Disturbance decoupling by behavioral feedback for linear differential-algebraic systems

Thomas Berger

Fachbereich Mathematik, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany

Abstract

We study disturbance decoupling for linear differential-algebraic systems which are not necessarily regular. Compared to previous approaches, where state feedback is used, we use the concept of behavioral feedback which allows to study a larger class of systems. We derive geometric characterizations for solvability of the disturbance decoupling problem following the classical approach. Exploiting the freedom in the choice of the behavioral feedback we show that whenever disturbance decoupling can be achieved by behavioral feedback we may additionally achieve autonomous zero dynamics. Finally we solve Lebret's twenty year old open problem concerning disturbance decoupling with output uniqueness using behavioral feedback.

Key words: differential-algebraic systems; descriptor systems; disturbance decoupling; behavioral feedback; Wong sequences; zero dynamics.

Nomenclature:

$\mathbb{N}$, $\mathbb{N}_0$	set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\mathbb{R}[s], \mathbb{R}(s)$	the ring of polynomials with coefficients in $\mathbb{R}$ and its quotient field, resp.
$R^{n\times m}$	the set of $n \times m$ matrices with entries in a ring R
$\mathbf{Gl}_n(R)$	the group of invertible matrices in $R^{n\times n}$
$M\mathcal{S}$	$= \left\{ Mx \in \mathbb{R}^l \;\middle|\; x \in \mathcal{S} \right\}$, the image of $\mathcal{S} \subseteq \mathbb{R}^n$ under $M \in \mathbb{R}^{l\times n}$
$M^{-1}\mathcal{S}$	$= \left\{ x \in \mathbb{R}^n \;\middle|\; Mx \in \mathcal{S} \right\}$, the preimage of $\mathcal{S} \subseteq \mathbb{R}^l$ under $M \in \mathbb{R}^{l\times n}$
$\mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^n)$	the set of infinitely times differentiable functions $f : \mathbb{R}\to\mathbb{R}^n$; f is also called *smooth*
$\mathcal{AC}(\mathbb{R}\to\mathbb{R}^n)$	the set of absolutely continuous functions $f : \mathbb{R}\to\mathbb{R}^n$
$\mathcal{L}^1_{\rm loc}(\mathbb{R}\to\mathbb{R}^n)$	the set of locally (Lebesgue) integrable functions $f : \mathbb{R}\to\mathbb{R}^n$
$f \overset{\text{a.e.}}{=} g$	means that $f, g \in \mathcal{L}^1_{\rm loc}(\mathbb{R} \to \mathbb{R}^n)$ are equal "almost everywhere", i.e., $f(t) = g(t)$ for almost all $t \in \mathbb{R}$
$f\vert_I$	restriction of the function $f : \mathbb{R}\to\mathbb{R}^n$ to $I \subseteq \mathbb{R}$

Email address: thomas.berger@uni-hamburg.de (Thomas Berger).

1 Introduction

We study linear time-invariant systems given by differential-algebraic equations (DAEs) of the form

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= Ax(t) + Bu(t)\,, \\ y(t) &= Cx(t)\,, \end{aligned} \tag{1}$$

where $E, A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{p\times n}$. Systems of that type are also called *descriptor systems*. The set of systems (1) is denoted by $\Sigma_{l,n,m,p}$ and we write $[E, A, B, C] \in \Sigma_{l,n,m,p}$. DAE systems of the form (1) occur for example when modeling dynamical systems subject to algebraic constraints; for a further motivation we refer to [6, 20, 28, 29, 33] and the references therein. In the present paper we put special emphasis on the non-regular case, i.e., we do not assume that $sE - A$ is *regular*, which would mean that $l = n$ and $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$.
The functions $u : \mathbb{R}\to\mathbb{R}^m$ and $y : \mathbb{R}\to\mathbb{R}^p$ are called *input* and *output* of the system (1), resp. The tuple $(x, u, y) : \mathbb{R}\to\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ is said to be a *solution* of (1), if it belongs to the *behavior* of (1):

$$\mathfrak{B}_{[E,A,B,C]} := \left\{ \begin{array}{c} (x,u,y) \in \\ \mathcal{L}^1_{\rm loc}(\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^p) \end{array} \;\middle|\; \begin{array}{l} Ex \in \mathcal{AC}(\mathbb{R}\to\mathbb{R}^l) \text{ and} \\ (x,u,y) \text{ satisfies (1)} \\ \text{for almost all } t \in \mathbb{R} \end{array} \right\}.$$

Based on the above behavior, DAE control systems have been studied in detail e.g. in [6]. We assume that the states, inputs and outputs of the systems in $\Sigma_{l,n,m,p}$ are fixed a priori by the designer. This is different from other approaches based on the behavioral setting, see [19,22]. Our aim in the present paper is to characterize the influence of disturbances on the system (1), i.e., for a given disturbance matrix $Q \in \mathbb{R}^{l\times q}$ we consider the disturbed system

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= Ax(t) + Bu(t) + Qw(t)\,, \\ y(t) &= Cx(t)\,, \end{aligned} \tag{2}$$

where $w \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^q)$ represents a smooth disturbance, which may be due to noise, modeling or measuring errors, or higher terms in linearization.

In the case of ODE systems, the *disturbance decoupling problem (DDP)* is the problem of finding, for a given system $[I, A, B, C]$ and disturbance matrix Q, a proportional state feedback $u = Fx$ with $F \in \mathbb{R}^{m\times n}$ for (2) such that the transfer function of the closed-loop system satisfies $C(sI - (A + BF))^{-1}Q = 0$. This problem has first been treated and solved by Wonham and Morse [45], see also the classical textbooks [3, 34, 44]; several other versions have been considered by Willems [38,39]. In order to pursue a similar approach for DAEs, it must be required that the closed-loop pencil $sE-(A+BF)$ is regular. Then the transfer function $C(sE - (A + BF))^{-1}Q$ exists and the DDP can be investigated; this has been done in [24] where additionally derivative feedback is allowed and it is required that the closed-loop system has index at most one. Extensions of this problem have been considered e.g. in [23] and [26].

However, the class of regular DAE systems is not closed under the action of a feedback group [1], thus requiring $sE-(A+BF)$ to be regular is a restriction in the choice of F. The first version of the DDP for DAEs, which has been formulated by Fletcher and Aasaraai [25], does not impose any regularity assumptions. However, it was assumed that the output is independent of the disturbance in the sense that there is a set of admissible initial conditions such that the output vanishes. But admissibility of an initial condition depends on the disturbance, which is usually unknown, and hence it is not known a priori if a given initial condition is admissible. The appropriate version of the DDP with proportional state feedback for DAEs has been introduced and solved by Banaszuk et al. [2]; some alternative characterizations have also been given in [30]. However, the definition of the DDP in [2] already requires a zero output, which does not reflect the intuitive notion of a disturbance not influencing the output.

To define disturbance decoupling we follow an intuitive approach. In the case $B = 0$ we may treat the disturbance w as the input of system (2) and define disturbance decoupling in terms of the set-valued input-output map of the system $[E, A, Q, C] \in \Sigma_{l,n,q,p}$.

Definition 1 *For a system $[E, A, B, C] \in \Sigma_{l,n,m,p}$, we call the set-valued map*

$$\begin{aligned} &\Phi_{[E,A,B,C]} : \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^m) \to \mathcal{P}\big(\mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^p)\big), \\ &u \mapsto \left\{ \, y \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^p) \;\middle|\; \begin{array}{l} \exists\, x \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^n) : \\ (x, u, y) \in \mathfrak{B}_{[E,A,B,C]} \end{array} \right\}, \end{aligned}$$

the input-output map *of $[E, A, B, C]$. Here $\mathcal{P}(\mathcal{M})$ denotes the power set of a set $\mathcal{M}$.*

Definition 2 *Let $[E, A, 0, C] \in \Sigma_{l,n,0,p}$ and $Q \in \mathbb{R}^{l\times q}$. Then we call $[E, A, Q, C]$* disturbance decoupled*, if*

$$\begin{aligned} \forall\, w_1, w_2 \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q) : \\ \Phi_{[E,A,Q,C]}(w_1) = \Phi_{[E,A,Q,C]}(w_2). \end{aligned}$$

Roughly speaking, $[E, A, Q, C]$ is disturbance decoupled, if any two disturbances cannot be distinguished using knowledge of the output. In Section 3 we show that this is equivalent to the concept introduced in [2].

Example 3 *The system*

$$\dot{x}_1(t) = x_2(t) + w(t), \quad y(t) = x_1(t)$$

is disturbance decoupled, since the output is given by

$$y(t) = x_1(0) + \int_0^t \big(x_2(s) + w(s)\big)\,\mathrm{d}s\,, \quad t \in \mathbb{R},$$

for fixed $w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R})$ and arbitrary (free) state variable $x_2 \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R})$. The non-uniqueness of the output for a fixed disturbance "hides" the disturbance, i.e., it is compensated by the free state variable. However, we stress that the disturbance is not completely rejected as a hidden dependence of the output on the disturbance is still present. Here disturbance rejection means disturbance decoupling with additional output uniqueness which is considered in Section 5.

Compared to the approaches in [2,25,30], we do not consider proportional state feedback of the form $u = Fx$ for the solution of the DDP. Taking the viewpoint of the behavioral approach due to Willems [37], see also [40,42], the variables of the system do not have the interpretation of states and inputs until an analysis of the system reveals the free variables. These free variables should then be interpreted as inputs, since "they can be viewed as unexplained by the model and imposed on the system by the environment" [32]. This approach obeys the physical meaning of the system variables and it may turn out that in the original model the choice of states and inputs was inappropriate. With this in mind, the variables labeled u by the designer are not necessarily the free variables, i.e., some of the them may be constrained, and on the other hand free state variables may be present. Therefore, the use of proportional state feedback is limited for DAE systems, and actually a feedback in terms of the free variables is needed. A setup where this is allowed is provided by the use of *behavioral feedback* of the form $K_1x + K_2u = 0$, where $K = [K_1, K_2] \in \mathbb{R}^{k\times n} \times \mathbb{R}^{k\times m}$, which is not always solvable for u in general. The concept of feedback in the behavioral sense has its origin in the works by Willems, Polderman and Trentelman [5, 32, 35, 41, 42], where differential behaviors and their stabilization via *control by interconnection* is considered. The latter means a systematic addition of some further equations such that a desired behavior is achieved. The interconnection of system (2) with the behavioral feedback is depicted in Figure 1.

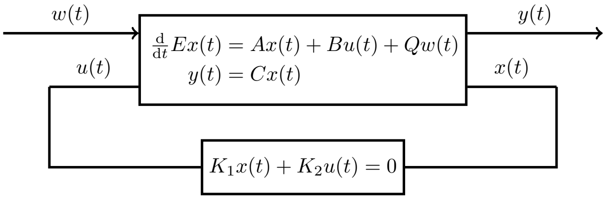

Fig. 1. Interconnection of system and behavioral feedback

The closed-loop system of (2) with the behavioral feedback $K_1x + K_2u = 0$ is given by

$$[E^K, A^K, Q^K, C^K] = \left[\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ K_1 & K_2 \end{bmatrix}, \begin{bmatrix} Q \\ 0 \end{bmatrix}, [C, 0]\right] \in \Sigma_{l+k,n+m,q,p} \quad (3)$$

with state $\binom{x}{u}$, input w and output y. If $[K_1, K_2] = [F, -I_m]$, then $[E^K, A^K, Q^K, C^K]$ is equivalent to $[E, A+BF, Q, C]$ and we are in the case of proportional state feedback. In Theorem 12 we derive a geometric characterization of solvability of the DDP with behavioral feedback. This result is displayed in Figure 2 and compared to the classical ODE result, see [44], and to the DAE result from [2], which both consider proportional state feedback.

The vast freedom in choosing the control matrix $K = [K_1, K_2]$ for the behavioral feedback can be exploited to obtain several additional properties such as autonomous zero dynamics of the (undisturbed) closed-loop system $[E^K, A^K, 0, C^K]$. Roughly speaking, the zero dynamics of a system are those dynamics which are not visible at the output; they are called autonomous, if every trajectory is uniquely determined by its values on any open interval, see Section 4 for more details.

Note that for the ODE system $\dot{x} = u + w,\ y = x$, where $w \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R})$ is a disturbance, the DDP is not solvable by state feedback $u = Fx$, but it is solvable by behavioral feedback choosing $[K_1, K_2] = 0 \in \mathbb{R}^{0\times 2}$ (i.e., x and u are not restricted) [1]; this results in a system as in Example 3 with free variable $x_2 = u$. However, as noted there, disturbance rejection is not achieved. This leads to Lebret's twenty year old open problem [30], that is disturbance decoupling with output uniqueness. In the present paper we present two possible solutions: One is to derive additional conditions on $[E, A, B, C]$ and Q which guarantee output uniqueness, see Theorem 21. For the second solution we relax the compatibility assumption on the control matrix K – a trade-off between requirements on the data and properties of the control – see Theorem 25. These results and their relations are depicted in Figure 3.

Example 4 *We consider an example where the DDP is not solvable by proportional state feedback, but the DDP with output uniqueness is solvable by behavioral feedback. Consider the system*

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + x_3(t), & 0 &= u(t) + w_2(t) \\ \dot{x}_2(t) &= w_1(t), & y(t) &= x_1(t), \end{aligned}$$

where $w_1, w_2 \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R})$ are disturbances. Seeking a solution of the DDP by state feedback $u = f_1x_1 + f_2x_2 + f_3x_3$, $f_i \in \mathbb{R}$, leads to, when $y = 0$,

$$\begin{aligned} 0 &= x_2(t) + x_3(t), \qquad 0 = f_2x_2(t) + f_3x_3(t) + w_2(t) \\ \dot{x}_2(t) &= w_1(t), \end{aligned}$$

and hence $w_2 = (1 - f_2)x_2$ and $\dot{x}_2 = w_1$, which gives $w_2 = 0$ if $f_2 = 1$, or $\dot{w}_2 = (1 - f_2)w_1$ if $f_2 \neq 1$. In any case, the disturbances cannot be chosen arbitrarily, i.e., the closed-loop system cannot be disturbance decoupled. However, choosing the behavioral feedback $x_3 = -x_2$ we arrive at the closed-loop system

$$\begin{aligned} \dot{x}_1(t) &= 0, & 0 &= u(t) + w_2(t) \\ \dot{x}_2(t) &= w_1(t), & y(t) &= x_1(t), \end{aligned}$$

and the output $y(t) \equiv x_1(0)$ is independent of the disturbances, i.e., the system is disturbance decoupled with unique output. Note that in this example the input variable u is not a free variable of the system.

The present paper is organized as follows: In Section 2 we introduce the generalized Wong sequences as the crucial geometric tool for the characterization of solvability of the DDP. Disturbance decoupled systems are characterized in Section 3 and we investigate when disturbance decoupling can be achieved by behavioral feedback. The considerations in Section 3 reveal that there is a lot of freedom in the choice of the behavioral feedback. In Section 4 we exploit this freedom. We recall the concept of zero dynamics and show that whenever disturbance decoupling can be achieved by behavioral feedback we may additionally achieve autonomous zero dynamics. In Section 5 we investigate Lebret's open problem [30] and solve it using the concept of behavioral feedback.

[1] See Remark 11 for the definition of $\mathbb{R}^{0\times 2}$.

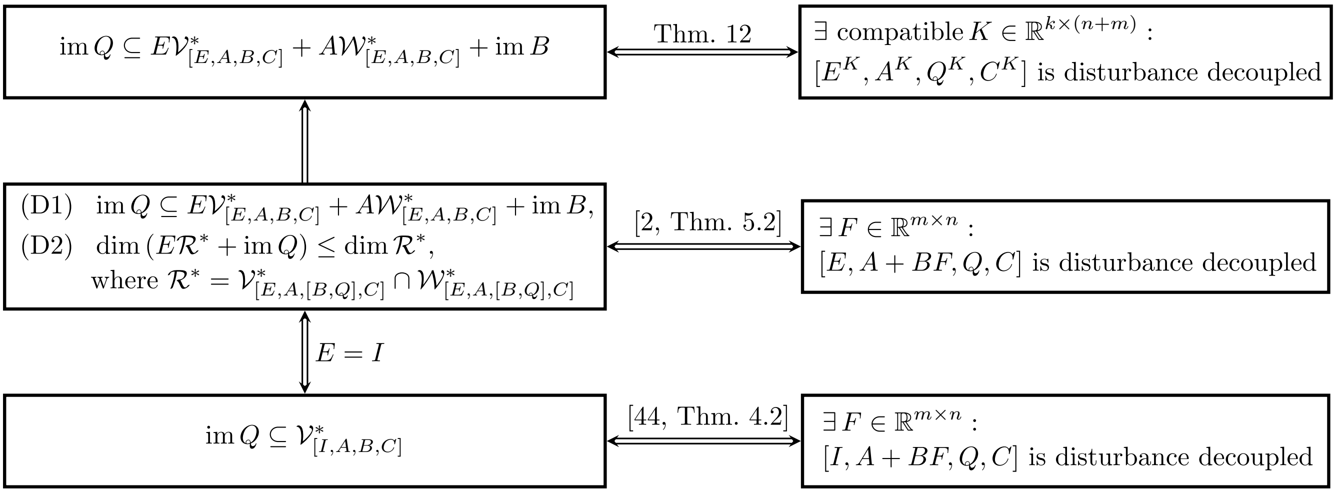

Fig. 2. Theorem 12 compared to earlier results on disturbance decoupling. The spaces $\mathcal{V}^*_{[E,A,B,C]}$ and $\mathcal{W}^*_{[E,A,B,C]}$ are introduced in Section 2. For compatibility of a control matrix K see Definition 10.

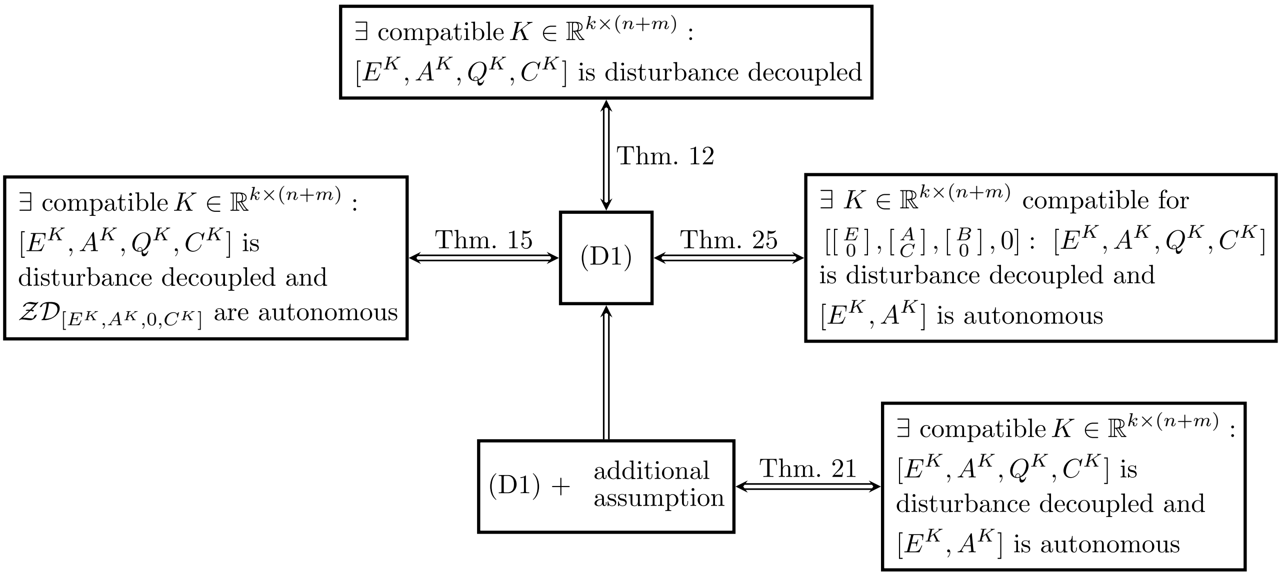

Fig. 3. Disturbance decoupling results and their relations. (Autonomous) zero dynamics $\mathcal{ZD}_{[E^K,A^K,0,C^K]}$ are defined in Section 4. Autonomy of DAEs $[E^K, A^K]$ is defined in Section 5.

2 Generalized Wong sequences

In this section we introduce the crucial geometric tools for the characterization of disturbance decoupling. Since the seminal work by Wonham and Morse [45] the existence of a feedback which achieves disturbance decoupling is usually characterized by a geometric condition involving the limits of certain subspace sequences. For DAE systems $[E, A, B, C] \in \Sigma_{l,n,m,p}$ we define the sequences

$$\begin{aligned}
\mathcal{V}^0_{[E,A,B,C]} &= \ker C, \\
\mathcal{V}^{i+1}_{[E,A,B,C]} &= A^{-1}(E\mathcal{V}^i_{[E,A,B,C]} + \mathrm{im}\, B) \cap \ker C, \\
\mathcal{W}^0_{[E,A,B,C]} &= \{0\}, \\
\mathcal{W}^{i+1}_{[E,A,B,C]} &= E^{-1}(A\mathcal{W}^i_{[E,A,B,C]} + \mathrm{im}\, B) \cap \ker C.
\end{aligned}$$

The sequence $(\mathcal{V}^i_{[E,A,B,C]})_{i\in\mathbb{N}_0}$ is non-increasing and $(\mathcal{W}^i_{[E,A,B,C]})_{i\in\mathbb{N}_0}$ is non-decreasing and both sequences terminate after finitely many steps, thus we may set

$$\mathcal{V}^*_{[E,A,B,C]} = \bigcap_{i\in\mathbb{N}_0} \mathcal{V}^i_{[E,A,B,C]},$$
$$\mathcal{W}^*_{[E,A,B,C]} = \bigcup_{i\in\mathbb{N}_0} \mathcal{W}^i_{[E,A,B,C]}.$$

We will call the sequences $(\mathcal{V}^i_{[E,A,B,C]})_{i\in\mathbb{N}_0}$ and $(\mathcal{W}^i_{[E,A,B,C]})_{i\in\mathbb{N}_0}$ *generalized Wong sequences*. In [11,16, 17] the Wong sequences for matrix pencils (i.e., $B = 0$ and $C = 0$) are investigated, the name chosen this way since Wong [43] was the first who used both sequences for the analysis of matrix pencils. In [12,14,18] the case $C = 0$ is considered and the sequences $(\mathcal{V}^i_{[E,A,B,0]})_{i\in\mathbb{N}_0}$ and $(\mathcal{W}^i_{[E,A,B,0]})_{i\in\mathbb{N}_0}$ are called *augmented Wong sequences*. Similarly, in [15] the sequences $(\mathcal{V}^i_{[E,A,0,C]})_{i\in\mathbb{N}_0}$ and $(\mathcal{W}^i_{[E,A,0,C]})_{i\in\mathbb{N}_0}$ (i.e., $B = 0$) are called *restricted Wong sequences*. Using the invariance concepts introduced in [31], $\mathcal{V}^*_{[E,A,B,C]}$ is the supremal (A,E,B)-invariant subspace of $\ker C$, i.e., the largest subspace $\mathcal{V} \subseteq \ker C$ with the property $A\mathcal{V} \subseteq E\mathcal{V} + \operatorname{im} B$. Furthermore, $\mathcal{W}^*_{[E,A,B,C]}$ is the infimal restricted (E,A,B)-invariant subspace of $\ker C$, i.e., the smallest subspace $\mathcal{W} \subseteq \ker C$ with the property $\mathcal{W} = E^{-1}(A\mathcal{W} + \operatorname{im} B) \cap \ker C$. For more details on the Wong sequences see the surveys [12,15] and the references therein.

Note that in geometric control theory for ODE systems (i.e., $E = I$), see e.g. [44], the sequence $(\mathcal{V}^i_{[I,A,B,C]})_{i\in\mathbb{N}_0}$ is called *invariant subspace algorithm*, and the sequence $(\mathcal{W}^i_{[I,A,B,C]})_{i\in\mathbb{N}_0}$ is called *controllability subspace algorithm*.

In the remainder of this section we derive some important relations for the generalized Wong sequences which are the basis for the results on disturbance decoupling.

Lemma 5 *Let $[E,A,B,C] \in \Sigma_{l,n,m,p}$ and choose $T \in \mathbb{R}^{n\times k}$ with $\operatorname{rk} T = k$ such that $\operatorname{im} T = \ker C$. Then*

$$(ET)\mathcal{V}^*_{[ET,AT,B,0]} = E\mathcal{V}^*_{[E,A,B,C]}$$
$$\text{and} \quad (AT)\mathcal{W}^*_{[ET,AT,B,0]} = A\mathcal{W}^*_{[E,A,B,C]}.$$

PROOF. First we prove that

$$\forall\, i\in\mathbb{N}_0:\ ET\mathcal{V}^i_{[ET,AT,B,0]} = E\mathcal{V}^i_{[E,A,B,C]}.$$

For $i = 0$ we have $E\mathcal{V}^0_{[E,A,B,C]} = E(\ker C) = \operatorname{im} ET = ET\mathcal{V}^0_{[ET,AT,B,0]}$. Suppose that the assertion is true for some $i \in \mathbb{N}_0$. Then

$$\begin{aligned}
&ET\mathcal{V}^{i+1}_{[ET,AT,B,0]} \\
&= ET(AT)^{-1}(ET\mathcal{V}^i_{[ET,AT,B,0]} + \operatorname{im} B) \\
&= ET(AT)^{-1}(E\mathcal{V}^i_{[E,A,B,C]} + \operatorname{im} B) \\
&= ET\left\{\ x\in\mathbb{R}^k \ \middle|\ ATx \in E\mathcal{V}^i_{[E,A,B,C]} + \operatorname{im} B\ \right\} \\
&= E\left\{\ y\in\operatorname{im} T \ \middle|\ Ay \in E\mathcal{V}^i_{[E,A,B,C]} + \operatorname{im} B\ \right\} \\
&= E(A^{-1}(E\mathcal{V}^i_{[E,A,B,C]} + \operatorname{im} B) \cap \ker C) = E\mathcal{V}^{i+1}_{[E,A,B,C]}.
\end{aligned}$$

The proof for $AT\mathcal{W}^i_{[ET,AT,B,0]} = A\mathcal{W}^i_{[E,A,B,C]}$ for all $i \in \mathbb{N}_0$ is analogous and omitted. $\square$

Lemma 6 *Let $[E,A,B,C] \in \Sigma_{l,n,m,p}$. Then*

$$[E,0]\mathcal{V}^*_{\left[[E,0],[A,B],0,0\right]} = E\mathcal{V}^*_{[E,A,B,0]}$$
$$\text{and} \quad [A,B]\mathcal{W}^*_{\left[[E,0],[A,B],0,0\right]} = A\mathcal{W}^*_{[E,A,B,0]} + \operatorname{im} B.$$

PROOF. First we prove that

$$\forall\, i\in\mathbb{N}_0:\ [E,0]\mathcal{V}^i_{\left[[E,0],[A,B],0,0\right]} = E\mathcal{V}^i_{[E,A,B,0]}.$$

For $i = 0$ we have $E\mathcal{V}^0_{[E,A,B,0]} = \operatorname{im} E = [E,0]\mathcal{V}^0_{\left[[E,0],[A,B],0,0\right]}$. Suppose that the assertion is true for some $i \in \mathbb{N}_0$. Then

$$\begin{aligned}
&[E,0]\mathcal{V}^{i+1}_{\left[[E,0],[A,B],0,0\right]} \\
&= [E,0]([A,B])^{-1}\left([E,0]\mathcal{V}^i_{\left[[E,0],[A,B],0,0\right]}\right) \\
&= [E,0]([A,B])^{-1}(E\mathcal{V}^i_{[E,A,B,0]}) \\
&= [E,0]\left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^{n+m} \ \middle|\ Ax_1 + Bx_2 \in E\mathcal{V}^i_{[E,A,B,0]}\right\} \\
&= E\left\{x_1\in\mathbb{R}^n \ \middle|\ \exists\, x_2\in\mathbb{R}^m:\ Ax_1 + Bx_2 \in E\mathcal{V}^i_{[E,A,B,0]}\right\} \\
&= EA^{-1}(E\mathcal{V}^i_{[E,A,B,0]} + \operatorname{im} B) = E\mathcal{V}^{i+1}_{[E,A,B,0]}.
\end{aligned}$$

Now we prove that

$$\begin{aligned}
\forall\, i\in\mathbb{N}_0:\ [A,B]\mathcal{W}^i_{\left[[E,0],[A,B],0,0\right]} &\subseteq A\mathcal{W}^i_{[E,A,B,0]} + \operatorname{im} B \\
&\subseteq [A,B]\mathcal{W}^{i+1}_{\left[[E,0],[A,B],0,0\right]}.
\end{aligned}$$

For $i = 0$ we have

$$\begin{aligned}
[A,B]\mathcal{W}^0_{\left[[E,0],[A,B],0,0\right]} &= \{0\} \subseteq A\mathcal{W}^0_{[E,A,B,0]} + \operatorname{im} B \\
&\subseteq A\ker E + \operatorname{im} B = [A,B]\mathcal{W}^1_{\left[[E,0],[A,B],0,0\right]}.
\end{aligned}$$

Suppose that the assertion is true for some $i \in \mathbb{N}_0$. Then

$$
\begin{aligned}
&[A, B]\mathcal{W}^{i+1}_{[[E,0],[A,B],0,0]} \\
&= [A, B]([E, 0])^{-1}\left([A, B]\mathcal{W}^{i}_{[[E,0],[A,B],0,0]}\right) \\
&\subseteq [A, B]([E, 0])^{-1}\left(A\mathcal{W}^{i}_{[E,A,B,0]} + \operatorname{im} B\right) \\
&= [A, B]\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^{n+m} \,\middle|\, Ex_1 \in A\mathcal{W}^{i}_{[E,A,B,0]} + \operatorname{im} B \right\} \\
&= \left\{ Ax_1 + Bx_2 \,\middle|\, \begin{array}{l} x_1 \in \mathbb{R}^n,\ x_2 \in \mathbb{R}^m, \\ x_1 \in E^{-1}(A\mathcal{W}^{i}_{[E,A,B,0]} + \operatorname{im} B) \\ \qquad\qquad = \mathcal{W}^{i+1}_{[E,A,B,0]} \end{array} \right\} \\
&= A\mathcal{W}^{i+1}_{[E,A,B,0]} + \operatorname{im} B
\end{aligned}
$$

and analogously, just with the opposite inclusion sign, we obtain $[A, B]\mathcal{W}^{i+2}_{[[E,0],[A,B],0,0]} \supseteq A\mathcal{W}^{i+1}_{[E,A,B,0]} + \operatorname{im} B$. □

3 Disturbance decoupling

In this section we derive a characterization of disturbance decoupled systems. Furthermore, we recall the concepts of behavioral feedback and compatible control and characterize solvability of the DDP by behavioral feedback.
Recall Definition 2 of disturbance decoupled systems. We derive the following characterization of disturbance decoupled systems which matches the definition for the DDP used in [2], but differs from the concepts used in [23–26] where regularity is required.

Proposition 7 *Let $[E, A, 0, C] \in \Sigma_{l,n,0,p}$ and $Q \in \mathbb{R}^{l\times q}$. Then $[E, A, Q, C]$ is disturbance decoupled if, and only if,*

$$
\begin{aligned}
&\forall\, w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)\ \exists\, x \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^n): \\
&\qquad Cx = 0 \text{ and } E\dot{x} = Ax + Qw\,.
\end{aligned}
\tag{4}
$$

PROOF. $\Rightarrow$: Let $w \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^q)$. Then $\Phi_{[E,A,Q,C]}(w) = \Phi_{[E,A,Q,C]}(0)$ and the assertion follows from $0 \in \Phi_{[E,A,Q,C]}(0)$.
$\Leftarrow$: It suffices to show that $\Phi_{[E,A,Q,C]}(w) = \Phi_{[E,A,Q,C]}(0)$ for all $w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)$. Let $w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)$ and observe that $\Phi_{[E,A,Q,C]}(w) \neq \emptyset$ by (4). We show $\Phi_{[E,A,Q,C]}(w) \subseteq \Phi_{[E,A,Q,C]}(0)$. Let $y \in \Phi_{[E,A,Q,C]}(w)$. Then there exists $z_1 \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^n)$ such that $E\dot{z}_1 = Az_1 + Qw$ and $y = Cz_1$. By assumption, there exists $z_2 \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^n)$ such that $E\dot{z}_2 = Az_2 + Qw$ and $Cz_2 = 0$. Setting $x := z_1 - z_2 \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^n)$ yields

$$E\dot{x} = Az_1 + Qw - Az_2 - Qw = Ax$$

and $Cx = Cz_1 - Cz_2 = y$. Therefore, $y \in \Phi_{[E,A,Q,C]}(0)$. The opposite inclusion can be shown analogously and this finishes the proof. □

Remark 8 *Consider $[E, A, 0, C] \in \Sigma_{n,n,0,p}$ with regular $sE - A$ and let $Q \in \mathbb{R}^{l\times q}$. Using Laplace transform and Proposition 7 it is immediate that $[E, A, Q, C]$ is disturbance decoupled if, and only if, the transfer function from the disturbance to the output is zero, i.e., $C(sE - A)^{-1}Q = 0$. In particular, the concept introduced in Definition 2 generalizes the classical concept of disturbance decoupled ODE systems, see e.g. [44].*

We are now in the position to derive a geometric characterization for $[E, A, Q, C]$ being disturbance decoupled.

Theorem 9 *Let $[E, A, 0, C] \in \Sigma_{l,n,0,p}$ and $Q \in \mathbb{R}^{l\times q}$. Then $[E, A, Q, C]$ is disturbance decoupled if, and only if,*

$$\operatorname{im} Q \subseteq E\mathcal{V}^*_{[E,A,0,C]} + A\mathcal{W}^*_{[E,A,0,C]}. \tag{5}$$

PROOF. *Step 1*: We reduce the problem to a problem of existence of solutions for a certain DAE. Let $T_1 \in \mathbb{R}^{n\times r}, T_2 \in \mathbb{R}^{n\times(n-r)}$ be such that $\operatorname{im} T_1 = \ker C$ and $T = [T_1, T_2]$ is invertible. Then $CT = [0, CT_2] = [0, C_2]$, where $C_2 \in \mathbb{R}^{p\times(n-r)}$ has full column rank: $C_2x = 0 = CT_2x$ for some $x \in \mathbb{R}^{n-r}$ implies $T_2x \in \operatorname{im} T_2 \cap \ker C = \operatorname{im} T_2 \cap \operatorname{im} T_1 = \{0\}$, thus $T_2x = 0$ and hence $x = 0$ by full column rank of T_2. By Proposition 7, $[E, A, Q, C]$ is disturbance decoupled if, and only if, (4) holds. Applying the coordinate transformation $z = T^{-1}x$, (4) is equivalent to

$$
\begin{aligned}
&\forall\, w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)\ \exists\, z \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^n): \\
&\qquad CTz = 0 \text{ and } ET\dot{z} = ATz + Qw.
\end{aligned}
$$

Partitioning $z = (z_1^\top, z_2^\top)^\top$ with $z_1 \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^r)$, $z_2 \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{n-r})$ and invoking that $0 = CTz = C_2z_2$ implies $z_2 = 0$, we find that, with $E_1 = ET_1$, $A_1 = AT_1$, $[E, A, Q, C]$ is disturbance decoupled if, and only if,

$$
\begin{aligned}
&\forall\, w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)\ \exists\, z_1 \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^r): \\
&\qquad E_1\dot{z}_1 = A_1z_1 + Qw.
\end{aligned}
\tag{6}
$$

Step 2: We prove that (6) is equivalent to (5). Choose full rank matrices $P_1 \in \mathbb{R}^{n\times n_1}, R_1 \in \mathbb{R}^{n\times n_2}, P_2 \in \mathbb{R}^{l\times l_1}, R_2 \in \mathbb{R}^{l\times l_2}$ such that

$$
\begin{aligned}
\operatorname{im} P_1 &= \mathcal{V}^*_{[E_1,A_1,0,0]} + \mathcal{W}^*_{[E_1,A_1,0,0]}, \\
\operatorname{im} P_1 \oplus \operatorname{im} R_1 &= \mathbb{R}^n, \\
\operatorname{im} P_2 &= E_1\mathcal{V}^*_{[E_1,A_1,0,0]} + A_1\mathcal{W}^*_{[E_1,A_1,0,0]}, \\
\operatorname{im} P_2 \oplus \operatorname{im} R_2 &= \mathbb{R}^l.
\end{aligned}
$$

Then, by [16, Thm. 2.3], with $V = [P_1, R_1]$ and $W = [P_2, R_2]^{-1}$ we have

$$W(sE_1 - A_1)V = \begin{bmatrix} sE_{11} - A_{11} & sE_{12} - A_{12} \\ 0 & sE_{22} - A_{22} \end{bmatrix},$$

where

(i) $E_{11}, A_{11} \in \mathbb{R}^{l_1 \times n_1}$, $l_1 \le n_1$, satisfy $\mathrm{rk}_{\mathbb{R}(s)}(sE_{11} - A_{11}) = l_1$,
(ii) $E_{22}, A_{22} \in \mathbb{R}^{l_2 \times n_2}$, $l_2 > n_2$ or $l_2 = n_2 = 0$, satisfy $\mathrm{rk}\, E_{22} = n_2$ and $\mathrm{rk}_{\mathbb{C}}(\lambda E_{22} - A_{22}) = n_2$ for all $\lambda \in \mathbb{C}$.

By [16, Lem. 3.1] there exists a unimodular matrix $\begin{bmatrix} M(s) \\ K(s) \end{bmatrix} \in \mathbb{R}[s]^{(n_2 + (l_2 - n_2)) \times l_2}$ (i.e., $\begin{bmatrix} M(s) \\ K(s) \end{bmatrix}$ is invertible over $\mathbb{R}[s]$) such that

$$\begin{bmatrix} M(s) \\ K(s) \end{bmatrix} (sE_{22} - A_{22}) = \begin{bmatrix} I_{n_2} \\ 0 \end{bmatrix}.$$

Now, [16, Thm. 3.2] yields that for given $w \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^q)$ the DAE $E_1 \dot{z}_1 = A_1 z_1 + Qw$ has a solution $z_1 \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^r)$ if, and only if,

$$K(\tfrac{\mathrm{d}}{\mathrm{d}t})\tilde{w} = 0, \quad \text{where } \tilde{w} = [0, I_{l_2}]WQw.$$

Therefore, (6) is equivalent to

$$\forall\, w \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^q): \; K(\tfrac{\mathrm{d}}{\mathrm{d}t})[0, I_{l_2}]WQw = 0 \tag{7}$$

and it remains to show that (7) is equivalent to (5). We prove that (7) is true if, and only if, $[0, I_{l_2}]WQ = 0$: Sufficiency is clear and we show necessity. Condition (7) implies that $K(s)[0, I_{l_2}]WQ = 0$ as a polynomial matrix. Writing $K(s) = K_1 + sK_2 + \ldots + s^{p-1}K_p$ we thus have $K_i[0, I_{l_2}]WQ = 0$ for all $i = 1, \ldots, p$, which gives

$$\mathrm{im}[0, I_{l_2}]WQ \subseteq \bigcap_{i=1}^{p} \ker K_i.$$

As shown in the proof of [16, Lem. 4.19] we have $\bigcap_{i=1}^{p} \ker K_i = \{0\}$ and hence $[0, I_{l_2}]WQ = 0$.
Finally, this shows that $[E, A, Q, C]$ is disturbance decoupled if, and only if,

$$\begin{aligned} \mathrm{im}\, Q \subseteq \ker[0, I_{l_2}]W = W^{-1}\,\mathrm{im} \begin{bmatrix} I_{l_1} \\ 0 \end{bmatrix} = \mathrm{im}\, P_2 \\ = E_1 \mathcal{V}^*_{[E_1, A_1, 0, 0]} + A_1 \mathcal{W}^*_{[E_1, A_1, 0, 0]}. \end{aligned}$$

From Lemma 5 we may deduce that $E_1 \mathcal{V}^*_{[E_1, A_1, 0, 0]} + A_1 \mathcal{W}^*_{[E_1, A_1, 0, 0]} = E\mathcal{V}^*_{[E, A, 0, C]} + A\mathcal{W}^*_{[E, A, 0, C]}$ and this concludes the proof. □

As mentioned in the introduction, the solution of the DDP with proportional state feedback has been derived in [2] for DAEs, where it is shown that for $[E, A, B, C] \in \Sigma_{l,n,m,p}$ and disturbance matrix $Q \in \mathbb{R}^{l \times q}$ there exists $F \in \mathbb{R}^{m \times n}$ such that $[E, A + BF, Q, C]$ is disturbance decoupled if, and only if, conditions (D1) and (D2) as depicted in Figure 2 hold true. However, as explained, the use of state feedback is limited for DAEs, since it does not take the physical meaning of the variables into account. In contrast to the approach in [2], we seek a feedback in the behavioral sense, i.e., a control $K = [K_1, K_2] \in \mathbb{R}^{k \times n} \times \mathbb{R}^{k \times m}$ such that the closed-loop system $[E^K, A^K, Q^K, C^K]$ as in (3) (cf. also Figure 1) is disturbance decoupled. In the undisturbed case $w = 0$, the control K has to be compatible with the system in a certain sense, cf. also [12] and the references therein. Here we introduce a slightly different notion of compatible control which uses smooth solutions only.

Definition 10 *Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$. A control matrix $K = [K_1, K_2] \in \mathbb{R}^{k \times n} \times \mathbb{R}^{k \times m}$, $k \in \mathbb{N}_0$, is called* compatible *for $[E, A, B, C]$, if*

$$\begin{aligned} &\forall\,(x, u, y) \in \mathfrak{B}_{[E,A,B,C]} \cap \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p) \\ &\exists\,(\tilde{x}, \tilde{u}) \in \mathfrak{B}_{[E^K, A^K, 0_{l \times 0}, 0_{0 \times n}]} \cap \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m): \\ &\qquad\qquad Ex(0) = E\tilde{x}(0). \end{aligned}$$

Remark 11 *We explain what $0_{0 \times n}$ for $n \in \mathbb{N}_0$ is.*

(i) *We deem 0_0 as the only element of $\mathbb{R}^0 = \{0_0\}$ and view the latter as a linear $\mathbb{R}$-vector space. It does not have a basis, so its element 0_0 cannot be represented by the coordinates of the basis vectors. $\mathbb{R}^0$ is generated by the empty set and its dimension is zero.*
(ii) *We may also consider the set of all homomorphisms mapping $\mathbb{R}^n$ to $\mathbb{R}^0$, that is*

$$\mathbb{R}^{0 \times n} := \left\{ L : \mathbb{R}^n \to \mathbb{R}^0 \;\middle|\; L \text{ a homomorphism} \right\},$$

$n \in \mathbb{N}_0$. The vector space $\mathbb{R}^{0 \times n}$ has also only one element, i.e., $\mathbb{R}^{0 \times n} = \{0_{0 \times n}\}$, and $0_{0 \times n} =: =\!=$ does not have a matrix representation in the usual sense, but it holds that $=\!= v = 0_0$ for all $v \in \mathbb{R}^n$.
(iii) *Analogous statements for $\mathbb{R}^{l \times 0} = \{0_{l \times 0}\}$, $l \in \mathbb{N}_0$, hold with $\| := 0_{l \times 0}$ and $\| 0_0 = 0_l \in \mathbb{R}^l$.*

The concept of a compatible control is important from a practical point of view. If we assume that the controller is switched on at time $t = 0$, then it must be guaranteed that there actually exists a closed-loop trajectory $(\tilde{x}, \tilde{u})$ such that the initial differential variables $E\tilde{x}(0)$ match those of the open-loop trajectory (x, u, y). Otherwise, a jump from $Ex(0)$ to $E\tilde{x}(0)$ would occur which must be avoided. Note that our concept of compatible control is a slight modification of the concept introduced in [27]. The following theorem is the analog of [2, Thm. 5.2] for the case of behavioral feedback.

Theorem 12 *Let $[E,A,B,C] \in \Sigma_{l,n,m,p}$ and $Q \in \mathbb{R}^{l\times q}$. Then there exists a control $K = [K_1, K_2] \in \mathbb{R}^{k\times n} \times \mathbb{R}^{k\times m}$ compatible for $[E,A,B,C]$ such that $[E^K, A^K, Q^K, C^K]$ is disturbance decoupled if, and only if,*

$$\operatorname{im} Q \subseteq E\mathcal{V}^*_{[E,A,B,C]} + A\mathcal{W}^*_{[E,A,B,C]} + \operatorname{im} B.$$

PROOF. Let $T_1 \in \mathbb{R}^{n\times r}, T_2 \in \mathbb{R}^{n\times(n-r)}$ be such that $\operatorname{im} T_1 = \ker C$ and $T = [T_1, T_2]$ is invertible. As in Step 1 of the proof of Theorem 9 we may show that for some compatible control $K \in \mathbb{R}^{k\times(n+m)}$, the system $[E^K, A^K, Q^K, C^K]$ is disturbance decoupled if, and only if,

$$\forall\, w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)\ \exists\,(z,u) \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^r\times\mathbb{R}^m):$$
$$ET_1\dot z = AT_1 z + Bu + Qw \text{ and } K_1T_1z + K_2u = 0. \quad (8)$$

Writing

$$[E_1^K, A_1^K] = \left[\begin{bmatrix} ET_1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix} AT_1 & B\\ K_1T_1 & K_2\end{bmatrix}\right],$$

we find that (8) is equivalent to

$$\forall\, w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)\ \exists\, v \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^{r+m}):$$
$$E_1^K\dot v = A_1^K v + \left[\begin{smallmatrix} Q\\ 0\end{smallmatrix}\right] w.$$

$\Leftarrow$: Choosing $k=0$ (i.e., $K = 0 \in \mathbb{R}^{0\times(n+m)}$) and invoking the above statement, $[E^K, A^K, Q^K, C^K]$ is disturbance decoupled if, and only if, $\big[[ET_1, 0], [AT_1, B], Q, 0\big]$ is disturbance decoupled. By Theorem 9, this is equivalent to

$$\operatorname{im} Q \subseteq [ET_1, 0]\mathcal{V}^*_{\left[[ET_1,0],[AT_1,B],0,0\right]} + [AT_1, B]\mathcal{W}^*_{\left[[ET_1,0],[AT_1,B],0,0\right]}, \quad (9)$$

and invoking Lemmas 5 and 6 we find that

$$E\mathcal{V}^*_{[E,A,B,C]} = [ET_1, 0]\mathcal{V}^*_{\left[[ET_1,0],[AT_1,B],0,0\right]},$$
$$A\mathcal{W}^*_{[E,A,B,C]} + \operatorname{im} B = [AT_1, B]\mathcal{W}^*_{\left[[ET_1,0],[AT_1,B],0,0\right]}, \quad (10)$$

which yields the claim.
$\Rightarrow$: If $K = [K_1, K_2] \in \mathbb{R}^{k\times n} \times \mathbb{R}^{k\times m}$ is a compatible control for $[E,A,B,C]$ such that the system $[E^K, A^K, Q^K, C^K]$ is disturbance decoupled, then (8) in particular implies

$$\forall\, w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)\ \exists\,(z,u) \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^r\times\mathbb{R}^m):$$
$$ET_1\dot z = AT_1z + Bu + Qw,$$

which means that $\big[[ET_1, 0], [AT_1, B], Q, 0\big]$ is disturbance decoupled. Then (9) and (10) hold true and the claim follows. □

Remark 13 *We consider an ODE system $[I,A,B,C] \in \Sigma_{n,n,m,p}$ with disturbance matrix $Q \in \mathbb{R}^{n\times q}$ and compare the classical result in [44, Thm. 4.2] to Theorem 12, see also Figure 2. The main difference is that in [44, Thm. 4.2] proportional state feedback $u = Fx$ is considered to achieve disturbance decoupling, whereas we consider behavioral feedback $K_1x + K_2u = 0$. Roughly speaking, in the latter the input variables are not completely determined by the state variables in general, but are free variables in the closed-loop system. These input variables can be used to cancel the disturbances in the closed-loop system. Exemplary, we consider the case $C = I$ in more detail. In this case, the condition $\operatorname{im} Q \subseteq \mathcal{V}^*_{[I,A,B,C]}$ from [44, Thm. 4.2] implies $Q = 0$, which may also be verified by investigating the solutions of*

$$\dot x = (A + BF)x + Qw, \quad y = x$$

for some feedback matrix $F \in \mathbb{R}^{m\times n}$. The output admits the representation

$$y(t) = e^{(A+BF)t}x(0) + \int_0^t e^{(A+BF)(t-s)}Qw(s)\,\mathrm{d}s$$

for all $t \in \mathbb{R}$, and is independent of w if, and only if, $Q = 0$. If we consider a behavioral feedback instead and choose $K = [K_1, K_2] = 0 \in \mathbb{R}^{0\times(n+m)}$, then the output of the corresponding closed-loop system $[I^K, A^K, Q^K, C^K]$, namely

$$\dot x = Ax + Bu + Qw, \quad y = x,$$

reads

$$y(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}\big(Bu(s) + Qw(s)\big)\,\mathrm{d}s, \quad t \in \mathbb{R},$$

and it is independent of w in the sence of Definition 2 if, and only if, for any w there exists u such that $Bu = -Qw$. This is equivalent to $\operatorname{im} Q \subseteq \operatorname{im} B$ or, what is the same, to the condition $\operatorname{im} Q \subseteq \mathcal{V}^*_{[I,A,B,C]} + A\mathcal{W}^*_{[I,A,B,C]} + \operatorname{im} B$ from Theorem 12.
However, this does not mean complete disturbance rejection, since a hidden dependence of y on w is still present, cf. also Example 3. This problem is treated in Section 5.

4 Disturbance decoupling and zero dynamics

The proof of Theorem 12 does not exploit any freedom in choosing the control $[K_1, K_2]$. In view of Proposition 7 the closed-loop system $[E^K, A^K, Q^K, C^K]$ has, for every smooth "input" w, a solution which generates zero output. We show that an appropriate additional condition yields uniqueness of this solution in the sense

$$\forall w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)\ \ \forall (z_1,w,0),(z_2,w,0) \in \mathfrak{B}_{[E^K,A^K,Q^K,C^K]} : \ E^K z_1(0) = E^K z_2(0) \implies z_1 \overset{\text{a.e.}}{=} z_2.$$

By linearity of the behavior the above statement is equivalent to

$$\forall (z,0,0) \in \mathfrak{B}_{[E^K,A^K,0,C^K]} : \ E^K z(0) = 0 \implies z \overset{\text{a.e.}}{=} 0$$

and therefore independent of the disturbance matrix Q. In fact, the above property means that the zero dynamics of $[E^K,A^K,0,C^K] \in \Sigma_{l+k,n+m,0,p}$ are autonomous. Loosely speaking, the zero dynamics are those dynamics which are not visible at the output. For ODE systems this concept has been introduced in [21]. The zero dynamics are, for $[E,A,B,C] \in \Sigma_{l,n,m,p}$, defined by

$$\mathcal{ZD}_{[E,A,B,C]} := \left\{ (x,u,y) \in \mathfrak{B}_{[E,A,B,C]} \ \middle|\ y \overset{\text{a.e.}}{=} 0 \right\}.$$

For linear DAE systems the zero dynamics have been well investigated, see [7–10]. The zero dynamics of (1) are called *autonomous*, if

$$\forall \zeta \in \mathcal{ZD}_{[E,A,B,C]}\ \forall I \subseteq \mathbb{R} \text{ open interval}: \ \zeta|_I \overset{\text{a.e.}}{=} 0 \implies \zeta \overset{\text{a.e.}}{=} 0.$$

The definition of autonomous zero dynamics is a special case of the definition of autonomy, as it has been introduced in [32, Sec. 3.2] for general behaviors.
Recall the following characterization of autonomous zero dynamics from [8, Prop. 3.5].

Lemma 14 *For $[E,A,B,C] \in \Sigma_{l,n,m,p}$ the zero dynamics $\mathcal{ZD}_{[E,A,B,C]}$ are autonomous if, and only if,*

$$\operatorname{rk}_{\mathbb{R}[s]} \begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix} = n+m.$$

In the following we show that under the condition in Theorem 12 we may choose $[K_1,K_2]$ such that, additionally to disturbance decoupling, we achieve autonomous zero dynamics of the undisturbed closed-loop system.

Theorem 15 *Let $[E,A,B,C] \in \Sigma_{l,n,m,p}$ and $Q \in \mathbb{R}^{l\times q}$. Then there exists a control $K = [K_1,K_2] \in \mathbb{R}^{k\times n}\times\mathbb{R}^{k\times m}$ compatible for $[E,A,B,C]$ such that $[E^K,A^K,Q^K,C^K]$ is disturbance decoupled and $\mathcal{ZD}_{[E^K,A^K,0,C^K]}$ are autonomous if, and only if,*

$$\operatorname{im} Q \subseteq E\mathcal{V}^*_{[E,A,B,C]} + A\mathcal{W}^*_{[E,A,B,C]} + \operatorname{im} B. \tag{11}$$

PROOF. $\Rightarrow$: Follows from Theorem 12.
$\Leftarrow$: *Step 1*: Let $T_1 \in \mathbb{R}^{n\times r}, T_2 \in \mathbb{R}^{n\times(n-r)}$ be such that $\operatorname{im} T_1 = \ker C$ and $[T_1,T_2]$ is invertible. As in the proof of Theorem 12 we may show that for any control $K = [K_1,K_2] \in \mathbb{R}^{k\times n}\times\mathbb{R}^{k\times m}$ compatible for $[E,A,B,C]$, the system $[E^K,A^K,Q^K,C^K]$ is disturbance decoupled if, and only if,

$$\forall w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)\ \exists v \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^{r+m}) : \ E_1^K \dot v = A_1^K v + \left[\begin{smallmatrix} Q \\ 0 \end{smallmatrix}\right] w, \tag{12}$$

where

$$[E_1^K, A_1^K] = \left[\begin{bmatrix} ET_1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} AT_1 & B \\ K_1T_1 & K_2 \end{bmatrix} \right].$$

In particular, condition (11) implies that

$$\forall w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)\ \exists (x_1,u) \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^r\times\mathbb{R}^m) : \ ET_1\dot x_1 = AT_1x_1 + Bu + Qw. \tag{13}$$

Furthermore, it is a straightforward calculation to see that for any $(z,u) \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^m)$ with $\left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right) := [T_1,T_2]^{-1}z$ we have

$$\begin{aligned} &\left(\left(\begin{smallmatrix} z \\ u \end{smallmatrix}\right),0,0\right) \in \mathcal{ZD}_{[E^K,A^K,0,C^K]} \\ \iff\ &\left(\left(\begin{smallmatrix} z_1 \\ u \end{smallmatrix}\right),0,0\right) \in \mathcal{ZD}_{[E_1^K,A_1^K,0,0]} \text{ and } z_2 = 0. \end{aligned}$$

Therefore, it suffices to find a control $K = [K_1,K_2] \in \mathbb{R}^{k\times n}\times\mathbb{R}^{k\times m}$ compatible for $[E,A,B,C]$ such that $[E_1^K,A_1^K,Q^K,0]$ is disturbance decoupled and $\mathcal{ZD}_{[E_1^K,A_1^K,0,0]}$ are autonomous.
Step 2: In order to construct K, let

$$E_1 := [ET_1, 0_{l\times m}], \quad A_1 := [AT_1, B],$$

and choose, according to [17, Cor. 2.3], $S \in \mathbf{Gl}_l(\mathbb{R})$ and $T \in \mathbf{Gl}_{r+m}(\mathbb{R})$ such that

$$S(sE_1 - A_1)T = \begin{bmatrix} sI_{n_1} - A_{11} & -A_{12} & 0 \\ 0 & 0 & sE_{23} - A_{23} \end{bmatrix},$$

where $A_{11} \in \mathbb{R}^{n_1\times n_1}, A_{12} \in \mathbb{R}^{n_1\times n_2}, E_{23}, A_{23} \in \mathbb{R}^{(l-n_1)\times n_3}$ with $\operatorname{rk}_{\mathbb{C}}(\lambda E_{23} - A_{23}) = n_3$ for all $\lambda \in \mathbb{C}$. Set

$$\begin{aligned} [\hat K_1, \hat K_2] &:= [0_{n_2\times n_1}, I_{n_2}, 0_{n_2\times n_3}]T^{-1} \in \mathbb{R}^{n_2\times r}\times\mathbb{R}^{n_2\times m}, \\ K_1 &:= [\hat K_1, 0_{n_2\times(n-r)}][T_1,T_2]^{-1}, \quad K_2 := \hat K_2. \end{aligned}$$

We show that $K = [K_1,K_2]$ satisfies the requirements.
Step 2a: To show compatibility of K, let

$$\begin{bmatrix} E_{14} \\ E_{24} \end{bmatrix} := SET_2, \quad \begin{bmatrix} A_{14} \\ A_{24} \end{bmatrix} := SAT_2,$$

and observe that K is compatible for $[E, A, B, C]$ if, and only if, $\tilde{K} := [0_{n_2 \times n_1}, I_{n_2}, 0_{n_2 \times n_3}, 0_{n_2 \times (n-r)}]$ is compatible for

$$[\tilde{E}, \tilde{A}, 0, C] := \left[\begin{bmatrix} I_{n_1} & 0 & 0 & E_{14} \\ 0 & 0 & E_{23} & E_{24} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & 0 & A_{14} \\ 0 & 0 & A_{23} & A_{24} \end{bmatrix}, 0, C \right].$$

To show the latter we let $z = (z_1, z_2, z_3, z_4) \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3} \times \mathbb{R}^{n-r})$ such that $\tilde{E}\dot{z} = \tilde{A}z$, i.e.,

$$\begin{aligned} \dot{z}_1 + E_{14}\dot{z}_4 &= A_{11}z_1 + A_{12}z_2 + A_{14}z_4, \\ E_{23}\dot{z}_3 + E_{24}\dot{z}_4 &= A_{23}z_3 + A_{24}z_4. \end{aligned}$$

Let $\tilde{z}_1 \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{n_1})$ be the solution of

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\tilde{z}_1 = A_{11}\tilde{z}_1 - E_{14}\dot{z}_4 + A_{14}z_4, \quad \tilde{z}_1(0) = z_1(0).$$

Then $\tilde{z} := (\tilde{z}_1, 0, z_3, z_4)$ solves $\frac{\mathrm{d}}{\mathrm{d}t}\tilde{E}\tilde{z} = \tilde{A}\tilde{z}$ under the constraint $\tilde{K}\tilde{z} = 0$, and we have

$$\tilde{E}z(0) = \begin{pmatrix} z_1(0) + E_{14}z_4(0) \\ E_{23}z_3(0) + E_{24}z_4(0) \end{pmatrix} = \tilde{E}\tilde{z}(0).$$

Step 2b: By Step 1 and Lemma 14 the zero dynamics $\mathcal{ZD}_{[E_1^K, A_1^K, 0, 0]}$ are autonomous if, and only if, $\mathrm{rk}_{\mathbb{R}[s]}(sE_1^K - A_1^K) = r + m$. To see that this holds, we observe that

$$\begin{aligned} &\mathrm{rk}_{\mathbb{R}[s]}(sE_1^K - A_1^K) \\ &= \mathrm{rk}_{\mathbb{R}[s]} \begin{bmatrix} sI_{n_1} - A_{11} & -A_{12} & 0 \\ 0 & 0 & sE_{23} - A_{23} \\ 0 & I_{n_2} & 0 \end{bmatrix} = r + m \end{aligned}$$

since

$$\mathrm{rk}_{\mathbb{R}[s]}(sE_{23} - A_{23}) = n_3.$$

Step 2c: We show (12) or, equivalently, that for all $w \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^q)$ there exists $(z_1, z_2, z_3) \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3})$ such that

$$\begin{aligned} \dot{z}_1 &= A_{11}z_1 + A_{12}z_2 + Q_1 w, \\ E_{23}\dot{z}_3 &= A_{23}z_3 + Q_2 w, \\ 0 &= z_2, \end{aligned}$$

where $\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} := SQ$. Since (13) holds due to assumption (11), it follows that there exists $z_3 \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{n_3})$ such that $E_{23}\dot{z}_3 = A_{23}z_3 + Q_2 w$. With z_1 being the solution of $\dot{z}_1 = A_{11}z_1 + Q_1 w$ we may conclude the proof of the theorem. □

Note that as a consequence of Theorem 15, for any $[E, A, B, C] \in \Sigma_{l,n,m,p}$ there always exists a control $K \in \mathbb{R}^{k \times (n+m)}$ compatible for $[E, A, B, C]$ such that $\mathcal{ZD}_{[E^K, A^K, 0, C^K]}$ are autonomous.

5 Lebret's open problem

Lebret [30] pointed out that solvability of the DDP with proportional state feedback does not guarantee disturbance rejection in general. This is still true when we consider behavioral feedback. The following example is taken from [30].

Example 16 *Consider the system*

$$[1, 0]\,\dot{x}(t) = [0, -1]\,x(t) + u(t) + w(t), \quad y(t) = [1, 0]x(t).$$

It is straightforward to check that the condition in Theorem 15 is satisfied and hence there exists a compatible control $[K_1, K_2]$ *which achieves disturbance decoupling and autonomous zero dynamics of the closed-loop system. For the present example we may choose, e.g.,*

$$K_1 = [0, 0], \quad K_2 = [1]$$

and hence the closed-loop system reads

$$\dot{x}_1(t) = -x_2(t) + w(t), \quad u(t) = 0, \quad y(t) = x_1(t).$$

However, y *still depends on* w *as*

$$y(t) = x_1(0) + \int_0^t \big(w(s) - x_2(s)\big)\,\mathrm{d}s\,, \quad t \in \mathbb{R},$$

but the disturbance is canceled by the free variable x_2 *in the sense that two different disturbances are not distinguishable at the output. The dependence of* y *on the disturbance is therefore hidden.*

To exclude a hidden dependence on the disturbance and achieve disturbance rejection, an additional assumption for disturbance decoupling suggested by Lebret [30] is uniqueness of the output of the closed-loop system. This justifies the following definition.

Definition 17 *Let* $[E, A, 0, C] \in \Sigma_{l,n,0,p}$ *and* $Q \in \mathbb{R}^{l \times q}$. *Then we call* $[E, A, Q, C]$ disturbance decoupled with output uniqueness (DDOU)*, if* $[E, A, Q, C]$ *is disturbance decoupled and*

$$\begin{aligned} &\forall\, w \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^q)\ \forall\, (x_1, w, y_1), (x_2, w, y_2) \\ &\qquad \in \mathfrak{B}_{[E,A,Q,C]} \cap \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^p) \\ &\forall\, I \subseteq \mathbb{R} \text{ open interval}: \quad y_1|_I = y_2|_I \implies y_1 = y_2. \end{aligned}$$

Note that by linearity, $[E,A,Q,C]$ is DDOU if, and only if, $[E,A,Q,C]$ is disturbance decoupled and

$$\forall\,(x,y)\in\mathfrak{B}_{[E,A,0_{l\times 0},C]}\cap\mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^p)\quad \forall\,I\subseteq\mathbb{R}\text{ open interval}:\quad y|_I=0\implies y=0.$$

Therefore, compared to disturbance decoupling, the additional condition of output uniqueness in the property DDOU is independent of Q.

In the context of feedback in the behavioral sense, and using the notation from Theorem 15, we may now seek a compatible control K such that $[E^K,A^K,Q^K,C^K]$ is DDOU and $\mathcal{ZD}_{[E^K,A^K,0,C^K]}$ are autonomous. Lebret [30] conjectures a characterization of this problem (without the additional property of the zero dynamics) where proportional state feedback $u=Fx$ is considered – a proof or counterexample to this conjecture has not been found so far. In Subsections 5.1 and 5.2 we derive two different solutions using the more general behavioral feedback. For an example where the DDP is not solvable by proportional state feedback, but the DDP with output uniqueness is solvable by behavioral feedback see Example 4.

First, we show that the problem of achieving a unique output is the same as the problem of achieving a unique state. The latter means that the underlying DAE is autonomous. To define this, we consider the set of homogeneous DAEs

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t)=Ax(t),\tag{14}$$

where $E,A\in\mathbb{R}^{l\times n}$, which is denoted by $\Sigma_{l,n}$ and we write $[E,A]\in\Sigma_{l,n}$. The behavior of $[E,A]\in\Sigma_{l,n}$ is given by

$$\mathfrak{B}_{[E,A]}:=\{x\in\mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}\to\mathbb{R}^n)\ \mid Ex\in\mathcal{AC}(\mathbb{R}\to\mathbb{R}^l),\ x\text{ satisfies (14) for almost all }t\in\mathbb{R}\ \},$$

Similar to autonomous zero dynamics, a DAE $[E,A]\in\Sigma_{l,n}$ is called *autonomous*, if

$$\forall\,x\in\mathfrak{B}_{[E,A]}\ \forall\,I\subseteq\mathbb{R}\text{ open interval}:\quad x|_I\overset{\text{a.e.}}{=}0\implies x\overset{\text{a.e.}}{=}0.$$

For characterizations of autonomy see also [14]. Here we need the following algebraic characterization which is an immediate consequence of [12, Cor. 5.2].

Lemma 18 *A DAE $[E,A]\in\Sigma_{l,n}$ is autonomous if, and only if,* $\mathrm{rk}_{\mathbb{R}[s]}(sE-A)=n$.

In the following result we show that a system is DDOU with autonomous zero dynamics if, and only if, it is disturbance decoupled and autonomous.

Proposition 19 *Let $[E,A,0,C]\in\Sigma_{l,n,0,p}$ and $Q\in\mathbb{R}^{l\times q}$. Then $[E,A,Q,C]$ is DDOU and $\mathcal{ZD}_{[E,A,0,C]}$ are autonomous if, and only if, $[E,A,Q,C]$ is disturbance decoupled and $[E,A]$ is autonomous.*

PROOF. $\Leftarrow$: First, by autonomy of $[E,A]$ and Lemma 18 we have

$$\mathrm{rk}_{\mathbb{R}[s]}(sE-A)=n,\quad\text{thus}\quad \mathrm{rk}_{\mathbb{R}[s]}\begin{bmatrix}sE-A\\-C\end{bmatrix}=n,$$

which, by Lemma 14 is equivalent to $\mathcal{ZD}_{[E,A,0,C]}$ being autonomous. Choose $V\in\mathbf{Gl}_p(\mathbb{R})$ such that $VC=\left[\begin{smallmatrix}C_1\\0\end{smallmatrix}\right]$ where $C_1\in\mathbb{R}^{p_1\times n}$ has full row rank. Then, invoking that $[E,A,Q,C]$ is disturbance decoupled, $[E,A,Q,C]$ is DDOU, if

$$\forall\,(x,y)\in\mathfrak{B}_{[E,A,0,C_1]}\cap\mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^{p_1})\quad \forall\,I\subseteq\mathbb{R}\text{ open interval}:\quad y|_I=0\Rightarrow y=0.$$

Clearly, $\mathcal{ZD}_{[E,A,0,C_1]}$ are autonomous as well, so we may apply [8, Thm. 4.3] to find $S\in\mathbf{Gl}_l(\mathbb{R})$, $T\in\mathbf{Gl}_n(\mathbb{R})$ such that

$$SET=\begin{bmatrix}I_{n_1}&0&0\\0&E_{32}&N\\0&E_{42}&E_{43}\end{bmatrix},\ SAT=\begin{bmatrix}A_{11}&A_{12}&0\\0&0&I_{n_3}\\0&A_{42}&0\end{bmatrix},\quad C_1T=[0,I_{p_1},0],\tag{15}$$

where $N\in\mathbb{R}^{n_3\times n_3}$ is nilpotent with $N^\nu=0$, $N^{\nu-1}\neq 0$. Seeking a contradiction, assume that there exists an open interval $I=(a,b)\subseteq\mathbb{R}$ and $(x,y)\in\mathfrak{B}_{[E,A,0,C_1]}\cap\mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^{p_1})$ such that $y|_I=0$ and $y\neq 0$. Let $(x_1^\top,y^\top,x_3^\top)^\top=T^{-1}x$, then

$$x_3=\sum_{i=0}^{\nu-1}E_{32}N^iy^{(i+1)}\quad\text{and}$$

$$x_1(t)=e^{A_{11}(t-a)}x_1(a)+\int_a^t e^{A_{11}(t-s)}A_{12}y(s)\,\mathrm{d}s$$

for all $t\in\mathbb{R}$, and hence $x_3|_I=0$. Since $\tilde{x}_1(t):=e^{A_{11}(t-a)}x_1(a)$, $t\in\mathbb{R}$, and $\tilde{x}:=T(x_1^\top,0,0)^\top$ satisfy $(\tilde{x},0)\in\mathfrak{B}_{[E,A,0,C_1]}$, by linearity of the behavior we find $(x-\tilde{x},y)\in\mathfrak{B}_{[E,A,0,C_1]}$ and we have $(x-\tilde{x})|_I=0$. Autonomy of $[E,A]$ now implies $x-\tilde{x}=0$, thus, in particular, $y=0$.

$\Rightarrow$: We only need to show that $[E,A]$ is autonomous. Again we assume that (15) holds for some invertible S and T. Then, invoking Lemma 18, $[E,A]$ is autonomous if, and only if,

$$\operatorname{rk}_{\mathbb{R}[s]} \begin{bmatrix} sI - A_{11} & -A_{12} & 0 \\ 0 & sE_{32} & sN - I \\ 0 & sE_{42} - A_{42} & sE_{43} \end{bmatrix} = n$$

$$\iff \operatorname{rk}_{\mathbb{R}[s]} \begin{bmatrix} sE_{32} & sN - I \\ sE_{42} - A_{42} & sE_{43} \end{bmatrix} = p_1 + n_3.$$

Seeking a contradiction, assume that $\operatorname{rk}_{\mathbb{R}[s]} \left[\begin{smallmatrix} sE_{32} & sN-I \\ sE_{42}-A_{42} & sE_{43} \end{smallmatrix}\right] < p_1 + n_3$ and hence the DAE $\left[\left[\begin{smallmatrix} E_{32} & N \\ E_{42} & E_{43} \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & I \\ A_{42} & 0 \end{smallmatrix}\right]\right]$ is not autonomous. Therefore, there exists $(y, x_3) \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{p_1} \times \mathbb{R}^{n_3})$ and an open interval $I \subseteq \mathbb{R}$ such that $(y, x_3)|_I = 0$ and $(y, x_3) \neq 0$. If $y = 0$, then $x_3 = \sum_{i=0}^{\nu-1} E_{32} N^i y^{(i+1)} = 0$ which cannot be, thus $y \neq 0$. Chosse $x_1 \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{n_1})$ such that $\dot{x}_1 = A_{11}x_1 + A_{12}y$, then, with $x := T(x_1^\top, y^\top, x_3^\top)^\top$ we have $(x, y) \in \mathfrak{B}_{[E,A,0,C_1]} \cap \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^{p_1})$. Since $y|_I = 0$ and $y \neq 0$ this contradicts the assumption of unique output. □

Using state feedback $u = Fx$, the problem of finding F such that the closed-loop system is disturbance decoupled and the output is unique has been called DDP in [30] and the problem of finding F such that the closed-loop system is autonomous and disturbance decoupled has been called DDPU in [2,30]. Proposition 19 shows that in the case of behavioral feedback the DDP (where we can always achieve autonomous zero dynamics) is equivalent to the DDPU.

In the remainder of this section we present two different solutions to the DDP with unique output. In the first approach we seek for an additional condition on $[E, A, B, C]$ and Q compared to that in Theorem 15 that ensures autonomy of the closed-loop system. In the second approach, we keep the condition from Theorem 15 and relax the assumption on the behavioral control K. Instead of requiring K to be compatible for $[E, A, B, C]$, we require K to be compatible for the system which consists of those trajectories which produce zero output.

5.1 Solution by additional assumption

We start with a characterization of all compatible controls K which render $[E^K, A^K]$ autonomous.

Proposition 20 *Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ and let $K \in \mathbb{R}^{k\times(n+m)}$. Choose, according to [17, Cor. 2.3], $S \in \mathbf{Gl}_l(\mathbb{R})$ and $T \in \mathbf{Gl}_{n+m}(\mathbb{R})$ such that*

$$S[sE - A, -B]T = \begin{bmatrix} sI_{n_1} - A_{11} & -A_{12} & 0 \\ 0 & 0 & sE_{22} - A_{22} \end{bmatrix}, \tag{16}$$

where $A_{11} \in \mathbb{R}^{n_1\times n_1}, A_{12} \in \mathbb{R}^{n_1\times n_2}, E_{22}, A_{22} \in \mathbb{R}^{(l-n_1)\times n_3}$ with $\operatorname{rk}_{\mathbb{C}}(\lambda E_{22} - A_{22}) = n_3$ for all $\lambda \in \mathbb{C}$, and let $KT = [K_1, K_2, K_3]$ according to the partitioning in (16). Then K is a compatible control for $[E, A, B, C]$ such that $[E^K, A^K]$ is autonomous if, and only if, $\operatorname{im} K_1 \subseteq \operatorname{im} K_2$ and K_2 has full column rank n_2.

PROOF. $\Rightarrow$: Let $x_1 \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^{n_1})$ be such that $\dot{x}_1 = A_{11}x_1$. Then $(x, u) := T(x_1^\top, 0, 0)^\top$ satisfies $(x, u, Cx) \in \mathfrak{B}_{[E,A,B,C]}$. Since K is compatible for $[E, A, B, C]$ there exists $(\tilde{x}, \tilde{u}) \in \mathfrak{B}_{[E^K,A^K,0,0]} \cap \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m)$ such that $Ex(0) = E\tilde{x}(0)$. Write $(\tilde{x}_1^\top, \tilde{x}_2^\top, \tilde{x}_3^\top)^\top = T^{-1}\tilde{x}$ according to the decomposition (16). If follows from the condition $\operatorname{rk}_{\mathbb{C}}(\lambda E_{22} - A_{22}) = n_3$ for all $\lambda \in \mathbb{C}$ that $x_3 = 0$, see e.g. [16, Thm. 3.2]. Then $Ex(0) = E\tilde{x}(0)$ if, and only if, $x_1(0) = \tilde{x}_1(0)$. As furthermore

$$K_1\tilde{x}_1(0) + K_2\tilde{x}_2(0) = 0$$

and $x_1(0)$ is arbitrary it follows that $\operatorname{im} K_1 \subseteq \operatorname{im} K_2$. Then, in particular, there exists $Z \in \mathbb{R}^{n_2\times n_1}$ such that $K_1 = K_2 Z$.

In order to show that K_2 has full column rank, we assume that $\operatorname{rk} K_2 < n_2$. Note that we have

$$\begin{bmatrix} S & 0 \\ 0 & I_k \end{bmatrix} (sE^K - A^K)T = \left[\begin{smallmatrix} sI_{n_1} - A_{11} & -A_{12} & 0 \\ 0 & 0 & sE_{22} - A_{22} \\ -K_2 Z & -K_2 & -K_3 \end{smallmatrix}\right].$$

If $\operatorname{rk} \left[\begin{smallmatrix} A_{12} \\ K_2 \end{smallmatrix}\right] < n_2$, then there exists $y \in \mathbb{R}^{n_2}$ such that $A_{12}y = 0$ and $K_2 y = 0$. Therefore,

$$\left[\begin{smallmatrix} sI_{n_1} - A_{11} & -A_{12} & 0 \\ 0 & 0 & sE_{22} - A_{22} \\ -K_2 Z & -K_2 & -K_3 \end{smallmatrix}\right] \left(\begin{smallmatrix} 0 \\ y \\ 0 \end{smallmatrix}\right) = 0,$$

which contradicts the fact that $\operatorname{rk}_{\mathbb{R}[s]}(sE^K - A^K) = n + m$ by autonomy of $[E^K, A^K]$ and Lemma 18. Therefore, assume that $\operatorname{rk} \left[\begin{smallmatrix} A_{12} \\ K_2 \end{smallmatrix}\right] = n_2$ and let $S = \left[\begin{smallmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{smallmatrix}\right] \in \mathbf{Gl}_{n_1+k}(\mathbb{R})$, where $S_{11} \in \mathbb{R}^{n_2\times n_1}, S_{22} \in \mathbb{R}^{(n_1+k-n_2)\times k}$ and S_{12}, S_{21} are of appropriate sizes, be such that

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} A_{12} \\ K_2 \end{bmatrix} = \begin{bmatrix} I_{n_2} \\ 0 \end{bmatrix}.$$

We show that $\ker S_{21} \neq \{0\}$ by contradiction. So assume that $\ker S_{21} = \{0\}$. Then $S_{21}A_{12} + S_{22}K_2 = 0$ implies

$$A_{12} = -(S_{21}^\top S_{21})^{-1} S_{21}^\top S_{22} K_2,$$

and, since $\ker K_2 \neq \{0\}$, we arrive at

$$\ker \begin{bmatrix} A_{12} \\ K_2 \end{bmatrix} = \ker \begin{bmatrix} -(S_{21}^\top S_{21})^{-1} S_{21}^\top S_{22} K_2 \\ K_2 \end{bmatrix} \neq \{0\},$$

a contradiction. Now let $v \in \ker S_{21} \setminus \{0\}$ and set $p_1(s) := \big(sI - (A_{11} - A_{12}Z)\big)^{-1} v \in \mathbb{R}(s)^{n_1} \setminus \{0\}$. Then

$$
\begin{aligned}
& S_{21}\big(sI - (A_{11} - A_{12}Z)\big)p_1(s) = 0 \\
\iff \quad & S_{21}(sI - A_{11})p_1(s) + S_{21}A_{12}Zp_1(s) = 0 \\
\overset{\substack{S_{21}A_{21} \\ +S_{22}K_2 = 0}}{\iff} \quad & S_{21}(sI - A_{11})p_1(s) - S_{22}K_2Zp_1(s) = 0 \\
\iff \quad & [S_{21}, S_{22}] \begin{bmatrix} sI - A_{11} \\ -K_2Z \end{bmatrix} p_1(s) = 0.
\end{aligned}
$$

Set $p_2(s) := [S_{11}, S_{12}] \left[\begin{smallmatrix} sI-A_{11} \\ -K_2Z \end{smallmatrix}\right] p_1(s) \in \mathbb{R}(s)^{n_2}$ and $p_3(s) := 0 \in \mathbb{R}(s)^{n_3}$. Then

$$
\begin{aligned}
& \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} sI - A_{11} \\ -K_2Z \end{bmatrix} p_1(s) - \begin{bmatrix} I_{n_2} \\ 0 \end{bmatrix} p_2(s) = 0 \\
\iff \quad & \begin{bmatrix} sI - A_{11} \\ -K_2Z \end{bmatrix} p_1(s) - \begin{bmatrix} A_{12} \\ K_2 \end{bmatrix} p_2(s) = 0 \\
\iff \quad & \begin{bmatrix} sI - A_{11} & -A_{12} & 0 \\ 0 & 0 & sE_{22} - A_{22} \\ -K_2Z & -K_2 & -K_3 \end{bmatrix} \begin{pmatrix} p_1(s) \\ p_2(s) \\ p_3(s) \end{pmatrix} = 0,
\end{aligned}
$$

and hence $\mathrm{rk}_{\mathbb{R}[s]}(sE^K - A^K) = \mathrm{rk}_{\mathbb{R}(s)}(sE^K - A^K) < n + m$, a contradiction.

$\Leftarrow$: Let $Z \in \mathbb{R}^{n_2 \times n_1}$ be such that $K_1 = K_2Z$. First we show that K is compatible for $[E, A, B, C]$. Let $(x_1, x_2) \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ be such that $\dot{x}_1 = A_{11}x_1 + A_{12}x_2$. Then, for the solution $\tilde{x}_1$ of

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\tilde{x}_1 = (A_{11} - A_{12}Z)\tilde{x}_1, \quad \tilde{x}_1(0) = x_1(0),$$

and $\tilde{x}_2 := -Z\tilde{x}_1$ we find

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\tilde{x}_1 = A_{11}\tilde{x}_1 + A_{12}\tilde{x}_2, \quad K_1\tilde{x}_1 + K_2\tilde{x}_2 = 0,$$

and, furthermore,

$$[I_{n_1}, 0] \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = [I_{n_1}, 0] \begin{pmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{pmatrix},$$

which proves that $[K_1, K_2, K_3]$ is compatible for $[E, A, B, C]$. For autonomy of $[E^K, A^K]$, by Lemma 18 it remains to show that

$$\mathrm{rk}_{\mathbb{R}[s]} \begin{bmatrix} sI - A_{11} & -A_{12} & 0 \\ 0 & 0 & sE_{22} - A_{22} \\ -K_2Z & -K_2 & -K_3 \end{bmatrix} = n + m$$

or, what is the same because of the rank property of $sE_{22} - A_{22}$, that

$$\ker_{\mathbb{R}[s]} \begin{bmatrix} sI - A_{11} & -A_{12} \\ -K_2Z & -K_2 \end{bmatrix} = \{0\}.$$

Let $p_1(s) \in \mathbb{R}[s]^{n_1}, p_2(s) \in \mathbb{R}[s]^{n_2}$ be such that

$$\begin{bmatrix} sI - A_{11} & -A_{12} \\ -K_2Z & -K_2 \end{bmatrix} \begin{pmatrix} p_1(s) \\ p_2(s) \end{pmatrix} = 0.$$

Since K_2 has full column rank it follows $p_2(s) = -Zp_1(s)$. Therefore,

$$0 = (sI - A_{11})p_1(s) + A_{12}p_2(s) = \big(sI - (A_{11} - A_{12}Z)\big)p_1(s)$$

and hence $p_1(s) = 0$ and $p_2(s) = 0$. This completes the proof. $\square$

We are now in the position to prove the main result of this section, which uses the conditions (D1) and (D2) from Figure 2 applied to the decomposition (16), where the free variables are revealed as those variables corresponding to the second column. Then a state feedback with respect to the free variables is used to obtain disturbance decoupling; this feedback can be reformulated as a behavioral feedback for the original system and renders the closed-loop system autonomous.

Theorem 21 *Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ and $Q \in \mathbb{R}^{l \times q}$. Choose $S \in \mathbf{Gl}_l(\mathbb{R})$, $T \in \mathbf{Gl}_{n+m}(\mathbb{R})$ such that (16) holds and let, accordingly, $SQ = \left[\begin{smallmatrix} Q_1 \\ Q_2 \end{smallmatrix}\right]$ and $[C, 0]T = [C_1, C_2, C_3]$. Define*

$$
[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{Q}] := \left[\begin{bmatrix} I_{n_1} & 0 \\ 0 & E_{22} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \\ C_1 & C_3 \end{bmatrix}, \begin{bmatrix} A_{12} \\ 0 \\ C_2 \end{bmatrix}, \begin{bmatrix} Q_1 \\ Q_2 \\ 0 \end{bmatrix} \right]
$$

and

$$\tilde{\mathcal{R}}^* := \mathcal{V}^*_{[\tilde{E}, \tilde{A}, [\tilde{B}, \tilde{Q}], 0]} \cap \mathcal{W}^*_{[\tilde{E}, \tilde{A}, [\tilde{B}, \tilde{Q}], 0]}.$$

Then there exists a control $K \in \mathbb{R}^{k \times (n+m)}$ compatible for $[E, A, B, C]$ such that $[E^K, A^K, Q^K, C^K]$ is disturbance decoupled and $[E^K, A^K]$ is autonomous if, and only if,

(i) $\mathrm{im}\, \tilde{Q} \subseteq \tilde{E}\mathcal{V}^*_{[\tilde{E}, \tilde{A}, \tilde{B}, 0]} + \tilde{A}\mathcal{W}^*_{[\tilde{E}, \tilde{A}, \tilde{B}, 0]} + \mathrm{im}\, \tilde{B}$,

(ii) $\dim \big(\tilde{E}\tilde{\mathcal{R}}^* + \mathrm{im}\, \tilde{Q}\big) \leq \dim \tilde{\mathcal{R}}^*$.

PROOF. $\Rightarrow$: Let $KT = [K_1, K_2, K_3]$ according to the decomposition (16). From Proposition 7 we may deduce that

$$\begin{aligned}
&\forall\, w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)\\
&\quad \exists\,(x_1,x_2,x_3) \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3}):\\
&\qquad\qquad \dot x_1 = A_{11}x_1 + A_{12}x_2 + Q_1 w\\
&\qquad\quad E_{22}\dot x_3 = A_{22}x_3 + Q_2 w\\
&\qquad\qquad\quad 0 = C_1x_1 + C_2x_2 + C_3x_3\\
&\qquad\qquad\quad 0 = K_1x_1 + K_2x_2 + K_3x_3.
\end{aligned} \tag{17}$$

Proposition 20 gives that $\operatorname{im} K_1 \subseteq \operatorname{im} K_2$ and K_2 has full column rank n_2. Therefore, there exists $V \in \mathbf{Gl}_k(\mathbb{R})$ such that

$$V[K_1,K_2,K_3] = \begin{bmatrix} F_1 & I_{n_2} & F_2\\ 0 & 0 & F_3\end{bmatrix}$$

with $F_1 \in \mathbb{R}^{n_2\times n_1}, F_2 \in \mathbb{R}^{n_2\times n_3}, F_3 \in \mathbb{R}^{(k-n_2)\times n_3}$. Hence,

$$x_2 = -F_1x_1 - F_2x_3, \quad F_3x_3 = 0,$$

and in particular

$$\begin{aligned}
&\forall\, w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)\ \exists\,(x_1,x_3) \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^{n_1}\times\mathbb{R}^{n_3}):\\
&\quad \dot x_1 = (A_{11}-A_{12}F_1)x_1 - A_{12}F_2x_3 + Q_1w\\
&E_{22}\dot x_3 = A_{22}x_3 + Q_2w\\
&\qquad 0 = (C_1 - C_2F_1)x_1 + (C_3 - C_2F_2)x_3.
\end{aligned} \tag{18}$$

This means that for $F := [F_1, F_2]$ the system $[\tilde E, \tilde A - \tilde B F, \tilde Q, 0]$ is disturbance decoupled. Then [2, Thm. 5.2] implies (i) and (ii), cf. also Figure 2.
$\Leftarrow$: By [2, Thm. 5.2], conditions (i) and (ii) imply that there exists $F = [F_1, F_2] \in \mathbb{R}^{n_2\times n_1}\times\mathbb{R}^{n_2\times n_3}$ such that (18) holds. Define

$$K_1 := F_1,\ K_2 := I_{n_2},\ K_3 := F_2,$$

then (17) holds and hence Proposition 7 yields that, for $K := [K_1,K_2,K_3]T^{-1}$, $[E^K, A^K, Q^K, C^K]$ is disturbance decoupled. Compatibility of K and autonomy of $[E^K, A^K]$ is an immediate consequence of the choice of K and Proposition 20. □

Remark 22

a) The conditions (i) and (ii) in Theorem 21 in particular imply condition (11).

b) The proof of Theorem 21 is constructive and the behavioral feedback K can be computed as follows: To obtain the transformation to the form (16) *the (numerically stable) staircase algorithm, see [4, 36], can be used. This algorithm in particular ensures that S and T are orthogonal. The state feedback matrix F for the replacement system $[\tilde E, \tilde A, \tilde B, 0]$ with disturbance matrix $\tilde Q$ can then be computed as described in [2, Rem. 5.1].*

5.2 Solution by relaxing compatibility

In this subsection we present a different solution for DDOU which only needs the condition (11). However, the drawback is that the control is not compatible in general. This is a trade-off between requirements on the data and properties of the control. For a motivation we revisit Example 16.

Example 23 *Use the notation from Example 16. By implementing the condition $y = 0$ as an additional constraint in the system itself, i.e., extending $[K_1, K_2]$ to*

$$K_1 = \begin{bmatrix} 0 & 0\\ 1 & 0\end{bmatrix}, \quad K_2 = \begin{bmatrix} 1\\ 0\end{bmatrix}$$

we achieve the closed-loop system

$$0 = x_2(t) + w(t), \quad x_1(t) = u(t) = 0, \quad y(t) = 0,$$

where now the output is independent of the disturbance. Furthermore, we still have existence of a solution for every $w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R})$. However, the control $[K_1, K_2]$ is not compatible anymore for the system $[E, A, B, C]$, but it is compatible for the system $[\left[\begin{smallmatrix}E\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}A\\C\end{smallmatrix}\right], \left[\begin{smallmatrix}B\\0\end{smallmatrix}\right], 0]$.

Example 23 motivates to relax the assumption of compatibility of the control K. This may be justified by the fact that for disturbance decoupling only solutions (x, u) of the disturbed system with $Cx = 0$ are considered, cf. Proposition 7. Therefore, it is sometimes sufficient to restrict the compatibility of K to those solution trajectories, i.e., only require K to be compatible for $[\left[\begin{smallmatrix}E\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}A\\C\end{smallmatrix}\right], \left[\begin{smallmatrix}B\\0\end{smallmatrix}\right], 0]$. In other words, this means that K is a compatible control for the zero dynamics $\mathcal{ZD}_{[E,A,B,C]}$. The above motivation justifies the following ansatz for K,

$$K = \begin{bmatrix} C & 0\\ Z_1 & Z_2\end{bmatrix},$$

for $Z_1 \in \mathbb{R}^{k\times n}, Z_2 \in \mathbb{R}^{k\times m}$. The idea is that putting C into the control K does not change solvability of the DDP since the constraint $Cx = 0$ is present anyway. Furthermore, this "superfluous" constraint makes it easier rather than harder to find Z_1 and Z_2 such that $[E^K, A^K]$ is autonomous. This structure of K allows to derive some crucial connections.

Proposition 24 *Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ and $Q \in \mathbb{R}^{l\times q}$. There exists a control $K \in \mathbb{R}^{k\times(n+m)}$ compatible for $[\left[\begin{smallmatrix}E\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}A\\C\end{smallmatrix}\right], \left[\begin{smallmatrix}B\\0\end{smallmatrix}\right], 0]$ such that $[E^K, A^K, Q^K, C^K]$ is disturbance decoupled and $[E^K, A^K]$ is autonomous if, and only if, there exists a control $[Z_1, Z_2] \in \mathbb{R}^{k\times n}\times\mathbb{R}^{k\times m}$ compatible for $[\left[\begin{smallmatrix}E\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}A\\C\end{smallmatrix}\right], \left[\begin{smallmatrix}B\\0\end{smallmatrix}\right], 0]$ such that, with $\tilde K =$*

$\left[\begin{smallmatrix} C & 0 \\ Z_1 & Z_2 \end{smallmatrix}\right]$, $[E^{\tilde K}, A^{\tilde K}, Q^{\tilde K}, 0]$ *is disturbance decoupled and* $[E^{\tilde K}, A^{\tilde K}]$ *is autonomous.*

PROOF. For $\Rightarrow$ set $[Z_1, Z_2] = K$ and for $\Leftarrow$ set $K = \left[\begin{smallmatrix} C & 0 \\ Z_1 & Z_2 \end{smallmatrix}\right]$. The remaining calculations are simple and straightforward. $\square$

Under this relaxed compatibility assumption on K the DDP with output uniqueness is solvable if, and only if, the DDP is solvable. That is, using this larger class of controls, the output uniqueness (in fact, the state uniqueness by Proposition 19) can always be satisfied when disturbance decoupling can be achieved. In this sense, the disturbance decoupling problem (called IDDP in [30]), the disturbance decoupling problem with state uniqueness (called DDPU in [30]) and the disturbance decoupling problem with output uniqueness (called DDP in [30]) are all equally hard problems.

Theorem 25 *Let* $[E, A, B, C] \in \Sigma_{l,n,m,p}$ *and* $Q \in \mathbb{R}^{l\times q}$. *Then there exists a control* $K \in \mathbb{R}^{k\times(n+m)}$ *compatible for* $[\left[\begin{smallmatrix} E \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right], \left[\begin{smallmatrix} B \\ 0 \end{smallmatrix}\right], 0]$ *such that* $[E^K, A^K, Q^K, C^K]$ *is disturbance decoupled and* $[E^K, A^K]$ *is autonomous if, and only if,*

$$\operatorname{im} Q \subseteq E\mathcal{V}^*_{[E,A,B,C]} + A\mathcal{W}^*_{[E,A,B,C]} + \operatorname{im} B.$$

PROOF. By Proposition 24 the problem of finding K is equivalent to finding a control $Z = [Z_1, Z_2] \in \mathbb{R}^{k\times n} \times \mathbb{R}^{k\times m}$ compatible for $[\left[\begin{smallmatrix} E \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right], \left[\begin{smallmatrix} B \\ 0 \end{smallmatrix}\right], 0]$ such that, with $\tilde K = \left[\begin{smallmatrix} C & 0 \\ Z_1 & Z_2 \end{smallmatrix}\right]$, $[E^{\tilde K}, A^{\tilde K}, Q^{\tilde K}, 0]$ is disturbance decoupled and $[E^{\tilde K}, A^{\tilde K}]$ is autonomous. Observe that $[E^{\tilde K}, A^{\tilde K}, Q^{\tilde K}, 0]$ is disturbance decoupled if, and only if, $[E^Z, A^Z, Q^Z, C^Z]$ is disturbance decoupled. Furthermore, $[E^{\tilde K}, A^{\tilde K}]$ is autonomous if, and only if,

$$\operatorname{rk}_{\mathbb{R}[s]} \begin{bmatrix} sE - A & -B \\ -C & 0 \\ -Z_1 & -Z_2 \end{bmatrix} = n + m$$

and this is equivalent to $\mathcal{ZD}_{[E^Z,A^Z,0,C^Z]}$ being autonomous. Therefore, the problem of finding K is equivalent to finding a compatible Z such that $[E^Z, A^Z, Q^Z, C^Z]$ is disturbance decoupled and $\mathcal{ZD}_{[E^Z,A^Z,0,C^Z]}$ are autonomous. The assertion now follows from Theorem 15. $\square$

6 Conclusions

In the present paper we have shown that the use of proportional state feedback is limited for DAE systems, as actually a feedback in terms of the free variables of the system is needed. This is possible when a behavioral feedback is used. We have derived a geometric characterization for solvability of the DDP by behavioral feedback. Exploiting the freedom in the choice of the behavioral feedback we have shown that whenever disturbance decoupling can be achieved we may additionally achieve autonomous zero dynamics. Furthermore, the behavioral feedback approach allowed us to solve Lebret's twenty year old open problem [30] of disturbance decoupling with output uniqueness.

The behavioral feedback approach to disturbance decoupling presented in this paper opens the door for the study of various related problems and extensions, among them disturbance decoupled state estimation and disturbance decoupling by dynamic feedback controllers using behavioral feedback as well as almost disturbance decoupling by behavioral feedback. One may consider cases where the disturbances influence the measurement and controlled output, resp., and study the additional stabilization of the closed-loop system. In the absence of disturbances, some of these problems have already been treated using the framework of behavioral feedback, see [7,12,13]. In the present paper we took the first step at incorporating disturbance decoupling.

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