually with respect to the spatial L^∞ norm.

Lemma 5.2 *If (1.2) is satisfied with some $K_f > 0$ and $\alpha > 0$, then there exists $C > 0$ fulfilling*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t \in (0,T_{max}). \tag{5.2}$$

PROOF. Using (1.2) and the fact that $D(A^\vartheta) \hookrightarrow L^\infty(\Omega)$ for all $\vartheta \in (\frac{1}{2},1)$ ([14], [17]), on the basis of (4.19), (4.20) and (5.1) we readily infer the existence of $r > 2$ and $C_1 > 0$ such that $h := f(|\nabla c|^2)\nabla c + nu$ satisfies

$$\|h(\cdot,t)\|_{L^r(\Omega)} \le C_1 \qquad \text{for all } t \in (0,T_{max}). \tag{5.3}$$

We then take any $\lambda \in (2,r)$ and employ the comparison principle along with a known smoothing estimate for the Neumann heat semigroup $(e^{t\Delta})_{t\ge 0}$ on Ω ([12]) to see that with some $C_2 > 0$ and $\mu > 0$ we have

$$\begin{aligned}
\|n(\cdot,t)\|_{L^\infty(\Omega)} &= \left\| e^{t\Delta} n_0 - \int_0^t e^{(t-s)\Delta} \nabla \cdot \Big\{ n(\cdot,s) h(\cdot,s) \Big\} ds \right\|_{L^\infty(\Omega)} \\
&\le \|n_0\|_{L^\infty(\Omega)} + C_2 \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{\lambda}} e^{-\mu(t-s)} \|n(\cdot,s)h(\cdot,s)\|_{L^\lambda(\Omega)} ds \qquad \text{for all } t \in (0,T_{max}).
\end{aligned}$$

Since according to (2.2) and (5.3) we can estimate

$$\begin{aligned}
\|n(\cdot,s)h(\cdot,s)\|_{L^\lambda(\Omega)} &\le \|n(\cdot,s)\|_{L^\infty(\Omega)}^\theta \|n(\cdot,s)\|_{L^1(\Omega)}^{1-\theta} \|h(\cdot,s)\|_{L^r(\Omega)} \\
&\le C_1 \cdot \Big\{ \int_\Omega n_0 \Big\}^{1-\theta} \cdot \|n(\cdot,s)\|_{L^\infty(\Omega)}^\theta \qquad \text{for all } s \in (0,T_{max})
\end{aligned}$$

with $\theta := \frac{r\lambda - r + \lambda}{r\lambda} \in (0,1)$, this implies that

$$\sup_{t\in(0,T)} \|n(\cdot,t)\|_{L^\infty(\Omega)} \le C_3 + C_3 \cdot \Big\{ \sup_{t\in(0,T)} \|n(\cdot,t)\|_{L^\infty(\Omega)} \Big\}^\theta \qquad \text{for all } T \in (0,T_{max}),$$

with

$$C_3 := \max\left\{ \|n_0\|_{L^\infty(\Omega)} \, , \, C_1 C_2 \cdot \Big\{ \int_\Omega n_0 \Big\}^{1-\theta} \cdot \int_0^\infty \sigma^{-\frac{1}{2}-\frac{1}{\lambda}} e^{-\mu\sigma} d\sigma \right\}$$

being finite due to the fact that $\lambda > 2$. As $\theta < 1$, we thus obtain that

$$\sup_{t\in(0,T)} \|n(\cdot,t)\|_{L^\infty(\Omega)} \le \max\Big\{ 1 \, , \, (2C_3)^{\frac{1}{1-\theta}} \Big\} \qquad \text{for all } T \in (0,T_{max}),$$

from which (5.2) follows upon letting $T \nearrow T_{max}$. $\Box$

We have thereby collected all ingredients needed to derive our main result from Lemma 2.1.

PROOF of Theorem 1.1. We only need to combine the bounds provided by Lemma 5.2, (4.20) and Lemma 5.1 with Lemma 2.1. $\Box$

Acknowledgement. The author acknowledges support of the *Deutsche Forschungsgemeinschaft* in the context of the project *Emergence of structures and advantages in cross-diffusion systems* (Project No. 411007140, GZ: WI 3707/5-1).

References

[1] Bellomo, N., Bellouquid, A., Nieto, J., Soler, J.: *Multiscale biological tissue models and flux-limited chemotaxis from binary mixtures of multicellular growing systems.* Math. Mod. Meth. Appl. Sci. **20**, 1675-1693 (2010)

[2] Bianchi, A., Painter, K.J., Sherratt, J.A.: *A mathematical model for lymphangiogenesis in normal and diabetic wounds.* J. Theor. Biol. **383**, 61-86 (2015)

[3] Bianchi, A., Painter., K.J., Sherratt, J.A.: *Spatio-temporal models of lymphangiogenesis in wound healing.* Bull. Math. Biol. **78**, 1904-1941 (2016) Math. Meth. Appl. Sci. **32**, 1704-1737 (2009)

[4] Biler, P.: *Local and global solvability of some parabolic systems modelling chemotaxis.* Adv. Math. Sci. Appl. **8**, 715-743 (1998)

[5] Black, T.: *Global very weak solutions to a chemotaxis-fluid system with nonlinear diffusion.* SIAM J. Math. Anal. **50**, 4087-4116 (2018)

[6] Cao, X.: *Fluid interaction does not affect the critical exponent in a three-dimensional Keller-Segel-Stokes model.* Z. Angew. Math. Phys. **71**, 61 (2020)

[7] Cieślak, T., Stinner, C.: *New critical exponents in a fully parabolic quasilinear Keller-Segel system and applications to volume filling models.* J. Differential Eq. **258**, 2080-2113 (2015)

[8] Coll, J., et al.: *Chemical aspects of mass spawning in corals. I. Sperm-attractant molecules in the eggs of the scleractinian coral montipora digitata.* Mar. Biol. **118**, 177-182 (1994)

[9] Deshmane, S.L., Kremlev, S., Amini, S., Sawaya, B.E.: *Monocyte chemoattractant protein-1 (mcp-1): an overview.* J. Interferon Cytokine Res. **29**, 313-326 (2009)

[10] Evje, S., Winkler, M.: *Mathematical Analysis of Two Competing Cancer Cell Migration Mechanisms Driven by Interstitial Fluid Flow.* J. Nonlin. Sci. **30**, 1809-1847 (2020)

[11] Friedman, A.: *Partial Differential Equations.* Holt, Rinehart & Winston, New York, 1969

[12] Fujie, K., Ito, A., Winkler, M., Yokota, T.: *Stabilization in a chemotaxis model for tumor invasion.* Discrete Cont. Dyn. Syst. **36**, 151-169 (2016)

[13] Fujiwara, D., Morimoto, H.: *An L^r-theorem of the Helmholtz decomposition of vector fields.* J. Fac. Sci. Univ. Tokyo **24**, 685-700 (1977)

[14] Giga, Y.: *The Stokes operator in L_r spaces.* Proc. Japan Acad. S. **2**, 85-89 (1981)

[15] Giga, Y.: *Solutions for Semilinear Parabolic Equations in L_p and Regularity of Weak Solutions of the Navier-Stokes System.* J. Differential Eq. **61**, 186-212 (1986)

[16] He, S., Tadmor, E.: *Suppressing Chemotactic Blow-Up Through a Fast Splitting Scenario on the Plane.* Arch. Ration. Mech. Anal. **232**, 951-986 (2019)

[17] Henry, D.: *Geometric Theory of Semilinear Parabolic Equations.* Springer, Berlin/Heidelberg, 1981

[18] Herrero, M.A., Velázquez, J.J.L.: *A blow-up mechanism for a chemotaxis model.* Ann. Scu. Norm. Sup. Pisa Cl. Sci. **24**, 633-683 (1997)

[19] Horstmann, D., Wang, G.: Blow-up in a chemotaxis model without symmetry assumptions. Eur. J. Appl. Math. **12**, 159-177 (2001)

[20] Ishida, S., Seki, K., Yokota, T.: *Boundedness in quasilinear Keller-Segel systems of parabolic-parabolic type on non-convex bounded domains.* J. Differential Eq. **256**, 2993-3010 (2014)

[21] Kiselev, A., Ryzhik, L.: *Biomixing by chemotaxis and enhancement of biological reactions.* Comm. Partial Differ. Eq. **37** (1-3), 298-318 (2012)

[22] Kiselev, A. Xu, X.: *Suppression of Chemotactic Explosion by Mixing.* Arch. Ration. Mech. Anal. **222**, 1077-1112 (2016)

[23] Lorz, A.: *Coupled Keller-Segel-Stokes model: global existence for small initial data and blow-up delay.* Comm. Math. Sci. **10**, 555-574 (2012)

[24] Miller, R.L.: *Demonstration of sperm chemotaxis in echinodermata: Asteroidea, holothuroidea, ophiuroidea.* J. Exp. Zool. **234**, 383-414 (1985)

[25] Nagai, T.: *Blow-up of radially symmetric solutions to a chemotaxis system.* Adv. Math. Sci. Appl. **5**, 581-601 (1995)

[26] Nagai, T.: *Blowup of Nonradial Solutions to Parabolic-Elliptic Systems Modeling Chemotaxis in Two-Dimensional Domains.* J. Inequal. Appl. **6**, 37-55 (2001)

[27] Nagai, T., Senba, T., Yoshida, K.: *Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis.* Funkc. Ekvacioj, Ser. Int. **40**, 411-433 (1997)

[28] Negreanu, M., Tello, J.I.: *On a parabolic-elliptic system with gradient dependent chemotactic coefficient.* J. Differential Eq. **265**, 733-751 (2018)

[29] Perthame, B., Yasuda, S.: *Stiff-response-induced instability for chemotactic bacteria and flux-limited Keller-Segel equation.* Nonlinearity **31**, 4065 (2018)

[30] Senba, T., Suzuki, T.: *A quasi-linear system of chemotaxis.* Abstr. Appl. Anal. 2006, 1-21 (2006)

[31] Sohr, H.: *The Navier-Stokes Equations. An Elementary Functional Analytic Approach.* Birkhäuser, Basel, 2001

[32] Tao, Y., Winkler, M.: *Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity.* J. Differential Eq. **252**, 692-715 (2012)

[33] Taub, D., Proost, P., Murphy, W., Anver, M., Longo, D., Van Damme, J., Oppenheim, J.: *Monocyte chemotactic protein-1 (mcp-1),-2, and-3 are chemotactic for human t lymphocytes.* J. Clin. Investig. **95**, 1370 (1995)

[34] Tuval, I., Cisneros, L., Dombrowski, C., Wolgemuth, C.W., Kessler, J.O., Goldstein, R.E.: *Bacterial swimming and oxygen transport near contact lines.* Proc. Nat. Acad. Sci. USA **102**, 2277-2282 (2005)

[35] Wang, Y., Winkler, M., Xiang, Z.: *Global classical solutions in a two-dimensional chemotaxis-Navier-Stokes system with subcritical sensitivity.* Ann. Scu. Norm. Sup. Pisa Cl. Sci. **18**, 421-466 (2018)

[36] Wang, Y., Xiang, Z.: *Global existence and boundedness in a Keller-Segel-Stokes system involving a tensor-valued sensitivity with saturation.* J. Differential Eq. **259**, 7578-7609 (2015)

[37] Wang, Y., Xiang, Z.: *Global existence and boundedness in a Keller-Segel-Stokes system involving a tensor-valued sensitivity with saturation: The 3D case.* J. Differ. Eq. **261**, 4944-4973 (2016)

[38] Winkler, M.: *Does a 'volume-filling effect' always prevent chemotactic collapse?* Math. Meth. Appl. Sci. **33**, 12-24 (2010)

[39] Winkler, M.: *Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops.* Comm. Partial Differ. Eq. **37**, 319-351 (2012)

[40] Winkler, M.: *Boundedness and large time behavior in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion and general sensitivity.* Calc. Var. Partial Differential Eq. **54**, 3789-3828 (2015)

[41] Winkler, M.: *Does Fluid Interaction Affect Regularity in the Three-Dimensional Keller-Segel System with Saturated Sensitivity?* J. Math. Fluid Mech. **20**, 1889-1909 (2018)

[42] Winkler, M.: *A three-dimensional Keller-Segel-Navier-Stokes system with logistic source: Global weak solutions and asymptotic stabilization.* J. Funct. Anal. **276**, 1339-1401 (2019)

[43] Winkler, M.: *Small-mass solutions in the two-dimensional Keller-Segel system coupled to the Navier-Stokes equations.* SIAM J. Math. Anal. **52**, 2041-2080 (2020)

[44] Winkler, M.: *Boundedness in a three-dimensional Keller-Segel-Stokes system with subcritical sensitivity.* Preprint

[45] Winkler, M.: *A critical blow-up exponent for flux limitation in a Keller-Segel system.* Preprint

[46] Winkler, M.: *A unifying approach toward boundedness in Keller-Segel type cross-diffusion systems via conditional L^∞ estimates for taxis gradients.* Preprint

[47] Xiang, T.: *Boundedness and global existence in the higher-dimensional parabolic-parabolic chemotaxis system with/without growth source.* J. Differential Eq. **258**, 4275-4323 (2015)