Zero dynamics and stabilization for linear DAEs

Thomas Berger

Abstract We study linear differential-algebraic multi-input multi-output systems which are not necessarily regular and investigate the asymptotic stability of the zero dynamics and stabilizability. To this end, the concepts of autonomous zero dynamics, transmission zeros, right-invertibility, stabilizability in the behavioral sense and detectability in the behavioral sense are introduced and algebraic characterizations are derived. It is then proved, for the class of right-invertible systems with autonomous zero dynamics, that asymptotic stability of the zero dynamics is equivalent to three conditions: stabilizability in the behavioral sense, detectability in the behavioral sense, and the condition that all transmission zeros of the system are in the open left complex half-plane. Furthermore, for the same class, it is shown that we can achieve, by a compatible control in the behavioral sense, that the Lyapunov exponent of the interconnected system equals the Lyapunov exponent of the zero dynamics.

Keywords Differential-algebraic equations · Zero dynamics · Transmission zeros · Right-invertibility · Stabilizability · Detectability · Lyapunov exponent

Mathematics Subject Classification (2010) 34A09 · 15A22 · 93B05 · 93B07 · 93B25

The author was supported by DFG grant IL 25/9.

Thomas Berger
Fachbereich Mathematik, Universität Hamburg,
Bundesstraße 55, 20146 Hamburg, Germany
e-mail: thomas.berger@uni-hamburg.de

Nomenclature

$\mathbb{N}$, $\mathbb{N}_0$	the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$L(\alpha), \lvert\alpha\rvert$	length $L(\alpha) = l$ and absolute value $\lvert\alpha\rvert = \sum_{i=1}^{l} \alpha_i$ of a multi-index $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l$
$\mathbb{C}_+ (\mathbb{C}_-)$	open set of complex numbers with positive (negative) real part, resp.
$\mathbb{R}[s]$	the ring of polynomials with coefficients in $\mathbb{R}$
$\mathbb{R}(s)$	the quotient field of $\mathbb{R}[s]$
$R^{n\times m}$	the set of $n \times m$ matrices with entries in a ring R
$\mathbf{Gl}_n(R)$	the group of invertible matrices in $R^{n\times n}$
$\sigma(M)$	the spectrum of $M \in \mathbb{R}^{n\times m}$
$\lVert x\rVert$	$= \sqrt{x^\top x}$, the Euclidean norm of $x \in \mathbb{R}^n$
$\lVert M\rVert$	$= \max\left\{ \lVert Mx\rVert \,\middle|\, x \in \mathbb{R}^m,\ \lVert x\rVert = 1 \right\}$, induced norm of $M \in \mathbb{R}^{n\times m}$
$M^{-1}\mathscr{S}$	$= \{\, x \in \mathbb{R}^m \mid Mx \in \mathscr{S} \,\}$, the pre-image of the set $\mathscr{S} \subseteq \mathbb{R}^n$ under $M \in \mathbb{R}^{n\times m}$
$\mathscr{C}^\infty(\mathbb{R};\mathbb{R}^n)$	the set of infinitely-times continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}^n$
$\mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$	the set of locally Lebesgue integrable functions $f : \mathbb{R} \to \mathbb{R}^n$, where $\int_K \lVert f(t)\rVert\, \mathrm{d}t < \infty$ for all compact $K \subseteq \mathbb{R}$
$\dot f$ ($f^{(i)}$)	the (i-th) weak derivative of $f \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$, $i \in \mathbb{N}_0$, see [1, Chap. 1]
$\mathscr{W}^{k,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$	$= \left\{ f \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) \,\middle|\, f^{(i)} \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n),\ i = 0, \ldots, k \right\}$, $k \in \mathbb{N}_0$
$f \overset{\text{a.e.}}{=} g$	means that $f, g \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$ are equal "almost everywhere", i.e., $f(t) = g(t)$ for almost all (a.a.) $t \in \mathbb{R}$
$\text{ess-sup}_I \lVert f\rVert$	the essential supremum of the measurable function $f : \mathbb{R} \to \mathbb{R}^n$ over $I \subseteq \mathbb{R}$
$f\vert_I$	the restriction of the function $f : \mathbb{R} \to \mathbb{R}^n$ to $I \subseteq \mathbb{R}$

1 Introduction

We consider linear constant coefficient DAEs of the form

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) &= Ax(t)+Bu(t) \\ y(t) &= Cx(t)\,, \end{aligned} \tag{1}$$

where $E,A \in \mathbb{R}^{\ell\times n}$, $B \in \mathbb{R}^{\ell\times m}$, $C \in \mathbb{R}^{p\times n}$. The set of these systems is denoted by $\Sigma_{\ell,n,m,p}$ and we write $[E,A,B,C] \in \Sigma_{\ell,n,m,p}$. In the present paper, we put special emphasis on the non-regular case, i.e., we do not assume that $sE-A$ is *regular*, which would mean that $\ell = n$ and $\det(sE-A) \in \mathbb{R}[s]\setminus\{0\}$.

The functions $u:\mathbb{R}\to\mathbb{R}^m$ and $y:\mathbb{R}\to\mathbb{R}^p$ are called *input* and *output* of the system, resp. A trajectory $(x,u,y):\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^p$ is said to be a *solution* of (1) if, and only if, it belongs to the *behavior* of (1):

$$\mathfrak{B}_{(1)} := \left\{ (x,u,y) \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^p) \;\middle|\; \begin{array}{l} Ex \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^\ell) \text{ and } (x,u,y) \\ \text{solves (1) for a.a. } t\in\mathbb{R} \end{array} \right\}.$$

Recall that any function $z \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^\ell)$ is in particular continuous.

Particular emphasis is placed on the *zero dynamics* of (1). These are, for $[E,A,B,C] \in \Sigma_{\ell,n,m,p}$, defined by

$$\mathscr{Z}\mathscr{D}_{(1)} := \left\{ (x,u,y) \in \mathfrak{B}_{(1)} \;\middle|\; y \overset{\text{a.e.}}{=} 0 \right\}.$$

By linearity of (1), $\mathscr{Z}\mathscr{D}_{(1)}$ is a real vector space.

The zero dynamics of (1) are called *autonomous* if, and only if,

$$\forall\, w_1,w_2 \in \mathscr{Z}\mathscr{D}_{(1)}\; \forall\, I\subseteq\mathbb{R} \text{ open interval}: \quad w_1|_I \overset{\text{a.e.}}{=} w_2|_I \implies w_1 \overset{\text{a.e.}}{=} w_2\,;$$

and *asymptotically stable* if, and only if,

$$\forall\,(x,u,y)\in\mathscr{Z}\mathscr{D}_{(1)}: \quad \lim_{t\to\infty} \operatorname{ess-sup}_{[t,\infty)} \|(x,u)\| = 0.$$

Note that the above definitions are within the spirit of the *behavioral approach* [20] and take into account that the zero dynamics $\mathscr{Z}\mathscr{D}_{(1)}$ are a linear behavior. In this framework the definition for autonomy of a general behavior is given in [20, Sec. 3.2] and the definition of asymptotic stability in [20, Def. 7.2.1].

(Asymptotically stable) zero dynamics are the vector space of those trajectories of the system which are, loosely speaking, not visible at the output (and tend to zero).

In the present paper, we show for the class of right-invertible systems with autonomous zero dynamics, that asymptotic stability of the zero dynamics is equivalent to the three conditions: stabilizability in the behavioral sense, detectability in the behavioral sense and the condition that all transmission zeros are in the open left complex half-plane. Furthermore, we show that we can achieve, by a compatible control in the behavioral sense, that the Lyapunov exponent of the interconnected system equals the Lyapunov exponent of the zero dynamics. In Section 2 we collect some basic control theoretic concepts such as transmission zeros, right-invertibility, stabilizability in the behavioral sense and detectability in the behavioral sense, and give algebraic characterizations of them. The first main result of the present paper, namely Theorem 3.1, is then stated and proved in Section 3 and some consequences for regular systems are derived. In Section 4 we introduce the concepts of compatible control (in the behavioral sense) and Lyapunov exponent for DAE systems and prove the second main result, namely Theorem 4.4.

For the application of compatible control it is necessary that the states and inputs of the DAE system are fixed a priori by the designer in order to establish the control law. This is different from other approaches based on the behavioral setting, see [12], where only the free variables in the system are viewed as inputs; this may require a reinterpretation of states as inputs and of inputs as states. In the present paper we will assume that such a reinterpretation of variables has already been done or is not feasible, and the given DAE system is fix.

2 Some control theoretic concepts

In this section we recall the concepts used in the present paper in a control theoretic way and give useful algebraic characterizations. These concepts include transmission zeros, right-invertibility, stabilizability in the behavioral sense and detectability in the behavioral sense. We start with characterizations of autonomous and asymptotically stable zero dynamics, which have been introduced in Section 1.

Lemma 2.1 (Autonomous and stable zero dynamics)**.** *Let* $[E,A,B,C] \in \Sigma_{\ell,n,m,p}$. *Then we have the following equivalences:*

(i) $\mathscr{Z}\mathscr{D}_{(1)}$ *are autonomous* $\iff \operatorname{rk}_{\mathbb{R}[s]} \begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix} = n+m.$

(ii) $\mathscr{Z}\mathscr{D}_{(1)}$ *are asymptotically stable* $\iff \forall\, \lambda \in \overline{\mathbb{C}}_+ :\ \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E-A & -B \\ -C & 0 \end{bmatrix} = n+m.$

Proof. (i) follows from [4, Prop. 4.1.5] and (ii) from [4, Lem. 4.3.9]. □

Note that the above cited results from [4] have been first reported in [5]; in the following, this holds true for all results cited from [4].

The autonomy of the zero dynamics allows for a decomposition of the system, provided that C has full row rank. The main result of the present paper (see Section 3) is based on this decomposition.

Lemma 2.2 (System decomposition)**.** *Let* $[E,A,B,C] \in \Sigma_{\ell,n,m,p}$ *with autonomous zero dynamics and* $\operatorname{rk} C = p$. *Then there exist* $S \in \mathbf{Gl}_\ell(\mathbb{R})$ *and* $T \in \mathbf{Gl}_n(\mathbb{R})$ *such that*

$$\begin{bmatrix} s\hat{E}-\hat{A} & \hat{B} \\ \hat{C} & 0 \end{bmatrix} = \begin{bmatrix} S & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} sE-A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix}, \tag{2}$$

where

$$\hat{E} = \begin{bmatrix} I_k & 0 & 0 \\ 0 & E_{22} & E_{23} \\ 0 & E_{32} & N \\ 0 & E_{42} & E_{43} \end{bmatrix}, \ \hat{A} = \begin{bmatrix} Q & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & I_{n_3} \\ 0 & A_{42} & 0 \end{bmatrix}, \ \hat{B} = \begin{bmatrix} 0 \\ I_m \\ 0 \\ 0 \end{bmatrix}, \ \hat{C} = [0, I_p, 0], \tag{3}$$

$$k = \dim \mathscr{ZD}_{(1)}, \tag{4}$$

and $N \in \mathbb{R}^{n_3 \times n_3}$, $n_3 = n - k - p$, *is nilpotent with* $N^\nu = 0$ *and* $N^{\nu-1} \neq 0$, $\nu \in \mathbb{N}$, $E_{22}, A_{22} \in \mathbb{R}^{m \times p}$ *and all other matrices are of appropriate sizes.*

Proof. This result can be found in [4, Thm. 4.2.7]. □

An important characterization of asymptotically stable zero dynamics is the following, which is from [4, Cor. 4.2.11].

Lemma 2.3 (Stable zero dynamics)**.** *Let* $[E, A, B, C] \in \Sigma_{\ell,n,m,p}$ *with autonomous zero dynamics and* $\operatorname{rk} C = p$*. Then, using the notation from Lemma 2.2, the zero dynamics* $\mathscr{ZD}_{(1)}$ *are asymptotically stable if, and only if,* $\sigma(Q) \subseteq \mathbb{C}_-$*.*

Next, in order to define transmission zeros, we introduce the Smith-McMillan form of a rational matrix function.

Definition 2.4 (Smith-McMillan form [16, Sec. 6.5.2])**.** Let $G(s) \in \mathbb{R}(s)^{m \times p}$ with $\operatorname{rk}_{\mathbb{R}(s)} G(s) = r$. Then there exist $U(s) \in \mathbf{Gl}_m(\mathbb{R}[s])$, $V(s) \in \mathbf{Gl}_p(\mathbb{R}[s])$ such that

$$U(s)G(s)V(s) = \operatorname{diag}\left(\frac{\varepsilon_1(s)}{\psi_1(s)}, \ldots, \frac{\varepsilon_r(s)}{\psi_r(s)}, 0_{(m-r)\times(p-r)} \right),$$

where $\varepsilon_i(s), \psi_i(s) \in \mathbb{R}[s]$ are monic, coprime and satisfy $\varepsilon_i(s) | \varepsilon_{i+1}(s)$, $\psi_{i+1}(s) | \psi_i(s)$ for $i = 1, \ldots, r-1$. The number $s_0 \in \mathbb{C}$ is called *zero* of $G(s)$ if, and only if, $\varepsilon_r(s_0) = 0$ and *pole* of $G(s)$ if, and only if, $\psi_1(s_0) = 0$.

In the following we give the definition of transmission zeros for the system $[E, A, B, C]$. In fact, there are many different possibilities to define transmission zeros of control systems, even in the ODE case, see [13]; and they are not equivalent. We follow the definition given by Rosenbrock [21]: For $[I, A, B, C] \in \Sigma_{n,n,m,p}$, the transmission zeros are the zeros of the transfer function $C(sI - A)^{-1}B$. This definition has been generalized to regular DAE systems with transfer function $C(sE - A)^{-1}B$ in [7, Def. 5.3]. In the present framework, we do not require regularity of $sE - A$ and so a transfer function does in general not exist. However, it is possible to give a generalization of the inverse transfer function if the zero dynamics of $[E, A, B, C] \in \Sigma_{\ell,n,m,p}$ are autonomous: Let $L(s)$ be a left inverse of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ over $\mathbb{R}(s)$ (which exists by Lemma 2.1) and define

$$H(s) := -[0, I_m] L(s) \begin{bmatrix} 0 \\ I_p \end{bmatrix} \in \mathbb{R}(s)^{m \times p}. \tag{5}$$

It can be shown that $H(s)$ is independent of the choice of the left inverse $L(s)$ [4, Lem. 4.3.2] and if $sE - A$ is regular and $m = p$, then $H(s) = \left(C(sE - A)^{-1}B\right)^{-1}$ [4, Rem. 4.3.3], i.e., $H(s)$ is indeed the inverse of the transfer function in case of regularity. The fact that the zeros of $H(s)^{-1}$ are the poles of $H(s)$ and vice versa motivates the following definition.

Definition 2.5 (Transmission zeros)**.** Let $[E,A,B,C] \in \Sigma_{\ell,n,m,p}$ with autonomous zero dynamics. Let $L(s)$ be a left inverse of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ over $\mathbb{R}(s)$ and let $H(s)$ be given as in (5). Then $s_0 \in \mathbb{C}$ is called *transmission zero* of $[E,A,B,C]$ if, and only if, s_0 is a pole $H(s)$.

Now we recall the definition of right-invertibility of a system from [22, Sec. 8.2].

Definition 2.6 (Right-invertibility)**.** $[E,A,B,C] \in \Sigma_{\ell,n,m,p}$ is called *right-invertible* if, and only if,

$$\forall y \in \mathscr{C}^\infty(\mathbb{R};\mathbb{R}^p)\ \exists (x,u) \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) \times \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^m):\ (x,u,y) \in \mathfrak{B}_{(1)}.$$

Right-invertibility may be characterized for systems with autonomous zero dynamics in terms of the form (3).

Lemma 2.7 (Right-invertibility and system decomposition)**.** *Let $[E,A,B,C] \in \Sigma_{\ell,n,m,p}$ with autonomous zero dynamics. Then, using the notation from Lemma 2.2,*

$$[E,A,B,C] \text{ is right-invertible} \quad \Longleftrightarrow \quad \begin{cases} \operatorname{rk} C = p,\ E_{42} = 0,\ A_{42} = 0 \text{ and} \\ E_{43} N^j E_{32} = 0 \text{ for } j = 0,\ldots,\nu-1. \end{cases}$$

Proof. A proof can be found in [4, Prop. 4.2.12]. □

We are now in a position to characterize the transmission zeros in terms of the form (3).

Corollary 2.8 (Transmission zeros in decomposition)**.** *Let $[E,A,B,C] \in \Sigma_{\ell,n,m,p}$ be right-invertible and have autonomous zero dynamics. Let $L(s)$ be a left inverse of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ over $\mathbb{R}(s)$ and let $H(s)$ be given as in* (5)*. Then, using the notation from Lemma 2.2,*

$$H(s) = sE_{22} - A_{22} - A_{21}(sI_k - Q)^{-1} A_{12} - s^2 E_{23}(sN - I_{n_3})^{-1} E_{32}$$

and $s_0 \in \mathbb{C}$ is a transmission zero of $[E,A,B,C]$ if, and only if, s_0 is a pole of

$$A_{21}(sI_k - Q)^{-1} A_{12}.$$

Proof. The representation of $H(s)$ follows from [4, Lem. 4.3.2] and the characterization of transmission zeros is then immediate since $sE_{22} - A_{22} - s^2 E_{23}(sN - I)^{-1} E_{32}$ is a polynomial as N is nilpotent and hence

$$(sN - I)^{-1} = -I - sN - \ldots - s^{\nu-1} N^{\nu-1}. \tag{6}$$

□

In the remainder of this section we introduce and characterize the concepts of stabilizability and detectability in the behavioral sense. (Behavioral) stabilizability for systems $[E,A,B,C] \in \Sigma_{\ell,n,m,p}$ is well-investigated, see e.g. the survey [8]. Detectability has been first defined and characterized for regular systems in [2]. For general DAE systems, a definition and characterization can be found in [14]; see also the equivalent definition in [20, Sec. 5.3.2]. The latter definition is given within the behavioral framework, however it is yet too restrictive for our purposes and it is not dual to the respective stabilizability concept. We use the following concepts of behavioral stabilizability and detectability.

Definition 2.9 (Stabilizability and detectability)**.** $[E,A,B,C] \in \Sigma_{\ell,n,m,p}$ is called

(i) *stabilizable in the behavioral sense* if, and only if,

$$\forall (x,u,y) \in \mathfrak{B}_{(1)}\ \exists (x_0,u_0,y_0) \in \mathfrak{B}_{(1)}: \\ \big(\forall t<0:\ (x(t),u(t)) = (x_0(t),u_0(t))\big)\ \wedge\ \lim_{t\to\infty} \text{ess-sup}_{[t,\infty)} \|(x_0,u_0)\| = 0.$$

(ii) *detectable in the behavioral sense* if, and only if,

$$\forall (x,0,0) \in \mathfrak{B}_{(1)}\ \exists (x_0,0,0) \in \mathfrak{B}_{(1)}: \\ (\forall t<0:\ x(t) = x_0(t))\ \wedge\ \lim_{t\to\infty} \text{ess-sup}_{[t,\infty)} \|x_0\| = 0.$$

In order to derive duality of the above concepts it is useful to consider, for $E,A \in \mathbb{R}^{\ell\times n}$, the DAE

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t) \tag{7}$$

without inputs and outputs. The behavior of (7) is given by

$$\mathfrak{B}_{(7)} := \left\{ x \in \mathscr{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n)\ \middle|\ \begin{array}{l} Ex \in \mathscr{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^\ell) \text{ and } x \\ \text{solves (7) for a.a. } t \in \mathbb{R} \end{array} \right\}.$$

Definition 2.10 (Stabilizability [8, Def. 5.1])**.** Let $E,A \in \mathbb{R}^{\ell\times n}$. Then $[E,A]$ is called *stabilizable in the behavioral sense* if, and only if,

$$\forall x \in \mathfrak{B}_{(7)}\ \exists x_0 \in \mathfrak{B}_{(7)}:\ (\forall t<0:\ x(t) = x_0(t))\ \wedge\ \lim_{t\to\infty} \text{ess-sup}_{[t,\infty)} \|x_0\| = 0.$$

We are now in a position to derive a duality result.

Lemma 2.11 (Duality)**.** *Let $[E,A,B,C] \in \Sigma_{\ell,n,m,p}$. Then the following statements are equivalent:*

(i) $[E,A,B,C]$ *is stabilizable in the behavioral sense.*

(ii) $[[E,0],[A,B]]$ *is stabilizable in the behavioral sense.*

(iii) $\left[\begin{bmatrix} E^\top \\ 0 \end{bmatrix}, \begin{bmatrix} A^\top \\ B^\top \end{bmatrix}\right]$ *is stabilizable in the behavioral sense.*

(iv) $[E^\top,A^\top,C^\top,B^\top]$ *is detectable in the behavioral sense.*

Proof. It follows from the definition that (i)⇔(ii) and (iii)⇔(iv). By [8, Cor. 5.2], (ii) is equivalent to

$$\forall \lambda \in \overline{\mathbb{C}}_+ : \ \mathrm{rk}_{\mathbb{C}}[\lambda E - A, -B] = \mathrm{rk}_{\mathbb{R}(s)}[sE - A, -B].$$

Since ranks are invariant under matrix transpose, we find that (ii) is equivalent to

$$\forall \lambda \in \overline{\mathbb{C}}_+ : \ \mathrm{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E^\top - A^\top \\ -B^\top \end{bmatrix} = \mathrm{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE^\top - A^\top \\ -B^\top \end{bmatrix},$$

which, again by [8, Cor. 5.2], is equivalent to (iv). This completes the proof. □

In view of Lemma 2.11 and [8, Cor. 5.2] we may infer the following.

Corollary 2.12 (Characterization of stabilizability and detectability)**.** *Let* $[E,A,B,C] \in \Sigma_{\ell,n,m,p}$. *Then the following holds true.*

(i) $[E,A,B,C]$ *is stabilizable in the behavioral sense if, and only if,*

$$\forall \lambda \in \overline{\mathbb{C}}_+ : \ \mathrm{rk}_{\mathbb{C}}[\lambda E - A, -B] = \mathrm{rk}_{\mathbb{R}(s)}[sE - A, -B].$$

(ii) $[E,A,B,C]$ *is detectable in the behavioral sense if, and only if,*

$$\forall \lambda \in \overline{\mathbb{C}}_+ : \ \mathrm{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A \\ -C \end{bmatrix} = \mathrm{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE - A \\ -C \end{bmatrix}.$$

3 Stable zero dynamics

In this section we state and prove one of the main results of the present paper and derive some consequences for regular systems.

Theorem 3.1 (Characterization of stable zero dynamics)**.** *Let* $[E,A,B,C] \in \Sigma_{\ell,n,m,p}$ *be right-invertible and have autonomous zero dynamics. Then the zero dynamics* $\mathscr{ZD}_{(1)}$ *are asymptotically stable if, and only if, the following three conditions hold:*

(i) $[E,A,B,C]$ *is stabilizable in the behavioral sense,*
(ii) $[E,A,B,C]$ *is detectable in the behavioral sense,*
(iii) $[E,A,B,C]$ *has no transmission zeros in* $\overline{\mathbb{C}}_+$.

Proof. Since right-invertibility of $[E,A,B,C]$ implies, by Lemma 2.7, that $\mathrm{rk}\, C = p$, the assumptions of Lemma 2.2 are satisfied and we may assume that, without loss of generality, $[E,A,B,C]$ is in the form (3).

⇒: *Step 1*: We show (i). Let

$$T_1(s) := \begin{bmatrix} I_k & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & I_{n_3} & 0 \\ -A_{21} & sE_{22} - A_{22} & sE_{23} & -I_m \end{bmatrix} \in \mathbf{Gl}_{n+m}(\mathbb{R}[s])$$

and observe that, since $E_{42} = A_{42} = 0$ by Lemma 2.7,

$$[sE-A,-B]T_1(s) = \begin{bmatrix} sI_k - Q & -A_{12} & 0 & 0 \\ 0 & 0 & 0 & I_m \\ 0 & sE_{32} & sN - I_{n_3} & 0 \\ 0 & 0 & sE_{43} & 0 \end{bmatrix}.$$

Then, with

$$T_2(s) := \begin{bmatrix} I_k & (sI_k - Q)^{-1}A_{12} & 0 & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & I_{n_3} & 0 \\ 0 & 0 & 0 & -I_m \end{bmatrix} \in \mathbf{Gl}_{n+m}(\mathbb{R}(s)),$$

and

$$T_3(s) := \begin{bmatrix} I_k & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \\ 0 & -s(sN - I_{n_3})^{-1}E_{32} & I_{n_3} & 0 \\ 0 & 0 & 0 & -I_m \end{bmatrix} \in \mathbf{Gl}_{n+m}(\mathbb{R}[s]),$$

where we note that it follows from (6) that $T_3(s)$ is a polynomial, we obtain

$$[sE-A,-B]T_1(s)T_2(s)T_3(s) = \begin{bmatrix} sI_k - Q & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m \\ 0 & 0 & sN - I_{n_3} & 0 \\ 0 & X(s) & sE_{43} & 0 \end{bmatrix},$$

where $X(s) = -s^2E_{43}(sN - I_{n_3})^{-1}E_{32} = 0$ by Lemma 2.7 and (6). Finally,

$$S_1(s) := \begin{bmatrix} I_k & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \\ 0 & & I_{n_3} & 0 \\ 0 & 0 & -sE_{43}(sN - I_{n_3})^{-1} & -I_m \end{bmatrix} \in \mathbf{Gl}_{n+m}(\mathbb{R}[s])$$

yields

$$S_1(s)[sE-A,-B]T_1(s)T_2(s)T_3(s) = \begin{bmatrix} sI_k - Q & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m \\ 0 & 0 & sN - I_{n_3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and hence $\mathrm{rk}_{\mathbb{R}(s)}[sE-A,-B] = k + n_3 + m = n + m - p$, since $n_3 = n - k - p$ by Lemma 2.2. Now let $\lambda \in \overline{\mathbb{C}}_+$ and observe that, by Lemma 2.3, $\lambda I_k - Q$ is invertible. Hence, the matrices $T_1(\lambda), T_2(\lambda), T_3(\lambda)$ and $S_1(\lambda)$ exist and are invertible. Thus, using the same transformations as above for fixed $\lambda \in \overline{\mathbb{C}}_+$ now, we find that $\mathrm{rk}_{\mathbb{C}}[\lambda E - A, -B] = n + m - p$. This proves (i).

Step 2: We show (ii). Similar to Step 1 it can be shown that

$$\forall \lambda \in \overline{\mathbb{C}}_+ : \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A \\ -C \end{bmatrix} = \operatorname{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE - A \\ -C \end{bmatrix} = n.$$

Step 3: We show (iii). By Corollary 2.8, the transmission zeros of $[E,A,B,C]$ are the poles of

$$F(s) := A_{21}(sI_k - Q)^{-1}A_{12}.$$

Every pole of $F(s)$ is also an eigenvalue of Q. In view of Lemma 2.3, we have that $\sigma(Q) \subseteq \mathbb{C}_-$ and so (iii) follows.

$\Leftarrow$: By Lemma 2.3, we have to show that if $\lambda \in \sigma(Q)$, then $\lambda \in \mathbb{C}_-$. Let $\lambda \in \sigma(Q)$. We distinguish two cases:

Case 1: λ is a pole of $F(s)$. Then, by Corollary 2.8, λ is a transmission zero of $[E,A,B,C]$ and by (iii) we obtain $\lambda \in \mathbb{C}_-$.

Case 2: λ is not a pole of $F(s)$. Then [7, Lem. 8.3] applied to $[I_k, Q, A_{12}, A_{21}]$ and λ yields that

$$\text{(a) } \operatorname{rk}_{\mathbb{C}}[\lambda I_k - Q, A_{12}] < k \quad \text{or} \quad \text{(b) } \operatorname{rk}_{\mathbb{C}}[\lambda I_k - Q^\top, A_{21}^\top] < k.$$

If (a) holds, then there exists $v_1 \in \mathbb{C}^k \setminus \{0\}$ such that

$$v_1^\top [\lambda I_k - Q, A_{12}] = 0.$$

Let $v_4 \in \mathbb{C}^{(\ell-n)+(p-m)}$ be arbitrary and define

$$v_3^\top := -\lambda v_4^\top E_{43}(\lambda N - I_{n_3})^{-1}.$$

Now observe that

$$(v_1^\top, 0, v_3^\top, v_4^\top) \begin{bmatrix} \lambda I_k - Q & -A_{12} & 0 & 0 \\ -A_{21} & \lambda E_{22} - A_{22} & \lambda E_{23} & I_m \\ 0 & \lambda E_{32} & \lambda N - I_{n_3} & 0 \\ 0 & 0 & \lambda E_{43} & 0 \end{bmatrix} = (0, w^\top, 0, 0),$$

where

$$w^\top = -v_1^\top A_{12} + \lambda v_3^\top E_{32} = -\lambda^2 v_4^\top E_{43}(\lambda N - I_{n_3})^{-1} E_{32} = 0$$

by Lemma 2.7 and (6). This implies that $\mathscr{K} := \ker [\lambda E - A, -B]^\top \subseteq \mathbb{C}^l$ has dimension $\dim \mathscr{K} \geq (\ell - n) + (p - m) + 1$. Therefore,

$$\begin{aligned} \operatorname{rk}_{\mathbb{C}}[\lambda E - A, -B] &\leq \ell - \dim \mathscr{K} \leq n + m - p - 1 \\ &= \operatorname{rk}_{\mathbb{R}(s)}[sE - A, -B] - 1 < \operatorname{rk}_{\mathbb{R}(s)}[sE - A, -B], \end{aligned} \tag{8}$$

where $\operatorname{rk}_{\mathbb{R}(s)}[sE - A, -B] = n + m - p$ has been proved in Step 1 of "$\Rightarrow$". Hence, (8) together with (i) implies that $\lambda \in \mathbb{C}_-$.

If (b) holds, then there exists $v_1 \in \mathbb{C}^k \setminus \{0\}$ such that $v_1^\top [\lambda I_k - Q^\top, A_{21}^\top] = 0$. Therefore,

$$\begin{bmatrix} \lambda I_k - Q & -A_{12} & 0 \\ -A_{21} & \lambda E_{22} - A_{22} & \lambda E_{23} \\ 0 & \lambda E_{32} & \lambda N - I_{n_3} \\ 0 & 0 & \lambda E_{43} \\ 0 & I_p & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix} = 0$$

and thus

$$\operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A \\ -C \end{bmatrix} < n = \operatorname{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE - A \\ -C \end{bmatrix}, \tag{9}$$

where $\operatorname{rk}_{\mathbb{R}(s)} \left[\begin{smallmatrix} sE-A \\ -C \end{smallmatrix}\right] = n$ has been proved in Step 2 of "⇒". Hence, (9) together with (ii) implies that $\lambda \in \mathbb{C}_-$. This completes the proof of the theorem. □

Remark 3.2. It might be surprising that in Step 1 and Step 2 it is calculated that

$$\operatorname{rk}_{\mathbb{R}(s)} [sE - A, -B] = n + m - p \quad \text{and} \quad \operatorname{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE - A \\ -C \end{bmatrix} = n. \tag{10}$$

Because of duality reasons it could be expected that the two ranks satisfy

$$\operatorname{rk}_{\mathbb{R}(s)} [sE - A, -B] = \min\{n + m - p, \ell\} \text{ and } \operatorname{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE - A \\ -C \end{bmatrix} = \min\{\ell + p - m, n\}.$$

Since it is assumed that the zero dynamics are autonomous, it follows that the system

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t) + Bu(t), \quad Cx(t) = 0$$

is autonomous and hence no free variables are present. As there are $n+m$ variables and $\ell + p$ equations, it is necessary that $n + m \le \ell + p$. This implies $n + m - p \le \ell$ and $n \le \ell + p - m$, and hence (10).

For regular systems with invertible transfer function we may characterize asymptotic stability of the zero dynamics by Hautus criteria for stabilizability and detectability and the absence of zeros of the transfer function in the closed right complex half-plane (recall Definition 2.4 for the definition of a zero of a rational matrix function).

Corollary 3.3 (Regular systems)**.** *Let $[E,A,B,C] \in \Sigma_{n,n,m,m}$ be such that $sE - A$ is regular and $G(s) := C(sE - A)^{-1}B$ is invertible over $\mathbb{R}(s)$. Then the zero dynamics $\mathscr{ZD}_{(1)}$ are asymptotically stable if, and only if, the following three conditions hold:*

(i) $\forall \lambda \in \overline{\mathbb{C}}_+ : \operatorname{rk}_{\mathbb{C}} [\lambda E - A, -B] = n$,

(ii) $\forall \lambda \in \overline{\mathbb{C}}_+ : \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A \\ -C \end{bmatrix} = n$,

(iii) *$G(s)$ has no zeros in $\overline{\mathbb{C}}_+$.*

Proof. Since $G(s) \in \mathbf{Gl}_m(\mathbb{R}(s))$ it follows from Lemma 2.1 that $\mathscr{ZD}_{(1)}$ are autonomous. Furthermore, $\operatorname{rk} C = m$ and hence we may infer from [4, Rem. 4.2.13] that $[E,A,B,C]$ is right-invertible. Now, we may apply Theorem 3.1 to deduce that $\mathscr{ZD}_{(1)}$ are asymptotically stable if, and only if,

(a) $[E,A,B,C]$ is stabilizable in the behavioral sense,
(b) $[E,A,B,C]$ is detectable in the behavioral sense,
(c) $[E,A,B,C]$ has no transmission zeros in $\overline{\mathbb{C}}_+$.

Since regularity of $sE-A$ gives that $\mathrm{rk}_{\mathbb{R}(s)}[sE-A,-B] = \mathrm{rk}_{\mathbb{R}(s)} \left[\begin{smallmatrix} sE-A \\ -C \end{smallmatrix}\right] = n$, we find that (i)⇔(a) and (ii)⇔(b). (iii)⇔(c) follows from the fact that by [4, Rem. 4.3.3] we have $H(s) = G(s)^{-1}$ for $H(s)$ as in (5) and that transmission zeros of $[E,A,B,C]$ are, by definition, exactly the poles of $H(s)$. □

4 Stabilization

In this section we consider stabilizing control for DAE systems. More precisely, we introduce the concepts of Lyapunov exponent and compatible control and show that for right-invertible systems with autonomous zero dynamics it is possible to assign, via a compatible control, the Lyapunov exponent of the system to a value specified by the zero dynamics.

The usual concept of feedback is the additional application of the relation $u(t) = Fx(t)$ to the system $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t)$; for instance, high-gain feedback has been successfully applied to DAEs in [6] in order to achieve stabilization. Feedback can therefore be seen as an additional algebraic constraint that can be resolved for the input. Control in the behavioral sense, or control via interconnection [24], generalizes this approach by also allowing further algebraic relations in which the state not necessarily uniquely determines the input (see also [8, Sec. 5.3]). That is, for given (or to be determined) $K = [K_x, K_u] \in \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ and $[E,A,B,C] \in \Sigma_{\ell,n,m,p}$ we consider

$$\mathfrak{B}^K_{[E,A,B]} = \left\{ (x,u) \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n \times \mathbb{R}^m) \;\middle|\; \begin{array}{l} Ex \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^\ell) \text{ and,} \\ \text{for a.a. } t \in \mathbb{R}, \\ \frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t) \\ \qquad\quad 0 = K_x x(t) + K_u u(t) \end{array} \right\}.$$

We call K the *control matrix*, since it induces the control law $K_x x + K_u u \overset{\text{a.e.}}{=} 0$. Note that, in principle, one could make the extreme choice $K = I_{n+m}$ to end up with a behavior

$$\mathfrak{B}^K_{[E,A,B]} \subseteq \left\{ (x,u) \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n \times \mathbb{R}^m) \;\middle|\; (x,u) \overset{\text{a.e.}}{=} 0 \right\},$$

which is obviously asymptotically stable. This, however, is not suitable from a practical point of view. If we assume that the controller is switched on at a certain time $t \in \mathbb{R}$, then this causes a jump from a solution trajectory of the original system $[E,A,B]$ onto a solution within the interconnected behavior $\mathfrak{B}^K_{[E,A,B]}$ (the trivial solution in this case) at time t. Hence, jumps occur in Ex. To avoid this, we use the concept of compatible control.

Definition 4.1 (Compatible control [8, Def. 5.2])**.** Let $[E,A,B,C] \in \Sigma_{\ell,n,m,p}$. The control matrix $K = [K_x, K_u] \in \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ is called *compatible* for $[E,A,B,C]$ if, and only if,

$$\forall x^0 \in \left\{ x^0 \in \mathbb{R}^n \;\middle|\; \exists (x,u,y) \in \mathfrak{B}_{(1)} : \ Ex(0) = Ex^0 \right\} \qquad \exists (x,u) \in \mathfrak{B}^K_{[E,A,B]} : \ Ex(0) = Ex^0.$$

We construct a compatible control which not only results in an asymptotically stable interconnected system, but also the Lyapunov exponent of the interconnected system is prescribed by the zero dynamics of the nominal system. In order to get a most general definition of the Lyapunov exponent, we use a definition similar to the Bohl exponent in [3, Def. 3.4], not requiring a fundamental solution matrix as in [18].

Definition 4.2 (Lyapunov exponent)**.** Let $E,A \in \mathbb{R}^{\ell\times n}$. The *Lyapunov exponent* of $[E,A]$ is defined as

$$k_L(E,A) := \inf \left\{ \mu \in \mathbb{R} \;\middle|\; \begin{array}{r} \exists M_\mu > 0 \ \forall x \in \mathfrak{B}_{(7)} \text{ for a.a. } t \geq s : \\ \|x(t)\| \leq M_\mu e^{\mu(t-s)} \|x(s)\| \end{array} \right\}.$$

Note that we use the convention $\inf \emptyset = +\infty$.

The (minimal) exponential decay rate of the (asymptotically stable) zero dynamics of a system can be determined by the Lyapunov exponent of the DAE $\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & B \\ C & 0 \end{smallmatrix}\right]\right]$.

Lemma 4.3 (Lyapunov exponent and stable zero dynamics)**.** *Let* $[E,A,B,C] \in \Sigma_{\ell,n,m,p}$ *with autonomous zero dynamics and* $\operatorname{rk} C = p$*. Then, using the notation from Lemma 2.2 and* k *as in* (4)*, we have*

$$\begin{aligned} k_L(\mathscr{ZD}_{(1)}) &:= \inf \left\{ \mu \in \mathbb{R} \;\middle|\; \begin{array}{r} \exists M_\mu > 0 \ \forall w \in \mathscr{ZD}_{(1)} \text{ for a.a. } t \geq s : \\ \|w(t)\| \leq M_\mu e^{\mu(t-s)} \|w(s)\| \end{array} \right\} \\ &= k_L\left(\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \right) \\ &= \begin{cases} \max\{ \operatorname{Re} \lambda \mid \lambda \in \sigma(Q) \}, & \text{if } k > 0 \\ -\infty, & \text{if } k = 0. \end{cases} \end{aligned}$$

Proof. The first equality follows from the fact that the trajectories in $\mathscr{ZD}_{(1)}$ can be identified with those in the behavior $\mathfrak{B}_{(7)}$ of the DAE system corresponding to $\left(\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & B \\ C & 0 \end{smallmatrix}\right]\right)$.

The second equality can be seen by using the decomposition (3): Since the Lyapunov exponent is invariant under transformation of the system (see e.g. [3, Prop. 3.17]) we may assume that, without loss of generality, $[E,A,B,C]$ is in the form (3). Now observe that $(x,u,y) \in \mathscr{ZD}_{(1)}$, where $x = (x_1, y, x_3)$, if, and only if, $y \stackrel{\text{a.e.}}{=} 0$, $x_3 \stackrel{\text{a.e.}}{=} 0$ and $x_1 \in \mathscr{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^k)$, $u \in \mathscr{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^m)$ satisfy

$$\tfrac{\mathrm{d}}{\mathrm{d}t}x_1 \overset{\text{a.e.}}{=} Qx_1, \quad u \overset{\text{a.e.}}{=} -A_{21}x_1.$$

This equivalence of solution trajectories yields the assertion. □

Note that it follows from Lemmas 2.3 and 4.3 that asymptotic stability of the zero dynamics implies exponential stability of the zero dynamics, i.e., any trajectory tends to zero exponentially.

We are now in a position to prove the main result of this section, which states that for right-invertible systems with autonomous zero dynamics there exists a compatible control such that the Lyapunov exponent of the interconnected system is equal to the Lyapunov exponent of the zero dynamics of the nominal system; in particular, this shows that asymptotic stability of the zero dynamics implies that the system can be asymptotically stabilized in the sense that every solution of the interconnected system tends to zero.

Theorem 4.4 (Compatible and stabilizing control)**.** *Let* $[E,A,B,C] \in \Sigma_{\ell,n,m,p}$ *be right-invertible with autonomous zero dynamics. If* $\dim \mathscr{ZD}_{(1)} > 0$*, then there exists a compatible control matrix* $K = [K_x, K_u] \in \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ *for* $[E,A,B,C]$ *such that*

$$k_L\left(\begin{bmatrix} E & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix} A & B\\ K_x & K_u\end{bmatrix}\right) = k_L(\mathscr{ZD}_{(1)}). \tag{11}$$

If $\dim \mathscr{ZD}_{(1)} = 0$*, then for all* $\mu \in \mathbb{R}$ *there exists a compatible control matrix* $K = [K_x, K_u] \in \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ *for* $[E,A,B,C]$ *such that*

$$k_L\left(\begin{bmatrix} E & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix} A & B\\ K_x & K_u\end{bmatrix}\right) \le \mu. \tag{12}$$

Proof. Since the Lyapunov exponent is invariant under transformation of the system (see e.g. [3, Prop. 3.17]) we may, similar to the proof of Theorem 3.1, assume that, without loss of generality, $[E,A,B,C]$ is in the form (3). Then, with similar transformations as in Step 1 of the proof of Theorem 3.1, it can be shown that

$$\forall\, \lambda \in \mathbb{C}:\ \mathrm{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E_{22} - A_{22} & \lambda E_{23} & I_m\\ \lambda E_{32} & \lambda N - I_{n_3} & 0\\ 0 & \lambda E_{43} & 0\end{bmatrix} = \mathrm{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE_{22} - A_{22} & sE_{23} & I_m\\ sE_{32} & sN - I_{n_3} & 0\\ 0 & sE_{43} & 0\end{bmatrix},$$

and hence, by [8, Cor. 4.3], the system

$$[\tilde E, \tilde A, \tilde B, \tilde C] := \left[\begin{bmatrix} E_{22} & E_{23}\\ E_{32} & N\\ 0 & E_{43}\end{bmatrix}, \begin{bmatrix} A_{22} & 0\\ 0 & I_{n_3}\\ 0 & 0\end{bmatrix}, \begin{bmatrix} I_m\\ 0\\ 0\end{bmatrix}, [I_p, 0]\right]$$

is controllable in the behavioral sense as in [8, Def. 2.1].

We will now mimic the proof of [8, Thm. 5.4] without repeating all of its arguments: It follows from the above controllability in the behavioral sense and [8, Cor. 3.4] that in the feedback form [8, (3.10)] of $[\tilde E, \tilde A, \tilde B]$ we have $n_{\overline{c}} = 0$. Therefore, for any given $\mu \in \mathbb{R}$ and $\varepsilon > 0$, it is possible to choose F_{11} and K_x in the proof of [8, Thm. 5.4] such that the resulting control matrix $\tilde K = [K_1, K_2] \in \mathbb{R}^{q\times(n-k)} \times \mathbb{R}^{q\times m}$ is compatible for $\left[\tilde{E},\tilde{A},\tilde{B},\tilde{C}\right]$ and satisfies

$$k_L\left(\begin{bmatrix}\tilde{E} & 0\\ 0 & 0\end{bmatrix},\begin{bmatrix}\tilde{A} & \tilde{B}\\ K_1 & K_2\end{bmatrix}\right) \le \mu-\varepsilon. \tag{13}$$

We show that

$$K = [K_x \,|\, K_u] := [K_2A_{21}, K_1 \,|\, K_2] \in \mathbb{R}^{q\times k}\times\mathbb{R}^{q\times(n-k)}\times\mathbb{R}^{q\times m},$$

is compatible for $[E,A,B,C]$ and satisfies (11) or (12), resp.

Step 1: We show compatibility. Let

$$x^0 \in \left\{\, x^0\in\mathbb{R}^n \;\middle|\; \exists\,(x,u,y)\in\mathfrak{B}_{(1)}:\ Ex(0)=Ex^0 \,\right\}$$

and partition $x^0 = \left((x_1^0)^\top,(x_2^0)^\top\right)^\top$ with $x_1^0\in\mathbb{R}^k$, $x_2^0\in\mathbb{R}^{n-k}$. Then there exist $x_1\in\mathscr{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^k)$, $x_2\in\mathscr{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^{n-k})$ and $u\in\mathscr{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^m)$ such that $\tilde{E}x_2\in\mathscr{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^{n-k})$ and

$$\left.\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}x_1 &\overset{\text{a.e.}}{=} Qx_1+[A_{12},0]x_2,\\ \tfrac{\mathrm{d}}{\mathrm{d}t}\tilde{E}x_2 &\overset{\text{a.e.}}{=} \begin{bmatrix}A_{21}\\0\\0\end{bmatrix}x_1+\tilde{A}x_2+\tilde{B}u,\\ x_1(0) &= x_1^0,\\ \tilde{E}x_2(0) &= \tilde{E}x_2^0.\end{aligned}\right\} \tag{14}$$

Therefore,

$$x_2^0 \in \left\{\, x_2^0\in\mathbb{R}^n \;\middle|\; \exists\,(x_2,u,\tilde{C}x_2)\in\mathfrak{B}_{[\tilde{E},\tilde{A},\tilde{B},\tilde{C}]}:\ \tilde{E}x_2(0)=\tilde{E}x_2^0 \,\right\},$$

where $\mathfrak{B}_{[\tilde{E},\tilde{A},\tilde{B},\tilde{C}]}$ denotes the behavior of (1) corresponding to the system $[\tilde{E},\tilde{A},\tilde{B},\tilde{C}]$, and by compatibility of $[K_1,K_2]$ for $\left[\tilde{E},\tilde{A},\tilde{B},\tilde{C}\right]$ there exists $(x_2,v)\in\mathfrak{B}^{[K_1,K_2]}_{[\tilde{E},\tilde{A},\tilde{B}]}$ such that

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\tilde{E}x_2 &\overset{\text{a.e.}}{=} \tilde{A}x_2+\tilde{B}v,\\ 0 &\overset{\text{a.e.}}{=} K_1x_2+K_2v,\end{aligned} \tag{15}$$

and $\tilde{E}x_2(0)=\tilde{E}x_2^0$. Define

$$x_1(t) := e^{Qt}x_1^0+\int_0^t e^{Q(t-s)}[A_{12},0]x_2(s)\,\mathrm{d}s,\quad t\in\mathbb{R},$$

which is well-defined since $x_2\in\mathscr{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^{n-k})$, and let $u:=v-A_{21}x_1$. Then (x_1,x_2,u) solves (14) and satisfies

$$K_2A_{21}x_1+K_1x_2+K_2u \overset{\text{a.e.}}{=} K_2A_{21}x_1+K_1x_2+K_2v-K_2A_{21}x_1 \overset{\text{a.e.}}{=} 0,$$

which proves that $[K_2A_{21}, K_1, K_2]$ is compatible for $[E,A,B,C]$.

Step 2: We show that (12) is satisfied in case that $k=0$ for k as in (4). This follows from (13) since

$$k_L\left(\begin{bmatrix} E & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix} A & B\\ K_x & K_u\end{bmatrix}\right) = k_L\left(\begin{bmatrix} \tilde{E} & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix} \tilde{A} & \tilde{B}\\ K_1 & K_2\end{bmatrix}\right) \le \mu - \varepsilon$$

with arbitrary $\mu \in \mathbb{R}$ and $\varepsilon > 0$.

Step 3: We show that (11) is satisfied in case that $k>0$. Denote

$$\mu := k_L(\mathscr{Z}\mathscr{D}_{(1)}) \overset{\text{Lem. 4.3}}{=} \max\{\, \operatorname{Re}\lambda \mid \lambda \in \sigma(Q) \,\}$$

and let $\rho > 0$ be arbitrary. Then there exists $M_\rho > 0$ such that, for all $t \ge 0$, $\|e^{Qt}\| \le M_\rho e^{(\mu+\rho)t}$.

Step 3a: We show "$\ge$" in (11). Since, for any solution $x_1 \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^k)$ of $\frac{\mathrm{d}}{\mathrm{d}t}x_1 = Qx_1$ we have

$$\big((x_1^\top, 0)^\top, -A_{21}x_1, 0\big) \in \mathfrak{B}^K_{[E,A,B]},$$

it follows that

$$k_L\left(\begin{bmatrix} E & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix} A & B\\ K_x & K_u\end{bmatrix}\right) \ge \mu.$$

Step 3b: We show "$\le$" in (11). Let $(x,u) \in \mathfrak{B}^K_{[E,A,B]}$ and write $x = \big(x_1^\top, x_2^\top\big)^\top$ with $x_1 \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^k)$ and $x_2 \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{n-k})$. Then we have

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}x_1 &\overset{\text{a.e.}}{=} Qx_1 + [A_{12}, 0]x_2,\\ \tfrac{\mathrm{d}}{\mathrm{d}t}\tilde{E}x_2 &\overset{\text{a.e.}}{=} \begin{bmatrix} A_{21}\\ 0\\ 0\end{bmatrix} x_1 + \tilde{A}x_2 + \tilde{B}u,\\ 0 &\overset{\text{a.e.}}{=} K_2A_{21}x_1 + K_1x_2 + K_2u. \end{aligned}$$

Observe that $(x_2, w := u + A_{21}x_1)$ solves (15) and hence, by (13) for μ and some $\varepsilon > 0$, there exists $M_1 > 0$ such that

$$\text{for a.a. } t \ge s: \quad \left\|\begin{pmatrix} x_2(t)\\ w(t)\end{pmatrix}\right\| \le M_1 e^{(\mu-\varepsilon)(t-s)} \left\|\begin{pmatrix} x_2(s)\\ w(s)\end{pmatrix}\right\|.$$

Therefore,

$$\begin{aligned} \|x_1(t)\| &\le \|e^{Q(t-s)}\| \cdot \|x_1(s)\| + \int_s^t \|e^{Q(t-\tau)}\| \cdot \|[A_{12},0]\| \cdot \left\|\begin{pmatrix} x_2(\tau)\\ w(\tau)\end{pmatrix}\right\| \,\mathrm{d}\tau\\ &\le M_\rho e^{(\mu+\rho)(t-s)}\|x_1(s)\|\\ &\quad + M_1M_\rho e^{(\mu+\rho)(t-s)} \cdot \|[A_{12},0]\| \cdot \left\|\begin{pmatrix} x_2(s)\\ w(s)\end{pmatrix}\right\| \underbrace{\int_s^t e^{-(\varepsilon+\rho)(t-\tau)}\,\mathrm{d}\tau}_{\le 1/\varepsilon} \end{aligned}$$

for almost all $t,s \in \mathbb{R}$ with $t \geq s$. This implies that

$$k_L\left(\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ K_x & K_u \end{bmatrix}\right) \leq \mu + \rho$$

and since $\rho > 0$ is arbitrary the claim is shown. $\square$

Remark 4.5 (Construction of the control)**.** The construction of the control K in the proof of Theorem 4.4 relies on the construction used in [8, Thm. 5.4]. Here we make it precise. We have split up the procedure into several steps.

(i) The first step is to transform the given system $[E,A,B,C] \in \Sigma_{\ell,n,m,p}$ into the form (3). The first transformation which has to be applied in order to achieve this is stated in [4, Thm. 4.1.7] and uses the maximal (E,A,B)-invariant subspace included in $\ker C$. This subspace can be obtained easily via a subspace iteration as described in [4, Lem. 4.1.2]. The second transformation which has to be applied is stated in [4, Thm. 4.2.7]. Denote the resulting system by

$$\begin{bmatrix} s\overline{E} - \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} sE - A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I_m \end{bmatrix}.$$

(ii) Next we have to consider the subsystem

$$[\tilde{E},\tilde{A},\tilde{B},\tilde{C}] := \left[\begin{bmatrix} E_{22} & E_{23} \\ E_{32} & N \\ 0 & E_{43} \end{bmatrix}, \begin{bmatrix} A_{22} & 0 \\ 0 & I_{n_3} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I_m \\ 0 \\ 0 \end{bmatrix}, [I_p, 0]\right]$$

and transform it into a feedback form. To this end we introduce the following notation: For $j \in \mathbb{N}$, we define the matrices

$$N_j = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{j \times j}, \quad K_j = \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \; L_j = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \in \mathbb{R}^{(j-1) \times j}.$$

Further, let $e_i^{[j]} \in \mathbb{R}^j$ be the ith canonical unit vector, and, for some multi-index $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l$, we define

$$\begin{aligned} N_\alpha &= \operatorname{diag}(N_{\alpha_1}, \ldots, N_{\alpha_l}) \in \mathbb{R}^{|\alpha| \times |\alpha|}, \\ K_\alpha &= \operatorname{diag}(K_{\alpha_1}, \ldots, K_{\alpha_l}) \in \mathbb{R}^{(|\alpha|-l) \times |\alpha|}, \\ L_\alpha &= \operatorname{diag}(L_{\alpha_1}, \ldots, L_{\alpha_l}) \in \mathbb{R}^{(|\alpha|-l) \times |\alpha|}, \\ E_\alpha &= \operatorname{diag}(e_{\alpha_1}^{[\alpha_1]}, \ldots, e_{\alpha_l}^{[\alpha_l]}) \in \mathbb{R}^{|\alpha| \times l}. \end{aligned}$$

Then it was shown in [19] that a given system can, via state-space, input-space and feedback transformation, be put into a feedback canonical form. Here we use the feedback form from [8, Thm. 3.3], which is not canonical. Since $[\tilde{E},\tilde{A},\tilde{B}]$ is controllable in the behavioral sense as in [8, Def. 2.1] and $\operatorname{rk}\tilde{B} = m$, there exist $S \in \mathbf{Gl}_{\ell-k}(\mathbb{R})$, $T \in \mathbf{Gl}_{n-k}(\mathbb{R})$, $V \in \mathbf{Gl}_m(\mathbb{R})$, $F \in \mathbb{R}^{m \times (n-k)}$ such that

$$[s\hat{E}-\hat{A},\hat{B}] = S[s\tilde{E}-\tilde{A},\tilde{B}]\begin{bmatrix} T & 0 \\ -F & V \end{bmatrix},$$

where

$$[\hat{E},\hat{A},\hat{B}] = \left[\begin{bmatrix} I_{|\alpha|} & 0 & 0 & 0 & 0 \\ 0 & K_\beta & 0 & 0 & 0 \\ 0 & 0 & L_\gamma^\top & 0 & 0 \\ 0 & 0 & 0 & K_\delta^\top & 0 \\ 0 & 0 & 0 & 0 & N_\kappa \end{bmatrix}, \begin{bmatrix} N_\alpha^\top & 0 & 0 & 0 & 0 \\ 0 & L_\beta & 0 & 0 & 0 \\ 0 & 0 & K_\gamma^\top & 0 & 0 \\ 0 & 0 & 0 & L_\delta^\top & 0 \\ 0 & 0 & 0 & 0 & I_{|\kappa|} \end{bmatrix}, \begin{bmatrix} E_\alpha & 0 \\ 0 & 0 \\ 0 & E_\gamma \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right],$$

for some multi-indices $\alpha,\beta,\gamma,\delta,\kappa$.

(iii) Let $\mu \in \mathbb{R}$ be arbitrary. We construct a compatible control in the behavioral sense for $[\hat{E},\hat{A},\hat{B}]$ such that the interconnected system has Lyapunov exponent smaller or equal to μ. Let $F_{11} \in \mathbb{R}^{L(\alpha)\times|\alpha|}$ be such that

$$\max\{\ \mathrm{Re}\,\lambda \ \mid \lambda \in \sigma(N_\alpha + E_\alpha F_{11})\ \} \le \mu.$$

This can be achieved as follows: For $j = 1,\ldots,L(\alpha)$, consider vectors

$$a_j = -[a_{j\alpha_j-1},\ldots,a_{j0}] \in \mathbb{R}^{1\times\alpha_j}.$$

Then, for

$$F_{11} = \mathrm{diag}\,(a_1,\ldots,a_{L(\alpha)}) \in \mathbb{R}^{L(\alpha)\times|\alpha|},$$

the matrix $N_\alpha + E_\alpha F_{11}$ is diagonally composed of companion matrices, whence, for

$$p_j(s) = s^{\alpha_j} + a_{j\alpha_j-1}s^{\alpha_j-1} + \ldots + a_{j0} \in \mathbb{R}[s]$$

the characteristic polynomial of $N_\alpha + E_\alpha F_{11}$ is given by

$$\det(sI_{|\alpha|} - (N_\alpha + E_\alpha F_{11})) = \prod_{j=1}^{L(\alpha)} p_j(s).$$

Hence, choosing the coefficients a_{ji}, $j = 1,\ldots,L(\alpha)$, $i = 0,\ldots,\alpha_j$ such that the roots of the polynomials $p_1(s),\ldots,p_{L(\alpha)}(s) \in \mathbb{R}[s]$ are all smaller or equal to μ yields the assertion.

Now we find that

$$k_L\left(\begin{bmatrix} I_{|\alpha|} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} N_\alpha & E_\alpha \\ F_{11} & -I_{L(\alpha)} \end{bmatrix}\right) \le \mu.$$

Furthermore, by the same reasoning as above, for

$$a_j = [a_{j\beta_j-2},\ldots,a_{j0},1] \in \mathbb{R}^{1\times\beta_j}$$

with the property that the roots of the polynomials

$$p_j(s) = s^{\beta_j} + a_{j\beta_j-1}s^{\beta_j-1} + \ldots + a_{j0} \in \mathbb{R}[s]$$

are all smaller or equal to μ for $j = 1,\dots,L(\alpha)$, the choice

$$K_x = \operatorname{diag}(a_1,\dots,a_{L(\beta)}) \in \mathbb{R}^{L(\beta)\times|\beta|}$$

leads to

$$k_L\left(\begin{bmatrix}K_\beta\\0\end{bmatrix},\begin{bmatrix}L_\beta\\K_x\end{bmatrix}\right) \le \mu.$$

Therefore, the control matrix

$$\hat{K} = [\hat{K}_1,\hat{K}_2] = \begin{bmatrix}F_{11} & 0 & 0\;0\;0 & -I_{L(\alpha)} & 0\\ 0 & K_x & 0\;0\;0 & 0 & 0\end{bmatrix} \in \mathbb{R}^{q\times(n-k)}\times\mathbb{R}^{q\times m},$$

where $q = L(\alpha)+L(\beta)$, establishes that

$$k_L\left(\begin{bmatrix}\hat{E} & 0\\0 & 0\end{bmatrix},\begin{bmatrix}\hat{A} & \hat{B}\\ \hat{K}_1 & \hat{K}_2\end{bmatrix}\right) \le \mu.$$

Since the differential variables can be arbitrarily initialized in any of the previously discussed subsystems, the constructed control $\hat{K}$ is also compatible for $[\hat{E},\hat{A},\hat{B}]$.

(iv) We show that $\hat{K}$ leads to a compatible control $\tilde{K}$ for $[\tilde{E},\tilde{A},\tilde{B}]$ such that the interconnected system has Lyapunov exponent smaller or equal to μ. Observe that

$$\begin{bmatrix}S^{-1} & 0\\0 & I_q\end{bmatrix}\begin{bmatrix}s\hat{E}-\hat{A} & \hat{B}\\ \hat{K}_1 & \hat{K}_2\end{bmatrix}\begin{bmatrix}T^{-1} & 0\\ V^{-1}FT^{-1} & V^{-1}\end{bmatrix} = \begin{bmatrix}s\tilde{E}-\tilde{A} & \tilde{B}\\ \hat{K}_1+\hat{K}_2V^{-1}FT^{1} & \hat{K}_2V^{-1}\end{bmatrix}$$

and hence, by invariance of the Lyapunov exponent under transformation of the system (see e.g. [3, Prop. 3.17]), we find that for

$$[K_1,K_2] := [\hat{K}_1+\hat{K}_2V^{-1}FT^{1},\hat{K}_2V^{-1}] \in \mathbb{R}^{q\times(n-k)}\times\mathbb{R}^{q\times m},$$

we have

$$k_L\left(\begin{bmatrix}\tilde{E} & 0\\0 & 0\end{bmatrix},\begin{bmatrix}\tilde{A} & \tilde{B}\\ K_1 & K_2\end{bmatrix}\right) \le \mu.$$

(v) If $k = \dim \mathscr{ZD}_{(1)} = 0$, then we can choose $\mu \in \mathbb{R}$ as we like and obtain

$$k_L\left(\begin{bmatrix}\overline{E} & 0\\0 & 0\end{bmatrix},\begin{bmatrix}\overline{A} & \overline{B}\\ \overline{K}_x := K_1 & \overline{K}_u := K_2\end{bmatrix}\right) = k_L\left(\begin{bmatrix}\tilde{E} & 0\\0 & 0\end{bmatrix},\begin{bmatrix}\tilde{A} & \tilde{B}\\ K_1 & K_2\end{bmatrix}\right) \le \mu.$$

If $k > 0$, then we can choose $\mu < k_L(\mathscr{ZD}_{(1)})$ and obtain, with

$$[\overline{K}_x\,|\,\overline{K}_u] := [K_2A_{21},K_1\,|\,K_2] \in \mathbb{R}^{q\times k}\times\mathbb{R}^{q\times(n-k)}\times\mathbb{R}^{q\times m},$$

that

$$k_L\left(\begin{bmatrix}\overline{E} & 0\\0 & 0\end{bmatrix},\begin{bmatrix}\overline{A} & \overline{B}\\ \overline{K}_x & \overline{K}_u\end{bmatrix}\right) = k_L(\mathscr{ZD}_{(1)}).$$

This is shown in the proof of Theorem 4.4.

(vi) The desired compatible control K for $[E,A,B,C]$ is now given by

$$K = [\overline{K}_x Q^{-1}, \overline{K}_u].$$

Note that the practical computation of the decompositions in (i) and (ii) is in general not numerically stable. This can be achieved by using orthogonal transformations and condensed forms as in [11]. It seems that with some effort the form (3) in (i) can also be obtained with orthogonal transformations, but this needs to be investigated in detail. Instead of the feedback form from [8, Thm. 3.3] in (ii) a condensed form from [11] could be used. However, in the present work we do not focus on the numerical aspect.

Remark 4.6 (Implementation of the control)**.** As explained in the beginning of this section, the control law

$$K_x x(t) + K_u u(t) = 0$$

cannot necessarily be solved for $u(t)$. This raises the question for the implementation of the controller. There are basically two perspectives in this regard:

(i) In order to implement the controller it is necessary that all free variables of the open-loop system $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t)$ can be manipulated. The free variables of the system can be identified via the quasi-Kronecker form [9,10] of the pencil $s[E,0] - [A,B]$; each underdetermined block $sK_{\beta_i} - L_{\beta_i}$ in the quasi-Kronecker form yields one free variable, i.e., there are $L(\beta)$ free variables in the system. The set of free variables may consist of input variables as well as state variables and not necessarily all input variables are free variables. For the implementation of the control, the free variables are treated as controls and the control law can be solved for the free variables. A similar approach has been discussed in [12].

(ii) For an alternative approach, where we do not wish to reinterpret variables, we use the fact that (cf. also Remark 4.5 (iii)) the control law can be rewritten in the form

$$\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} x(t) + \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = 0,$$

where a suitable input space transformation has been performed. Then we may solve the first row for $u_1(t)$ and implement this control. It only remains to implement the algebraic condition $K_2 x(t) = 0$. In practice, this relation can be implemented by integrating appropriate components (such as dampers or resistors) into the given plant. In particular, it is not necessary to (actively) manipulate specific state variables, only the implementation of an algebraic relation between some of the state variables is necessary.

Theorem 4.4 shows that right-invertible systems $[E,A,B,C] \in \Sigma_{\ell,n,m,p}$ with asymptotically stable zero dynamics can be stabilized by a compatible control so that any solution of the interconnected system tends to zero. It is well known [15, Rem. 6.1.3] that any linear ODE system with asymptotically stable zero dynamics (and $p = m$)

is stabilizable by *state feedback*, i.e., the compatible control is of the form $u = Fx$. While there is a lot of literature on the state feedback stabilization of linear DAEs, see e.g. [8,17,23,25], it seems that the stabilization problem for systems with asymptotically stable zero dynamics has not been investigated yet. For regular DAE systems we obtain the following result.

Proposition 4.7 (Regular systems and state feedback)**.**
If $sE - A \in \mathbb{R}[s]^{n\times n}$ in Theorem 4.4 is regular, then the compatible control K can be chosen as a state feedback in each case, i.e., $K = [K_x, -I_m] \in \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times m}$.

Proof. We use the procedure presented in Remark 4.5 and modify it at some instances.

(i) Consider the system $[\hat{E}, \hat{A}, \hat{B}, \hat{C}]$ from Remark 4.5 (ii) and observe that, using the same argument as in [8, Rem. 5.3 (i)], we obtain $L(\delta) = 0$ and $L(\beta) = L(\gamma)$.

(ii) For any multi-index $\eta \in \mathbb{N}^l$ let

$$F_\eta := \operatorname{diag}\left(e_1^{[\eta_1]}, \ldots, e_1^{[\eta_l]}\right) \in \mathbb{R}^{|\eta|\times l}.$$

A straightforward calculation yields that there exists a permutation matrix $P \in \mathbb{R}^{\xi\times\xi}$, $\xi := |\beta| + |\gamma| - L(\gamma)$, such that

$$P\left(s\begin{bmatrix} K_\beta & 0 \\ 0 & L_\gamma^\top \end{bmatrix} - \begin{bmatrix} L_\beta & 0 \\ E_\gamma F_\beta^\top & K_\gamma^\top \end{bmatrix}\right) = s\tilde{N} - I_\xi,$$

where $\tilde{N} \in \mathbb{R}^{\xi\times\xi}$ is nilpotent.

(iii) Changing the control matrix $\hat{K}$ in Remark 4.5 (iii) to

$$\hat{K} = [\hat{K}_1, \hat{K}_2] = \begin{bmatrix} F_{11} & 0 & 0 & 0 & 0 & -I_{L(\alpha)} & 0 \\ 0 & F_\beta^\top & 0 & 0 & 0 & 0 & -I_{L(\gamma)} \end{bmatrix} \in \mathbb{R}^{m\times(n-k)} \times \mathbb{R}^{m\times m},$$

where it is worth noting that $L(\alpha) + L(\gamma) = m$, and invoking the observation in (ii), we obtain the same result for the Lyapunov exponent, and the control can be equivalently expressed as a state feedback

$$u_1 = F_{11}x_1, \quad u_2 = F_\beta^\top x_2.$$

Since $\hat{K}_2 = -I_m$ we can write $[K_1, K_2]$ in Remark 4.5 (iv) as

$$[K_1, K_2] = [V\hat{K}_1 - FT^{-1}, -I_m]$$

and, furthermore, we have $\overline{K}_u = -I_m$ in Remark 4.5 (vi). Therefore, the compatible control K is a state feedback. □

Acknowledgements I am indebted to Achim Ilchmann (Ilmenau University of Technology) for several constructive discussions.

References

1. Adams, R.A.: Sobolev Spaces. No. 65 in Pure and Applied Mathematics. Academic Press, New York, London (1975)
2. Bender, D., Laub, A.: The linear quadratic optimal regulator problem for descriptor systems. IEEE Trans. Autom. Control **32**, 672–688 (1987)
3. Berger, T.: Bohl exponent for time-varying linear differential-algebraic equations. Int. J. Control **85**(10), 1433–1451 (2012).
4. Berger, T.: On differential-algebraic control systems. Ph.D. thesis, Institut für Mathematik, Technische Universität Ilmenau, Universitätsverlag Ilmenau, Ilmenau, Germany (2013)
5. Berger, T.: Zero dynamics and funnel control of general linear differential-algebraic systems (2013). Submitted to *Math. Control Signals Syst.*, preprint available from the website of the author
6. Berger, T., Ilchmann, A., Reis, T.: Normal forms, high-gain, and funnel control for linear differential-algebraic systems. In: L.T. Biegler, S.L. Campbell, V. Mehrmann (eds.) Control and Optimization with Differential-Algebraic Constraints, *Advances in Design and Control*, vol. 23, pp. 127–164. SIAM, Philadelphia (2012)
7. Berger, T., Ilchmann, A., Reis, T.: Zero dynamics and funnel control of linear differential-algebraic systems. Math. Control Signals Syst. **24**(3), 219–263 (2012)
8. Berger, T., Reis, T.: Controllability of linear differential-algebraic systems - a survey. In: A. Ilchmann, T. Reis (eds.) Surveys in Differential-Algebraic Equations I, Differential-Algebraic Equations Forum, pp. 1–61. Springer-Verlag, Berlin-Heidelberg (2013)
9. Berger, T., Trenn, S.: The quasi-Kronecker form for matrix pencils. SIAM J. Matrix Anal. & Appl. **33**(2), 336–368 (2012)
10. Berger, T., Trenn, S.: Addition to “The quasi-Kronecker form for matrix pencils”. SIAM J. Matrix Anal. & Appl. **34**(1), 94–101 (2013).
11. Bunse-Gerstner, A., Byers, R., Mehrmann, V., Nichols, N.K.: Feedback design for regularizing descriptor systems. Lin. Alg. Appl. **299**, 119–151 (1999)
12. Campbell, S.L., Kunkel, P., Mehrmann, V.: Regularization of linear and nonlinear descriptor systems. In: L.T. Biegler, S.L. Campbell, V. Mehrmann (eds.) Control and Optimization with Differential-Algebraic Constraints, *Advances in Design and Control*, vol. 23, pp. 17–36. SIAM, Philadelphia (2012)
13. Francis, B.A., Wonham, W.M.: The role of transmission zeros in linear multivariable regulators. Int. J. Control **22**(5), 657–681 (1975)
14. Hou, M., Müller, P.C.: Observer design for descriptor systems. IEEE Trans. Autom. Control **44**(1), 164–168 (1999)
15. Isidori, A.: Nonlinear Control Systems, 3rd edn. Communications and Control Engineering Series. Springer-Verlag, Berlin (1995)
16. Kailath, T.: Linear Systems. Prentice-Hall, Englewood Cliffs, NJ (1980)
17. Lewis, F.L.: A survey of linear singular systems. IEEE Proc. Circuits, Systems and Signal Processing **5**(1), 3–36 (1986)
18. Linh, V.H., Mehrmann, V.: Lyapunov, Bohl and Sacker-Sell spectral intervals for differential-algebraic equations. J. of Dynamics and Differential Equations **21**, 153–194 (2009)
19. Loiseau, J.J., Özçaldiran, K., Malabre, M., Karcanias, N.: Feedback canonical forms of singular systems. Kybernetika **27**(4), 289–305 (1991)
20. Polderman, J.W., Willems, J.C.: Introduction to Mathematical Systems Theory. A Behavioral Approach. Springer-Verlag, New York (1998)
21. Rosenbrock, H.H.: State Space and Multivariable Theory. John Wiley and Sons Inc., New York, NY (1970)
22. Trentelman, H.L., Stoorvogel, A.A., Hautus, M.L.J.: Control Theory for Linear Systems. Communications and Control Engineering. Springer-Verlag, London (2001)
23. Varga, A.: On stabilization methods of descriptor systems. Syst. Control Lett. **24**, 133–138 (1995)

24. Willems, J.C.: The behavioral approach to open and interconnected systems. IEEE Control Systems Magazine **27**(6), 46–99 (2007)
25. Xu, S., Van Dooren, P., Ştefan, R., Lam, J.: Robust stability and stabilization for singular systems with state delay and parameter uncertainty. IEEE Trans. Autom. Control **47**(7), 1122–1128 (2002)