

Zero dynamics of time-varying linear systems

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Dedicated to the memory of Christopher I. Byrnes

Abstract

The Byrnes-Isidori form with respect to the relative degree is studied for time-varying linear multi-input, multi-output systems. It is clarified in which sense this form is a normal form. (A, B) -invariant time-varying subspaces are defined and the maximal (A, B) -invariant time-varying subspace included in the kernel of C is characterized. This is exploited to characterize the zero dynamics of the system. Finally, a high-gain derivative output feedback controller is introduced for the class of systems with higher relative degree and stable zero dynamics. All results are also new for time-invariant linear systems.

Keywords: Time-varying systems, linear systems, strict relative degree, zero dynamics, Byrnes-Isidori form, (A, B) -invariance, output feedback stabilization

1 Introduction

We study time-varying multi-input, multi-output linear systems of the form

$$\left. \begin{aligned} \dot{x} &= A(t)x + B(t)u(t) \\ y(t) &= C(t)x(t), \end{aligned} \right\} \quad (1.1)$$

where $A \in \mathcal{C}^\ell(\mathcal{T}; \mathbb{R}^{n \times n})$, $B, C^\top \in \mathcal{C}^\ell(\mathcal{T}; \mathbb{R}^{n \times m})$ on some open set $\mathcal{T} \subseteq \mathbb{R}$ and $\ell \in \mathbb{N}$.

After we recall (see [IM07]) the definition of strict and uniform relative degree on \mathcal{T} for systems (1.1) in Section 2, this concept is used to derive a Byrnes-Isidori form for the system. The latter is well-known; for time-invariant (nonlinear) systems see [Isi95, p. 137,220] and for time-varying systems see [IM07]. However, to the best of our knowledge, it has not yet been investigated in which sense the Byrnes-Isidori form is a normal form. This is clarified in Section 2. The Byrnes-Isidori form, certainly of theoretical interest in its own right, is also a main tool for the following sections: in Section 3, the vector space (or dynamical system) of zero dynamics (as defined in [IM07]) is characterized; the maximal (A, B) -invariant time-varying subspace included in the kernel of C (as defined in [Ilc89]) is characterized in Section 4; finally, in Section 5 a simple high-gain derivative output feedback controller is presented; this result generalizes well known results for time-invariant systems which have relative degree one (see e.g. [Ilc93]) and simplifies the controller [IM07, (4.2)] for time-varying systems which have higher relative degree. All main results of Sections 2-5 are also new for time-invariant linear systems.

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Nomenclature

\mathbb{N}, \mathbb{N}_0	the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\mathbb{C}_+, \mathbb{C}_-$	the sets of complex numbers with positive and negative real parts, resp.
$\ x\ $	$= \sqrt{x^\top x}$, the Euclidean norm of $x \in \mathbb{R}^n$
$\ M\ $	$= \max \{ \ Mx\ \mid x \in \mathbb{R}^m, \ x\ = 1 \}$, induced matrix norm of $M \in \mathbb{R}^{n \times m}$
$\mathbf{GL}_n(\mathbb{R})$	the general linear group of degree n , i.e. the set of invertible $n \times n$ matrices
$\mathcal{C}^\ell(\mathcal{T}; \mathbb{R}^n)$	the set of ℓ -times continuously differentiable functions $f : \mathcal{T} \rightarrow \mathbb{R}^n$ on the open set $\mathcal{T} \subseteq \mathbb{R}$
$\mathcal{AC}(\mathcal{T}; \mathbb{R}^n)$	the set of absolutely continuous functions $f : \mathcal{T} \rightarrow \mathbb{R}^n$ on the open set $\mathcal{T} \subseteq \mathbb{R}$, see [HP05, Def. A.3.12]
$\mathcal{PC}(\mathcal{T}; \mathbb{R}^n)$	the set of piecewise continuous functions $f : \mathcal{T} \rightarrow \mathbb{R}^n$ on the open set $\mathcal{T} \subseteq \mathbb{R}$, i.e. f is left continuous and has only finitely many discontinuities on any compact subset of \mathcal{T}
$\mathcal{L}^\infty(\mathcal{T}; \mathbb{R}^n)$	the set of essentially bounded functions $f : \mathcal{T} \rightarrow \mathbb{R}^n$ on the open set $\mathcal{T} \subseteq \mathbb{R}$

2 Relative degree and Byrnes-Isidori form

In this section we recall the concept of strict and uniform relative degree for time-varying linear systems. This allows to transform the system into Byrnes-Isidori form. The latter has been derived for time-varying linear systems in [IM07]. However, even for time-invariant systems, it was not clarified so far in which sense the Byrnes-Isidori form is a normal form. This will be done here. Moreover, we stress that the form allows to write the system as a decomposition into subsystems as depicted in Figure 1.

The following operator $(\frac{d}{dt} + A(t)_r)$, where the sub-script r in $A_r(C)$ indicates that A acts on C by multiplication from the right, has already been proved an advantageous notation for time-varying linear systems in [Fre71, IM07, Por69, Sil68].

Notation 2.1 (The operator $(\frac{d}{dt} + A(t)_r)^k$).

Let $\ell \in \mathbb{N}_0$, $\mathcal{T} \subseteq \mathbb{R}$ an open set, $A \in \mathcal{C}^\ell(\mathcal{T}; \mathbb{R}^{n \times n})$ and $C \in \mathcal{C}^\ell(\mathcal{T}; \mathbb{R}^{m \times n})$. Set

$$\begin{aligned} \forall t \in \mathcal{T} : \quad & \left(\frac{d}{dt} + A(t)_r\right)^0 (C(t)) := C(t), \\ \forall t \in \mathcal{T} : \quad & \left(\frac{d}{dt} + A(t)_r\right) (C(t)) := \dot{C}(t) + C(t)A(t), \\ \forall t \in \mathcal{T} \forall k \in \{1, \dots, \ell\} : \quad & \left(\frac{d}{dt} + A(t)_r\right)^k (C(t)) := \left(\frac{d}{dt} + A(t)_r\right) \left(\left(\frac{d}{dt} + A(t)_r\right)^{k-1} (C(t)) \right). \end{aligned}$$

The concept of relative degree is well known for time-invariant nonlinear SISO systems [Isi95, p. 137], time-invariant nonlinear MIMO systems [Isi95, p. 220], [LMS02], and for time-varying nonlinear MIMO systems [IM07, Def. 2.2]. When restricting the latter to time-varying linear systems, it is shown in [IM07, Thm. 2.7] that the definition of relative degree becomes as follows.

Definition 2.2 (Relative degree).

Let $\rho, \ell \in \mathbb{N}$ with $\rho \leq \ell$. Then the time-varying linear system (1.1) has *strict and uniform relative*

degree ρ (on \mathcal{T}) if, and only if,

$$\left. \begin{aligned} \forall t \in \mathcal{T} \quad \forall k = 0, \dots, \rho - 2 : \quad \left(\frac{d}{dt} + A(t)_r \right)^k (C(t)) B(t) &= 0_{m \times m} \\ \forall t \in \mathcal{T} : \quad \left(\frac{d}{dt} + A(t)_r \right)^{\rho-1} (C(t)) B(t) &\in \mathbf{GL}_m(\mathbb{R}). \end{aligned} \right\} \quad (2.1)$$

Remark 2.3.

(i) If (1.1) is a time-invariant system, then it is straightforward to see that

$$\forall k \in \mathbb{N}_0 : \left(\frac{d}{dt} + A(\cdot)_r \right)^k (C(\cdot)) B(\cdot) = CA^k B$$

and hence the conditions in (2.1) are equivalent to

$$CA^{\rho-1} B \in \mathbf{GL}_m(\mathbb{R}) \quad \text{and} \quad \forall k = 0, \dots, \rho - 2 : CA^k B = 0.$$

(ii) The notion ‘uniform’ refers to the time set \mathcal{T} in the sense that the conditions in (2.1) have to hold for all $t \in \mathcal{T}$.

(iii) The notion ‘strict’ is superfluous for single-input, single-output systems. However, for multivariable systems we may have $CA^k B = 0$ for all $k = 0, \dots, \rho - 2$ and $CA^{\rho-1} B \neq 0$ but $CA^{\rho-1} B \notin \mathbf{GL}_m(\mathbb{R})$. In this case, one may introduce the concept of a vector relative degree: the vector $(\rho_1, \dots, \rho_m) \in \mathbb{N}^m$ collects the smallest number of times ρ_j one has to differentiate $y_j(\cdot)$ so that the input occurs explicitly in $y_j^{(\rho_j)}(\cdot)$. This is not considered in the present note, for further details see [Isi95, Sec. 5.1] and [Mue09]. \diamond

As known for time-invariant systems, the relative degree ρ is the least number of times one has to differentiate the output $y(\cdot)$ so that the input occurs explicitly in $y^{(\rho)}(\cdot)$. That this also holds for time-varying systems is made explicit in the following proposition.

Proposition 2.4.

Let $\rho, \ell \in \mathbb{N}$ such that $\rho \leq \ell$ and consider a time-varying linear system (1.1) which has strict and uniform relative degree ρ . Then every solution $(x, u, y) \in \mathcal{AC}(\mathcal{T}; \mathbb{R}^n) \times \mathcal{PC}(\mathcal{T}; \mathbb{R}^m) \times \mathcal{AC}(\mathcal{T}; \mathbb{R}^m)$ of (1.1) satisfies the following:

$$\forall j = 0, \dots, \rho - 1 : y^{(j)} = \left(\frac{d}{dt} + A_r \right)^j (C) x \quad \text{a.e. on } \mathcal{T}, \quad (2.2)$$

$$y^{(\rho)} = \left(\frac{d}{dt} + A_r \right)^\rho (C) x + \left[\left(\frac{d}{dt} + A_r \right)^{\rho-1} (C) B \right] u \quad \text{a.e. on } \mathcal{T}. \quad (2.3)$$

Proof: We show (2.2) by induction over $j = 0, \dots, \rho - 1$. For $j = 0$ the statement is clear. Suppose it holds for some $j \in \{0, \dots, \rho - 2\}$. Then, invoking Definition 2.2 we have, for almost all $t \in \mathcal{T}$,

$$\begin{aligned} y^{(j+1)}(t) &= \frac{d}{dt} \left[\left(\frac{d}{dt} + A(t)_r \right)^j (C(t)) x(t) \right] \\ &= \left[\frac{d}{dt} \left(\frac{d}{dt} + A(t)_r \right)^j (C(t)) \right] x(t) + \left(\frac{d}{dt} + A(t)_r \right)^j (C(t)) (A(t)x(t) + B(t)u(t)) \\ &= \left[\frac{d}{dt} \left(\frac{d}{dt} + A(t)_r \right)^j (C(t)) + \left(\frac{d}{dt} + A(t)_r \right)^j (C(t)) A(t) \right] x(t) + \left(\frac{d}{dt} + A(t)_r \right)^j (C(t)) B(t) u(t) \\ &\stackrel{(2.1)}{=} \left(\frac{d}{dt} + A(t)_r \right)^{j+1} (C(t)) x(t). \end{aligned}$$

Now we may derive that

$$\text{for almost all } t \in \mathcal{T} : y^{(\rho)}(t) = \left(\frac{d}{dt} + A(t)_r \right)^\rho (C(t)) x(t) + \left(\frac{d}{dt} + A(t)_r \right)^{\rho-1} (C(t)) B(t) u(t). \quad \square$$

Note that we consider behaviours $(x, u, y) \in \mathcal{AC}(\mathcal{T}; \mathbb{R}^n) \times \mathcal{PC}(\mathcal{T}; \mathbb{R}^m) \times \mathcal{AC}(\mathcal{T}; \mathbb{R}^m)$. This is to some extent a matter of choice. We could choose the inputs more smooth, e.g. $u \in \mathcal{C}^k(\mathcal{T}; \mathbb{R}^m)$, then the solution x would become more smooth, $x \in \mathcal{C}^{k+1}(\mathcal{T}; \mathbb{R}^n)$.

The following theorem states a normal form for time-varying linear systems (1.1). The advantage of this form is that it expresses the dynamical properties of the system by allowing u only to effect the ρ^{th} derivative (ρ the relative degree) of the output and separating another part of the dynamics which is only influenced by y . See Figure 1.

The Byrnes-Isidori form has been derived for time-varying linear systems in [IM07, Th. 3.5]. Here we recall this form for later use and also clarify in which sense the form is a normal form. The latter is also new for time-invariant systems. We show that the entries R_1, \dots, R_ρ, S and Γ in (2.5) are uniquely defined; whereas Q and P are unique modulo a coordinate transformation. For time-invariant systems, the Byrnes-Isidori form is implicitly contained in [Isi95, Sec. 5.1]. The form decouples the zero dynamics from the rest of the system, see Remark 5.2 and Figure 1.

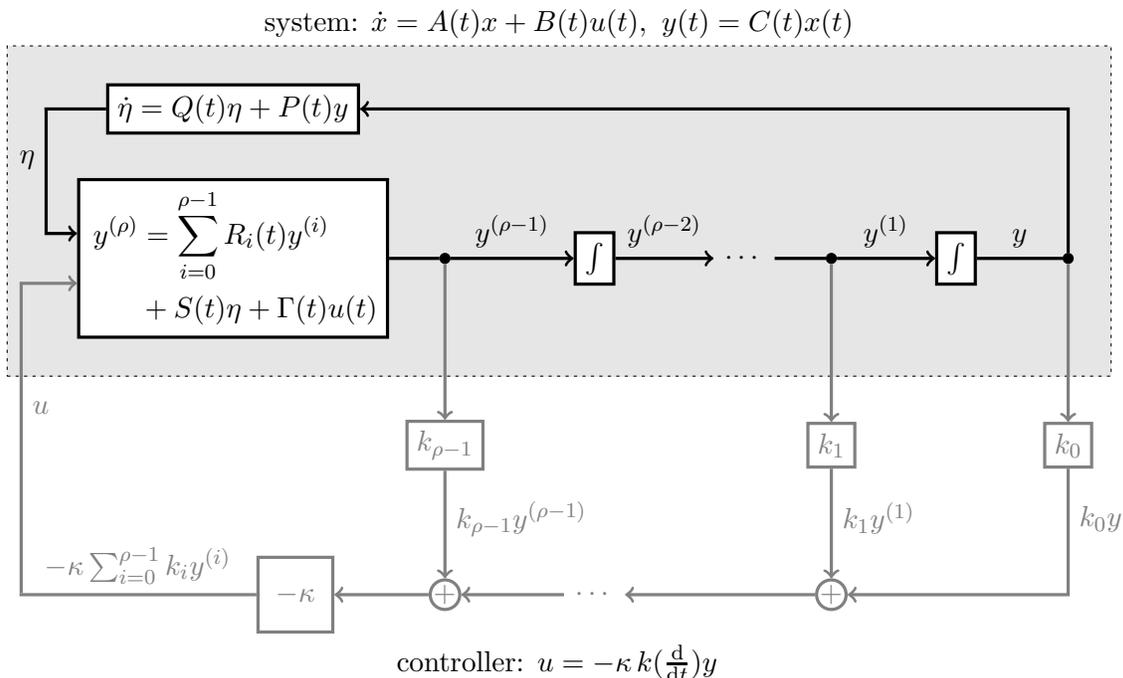


Figure 1: Byrnes-Isidori form and derivative output feedback controller

Theorem 2.5 (Byrnes-Isidori form).

Let $\rho, \ell \in \mathbb{N}$ such that $\rho \leq \ell$ which has strict and uniform relative degree ρ . Then there exists a coordinate transformation $U \in \mathcal{C}^1(\mathcal{T}, \mathbf{GL}_n(\mathbb{R}))$ such that

$$\begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} := \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \\ \vdots \\ \xi_\rho(t) \\ \eta(t) \end{pmatrix} := \begin{pmatrix} y(t) \\ y^{(1)}(t) \\ \vdots \\ y^{(\rho-1)}(t) \\ \eta(t) \end{pmatrix} = U(t)x(t), \quad (2.4)$$

transforms (1.1) into *Byrnes-Isidori form*

$$\left. \begin{aligned}
\dot{\xi}(t) &= \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ 0 & 0 & I_m & & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_m \\ R_1(t) & R_2(t) & \cdots & R_{\rho-1}(t) & R_\rho(t) \end{bmatrix} \xi(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ S(t) \end{bmatrix} \eta(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Gamma(t) \end{bmatrix} u(t), \\
\dot{\eta}(t) &= \begin{bmatrix} P(t) & 0 & \cdots & 0 & 0 \end{bmatrix} \xi(t) + Q(t) \eta(t) \\
y(t) &= \begin{bmatrix} I_m & 0 & \cdots & 0 & | & 0 \end{bmatrix} \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix}
\end{aligned} \right\} \quad (2.5)$$

with initial condition

$$\begin{pmatrix} \xi(0) \\ \eta(0) \end{pmatrix} = \begin{pmatrix} \xi^0 \\ \eta^0 \end{pmatrix} = \begin{pmatrix} y(0) \\ \vdots \\ y^{(\rho-1)}(0) \\ \eta^0 \end{pmatrix} = U(0) x^0; \quad (2.6)$$

and

- (i) $\Gamma = \left(\frac{d}{dt} + A_r\right)^{\rho-1} (C)B \in \mathcal{C}^1(\mathcal{T}; \mathbf{G}\mathbf{l}_m(\mathbb{R}))$,
- (ii) $[R_1, \dots, R_\rho, S] = \left(\frac{d}{dt} + A_r\right)^\rho (C)U^{-1} \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{m \times n})$,
- (iii) $(P, Q) \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{(n-\rho m) \times m}) \times \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{(n-\rho m) \times (n-\rho m)})$ is unique up to $(\tilde{Y}^{-1}P, \tilde{Y}^{-1}Q\tilde{Y})$ for some $\tilde{Y} \in \mathcal{C}^1(\mathcal{T}; \mathbf{G}\mathbf{l}_{n-\rho m}(\mathbb{R}))$.
- (iv) A possible transformation is given by $U = \begin{bmatrix} \mathcal{C} \\ N \end{bmatrix}$, where

$$\begin{aligned}
\mathcal{C} &:= \begin{bmatrix} C \\ \left(\frac{d}{dt} + A_r\right)(C) \\ \vdots \\ \left(\frac{d}{dt} + A_r\right)^{\rho-1}(C) \end{bmatrix} \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{\rho m \times n}) \\
\mathcal{B} &:= \left[B, \left(\frac{d}{dt} - A\right)(B), \dots, \left(\frac{d}{dt} - A\right)^{\rho-1}(B) \right] \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{n \times \rho m}) \\
N &:= (V^\top V)^{-1}V^\top [I - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}] \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{(n-\rho m) \times n}).
\end{aligned}$$

and $V \in \mathcal{L}^\infty(\mathcal{T}; \mathbb{R}^{n \times (n-\rho m)}) \cap \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{n \times (n-\rho m)})$ such that $(V^\top V)^{-1}V^\top \in \mathcal{L}^\infty(\mathcal{T}; \mathbb{R}^{(n-\rho m) \times n})$ and

$$\forall t \in \mathcal{T} : \text{im } V(t) = \ker \mathcal{C}(t) \quad \text{and} \quad \text{rk } V(t)^\top V(t) = n - \rho m.$$

Proof: The Byrnes-Isidori form (2.5) and the statements (i) and (ii) are proved in [IM07, Thm. 3.5]. The existence of V in statement (iv) is shown in [IM07, Rem. 3.4]. It is straightforward to show that $U^{-1} = [\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}, V]$. So it remains to prove uniqueness in statement (iii). Note that Γ in (i) is unique, it depends on (A, B, C) only.

Let

$$(\hat{A}, \hat{B}, \hat{C}) := ((UA + \dot{U})U^{-1}, UB, CU^{-1}) \quad (2.7)$$

for U as in (iv). Then

$$\hat{A} = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_m & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & I_m & 0 \\ R_1 & R_2 & \cdots & R_{\rho-1} & R_\rho & S \\ P & 0 & \cdots & 0 & 0 & Q \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Gamma \\ 0 \end{bmatrix}, \quad \hat{C} = [I_m, 0, \dots, 0] \quad (2.8)$$

holds (see [IM07, Thm. 3.5]) for (P, Q) given by [IM07, (3.9)-(3.12)].

Consider next

$$(\tilde{A}, \tilde{B}, \tilde{C}) = ((WA + \dot{W})W^{-1}, WB, CW^{-1}) \quad (2.9)$$

for any $W \in \mathcal{C}^1(\mathcal{T}; \mathbf{GL}_n(\mathbb{R}))$ such that

$$\tilde{A} = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_m & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & I_m & 0 \\ \tilde{R}_1 & \tilde{R}_2 & \cdots & \tilde{R}_{\rho-1} & \tilde{R}_\rho & \tilde{S} \\ \tilde{P} & 0 & \cdots & 0 & 0 & \tilde{Q} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Gamma \\ 0 \end{bmatrix}, \quad \tilde{C} = [I_m, 0, \dots, 0]. \quad (2.10)$$

We show that

$$\left. \begin{aligned} \tilde{S} &= S, & \tilde{R}_i &= R_i, & \forall i &= 1, \dots, \rho, \\ \tilde{P} &= \tilde{Y}P, & \tilde{Q} &= \tilde{Y}Q\tilde{Y}^{-1} & \text{for some } \tilde{Y} &\in \mathcal{C}^1(\mathcal{T}; \mathbf{GL}_{n-\rho m}(\mathbb{R})). \end{aligned} \right\} \quad (2.11)$$

Set

$$WU^{-1} =: Y = \begin{bmatrix} Y^1 \\ \vdots \\ Y^{\rho+1} \end{bmatrix} = [Y_1, \dots, Y_{\rho+1}] \quad (2.12)$$

for $Y^i, (Y^i)^\top \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{m \times n})$, $i = 1, \dots, \rho$, and $Y^{\rho+1}, (Y^{\rho+1})^\top \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{(n-\rho m) \times n})$. Then (2.7) and (2.9) together with $\frac{d}{dt}(U^{-1}) = -U^{-1}\dot{U}U^{-1}$ yield

$$\begin{aligned} (Y\hat{A} + \dot{Y})Y^{-1} &= (WAU^{-1} + WU^{-1}\dot{U}U^{-1} + \dot{W}U^{-1} + W\frac{d}{dt}(U^{-1}))UW^{-1} \\ &= WAW^{-1} + \dot{W}W^{-1} = \tilde{A}. \end{aligned}$$

Thus

$$(Y\hat{A} + \dot{Y})Y^{-1} = \tilde{A}, \quad (2.13)$$

$$Y\hat{B} = \tilde{B}, \quad (2.14)$$

$$\hat{C} = \tilde{C}Y. \quad (2.15)$$

This gives

$$Y^1 \stackrel{(2.15)}{=} [I_m, 0, \dots, 0] \quad \text{and} \quad Y_\rho \stackrel{(2.14)}{=} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_m \\ 0 \end{bmatrix}, \quad (2.16)$$

and we proceed

$$\begin{aligned}
[0, I_m, 0, \dots, 0] &\stackrel{(2.8)}{=} Y^1 \hat{A} + \frac{d}{dt} Y^1 &\stackrel{(2.16)}{=} Y^1 (Y \hat{A} + \dot{Y}) &\stackrel{(2.13)}{=} Y^1 \tilde{A} Y &\stackrel{(2.12)}{=} Y^2 \\
[0, 0, I_m, 0, \dots, 0] &\stackrel{(2.8)}{=} Y^2 \hat{A} + \frac{d}{dt} Y^2 &= Y^2 (Y \hat{A} + \dot{Y}) &\stackrel{(2.13)}{=} Y^2 \tilde{A} Y &\stackrel{(2.12)}{=} Y^3 \\
&\vdots &&&& \\
[0, \dots, 0, I_m, 0] &\stackrel{(2.8)}{=} Y^{\rho-1} \hat{A} + \frac{d}{dt} Y^{\rho-1} &= Y^{\rho-1} (Y \hat{A} + \dot{Y}) &\stackrel{(2.13)}{=} Y^{\rho-1} \tilde{A} Y &\stackrel{(2.12)}{=} Y^\rho.
\end{aligned}$$

Therefore, Y is of the form

$$Y = \begin{bmatrix} I_m & 0 & \dots & 0 & 0 \\ 0 & I_m & & & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & I_m & 0 \\ Y_{\rho+1,1} & \dots & Y_{\rho+1,\rho-1} & 0 & \tilde{Y} \end{bmatrix} \quad \text{for some } \tilde{Y} \in \mathcal{C}^1(\mathcal{T}; \mathbf{G}\mathbf{l}_{n-\rho m}(\mathbb{R})).$$

Now consider the last $n - \rho m$ rows in $Y \hat{A} + \dot{Y} = \tilde{A} Y$, which read

$$\begin{aligned}
[\tilde{Y} P + \frac{d}{dt} Y_{\rho+1,1}, Y_{\rho+1,1} + \frac{d}{dt} Y_{\rho+1,2}, \dots, Y_{\rho+1,\rho-2} + \frac{d}{dt} Y_{\rho+1,\rho-1}, Y_{\rho+1,\rho-1}, \tilde{Y} Q] \\
= [\tilde{P} + \tilde{Q} Y_{\rho+1,1}, \tilde{Q} Y_{\rho+1,2}, \dots, \tilde{Q} Y_{\rho+1,\rho-1}, 0, \tilde{Q} \tilde{Y}],
\end{aligned}$$

and comparing successively the ρ^{th} block, \dots , 1^{st} block yields $Y_{\rho+1,\rho-1} = 0, \dots, Y_{\rho+1,1} = 0$. Finally, $Y = \text{diag}\{I_m, \dots, I_m, \tilde{Y}\}$ applied to (2.13)-(2.15) gives (2.11). \square

Remark 2.6.

- (i) In the time-invariant case $(A, B, C) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{m \times n}$, all matrices in Theorem 2.5(i)–(iv) are constant matrices over \mathbb{R} .
- (ii) The uniqueness of (P, Q) in Theorem 2.5 up to $(\tilde{Y}^{-1} P, \tilde{Y}^{-1} Q \tilde{Y})$ for some $\tilde{Y} \in \mathcal{C}^1(\mathcal{T}; \mathbf{G}\mathbf{l}_{n-\rho m}(\mathbb{R}))$ corresponds to the freedom in choosing V such that (iv) holds. If V is replaced by $V \tilde{Y}$ for arbitrary $\tilde{Y} \in \mathcal{C}^1(\mathcal{T}; \mathbf{G}\mathbf{l}_{n-\rho m}(\mathbb{R}))$, then an easy calculation shows that N becomes $\tilde{Y}^{-1} N$ and therefore P and Q become $\tilde{Y}^{-1} P$ and $\tilde{Y}^{-1} Q \tilde{Y}$, resp.
- (iii) A formula for (P, Q) is given in [IM07, (3.9)-(3.12)] but unfortunately with a typo in formula [IM07, (3.10)] for Q ; the correct formula in terms of the notation from Theorem 2.5(iv) is

$$Q = -(V^\top V)^{-1} V^\top \left[\left(\frac{d}{dt} - A \right) V + B \Gamma^{-1} \left(\frac{d}{dt} + A_r \right)^\rho (C) V \right]. \quad (2.17)$$

For its proof see the proof of [IM07, Thm. 3.5].

- (iv) As a technicality, quite useful in the following, we collect the fact that Theorem 2.5(iv) gives

$$\forall t \in \mathcal{T} : \mathcal{C}(t) U(t)^{-1} = [I_{\rho m}, 0_{\rho m \times (n-\rho m)}] \quad \text{and} \quad \ker(\mathcal{C}(t) U(t)^{-1}) = \text{im} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{n-\rho m} \end{bmatrix}. \quad (2.18)$$

3 Zero dynamics

In this section we investigate the concept of zero dynamics. The zero dynamics of a system are, loosely speaking, those dynamics which are not visible at the output. We will use them to characterize (A, B) -invariant subspaces and to design an output feedback controller.

Definition 3.1 (Zero dynamics).

Let $\ell \in \mathbb{N}_0$ and $\mathcal{T} \subseteq \mathbb{R}$ be an open set. The *zero dynamics* of system (1.1) on \mathcal{T} is defined as the set of trajectories

$$\mathcal{ZD}(A, B, C) := \left\{ (x, u, y) \in \mathcal{AC}(\mathcal{T}; \mathbb{R}^n) \times \mathcal{PC}(\mathcal{T}; \mathbb{R}^m) \times \mathcal{AC}(\mathcal{T}; \mathbb{R}^m) \mid \begin{array}{l} (x, u, y) \text{ solves (1.1) on} \\ \mathcal{T} \text{ and } y \equiv 0 \end{array} \right\}.$$

◇

By linearity of (1.1), the set $\mathcal{ZD}(A, B, C)$ is a real vector space and, roughly speaking and made more precise in the following remark, it is also a dynamical system. We are indebted to our colleague Fabian Wirth (Würzburg) for pointing out this observation to us.

Remark 3.2 (Zero dynamics are a dynamical system).

Let the *state transition map* of (1.1) be denoted by the unique solution

$$\varphi(\cdot; t_0, x^0, u(\cdot)) : \mathcal{T} \rightarrow \mathbb{R}^n$$

of the initial value problem (1.1), $x(t_0) = x^0$ for any $(t_0, x^0, u(\cdot)) \in \mathcal{T} \times \mathbb{R}^n \times \mathcal{PC}(\mathcal{T}; \mathbb{R}^m)$, and the *output map* by

$$\eta(\cdot; t_0, x^0, u(\cdot)) : \mathcal{T} \rightarrow \mathbb{R}^m, \quad t \mapsto \eta(t; t_0, x^0, u(\cdot)) = C(t) \varphi(t; t_0, x^0, u(\cdot)).$$

Now it is readily verified that the axioms of a dynamical system as defined, e.g. in [HP05, Def. 2.1.1], are satisfied for the structure $(\mathcal{T}; \mathbb{R}^m, \mathcal{PC}(\mathcal{T}; \mathbb{R}^m), \mathbb{R}^n, \mathbb{R}^m, \varphi, \eta)$ and the set

$$\mathcal{D}_\varphi := \left\{ (t, t_0, x^0, u(\cdot)) \in \mathcal{T}^2 \times \mathbb{R}^n \times \mathcal{PC}(\mathcal{T}; \mathbb{R}^m) \mid \forall t \in \mathcal{T} : \eta(t; t_0, x^0, u(\cdot)) = 0 \right\}.$$

Next we show that the vector space of orbits induced by \mathcal{D}_φ fixed at $t_0 \in \mathcal{T}$ in the sense

$$\mathcal{D}_{\text{orb}, t_0} := \left\{ \varphi(\cdot; t_0, x^0, u(\cdot)) : \mathcal{T} \rightarrow \mathbb{R}^n \mid (t_0, t_0, x^0, u(\cdot)) \in \mathcal{D}_\varphi \right\}$$

is isomorphic to $\mathcal{ZD}(A, B, C)$. This is a consequence of uniqueness and global existence of the solution of the initial value problem (1.1), $x(t_0) = x^0$ which gives that the map

$$\xi_{t_0} : \mathcal{D}_{\text{orb}, t_0} \rightarrow \mathcal{ZD}(A, B, C), \quad \varphi(\cdot; t_0, x^0, u(\cdot)) \mapsto (\varphi(\cdot; t_0, x^0, u(\cdot)), u(\cdot), \eta(\cdot; t_0, x^0, u(\cdot)))$$

is a vector space isomorphism.

◇

The next corollary is an immediate consequence of Proposition 2.4.

Corollary 3.3 (Characterization of zero dynamics).

Let $\rho, \ell \in \mathbb{N}$ such that $\rho \leq \ell$ and consider a time-varying linear system (1.1) which has strict and uniform relative degree ρ . Then $(x, u, y) \in \mathcal{ZD}(A, B, C)$ if, and only if, the following three conditions are satisfied on \mathcal{T} :

- (i) $y(\cdot) = 0$,

$$(ii) \quad u(\cdot) = - \left[\left(\frac{d}{dt} + A_r \right)^{\rho-1} (C) B \right]^{-1} \left(\frac{d}{dt} + A_r \right)^{\rho} (C) x(\cdot),$$

$$(iii) \quad x(\cdot) \text{ solves } \dot{x} = \left[A - B \left[\left(\frac{d}{dt} + A_r \right)^{\rho-1} (C) B \right]^{-1} \left(\frac{d}{dt} + A_r \right)^{\rho} (C) \right] x. \quad \diamond$$

Remark 3.4 (Characterization of zero dynamics for time-invariant systems).

If (1.1) is time-invariant, then Corollary 3.3(i)-(iii) reads

$$(i) \quad y(\cdot) \equiv 0, \quad (ii) \quad u(\cdot) = -(CA^{\rho-1}B)^{-1} CA^{\rho} x(\cdot), \quad (iii) \quad x(\cdot) \text{ solves } \dot{x} = [A - B(CA^{\rho-1}B)^{-1}CA^{\rho}]x.$$

\diamond

4 (A, B) -invariant subspaces

In this section we show that the function space of the zero dynamics of a system (1.1) which has some relative degree on \mathcal{T} is isomorphic to the supremal (in fact maximal) (A, B) -invariant time-varying subspace included in $\ker C$ at some initial time. First we have to introduce some notations.

Let, for any open set $\mathcal{T} \subseteq \mathbb{R}$,

$$\mathcal{W}_n(\mathcal{T}) := \left\{ \mathcal{V} = (\mathcal{V}(t))_{t \in \mathcal{T}} \mid \exists k \in \mathbb{N} \exists V \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{n \times k}) \forall t \in \mathcal{T} : \mathcal{V}(t) = \text{im } V(t) \right\}$$

denote the set of all time-varying subspaces \mathcal{V} , generated by some continuously differentiable V .

Definition 4.1 ((A, B) -invariance).

Let $\mathcal{T} \subseteq \mathbb{R}$ be an open set, $(A, B) \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{n \times n}) \times \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{n \times m})$ and $\mathcal{V} \in \mathcal{W}_n(\mathcal{T})$ be generated by $V \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{n \times k})$ for some $k \in \mathbb{N}$. Then \mathcal{V} is called (A, B) -invariant if, and only if, there exists a discrete set $\mathcal{I} \subseteq \mathcal{T}$ and $N \in \mathcal{C}^1(\mathcal{T} \setminus \mathcal{I}; \mathbb{R}^{k \times k})$, $M \in \mathcal{C}^1(\mathcal{T} \setminus \mathcal{I}; \mathbb{R}^{m \times k})$, such that

$$\forall t \in \mathcal{T} \setminus \mathcal{I} : \left(\frac{d}{dt} - A(t) \right) (V(t)) = V(t)N(t) + B(t)M(t). \quad (4.1)$$

\diamond

The concept of (A, B) -invariance has been introduced by [BM69, WM70] and generalized in various directions, see the excellent textbook [Won85]. For time-varying linear systems see [Ilc89, Def. 4.1].

Remark 4.2.

If (A, B) has real analytic coefficients, then, as shown in [Ilc89, Rem. 4.4], Definition 4.1 is equivalent to

$$\left(\frac{d}{dt} - A(t) \right) (V(t)) \subset \text{im } V(t) + \text{im } B(t) \quad \text{for almost all } t \in \mathcal{T}.$$

For time-invariant (A, B) , ‘almost all’ is redundant. The following example shows that ‘almost all’ is not redundant for time-varying systems. Set $\mathcal{T} = \mathbb{R}$ and, for all $t \in \mathbb{R}$,

$$V(t) = \begin{bmatrix} 0 \\ t \end{bmatrix}, \quad A = 0_{2 \times 2}, \quad B(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then

$$\forall t \in \mathbb{R} \setminus \{0\} : \left(\frac{d}{dt} - A(t) \right) (V(t)) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} \cdot t^{-1} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot 0.$$

\diamond

Obviously, the elements of $\mathcal{W}_n(\mathcal{T})$ are partially ordered by the relation \subseteq^* , defined by

$$\mathcal{V} \subseteq^* \hat{\mathcal{V}} \iff \forall t \in \mathcal{T} : \mathcal{V}(t) \subseteq \hat{\mathcal{V}}(t),$$

for any $\mathcal{V}, \hat{\mathcal{V}} \in \mathcal{W}_n(\mathcal{T})$. Hence, for any

$$(A, B, C) \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{n \times n}) \times \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{n \times m}) \times \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{m \times n}),$$

the existence of

$$\mathcal{V}^*(A, B; \ker C) := \sup \{ \mathcal{V} \in \mathcal{W}_n(\mathcal{T}) \mid \mathcal{V} \text{ is } (A, B)\text{-invariant and } \mathcal{V}(t) \subseteq \ker C(t) \text{ for all } t \in \mathcal{T} \}$$

follows since the elements of $\mathcal{W}_n(\mathcal{T})$ are finite dimensional subspaces for all $t \in \mathcal{T}$, and the sum of (A, B) -invariant time-varying subspaces included in $\ker C$ is (A, B) -invariant and included in $\ker C$. Actually, the supremum in the definition of $\mathcal{V}^*(A, B; \ker C)$ is a maximum.

The following proposition shows that $\mathcal{V}^*(A, B; \ker C)$ has a simple representation if (1.1) has a strict and uniform relative degree. Note also that (4.2) becomes even simpler after the coordinate transformation (2.4) introduced in Theorem 2.5, see (4.5).

Proposition 4.3 (Representation of $\mathcal{V}^*(A, B; \ker C)$).

Let $\rho, \ell \in \mathbb{N}$ with $\rho \leq \ell$ and suppose system (1.1) has strict and uniform relative degree ρ on \mathcal{T} . Then, for \mathcal{C} as in Theorem 2.5(iv),

$$\forall t \in \mathcal{T} : \mathcal{V}^*(A, B; \ker C)(t) = \ker \mathcal{C}(t). \quad (4.2)$$

Proof: Let U be as in Theorem 2.5(iv) and $(\hat{A}, \hat{B}, \hat{C})$ as in (2.7). At several occasions, we will make use of the fact that

$$\forall t \in \mathcal{T} : U(t) \ker \mathcal{C}(t) = \ker (\mathcal{C}(t)U(t)^{-1}). \quad (4.3)$$

Step 1: We first show

$$\forall t \in \mathcal{T} : \mathcal{V}^*(A, B; \ker C)(t) = U(t)^{-1} \mathcal{V}^*(\hat{A}, \hat{B}; \ker \hat{C})(t)$$

which is equivalent to

$$\forall t \in \mathcal{T} : U(t) \mathcal{V}^*(A, B; \ker C)(t) = \mathcal{V}^*(\hat{A}, \hat{B}; \ker \hat{C})(t). \quad (4.4)$$

Step 1a: Let $\mathcal{V} \in \mathcal{W}_n(\mathcal{T})$ be any (A, B) -invariant time-varying subspace included in $\ker C$ and generated by $V \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{n \times k})$, $k \in \mathbb{N}$. Then (4.1) holds for some $N \in \mathcal{C}^1(\mathcal{T} \setminus \mathcal{I}; \mathbb{R}^{k \times k})$, $M \in \mathcal{C}^1(\mathcal{T} \setminus \mathcal{I}; \mathbb{R}^{m \times k})$, \mathcal{I} a discrete set, and hence, for all $t \in \mathcal{T} \setminus \mathcal{I}$, we have

$$\left(\frac{d}{dt} - \hat{A}\right)(UV) = \dot{U}V + U\dot{V} - \hat{A}UV \stackrel{(2.7)}{=} \dot{U}V + U\dot{V} - (UA + \dot{U})V = U\left(\frac{d}{dt} - A\right)(V) \stackrel{(4.1)}{\stackrel{(2.7)}}{=} (UV)N + \hat{B}M.$$

Therefore, $UV = (U(t)\mathcal{V}(t))_{t \in \mathcal{T}}$ is (\hat{A}, \hat{B}) -invariant. Furthermore,

$$\forall t \in \mathcal{T} : \text{im}(U(t)\mathcal{V}(t)) = U(t) \text{im } \mathcal{V}(t) \subseteq U(t) \ker C(t) \stackrel{(4.3)}{=} \ker (\mathcal{C}(t)U(t)^{-1}) \stackrel{(2.7)}{=} \ker \hat{C}(t)$$

and so “ \subseteq^* ” in (4.4) follows.

Step 1b: The proof of “ \supseteq^* ” in (4.4) is analogous and omitted.

Step 2: We show that

$$\forall t \in \mathcal{T} : U(t)^{-1} \mathcal{V}^*(\hat{A}, \hat{B}; \ker \hat{C})(t) = \ker \mathcal{C}(t)$$

which, in view of (4.3) and (2.18), is equivalent to

$$\mathcal{V}^*(\hat{A}, \hat{B}; \ker \hat{C}) = \mathcal{X}, \quad (4.5)$$

where

$$\mathcal{X} = (\mathcal{X}(t))_{t \in \mathcal{T}} \in \mathcal{W}_n(\mathcal{T}) \text{ is generated by the constant matrix } \mathcal{X}(\cdot) := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{n-\rho m} \end{bmatrix} \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{n \times (n-\rho m)}).$$

Step 2a: We show “ \supseteq^* ” in (4.5). The family \mathcal{X} is (\hat{A}, \hat{B}) -invariant, since

$$\left(\frac{d}{dt} - \hat{A}\right) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{n-\rho m} \end{bmatrix} \stackrel{(2.8)}{=} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ S \\ Q \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{n-\rho m} \end{bmatrix} (-Q) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \Gamma \end{bmatrix} (-\Gamma^{-1}S) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{n-\rho m} \end{bmatrix} N + \hat{B}M,$$

where $N := -Q \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{(n-\rho m) \times (n-\rho m)})$ and $M := -\Gamma^{-1}S \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{m \times (n-\rho m)})$; furthermore,

$$\forall t \in \mathcal{T} : \text{im} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{n-\rho m} \end{bmatrix} \stackrel{(2.18)}{=} \ker (C(t)U(t)^{-1}) \stackrel{(2.7)}{=} \ker \begin{bmatrix} \hat{C}(t) \\ (\frac{d}{dt} + A(t)_r) (C(t))U(t)^{-1} \\ \vdots \\ (\frac{d}{dt} + A(t)_r)^{\rho-1} (C(t))U(t)^{-1} \end{bmatrix} \subseteq \ker \hat{C}(t)$$

and therefore $\mathcal{X} \subseteq^* \mathcal{V}^*(\hat{A}, \hat{B}; \ker \hat{C})$.

Step 2b: We show “ \subseteq^* ” in (4.5), i.e. that any (\hat{A}, \hat{B}) -invariant time-varying subspace $\hat{\mathcal{V}} \in \mathcal{W}_n(\mathcal{T})$ included in $\ker \hat{C}$ fulfills $\hat{\mathcal{V}} \subseteq^* \mathcal{X}$. Let $\hat{V} \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{n \times k})$, $k \in \mathbb{N}$, and $N \in \mathcal{C}^1(\mathcal{T} \setminus \mathcal{I}; \mathbb{R}^{k \times k})$, $M \in \mathcal{C}^1(\mathcal{T} \setminus \mathcal{I}; \mathbb{R}^{m \times k})$, \mathcal{I} a discrete set, such that

$$\forall t \in \mathcal{T} : \text{im } \hat{V}(t) \subseteq \ker \hat{C}(t) \quad \text{and} \quad \forall t \in \mathcal{T} \setminus \mathcal{I} : \left(\frac{d}{dt} - \hat{A}(t)\right)(\hat{V}(t)) = \hat{V}(t)N(t) + \hat{B}(t)M(t). \quad (4.6)$$

It suffices to show that

$$\forall j = 1, \dots, \rho : S_j \hat{V} = 0, \quad (4.7)$$

where

$$S_j := \text{diag} \underbrace{\{I_m, \dots, I_m, 0, \dots, 0\}}_{j\text{-times}} \in \mathbb{R}^{n \times n}, \quad j = 1, \dots, \rho.$$

(4.7) is shown by induction. If $j = 1$, then

$$\forall t \in \mathcal{T} : S_1 \hat{V}(t) \stackrel{(2.8)}{=} \hat{C}(t)^\top \hat{C}(t) \hat{V}(t) \stackrel{(4.6)}{=} 0.$$

Suppose $S_j \hat{V} = 0$ holds for some $j \in \{1, \dots, \rho - 1\}$ and set

$$\hat{V}_i = \text{diag} \underbrace{\{0_{m \times m}, \dots, 0_{m \times m}, I_m, 0, \dots, 0\}}_{(i-1)\text{-times}} \hat{V}, \quad i = 2, \dots, j + 1.$$

Then $\text{im } \hat{V}(t) \subseteq \ker S_j$ for all $t \in \mathcal{T}$, and hence $\text{im } \frac{d}{dt} \hat{V}(t) \subseteq \ker S_j$ for all $t \in \mathcal{T}$, thus $S_j \frac{d}{dt} \hat{V} = 0$ and

$$\begin{bmatrix} \hat{V}_2 \\ \vdots \\ \hat{V}_{j+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \stackrel{(2.8)}{\underset{j \leq \rho-1}{=}} S_j \hat{A} \hat{V} = S_j \left(\frac{d}{dt} - \hat{A} \right) (\hat{V}) \stackrel{\text{a.e.}}{\underset{(4.6)}{=}} S_j \hat{V} N + S_j \hat{B} M \stackrel{(2.8)}{=} S_j \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \Gamma \\ 0 \end{bmatrix} M \stackrel{j \leq \rho-1}{=} 0,$$

where “a.e.” means “on $\mathcal{T} \setminus \mathcal{I}$ ” in this case. Hence we find $S_{j+1} \hat{V}(t) = 0$ for all $t \in \mathcal{T} \setminus \mathcal{I}$ and $\hat{V} \in \mathcal{C}^1(\mathcal{T}; \mathbb{R}^{n \times k})$ gives $S_{j+1} \hat{V} = 0$. So the proof of Step 2 is complete, and the proof of the proposition follows from Step 1 and Step 2. \square

We are now in a position to characterize the zero dynamics of systems which have strict and uniform relative degree. This result, motivated by some comments in [BLGS98] in the context of distributed parameter systems, seems, to the best of our knowledge, also new for time-invariant linear systems.

Proposition 4.4.

Let $\rho, \ell \in \mathbb{N}$ with $\rho \leq \ell$ and suppose system (1.1) has strict and uniform relative degree ρ on \mathcal{T} . Let $(x, u, y) \in \mathcal{AC}(\mathcal{T}; \mathbb{R}^n) \times \mathcal{PC}(\mathcal{T}; \mathbb{R}^m) \times \mathcal{AC}(\mathcal{T}; \mathbb{R}^m)$ be a solution of (1.1). Then

$$(x, u, y) \in \mathcal{ZD}(A, B, C) \iff \left[\forall t \in \mathcal{T} : x(t) \in \mathcal{V}^*(A, B; \ker C)(t) \right].$$

Proof: “ \Rightarrow ”: Let $(x, u, y) \in \mathcal{ZD}(A, B, C)$. Applying the coordinate transformation (2.4), where U is as in Theorem 2.5(iv), it follows from $y = 0$ that $\xi = 0$ and therefore, for all $t \in \mathcal{T}$,

$$x(t) = U(t)^{-1} \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} = U(t)^{-1} \begin{pmatrix} 0 \\ \eta(t) \end{pmatrix} \in U(t)^{-1} \text{im} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{n-\rho m} \end{bmatrix} \stackrel{(2.18)}{=} \ker \mathcal{C}(t) \stackrel{(4.2)}{=} \mathcal{V}^*(A, B; \ker C)(t).$$

“ \Leftarrow ”: Since

$$\forall t \in \mathcal{T} : x(t) \in \mathcal{V}^*(A, B; \ker C)(t) \stackrel{(4.2)}{=} \ker \mathcal{C}(t) \stackrel{(2.18)}{=} U(t)^{-1} \text{im} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{n-\rho m} \end{bmatrix},$$

it follows that

$$\forall t \in \mathcal{T} : \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} = U(t)x(t) \in \text{im} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{n-\rho m} \end{bmatrix},$$

and therefore (cf. (2.4)) $y = \xi_1 = 0$, whence $(x, u, y) \in \mathcal{ZD}(A, B, C)$. \square

We now state the main result of this section: for any system (1.1) which has strict and uniform relative degree on \mathcal{T} , the zero dynamics is isomorphic to the maximal (A, B) -invariant time-varying subspace included in $\ker C$ at any initial time.

Theorem 4.5 (Vector space isomorphism).

Let $\rho, \ell \in \mathbb{N}$ with $\rho \leq \ell$ and suppose (1.1) has strict and uniform relative degree ρ . Then, for every $t_0 \in \mathcal{T}$, the linear mapping

$$L_{t_0} : \mathcal{V}^*(A, B; \ker C)(t_0) \rightarrow \mathcal{ZD}(A, B, C),$$

$$x^0 \mapsto \left(x(\cdot), - \left[\left(\frac{d}{dt} + A_r \right)^{\rho-1} (C) B \right]^{-1} \left(\frac{d}{dt} + A_r \right)^\rho (C) x(\cdot), Cx(\cdot) \right),$$

where $x : \mathcal{T} \rightarrow \mathbb{R}^n$ solves

$$\dot{x} = \left(A - B \left[\left(\frac{d}{dt} + A_r \right)^{\rho-1} (C) B \right]^{-1} \left(\frac{d}{dt} + A_r \right)^\rho (C) \right) x, \quad x(t_0) = x^0,$$

is a vector space isomorphism. In particular, $t \mapsto \dim \mathcal{V}^*(A, B; \ker C)(t)$ is constant on \mathcal{T} .

Proof: *Step 1:* We show that L_{t_0} is well-defined, that means to show that for arbitrary $x^0 \in \mathcal{V}^*(A, B; \ker C)(t_0)$, the solution of

$$\dot{x} = \left(A - B \left[\left(\frac{d}{dt} + A_r \right)^{\rho-1} (C) B \right]^{-1} \left(\frac{d}{dt} + A_r \right)^\rho (C) \right) x, \quad x(t_0) = x^0 \quad (4.8)$$

on \mathcal{T} satisfies

$$(x, u, y) := \left(x, - \left[\left(\frac{d}{dt} + A_r \right)^{\rho-1} (C) B \right]^{-1} \left(\frac{d}{dt} + A_r \right)^\rho (C) x, Cx \right) \in \mathcal{ZD}(A, B, C). \quad (4.9)$$

It is an immediate consequence of (4.8) that (x, u, y) solves (1.1) on \mathcal{T} . In view of Corollary 3.3, it remains to show that $y = 0$.

Applying the coordinate transformation (2.4) and the notation as in (2.5) yields

$$\begin{aligned} \dot{\xi}(t) &= \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ 0 & 0 & I_m & & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_m \\ R_1(t) & R_2(t) & \cdots & R_{\rho-1}(t) & R_\rho(t) \end{bmatrix} \xi(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ S(t) \end{bmatrix} \eta(t) \\ &+ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Gamma(t) \end{bmatrix} \left(- \left[\left(\frac{d}{dt} + A(t)_r \right)^{\rho-1} (C(t)) B(t) \right]^{-1} \left(\frac{d}{dt} + A(t)_r \right)^\rho (C(t)) U(t)^{-1} \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} \right), \\ \dot{\eta}(t) &= P(t) \xi_1(t) + Q(t) \eta(t). \end{aligned}$$

It follows from (i) and (ii) of Theorem 2.5 that

$$\frac{d}{dt} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \left[\begin{array}{ccc|ccc} 0 & I_m & & & & 0 \\ & & \ddots & \ddots & & \vdots \\ & & & & 0 & 0 \\ \hline & & & 0 & I_m & \\ \hline [R_1, \dots, R_\rho, S] - \left(\frac{d}{dt} + A_r \right)^\rho (C) U^{-1} & & & & & \\ \hline P, & 0, & \dots, & 0 & & Q \end{array} \right] \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \left[\begin{array}{ccc|ccc} 0 & I_m & & & & 0 \\ & & \ddots & \ddots & & \vdots \\ & & & & 0 & 0 \\ \hline & & & 0 & I_m & \\ \hline 0, & 0, & \dots, & 0 & & 0 \\ \hline P, & 0, & \dots, & 0 & & Q \end{array} \right] \begin{pmatrix} \xi \\ \eta \end{pmatrix},$$

and the initial value satisfies

$$\begin{pmatrix} \xi(t_0) \\ \eta(t_0) \end{pmatrix} = U(t_0)x^0 \in U(t_0)\mathcal{V}^*(A, B; \ker C)(t_0) \stackrel{(4.2)}{=} U(t_0) \ker \mathcal{C}(t_0) \stackrel{(2.18)}{=} \text{im} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{n-\rho m} \end{bmatrix},$$

thus $y = \xi_1 = 0$.

Step 2: We show that L_{t_0} is injective. Let $x^1, x^2 \in \mathcal{V}^*(A, B; \ker C)(t_0)$ so that $L_{t_0}(x^1)(\cdot) = L_{t_0}(x^2)(\cdot)$. Then

$$(x^1, *, *) = L_{t_0}(x^1)(\cdot)|_{t=t_0} = L_{t_0}(x^2)(\cdot)|_{t=t_0} = (x^2, *, *).$$

Step 3: Surjectivity of L_{t_0} follows immediately from Proposition 4.4 and Corollary 3.3. \square

5 High-gain stabilization by output derivative feedback

In this section we design a simple high-gain output derivative feedback controller (see Figure 1) for systems (1.1), which have strict and uniform relative degree and uniformly asymptotically stable zero dynamics, so that the closed-loop system is uniformly exponentially stable. Throughout the section we set $\mathcal{T} = (0, \infty)$.

Recall (see e.g. [HP05, p. 257]) that a system $\dot{x} = A(t)x$, where $A \in \mathcal{PC}((0, \infty); \mathbb{R}^{n \times n})$, is called *uniformly exponentially stable* if, and only if, its transition matrix $\Phi_A(\cdot, \cdot)$ satisfies

$$\exists M, \lambda > 0 \forall t \geq t_0 > 0 : \|\Phi_A(t, t_0)\| \leq M e^{-\lambda(t-t_0)},$$

and this is equivalent (see e.g. [Rug96, Th. 6.7]) to

$$\exists M, \lambda > 0 \forall \text{sln. } x(\cdot) \text{ of } \dot{x} = A(t)x \forall t \geq t_0 > 0 : \|x(t)\| \leq M e^{-\lambda(t-t_0)} \|x(t_0)\|.$$

Stability of the zero dynamics is defined as follows:

Definition 5.1 (Stability of zero dynamics).

The zero dynamics of system (1.1) is called

uniformly stable if, and only if,

$$\begin{aligned} \forall \varepsilon > 0 \exists \delta > 0 \forall t_0 > 0 \forall (x, u, y) \in \mathcal{ZD}(A, B, C) \text{ s.t. } \|(x(t_0), u(t_0))\| < \delta \\ \forall t \geq t_0 : \|(x(t), u(t))\| < \varepsilon; \end{aligned}$$

attractive if, and only if,

$$\forall (x, u, y) \in \mathcal{ZD}(A, B, C) : \lim_{t \rightarrow \infty} (x(t), u(t)) = 0;$$

uniformly asymptotically stable or *exponentially stable* if, and only if, $\mathcal{ZD}(A, B, C)$ is uniformly stable and attractive. \diamond

The notions of uniformly stable and attractive zero dynamics introduced in Definition 5.1 are, precisely speaking, those of uniform stability and attractivity of the zero trajectory $(0, 0, 0) \in \mathcal{ZD}(A, B, C)$, resp. However, $\mathcal{ZD}(A, B, C)$ is a linear space and so the zero solution is uniformly stable (attractive) if, and only if, every solution $(x, u, y) \in \mathcal{ZD}(A, B, C)$ is uniformly stable (attractive), resp. Therefore, the abuse of terminology may be tolerated.

Remark 5.2 (Zero dynamics).

Let $\rho, \ell \in \mathbb{N}$ with $\rho \leq \ell$ and suppose (1.1) has strict and uniform relative degree ρ on $\mathcal{T} = (0, \infty)$ and Byrnes-Isidori form (2.7), (2.8) as in Theorem 2.5. Then in [IM07, Prop. 4.2] it is shown that

$$\mathcal{ZD}(A, B, C) = \{ (V\eta, -\Gamma^{-1}S\eta, 0) \mid \dot{\eta} = Q(t)\eta \}.$$

Furthermore, [IM07, Prop. 4.4] yields that the zero dynamics $\mathcal{ZD}(A, B, C)$ are uniformly asymptotically stable if, and only if, $\dot{\eta} = Q(t)\eta$ is a uniformly exponentially stable system. \diamond

We are now in a position to show that a system (1.1) which has some strict and uniform relative degree and uniformly asymptotically stable zero dynamics can be exponentially stabilized by a high-gain output feedback controller as depicted in Figure 1. We stress that the controller (5.1) is simpler than the controller [IM07, (4.2)]. Moreover, this result is also new for time-invariant systems.

Theorem 5.3 (High-gain derivative feedback stabilization).

Suppose

- (i) system (1.1) has strict and uniform relative degree $\rho \in \mathbb{N}$ on $(0, \infty)$,
- (ii) system (1.1) has uniformly asymptotically stable zero dynamics,
- (iii) the matrix functions (A, B, C) , \mathcal{B} , \mathcal{C} , $(\mathcal{CB})^{-1}$ defined in Theorem 2.5(iv) are bounded on $(0, \infty)$,
- (iv) the high-frequency gain matrix $\Gamma(t) := \left(\frac{d}{dt} + A(t)_r\right)^{\rho-1} (C(t))B(t)$ is uniformly positive definite in the sense:

$$\exists \alpha > 0 \forall t > 0 : \Gamma(t) + \Gamma(t)^\top \geq \alpha I_m.$$

Choose a Hurwitz polynomial $k(s) = \sum_{i=0}^{\rho-1} k_i s^i \in \mathbb{R}[s]$ such that $k_{\rho-1} > 0$. Then there exists $\kappa^* > 0$ such that, for all $\kappa \geq \kappa^*$, the derivative output feedback controller

$$u(t) = -\kappa k\left(\frac{d}{dt}\right)(y(t)) = -\kappa \sum_{i=0}^{\rho-1} k_i y^{(i)}(t), \quad (5.1)$$

applied to (1.1) yields a uniformly exponentially stable closed-loop system.

Remark 5.4 (Time-invariant systems).

For purpose of illustration, we discuss Theorem 5.3 for *time-invariant* systems (1.1). Suppose first that (1.1) has strict and uniform relative degree 1, i.e. $\det CB \neq 0$. Then the Byrnes-Isidori form of (1.1) is, in view of Remark 2.3(i), given by

$$\frac{d}{dt} \begin{bmatrix} y(t) \\ \eta(t) \end{bmatrix} = \begin{bmatrix} R & S \\ P & Q \end{bmatrix} \begin{bmatrix} y(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} CB \\ 0 \end{bmatrix} u(t) \quad (5.2)$$

for some $R \in \mathbb{R}^{m \times m}$, $S, P^\top \in \mathbb{R}^{m \times (n-m)}$, $Q \in \mathbb{R}^{(n-m) \times (n-m)}$. Suppose further that the high-frequency gain matrix is positive in the sense $\sigma(CB) \subset \mathbb{C}_+$ and that (1.1) has asymptotically stable zero dynamics; the latter means, in view of Remark 5.2, $\sigma(Q) \subset \mathbb{C}_-$. Now apply, for $\kappa > 0$, the feedback

$$u(t) = -\kappa y(t), \quad (5.3)$$

to (5.2). Then it is easy to see (see e.g. [Ilc93, Lem. 2.2.7]) that the closed-loop system

$$\frac{d}{dt} \begin{bmatrix} y(t) \\ \eta(t) \end{bmatrix} = \begin{bmatrix} R - \kappa CB & S \\ P & Q \end{bmatrix} \begin{bmatrix} y(t) \\ \eta(t) \end{bmatrix}$$

is exponentially stable for all $\kappa \geq \kappa^*$ and some $\kappa^* > 0$ sufficiently large.

We would like to generalize this result to time-invariant systems with relative degree $\rho > 1$. The obstacle of the higher relative degree can be circumvented by introducing the compensator

$$u(t) = -\kappa k\left(\frac{d}{dt}\right)v(t), \quad (5.4)$$

for some polynomial $k(s) = \sum_{i=0}^{\rho-1} k_i s^i \in \mathbb{R}[s]$ such that $k_{\rho-1} > 0$. Then the transfer function of the series interconnection (1.1), (5.4) is given by

$$C(sI_n - A)^{-1}Bk(s) = Ck(A)(sI_n - A)^{-1}B,$$

where equality follows from Remark 2.3(i), and the high-frequency gain matrix is $Ck(A)B = k_{\rho-1}CA^{\rho-1}B$. Therefore, the realization $(A, B, Ck(A))$ of the interconnection (1.1), (5.4) has strict relative degree 1. If system (1.1) has asymptotically stable zero dynamics and $k(\cdot)$ is Hurwitz, then $(A, B, Ck(A))$ has asymptotically stable zero dynamics, too; this follows from

$$\det \begin{bmatrix} sI - A & B \\ Ck(A) & 0 \end{bmatrix} = k(s) \det \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix}$$

which is easy to see by invoking the Byrnes-Isidori form.

Finally, applying the feedback (5.1) to (1.1) is equivalent to applying

$$v(t) = -\kappa y(t)$$

to (1.1), (5.4), equivalently to $(A, B, Ck(A))$, and hence the findings from above, concerning relative degree one systems, yield that the closed-loop system interconnection (1.1), (5.1) is exponentially stable for all $\kappa \geq \kappa^*$ and some $\kappa^* > 0$ sufficiently large.

The analogous result for time-varying systems is the content of Theorem 5.3 but the proof is much more involved. \diamond

Proof of Theorem 5.3: We proceed in several steps.

Step 1: Coordinate transformation of (1.1), (5.1).

By Theorem 2.5, the closed-loop system (1.1), (5.1) may be written, in terms of the coordinate transformation (2.4) and the transformed system matrices (2.7), (2.8) as

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} &= \hat{A}(t) \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} - \kappa \hat{B}(t) \sum_{i=0}^{\rho-1} k_i y(t)^{(i)} \\ \stackrel{(2.2)}{=} &\hat{A}(t) \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} - \kappa \hat{B}(t) \sum_{i=0}^{\rho-1} k_i \left(\frac{d}{dt} + \hat{A}(t)_r\right)^i (\hat{C}) \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} \\ \stackrel{(2.8)}{=} &\hat{A}(t) \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} - \kappa \hat{B}(t) \sum_{i=0}^{\rho-1} k_i \hat{C} \hat{A}(t)^i \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} \\ \stackrel{(2.8)}{=} &\hat{A}(t) \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} - \kappa \hat{B}(t) \hat{C} k(\hat{A}(t)) \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix}. \end{aligned}$$

Clearly,

$$\hat{C} k(\hat{A}(t)) = [C_k, 0] \quad \text{where} \quad C_k := [k_0 I_m, \dots, k_{\rho-1} I_m].$$

Then the closed-loop system (1.1), (5.1) is equivalent to

$$\frac{d}{dt} \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} = \hat{A}(t) \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} + \hat{B}(t) v(t) \quad (5.5a)$$

$$\tilde{y}(t) = [C_k, 0] \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} \quad (5.5b)$$

$$v(t) = -\kappa \tilde{y}(t). \quad (5.5c)$$

Furthermore, Assertion (iii) together with [IM07, Thm. 4.5] yields that the transformation matrix U , its inverse U^{-1} and all entries in (5.5) are bounded on $(0, \infty)$.

Step 2: Byrnes-Isidori form of (5.5a), (5.5b).

Note that

$$[C_k, 0]\hat{B}(t) = k_{\rho-1}\Gamma(t), \quad t > 0,$$

is the high-frequency gain matrix of the system (5.5a),(5.5b) and so it follows from Assertion (iv) that (5.5a), (5.5b) has strict and uniform relative degree one. Therefore, in view of

$$\text{im } \tilde{V} = \ker[C_k, 0], \quad \text{where} \quad \tilde{V} := \begin{bmatrix} 0 & k_{\rho-1}I_m & \cdots & k_2I_m & k_1I_m \\ 0 & 0 & & 0 & -k_0I_m \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & & \vdots \\ 0 & -k_0I_m & & & \\ I_{n-\rho m} & 0 & \cdots & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n \times (n-m)}, \quad (5.6)$$

we may apply Theorem 2.5 for

$$\tilde{U} = \begin{bmatrix} [C_k, 0] \\ \tilde{N} \end{bmatrix}, \quad \tilde{N} = (\tilde{V}^\top \tilde{V})^{-1} \tilde{V}^\top \left(I - \hat{B}([C_k, 0]\hat{B})^{-1}[C_k, 0] \right)$$

to transform system (5.5a), (5.5b) into Byrnes-Isidori form

$$\left((\tilde{U}A + \frac{d}{dt}\tilde{U})\tilde{U}^{-1}, \tilde{U}B, C\tilde{U}^{-1} \right) = \left(\begin{bmatrix} \tilde{R} & \tilde{S} \\ \tilde{P} & \tilde{Q} \end{bmatrix}, \begin{bmatrix} k_{\rho-1}\Gamma \\ 0_{(n-m) \times m} \end{bmatrix}, [I_m, 0_{(n-m) \times (n-m)}] \right) \quad (5.7)$$

or, equivalently,

$$\left. \begin{aligned} \left(\begin{array}{c} \frac{d}{dt}\tilde{y}(t) \\ \dot{z}(t) \end{array} \right) &= \begin{bmatrix} \tilde{R}(t) & \tilde{S}(t) \\ \tilde{P}(t) & \tilde{Q}(t) \end{bmatrix} \begin{pmatrix} \tilde{y}(t) \\ z(t) \end{pmatrix} + \begin{bmatrix} k_{\rho-1}\Gamma(t) \\ 0_{(n-m) \times m} \end{bmatrix} v(t) \\ \tilde{y}(t) &= [I_m, 0] \begin{pmatrix} \tilde{y}(t) \\ z(t) \end{pmatrix}. \end{aligned} \right\} \quad (5.8)$$

In view of Assertion (iii), the transformation matrix \tilde{U} , \tilde{U}^{-1} and all matrices in (5.7) are bounded on $(0, \infty)$.

Step 3: Zero dynamics of (5.5a), (5.5b).

By Remark 3.2, the zero dynamics of (5.8) are determined by $\dot{z} = \tilde{Q}(t)z$. We show that, for all $t > 0$,

$$\tilde{Q}(t) = \begin{bmatrix} Q(t) & E(t) \\ 0 & H_k \end{bmatrix}, \quad \text{where} \quad E(t) := [k_{\rho-1}P(t), k_{\rho-2}P(t), \dots, k_1P(t)],$$

$$H_k := \begin{bmatrix} -\frac{k_{\rho-2}}{k_{\rho-1}}I_m & -\frac{k_{\rho-3}}{k_{\rho-1}}I_m & \cdots & -\frac{k_1}{k_{\rho-1}}I_m & -\frac{k_0}{k_{\rho-1}}I_m \\ I_m & 0 & \cdots & 0 & 0 \\ 0 & I_m & & 0 & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & I_m & 0 \end{bmatrix}. \quad (5.9)$$

Formula (2.17) yields, for \tilde{V} as in (5.6),

$$\begin{aligned} \tilde{Q} &= -(\tilde{V}^\top \tilde{V})^{-1} \tilde{V}^\top \left[-\hat{A}\tilde{V} + \hat{B}(k_{\rho-1}\Gamma)^{-1} \left(\frac{d}{dt} + \hat{A}_r \right) ([C_k, 0])\tilde{V} \right] \\ &= (\tilde{V}^\top \tilde{V})^{-1} \tilde{V}^\top \left[I_n - \frac{1}{k_{\rho-1}} \hat{B}\Gamma^{-1}[C_k, 0] \right] \hat{A}\tilde{V}. \end{aligned} \quad (5.10)$$

and, for $\gamma := k_0^2 \sum_{j=0}^{\rho-1} k_j^2$,

$$(\tilde{V}^\top \tilde{V})^{-1} = \gamma^{-1} \begin{bmatrix} \gamma I_m & 0 & 0 & \cdots & 0 \\ 0 & \left(\sum_{\substack{j=0 \\ j \neq \rho-1}}^{\rho-1} k_j^2 \right) I_m & -k_{\rho-1} k_{\rho-2} I_m & -k_{\rho-1} k_{\rho-3} I_m & \cdots & -k_{\rho-1} k_1 I_m \\ 0 & -k_{\rho-1} k_{\rho-2} I_m & \left(\sum_{\substack{j=0 \\ j \neq \rho-2}}^{\rho-1} k_j^2 \right) I_m & -k_{\rho-2} k_{\rho-3} I_m & \cdots & -k_{\rho-2} k_1 I_m \\ 0 & -k_{\rho-1} k_{\rho-3} I_m & -k_{\rho-2} k_{\rho-3} I_m & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & -k_2 k_1 I_m \\ 0 & -k_{\rho-1} k_1 I_m & -k_{\rho-2} k_1 I_m & \cdots & -k_2 k_1 I_m & \left(\sum_{\substack{j=0 \\ j \neq 1}}^{\rho-1} k_j^2 \right) I_m \end{bmatrix}.$$

Inserting these findings into (5.10) yields (5.9). Note that E is bounded since P is bounded.

Step 4: Stability of the zero dynamics.

By Remark 3.2, the zero dynamics of (5.8) are uniformly asymptotically stable if, and only if, $\dot{z} = \tilde{Q}(t) z$ is uniformly exponentially stable.

Step 4a: We show that for H_k as defined in (5.9) we have

$$\exists M_1, \lambda_1 > 0 \forall t > 0: \|e^{H_k t}\| \leq M_1 e^{-\lambda_1 t}. \quad (5.11)$$

If we had shown that

$$\det(sI - H_k) = k_{\rho-1}^{-m} k(s)^m \in \mathbb{R}[s], \quad (5.12)$$

then (5.11) would be a consequence of the assumption that $k(\cdot)$ is Hurwitz.

To simplify the notation, set

$$\tilde{k}_i := \frac{k_i}{k_{\rho-1}} \quad \text{for } i = 0, \dots, \rho-2 \quad (5.13)$$

and

$$\underbrace{\begin{bmatrix} sI_m & & & \\ -I_m & sI_m & & 0 \\ & \ddots & \ddots & \\ 0 & & -I_m & sI_m \end{bmatrix}}_{=: F_{\rho-2} \in \mathbb{R}(s)^{(\rho-2)m \times (\rho-2)m}} \underbrace{\begin{bmatrix} s^{-1}I_m & & & \\ s^{-2}I_m & s^{-1}I_m & & 0 \\ \vdots & \ddots & \ddots & \\ s^{-(\rho-2)}I_m & \dots & s^{-2}I_m & s^{-1}I_m \end{bmatrix}}_{=: F_{\rho-2}^{-1}} = I_{(\rho-2)m}.$$

An application of the Schur complement formula (see e.g. [HP05, Lemma A.1.17]) yields

$$\det(sI - H_k) = \det \left[\begin{array}{c|ccc} (s + \tilde{k}_{\rho-2})I_m & \tilde{k}_{\rho-3}I_m & \dots & \tilde{k}_0I_m \\ \hline -I_m & sI_m & & \\ 0 & -I_m & sI_m & \\ \vdots & & \ddots & \ddots \\ 0 & & & -I_m & sI_m \end{array} \right]$$

$$\begin{aligned}
&= \det \left[\begin{array}{c|ccc} (s + \tilde{k}_{\rho-2})I_m & \tilde{k}_{\rho-3}I_m & \dots & \tilde{k}_0I_m \\ \hline -I_m & & & \\ 0 & & & \\ \vdots & & & \\ 0 & & F_{\rho-2} & \end{array} \right] \\
&= \det(F_{\rho-2}) \cdot \det \left(\begin{array}{c} (s + \tilde{k}_{\rho-2})I_m - [\tilde{k}_{\rho-3}I_m \ \dots \ \tilde{k}_0I_m] F_{\rho-2}^{-1} \\ \begin{bmatrix} -I_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} \right) \\
&= s^{(\rho-2)m} \cdot \det \left((s + \tilde{k}_{\rho-2} + \tilde{k}_{\rho-3}s^{-1} + \dots + \tilde{k}_0s^{-(\rho-2)})I_m \right) \\
&= \left[s^{\rho-1} + \tilde{k}_{\rho-2}s^{\rho-2} + \dots + \tilde{k}_0 \right]^m \\
&\stackrel{(5.13)}{=} \left[\frac{1}{k_{\rho-1}} k(s) \right]^m.
\end{aligned}$$

This completes the proof of (5.12).

Step 4b: We show that the transition matrix $\Phi_Q(\cdot, \cdot)$ of $\dot{x} = Q(t)x$ satisfies

$$\exists M_2, \lambda_2 > 0 \ \forall t \geq t_0 > 0 : \|\Phi_Q(t, s)\| \leq M_2 e^{-\lambda_2(t-t_0)}; \quad (5.14)$$

and we may assume that $\lambda_2 < \lambda_1$.

The zero dynamics of system (1.1) are uniformly asymptotically stable by Assertion (ii); they are determined by the system $\dot{\eta} = Q(t)\eta$ for Q as in (2.8). By Remark 5.2, system $\dot{\eta} = Q(t)\eta$ is uniformly exponentially stable, and so the transition matrix $\Phi_Q(\cdot, \cdot)$ of $\dot{\eta} = Q(t)\eta$ satisfies (5.14).

Step 4c: We show that $\dot{z} = \tilde{Q}(t)z$ is uniformly exponentially stable.

Consider the solution $(z_1(\cdot), z_2(\cdot))$ of the initial value problem

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \tilde{Q}(t) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} Q(t)z_1 + E(t)z_2 \\ H_k z_2 \end{pmatrix}, \quad \begin{pmatrix} z_1(t_0) \\ z_2(t_0) \end{pmatrix} = \begin{pmatrix} z_1^0 \\ z_2^0 \end{pmatrix} \quad (5.15)$$

for fixed but arbitrary $(z_1^0, z_2^0) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ and $t_0 > 0$. By boundedness of E we have

$$\exists c_1 > 0 \ \forall t > 0 : \|E(t)\| \leq c_1. \quad (5.16)$$

An application of the Variation of Constants formula to the first equation in (5.15) and taking norms yields, for all $t \geq t_0$,

$$\begin{aligned}
\|z_1(t)\| &= \left\| \Phi_Q(t, t_0)z_1^0 + \int_{t_0}^t \Phi_Q(t, s)E(s)z_2(s) \, ds \right\| \\
&\stackrel{(5.14)}{\leq} M_2 e^{-\lambda_2(t-t_0)} \|z_1^0\| + \int_{t_0}^t M_2 e^{-\lambda_2(t-s)} c_1 \|z_2(s)\| \, ds \\
&\stackrel{(5.16)}{\leq} M_2 e^{-\lambda_2(t-t_0)} \|z_1^0\| + c_1 M_1 M_2 \|z_2^0\| \int_{t_0}^t e^{-\lambda_2(t-s)} e^{-\lambda_1(s-t_0)} \, ds
\end{aligned}$$

$$\begin{aligned}
&= M_2 e^{-\lambda_2(t-t_0)} \|z_1^0\| + \frac{c_1 M_1 M_2 \|z_2^0\|}{\lambda_1 - \lambda_2} \left(e^{-\lambda_2(t-t_0)} - e^{-\lambda_1(t-t_0)} \right) \\
&\leq M_2 e^{-\lambda_2(t-t_0)} \|z_1^0\| + \frac{c_1 M_1 M_2}{\lambda_1 - \lambda_2} e^{-\lambda_2(t-t_0)} \|z_2^0\|.
\end{aligned} \tag{5.17}$$

Now it is a straightforward calculation to see that (5.11) together with (5.17) yields

$$\exists M_3 > 0 \forall t \geq t_0 > 0 : \left\| \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \right\| \leq M_3 e^{-\frac{\lambda_2}{2}(t-t_0)} \left\| \begin{pmatrix} z_1^0 \\ z_2^0 \end{pmatrix} \right\|$$

and therefore $\dot{z} = \tilde{Q}(t)z$ is uniformly exponentially stable.

Step 5: We show that there exists $\kappa^*, M_4, \lambda_4 > 0$ such that, for all $\kappa \geq \kappa^*$, every solution $(\tilde{y}(\cdot), z(\cdot))$ of the closed-loop system (5.8), $v = -\kappa\tilde{y}$ satisfies

$$\forall t \geq t_0 > 0 : \left\| \begin{pmatrix} \tilde{y}(t) \\ z(t) \end{pmatrix} \right\| \leq M_4 e^{-\lambda_4(t-t_0)} \left\| \begin{pmatrix} \tilde{y}(t_0) \\ z(t_0) \end{pmatrix} \right\|.$$

Note that boundedness of E and Q yields boundedness of \tilde{Q} and therefore we may apply [HP05, Th. 3.3.38] to conclude existence of a symmetric solution $P_{\tilde{Q}} \in \mathcal{C}^1((0, \infty), \mathbb{R}^{(n-m) \times (n-m)})$ to

$$\forall t > 0 : \tilde{Q}(t)^\top P_{\tilde{Q}}(t) + P_{\tilde{Q}}(t) \tilde{Q}(t) + \dot{P}_{\tilde{Q}}(t) = -I_{n-m} \tag{5.18}$$

which is bounded from above and below in the sense

$$\exists \beta_1, \beta_2 > 0 \forall t > 0 : \beta_1 I_{n-m} \leq P_{\tilde{Q}}(t) \leq \beta_2 I_{n-m}. \tag{5.19}$$

Now differentiation of

$$V : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \rightarrow \mathbb{R}, \quad (t, \tilde{y}, z) \mapsto \tilde{y}^\top \tilde{y} + z^\top P_{\tilde{Q}}(t)z,$$

along any solution $(\tilde{y}(\cdot), z(\cdot))$ of the closed-loop system (5.8), $v = -\kappa\tilde{y}$ yields, for all $t > 0$,

$$\begin{aligned}
\frac{d}{dt} V(t, \tilde{y}(t), z(t)) &= 2\tilde{y}(t)^\top \left(\tilde{R}(t)\tilde{y}(t) + \tilde{S}(t)z(t) - \kappa k_{\rho-1} \Gamma(t)\tilde{y}(t) \right) \\
&\quad + 2z(t)^\top \left(P_{\tilde{Q}}(t)\tilde{P}(t)\tilde{y}(t) + P_{\tilde{Q}}(t)\tilde{Q}(t)z(t) \right) + z(t)^\top \dot{P}_{\tilde{Q}}(t)z(t) \\
&\stackrel{(5.18)}{=} 2\tilde{y}(t)^\top \left(\tilde{R}(t)\tilde{y}(t) + \tilde{S}(t)z(t) - \kappa k_{\rho-1} \Gamma(t)\tilde{y}(t) \right) \\
&\quad + 2z(t)^\top P_{\tilde{Q}}(t)\tilde{P}(t)\tilde{y}(t) - \|z(t)\|^2 \\
&\stackrel{\text{Ass. (iv)}}{\leq} 2\tilde{y}(t)^\top \left(\tilde{R}(t)\tilde{y}(t) + \tilde{S}(t)z(t) \right) + 2z(t)^\top P_{\tilde{Q}}(t)\tilde{P}(t)\tilde{y}(t) \\
&\quad - \|z(t)\|^2 - \kappa k_{\rho-1} \alpha \|\tilde{y}(t)\|^2,
\end{aligned}$$

and by boundedness of $\tilde{R}, \tilde{S}, \tilde{P}$ (see Step 2) and $P_{\tilde{Q}}$ (see (5.19)) and making use of the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ for all $a, b \in \mathbb{R}$, it follows that there exists $M_4 > 0$ such that, for all $t > 0$,

$$\begin{aligned}
\frac{d}{dt} V(t, \tilde{y}(t), z(t)) &\leq (M_4 - \kappa k_{\rho-1} \alpha) \|\tilde{y}(t)\|^2 - \|z(t)\|^2 + M_4 \|\tilde{y}(t)\| \cdot \|z(t)\| \\
&\leq \left(M_4 + \frac{1}{2}M_4^2 - \kappa k_{\rho-1} \alpha \right) \|\tilde{y}(t)\|^2 - \frac{1}{2}\|z(t)\|^2 \\
&\stackrel{(5.19)}{\leq} - \left(\kappa k_{\rho-1} \alpha - M_4 - \frac{1}{2}M_4^2 \right) \|\tilde{y}(t)\|^2 - \frac{1}{2\beta_2} z(t)^\top P_{\tilde{Q}}(t)z(t).
\end{aligned}$$

Now set

$$\kappa^* := \frac{1 + M_4 + M_4^2/2}{k_{\rho-1}\alpha} \quad \text{and} \quad M_5 := \min \left\{ 1, \frac{1}{2\beta_2} \right\};$$

then

$$\forall \kappa \geq \kappa^* \quad \forall t > 0 : \frac{d}{dt} V(t, \tilde{y}(t), z(t)) \leq -M_5 V(t, \tilde{y}(t), z(t))$$

and separation of variables yields

$$\forall t \geq t_0 > 0 : V(t, \tilde{y}(t), z(t)) \leq e^{-M_5(t-t_0)} V(t_0, \tilde{y}(t_0), z(t_0)). \quad (5.20)$$

Finally, the claim of Step 5 follows by an application of (5.19) to (5.20) and standard arguments, see e.g. the proof of [Rug96, Th. 7.4].

Step 6: Since

$$x = U^{-1} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = U^{-1} \tilde{U}^{-1} \begin{pmatrix} \tilde{y} \\ z \end{pmatrix}$$

and U^{-1} , \tilde{U}^{-1} are bounded (see Step 1 and Step 2) it follows that the closed-loop system (1.1), (5.1) is uniformly exponentially stable. This completes the proof of the theorem. \square

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