On observers for nonlinear differential-algebraic systems

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Abstract—**We extend a recent approach to observer design for linear differential-algebraic systems to systems with Lipschitz nonlinearities. The observer design further extends the standard Luenberger type observer design. We show that the design parameters for the observer can be obtained by the solution of a Riccati type inequality. The solutions of the latter can in turn be obtained by solving a set of LMIs and BMIs which provides a computational procedure. A feature of the presented observer design is the possibility of reformulation as an ordinary differential equation.**

Index Terms—**Differential-algebraic systems, nonlinear systems, observers, Riccati inequality, LMIs.**

Nomenclature:

$\mathbb{R}^{n\times m}$	the set of real $n\times m$ matrices
$\mathrm{rk}A$, $\mathrm{im}A$	rank and image of $A\in\mathbb{R}^{n\times m}$
$\mathbf{Gl}_n(\mathbb{R})$	the group of invertible matrices in $\mathbb{R}^{n\times n}$
$M>_{\mathscr{V}}0$	$:\iff \forall x\in\mathscr{V}\setminus\{0\}:\ x^\top Mx>0$, for a matrix $M\in\mathbb{R}^{n\times n}$ and a subspace $\mathscr{V}\subseteq\mathbb{R}^n$
$\mathscr{C}^k(X\to Y)$	set of k-times continuously differentiable functions $f:X\to Y$, $k\in\mathbb{N}_0$; $\mathscr{C}(X\to Y):=\mathscr{C}^0(X\to Y)$
dom f	the domain of the function f
$f\|_I$	restriction of the function f to the set I

I. Introduction

We study observer design for nonlinear systems governed by differential-algebraic equations (DAEs). We follow the recent approach to observer design developed in [1] for linear DAE systems. In the main result in Theorem III.1 we show that an asymptotic observer can be designed whenever a certain Riccati inequality is solvable. We later show that solvability of certain linear and bilinear matrix inequalities (LMIs and BMIs) is sufficient for solvability of the Riccati inequality.

We consider DAE systems of the form

$$\begin{aligned}\dot{x}_1(t)&=Ax_1(t)+Bx_2(t)+f_1\big(x_1(t),x_2(t),u(t),y(t)\big),\\ 0&=Cx_1(t)+Dx_2(t)+f_2\big(u(t),y(t)\big),\\ y(t)&=Fx_1(t)+Gx_2(t)+h\big(u(t)\big),\end{aligned}\tag{1}$$

with $A\in\mathbb{R}^{r\times r}$ and all other matrices of appropriate sizes so that

$$\begin{bmatrix}A & B\\ C & D\end{bmatrix}\in\mathbb{R}^{l\times n},\quad [F,G]\in\mathbb{R}^{p\times n},$$

where we assume that

$$\mathrm{rk}\begin{bmatrix}D\\ G\end{bmatrix}=n-r.\tag{2}$$

Furthermore, $f_1\in\mathscr{C}^1(X_1\times X_2\times\mathbb{R}^m\times\mathbb{R}^p\to\mathbb{R}^r)$, $f_2\in\mathscr{C}^1(\mathbb{R}^m\times\mathbb{R}^p\to\mathbb{R}^{l-r})$, $h\in\mathscr{C}^1(\mathbb{R}^m\to\mathbb{R}^p)$, where $X_1\subseteq\mathbb{R}^r, X_2\subseteq\mathbb{R}^{n-r}$ are open, such that the following Lipschitz condition is satisfied:

$$\begin{aligned}&\exists L>0\ \forall\ (x_1^i,x_2^i)\in X_1\times X_2,\ i=1,2\ \forall\ (u,y)\in\mathbb{R}^m\times\mathbb{R}^p:\\ &\|f_1(x_1^1,x_2^1,u,y)-f_1(x_1^2,x_2^2,u,y)\|\le L\|(x_1^1-x_1^2,x_2^1-x_2^2)\|.\end{aligned}\tag{3}$$

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The functions $u:I\to\mathbb{R}^m$ and $y:I\to\mathbb{R}^p$ are called *input* and *output* of the system, resp. Note that although y from the last equation in (1) may be inserted in the first equation which may hence be written in the form

$$\dot{x}_1(t)=Ax_1(t)+Bx_2(t)+\tilde{f}_1\big(x_1(t),x_2(t),u(t)\big),$$

this would be a smaller class of systems since we would need to require $\tilde{f}_1$ to be Lipschitz continuous w.r.t. x_1 and x_2, while f_1 does not need to be Lipschitz w.r.t. y.

The system class (1) includes any linear DAE system and numerous important classes of nonlinear DAE systems (e.g. chemical process systems [2], mechanical systems [3], [4] and modified nodal analysis models of electrical circuits [5]). Nonlinear DAE systems seem to have been first considered by Luenberger [6]; see also the textbooks [7], [8] and the recent works [9], [10].

Observer design for systems of ordinary differential equations (ODEs) with Lipschitz nonlinearities is well studied, see e.g. [11]–[14] and also the recent paper [15]. The design of (asymptotic) observers for DAE systems similar to (1) has been studied in [16]–[19]. In [16] the regularity of the linear part is assumed, while in [17] it only needs to be square and in this way quite general results are obtained. A unified approach is presented, where existence of the designed observer is shown to depend on the solvability of certain LMIs. Due to the allowed Lipschitz continuity of the nonlinearities it is clear that, compared to the linear part, the Lipschitz constant must be small enough for an asymptotic observer to exist (e.g. when the linear part is detectable, but unobservable); this can be made precise in terms of the solvability of LMIs. A similar approach has been taken before in [20], [21] for nonlinear ODE systems with unknown inputs, which may be treated as DAE systems (1) as well. Recently, the approach from [17] has been extended in [18], where actuator and sensor faults (similar to [16]) as well as uncertainties are incorporated, and in [19], where the Lipschitz nonlinearities are also allowed in the output equation. Furthermore, the observer design in [18] and [19] additionally requires the solvability of certain BMIs. Different approaches are taken in [22], where the system is completely nonlinear, but semi-explicit and of index 1, and a nonlinear observer is constructed, and in [23] where a nonlinear generalized PI observer design is used, see also the references therein.

A drawback of the approach in [17] is that, as pointed out in [24], the constructed observer is again a DAE system which cannot necessarily be reformulated as an ODE. As discussed in [25], it is then possible that the observer dynamics involve derivatives of the inputs and/or outputs, which would lead to an ill-posed problem; this is not desirable from a practical point of view. Furthermore, ODE observers have the advantage that they can be initialized without any further restriction. The existence of ODE observers for general linear DAE systems is investigated in [25], [26].

In the present paper we present a novel observer design which is based on the recent approach in [1] and extends the standard Luenberger type observer design presented in [17], [24]. Compared to [17] we allow for systems which are not necessarily square (i.e., $l\neq n$ is possible) and at the same time we guarantee that the observer can be reformulated as an ODE system. Compared to [24] we present a more general observer design. We stress that there exist systems for which the Luenberger type observer presented in [17] cannot be reformulated as an ODE system, but the observer design presented here can be reformulated as an ODE; an example is given in Section VI-A. Therefore, we present a novel observer design that resolves the limitations of Luenberger type observers.

The features of our approach are as follows:

- we do not restrict ourselves to square systems (a different number of state variables and equations, i.e., $l \neq n$, is allowed),
- the observers can always be reformulated as ODE systems, even when the standard Luenberger type observer cannot be reformulated as an ODE,
- the observer design reduces to Luenberger type observers only in special cases.

In order to achieve an observer design which can be reformulated as an ODE, while at the same time the system does not need to be square, the rank condition (2) will be of importance. Applications of the observer design are for instance error detection and fault diagnosis, disturbance (or unknown input) estimation and feedback control, see e.g. [27], [28].

The present paper is organized as follows: In Section II we recall some basic definitions and concepts. The novel observer design is presented in Section III and we prove in the main result Theorem III.1 that it works provided a certain Riccati type inequality is solvable. In Section IV we show that solvability of certain LMIs and BMIs is sufficient for solvability of the Riccati inequality. The case where the presented observer design reduces to standard Luenberger type observers is discussed in Section V along with consequences thereof. Some illustrative examples are given in Section VI.

II. Preliminaries

In order to define (asymptotic) observers for nonlinear DAE systems we consider the general class of nonlinear systems governed by DAEs of the form

$$\begin{aligned} E\dot{x}(t) &= f\big(x(t),u(t),y(t)\big), \\ y(t) &= h\big(x(t),u(t)\big), \end{aligned} \tag{4}$$

where $X \subseteq \mathbb{R}^n$ is open, $f \in \mathscr{C}^1(X \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^l)$, $h \in \mathscr{C}^1(X \times \mathbb{R}^m \to \mathbb{R}^p)$ and $E \in \mathbb{R}^{l\times n}$. Since solutions not necessarily exist globally (e.g. finite escape times may arise) we consider local solutions of (4). A trajectory $(x,u,y) \in \mathscr{C}(I \to X \times \mathbb{R}^m \times \mathbb{R}^p)$ is called a *solution* of (4), if $I = \operatorname{dom} x \subseteq \mathbb{R}$ is an open interval, $x \in \mathscr{C}^1(I \to \mathbb{R}^l)$ and (x,u,y) solves (4) for all $t \in I$. Note that the interval of definition I of a solution of (4) depends on the choice of the input u and that a solution does not need to be maximal. Furthermore, for a given initial condition $Ex(t_0) = Ex^0$ a solution does not necessarily exist, and even if it exists it does not need to be unique. The *behavior* of (4) is defined as the set of all possible solution trajectories

$$\begin{aligned} \mathfrak{B}_{(4)} := \{\, (x,u,y) \in \mathscr{C}(I \to X \times \mathbb{R}^m \times \mathbb{R}^p) \mid I \subseteq \mathbb{R} \text{ open interval}, \\ (x,u,y) \text{ is a solution of (4)} \,\}. \end{aligned}$$

A novel observer design for linear DAE systems has been introduced in [1]. Here, we extend this approach and the accompanying concepts of (asymptotic) observers to nonlinear DAE systems.

Definition II.1. Consider a system (4). A system

$$\begin{aligned} E_o\dot{x}_o(t) &= f_o\big(x_o(t),u(t),y(t)\big), \\ z(t) &= h_o\big(x_o(t),u(t),y(t)\big), \end{aligned} \tag{5}$$

where $E_o \in \mathbb{R}^{l_o \times n_o}$, $f_o \in \mathscr{C}^1(X_o \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^{l_o})$, $h_o \in \mathscr{C}^1(X_o \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^{p_o})$, $X_o \subseteq \mathbb{R}^{n_o}$ open, is called an *acceptor for* (4), if for all $(x,u,y) \in \mathfrak{B}_{(4)}$ with $I = \operatorname{dom} x$, there exist $x_o \in \mathscr{C}^1(I \to \mathbb{R}^{n_o})$, $z \in \mathscr{C}(I \to \mathbb{R}^{p_o})$ such that

$$\big(x_o, \tbinom{u}{y}, z\big) \in \mathfrak{B}_{(5)}.$$

We stress that there is a directed signal flow from (4) to its acceptor (5) via input and output, see Fig. 1. That is, (4) may influence (5) but not vice-versa.

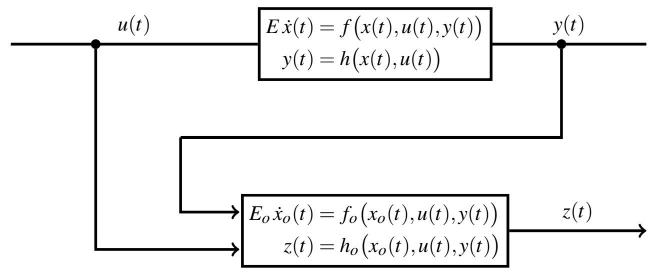

Fig. 1: Interconnection with an acceptor

Definition II.2. Consider a system (4). Then a system (5) with $p_o = n$ is called

a) an *observer for* (4), if it is an acceptor for (4), and

$$\begin{aligned} &\forall I \subseteq \mathbb{R} \text{ open intvl. } \forall t_0 \in I \\ &\forall (x,u,y,x_o,z) \in \mathscr{C}(I \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^{n_o} \times \mathbb{R}^n): \\ &\Big((x,u,y) \in \mathfrak{B}_{(4)} \ \wedge\ \big(x_o, \tbinom{u}{y}, z\big) \in \mathfrak{B}_{(5)} \ \wedge\ Ez(t_0) = Ex(t_0)\Big) \\ &\qquad\qquad \Longrightarrow\ z = x. \end{aligned}$$

b) an *asymptotic observer for* (4), if it is an observer for (4), and

$$\begin{aligned} &\forall t_0 \in \mathbb{R}\ \forall (x,u,y,x_o,z) \in \mathscr{C}([t_0,\infty) \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^{n_o} \times \mathbb{R}^n): \\ &\Big((x,u,y) \in \mathfrak{B}_{(4)} \ \wedge\ \big(x_o, \tbinom{u}{y}, z\big) \in \mathfrak{B}_{(5)}\Big) \\ &\qquad\qquad \Longrightarrow\ \lim_{t\to\infty} z(t) - x(t) = 0. \end{aligned}$$

We like to note that for linear DAE systems the rank condition (2) is equivalent to the concept of *impulse observability*, see e.g. the survey [29], which is introduced using a distributional solution framework and, roughly speaking, means that any Dirac impulses in the state variable can be uniquely determined from the measurement of the external signals. Our motivation to consider this condition here is that for linear impulse observable DAE systems there always exists an observer as shown in [1] (and it can even be reformulated as an ODE, cf. [25]). Therefore, we restrict ourselves to nonlinear systems of the form (1) which satisfy (2).

Concluding this section, we consider linear DAE systems

$$\bar{E}\dot{x}(t) = \bar{A}x(t), \quad y(t) = \bar{C}x(t) \tag{6}$$

with $\bar{E}, \bar{A} \in \mathbb{R}^{l\times n}$, $\bar{C} \in \mathbb{R}^{p\times n}$, and recall from [29] that (6) is called *behaviorally detectable*, if any continuous solution x of (6) with $y = 0$ satisfies $\lim_{t\to\infty} x(t) = 0$, and this is equivalent to $\operatorname{rk}\begin{bmatrix}\lambda\bar{E}-\bar{A}\\ \bar{C}\end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$. In Section III we show that this property can be generalized to a solvability condition for a Riccati type inequality for nonlinear systems (1) which satisfy (2).

III. Observer design by Riccati inequality

In this section we propose a novel design of asymptotic observers for systems (1). We improve upon earlier approaches by allowing a larger class of systems and we use an observer design which extends the Luenberger type observer design. We present a Riccati type inequality whose solutions are used for the observer design. In the subsequent Section IV we show that the solution of certain LMIs and BMIs yields a solution to this Riccati inequality. The LMIs and BMIs then yield a computational procedure for obtaining the observer.

Motivated by the observer design in [1] (which is closely related to that in [30] for behavioral systems) we propose the following observer, which consists of an internal model of the system (1) driven by additional "innovations" terms:

$$\begin{aligned}\dot z_1(t) &= Az_1(t)+Bz_2(t)+f_1\big(z_1(t),z_2(t),u(t),y(t)\big)+L_1d(t),\\ 0 &= Cz_1(t)+Dz_2(t)+f_2\big(u(t),y(t)\big)+L_2d(t),\\ 0 &= Fz_1(t)+Gz_2(t)+h\big(u(t)\big)-y(t)+L_3d(t),\\ z(t) &= \begin{pmatrix} z_1(t)\\ z_2(t)\end{pmatrix},\end{aligned} \tag{7}$$

where $L_1\in\mathbb{R}^{r\times k}, L_2\in\mathbb{R}^{(l-r)\times k}, L_3\in\mathbb{R}^{p\times k}$, $k\in\mathbb{N}_0$, and the additional observer state $d(t)$ represents the innovations; the complete observer state is $x_o(t)=\begin{pmatrix} z_1(t)\\ z_2(t)\\ d(t)\end{pmatrix}$. The innovations in the observer design have been first introduced by Polderman and Willems [31, p. 351] in order to "express how far the actual observed output differs from what we would have expected to observe". Set

$$\mathscr{E}=\begin{bmatrix} I_r & 0 & 0\\ 0&0&0\\ 0&0&0\end{bmatrix},\quad \mathscr{A}=\begin{bmatrix} A & B & L_1\\ C & D & L_2\\ F & G & L_3\end{bmatrix} \tag{8}$$

and

$$\mathscr{V}=\operatorname{im}\begin{bmatrix} I_r\\ M\end{bmatrix},\quad M=-\begin{bmatrix} D & L_2\\ G & L_3\end{bmatrix}^{-1}\begin{bmatrix} C\\ F\end{bmatrix}, \tag{9}$$

provided that $\left[\begin{smallmatrix} D & L_2\\ G & L_3\end{smallmatrix}\right]$ is invertible. In the following main result of the paper we show that there exists an asymptotic observer of the form (7), if the Riccati type inequality

$$\mathscr{A}^\top P\mathscr{E}+\mathscr{E}^\top P\mathscr{A}+\frac{1}{\delta}\mathscr{E}^\top P^2\mathscr{E}+(\delta L^2)I_{n+k}<_{\mathscr{V}}0 \tag{10}$$

has a solution L_1,L_2,L_3, $\delta>0$ and $P=P^\top\in\mathbb{R}^{(n+k)\times(n+k)}$ such that $[I_r,0]P\left[\begin{smallmatrix} I_r\\ 0\end{smallmatrix}\right]>0$. Riccati inequalities of the form (10), i.e., restricted to certain subspaces, have been studied before, see e.g. [32] and the references therein, but not within the context of observer design. Note that condition (2) implies $n\le l+p$, since otherwise $n-r>l+p-r$ which contradicts (2).

Theorem III.1. *Consider a system* (1) *which satisfies* (2) *and* (3). *Let* $k=l+p-n$ *and assume that* $L_1\in\mathbb{R}^{r\times k}, L_2\in\mathbb{R}^{(l-r)\times k}, L_3\in\mathbb{R}^{p\times k}$, $\delta>0$ *and* $P=P^\top\in\mathbb{R}^{(n+k)\times(n+k)}$ *solve* (10) *such that*

$$\begin{bmatrix} D & L_2\\ G & L_3\end{bmatrix}\in\mathbf{Gl}_{n-r+k}(\mathbb{R})\quad\text{and}\quad [I_r,0]P\begin{bmatrix} I_r\\ 0\end{bmatrix}>0. \tag{11}$$

Then (7) *is an asymptotic observer for* (1).

Proof. System (7) is an acceptor for (1) since for any $(x,u,y)\in\mathfrak{B}_{(1)}$ we have that $\left(\left(\begin{smallmatrix} x_1\\ x_2\\ 0\end{smallmatrix}\right),\left(\begin{smallmatrix} u\\ y\end{smallmatrix}\right),x\right)\in\mathfrak{B}_{(7)}$.

Step 1: We show that (7) is an observer for (1). To this end, let $I\subseteq\mathbb{R}$ be an open interval, $t_0\in I$ and $(\left(\begin{smallmatrix} x_1\\ x_2\end{smallmatrix}\right),u,y)\in\mathfrak{B}_{(1)}$, $\left(\left(\begin{smallmatrix} z_1\\ z_2\\ d\end{smallmatrix}\right),\left(\begin{smallmatrix} u\\ y\end{smallmatrix}\right),z\right)\in\mathfrak{B}_{(7)}$ be defined on I such that $x_1(t_0)=z_1(t_0)$. From (7) we have that

$$\begin{pmatrix} z_2(t)\\ d(t)\end{pmatrix}=-\begin{bmatrix} D & L_2\\ G & L_3\end{bmatrix}^{-1}\left(\begin{bmatrix} C\\ F\end{bmatrix}z_1(t)+\begin{pmatrix} f_2\big(u(t),y(t)\big)\\ h\big(u(t)\big)-y(t)\end{pmatrix}\right)$$

for all $t\in I$, and from (1), in a similar way,

$$\begin{pmatrix} x_2(t)\\ 0\end{pmatrix}=-\begin{bmatrix} D & L_2\\ G & L_3\end{bmatrix}^{-1}\left(\begin{bmatrix} C\\ F\end{bmatrix}x_1(t)+\begin{pmatrix} f_2\big(u(t),y(t)\big)\\ h\big(u(t)\big)-y(t)\end{pmatrix}\right).$$

With

$$\begin{pmatrix} g_1\big(x_1,u,y\big)\\ g_2\big(x_1,u,y\big)\end{pmatrix}:=-\begin{bmatrix} D & L_2\\ G & L_3\end{bmatrix}^{-1}\left(\begin{bmatrix} C\\ F\end{bmatrix}x_1+\begin{pmatrix} f_2\big(u,y\big)\\ h\big(u\big)-y\end{pmatrix}\right) \tag{12}$$

for $(x_1,u,y)\in X_1\times\mathbb{R}^m\times\mathbb{R}^p$ we thus have

$$\begin{aligned}\dot x_1(t) &= Ax_1(t)+Bg_1\big(x_1(t),u(t),y(t)\big)\\ &\quad+f_1\big(x_1(t),g_1\big(x_1(t),u(t),y(t)\big),u(t),y(t)\big)+L_1g_2\big(x_1(t),u(t),y(t)\big)\end{aligned}$$

for all $t\in I$, and z_1 solves the same ODE with the same initial value $z_1(t_0)=x_1(t_0)$. Therefore, since g_1 and g_2 are linear in x_1 and

$$X_1\ni x_1\mapsto f_1\big(x_1,g(x_1,u,y),u,y\big)$$

is Lipschitz in x_1 for all $(u,y)\in\mathbb{R}^m\times\mathbb{R}^p$, the uniqueness theorem for ODEs (see [33, Thm. 4.17]) yields that $x_1(t)=z_1(t)$ for all $t\in I$. Moreover,

$$z_2(t)=g_1\big(z_1(t),u(t),y(t)\big)=g_1\big(x_1(t),u(t),y(t)\big)=x_2(t)$$

for all $t\in I$, and this shows that (7) is an observer.

Step 2: We determine the observation error dynamics. Let $e_1:=z_1-x_1$ and $e_2:=z_2-x_2$, then

$$\begin{aligned}\dot e_1(t) &= Ae_1(t)+Be_2(t)+L_1d(t)\\ &\quad+f_1\big(z_1(t),z_2(t),u(t),y(t)\big)-f_1\big(x_1(t),x_2(t),u(t),y(t)\big),\\ 0 &= Ce_1(t)+De_2(t)+L_2d(t),\\ 0 &= Fe_1(t)+Ge_2(t)+L_3d(t).\end{aligned} \tag{13}$$

Let M and $\mathscr{V}$ be as in (9), then $\begin{pmatrix} e_2(t)\\ d(t)\end{pmatrix}=Me_1(t)$, and hence any solution (e_1,e_2,d) of (13) evolves in $\mathscr{V}$.

Step 3: We show that the observer (7) is asymptotic. To this end, let $t_0\in\mathbb{R}$ and $(\left(\begin{smallmatrix} x_1\\ x_2\end{smallmatrix}\right),u,y)\in\mathfrak{B}_{(1)}$, $\left(\left(\begin{smallmatrix} z_1\\ z_2\\ d\end{smallmatrix}\right),\left(\begin{smallmatrix} u\\ y\end{smallmatrix}\right),z\right)\in\mathfrak{B}_{(7)}$ be defined on $[t_0,\infty)$. The corresponding observation errors solve (13) and hence in particular, by Step 2,

$$\forall\, t\ge t_0:\ w(t):=\begin{pmatrix} e_1(t)\\ e_2(t)\\ d(t)\end{pmatrix}\in\mathscr{V}. \tag{14}$$

Use $\mathscr{E}$ and $\mathscr{A}$ from (8) and let

$$V:\mathbb{R}^{n+k}\to\mathbb{R},\ w\mapsto w^\top\mathscr{E}^\top P\mathscr{E}w.$$

Set $\chi_z(t):=\big(z_1(t),z_2(t),u(t),y(t)\big)$ and $\chi_x(t):=\big(x_1(t),x_2(t),u(t),y(t)\big)$, then by (13) we have

$$\begin{aligned}&\tfrac{\mathrm{d}}{\mathrm{d}t}V\big(w(t)\big)\\ &=w(t)^\top\big(\mathscr{A}^\top P\mathscr{E}+\mathscr{E}^\top P\mathscr{A}\big)w(t)\\ &\quad+w(t)^\top\mathscr{E}^\top P\begin{bmatrix} I_r\\ 0\\ 0\end{bmatrix}\Big(f_1\big(\chi_z(t)\big)-f_1\big(\chi_x(t)\big)\Big)\\ &\quad+\left(\begin{bmatrix} I_r\\ 0\\ 0\end{bmatrix}\Big(f_1\big(\chi_z(t)\big)-f_1\big(\chi_x(t)\big)\Big)\right)^\top P\mathscr{E}w(t)\\ &=w(t)^\top\big(\mathscr{A}^\top P\mathscr{E}+\mathscr{E}^\top P\mathscr{A}\big)w(t)+w(t)^\top\mathscr{E}^\top P\tilde f(t)+\tilde f(t)^\top P\mathscr{E}w(t),\end{aligned}$$

where

$$\tilde f(t):=\begin{bmatrix} I_r\\ 0\\ 0\end{bmatrix}\Big(f_1\big(\chi_z(t)\big)-f_1\big(\chi_x(t)\big)\Big).$$

Since for any vectors $u,v\in\mathbb{R}^q$ we have

$$u^\top v+v^\top u\le\delta u^\top u+\tfrac{1}{\delta}v^\top v$$

we obtain the inequality

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}V\big(w(t)\big) &\le w(t)^\top\big(\mathscr{A}^\top P\mathscr{E}+\mathscr{E}^\top P\mathscr{A}\big)w(t)+\tfrac{1}{\delta}w(t)^\top\mathscr{E}^\top P^2\mathscr{E}w(t)\\ &\quad+\delta\tilde f(t)^\top\tilde f(t)\\ &\overset{(3)}{\le} w(t)^\top\big(\mathscr{A}^\top P\mathscr{E}+\mathscr{E}^\top P\mathscr{A}+\tfrac{1}{\delta}\mathscr{E}^\top P^2\mathscr{E}+\delta L^2I_{n+k}\big)w(t)\\ &=-w(t)^\top Qw(t)\end{aligned}$$

for all $t\ge t_0$, where

$$Q:=-\big(\mathscr{A}^\top P\mathscr{E}+\mathscr{E}^\top P\mathscr{A}+\tfrac{1}{\delta}\mathscr{E}^\top P^2\mathscr{E}+\delta L^2I_{n+k}\big).$$

By (10) we have $Q>_{\mathscr{V}}0$ and thus, invoking (14), it follows that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}V\big(w(t)\big)\le -cV\big(w(t)\big),\quad t\ge t_0,$$

for some $c>0$, hence an application of Gronwall's lemma yields $\lim_{t\to\infty} V\big(w(t)\big)=0$. Invoking that $[I_r,0]P\left[\begin{smallmatrix} I_r\\ 0\end{smallmatrix}\right]>0$ we obtain that e_1 converges to zero, and hence $\left(\begin{smallmatrix} e_2\\ d\end{smallmatrix}\right)=Me_1$ converges to zero, too. Therefore,

$$\lim_{t\to\infty} z(t)-x(t)=0,$$

and this completes the proof of the theorem. □

As shown in Theorem III.1, the solvability of the Riccati type inequality (10) is a sufficient condition for the existence of an asymptotic observer for a nonlinear system (1). This generalizes results obtained in [1] for linear DAE systems, where it is shown that behavioral detectability (see Section II) is equivalent to the existence of an asymptotic observer. Indeed, if the Lipschitz constant $L=0$, then solvability of (10) is equivalent to behavioral detectability of the linear part of (1) as shown in the following.

Lemma III.2. *Consider a system* (1) *which satisfies* (2) *and* (3) *with* $L=0$. *Then, for* $k=l+p-n$ *there exist* $L_1\in\mathbb{R}^{r\times k}, L_2\in\mathbb{R}^{(l-r)\times k}, L_3\in\mathbb{R}^{p\times k}$, $\delta>0$ *and* $P=P^\top\in\mathbb{R}^{(n+k)\times(n+k)}$ *such that* (10) *and* (11) *hold if, and only if, the linear DAE* (6) *with*

$$\bar{E}=\begin{bmatrix} I_r & 0\\ 0 & 0\end{bmatrix},\quad \bar{A}=\begin{bmatrix} A & B\\ C & D\end{bmatrix},\quad \bar{C}=[F,G],$$

is behaviorally detectable.

Proof. $\Rightarrow$: From (10) it follows that the Lyapunov type inequality

$$\mathscr{A}^\top P\mathscr{E}+\mathscr{E}^\top P\mathscr{A} <_{\mathscr{V}} 0 \tag{15}$$

holds. Then [34, Thm. 5.10] implies that the linear DAE $\mathscr{E}\dot{x}(t)=\mathscr{A}x(t)$ is *behaviorally stable*, i.e., we have $\lim_{t\to\infty} x(t)=0$ for any continuous solution x of it. As a direct consequence, (6) is behaviorally detectable.
$\Leftarrow$: Invoking that (6) is behaviorally detectable and satisfies (2), it follows from [1, Thm. 3.8 (d)] that for $k=l+p-n$ there exist $L_1\in\mathbb{R}^{r\times k}, L_2\in\mathbb{R}^{(l-r)\times k}, L_3\in\mathbb{R}^{p\times k}$ such that $\left[\begin{smallmatrix} D & L_2\\ G & L_3\end{smallmatrix}\right]$ is invertible and $\mathscr{E}\dot{x}(t)=\mathscr{A}x(t)$ is behaviorally stable. Then [34, Thm. 5.10] yields that there exists $P=P^\top\in\mathbb{R}^{(n+k)\times(n+k)}$ with $[I_r,0]P\left[\begin{smallmatrix} I_r\\ 0\end{smallmatrix}\right]>0$ such that (15) holds. Choosing $\delta>0$ large enough we then obtain that

$$\mathscr{A}^\top P\mathscr{E}+\mathscr{E}^\top P\mathscr{A}+\tfrac{1}{\delta}\mathscr{E}^\top P^2\mathscr{E} <_{\mathscr{V}} 0,$$

i.e., we have (10). □

We like to stress that since $\left[\begin{smallmatrix} D & L_2\\ G & L_3\end{smallmatrix}\right]$ is invertible the asymptotic observer (7) can be reformulated as an ODE system as follows:

$$\begin{aligned}\dot{z}_1(t)&=\left(A-[B,L_1]\begin{bmatrix} D & L_2\\ G & L_3\end{bmatrix}^{-1}\begin{bmatrix} C\\ F\end{bmatrix}\right)z_1(t)\\ &\quad+f_1\big(z_1(t),g_1\big(z_1(t),u(t),y(t)\big),u(t),y(t)\big)\\ &\quad-[B,L_1]\begin{bmatrix} D & L_2\\ G & L_3\end{bmatrix}^{-1}\begin{pmatrix} f_2\big(u(t),y(t)\big)\\ h\big(u(t)\big)-y(t)\end{pmatrix},\\ z(t)&=\begin{pmatrix} z_1(t)\\ g_1\big(z_1(t),u(t),y(t)\big)\end{pmatrix},\end{aligned}\tag{16}$$

where g_1 is as in (12). However, since this structure is quite complicated, (7) may be preferred for implementation and numerical computations. We also stress that (16) is not of Luenberger type in general.

IV. Observer design by solution of LMIs and BMIs

In this section we derive a set of LMIs and BMIs and show how their solution yields a solution of the Riccati inequality (10). LMIs impose convex problems and can be solved efficiently with standard algorithms, however the drawback of the general approach taken here is that the Riccati inequality (10) cannot be completely reformulated as an LMI, but we show that it is possible to obtain its solution from a set of LMIs with an additional BMI constraint. This is the basis for a computational procedure in order to construct the asymptotic observer (7).

Lemma IV.1. *Let* $A\in\mathbb{R}^{r\times r}, B\in\mathbb{R}^{r\times(n-r)}, C\in\mathbb{R}^{(l-r)\times r}, D\in\mathbb{R}^{(l-r)\times(n-r)}, F\in\mathbb{R}^{p\times r}, G\in\mathbb{R}^{p\times(n-r)}$ *and* $L>0$ *be such that* (2) *holds. Assume that* $V\in\mathbf{Gl}_{l+p-r}(\mathbb{R}), W\in\mathbb{R}^{r\times r}, P=P^\top\in\mathbb{R}^{(n+k)\times(n+k)}$ *and* $\delta>0$, *where* $k=l+p-n$, *solve the following set of LMIs*

$$\begin{bmatrix} W^\top+W+(\delta L^2)I_r & [C^\top,F^\top]V^\top & [I_r,0]P^\top\\ V\left[\begin{smallmatrix} C\\ F\end{smallmatrix}\right] & -\frac{1}{\delta L^2}I_{l+p-r} & 0\\ P\left[\begin{smallmatrix} I_r\\ 0\end{smallmatrix}\right] & 0 & -\delta I_{n+k}\end{bmatrix}<0, \tag{17a}$$

$$V\begin{bmatrix} D\\ G\end{bmatrix}=\begin{bmatrix} I_{n-r}\\ 0\end{bmatrix}, \tag{17b}$$

$$P_{11}:=[I_r,0]P\begin{bmatrix} I_r\\ 0\end{bmatrix}>0, \tag{17c}$$

under the additional BMI constraint that

$$A-[B,L_1]V\begin{bmatrix} C\\ F\end{bmatrix}=P_{11}^{-\top}W \tag{18}$$

for some $L_1\in\mathbb{R}^{r\times k}$. *Then, with*

$$\begin{bmatrix} L_2\\ L_3\end{bmatrix}:=V^{-1}\begin{bmatrix} 0\\ I_k\end{bmatrix},$$

we have that the Riccati inequality (10) *is satisfied, where we use the notation from* (8) *and* (9).

Proof. With $K:=P_{11}^{-\top}W$ and $Z:=P\left[\begin{smallmatrix} I_r\\ 0\end{smallmatrix}\right]$ we find that, by (17a),

$$\begin{bmatrix} [K^\top,0]Z+Z^\top\left[\begin{smallmatrix} K\\ 0\end{smallmatrix}\right]+(\delta L^2)I_r & [C^\top,F^\top]V^\top & Z^\top\\ V\left[\begin{smallmatrix} C\\ F\end{smallmatrix}\right] & -\frac{1}{\delta L^2}I_{l+p-r} & 0\\ Z & 0 & -\delta I_{n+k}\end{bmatrix}<0.$$

Using the Schur complement lemma this is equivalent to

$$\begin{bmatrix} [K^\top,0]Z+Z^\top\left[\begin{smallmatrix} K\\ 0\end{smallmatrix}\right]+(\delta L^2)\left(I_r+[C^\top,F^\top]V^\top V\left[\begin{smallmatrix} C\\ F\end{smallmatrix}\right]\right) & Z^\top\\ Z & -\delta I_{n+k}\end{bmatrix}<0. \tag{19}$$

Now, invoking (17b), we have $V=\left[\begin{smallmatrix} D & L_2\\ G & L_3\end{smallmatrix}\right]^{-1}$ and hence we may calculate

$$\begin{aligned}\begin{bmatrix} K\\ 0_{(l+p-r)\times r}\end{bmatrix}&\overset{(18)}{=}\begin{bmatrix} A-[B,L_1]\left[\begin{smallmatrix} D & L_2\\ G & L_3\end{smallmatrix}\right]^{-1}\left[\begin{smallmatrix} C\\ F\end{smallmatrix}\right]\\ 0_{(l+p-r)\times r}\end{bmatrix}\\ &=\begin{bmatrix} A\\ C\\ F\end{bmatrix}-\begin{bmatrix} B & L_1\\ D & L_2\\ G & L_3\end{bmatrix}\begin{bmatrix} D & L_2\\ G & L_3\end{bmatrix}^{-1}\begin{bmatrix} C\\ F\end{bmatrix}=\begin{bmatrix} A & B & L_1\\ C & D & L_2\\ F & G & L_3\end{bmatrix}\begin{bmatrix} I_r\\ M\end{bmatrix},\end{aligned}$$

where M is defined in (9). Furthermore,

$$[C^\top,F^\top]V^\top V\left[\begin{smallmatrix} C\\ F\end{smallmatrix}\right]=M^\top M,$$

and hence (19) becomes

$$\begin{bmatrix} [I_r,M^\top]\mathscr{A}^\top Z+Z^\top\mathscr{A}\left[\begin{smallmatrix} I_r\\ M\end{smallmatrix}\right]+(\delta L^2)\left(I_r+M^\top M\right) & Z^\top\\ Z & -\delta I_{n+k}\end{bmatrix}<0. \tag{20}$$

Applying the Schur complement lemma to (20) yields

$$[I_r,M^\top]\mathscr{A}^\top Z+Z^\top\mathscr{A}\left[\begin{smallmatrix} I_r\\ M\end{smallmatrix}\right]+\frac{1}{\delta}Z^\top Z+(\delta L^2)\Big(I_r+M^\top M\Big)<0$$

and this is equivalent to

$$[I_r,M^\top]\left(\mathscr{A}^\top[Z,0]+\begin{bmatrix} Z^\top\\ 0\end{bmatrix}\mathscr{A}+\frac{1}{\delta}\begin{bmatrix} Z^\top\\ 0\end{bmatrix}[Z,0]+(\delta L^2)I_{l+p}\right)\begin{bmatrix} I_r\\ M\end{bmatrix}<0.$$

Invoking (9) this is in turn the same as

$$\mathscr{A}^\top [Z,0] + \begin{bmatrix} Z^\top \\ 0 \end{bmatrix} \mathscr{A} + \frac{1}{\delta} \begin{bmatrix} Z^\top \\ 0 \end{bmatrix} [Z,0] + (\delta L^2) I_{l+p} <_{\mathscr{V}} 0.$$

Observing that $[Z,0] = P\mathscr{E}$ yields (10) and this finishes the proof. □

Remark IV.2.

(i) A careful inspection of the proof of Lemma IV.1 reveals that the opposite implication is true as well, that is if $L_1 \in \mathbb{R}^{r\times k}, L_2 \in \mathbb{R}^{(l-r)\times k}, L_3 \in \mathbb{R}^{p\times k}$, $P = P^\top \in \mathbb{R}^{(n+k)\times(n+k)}$ and $\delta > 0$ solve (10) such that (17c) holds and $\begin{bmatrix} D & L_2 \\ G & L_3 \end{bmatrix}$ is invertible, then there exists $W \in \mathbb{R}^{r\times r}$ such that $V = \begin{bmatrix} D & L_2 \\ G & L_3 \end{bmatrix}^{-1}, W, P, L_1$ and δ solve (17a)–(17c) and (18). Therefore, solvability of the LMIs and BMIs is necessary and sufficient for solvability of the Riccati inequality (10).

(ii) Note that we refer to (18) as a BMI since the equality can be equivalently written as two inequalities.

(iii) While the equality constraint in (17b) is not a LMI at first sight, it can be incorporated into the other LMIs and BMIs as follows. Let $V_1, \ldots, V_q$ be a basis of the linear subspace

$$\operatorname{span}\left\{ V \in \mathbb{R}^{(l+p-r)\times(l+p-r)} \;\middle|\; V \begin{bmatrix} D \\ G \end{bmatrix} = 0 \right\}$$

and let $V_0 \in \mathbb{R}^{(l+p-r)\times(l+p-r)}$ be such that

$$V_0 \begin{bmatrix} D \\ G \end{bmatrix} = \begin{bmatrix} I_{n-r} \\ 0 \end{bmatrix}.$$

Then solve (17a), (17c) and (18) with $V = V_0 + \sum_{i=1}^q \alpha_i V_i$ for W, P, L_1, δ and $\alpha_1, \ldots, \alpha_q \in \mathbb{R}$. Therefore, it is common to refer to an equality constraint as in (17b) as a LMI as well.

(iv) We like to stress that the solutions of (17a)–(17c) and (18) have to satisfy the additional constraint that V must be invertible. This condition is equivalent to the non-convex quadratic inequality $V^\top V > 0$. By introducing a new variable $J = V$, this inequality can also be equivalently written as a BMI as follows

$$V^\top J > 0, \quad V - J \geq 0, \quad J - V \geq 0.$$

Therefore, the problems (17a)–(17c), (18) consist only of LMIs and BMIs which may be solved by standard MATLAB toolboxes like YALMIP [35] and PENLAB [36]. For other algorithmic approaches see e.g. the tutorial paper [37].

(v) Consider the case $C = 0$ and $F = 0$. Choose V such that (17b) is satisfied. Then the BMI (18) yields $W = P_{11}A$ and inserting this into the LMI (17a) gives, using the Schur complement lemma,

$$A^\top P_{11} + P_{11} A + \tfrac{1}{\delta} P_{11}^2 < -(\delta L^2) I_r,$$

and a solution to (17a), (17c) exists if, and only if, the above Riccati inequality has a solution $P_{11} = P_{11}^\top > 0$ and $\delta > 0$. It is clear that for general A and L a solution does not necessarily exist, not even if A is Hurwitz, i.e., all its eigenvalues have negative real part; consider $A = [-1]$ and $L = 2$ as a counterexample. If we fix $P = I_{n+k}$ and choose $\delta = \frac{1}{L}$, then we obtain the sufficient condition

$$A + A^\top < -2LI_r$$

on the system data in this case. This is a reasonable condition from the point of view that then the error dynamics

$$\begin{aligned} \dot{e}_1(t) &= Ae_1(t) + f_1\big(z_1(t), z_2(t), u(t), y(t)\big) \\ &\quad - f_1\big(x_1(t), x_2(t), u(t), y(t)\big), \\ \begin{pmatrix} e_2(t) \\ d(t) \end{pmatrix} &= 0, \end{aligned}$$

asymptotically stable for all f_1 which satisfy (3).

V. Observer of Luenberger type

In order to illustrate the observer (7) and Lemma IV.1 we consider the question as to when the observer (7) is of Luenberger type as considered e.g. in [17]. If we would have $L_3 = I_k$, then we can eliminate the variable d in (7) and reformulate it as

$$\begin{aligned} \dot{z}_1(t) &= ([A,B] - L_1[F,G]) \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} + L_1\big(y(t) - h\big(u(t)\big)\big) \\ &\quad + f_1\big(z_1(t), z_2(t), u(t), y(t)\big), \\ 0 &= ([C,D] - L_2[F,G]) \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} + L_2\big(y(t) - h\big(u(t)\big)\big) \\ &\quad + f_2\big(u(t), y(t)\big), \\ z(t) &= \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}, \end{aligned}$$

and the first two equations are equivalent to

$$\begin{aligned} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} &= \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} [F,G] \right) \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \\ &\quad + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \big(y(t) - h\big(u(t)\big)\big) + \begin{pmatrix} f_1\big(z_1(t), z_2(t), u(t), y(t)\big) \\ f_2\big(u(t), y(t)\big) \end{pmatrix}. \end{aligned}$$

This is a Luenberger type observer for system (1) with gain $\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$. With $e_1 := z_1 - x_1$ and $e_2 := z_2 - x_2$ the error dynamics (13) become

$$\begin{aligned} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \end{pmatrix} &= \begin{bmatrix} A - L_1F & B - L_1G \\ C - L_2F & D - L_2G \end{bmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} \\ &+ \begin{pmatrix} f_1\big(z_1(t), z_2(t), u(t), y(t)\big) - f_1\big(x_1(t), x_2(t), u(t), y(t)\big) \\ 0 \end{pmatrix}. \end{aligned} \tag{21}$$

Having a look at dimension, $L_3 = I_k$ can only be true if $k = p$ or, equivalently, $l = n$. If the latter is the case, then the matrix D is square, i.e., $D \in \mathbb{R}^{(n-r)\times(n-r)}$ and hence condition (2) implies existence of $L_2 \in \mathbb{R}^{(n-r)\times k}$ such that $D - L_2G$ is invertible. Therefore, the matrix $\begin{bmatrix} D & L_2 \\ G & I_k \end{bmatrix}$ is invertible, i.e., it is always possible to choose $L_3 = I_k$. In this case, the second equation in (21) can be solved for e_2 and with $Y := (D - L_2G)^{-1}(C - L_2F)$ we find

$$\begin{aligned} \dot{e}_1(t) &= \Big((A - L_1F) - (B - L_1G)Y\Big) e_1(t) \\ &\quad + f_1\big(z_1(t), z_2(t), u(t), y(t)\big) - f_1\big(x_1(t), x_2(t), u(t), y(t)\big), \\ e_2(t) &= -Ye_1(t). \end{aligned} \tag{22}$$

Summarizing, we find that if $l = n$, then the observer (7) can be chosen to be of Luenberger type by $k = p$, $L_3 = I_k$ and L_2 such that $D - L_2G$ is invertible.

Finally, we have a look at the solvability of the LMIs (17a)–(17c) together with the BMI (18) in this case. Use the notation from Lemma IV.1, then

$$V = \begin{bmatrix} D & L_2 \\ G & I_k \end{bmatrix}^{-1} = \begin{bmatrix} (D - L_2G)^{-1} & -(D - L_2G)^{-1}L_2 \\ -G(D - L_2G)^{-1} & I_k + G(D - L_2G)^{-1}L_2 \end{bmatrix}.$$

We restrict ourselves to $P = I_{n+k}$ and $W = W^\top$. Then (17a) reads

$$\begin{bmatrix} 2W + \delta L^2 I_r & Y^\top & (F - GY)^\top & [I_r, 0] \\ Y & -\frac{1}{\delta L^2} I_{n-r} & 0 & 0 \\ F - GY & 0 & -\frac{1}{\delta L^2} I_k & 0 \\ \begin{bmatrix} I_r \\ 0 \end{bmatrix} & 0 & 0 & -\delta I_{n+k} \end{bmatrix} < 0.$$

Successively applying the Schur complement lemma yields the equivalent inequality

$$2W + \delta L^2 \Big(I_r + (F - GY)^\top (F - GY) + Y^\top Y \Big) + \tfrac{1}{\delta} I_r < 0. \tag{23}$$

The equation (18) becomes

$$A - L_1F + L_1GY - BY = W,$$

which needs to be solved for L_1. Together with (23) we obtain the condition

$$A - L_1F + L_1GY - BY < -\tfrac{\delta L^2}{2}\left(I_r + (F-GY)^\top (F-GY) + Y^\top Y\right) - \tfrac{1}{2\delta} I_r. \quad (24)$$

on L_1, where Y depends on the choice of L_2. Choosing $\delta = \frac{1}{L}$ we obtain

$$A - L_1F + L_1GY - BY < -LI_r - \frac{L}{2}\left((F-GY)^\top (F-GY) + Y^\top Y\right). \quad (25)$$

This condition immediately implies the asymptotic stability of the error dynamics (22) for any f_1 satisfying (3) since

$$\begin{aligned} &\|f_1\big(z_1(t), z_2(t), u(t), y(t)\big) - f_1\big(x_1(t), x_2(t), u(t), y(t)\big)\| \\ &\le L\|(e_1(t), e_2(t))\| \le L\left\|\begin{bmatrix} I_r \\ -Y \end{bmatrix}\right\| \|e_1(t)\| \end{aligned}$$

which explains the term $Y^\top Y$ on the right hand side of (25). The term $(F-GY)^\top(F-GY)$ is due to the fact that the original Riccati inequality (10) is formulated on the space $\mathscr{V}$ for the variables (e_1, e_2, d), so it is actually possible to allow f_1 to depend on the "innovations" $d = (F-GY)e_1$, which is expected by the inequality (25) in some sense.

Different special cases may be discussed in terms of (24). For instance, if $F = 0$ and $C = 0$, then $Y = 0$ and (24) reduces to the inequality discussed in Remark IV.2 (v).

VI. Examples

We consider two illustrative examples with $l \neq n$, i.e., the corresponding observer cannot be reformulated as an observer of Luenberger type as discussed in Section V. In particular, this shows that the presented observer design is applicable to a larger class of systems than those presented in [16], [17] for instance.

A. Example

Consider (1) with $r = n = 2$, $l = 3$, $p = 1$,

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad C = [0,1], \quad F = [1,1],$$

and some functions f_1, f_2, h such that (3) is satisfied; note that B, D, G are not present since $n - r = 0$. Then $k = l + p - n = 2$ and in order to find a solution to (17a)–(17c) and (18) we set $P = I_4$ and

$$L_2 = [1,0], \quad L_3 = [0,1],$$

thus

$$\begin{bmatrix} L_2 \\ L_3 \end{bmatrix} = I_2 = V^{-1}.$$

The LMIs (17b) and (17c) are already satisfied and (17a) reads

$$\begin{bmatrix} W + W^\top + \delta L^2 I_2 & \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right] & [I_2, 0] \\ \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right] & -\frac{1}{\delta L^2} I_2 & 0 \\ \left[\begin{smallmatrix} I_2 \\ 0 \end{smallmatrix}\right] & 0 & -\delta I_4 \end{bmatrix} < 0$$

which is equivalent to

$$W + W^\top + \delta L^2 \left(I_2 + \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]\right) + \tfrac{1}{\delta} I_2 < 0.$$

Choosing $\delta = \frac{1}{L}$ gives

$$W + W^\top + L\begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix} < 0. \quad (26)$$

We need to find W and L_1 such that (26) is satisfied together with (18), which reads

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - L_1 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = W.$$

Inserting this into (26) yields

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + L\begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix} - L_1 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} L_1^\top < 0$$

One possible choice is to set the left hand side of the inequality equal to $-I_2$ and then solve for the coefficients of L_1. This leads to

$$L_1 = \begin{bmatrix} -\frac{5}{2}L & \frac{3}{2}L + \frac{1}{2} \\ 0 & 2L + \frac{1}{2} \end{bmatrix}$$

and hence

$$W = \tfrac{1}{2}\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} + L\begin{bmatrix} -\frac{3}{2} & 1 \\ -2 & -2 \end{bmatrix}.$$

Now, the observer (7) reads

$$\begin{aligned} \dot z(t) &= \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right] z(t) + L_1 \left(\begin{smallmatrix} d_1(t) \\ d_2(t) \end{smallmatrix}\right) + f_1\big(z(t), u(t), y(t)\big), \\ 0 &= [0,1]z(t) + d_1(t) + f_2\big(u(t), y(t)\big), \\ 0 &= [1,1]z(t) + d_2(t) + h\big(u(t)\big) - y(t), \end{aligned}$$

and we stress that this is not a Luenberger type observer. Nevertheless, it can be simplified to the ODE observer (16) given by

$$\dot z(t) = Wz(t) + f_1\big(z(t), u(t), y(t)\big) - L_1 \begin{pmatrix} f_2\big(u(t), y(t)\big) \\ h\big(u(t)\big) - y(t) \end{pmatrix}.$$

Note that, since $\left[\begin{smallmatrix} C \\ F \end{smallmatrix}\right]$ is invertible, one could, in principle, obtain the state x directly from (1) as

$$x(t) = -\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{pmatrix} f_2\big(u(t), y(t)\big) \\ h\big(u(t)\big) - y(t) \end{pmatrix}.$$

However, for large systems it may not be easy to directly see which parts of the state can be resolved, and in general none of the matrices C, D, F, G, $[C,D]$, $[F,G]$, $\left[\begin{smallmatrix} C \\ F \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} D \\ G \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} C & D \\ F & G \end{smallmatrix}\right]$ is invertible. The following example of such a case illustrates that the presented observer design is still feasible.

B. Example

Consider system (1) with $r = 1$, $n = 3$, $l = p = 2$,

$$A = [1], \quad B = [-2,1], \quad C = [1], \quad D = [-1,0],$$
$$F = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and some functions f_1, f_2, h such that (3) is satisfied. Then $k = l + p - n = 1$ and in order to find a solution to (17a)–(17c) and (18) we set $P = I_4$ and

$$L_2 = [1], \quad L_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

thus

$$V = \begin{bmatrix} D & L_2 \\ G & L_3 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The LMIs (17b) and (17c) are already satisfied and (17a) is obviously equivalent to

$$2W + 2\delta L^2 + \tfrac{1}{\delta} < 0.$$

Furthermore, with $L_1 = 1$, from (18) we find that

$$W = 1 - [-2,1,1]\left[\begin{smallmatrix} -1 \\ 0 \\ 0 \end{smallmatrix}\right] = -1,$$

and we obtain

$$-2 + 2\delta L^2 + \tfrac{1}{\delta} < 0.$$

The expression on the left hand side is minimal for $\delta = \frac{1}{2L^2}$, with which we find that

$$L < \frac{1}{\sqrt{2}},$$

which is a sufficient condition for the existence of an asymptotic observer; this condition reveals that the linear part of the system is (behaviorally) detectable, but unobservable. If this condition is satisfied, then the observer (7) reads

$$\begin{aligned}
\dot{z}_1(t) &= z_1(t) + [-2,1]\, z_2(t) + d(t) + f_1\big(z_1(t), z_2(t), u(t), y(t)\big),\\
0 &= z_1(t) + [-1,0]\, z_2(t) + d(t) + f_2\big(u(t), y(t)\big),\\
0 &= \left[\begin{smallmatrix}-1\\ 1\end{smallmatrix}\right] z_1(t) + \left[\begin{smallmatrix}1 & -1\\ -1 & 1\end{smallmatrix}\right] z_2(t) + \left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right] d(t) + h\big(u(t)\big) - y(t),\\
z(t) &= \left(\begin{smallmatrix}z_1(t)\\ z_2(t)\end{smallmatrix}\right)
\end{aligned}$$

and it can be simplified to the ODE observer (16) given by

$$\begin{aligned}
\dot{z}_1(t) &= -z_1(t) + f_1\big(z_1(t), z_2(t), u(t), y(t)\big)\\
&\quad + [-1,1,0] \begin{pmatrix} f_2\big(u(t), y(t)\big)\\ h\big(u(t)\big) - y(t)\end{pmatrix},\\
z_2(t) &= \begin{pmatrix} z_1(t) + f_2\big(u(t), y(t)\big)\\ f_2\big(u(t), y(t)\big)\end{pmatrix} - \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \big(h\big(u(t)\big) - y(t)\big),\\
z(t) &= \left(\begin{smallmatrix}z_1(t)\\ z_2(t)\end{smallmatrix}\right).
\end{aligned}$$

VII. Conclusion

In the present paper we developed a novel observer design for DAE systems with Lipschitz nonlinearities. The design parameters of the asymptotic observer are constructed from the solutions of a Riccati type inequality. We have further shown that the solution of certain LMIs and BMIs yields a solution to this Riccati inequality. The solvability of the LMIs and BMIs depends on the magnitude of the Lipschitz constant in general, e.g. when the linear part of the system is detectable, but unobservable.

The present work is the basis for extensions in several directions such as systems which do not satisfy the rank condition (2). Incorporating the presence of actuator and sensor faults as well as nonlinearities in the output equation (as discussed in [18], [19] for Luenberger type observers) is another interesting extension for future work.

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