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Structural properties of positive real and reciprocal rational matrices

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***Abstract*— We consider positive real matrix-valued rational functions. We show that the pointwise kernel as well as the pointwise kernel of the Hermitian part is constant in the right complex half plane. These results are the basis for a decomposition for positive real matrices under orthogonal similarity transformation.**

We further consider positive real matrices which have a certain symmetry property that is known as "reciprocity". A decomposition for reciprocal and positive real matrices under block orthogonal transformation is derived.

We illustrate our results by applying them to transfer functions arising in electrical circuit theory.

***Index Terms*— Passivity, positive realness, reciprocity, transfer function, electrical circuit, modified nodal analysis.**

Nomenclature:

$\mathbb{N}$, $\mathbb{N}_0$		set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\mathbb{C}_+$		open set of complex numbers with positive real part
$\mathbb{R}[s]$		the ring of real polynomials
$\mathbb{R}(s)$		the quotient field of $\mathbb{R}[s]$
$R^{n,m}$		the set of $n \times m$ matrices with entries in a ring R
$\operatorname{rk} A$, $\ker A$, $\operatorname{im} A$		rank, kernel and image of $A \in R^{n,m}$
$\mathbf{Gl}_n(R)$		the group of invertible matrices in $R^{n,n}$
M^*	=	$\overline{M}^\top$, the conjugate transpose of $M \in \mathbb{C}^{n,m}$
I_n		identity matrix of size $n \times n$

Note that we neglect the subscripts in the case where the sizes of the identity and zero matrices are clear from context.

I. Introduction

We study positive real rational matrix functions $G(s) \in \mathbb{R}(s)^{m,m}$, that is

a) $G(s)$ does not have poles in $\mathbb{C}_+$, and
b) $G(\lambda) + G(\lambda)^* \geq 0 \quad \forall\, \lambda \in \mathbb{C}_+$.

This class of rational matrices plays an important role in linear systems theory, since they are transfer functions of passive linear time-invariant systems [1]. In particular, they are important for the analysis and synthesis of electrical circuits and mechanical systems, see [2]–[4] and the references therein.

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Another crucial concept studied in the present article is reciprocity. Loosely speaking, this is the property of a transfer function to satisfy a certain symmetry property.

Definition 1.1 (Signature matrix, reciprocal matrix): Let $m_1, m_2, m \in \mathbb{N}_0$ be such that $m_1 + m_2 = m$. Then

$$S = \operatorname{diag}(I_{m_1}, -I_{m_2}) \in \mathbf{Gl}_m(\mathbb{R}) \tag{1}$$

is called a *signature matrix*.
A rational function $G(s) \in \mathbb{R}(s)^{m,m}$ is called *reciprocal (with signature (m_1, m_2))*, if for S as in (1) it holds

$$G(s)S = SG(s)^\top.$$

Reciprocity means that the transfer function can be partitioned as

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ -G_{12}(s)^\top & G_{22}(s) \end{bmatrix}, \tag{2}$$

where $G_{11}(s) \in \mathbb{R}(s)^{m_1,m_1}$, $G_{22}(s) \in \mathbb{R}(s)^{m_2,m_2}$ are symmetric. This class of rational matrices occurs in electrical circuit theory, where the signature is determined by the numbers of voltage and current sources [1], [2], [5].

In Section II we focus on positive real matrices. We show that $\ker G(\lambda) + G(\lambda)^*$ is independent of $\lambda \in \mathbb{C}_+$. We further prove that $G(s)v$ is constant for all $v \in \ker G(\lambda) + G(\lambda)^*$. Upon these facts we derive a decomposition for positive real matrices under orthogonal similarity transformation. This form decomposes $G(s)$ into a part which has a positive definite Hermitian part in $\mathbb{C}_+$, and some constant skew-Hermitian part.

Reciprocal matrices are then investigated in Section III and another decomposition is derived for positive real and reciprocal rational matrices under similarity transformation with block-diagonal orthogonal matrices.

Finally, in Section IV we consider differential-algebraic models of passive electrical circuits. The transfer functions of these systems are positive real and reciprocal. We derive some consequences of the results from the previous sections.

II. Positive real transfer functions

In this section we investigate positive real matrices and derive a decomposition under constant orthogonal transformations. First, we need the standard representation of positive real matrices. To this end, recall that $G(s) \in \mathbb{R}(s)^{m,m}$ is called *proper*, if $\lim_{\lambda\to\infty} G(\lambda) \in \mathbb{R}^{m,m}$ exists; $G(s)$ is called *strictly proper*, if $\lim_{\lambda\to\infty} G(\lambda) = 0$.

Lemma 2.1 (Positive real functions [6, Sec. 2.7]): Let $G(s) \in \mathbb{R}(s)^{m,m}$ be positive real. Then $G(s)$ has no poles in

$\mathbb{C}_+$. Furthermore, $G(s)$ has a representation

$$G(s) = G_s(s) + M_0 + sM_1 + \frac{R_0}{s} + \sum_{j=1}^{k} \frac{R_k}{s - i\omega_j} + \frac{\overline{R_k}}{s + i\omega_j}, \quad (3)$$

where

- $k \in \mathbb{N}$,
- $\omega_1, \dots, \omega_k \in \mathbb{R}$,
- $R_1, \dots, R_k \in \mathbb{C}^{m,m}$ are Hermitian and positive semi-definite,
- $R_0, M_1 \in \mathbb{R}^{m,m}$ are symmetric and positive semi-definite,
- $M_0 \in \mathbb{R}^{m,m}$ such that $M_0 + M_0^\top$ is positive semi-definite,
- $G_s(s) \in \mathbb{R}(s)^{m,m}$ is strictly proper.

Furthermore, the proper transfer function $M_0 + G_s(s) \in \mathbb{R}(s)^{m,m}$ is positive real.

We show that, for a positive real matrix $G(s) \in \mathbb{R}(s)^{m,m}$ and $\mu, \lambda \in \mathbb{C}_+$, the kernels of $G(\lambda) + G(\lambda)^*$ and $G(\mu) + G(\mu)^*$ coincide. We will conclude that the kernel of $G(s)$ is constant.

Proposition 2.2: Let $G(s) \in \mathbb{R}(s)^{m,m}$ be positive real and $\lambda \in \mathbb{C}_+$, $u \in \mathbb{C}^m$ be such that $u^*G(\lambda)u = 0$. Then $u^*G(s)u = 0 \in \mathbb{R}(s)$. Furthermore, $G(s)u$ is constant.

Proof: Consider the scalar rational function $g(s) = u^*G(s)u \in \mathbb{R}(s)$. Then $g(\lambda) = 0$ and, moreover, $g(s)$ is positive real. Assuming that $g(s) \neq 0$ gives that $1/g(s)$ is again positive real, cf. [7]. However, this contradicts the fact that $1/g(s)$ has a pole at $\lambda \in \mathbb{C}_+$. As a consequence, $g(s) = 0$, i.e., $u^*G(\mu)u = 0$ for all $\mu \in \mathbb{C}_+$. Thus, for all $\mu \in \mathbb{C}_+$, we have

$$u^*(G(\mu)^* + G(\mu))u = (G(\mu)u)^*u + u^*(G(\mu)u) = 0.$$

Since $G(\mu)^* + G(\mu)$ is positive semi-definite, we find $(G(\mu)^* + G(\mu))u = 0$, which is equivalent to

$$G(\mu)^*u = -G(\mu)u \quad \text{for all } \mu \in \mathbb{C}_+.$$

As a consequence, the entries of $G(\cdot)u : \mathbb{C}_+ \to \mathbb{C}$ are holomorphic and their complex conjugates are holomorphic as well. The Cauchy-Riemann equations [8, pp. 231] now imply that $G(s)u$ is constant. ∎

We may immediately conclude that the kernel of a positive real function is constant.

Corollary 2.3: Let $G(s) \in \mathbb{R}(s)^{m,m}$ be positive real and $\lambda \in \mathbb{C}_+$, $u \in \mathbb{C}^m$ be such that $G(\lambda)u = 0$. Then $G(s)u = 0 \in \mathbb{R}(s)^m$.

We now show that, via a similarity transformation with an orthogonal matrix, any positive real rational transfer function may be decomposed into a part which has positive definite Hermitian part in $\mathbb{C}_+$, and some constant skew-symmetric part.

Theorem 2.4: Let $G(s) \in \mathbb{R}(s)^{m,m}$ be positive real. Then there exist numbers and matrices

- $p \in \{0, \dots, m\}$,
- orthogonal $U \in \mathbb{R}^{m,m}$,
- positive real $G_1(s) \in \mathbb{R}(s)^{p,p}$ with $\ker G_1(\lambda) + G_1(\lambda)^* = \{0\}$ for all $\lambda \in \mathbb{C}_+$,
- $L_{12} \in \mathbb{R}^{p,m-p}$,
- skew-symmetric matrix $L_{22} \in \mathbb{R}^{m-p,m-p}$,

such that

$$U^\top G(s)U = \begin{bmatrix} G_1(s) & L_{12} \\ -L_{12}^\top & L_{22} \end{bmatrix}. \quad (4)$$

Furthermore, we have

$$\forall \lambda \in \mathbb{C}_+ : \ \ker G(\lambda) = U \cdot \left(\{0\}^p \times \ker \begin{bmatrix} L_{12} \\ L_{22} \end{bmatrix} \right). \quad (5)$$

Proof: Step 1: We show (4). Let $\{u_1, \dots, u_k\} \subseteq \mathbb{R}^m$ be an orthonormal basis of $\ker G(1) + G(1)^*$. By Corollary 2.3, we have

$$\operatorname{span}\{u_1, \dots, u_k\} = \ker G(\lambda) + G(\lambda)^* \quad \text{for all } \lambda \in \mathbb{C}_+.$$

Let $p = m - k$ and extend $\{u_1, \dots, u_k\}$ by $\{v_1, \dots, v_p\}$ to an orthonormal basis of $\mathbb{R}^m$, and set

$$U = \begin{bmatrix} v_1 & \dots & v_p & u_1 & \dots & u_k \end{bmatrix}.$$

By Proposition 2.2, we have that $G(s)u_i$ is constant for all $i \in \{1, \dots, m\}$. Since $G(s)^\top$ is positive real as well, we further obtain that $G(s)^\top u_i$ is constant for all $i \in \{1, \dots, m\}$. Consequently, $U^\top G(s)U$ is of the form

$$U^\top G(s)U = \begin{bmatrix} G_1(s) & L_{12} \\ L_{21} & L_{22} \end{bmatrix}.$$

Step 2: We show the properties of the blocks in (4). By construction of U, we have, for all $\lambda \in \mathbb{C}_+$,

$$\begin{aligned} \begin{bmatrix} \star & 0 \\ 0 & 0 \end{bmatrix} &= U^\top (G(\lambda) + G(\lambda)^*)U \\ &= U^\top G(\lambda)U + (U^\top G(\lambda)U)^* \\ &= \begin{bmatrix} G_1(\lambda) + G_1(\lambda)^* & L_{12} + L_{21}^\top \\ L_{21} + L_{12}^\top & L_{22} + L_{22}^\top \end{bmatrix}, \end{aligned}$$

Hence, we obtain that L_{22} is skew-symmetric and $L_{12} = -L_{21}^\top$. Assuming that $w \in \mathbb{C}^p \setminus \{0\}$ is such that $(G_1(\lambda) + G_1(\lambda)^*)w = 0$ implies that $(G(\lambda) + G(\lambda)^*)u = 0$ for $u := U(w^\top, 0)^\top$. Therefore,

$$u \in \ker G(\lambda) + G(\lambda)^* = \operatorname{span}\{u_1, \dots, u_k\} = \operatorname{im} U \begin{bmatrix} 0 \\ I_k \end{bmatrix},$$

which yields $u = 0$ and hence $w = 0$, a contradiction.

Step 3: It remains to prove (5). The inclusion "$\supseteq$" is obvious. To show "$\subseteq$", assume that $u \in \ker G(1)$. Then $u^*(G(1) + G(1)^*)u = 0$ and, by semi-definiteness of $G(1)$, we have $(G(1) + G(1)^*)u = 0$. Partitioning

$$\begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = U^\top u,$$

we obtain that $u^1 \in \ker G_1(1) + G_1(1)^*$ and hence $u_1 = 0$. Consequently

$$u_2 \in \ker \begin{bmatrix} L_{12} \\ L_{22} \end{bmatrix}.$$

∎

III. Reciprocal transfer functions

Passive electrical circuits modelled by the MNA method feature a special symmetry property, that is their transfer matrix is reciprocal. In this section we aim for decomposition of positive real and reciprocal rational matrices. First, we have a closer look at the representation (3) of a reciprocal matrix.

Theorem 3.1: Let $G(s) \in \mathbb{R}(s)^{m,m}$ be positive real and reciprocal with signature (m_1, m_2). Then the matrices in (3) are of the form

$$\begin{aligned} M_0 &= \begin{bmatrix} M_{0,11} & M_{0,12} \\ -M_{0,12}^\top & M_{0,22} \end{bmatrix}, \quad M_1 = \begin{bmatrix} M_{1,11} & 0 \\ 0 & M_{1,22} \end{bmatrix}, \\ R_0 &= \begin{bmatrix} R_{0,11} & 0 \\ 0 & R_{0,22} \end{bmatrix}, \\ R_j &= \begin{bmatrix} R_{j,11} & iR_{j,12} \\ -iR_{j,12}^\top & R_{j,22} \end{bmatrix} \text{ for } j \in \{1,\dots,k\}, \end{aligned} \tag{6}$$

where

- $M_{0,11}, M_{1,11}, R_{0,11}, R_{1,11}, \dots, R_{k,11} \in \mathbb{R}^{m_1,m_1}$ and $M_{0,22}, M_{1,22}, R_{0,22}, R_{1,22}, \dots, R_{k,22} \in \mathbb{R}^{m_2,m_2}$ are symmetric and positive semi-definite,
- $M_{0,12}, R_{1,12}, \dots, R_{k,12} \in \mathbb{R}^{m_1,m_2}$.

Furthermore, the matrices

$$\begin{bmatrix} R_{j,11} & R_{j,12} \\ R_{j,12}^\top & R_{j,22} \end{bmatrix} \tag{7}$$

are positive semi-definite for all $j \in \{1,\dots,k\}$.

Proof: By reciprocity of $G(s)$, we find

$$M_1 S = \lim_{\lambda\to\infty} \frac{1}{\lambda} G(\lambda) S = \lim_{\lambda\to\infty} \frac{1}{\lambda} S G(\lambda)^\top = S M_1^\top.$$

Together with the symmetry property $M_1 = M_1^\top$, we now obtain that this matrix is of the form

$$M_1 = \operatorname{diag}(M_{1,11}, M_{1,22}).$$

By furthermore using the limit representations

$$R_0 = \lim_{\lambda\to 0} \lambda G(\lambda), \quad R_j = \lim_{\lambda\to i\omega_j} (\lambda - i\omega_j) G(\lambda),$$

we can infer from the reciprocity of $G(s)$ that

$$\begin{aligned} R_0 S &= S R_0^\top, \\ \text{and} \quad R_j S &= S R_j^\top \text{ for all } j \in \{1,\dots,k\}. \end{aligned}$$

Hence, R_0 has a block diagonal structure by the same reason as for M_1. We can furthermore apply this argumentation to the symmetric and positive semi-definite matrices $\operatorname{Re}(R_1), \dots, \operatorname{Re}(R_k)$ to see that they are of a block diagonal structure. On the other hand the imaginary parts $\operatorname{Im}(R_1), \dots, \operatorname{Im}(R_k)$ are skew-symmetric and reciprocal with signature (m_1, m_2). This implies that $\operatorname{Im}(R_j)$ has a block structure

$$\operatorname{Im}(R_j) = \begin{bmatrix} 0 & R_{j,12} \\ -R_{j,12}^\top & 0 \end{bmatrix} \text{ for } j \in \{1,\dots,k\},$$

where $R_{1,12}, \dots, R_{k,12} \in \mathbb{R}^{m_1,m_2}$.
To verify the block structure of M_0, observe that by Lemma 2.1 the matrix $M_0 + M_0^\top$ is positive semi-definite. Furthermore,

$$M_0 = \lim_{\lambda\to\infty} (G(\lambda) - \lambda M_1),$$

hence, using $M_1 S = S M_1^\top$, we find $M_0 S = S M_0^\top$.
It remains to show that (7) is positive semi-definite. This follows from

$$\begin{aligned} &\begin{bmatrix} R_{j,11} & R_{j,12} \\ R_{j,12}^\top & R_{j,22} \end{bmatrix} \\ &\quad = \begin{bmatrix} I_{m_1} & 0 \\ 0 & iI_{m_2} \end{bmatrix} \underbrace{\begin{bmatrix} R_{j,11} & iR_{j,12} \\ -iR_{j,12}^\top & R_{j,22} \end{bmatrix}}_{=R_j} \begin{bmatrix} I_{m_1} & 0 \\ 0 & iI_{m_2} \end{bmatrix}^*. \end{aligned}$$

■

Since the matrices M_1 and R_0 vanish outside the diagonal blocks, we can conclude the following.

Corollary 3.2: Let $G(s) \in \mathbb{R}(s)^{m,m}$ be positive real and reciprocal with signature (m_1, m_2) and let $G(s)$ be partitioned as in (2). Then the rational function $G_{12}(s)$ is proper and has no pole at zero.

If $G(s) \in \mathbb{R}(s)^{m,m}$ is positive real and reciprocal with signature (m_1, m_2), then the functions $G_{11}(s) \in \mathbb{R}(s)^{m_1,m_1}$, $G_{22}(s) \in \mathbb{R}(s)^{m_2,m_2}$ are both symmetric and positive real. By Proposition 2.2, the latter property implies that the kernels of $G_{11}(\lambda)$ and $G_{22}(\lambda)$ are independent of $\lambda \in \mathbb{C}_+$. In the following result, we show that $G_{12}^\top(s)$ and $G_{12}(s)$ are constant on these kernels.

Proposition 3.3: Let $G(s) \in \mathbb{R}(s)^{m,m}$ be positive real and reciprocal with signature (m_1, m_2) and let $G(s)$ be partitioned as in (2). If $u_1 \in \mathbb{R}^m$ is such that $u_1^\top G_{11}(s) u_1 = 0$, then $G_{11}(s) u_1 = 0$ and $G_{12}(s)^\top u_1$ is constant. If $u_2 \in \mathbb{R}^m$ is such that $u_2^\top G_{22}(s) u_2 = 0$, then $G_{22}(s) u_2 = 0$ and $G_{12}(s) u_2$ is constant.

Proof: By Theorem 2.4, there exists some orthogonal matrix $U_1 \in \mathbb{R}^{m_1,m_1}$, such that

$$U_1^\top G_{11}(s) U_1 = \begin{bmatrix} H_{11}(s) & L_{12} \\ -L_{12}^\top & L_{22} \end{bmatrix},$$

where $H_{11}(s) \in \mathbb{R}(s)^{p_1,p_1}$ has the property that $H_{11}(\lambda) + H_{11}(\lambda)^*$ is positive definite for all $\lambda \in \mathbb{C}_+$, and $L_{12} \in \mathbb{R}^{p_1,m_1-p_1}$, $L_{22} \in \mathbb{R}^{m_1-p_1,m_1-p_1}$ is such that $L_{22}^\top = -L_{22}$. The symmetry of $G_{11}(s)$ now implies that $L_{12} = 0$ and $L_{22} = 0$. Assuming that $u_1^\top G_{11}(s) u_1 = 0$, we obtain from the above matrix decomposition that

$$u_1 = U_1 \begin{bmatrix} 0 \\ u_{12} \end{bmatrix}$$

for some $u_{12} \in \mathbb{R}^{p_1 - m_1}$. Consequently, we obtain

$$G_{11}(s) u = U_1 \begin{bmatrix} H_{11}(s) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ u_{12} \end{bmatrix} = 0.$$

It remains to show that $G_{12}(s)^\top u_1$ is constant: By defining $u = (u_1^\top, 0)^\top$, we have

$$u^\top G(s) u = u_1^\top G_{11}(s) u_1 = 0.$$

By Proposition 2.2, we obtain the desired result from

$$G(s) u = \begin{bmatrix} 0 \\ -G_{12}(s)^\top u_1 \end{bmatrix}.$$

The proof of the last statement is analogous. ■

As a consequence of Proposition 3.3, we can infer that a positive real and reciprocal rational matrix admits a certain form under similarity transformation with orthogonal matrices.

Theorem 3.4: Let $G(s) \in \mathbb{R}(s)^{m,m}$ be positive real and reciprocal with signature (m_1, m_2). Then there exist numbers and matrices

- $p_1 \in \{0, \dots, m_1\}$, $p_2 \in \{0, \dots, m_2\}$,
- orthogonal $U_1 \in \mathbb{R}^{m_1,m_1}$, $U_2 \in \mathbb{R}^{m_2,m_2}$,
- positive real and symmetric $H_{11}(s) \in \mathbb{R}(s)^{p_1,p_1}$, $H_{33}(s) \in \mathbb{R}(s)^{p_2,p_2}$ such that $H_{11}(\lambda) + H_{11}(\lambda)^* > 0$ and $H_{33}(\lambda) + H_{33}(\lambda)^* > 0$ for all $\lambda \in \mathbb{C}_+$,
- $H_{13}(s) \in \mathbb{R}(s)^{p_1,p_2}$, $H_{31}(s) \in \mathbb{R}(s)^{p_2,p_1}$, $L_{14} \in \mathbb{R}^{p_1,m_2-p_2}$, $L_{23} \in \mathbb{R}^{m_1-p_1,p_2}$, $L_{24} \in \mathbb{R}^{m_1-p_1,m_2-p_2}$,

such that, for $U = \operatorname{diag}(U_1, U_2)$, it holds

$$U^\top G(s) U = \begin{bmatrix} H_{11}(s) & 0 & H_{13}(s) & L_{14} \\ 0 & 0 & L_{23} & L_{24} \\ -H_{13}(s)^\top & -L_{23}^\top & H_{33}(s) & 0 \\ -L_{14}^\top & -L_{24}^\top & 0 & 0 \end{bmatrix}. \tag{8}$$

Furthermore, for all $\lambda \in \mathbb{C}_+$, we have

$$\ker G(\lambda) = U^\top \cdot \left(\{0\}^{p_1} \times \ker \begin{bmatrix} L_{23}^\top \\ L_{24}^\top \end{bmatrix} \times \{0\}^{p_2} \times \ker \begin{bmatrix} L_{14} \\ L_{24} \end{bmatrix} \right). \tag{9}$$

Proof: *Step 1*: We show the form (8). Let $\{u_{11}, \dots, u_{k1}\} \subseteq \mathbb{R}^{m_1}$, $\{u_{12}, \dots, u_{l2}\} \subseteq \mathbb{R}^{m_2}$ be orthonormal bases of $\ker G_{11}(1) + G_{11}(1)^*$ and $\ker G_{22}(1) + G_{22}(1)^*$, resp. We can infer from Proposition 3.3, that for all $\lambda \in \mathbb{C}_+$

$$\begin{aligned} \operatorname{span}\{u_{11}, \dots, u_{k1}\} &= \ker G_{11}(\lambda) = \ker G_{11}(\lambda) + G_{11}(\lambda)^*, \\ \operatorname{span}\{u_{12}, \dots, u_{l2}\} &= \ker G_{22}(\lambda) = \ker G_{22}(\lambda) + G_{22}(\lambda)^*. \end{aligned}$$

Let $p_1 = m_1 - k$, $p_2 = m_2 - l$ and extend $\{u_{11}, \dots, u_{k1}\}$ by $\{v_{11}, \dots, v_{p_1 1}\}$ and $\{u_{12}, \dots, u_{l2}\}$ by $\{v_{12}, \dots, v_{p_2 2}\}$ to orthonormal bases of $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$, resp. Define the orthogonal matrices

$$\begin{aligned} U_1 &= \begin{bmatrix} v_{11} & \dots & v_{p_1 1} & u_{11} & \dots & u_{k1} \end{bmatrix}, \\ U_2 &= \begin{bmatrix} v_{12} & \dots & v_{p_2 2} & u_{12} & \dots & u_{l2} \end{bmatrix}. \end{aligned}$$

By Proposition 3.3, we have that $G_{11}(s) u_{j1} = 0$ and $G_{12}(s)^\top u_{j1}$ is constant for all $j \in \{1, \dots, k\}$. Analogously we find that $G_{22}(s) u_{j2} = 0$ and $G_{12}(s) u_{j2}$ is constant for all $j \in \{1, \dots, l\}$. Consequently, $U^\top G(s) U$ is of the form

$$U^\top G(s) U = \begin{bmatrix} H_{11}(s) & 0 & H_{13}(s) & L_{14} \\ 0 & 0 & L_{23} & L_{24} \\ H_{31}(s) & L_{32} & H_{33}(s) & 0 \\ L_{41} & L_{42} & 0 & 0 \end{bmatrix}$$

for some

$$\begin{aligned} &H_{11}(s) \in \mathbb{R}(s)^{p_1,p_1}, && H_{13}(s) \in \mathbb{R}(s)^{p_1,p_2}, \\ &H_{31}(s) \in \mathbb{R}(s)^{p_2,p_1}, && H_{33}(s) \in \mathbb{R}(s)^{p_2,p_2}, \\ &L_{14} \in \mathbb{R}^{p_1,m_2-p_2}, && L_{23} \in \mathbb{R}^{m_1-p_1,p_2}, \\ &L_{24} \in \mathbb{R}^{m_1-p_1,m_2-p_2}, && L_{32} \in \mathbb{R}^{p_2,m_1-p_1}, \\ &L_{41} \in \mathbb{R}^{m_2-p_2,p_1}, && L_{42} \in \mathbb{R}^{m_2-p_2,m_1-p_1}. \end{aligned}$$

Reciprocity of $G(s)$ means that $G(s)S = SG(s)^\top$. Since $U = \operatorname{diag}(U_1, U_2)$, we obtain that $US = SU$ and $U^\top S = SU^\top$. Therefore,

$$U^\top G(s) US = U^\top G(s) SU = U^\top SG(s)^\top U = SU^\top G(s)^\top U,$$

which means that $U^\top G(s) U$ is reciprocal with signature (m_1, m_2). This implies

$$\begin{aligned} H_{31}(s) &= -H_{13}(s)^\top, & L_{32} &= -L_{23}^\top, \\ L_{41} &= -L_{14}^\top, & L_{42} &= -L_{24}^\top, \end{aligned}$$

and that $H_{11}(s), H_{33}(s)$ are symmetric.

Step 2: We show that $H_{11}(\lambda) + H_{11}(\lambda)^* > 0$ and $H_{33}(\lambda) + H_{33}(\lambda)^* > 0$ for all $\lambda \in \mathbb{C}_+$. Since $H_{11}(s)$ and $H_{33}(s)$ are positive real, positive semi-definiteness of the aforementioned matrices is clear. Suppose there exists $u_1 \in \mathbb{C}^{p_1} \setminus \{0\}$ such that $u_1^\top (H_{11}(\lambda) + H_{11}(\lambda)^*) u_1 = 0$. Then $u := U_1 (u_1^\top, 0,)^\top$ satisfies $u^\top (G_{11}(\lambda) + G_{11}(\lambda)^*) u = 0$, hence

$$u \in \ker G_{11}(\lambda) = \operatorname{span}\{u_{11}, \dots, u_{k1}\} = \operatorname{im} U_1 \begin{bmatrix} 0 \\ I_k \end{bmatrix},$$

which implies that $u_1 = 0$, a contradiction. The proof for $H_{33}(\lambda) + H_{33}(\lambda)^*$ is analogous.

Step 3: It remains to prove (9). The inclusion "$\supseteq$" is obvious. To show "$\subseteq$", let $u \in \ker G(1)$. Then we have $u^\top (G(1) + G(1)^\top) u = 0$ and, by the semi-definiteness of $G(1)$, we find $(G(1) + G(1)^\top) u = 0$. Partitioning

$$(u_1^\top, u_2^\top, u_3^\top, u_4^\top)^\top = U^\top u,$$

we obtain

$$\begin{aligned} &u^\top (G(1) + G(1)^\top) u \\ =\,& u_1^\top (H_{11}(1) + H_{11}(1)^\top) u_1 + u_3^\top (H_{33}(1) + H_{33}(1)^\top) u_3, \end{aligned}$$

and thus $u_1 = 0$ and $u_3 = 0$ by the findings of Step 2. The equation $G(1) u = 0$ therefore leads to

$$u_2 \in \ker \begin{bmatrix} L_{23}^\top \\ L_{24}^\top \end{bmatrix}, \quad u_4 \in \ker \begin{bmatrix} L_{14} \\ L_{24} \end{bmatrix}.$$

■

Remark 3.5: With the orthogonal block-diagonal matrix U as in Theorem 3.4, the residual matrices from (6) have, for all $j \in \{1, \dots, k\}$, the special form

$$U^\top M_1 U = \begin{bmatrix} \tilde{M}_{11,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{M}_{33,1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$U^\top R_0 U = \begin{bmatrix} \tilde{R}_{11,0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{R}_{33,0} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$U^\top M_0 U = \begin{bmatrix} \tilde{M}_{11,0} & 0 & \tilde{M}_{13,0} & L_{14} \\ 0 & 0 & L_{23} & L_{24} \\ -\tilde{M}_{13,0}^\top & -L_{23}^\top & \tilde{M}_{33,0} & 0 \\ -L_{14}^\top & -L_{24}^\top & 0 & 0 \end{bmatrix},$$

$$U^\top R_j U = \begin{bmatrix} \tilde{R}_{11,j} & 0 & i\tilde{R}_{12,j} & 0 \\ 0 & 0 & 0 & 0 \\ -i\tilde{R}_{12,j}^\top & 0 & \tilde{R}_{33,j} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

IV. Transfer functions of electrical circuits

Electrical circuits with linear time-invariant resistances, capacitances and inductances can be modeled by linear differential-algebraic systems of the form

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned} \tag{10}$$

where $E, A \in \mathbb{R}^{n,n}$, $B, C^\top \in \mathbb{R}^{n,m}$.

The functions $u, y : \mathbb{R} \to \mathbb{R}^m$ are called *input* and *output* of the system, resp. If the matrix pencil $sE - A \in \mathbb{R}[s]$ is *regular* (that is, $sE - A \in \mathbf{Gl}_n(\mathbb{R}(s))$), then the frequency domain behavior is described by the *transfer function*, which is given by

$$G(s) = C(sE - A)^{-1}B \in \mathbb{R}(s)^{m,m}.$$

We assume that the input is formed by the currents of current sources and the voltages of voltage sources, and the output is given by the currents of voltage sources together with the voltages of voltage sources. Modified nodal analysis (MNA) [9] leads to

$$\begin{aligned} sE - A &= \begin{bmatrix} s\mathrm{A}_\mathcal{C}\mathcal{C}\mathrm{A}_\mathcal{C}^\top + \mathrm{A}_\mathcal{R}\mathcal{G}\mathrm{A}_\mathcal{R}^\top & \mathrm{A}_\mathcal{L} & \mathrm{A}_\mathcal{V} \\ -\mathrm{A}_\mathcal{L}^\top & s\mathcal{L} & 0 \\ -\mathrm{A}_\mathcal{V}^\top & 0 & 0 \end{bmatrix}, \\ B^\top = C &= \begin{bmatrix} -\mathrm{A}_\mathcal{I}^\top & 0 & 0 \\ 0 & 0 & -I_{n_\mathcal{V}} \end{bmatrix}, \end{aligned} \tag{11}$$

$$x = (\eta^\top, i_\mathcal{L}^\top, i_\mathcal{V}^\top)^\top, \quad u = (i_\mathcal{I}^\top, v_\mathcal{V}^\top)^\top, \quad y = (-v_\mathcal{I}^\top, -i_\mathcal{V}^\top)^\top, \tag{12}$$

where

$$\begin{aligned} &\mathcal{C} \in \mathbb{R}^{n_\mathcal{C}, n_\mathcal{C}}, \mathcal{G} \in \mathbb{R}^{n_\mathcal{G}, n_\mathcal{G}}, \mathcal{L} \in \mathbb{R}^{n_\mathcal{L}, n_\mathcal{L}}, \mathrm{A}_\mathcal{C} \in \mathbb{R}^{n_e, n_\mathcal{C}}, \\ &\mathrm{A}_\mathcal{R} \in \mathbb{R}^{n_e, n_\mathcal{G}}, \mathrm{A}_\mathcal{L} \in \mathbb{R}^{n_e, n_\mathcal{L}}, \mathrm{A}_\mathcal{V} \in \mathbb{R}^{n_e, m_\mathcal{V}}, \mathrm{A}_\mathcal{I} \in \mathbb{R}^{n_e, m_\mathcal{I}}, \\ &n = n_e + n_\mathcal{L} + n_\mathcal{V}, \quad m = m_\mathcal{I} + m_\mathcal{V}. \end{aligned} \tag{13}$$

Here $\mathrm{A}_\mathcal{C}$, $\mathrm{A}_\mathcal{R}$, $\mathrm{A}_\mathcal{L}$, $\mathrm{A}_\mathcal{V}$ and $\mathrm{A}_\mathcal{I}$ denote the element-related incidence matrices, $\mathcal{C}$, $\mathcal{G}$ and $\mathcal{L}$ are the matrices expressing the consecutive relations of capacitances, resistances and inductances, $\eta(t)$ is the vector of node potentials, $i_\mathcal{L}(t)$, $i_\mathcal{V}(t)$, $i_\mathcal{I}(t)$ are the vectors of currents through inductances, voltage and current sources, and $v_\mathcal{V}(t)$, $v_\mathcal{I}(t)$ are the voltages of voltage and current sources.

We assume that the given circuit is connected and passive, which is guaranteed by the assumptions

(A1) $\mathrm{rk}[\mathrm{A}_\mathcal{C}, \mathrm{A}_\mathcal{R}, \mathrm{A}_\mathcal{L}, \mathrm{A}_\mathcal{V}, \mathrm{A}_\mathcal{I}] = n_e$,

(A2) $\mathcal{C} = \mathcal{C}^\top > 0, \mathcal{G} = \mathcal{G}^\top > 0, \mathcal{L} = \mathcal{L}^\top > 0.$

Note that regularity of $sE - A$ is equivalent to

$$\ker \mathrm{A}_\mathcal{V} = \{0\}, \quad \text{and} \tag{14}$$

$$\ker[\mathrm{A}_\mathcal{C}, \mathrm{A}_\mathcal{R}, \mathrm{A}_\mathcal{L}, \mathrm{A}_\mathcal{V}]^\top = \{0\}. \tag{15}$$

This is equivalent to the absence of loops of voltage sources and cutsets of current sources (basically, this means that the circuit does not contain any short circuits) [10], [11]. The rational function $G(s) = C(sE - A)^{-1}B$ is moreover positive real and reciprocal with signature $(m_\mathcal{I}, m_\mathcal{V})$ [5]. Consequently, the overall transfer function has the form

$$G(s) = \begin{bmatrix} G_{\mathcal{I}\mathcal{I}}(s) & G_{\mathcal{I}\mathcal{V}}(s) \\ -G_{\mathcal{I}\mathcal{V}}(s)^\top & G_{\mathcal{V}\mathcal{V}}(s) \end{bmatrix}, \tag{16}$$

where

$$\begin{aligned} &G_{\mathcal{I}\mathcal{I}}(s) = G_{\mathcal{I}\mathcal{I}}(s)^\top \in \mathbb{R}(s)^{m_\mathcal{I}, m_\mathcal{I}}, \\ &G_{\mathcal{V}\mathcal{V}}(s) = G_{\mathcal{V}\mathcal{V}}(s)^\top \in \mathbb{R}(s)^{m_\mathcal{V}, m_\mathcal{V}}, \\ &G_{\mathcal{I}\mathcal{V}}(s) \in \mathbb{R}(s)^{m_\mathcal{I}, m_\mathcal{V}}. \end{aligned}$$

Next we determine some expressions for $G_{\mathcal{I}\mathcal{I}}(s)$, $G_{\mathcal{I}\mathcal{V}}(s)$, $G_{\mathcal{V}\mathcal{V}}(s)$. Define the rational function

$$H_{\mathcal{CRL}}(s) = s\mathrm{A}_\mathcal{C}\mathcal{C}\mathrm{A}_\mathcal{C}^\top + \mathrm{A}_\mathcal{R}\mathcal{G}\mathrm{A}_\mathcal{R}^\top + \tfrac{1}{s}\mathrm{A}_\mathcal{L}\mathcal{L}^{-1}\mathrm{A}_\mathcal{L}^\top \in \mathbb{R}(s)^{n_e, n_e}.$$

It follows by simple arithmetics that

$$G(s) = \begin{bmatrix} \mathrm{A}_\mathcal{I}^\top & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} H_{\mathcal{CRL}}(s) & \mathrm{A}_\mathcal{V} \\ -\mathrm{A}_\mathcal{V}^\top & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{A}_\mathcal{I} & 0 \\ 0 & I \end{bmatrix}. \tag{17}$$

Next we determine

$$\underbrace{\begin{bmatrix} X_{11}(s) & X_{12}(s) \\ X_{21}(s) & X_{22}(s) \end{bmatrix}}_{X(s)} = \begin{bmatrix} H_{\mathcal{CRL}}(s) & \mathrm{A}_\mathcal{V} \\ -\mathrm{A}_\mathcal{V}^\top & 0 \end{bmatrix}^{-1}. \tag{18}$$

Let $Z_\mathcal{V}$ be a matrix with full column rank and $\mathrm{im}\, Z_\mathcal{V} = \ker \mathrm{A}_\mathcal{V}^\top$. Then the equation

$$\begin{aligned} I &= H_{\mathcal{CRL}}(s)X_{11}(s) + \mathrm{A}_\mathcal{V}X_{21}(s), \\ 0 &= -\mathrm{A}_\mathcal{V}^\top X_{11}(s) \end{aligned} \tag{19}$$

leads to $X_{11}(s) = Z_\mathcal{V}Y_{11}(s)$ for some real rational matrix $Y_{11}(s)$. A multiplication of the first equation in (19) from the left with $Z_\mathcal{V}^\top$ gives rise to

$$Z_\mathcal{V}^\top = Z_\mathcal{V}^\top H_{\mathcal{CRL}}(s)Z_\mathcal{V}Y_{11}(s).$$

Condition (15) together with $\ker Z_\mathcal{V} = \{0\}$ implies

$$\ker[\mathrm{A}_\mathcal{C}, \mathrm{A}_\mathcal{R}, \mathrm{A}_\mathcal{L}]^\top Z_\mathcal{V} = \{0\}. \tag{20}$$

The positive definiteness of $\mathcal{C}$, $\mathcal{G}$ and $\mathcal{L}$ hence gives rise to the positive definiteness of $Z_\mathcal{V}^\top H_{\mathcal{CRL}}(1)Z_\mathcal{V}$. Using that $H_{\mathcal{CRL}}(s)$ is positive real, we can apply Corollary 2.3 to see that $Z_\mathcal{V}^\top H_{\mathcal{CRL}}(\lambda)Z_\mathcal{V}$ is invertible for all $\lambda \in \mathbb{C}_+$. Therefore, we obtain

$$X_{11}(s) = Z_\mathcal{V}Y_{11}(s) = Z_\mathcal{V}(Z_\mathcal{V}^\top H_{\mathcal{CRL}}(s)Z_\mathcal{V})^{-1}Z_\mathcal{V}^\top. \tag{21}$$

Inserting this expression into (19), and observing that $\mathrm{A}_\mathcal{V}^\top\mathrm{A}_\mathcal{V}$ is invertible by (14), a multiplication of the resulting expression with $(\mathrm{A}_\mathcal{V}^\top\mathrm{A}_\mathcal{V})^{-1}\mathrm{A}_\mathcal{V}^\top$ gives rise to

$$X_{21}(s) = (\mathrm{A}_\mathcal{V}^\top\mathrm{A}_\mathcal{V})^{-1}\mathrm{A}_\mathcal{V}^\top(I - H_{\mathcal{CRL}}(s)Z_\mathcal{V}(Z_\mathcal{V}^\top H_{\mathcal{CRL}}(s)Z_\mathcal{V})^{-1}Z_\mathcal{V}^\top). \tag{22}$$

Since its inverse is reciprocal with signature $(n_e, m_\mathcal{V})$, the rational matrix $X(s)$ has this property, too. Thus we have

$$X_{12}(s) = (Z_{\mathcal{V}}(Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s)Z_{\mathcal{V}})^{-1}Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s) - I)\mathrm{A}_{\mathcal{V}}(\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}. \tag{23}$$

The equation

$$0 = H_{\mathcal{CRL}}(s)X_{12}(s) + \mathrm{A}_{\mathcal{V}}X_{22}(s), \tag{24}$$

then leads to

$$\begin{aligned} X_{22}(s) = &(\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}\mathrm{A}_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s) \\ &\cdot (I - Z_{\mathcal{V}}(Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s)Z_{\mathcal{V}})^{-1}Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s)) \\ &\cdot \mathrm{A}_{\mathcal{V}}(\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}. \end{aligned} \tag{25}$$

Note that, for any matrix $Z_{\mathcal{V}}$ with full column rank such that $\operatorname{im} Z_{\mathcal{V}} = \ker \mathrm{A}_{\mathcal{V}}^\top$, the rational matrices $X_{11}(s)$, $X_{12}(s)$, $X_{21}(s)$, $X_{22}(s)$ defined in (21)–(25) solve (18). Consequently, the transfer function in (16) is given by

$$\begin{aligned} G_{\mathcal{I}\mathcal{I}}(s) &= \mathrm{A}_{\mathcal{I}}^\top Z_{\mathcal{V}}(Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s)Z_{\mathcal{V}})^{-1}Z_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{I}}, \\ G_{\mathcal{I}\mathcal{V}}(s) &= \mathrm{A}_{\mathcal{I}}^\top (Z_{\mathcal{V}}(Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s)Z_{\mathcal{V}})^{-1}Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s) - I) \\ &\quad\cdot \mathrm{A}_{\mathcal{V}}(\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}, \\ G_{\mathcal{V}\mathcal{V}}(s) &= (\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}\mathrm{A}_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s) \\ &\quad\cdot (I - Z_{\mathcal{V}}(Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s)Z_{\mathcal{V}})^{-1}Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s)) \\ &\quad\cdot \mathrm{A}_{\mathcal{V}}(\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}, \end{aligned} \tag{26}$$

parametrized by $Z_{\mathcal{V}}$. By the findings in Corollary 3.2, the function $G_{\mathcal{I}\mathcal{V}}(s)$ is proper and has not pole at zero.

Now, we determine the kernels of $G_{\mathcal{I}\mathcal{I}}(s)$ and $G_{\mathcal{V}\mathcal{V}}(s)$.

Proposition 4.1: Let $[E,A,B,C]$ as in (11) be given such that **(A1)**,**(A2)**, (14) and (15) hold. Then, for all $\lambda \in \mathbb{C}_+$, the transfer matrices as in+(26) satisfy

$$\ker G_{\mathcal{I}\mathcal{I}}(\lambda) = \{\ x \in \mathbb{R}^{n_{\mathcal{I}}}\ \mid \mathrm{A}_{\mathcal{I}}x \in \operatorname{im} \mathrm{A}_{\mathcal{V}}\ \}, \tag{27}$$

$$\ker G_{\mathcal{V}\mathcal{V}}(\lambda) = \mathrm{A}_{\mathcal{V}}^\top \cdot \ker[\mathrm{A}_{\mathcal{C}}\,,\,\mathrm{A}_{\mathcal{R}}\,,\,\mathrm{A}_{\mathcal{L}}]^\top. \tag{28}$$

Proof: Since

$$\ker Z_{\mathcal{V}}^\top = (\operatorname{im} Z_{\mathcal{V}})^\perp = (\ker \mathrm{A}_{\mathcal{V}}^\top)^\perp = \operatorname{im} \mathrm{A}_{\mathcal{V}},$$

we find

$$\{\ x \in \mathbb{R}^{n_{\mathcal{I}}}\ \mid \mathrm{A}_{\mathcal{I}}x \in \operatorname{im} \mathrm{A}_{\mathcal{V}}\ \} = \ker Z_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{I}}.$$

Then relation (27) hence follows from (20) and the fact that $H_{\mathcal{CRL}}(\lambda) + H_{\mathcal{CRL}}(\lambda)^*$ is positive semi-definite with

$$\ker H_{\mathcal{CRL}}(\lambda) + H_{\mathcal{CRL}}(\lambda)^* = \ker[\mathrm{A}_{\mathcal{C}}\,,\mathrm{A}_{\mathcal{R}}\,,\,\mathrm{A}_{\mathcal{L}}]^\top$$

for all $\lambda \in \mathbb{C}_+$.
Since $G_{\mathcal{V}\mathcal{V}}(s) \in \mathbb{R}(s)^{n_{\mathcal{V}},n_{\mathcal{V}}}$ is symmetric and positive real, we can apply Proposition 2.2 to see that, for proving the inclusion "$\subseteq$" in (28), it suffices to show that $u \in \mathbb{R}^{m_{\mathcal{V}}}$ with $u^\top G_{\mathcal{V}\mathcal{V}}(1)u = 0$ implies $u \in \mathrm{A}_{\mathcal{V}}^\top \cdot \ker[\mathrm{A}_{\mathcal{C}}\,,\mathrm{A}_{\mathcal{R}}\,,\mathrm{A}_{\mathcal{L}}]^\top$: Since

$$P = I - Z_{\mathcal{V}}(Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(1)Z_{\mathcal{V}})^{-1}Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(1) \in \mathbb{R}^{n_e,n_e}$$

is a projector (onto $\ker Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(1)$ and along $\operatorname{im} Z_{\mathcal{V}}$) with the property that

$$\begin{aligned} H_{\mathcal{CRL}}(1)P &= H_{\mathcal{CRL}}(1) - H_{\mathcal{CRL}}(1)Z_{\mathcal{V}}(Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(1)Z_{\mathcal{V}})^{-1} \\ &\quad\cdot Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(1) \\ &= P^\top H_{\mathcal{CRL}}(1), \end{aligned}$$

it follows that

$$H_{\mathcal{CRL}}(1)P = P^\top H_{\mathcal{CRL}}(1)P.$$

In particular, $H_{\mathcal{CRL}}(1)P$ is positive semi-definite. As a consequence, $u^\top G_{\mathcal{V}\mathcal{V}}(1)u = 0$ implies that

$$P\mathrm{A}_{\mathcal{V}}(\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}u \in \ker H_{\mathcal{CRL}}(1) = \ker[\mathrm{A}_{\mathcal{C}}\,,\,\mathrm{A}_{\mathcal{R}}\,,\mathrm{A}_{\mathcal{L}}]^\top.$$

Since P is a projector, we have

$$\begin{aligned} &\mathrm{A}_{\mathcal{V}}(\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}u \\ \in\ &\underbrace{\ker P}_{=\operatorname{im} Z_{\mathcal{V}} = \ker \mathrm{A}_{\mathcal{V}}^\top} + (\underbrace{\operatorname{im} P}_{=\ker Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(1)} \cap \ker[\mathrm{A}_{\mathcal{C}}\,,\,\mathrm{A}_{\mathcal{R}}\,,\mathrm{A}_{\mathcal{L}}]^\top) \\ =\ &\ker \mathrm{A}_{\mathcal{V}}^\top + (\underbrace{\ker Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(1)}_{=\ker H_{\mathcal{CRL}}(1) = \ker[\mathrm{A}_{\mathcal{C}},\mathrm{A}_{\mathcal{R}},\mathrm{A}_{\mathcal{L}}]^\top} \cap \ker[\mathrm{A}_{\mathcal{C}}\,,\,\mathrm{A}_{\mathcal{R}}\,,\mathrm{A}_{\mathcal{L}}]^\top) \\ =\ &\ker \mathrm{A}_{\mathcal{V}}^\top + \ker[\mathrm{A}_{\mathcal{C}}\,,\,\mathrm{A}_{\mathcal{R}}\,,\mathrm{A}_{\mathcal{L}}]^\top. \end{aligned}$$

A multiplication from the left with $\mathrm{A}_{\mathcal{V}}^\top$ leads to

$$u \in \mathrm{A}_{\mathcal{V}}^\top \cdot \ker[\mathrm{A}_{\mathcal{C}}\,,\,\mathrm{A}_{\mathcal{R}}\,,\mathrm{A}_{\mathcal{L}}]^\top. \tag{29}$$

We show "$\supseteq$" in (28): If (29) holds, then $u = \mathrm{A}_{\mathcal{V}}^\top z$ for some $z \in \ker[\mathrm{A}_{\mathcal{C}}\,,\,\mathrm{A}_{\mathcal{R}}\,,\mathrm{A}_{\mathcal{L}}]^\top$. Since $\mathrm{A}_{\mathcal{V}}(\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}\mathrm{A}_{\mathcal{V}}^\top$, $Z_{\mathcal{V}}(Z_{\mathcal{V}}^\top Z_{\mathcal{V}})^{-1}Z_{\mathcal{V}}^\top$ are both orthogonal projectors, and their ranges orthogonally sum up to $\mathbb{R}^{n_e}$, we have

$$\mathrm{A}_{\mathcal{V}}(\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}\mathrm{A}_{\mathcal{V}}^\top = I - Z_{\mathcal{V}}(Z_{\mathcal{V}}^\top Z_{\mathcal{V}})^{-1}Z_{\mathcal{V}}^\top. \tag{30}$$

Thus we obtain

$$\begin{aligned} &H_{\mathcal{CRL}}(1)P\mathrm{A}_{\mathcal{V}}(\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}u \\ &= H_{\mathcal{CRL}}(1)P\mathrm{A}_{\mathcal{V}}(\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}\mathrm{A}_{\mathcal{V}}^\top z \\ &= H_{\mathcal{CRL}}(1)\underbrace{Pz}_{=z} - H_{\mathcal{CRL}}(1)\underbrace{PZ_{\mathcal{V}}(Z_{\mathcal{V}}^\top Z_{\mathcal{V}})^{-1}Z_{\mathcal{V}}^\top}_{=0}z \\ &= H_{\mathcal{CRL}}(1)z = 0, \end{aligned}$$

since $z \in \ker[\mathrm{A}_{\mathcal{C}}\,,\,\mathrm{A}_{\mathcal{R}}\,,\mathrm{A}_{\mathcal{L}}]^\top = \ker H_{\mathcal{CRL}}(1)$. Thus we have

$$G_{\mathcal{V}\mathcal{V}}(1)u = (\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}\mathrm{A}_{\mathcal{V}}^\top H_{\mathcal{CRL}}(1)P\mathrm{A}_{\mathcal{V}}(\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}u = 0.$$

Corollary 2.3 then implies $G_{\mathcal{V}\mathcal{V}}(\lambda)u = 0$ for all $\lambda \in \mathbb{C}_+$. ■

Proposition 3.3 shows that $G_{\mathcal{I}\mathcal{V}}(s)^\top$ and $G_{\mathcal{I}\mathcal{V}}(s)$ are constant on $\ker G_{\mathcal{I}\mathcal{I}}(\lambda)$ and $\ker G_{\mathcal{V}\mathcal{V}}(\lambda)$, resp. To verify this fact, let $u_1 \in \ker Z_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{I}}$, $u_2 \in \mathrm{A}_{\mathcal{V}}^\top \cdot \ker[\mathrm{A}_{\mathcal{C}}\,,\,\mathrm{A}_{\mathcal{R}}\,,\mathrm{A}_{\mathcal{L}}]^\top$. Then

$$\begin{aligned} G_{\mathcal{I}\mathcal{V}}(s)^\top u_1 &= (\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}\mathrm{A}_{\mathcal{V}}^\top \\ &\quad\cdot (H_{\mathcal{CRL}}(s)Z_{\mathcal{V}}(Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s)Z_{\mathcal{V}})^{-1}Z_{\mathcal{V}}^\top - I)\mathrm{A}_{\mathcal{I}}u_1 \\ &= -(\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{I}}u_1. \end{aligned}$$

Further, let $z \in \ker[\mathrm{A}_{\mathcal{C}}\,,\,\mathrm{A}_{\mathcal{R}}\,,\mathrm{A}_{\mathcal{L}}]^\top$ be such that $u_2 = \mathrm{A}_{\mathcal{V}}^\top z$.

Then, again using (30), we obtain

$$\begin{aligned}
&G_{\mathscr{I}\mathcal{V}}(s)u_2\\
&= \mathrm{A}_{\mathscr{I}}^\top(Z_{\mathcal{V}}(Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s)Z_{\mathcal{V}})^{-1}Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s)-I)\\
&\quad\cdot \mathrm{A}_{\mathcal{V}}(\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}\mathrm{A}_{\mathcal{V}}^\top z\\
&= -\mathrm{A}_{\mathscr{I}}^\top \mathrm{A}_{\mathcal{V}}(\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}\mathrm{A}_{\mathcal{V}}^\top z\\
&\quad+\mathrm{A}_{\mathscr{I}}^\top Z_{\mathcal{V}}(Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s)Z_{\mathcal{V}})^{-1}Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s)\\
&\quad\cdot(I-Z_{\mathcal{V}}(Z_{\mathcal{V}}^\top Z_{\mathcal{V}})^{-1}Z_{\mathcal{V}}^\top)z\\
&= -\mathrm{A}_{\mathscr{I}}^\top \mathrm{A}_{\mathcal{V}}(\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}\mathrm{A}_{\mathcal{V}}^\top z\\
&\quad+\mathrm{A}_{\mathscr{I}}^\top Z_{\mathcal{V}}(Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s)Z_{\mathcal{V}})^{-1}Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s)z\\
&\quad-\mathrm{A}_{\mathscr{I}}^\top Z_{\mathcal{V}}(Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s)Z_{\mathcal{V}})^{-1}Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s)Z_{\mathcal{V}}(Z_{\mathcal{V}}^\top Z_{\mathcal{V}})^{-1}Z_{\mathcal{V}}^\top z\\
&\qquad= -\mathrm{A}_{\mathscr{I}}^\top \mathrm{A}_{\mathcal{V}}(\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}\mathrm{A}_{\mathcal{V}}^\top z\\
&\qquad\quad+\mathrm{A}_{\mathscr{I}}^\top Z_{\mathcal{V}}(Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s)Z_{\mathcal{V}})^{-1}Z_{\mathcal{V}}^\top \underbrace{H_{\mathcal{CRL}}(s)z}_{=0}\\
&\qquad\quad-\mathrm{A}_{\mathscr{I}}^\top Z_{\mathcal{V}}(Z_{\mathcal{V}}^\top Z_{\mathcal{V}})^{-1}Z_{\mathcal{V}}^\top z = -\mathrm{A}_{\mathscr{I}}^\top z.
\end{aligned}$$

From the equation

$$\begin{pmatrix}0\\0\\0\\u_2\end{pmatrix}=\begin{bmatrix}\mathrm{A}_{\mathcal{C}}^\top\\ \mathrm{A}_{\mathcal{R}}^\top\\ \mathrm{A}_{\mathcal{L}}^\top\\ \mathrm{A}_{\mathcal{V}}^\top\end{bmatrix}z,$$

we obtain, by multiplying from the left with $[\mathrm{A}_{\mathcal{C}}, \mathrm{A}_{\mathcal{R}}, \mathrm{A}_{\mathcal{L}}, \mathrm{A}_{\mathcal{V}}]$ and using (15), that

$$z=(\mathrm{A}_{\mathcal{C}}\mathrm{A}_{\mathcal{C}}^\top+\mathrm{A}_{\mathcal{R}}\mathrm{A}_{\mathcal{R}}^\top+\mathrm{A}_{\mathcal{L}}\mathrm{A}_{\mathcal{L}}^\top+\mathrm{A}_{\mathcal{V}}\mathrm{A}_{\mathcal{V}}^\top)^{-1}\mathrm{A}_{\mathcal{V}}u_2.$$

This gives rise to

$$G_{\mathscr{I}\mathcal{V}}(s)u_2=-\mathrm{A}_{\mathscr{I}}^\top(\mathrm{A}_{\mathcal{C}}\mathrm{A}_{\mathcal{C}}^\top+\mathrm{A}_{\mathcal{R}}\mathrm{A}_{\mathcal{R}}^\top+\mathrm{A}_{\mathcal{L}}\mathrm{A}_{\mathcal{L}}^\top+\mathrm{A}_{\mathcal{V}}\mathrm{A}_{\mathcal{V}}^\top)^{-1}\mathrm{A}_{\mathcal{V}}u_2.$$

Using the above computations, we can determine the form (8). Consider a matrix $Z_{\mathcal{CRL}}$ with $\operatorname{im} Z_{\mathcal{CRL}} = \ker[\mathrm{A}_{\mathcal{C}}, \mathrm{A}_{\mathcal{R}}, \mathrm{A}_{\mathcal{L}}]^\top$. Further, let $\overline{Z}_{\mathscr{I}-\mathcal{V}}$, $\overline{Z}'_{\mathscr{I}-\mathcal{V}}$, $\overline{Z}_{\mathcal{V}-\mathcal{CRL}}$, $\overline{Z}'_{\mathcal{V}-\mathcal{CRL}}$ be matrices with orthonormal columns and

$$\begin{aligned}
\operatorname{im}\overline{Z}_{\mathscr{I}-\mathcal{V}} &= \ker Z_{\mathcal{V}}^\top \mathrm{A}_{\mathscr{I}}, &\quad \operatorname{im}\overline{Z}'_{\mathscr{I}-\mathcal{V}} &= \operatorname{im}\mathrm{A}_{\mathscr{I}}^\top Z_{\mathcal{V}},\\
\operatorname{im}\overline{Z}_{\mathcal{V}-\mathcal{CRL}} &= \ker Z_{\mathcal{CRL}}^\top \mathrm{A}_{\mathcal{V}}, &\quad \operatorname{im}\overline{Z}'_{\mathcal{V}-\mathcal{CRL}} &= \operatorname{im}\mathrm{A}_{\mathcal{V}}^\top Z_{\mathcal{CRL}}.
\end{aligned}$$

Then the matrix

$$U=\begin{bmatrix}\overline{Z}'_{\mathscr{I}-\mathcal{V}} & \overline{Z}_{\mathscr{I}-\mathcal{V}} & 0 & 0\\ 0 & 0 & \overline{Z}_{\mathcal{V}-\mathcal{CRL}} & \overline{Z}'_{\mathcal{V}-\mathcal{CRL}}\end{bmatrix}$$

is orthogonal. The previous calculations further give rise to $U^\top G(s)U$ being in the form (8) with

$$\begin{aligned}
L_{14} &= -\overline{Z}'^\top_{\mathscr{I}-\mathcal{V}}\mathrm{A}_{\mathscr{I}}^\top(\mathrm{A}_{\mathcal{C}}\mathrm{A}_{\mathcal{C}}^\top+\mathrm{A}_{\mathcal{R}}\mathrm{A}_{\mathcal{R}}^\top+\mathrm{A}_{\mathcal{L}}\mathrm{A}_{\mathcal{L}}^\top+\mathrm{A}_{\mathcal{V}}\mathrm{A}_{\mathcal{V}}^\top)^{-1}\\
&\quad\cdot \mathrm{A}_{\mathcal{V}}\overline{Z}'_{\mathcal{V}-\mathcal{CRL}},\\
L_{23} &= -\overline{Z}^\top_{\mathscr{I}-\mathcal{V}}\mathrm{A}_{\mathscr{I}}^\top \mathrm{A}_{\mathcal{V}}(\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}\overline{Z}_{\mathcal{V}-\mathcal{CRL}},\\
L_{24} &= -\overline{Z}^\top_{\mathscr{I}-\mathcal{V}}\mathrm{A}_{\mathscr{I}}^\top(\mathrm{A}_{\mathcal{C}}\mathrm{A}_{\mathcal{C}}^\top+\mathrm{A}_{\mathcal{R}}\mathrm{A}_{\mathcal{R}}^\top+\mathrm{A}_{\mathcal{L}}\mathrm{A}_{\mathcal{L}}^\top+\mathrm{A}_{\mathcal{V}}\mathrm{A}_{\mathcal{V}}^\top)^{-1}\\
&\quad\cdot \mathrm{A}_{\mathcal{V}}\overline{Z}'_{\mathcal{V}-\mathcal{CRL}},
\end{aligned}$$

$$\begin{aligned}
H_{11}(s) &= \overline{Z}'^\top_{\mathscr{I}-\mathcal{V}}\mathrm{A}_{\mathscr{I}}^\top Z_{\mathcal{V}}(Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s)Z_{\mathcal{V}})^{-1}Z_{\mathcal{V}}^\top \mathrm{A}_{\mathscr{I}}\overline{Z}'_{\mathscr{I}-\mathcal{V}},\\
H_{13}(s) &= \overline{Z}'^\top_{\mathscr{I}-\mathcal{V}}\mathrm{A}_{\mathscr{I}}^\top(Z_{\mathcal{V}}(Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s)Z_{\mathcal{V}})^{-1}Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s)-I)\\
&\quad\cdot \mathrm{A}_{\mathcal{V}}(\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}\overline{Z}_{\mathcal{V}-\mathcal{CRL}},\\
H_{33}(s) &= \overline{Z}^\top_{\mathcal{V}-\mathcal{CRL}}(\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}\mathrm{A}_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s)\\
&\quad\cdot(I-Z_{\mathcal{V}}(Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s)Z_{\mathcal{V}})^{-1}Z_{\mathcal{V}}^\top H_{\mathcal{CRL}}(s))\\
&\quad\cdot \mathrm{A}_{\mathcal{V}}(\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}\overline{Z}_{\mathcal{V}-\mathcal{CRL}}.
\end{aligned}$$

We finally give an expression for the kernels of the circuit transfer function.

Proposition 4.2: Let $[E,A,B,C]$ as in (11) be given such that **(A1)**, **(A2)**, (14) and (15) hold. Then the rational function $G(s)=C(sE-A)^{-1}B\in\mathbb{R}(s)^{m,m}$ satisfies, for all $\lambda\in\mathbb{C}_+$,

$$\ker G(\lambda)=\ker \mathrm{A}_{\mathscr{I}}\times\left(\mathrm{A}_{\mathcal{V}}^\top\cdot\ker[\mathrm{A}_{\mathcal{C}}, \mathrm{A}_{\mathcal{R}}, \mathrm{A}_{\mathcal{L}}, \mathrm{A}_{\mathscr{I}}]^\top\right).$$

Proof: Using (9) and the above expressions for L_{14}, L_{23} and L_{24}, it suffices to prove that

(i)
$$\left\{\, x\in\ker Z_{\mathcal{V}}^\top \mathrm{A}_{\mathscr{I}} \;\middle|\; (\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathscr{I}}x=0 \,\right\} = \ker \mathrm{A}_{\mathscr{I}},$$

(ii)
$$\begin{aligned}
&\Big\{x\in\ker[\mathrm{A}_{\mathcal{C}}, \mathrm{A}_{\mathcal{R}}, \mathrm{A}_{\mathcal{L}}]^\top \;\Big|\; \mathrm{A}_{\mathscr{I}}^\top(\mathrm{A}_{\mathcal{C}}\mathrm{A}_{\mathcal{C}}^\top+\mathrm{A}_{\mathcal{R}}\mathrm{A}_{\mathcal{R}}^\top\\
&\qquad+\mathrm{A}_{\mathcal{L}}\mathrm{A}_{\mathcal{L}}^\top+\mathrm{A}_{\mathcal{V}}\mathrm{A}_{\mathcal{V}}^\top)^{-1}A_{\mathcal{V}}A_{\mathcal{V}}^\top x=0\Big\}.\\
&=\ker[\mathrm{A}_{\mathcal{C}}, \mathrm{A}_{\mathcal{R}}, \mathrm{A}_{\mathcal{L}}, \mathrm{A}_{\mathscr{I}}]^\top.
\end{aligned}$$

(i) The inclusion "⊆" follows from invertibility of $\begin{bmatrix}(\mathrm{A}_{\mathcal{V}}^\top \mathrm{A}_{\mathcal{V}})^{-1}\mathrm{A}_{\mathcal{V}}^\top\\ Z_{\mathcal{V}}^\top\end{bmatrix}$. The converse inclusion is obvious.

(ii) To prove "⊆", let $x\in\ker[\mathrm{A}_{\mathcal{C}}, \mathrm{A}_{\mathcal{R}}, \mathrm{A}_{\mathcal{L}}]^\top$ satisfy

$$\mathrm{A}_{\mathscr{I}}^\top(\mathrm{A}_{\mathcal{C}}\mathrm{A}_{\mathcal{C}}^\top+\mathrm{A}_{\mathcal{R}}\mathrm{A}_{\mathcal{R}}^\top+\mathrm{A}_{\mathcal{L}}\mathrm{A}_{\mathcal{L}}^\top+\mathrm{A}_{\mathcal{V}}\mathrm{A}_{\mathcal{V}}^\top)^{-1}\mathrm{A}_{\mathcal{V}}\mathrm{A}_{\mathcal{V}}^\top x=0.$$

Then

$$\begin{aligned}
0 &= \mathrm{A}_{\mathscr{I}}^\top(\mathrm{A}_{\mathcal{C}}\mathrm{A}_{\mathcal{C}}^\top+\mathrm{A}_{\mathcal{R}}\mathrm{A}_{\mathcal{R}}^\top+\mathrm{A}_{\mathcal{L}}\mathrm{A}_{\mathcal{L}}^\top+\mathrm{A}_{\mathcal{V}}\mathrm{A}_{\mathcal{V}}^\top)^{-1}\\
&\quad\cdot(\mathrm{A}_{\mathcal{C}}\mathrm{A}_{\mathcal{C}}^\top+\mathrm{A}_{\mathcal{R}}\mathrm{A}_{\mathcal{R}}^\top+\mathrm{A}_{\mathcal{L}}\mathrm{A}_{\mathcal{L}}^\top+\mathrm{A}_{\mathcal{V}}\mathrm{A}_{\mathcal{V}}^\top)x\\
&= \mathrm{A}_{\mathscr{I}}^\top x,
\end{aligned}$$

and thus $x\in\ker[\mathrm{A}_{\mathcal{C}}, \mathrm{A}_{\mathcal{R}}, \mathrm{A}_{\mathcal{L}}, \mathrm{A}_{\mathscr{I}}]^\top$.
Assuming conversely that $x\in\ker[\mathrm{A}_{\mathcal{C}}, \mathrm{A}_{\mathcal{R}}, \mathrm{A}_{\mathcal{L}}, \mathrm{A}_{\mathscr{I}}]^\top$, we clearly have $x\in\ker[\mathrm{A}_{\mathcal{C}}, \mathrm{A}_{\mathcal{R}}, \mathrm{A}_{\mathcal{L}}]^\top$, and further

$$\begin{aligned}
&\mathrm{A}_{\mathscr{I}}^\top(\mathrm{A}_{\mathcal{C}}\mathrm{A}_{\mathcal{C}}^\top+\mathrm{A}_{\mathcal{R}}\mathrm{A}_{\mathcal{R}}^\top+\mathrm{A}_{\mathcal{L}}\mathrm{A}_{\mathcal{L}}^\top+\mathrm{A}_{\mathcal{V}}\mathrm{A}_{\mathcal{V}}^\top)^{-1}\mathrm{A}_{\mathcal{V}}\mathrm{A}_{\mathcal{V}}^\top x\\
&= \mathrm{A}_{\mathscr{I}}^\top(\mathrm{A}_{\mathcal{C}}\mathrm{A}_{\mathcal{C}}^\top+\mathrm{A}_{\mathcal{R}}\mathrm{A}_{\mathcal{R}}^\top+\mathrm{A}_{\mathcal{L}}\mathrm{A}_{\mathcal{L}}^\top+\mathrm{A}_{\mathcal{V}}\mathrm{A}_{\mathcal{V}}^\top)^{-1}\\
&\quad\cdot(\mathrm{A}_{\mathcal{C}}\mathrm{A}_{\mathcal{C}}^\top+\mathrm{A}_{\mathcal{R}}\mathrm{A}_{\mathcal{R}}^\top+\mathrm{A}_{\mathcal{L}}\mathrm{A}_{\mathcal{L}}^\top+\mathrm{A}_{\mathcal{V}}\mathrm{A}_{\mathcal{V}}^\top)x\\
&= \mathrm{A}_{\mathscr{I}}^\top x=0.
\end{aligned}$$

■

References

[1] J. C. Willems, "Dissipative dynamical systems. Part II: Linear systems with quadratic supply rates," *Archive for Rational Mechanics and Analysis*, vol. 45, pp. 352–393, 1972.
[2] T. Reis, "Circuit synthesis of passive descriptor systems - a modified nodal approach," *Int. J. Circ. Theor. Appl.*, vol. 38, pp. 44–68, 2010.
[3] M. C. Smith, "Synthesis of mechanical networks: The inerter," *IEEE Trans. Autom. Control*, vol. 47, no. 10, pp. 1648–1662, 2002.

[4] W. Cauer, “Die Verwirklichung der Wechselstromwiderstände vorgeschriebener Frequenzabhängigkeit,” *Archiv für Elektrotechnik*, vol. 17, pp. 355–388, 1926.

[5] T. Reis and T. Stykel, “PABTEC: Passivity-Preserving Balanced Truncation for Electrical Circuits,” *IEEE Trans. on Computer-Aided Design of Integrated Circuits and Systems*, vol. 29, no. 10, pp. 1354–1367, 2010.

[6] B. D. O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis – A Modern Systems Theory Approach*. Englewood Cliffs, NJ: Prentice-Hall, 1973.

[7] P. I. Richards, “A special class of functions with positive real part in a half-plane,” *Duke Math. J.*, vol. 14, no. 3, pp. 777–786, 1947.

[8] W. Rudin, *Real and Complex Analysis*, 3rd ed. New York: McGraw-Hill, 1987.

[9] C.-W. Ho, A. E. Ruehli, and P. A. Brennan, “The modified nodal approach to network analysis,” *IEEE Trans. Circuits Syst.*, vol. CAS-22, no. 6, pp. 504–509, 1975.

[10] D. Estévez Schwarz and C. Tischendorf, “Structural analysis for electric circuits and consequences for MNA,” *Int. J. Circuit Theory Appl.*, vol. 28, no. 2, pp. 131–162, 2000.

[11] T. Berger and T. Reis, “Zero dynamics and funnel control for linear electrical circuits,” 2013, submitted for publication, preprint available from the website of the authors.