Funnel control for nonlinear functional differential-algebraic systems*

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Abstract— We consider tracking control for a class of nonlinear functional differential-algebraic systems. Funnel control, that is a static nonlinear proportional output error feedback, is applied to achieve tracking of a reference signal by the output signal with prescribed transient behavior.

Index Terms— Differential-algebraic equations, nonlinear systems, functional differential equations, funnel control.

I. INTRODUCTION

Differential-algebraic equations (DAEs) are an appropriate tool to model systems coming from applications such as multibody dynamics [1] and electrical networks [2]. The dynamics and constraints of the system are modeled as a set of differential and algebraic equations. If the internal dynamics of the system are autonomous and the input does affect at most the first derivative of the output (roughly speaking, the largest relative degree part is one), then the DAE model may be written in the form

$$
\begin{aligned}
\Gamma\big(y(t)\big)\, \dot{y}(t) &= f_1\big(y(t)\big) + f_2\big(d_1(t), x(t)\big) \\
&\quad + f_3\big(d_2(t), x(t)\big)\, u(t), \\
\dot{x}(t) &= f_4\big(x(t), y(t), d_3(t)\big).
\end{aligned}
\tag{1.1}
$$

The functions $u : \mathbb{R} \to \mathbb{R}^m$ and $y : \mathbb{R} \to \mathbb{R}^m$ are called *input* and *output* of the system, resp., and d_1, d_2, and d_3 are bounded disturbances. The second equation in (1.1) represents the internal dynamics, governed by the *state* $x : \mathbb{R} \to \mathbb{R}^q$. It is possible, that there are also algebraic variables in the system which depend on x and y and their derivatives, but these do not affect the input-output behavior of the system and hence we omit them in the model (1.1). The differentiable functions f_1, f_2 and f_4 are vector valued, f_3 is scalar valued (for simplicity, cf. Remark 4.2) and Γ is matrix valued; for more details see Section II.

If the internal dynamics of (1.1) are input-to-state stable (ISS) [3], then system (1.1) can be rewritten, by the choice of an appropriate operator T (which depends on f_4, d_3 and the initial value $x(0)$) explained in [4, Sect. 2.3], as a nonlinear functional differential-algebraic multi-input, multi-output systems of the form

$$
\begin{aligned}
\Gamma\big(y(t)\big)\, \dot{y}(t) &= f_1\big(y(t)\big) + f_2\big(d_1(t), (Ty)(t)\big) \\
&\quad + f_3\big(d_2(t), (Ty)(t)\big)\, u(t).
\end{aligned}
\tag{1.2}
$$

In the present paper we consider DAE systems (1.2) which may arise from different models or applications, i.e., T is not necessarily a solution operator as in the motivation above but some causal operator with a bounded input, bounded output property, see Section II for details. This allows for (1.2) to encompass various nonlinear and infinite-dimensional effects, including delays and hysteresis phenomena.

We consider tracking control for systems (1.2). It is the aim of the present paper to prove that the application of the *funnel controller*

$$
\begin{aligned}
u(t) &= -k(t)\, e(t), \quad \text{where} \quad e(t) = y(t) - y_{\text{ref}}(t), \\
k(t) &= \frac{\hat{k}}{1 - \varphi(t)^2 \|e(t)\|^2}\,.
\end{aligned}
\tag{1.3}
$$

to the system (1.2) achieves tracking of the reference signal y_{ref} by the output signal y within the pre-specified performance funnel

$$
\mathcal{F}_\varphi := \big\{\, (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|e\| < 1 \,\big\}, \tag{1.4}
$$

where φ is a nonnegative bounded function with $\varphi(0) = 0$ and otherwise φ is bounded away from 0, see Section IV. Note that no exact tracking is pursued, but a tracking error evolving in $\mathcal{F}_\varphi$. In contrast to approximate tracking, funnel control achieves arbitrarily given transient behavior and the funnel boundary is not necessarily monotonically decreasing or even constant.

The concept of funnel control as a simple strategy for trajectory tracking has been developed in [5] for ODEs, see also the survey [6] and the references therein. Funnel control for linear DAE systems has been investigated in the recent papers [7], [8], [9], [10]. In the present paper we study funnel control for nonlinear DAE systems. This generalizes the results for nonlinear ODE systems obtained in [5], [11] and the results for linear DAE systems obtained in [9]. The essential difficulty, compared to our earlier work, was to derive appropriate assumptions on the nonlinear functions in (1.2), to treat the index of the closed-loop system together with the assumption on the minimal gain, and to derive the set of consistent initial values explicitly.

Due to space limitations, the following issues are not addressed in the present paper, but similar results can be expected for DAEs: Robustness of funnel control (see [12] for ODEs) and funnel control in conjunction with a high-pass filter or a PI-controller as pre-compensator (see [13] for ODEs).

The paper is organized as follows: We introduce the class of systems (1.2) considered in the present paper and, in particular, the class of operators T allowed in (1.2) in

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Section II. Two preliminary results for the proof of our main result - Theorem 4.1 - are provided in Section III. In Section IV the concept of funnel control is introduced and it is proved that the funnel controller achieves tracking with prescribed transient behavior. This result is illustrated by an academic example in Section V.

II. SYSTEM CLASS

We study nonlinear functional DAEs (1.2), where T is a causal operator and d_1, d_2 are extraneous disturbances. We extend a class of operators which has been introduced in [11].

Definition 2.1 (Operator class $\mathcal{T}_{m,q}$): For $t \geq 0$, $w \in \mathcal{C}([0,t]; \mathbb{R}^m)$, $\tau > t$ and $\delta > 0$, define the following set of extensions of w:

$$\mathcal{C}(w; t, \tau, \delta)$$
$$:= \left\{ v \in \mathcal{C}([0,\tau]; \mathbb{R}^m) \left| \begin{array}{l} v|_{[0,t]} = w \wedge \forall s \in [t, \tau] : \\ \|v(s) - w(t)\| \leq \delta \end{array} \right. \right\}.$$

An operator $T : \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^m) \to \mathcal{C}^1(\mathbb{R}_{\geq 0}; \mathbb{R}^q)$ is said to be of class $\mathcal{T}_{m,q}$ if, and only if,

(i) T is a causal operator,
(ii) $\forall t \geq 0 \ \forall w \in \mathcal{C}([0,t]; \mathbb{R}^m) \ \exists \tau > t \ \exists \delta > 0$
   $\exists c_0 > 0 \ \forall u, v \in \mathcal{C}(w; t, \tau, \delta)$:

$$\max_{s \in [t, \tau]} \|(Tu)(s) - (Tv)(s)\| \leq c_0 \max_{s \in [t, \tau]} \|u(s) - v(s)\|,$$

(iii) $\forall c_1 > 0 \ \exists c_2 > 0 \ \forall v \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$:

$$\sup_{s \in \mathbb{R}_{\geq 0}} \|v(s)\| \leq c_1 \implies \sup_{t \in \mathbb{R}_{\geq 0}} \|(Tv)(t)\| \leq c_2,$$

(iv) $\exists h \in \mathcal{C}(\mathbb{R}^m \times \mathbb{R}^q; \mathbb{R}^q) \ \exists \tilde{T} : \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^m) \to \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^q)$
   with Properties (i)–(iii) $\forall v \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^m) \ \forall t \geq 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(Tv)(t) = h\big(v(t), (\tilde{T}v)(t)\big).$$

By allowing operators $T \in \mathcal{T}_{m,q}$ in (1.2), these systems encompass infinite-dimensionality (e.g. delays, both point and distributed) and hysteretic effects (e.g. backlash, Prandtl and Preisach hysteresis), see also [11] for more details. Property (i) in Definition 2.1 is clearly indispensible and Property (ii) is a technical assumption of local Lipschitz type which is required for well-posedness of the closed-loop system. Property (iii) of the operators in $\mathcal{T}_{m,q}$ is a bounded-input, bounded-output assumption and is the counterpart to the assumption of asymptotically stable zero dynamics used for linear systems e.g. in [6], [14].

Compared to [11], we have added Property (iv) which is needed for the case where equation (1.2) has parts with relative degree smaller or equal to zero - a differentiation of these parts is required for the solvability of the closed-loop system (1.2), (1.3). Assumption (iv) is not very restrictive since usually T is an integral operator or a solution operator of a differential equation.

Remark 2.2: Linear ODE minimum-phase systems with positive definite high-frequency gain matrix can be written in the form

$$\dot{y}(t) = p(t) + (Ty)(t) + Bu(t), \qquad (2.1)$$

where $p \in \mathcal{C}^1(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$, $B \in \mathbb{R}^{m \times m}$ satisfies $B = B^\top > 0$ and

$$(Ty)(t) := A_1 y(t) + A_2 \int_0^t e^{A_4(t-s)} A_3 y(s) \, \mathrm{d}s, \quad (2.2)$$

where $A_1 \in \mathbb{R}^{m \times m}$, $A_2 \in \mathbb{R}^{m \times p}$, $A_3 \in \mathbb{R}^{p \times m}$, $A_4 \in \mathbb{R}^{p \times p}$, defines an operator T which satisfies properties (i) and (ii) in Definition 2.1. Property (iii) is satisfied if A_4 has spectrum in the open left complex half plane. For property (iv) to hold, the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}(Ty)(t) = A_1 \dot{y}(t) + A_2 A_3 y(t)$$
$$+ A_2 A_4 \int_0^t e^{A_4(t-s)} A_3 y(s) \, \mathrm{d}s = h\big(y(t), (\tilde{T}y)(t)\big)$$

needs to be satisfied for some $h \in \mathcal{C}(\mathbb{R}^m \times \mathbb{R}^q; \mathbb{R}^q)$ and $\tilde{T} : \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^m) \to \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^q)$ with properties (i)–(iii) and for all $y \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ and all $t \geq 0$. Hence, property (iv) holds if, and only if, $A_1 = 0$. Therefore, the term $A_1 y(t)$ in the definition of T has to be delegated into the function f_1 in the formulation (1.2). Then T belongs to $\mathcal{T}_{m,m}$.

The above observation for linear ODE systems implies that the initial points of Ty are uniquely determined by $y(0)$. In the general case, by causality of $T \in \mathcal{T}_{m,q}$ there exists $j : \mathbb{R}^m \to \mathbb{R}^q$ such that

$$\forall v \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^m) : \ (Tv)(0) = j\big(v(0)\big). \qquad (2.3)$$

Definition 2.3 (System class $\Sigma_{m,p,q,r}$): The functional differential-algebraic equation (1.2) is said to define a *system of class* $\Sigma_{m,p,q,r}$, and we write $(\Gamma, f_1, f_2, f_3, T, d_1, d_2) \in \Sigma_{m,p,q,r}$, if, and only if,

(i) $\exists R \in \mathbb{R}^{m \times r} \ \exists G \in \mathcal{C}(\mathbb{R}^m; \mathbb{R}^{r \times r}) \ \forall y \in \mathbb{R}^m$:
   $G(y) > 0 \ \wedge \ \Gamma(y) = R G(y) R^\top$,
(ii) $f_1 \in \mathcal{C}^1(\mathbb{R}^m; \mathbb{R}^m)$ and, for any basis matrix K of $\ker R^\top$, it holds that $K^\top f_1' K$ is bounded,
(iii) $f_2 \in \mathcal{C}^1(\mathbb{R}^p \times \mathbb{R}^q; \mathbb{R}^m)$,
(iv) $f_3 \in \mathcal{C}^1(\mathbb{R}^p \times \mathbb{R}^q; \mathbb{R}) \ \wedge \ \exists \alpha > 0 \ \forall (d, v) \in \mathbb{R}^p \times \mathbb{R}^q$:
   $f_3(d, v) \geq \alpha$,
(v) $T \in \mathcal{T}_{m,q}$,
(vi) $d_1, d_2 \in \mathcal{C}^1(\mathbb{R}_{\geq 0}; \mathbb{R}^p)$ are bounded.

Linear ODE systems of the form (2.1) belong to $\Sigma_{m,m,m,m}$ with $\Gamma = B^{-1}$. Furthermore, the system class $\Sigma_{m,p,q,r}$ encompasses even singular DAE systems (descriptor systems). In (ii), the assumption on the derivative of f_1 is essential for the solvability of the closed-loop system (1.2), (1.3). More precise, we will require that $\hat{k}$ in (1.3) is larger than the infimum norm of $K^\top f_1' K$ multiplied with $\|(K^\top K)^{-1}\|$ and divided by α from (iv) in order to guarantee invertibility of $\alpha \hat{k} I - \|(K^\top K)^{-1}\| K^\top f_1' K$. The latter is crucial for the explicit solution of the hidden algebraic constraint on the output error in the closed-loop system (1.2), (1.3), i.e., it guarantees that this system is index-1, cf. [15], [16].

III. PRELIMINARY RESULTS

We show three lemmata which are important for the proof of our main result.

Lemma 3.1: Let $D \in \mathcal{C}(\mathbb{R}^m; \mathbb{R}^{l \times l})$ be such that $D(y) > 0$ for all $y \in \mathbb{R}^m$ and let $K \subseteq \mathbb{R}^m$ be compact. Then

$$\exists \beta > 0 \ \forall y \in K : \ \min \sigma\big(D(y) + D(y)^\top\big) \geq \beta.$$

Proof: Since $D + D^\top$ is pointwise symmetric and positive definite, there exists a pointwise eigenvalue decomposition of the form

$$D(y) + D(y)^\top = V(y)J(y)V(y)^\top,$$
$$J(y) = \mathrm{diag}\,(\lambda_1(y), \ldots, \lambda_l(y)),$$

where $V(y)$ is orthogonal and $\lambda_i(y) > 0$ for all $i = 1, \ldots, l$ and all $y \in \mathbb{R}^m$. Seeking a contradiction, we assume that

$$\forall \beta > 0 \ \exists y \in K \ \exists i \in \{1, \ldots, l\} : \ \lambda_i(y) < \beta.$$

Let $\beta > 0$ be arbitrary and choose $y \in K$ and $i \in \{1, \ldots, l\}$ such that $\lambda_i(y) < \beta$. Define $x(y) := V(y)e_i$ and observe that

$$\big(D(y) + D(y)^\top\big)x(y) = V(y)J(y)e_i$$
$$= \lambda_i(y)V(y)e_i = \lambda_i(y)x(y).$$

This implies that

$$x(y)^\top \big(D(y) + D(y)^\top\big)x(y) = \lambda_i(y) < \beta,$$

and, since $x(y) \in \mathcal{S}_m^1 = \{ \, x \in \mathbb{R}^m \mid \|x\| = 1 \, \}$, we have

$$\forall \beta > 0 \ \exists y \in K \ \exists x \in \mathcal{S}_m^1 : \ x^\top\big(D(y) + D(y)^\top\big)x < \beta.$$

We may hence choose sequences $(y_n) \in K^\mathbb{N}$ and $(x_n) \in (\mathcal{S}_m^1)^\mathbb{N}$ such that

$$\lim_{n \to \infty} x_n^\top \big(D(y_n) + D(y_n)^\top\big)x_n = 0.$$

However, this contradicts the fact that the continuous map

$$\mathbb{R}^l \times \mathbb{R}^m \ni (x, y) \mapsto x^\top\big(D(y) + D(y)^\top\big)x$$

has a positive minimum on the compact set $\mathcal{S}_m^1 \times K$. ∎

Lemma 3.2: Let $D \in \mathbb{R}^{l \times l}$ be positive definite. Then

$$\forall z \in \mathbb{R}^l : \ z^\top D^{-1}z \geq \frac{\sigma_{\min}(D + D^\top)}{2\sigma_{\max}(DD^\top)}\|z\|^2,$$

where $\sigma_{\min}(D + D^\top)$ denotes the smallest eigenvalue of $D + D^\top$ and $\sigma_{\max}(DD^\top)$ denotes the largest eigenvalue of $DD^\top$, which are both real and positive.

Proof:

$$z^\top D^{-1}z = \frac{1}{2}z^\top D^{-\top}(D + D^\top)D^{-1}z$$
$$\geq \frac{1}{2}\sigma_{\min}(D + D^\top)\|D^{-1}z\|^2$$
$$= \frac{1}{2}\sigma_{\min}(D + D^\top)z^\top(DD^\top)^{-1}z$$
$$\geq \frac{\sigma_{\min}(D + D^\top)}{2\sigma_{\max}(DD^\top)}\|z\|^2.$$

∎

Lemma 3.3: Let $K \in \mathbb{R}^{m \times r}$ with $\mathrm{rk}\, K = r$ and $M \in \mathbb{R}^{m \times m}$ such that $M = M^\top \geq 0$. Then

$$\big\|(K^\top K + K^\top MK)^{-1}\big\| \leq \big\|(K^\top K)^{-1}\big\|.$$

Proof: It follows from [17, Prop. 8.6.6] that

$$0 < (K^\top K + K^\top MK)^{-1} \leq (K^\top K)^{-1},$$

and this implies the assertion of the lemma. ∎

IV. FUNNEL CONTROL

In this section we prove the main result of the paper: the funnel controller (1.3) achieves tracking of a reference trajectory by the output signal with prescribed transient behavior. Let

$$\mathcal{B}^1(\mathbb{R}_{\geq 0}; \mathbb{R}^m) := \big\{\eta \in \mathcal{C}^1(\mathbb{R}_{\geq 0}; \mathbb{R}^m) \,|\, \eta, \dot{\eta} \text{ are bounded}\big\}$$

and associate, for any function φ belonging to

$$\Phi := \left\{\varphi \in \mathcal{B}^1(\mathbb{R}_{\geq 0}; \mathbb{R}) \,\middle|\, \begin{array}{l} \varphi(0) = 0, \ \varphi(s) > 0 \text{ for all } s > 0 \\ \text{and } \liminf_{s \to \infty} \varphi(s) > 0 \end{array}\right\}$$

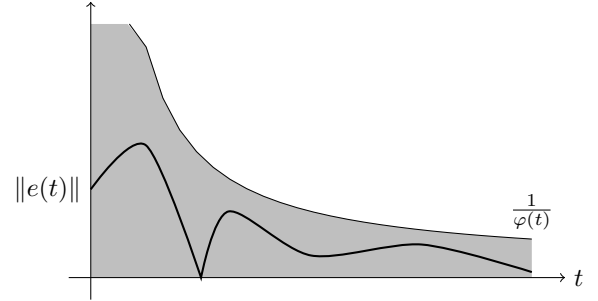the performance funnel $\mathcal{F}_\varphi$, see (1.4) and Figure 1.

Fig. 1: Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $1/\varphi(t)$ for $t > 0$ and a pole at $t = 0$.

The control objective is feedback control so that the tracking error $e = y - y_{\mathrm{ref}}$, where y_{ref} is the reference signal, evolves within $\mathcal{F}_\varphi$ and all variables are bounded. More specific, the transient behavior is supposed to satisfy

$$\forall t > 0 : \ \|e(t)\| < 1/\varphi(t).$$

The bounded-input, bounded-output property of the operator $T \in \mathcal{T}_{m,q}$ can be exploited for an inherent high-gain property of the system (1.2) and hence to maintain error evolution within the funnel: by the design of the controller (1.3), the gain $k(t)$ increases if the norm of the error $\|e(t)\|$ approaches the funnel boundary $1/\varphi(t)$. The control design (1.3) has two advantages: k is non-monotone and (1.3) is a simple static time-varying proportional output feedback.

Theorem 4.1 (Funnel control): Let the system $(\Gamma, f_1, f_2, f_3, T, d_1, d_2) \in \Sigma_{m,p,q,r}$ be given, let $\varphi \in \Phi$ define a performance funnel $\mathcal{F}_\varphi$, and use the notation from Definition 2.3. Furthermore, let $y_{\mathrm{ref}} \in \mathcal{B}^1(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ be any reference trajectory, K be a basis matrix of $\ker R^\top$, and assume that the initial gain satisfies

$$\hat{k} > \alpha^{-1}\|(K^\top K)^{-1}\| \sup_{y \in \mathbb{R}^m} \|K^\top f_1'(y)K\|. \quad (4.1)$$

Then, using the function j from (2.3), for any initial value

$$y^0 \in \left\{ w \in \mathbb{R}^m \;\middle|\; \begin{array}{l} K^\top \Big(f_1(w) + f_2\big(d_1(0), j(w)\big) \\ -\hat{k}\, f_3\big(d_2(0), j(w)\big)\big(w - y_{\mathrm{ref}}(0)\big) \Big) = 0 \end{array} \right\},$$

the application of the funnel controller (1.3) to (1.2), $y(0) = y^0$, yields a closed-loop initial-value problem that has a solution and every solution can be extended to a global solution. Furthermore, for every global solution y,

(i) the corresponding tracking error $e = y - y_{\mathrm{ref}}$ evolves uniformly within the performance funnel $\mathcal{F}_\varphi$; more precisely,

$$\exists\, \varepsilon > 0 \;\; \forall\, t > 0: \; \|e(t)\| \leq \varphi(t)^{-1} - \varepsilon. \qquad (4.2)$$

(ii) the corresponding gain function k given by (1.3) is bounded:

$$\forall\, t_0 > 0: \; \sup_{t \geq t_0} |k(t)| \leq \frac{\hat{k}}{1 - (1 - \varepsilon \lambda_{t_0})^2},$$

where $\lambda_{t_0} := \inf_{t \geq t_0} \varphi(t) > 0$ for all $t_0 > 0$.

Proof: We proceed in several steps.

Step 1: We show existence of a local solution of the closed-loop system (1.2), (1.3). Set

$$\tilde{\mathcal{D}} := \left\{ (t, e, k) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{m+1} \;\middle|\; \varphi(t)\|e\| < 1 \right\}.$$

The closed-loop system (1.2), (1.3) may be written in the form

$$\begin{aligned}
\Gamma\big(y(t)\big)\dot{e}(t) &= f_1\big(y(t)\big) - k(t) f_3\big(d_2(t), (Ty)(t)\big) e(t) \\
&\quad + f_2\big(d_1(t), (Ty)(t)\big) - \Gamma\big(y(t)\big)\dot{y}_{\mathrm{ref}}(t), \\
k(t) &= \hat{k}\,\big(1 - \varphi(t)^2 \|e(t)\|^2\big)^{-1},
\end{aligned}$$
$$(4.3)$$

where $y = e + y_{\mathrm{ref}}$. If $\Gamma\big(y^0 - y_{\mathrm{ref}}(0)\big)$ were invertible, then the solution theory of functional differential equations (see [11, Thm. B.1]) would guarantee the existence of a local solution with $(t, e(t), k(t)) \in \tilde{\mathcal{D}}$ and $(t, e(t)) \in \mathcal{F}_\varphi$ at initial data

$$\begin{pmatrix} e \\ k \end{pmatrix}(0) = \begin{pmatrix} y^0 - y_{\mathrm{ref}}(0) \\ \hat{k} \end{pmatrix}. \qquad (4.4)$$

In the present case, we need to decompose equation (4.3) into an ODE part and an algebraic constraint. By differentiating the algebraic constraint we may obtain an ODE in all system variables, the solution of which satisfies the algebraic constraint. In this sense, equation (4.3) is an index-1 DAE, cf. [15], [16].

Step 2: We will now rewrite (4.3) as an explicit functional differential equation. Observe that, by the singular value decomposition, there exists an orthogonal matrix $U \in \mathbb{R}^{m \times m}$ such that $UR = [\tilde{R}^\top, 0]^\top$, where $\tilde{R} \in \mathbb{R}^{l \times r}$ has full row rank. This implies that, for all $w \in \mathbb{R}^m$,

$$U\Gamma(w)U^\top = \begin{bmatrix} D(w) & 0 \\ 0 & 0 \end{bmatrix}, \; D(w) = \tilde{R}G(w)\tilde{R}^\top > 0.$$

We introduce new variables $e_1 := [I_l, 0]Ue$ and $e_2 := [0, I_{m-l}]Ue$. Then (4.3) may be written, by the use of

$\|e(t)\|^2 = \|Ue(t)\|^2 = \|e_1(t)\|^2 + \|e_2(t)\|^2$, as the system

$$\begin{aligned}
\dot{e}_1(t) &= [D(y(t))^{-1}, 0]U\Big(f_1\big(y(t)\big) + f_2\big(d_1(t), (Ty)(t)\big) \Big) \\
&\quad - k(t) f_3\big(d_2(t), (Ty)(t)\big) D\big(y(t)\big)^{-1} e_1(t) \\
&\quad - [I_l, 0]U\dot{y}_{\mathrm{ref}}(t), \\
0 &= [0, I_{m-l}]U\Big(f_1\big(y(t)\big) + f_2\big(d_1(t), (Ty)(t)\big) \Big) \\
&\quad - k(t) f_3\big(d_2(t), (Ty)(t)\big) e_2(t), \\
k(t) &= \hat{k}\,\big(1 - \varphi(t)^2(\|e_1(t)\|^2 + \|e_2(t)\|^2)\big)^{-1}.
\end{aligned}$$
$$(4.5)$$

Note that, since D is continuous, the map $w \mapsto D(w)^{-1}$ is continuous as well. Now, differentiation of the second equation in (4.5), and using

$$\begin{aligned}
F_{21}(w) &:= [0, I_{m-l}]Uf_1'(w)U^\top[I_l, 0]^\top, \\
F_{22}(w) &:= [0, I_{m-l}]Uf_1'(w)U^\top[0, I_{m-l}]^\top, \quad w \in \mathbb{R}^m,
\end{aligned}$$

yields

$$\begin{aligned}
0 &= F_{21}\big(y(t)\big)\dot{e}_1(t) + F_{22}\big(y(t)\big)\dot{e}_2(t) \\
&\quad + [0, I_{m-l}]Uf_2'\big(d_1(t), (Ty)(t)\big)\begin{pmatrix} \dot{d}_1(t) \\ h\big(y(t), (\tilde{T}y)(t)\big) \end{pmatrix} \\
&\quad - k(t)e_2(t) f_3'\big(d_2(t), (Ty)(t)\big)\begin{pmatrix} \dot{d}_2(t) \\ h\big(y(t), (\tilde{T}y)(t)\big) \end{pmatrix} \\
&\quad - f_3\big(d_2(t), (Ty)(t)\big)\big(\dot{k}(t)e_2(t) + k(t)\dot{e}_2(t)\big). \qquad (4.6)
\end{aligned}$$

Observe that the derivative of k is given by

$$\begin{aligned}
\dot{k}(t) &= 2k(t)\,\big(1 - \varphi(t)^2(\|e_1(t)\|^2 + \|e_2(t)\|^2)\big)^{-1} \\
&\quad \times \big(\varphi(t)\dot{\varphi}(t)(\|e_1(t)\|^2 + \|e_2(t)\|^2) \\
&\quad + \varphi(t)^2(e_1(t)^\top \dot{e}_1(t) + e_2(t)^\top \dot{e}_2(t))\big). \qquad (4.7)
\end{aligned}$$

Introduce the set

$$\mathcal{D} := \Big\{ (t, k, e_1, e_2) \in \mathbb{R}_{\geq 0} \times [\hat{k}, \infty) \times \mathbb{R}^l \times \mathbb{R}^{m-l} \;\Big| \\ \varphi(t)^2(\|e_1\|^2 + \|e_2\|^2) < 1 \Big\}$$

and define

$$\begin{aligned}
\xi : \mathbb{R}_{\geq 0} \times \mathbb{R}^l \times \mathbb{R}^{m-l} &\to \mathbb{R}^m, \\
(t, e_1, e_2) &\mapsto U^\top \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + y_{\mathrm{ref}}(t), \\
\Theta_1 : \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^l) \times \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^l) &\to \mathcal{C}^1(\mathbb{R}_{\geq 0}; \mathbb{R}^q), \\
(e_1, e_2) &\mapsto T(\xi(\cdot, e_1(\cdot), e_2(\cdot))), \\
\Theta_2 : \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^l) \times \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^l) &\to \mathcal{C}^1(\mathbb{R}_{\geq 0}; \mathbb{R}^q), \\
(e_1, e_2) &\mapsto \tilde{T}(\xi(\cdot, e_1(\cdot), e_2(\cdot))),
\end{aligned}$$

$$g_1 : \mathcal{D} \times \mathbb{R}^q \to \mathbb{R}^l, \qquad (t, k, e_1, e_2, \eta) \mapsto$$
$$[D\big(\xi(t, e_1, e_2)\big)^{-1}, 0]U\Big(f_1\big(\xi(t, e_1, e_2)\big) + f_2\big(d_1(t), \eta\big) \Big)$$
$$- kf_3\big(d_2(t), \eta\big) D\big(\xi(t, e_1, e_2)\big)^{-1} e_1(t) - [I_l, 0]U\dot{y}_{\mathrm{ref}}(t).$$

Now, the first equation in (4.5) can be written as

$$\dot{e}_1(t) = g_1\big(t, k(t), e_1(t), e_2(t), \Theta(e_1, e_2)(t)\big).$$

Further define

$$M : \mathcal{D} \times \mathbb{R}^q \to \mathbf{Gl}_{m-l}(\mathbb{R}), \ (t, k, e_1, e_2, \eta) \mapsto$$
$$F_{22}\big(\xi(t, e_1, e_2)\big) - k\, f_3\big(d_2(t), \eta\big)\Big(I_{m-l}+$$
$$2\varphi(t)^2\big(1 - \varphi(t)(\|e_1\|^2 + \|e_2\|^2)\big)^{-1} e_2 e_2^\top\Big)$$

and

$$g_2 : \mathcal{D} \times \mathbb{R}^l \times \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^{m-l},$$
$$(t, k, e_1, e_2, \tilde{e}_1, \eta_1, \eta_2) \mapsto F_{21}\big(\xi(t, e_1, e_2)\big)\tilde{e}_1$$
$$+ [0, I_{m-l}]U f_2'\big(d_1(t), \eta_1\big)\begin{pmatrix} \dot{d}_1(t) \\ h\big(\xi(t, e_1, e_2), \eta_2\big) \end{pmatrix}$$
$$- k\, e_2\, f_3'\big(d_2(t), \eta_1\big)\begin{pmatrix} \dot{d}_2(t) \\ h\big(\xi(t, e_1, e_2), \eta_2\big) \end{pmatrix}$$
$$- 2k\, f_3\big(d_2(t), \eta_1\big)\big(1 - \varphi(t)^2(\|e_1\|^2 + \|e_2\|^2)\big)^{-1}$$
$$\times \big(\varphi(t)\dot{\varphi}(t)(\|e_1\|^2 + \|e_2\|^2) + \varphi(t)^2 e_1^\top \tilde{e}_1\big)e_2.$$

If M is well defined, then inserting $\dot{k}$ from (4.7) into (4.6) and rearranging according to $\dot{e}_2$ gives

$$M\big(t, k(t), e_1(t), e_2(t), \Theta_1(e_1, e_2)(t)\big)\dot{e}_2(t) =$$
$$g_2\big(t, k(t), e_1(t), e_2(t), \dot{e}_1(t), \Theta_1(e_1, e_2)(t), \Theta_2(e_1, e_2)(t)\big).$$

Now we show that M is well defined, i.e., that M is invertible everywhere on its domain. To this end, let K be the basis matrix of $\ker R^\top$ as in the statement of the theorem. Since $\tilde{K} := U^\top[0, I_{m-l}]^\top$ is another basis matrix of $\ker R^\top$, there exists $T \in \mathbf{Gl}_{m-l}(\mathbb{R})$ such that $\tilde{K} = KT$. We may now write

$$I_{m-l} = \tilde{K}^\top \tilde{K} = T^\top K^\top KT,$$
$$\tilde{e}(e_1, e_2) := U^\top(e_1, e_2^\top)^\top, \ e_1 \in \mathbb{R}^l, e_2 \in \mathbb{R}^{m-l},$$
$$e_2 = \tilde{K}^\top \tilde{e}(e_1, e_2) = T^\top K^\top \tilde{e}(e_1, e_2).$$

The matrix-valued function

$$E : \tilde{\mathcal{D}} \to \mathbb{R}^{m \times m}, \ (t, e, k) \mapsto 2\varphi(t)^2\big(1 - \varphi(t)^2\|e\|^2\big)^{-1} ee^\top$$

is symmetric and positive semi-definite everywhere. Then we have that

$$M(t, k, e_1, e_2, \eta) = T^\top K^\top f_1'\big(\xi(t, e_1, e_2)\big)KT$$
$$- k f_3\big(d_2(t), \eta\big)T^\top\Big(K^\top K + K^\top E\big(t, \tilde{e}(e_1, e_2), k\big)K\Big)T$$

and clearly M is invertible everywhere if, and only if, $T^{-\top}MT^{-1}$ is invertible everywhere. By Lemma 3.3 and (4.1) we obtain, for all $(t, k, e_1, e_2, \eta) \in \mathcal{D} \times \mathbb{R}^q$, that

$$\|k^{-1}f_3\big(d_2(t), \eta\big)^{-1}\big(K^\top K + K^\top E(t, \tilde{e}(e_1, e_2), k)K\big)^{-1}$$
$$\times K^\top f_1'\big(\xi(t, e_1, e_2)\big)K\|$$
$$\leq \hat{k}^{-1}\alpha^{-1}\|(K^\top K)^{-1}\|\|K^\top f_1'\big(\xi(t, e_1, e_2)\big)K\| < 1.$$

This implies that M is invertible everywhere. Now, with

$$\tilde{g}_2 : \mathcal{D} \times \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^{m-l}, \ (t, k, e_1, e_2, \eta_1, \eta_2) \mapsto$$
$$M(t, k, e_1, e_2, \eta_1)^{-1}g_2\big(t, k, e_1, e_2, g_1(t, k, e_1, e_2, \eta_1), \eta_1, \eta_2\big),$$

and

$$g_3 : \mathcal{D} \times \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}, \ (t, k, e_1, e_2, \eta_1, \eta_2) \mapsto$$
$$2k\big(1 - \varphi(t)^2(\|e_1\|^2 + \|e_2\|^2)\big)^{-1}\Big(\varphi(t)\dot{\varphi}(t)(\|e_1\|^2 + \|e_2\|^2)$$
$$+ \varphi(t)^2\big(e_1^\top g_1(t, k, e_1, e_2, \eta_1) + e_2^\top \tilde{g}_2(t, k, e_1, e_2, \eta_1, \eta_2)\big)\Big)$$

we obtain the system

$$\dot{e}_1(t) = g_1\big(t, k(t), e_1(t), e_2(t), \Theta_1(e_1, e_2)(t)\big)$$
$$\dot{e}_2(t) = \tilde{g}_2\big(t, k(t), e_1(t), e_2(t), \Theta_1(e_1, e_2)(t), \Theta_2(e_1, e_2)(t)\big)$$
$$\dot{k}(t) = g_3\big(t, k(t), e_1(t), e_2(t), \Theta_1(e_1, e_2)(t), \Theta_2(e_1, e_2)(t)\big)$$
$$\tag{4.8}$$

with initial data

$$(k, e_1, e_2)(0) = \Big(\hat{k}, U(y^0 - y_{\text{ref}}(0))\Big) =: \zeta \tag{4.9}$$

Step 3: We show existence of a maximal local solution of (4.8), (4.9) which evolves in $\mathcal{D}$ and leaves every compact subset of $\mathcal{D}$. We may write (4.8), (4.9) for appropriate $F : \mathcal{D} \times \mathbb{R}^{2q} \to \mathbb{R}^{m+1}$ in the form

$$\dot{z}(t) = F\big(t, z(t), (Sz)(t)\big), \quad z(0) = \zeta, \tag{4.10}$$

where $Sz = \big(\Theta_1(e_1, e_2)^\top, \Theta_2(e_1, e_2)^\top\big)^\top$ and $S : \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{m+1}) \to \mathcal{C}^1(\mathbb{R}_{\geq 0}; \mathbb{R}^{2q})$ is an operator with the properties as in [11, Def. 2.1] (note that in [11] only operators with domain $\mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R})$ are considered, but the generalization to domain $\mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{m+1})$ is straightforward), which follows by invoking that y_{ref} and $\dot{y}_{\text{ref}}$ are bounded.

It is clear that g_1, $\tilde{g}_2$ and g_3 are continuous and hence F is continuous.

Then [11, Thm. B.1] is applicable to the system (4.10) (note that in [11] a domain $\mathcal{D} \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}$ is considered, but the generalization to the higher dimensional case is only a technicality) and we may conclude that

(a) there exists a solution of (4.10), i.e., a function $z \in \mathcal{C}([0, \rho); \mathbb{R}^{m+1})$ for some $\rho \in (0, \infty]$ such that z is locally absolutely continuous, $z(0) = \zeta$, $(t, z(t)) \in \mathcal{D}$ for all $t \in [0, \rho)$ and (4.10) holds for almost all $t \in [0, \rho)$,

(b) every solution can be extended to a maximal solution $z \in \mathcal{C}([0, \omega); \mathbb{R}^{m+1})$, i.e., z has no proper right extension that is also a solution,

(c) if $z \in \mathcal{C}([0, \rho); \mathbb{R}^{m+1})$ is a maximal solution, then the closure of $\operatorname{graph} z$ is not a compact subset of $\mathcal{D}$.

Property (c) follows since F is locally essentially bounded, as it is continuous. Let $z \in \mathcal{C}([0, \omega); \mathbb{R}^{m+1})$ be a maximal solution of (4.10) and observe that z is continuously differentiable since F is continuous. It is clear that z is a maximal solution of (4.8), (4.9) which leaves every compact subset of $\mathcal{D}$.

Step 4: We show that there exists a maximal solution of (4.3), (4.4) which evolves in $\tilde{\mathcal{D}}$ and leaves every compact

subset of $\tilde{\mathcal{D}}$. The solution $(e_1, e_2, k) \in \mathcal{C}([0,\omega); \mathbb{R}^{m+1})$ of (4.8) in particular satisfies (4.6). Integration gives, for all $t \in [0,\omega)$,

$$
\begin{aligned}
&[0, I_{m-l}]U\Big(f_1\big(y(t)\big) + f_2\big(d_1(t), (Ty)(t)\big)\Big) \\
&- k(t)f_3\big(d_2(t), (Ty)(t)\big)\, e_2(t) \\
&- [0, I_{m-l}]U\Big(f_1\big(y^0\big) + f_2\big(d_1(0), j(y^0)\big)\Big) \\
&+ \hat{k}f_3\big(d_2(0), j(y^0)\big)[0, I_{m-l}]U\big(y^0 - y_{\mathrm{ref}}(0)\big) = 0,
\end{aligned}
$$

where $y = e + y_{\mathrm{ref}}$ and j is from (2.3). Since $\ker[0, I_{m-l}]U = \ker K$ it follows from the choice of y^0 that (e_1, e_2, k) satisfies the second equation in (4.5), and hence all equations in (4.5) are satisfied on $[0,\omega)$. This leads to a maximal solution $(e, k) \in \mathcal{C}^1([0,\omega); \mathbb{R}^{n+1})$ of (4.3) with $\mathrm{graph}\,(e,k) \subseteq \tilde{\mathcal{D}}$. Note that the solution is maximal, since the existence of a right extension would lead to a right extension of z, a contradiction. Furthermore, by (c) we have

the closure of $\mathrm{graph}\,(e,k)$ is not a compact subset of $\tilde{\mathcal{D}}$. (4.11)

Step 5: We show that k is bounded. Seeking a contradiction, assume that $k(t) \to \infty$ for $t \to \omega$.

Step 5a: We show that $e_2(t) \to 0$ for $t \to \omega$. Seeking a contradiction, assume that there exist $\kappa > 0$ and a sequence $(t_n) \subseteq \mathbb{R}_{\geq 0}$ with $t_n \nearrow \omega$ such that $\|e_2(t_n)\| \geq \kappa$ for all $n \in \mathbb{N}$. Let $H(t) := [0, I_{m-l}]U f_2\big(d_1(t), (Ty)(t)\big)$, $t \geq 0$, and $\tilde{f}_1(y) := [0, I_{m-l}]U f_1(y)$, $y \in \mathbb{R}^m$. Then, from (4.5) we obtain, for all $t \geq 0$,

$$
\begin{aligned}
\|H(t)\| &= \|\tilde{f}_1\big(y(t)\big) - k(t)f_3\big(d_2(t), (Ty)(t)\big)e_2(t)\| \\
&\geq \Big| \|\tilde{f}_1\big(y(t)\big)\| - k(t)f_3\big(d_2(t), (Ty)(t)\big)\|e_2(t)\| \Big|.
\end{aligned}
$$

Since y is bounded and f_1 is continuous, there exists $\gamma > 0$ such that $\sup_{t \geq 0} \|\tilde{f}_1\big(y(t)\big)\| \leq \gamma$. Since $k(t) \to \infty$, $\|e_2(t_n)\| \geq \kappa$ and $f_3\big(d_2(t_n), (Ty)(t_n)\big) \geq \alpha$, we find that for $n \in \mathbb{N}$ large enough

$$
\|\tilde{f}_1\big(y(t_n)\big)\| < k(t_n)f_3\big(d_2(t_n), (Ty)(t_n)\big)\|e_2(t_n)\|
$$

and hence

$$
\|H(t_n)\| \geq \alpha\kappa k(t_n) - \beta \to \infty \quad \text{for } n \to \infty.
$$

However, this contradicts the fact that H is bounded, as d_1 and Ty are bounded (the latter follows from boundedness of y and Property (iii) in Definition 2.1).

Step 5b: Now, if $l = 0$ then $e = e_2$ and we have $\lim_{t \to \omega} \|e(t)\| = 0$, which implies, by boundedness of φ, $\lim_{t \to \omega} \varphi(t)^2 \|e(t)\|^2 = 0$, hence $\lim_{t \to \omega} k(t) = \hat{k}$, a contradiction. Hence, in the following we assume that $l > 0$.

Let $\delta \in (0, \omega)$ be arbitrary but fixed and $\lambda := \inf_{t \in (0,\omega)} \varphi(t)^{-1} > 0$. Since $\dot{\varphi}$ is bounded and $\liminf_{t \to \infty} \varphi(t) > 0$ we find that $\frac{\mathrm{d}}{\mathrm{d}t}\varphi|_{[\delta,\infty)}(\cdot)^{-1}$ is bounded and hence there exists a Lipschitz bound $L > 0$ of $\varphi|_{[\delta,\infty)}(\cdot)^{-1}$. Furthermore, observe that by continuity of $D(\cdot)$, $D(\cdot)^{-1}$ is continuous as well and since y, Ty, d_1 and

$\dot{y}_{\mathrm{ref}}$ are bounded, the number

$$
\begin{aligned}
\mu := \sup_{t \in [0,\omega)} \Big\| &[D(y(t))^{-1}, 0]U\Big(f_1\big(y(t)\big) \\
&+ f_2\big(d_1(t), (Ty)(t)\big)\Big) - [I_l, 0]U \dot{y}_{\mathrm{ref}}(t) \Big\|
\end{aligned}
$$

is well defined. Moreover, Lemma 3.1 implies

$$
\exists \beta_1 > 0 \; \forall t \in [0,\omega): \; \min \sigma\big(D(y(t)) + D(y(t))^\top\big) \geq \beta_1.
$$

Continuity of D and boundedness of y give

$$
\exists \beta_2 > 0 \; \forall t \in [0,\omega): \; \max \sigma\big(D(y(t))D(y(t))^\top\big) \leq \beta_2.
$$

We may now conclude from Lemma 3.2 that

$$
\forall t \in [0,\omega) \; \forall z \in \mathbb{R}^l: \; z^\top D\big(y(t)\big)^{-1} z \geq \frac{\beta_1}{2\beta_2}\|z\|^2. \quad (4.12)
$$

Define $\nu := \frac{\lambda^2 \hat{k}\alpha\beta_1}{8\beta_2}$. Now, choose $\varepsilon > 0$ small enough so that

$$
\varepsilon \leq \min\left\{ \frac{\lambda}{2}, \min_{t \in [0,\delta]}(\varphi(t)^{-1} - \|e_1(t)\|) \right\}
$$

and

$$
L \leq -\mu + \frac{\nu}{\varepsilon}. \quad (4.13)
$$

We show that

$$
\forall t \in (0,\omega): \; \varphi(t)^{-1} - \|e_1(t)\| \geq \varepsilon. \quad (4.14)
$$

By definition of ε this holds on $(0,\delta]$. Seeking a contradiction suppose that

$$
\exists t_1 \in [\delta, \omega): \; \varphi(t_1)^{-1} - \|e_1(t_1)\| < \varepsilon.
$$

Then for

$$
t_0 := \max\left\{ t \in [\delta, t_1) \; \middle| \; \varphi(t)^{-1} - \|e_1(t)\| = \varepsilon \right\}
$$

we have for all $t \in [t_0, t_1]$ that

$$
\varphi(t)^{-1} - \|e_1(t)\| \leq \varepsilon \quad \text{and}
$$

$$
\|e_1(t)\| \geq \varphi(t)^{-1} - \varepsilon \geq \lambda - \varepsilon \geq \frac{\lambda}{2}
$$

and

$$
\begin{aligned}
k(t) &= \frac{\hat{k}}{1 - \varphi(t)^2\|e(t)\|^2} \geq \frac{\hat{k}}{1 - \varphi(t)^2\|e_1(t)\|^2} \\
&= \frac{\hat{k}}{(1 - \varphi(t)\|e_1(t)\|)(1 + \varphi(t)\|e_1(t)\|)} \geq \frac{\hat{k}}{2\varepsilon\varphi(t)} \geq \frac{\lambda\hat{k}}{2\varepsilon}.
\end{aligned}
$$

Now we have, for all $t \in [t_0, t_1]$,

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|^2 &= e_1(t)^\top \dot{e}_1(t) \\
&\overset{(4.5)}{\leq} \mu\|e_1(t)\| - \alpha k(t)e_1(t)^\top D\big(y(t)\big)^{-1} e_1(t)^\top \\
&\overset{(4.12)}{\leq} \mu\|e_1(t)\| - \frac{\lambda\hat{k}\alpha\beta_1}{4\varepsilon\beta_2}\|e_1(t)\|^2
\end{aligned}
$$

This yields that

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|^2 \leq \left(\mu - \frac{\nu}{\varepsilon}\right)\|e_1(t)\| \overset{(4.13)}{\leq} -L\|e_1(t)\|.
$$

Therefore, using

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|^2 = \|e_1(t)\|\frac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|,$$

we find that

$$\|e_1(t_1)\| - \|e_1(t_0)\| = \int_{t_0}^{t_1} \frac{1}{2}\|e_1(t)\|^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|^2 \,\mathrm{d}t$$
$$\leq -L(t_1 - t_0) \leq -|\varphi(t_1)^{-1} - \varphi(t_0)^{-1}|$$
$$\leq \varphi(t_1)^{-1} - \varphi(t_0)^{-1},$$

and hence

$$\varepsilon = \varphi(t_0)^{-1} - \|e_1(t_0)\| \leq \varphi(t_1)^{-1} - \|e_1(t_1)\| < \varepsilon,$$

a contradiction.

Therefore, (4.14) holds and as $e_2(t) \to 0$ there exists $\tilde{t} \in [0,\omega)$ such that $\|e_2(t)\| \leq \varepsilon$ for all $t \in [\tilde{t}, \omega)$. Then, invoking $\varepsilon \leq \frac{\lambda}{2}$, we obtain for all $t \in [\tilde{t}, \omega)$

$$\|e(t)\|^2 = \|e_1(t)\|^2 + \|e_2(t)\|^2 \leq (\varphi(t)^{-1} - \varepsilon)^2 + \varepsilon^2$$
$$\leq \varphi(t)^{-2} - 2\varepsilon\lambda + 2\varepsilon^2 \leq \varphi(t)^{-2} - 2\varepsilon^2.$$

This implies boundedness of k, a contradiction.

Step 6: We show that $\omega = \infty$. First note that by Step 2 and Step 3 we have that $(e,k) : [0,\omega) \to \mathbb{R}^{m+1}$ is bounded. Further noting that boundedness of k is equivalent to (4.2) (for $t \in [0,\omega)$), the assumption $\omega < \infty$ implies existence of a compact subset $\mathcal{K} \subseteq \tilde{\mathcal{D}}$ such that $\mathrm{graph}\,(x_1, e, x_3, k) \subseteq \mathcal{K}$. This contradicts (4.11).

Step 7: It remains to show (ii). This follows from

$$\forall\, t > 0 : \; k(t) = \hat{k} + k(t)\varphi(t)^2\|e(t)\|^2$$
$$\overset{(4.2)}{\leq} \hat{k} + k(t)\varphi(t)^2(\varphi(t)^{-1} - \varepsilon)^2 = \hat{k} + k(t)(1 - \varphi(t)\varepsilon)^2.$$

This completes the proof of the theorem. ∎

We like to stress again that the condition (4.1) in Theorem 4.1 is sufficient for the closed-loop system to be index-1. It is an open problem as to whether (4.1) is also necessary for the index-1 property, although it seems that this is the case.

Remark 4.2: It is only a technicality to extend the proof of Theorem 4.1 to incorporate disturbances and the operator dependency in Γ, i.e., on the left hand side of (1.2) we have

$$\Gamma\big(y(t), (Ty)(t), d_4(t)\big)\,\dot{y}(t).$$

Furthermore, one could allow for f_3 to be matrix valued, incorporate disturbances in f_1 and replace $u(t)$ by $f_5\big(u(t) + d_5(t)\big)$ for some appropriate function f_5 and disturbance d_5.

V. SIMULATION

In this section we illustrate Theorem 4.1 by an academic example. We consider the 2-input 2-output system

$$0 = \sin(t)\cdot(Ty)(t)^2 + u_1(t),$$
$$(y_1(t)^2 + y_2(t)^2 + 1)\,\dot{y}_2(t) = y_1(t)^3 - y_2(t)^2$$
$$+ \cos(t)\cdot(Ty)(t) + u_2(t), \quad (5.1)$$

which is of class $\Sigma_{2,2,1,1}$ with

$$R = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \;\; G(y_1,y_2) = y_1^2 + y_2^2 + 1, \;\; f_1(y_1,y_2) = \begin{pmatrix} 0 \\ y_1^3 - y_2^2 \end{pmatrix}$$

$$f_2(d_1^1, d_1^2, v) = \begin{pmatrix} d_1^1 v^2 \\ d_1^2 v \end{pmatrix}, \;\; f_3(d_1^1, d_1^2, v) = 1,$$

operator

$$\big(T(y_1,y_2)\big)(t) := e^{-t}x^0 + \int_0^t e^{-(t-s)}y_1(s)\,\mathrm{d}s, \;\; t \geq 0,$$

for any fix $x^0 \in \mathbb{R}$, and bounded disturbance

$$d_1(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}, \;\; t \geq 0.$$

It is easy to verify that T satisfies the conditions in Definition 2.1 and is therefore in class $\mathcal{T}_{2,1}$.

Note that the first equation in (5.1) can be written as the system

$$\dot{x}(t) = -x(t) + y_1(t), \qquad x(0) = x^0$$
$$0 = \sin(t)\cdot x(t)^2 + u_1(t),$$

and T is the solution operator of $\dot{x}(t) = -x(t) + y_1(t)$, $x(0) = x^0$, for fix $x^0 \in \mathbb{R}$ and given $y_1 \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R})$.

As reference signal we take $y_{\mathrm{ref}} = (\sin, \cos)^\top \in \mathcal{B}^1(\mathbb{R}_{\geq 0}; \mathbb{R}^2)$. We may choose $\hat{k} = 1$, since with $K = [1,0]^\top$ we have that $\mathrm{im}\,K = \ker R^\top$ and

$$\sup_{y \in \mathbb{R}^2} \|K^\top f_1'(y)K\| = \sup_{y \in \mathbb{R}^2} \left\| [1,0]\begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\| = 0.$$

Furthermore, the function j from (2.3) is given by $j(v) = x^0$ for $v \in \mathbb{R}^2$. Then an initial value $y^0 \in \mathbb{R}^2$ is consistent for the closed-loop system (5.1), (1.3) (i.e., the condition in Theorem 4.1 is satisfied) if, and only if, $y_1^0 = y_{\mathrm{ref},1}(0) = 0$. We may hence choose $y^0 = (0, -1)^\top$. We also choose $x^0 = 1$.

The funnel $\mathcal{F}_\varphi$ is determined by the function

$$\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad t \mapsto 0.5\,te^{-t} + 2\,\arctan t. \quad (5.2)$$

Note that this prescribes an exponentially (exponent 1) decaying funnel in the transient phase $[0, T]$, where $T \approx 3$, and a tracking accuracy quantified by $\lambda = 1/\pi$ thereafter, see Fig. 2d.

The simulation has been performed in MATLAB (solver: ode15s, relative tolerance: 10^{-14}, absolute tolerance: 10^{-10}). In Figure 2 the simulation, over the time interval $[0, 10]$, of the funnel controller (1.3) with funnel boundary specified in (5.2) and reference signal $y_{\mathrm{ref}} = (\sin, \cos)^\top$ applied to system (5.1) with initial data $y^0 = (0, -1)^\top$ and $x^0 = 1$ is depicted. Fig. 2a shows the output components y_1 and y_2 tracking the reference signal y_{ref} within the funnel shown in Fig. 2d. Note that an action of the input components u_1 and u_2 in Fig. 2b and the gain function k in Fig. 2c is required only if the error $\|e(t)\|$ is close to the funnel boundary $\varphi(t)^{-1}$. This in particular shows that the gain function k is non-monotone.
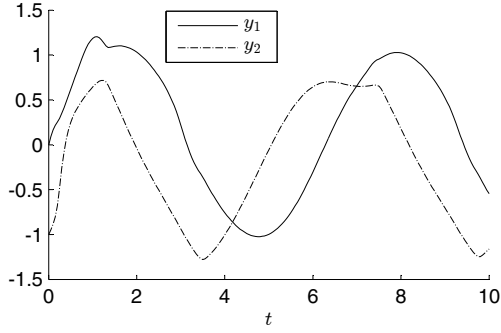
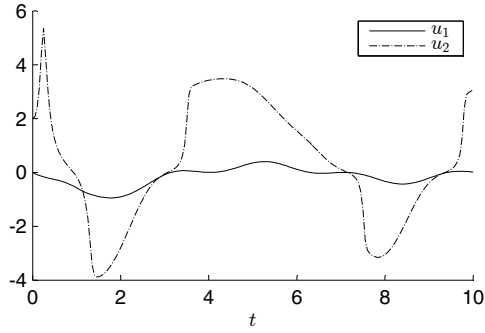Fig. a: Solution components y_1 and y_2

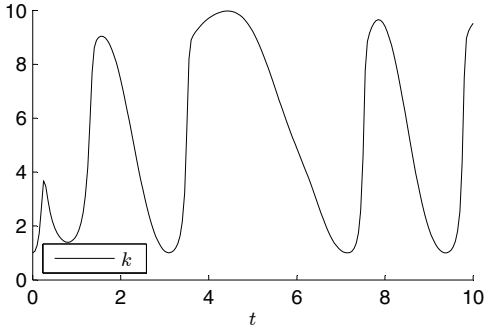
Fig. b: Input components u_1 and u_2
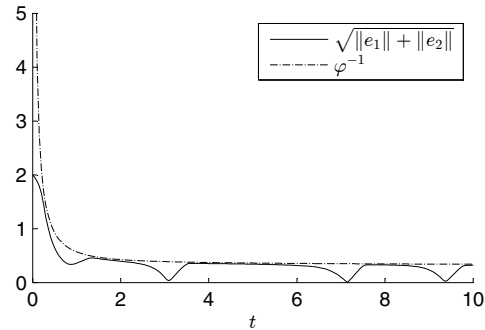

Fig. c: Gain k

Fig. d: Norm of error $\|e(\cdot)\|$ and funnel boundary $\varphi(\cdot)^{-1}$

Fig. 2: Simulation of the funnel controller (1.3) with funnel boundary specified in (5.2) and reference signal $y_{\mathrm{ref}} = (\sin, \cos)^\top$ applied to system (5.1) with initial data $y^0 = (0, -1)^\top$ and $x^0 = 1$.

VI. Conclusions

We have shown that funnel control is feasible for a class of nonlinear functional differential-algebraic systems which in particular encompasses delays and hysteretic effects. Tracking of a reference signal by the output signal with prescribed transient behavior of the tracking error is achieved. The required properties of the system class, most important the bounded-input bounded output property of the of the operator T and the uniform positive semi-definiteness of Γ, are discussed. The condition (4.1) on the minimal gain is characterized as an index-1 condition on the closed-loop system and the set of consistent initial values for the latter is derived explicitly. The results are illustrated by an academic example.

References

[1] E. Eich-Soellner and C. Führer, *Numerical Methods in Multibody Dynamics*. Stuttgart: Teubner, 1998.

[2] R. Riaza, *Differential-Algebraic Systems. Analytical Aspects and Circuit Applications*. Basel: World Scientific Publishing, 2008.

[3] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Trans. Autom. Control*, vol. 34, no. 4, pp. 435–443, 1989.

[4] A. Ilchmann, E. P. Ryan, and C. J. Sangwin, "Systems of controlled functional differential equations and adaptive tracking," *SIAM J. Control Optim.*, vol. 40, no. 6, pp. 1746–1764, 2002. [Online]. Available: http://link.aip.org/link/?SJC/40/1746/1

[5] ——, "Tracking with prescribed transient behaviour," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 7, pp. 471–493, 2002.

[6] A. Ilchmann and E. P. Ryan, "High-gain control without identification: a survey," *GAMM Mitt.*, vol. 31, no. 1, pp. 115–125, 2008.

[7] T. Berger, A. Ilchmann, and T. Reis, "Zero dynamics and funnel control of linear differential-algebraic systems," *Math. Control Signals Syst.*, vol. 24, no. 3, pp. 219–263, 2012.

[8] ——, "Normal forms, high-gain, and funnel control for linear differential-algebraic systems," in *Control and Optimization with Differential-Algebraic Constraints*, ser. Advances in Design and Control, L. T. Biegler, S. L. Campbell, and V. Mehrmann, Eds. Philadelphia: SIAM, 2012, vol. 23, pp. 127–164.

[9] T. Berger, "Zero dynamics and funnel control of general linear differential-algebraic systems," 2013, submitted to *Math. Control Signals Syst.*, preprint available from the website of the author.

[10] T. Berger and T. Reis, "Zero dynamics and funnel control for linear electrical circuits," 2013, submitted to *J. Franklin Inst.*, preprint available from the website of the authors.

[11] A. Ilchmann and E. P. Ryan, "Performance funnels and tracking control," *Int. J. Control*, vol. 82, no. 10, pp. 1828–1840, 2009.

[12] A. Ilchmann and M. Mueller, "Robustness of funnel control in the gap metric," *SIAM J. Control Optim.*, vol. 48, no. 5, pp. 3169–3190, 2009. [Online]. Available: http://link.aip.org/link/?SJC/48/3169/1

[13] A. Ilchmann and H. Schuster, "PI-funnel control for two mass systems," *IEEE Trans. Autom. Control*, vol. 54, no. 4, pp. 918–923, 2009.

[14] T. Berger, "On differential-algebraic control systems," Ph.D. dissertation, Institut für Mathematik, Technische Universität Ilmenau, Universitätsverlag Ilmenau, Ilmenau, Germany, 2014.

[15] P. Kunkel and V. Mehrmann, *Differential-Algebraic Equations. Analysis and Numerical Solution*. Zürich, Switzerland: EMS Publishing House, 2006.

[16] R. Lamour, R. März, and C. Tischendorf, *Differential Algebraic Equations: A Projector Based Analysis*, ser. Differential-Algebraic Equations Forum. Heidelberg-Berlin: Springer-Verlag, 2013, vol. 1.

[17] D. S. Bernstein, *Matrix Mathematics: Theory, Facts, and Formulas*, 2nd ed. Princeton and Oxford: Princeton University Press, 2009.