Funnel control for a moving water tank

Thomas Berger, Marc Puche and Felix L. Schwenninger

***Abstract*—We study tracking control for a moving water tank system modelled by the (linearized) Saint-Venant equations. The output is given by the position of the tank and the control input is the force acting on it. For a given reference signal, the objective is to achieve that the tracking error evolves within a prespecified performance funnel. Exploiting recent results in funnel control it suffices to show that the operator associated with the internal dynamics of the system is causal, locally Lipschitz continuous and maps bounded functions to bounded functions.**

***Index Terms*—Shallow water equations, Saint-Venant equations, sloshing, well-posed systems, adaptive control, funnel control.**

I. Introduction

When a liquid-filled containment is subject to movement, the motion of the fluid may have a significant effect on the dynamics of the overall system and is known as *sloshing*. The latter phenomenon can be understood as internal dynamics of the system and it is of great importance in a range of applications such as aeronautics and control of containers and vehicles, and has been studied in engineering for a long time, see e.g. [1]–[6].

The standard model for the one-dimensional movement of a fluid is given by the Saint-Venant equations, which is a system of nonlinear hyperbolic partial differential equations (PDEs). Models of a moving water tank involving these equations without friction have been studied in various articles. The first approach appears in [7] where a flat output for the linearized model is constructed. Several additional control problems related to this model are studied in [8] and it is proved that the linearization is steady-state controllable. Even more so, the seminal work [9] shows that the (nonlinear) model is locally controllable around any steady state. Different stabilization approaches by state and output feedback using Lyapunov functions are studied in [10]. In [11] observers are designed to estimate the horizontal currents by exploiting the symmetries in the Saint-Venant equations. Convergence of the estimates to the actual states is studied for the linearized model. In [1] a port-Hamiltonian formulation of the system is provided as a mixed finite-infinite dimensional port-Hamiltonian system. For a recent numerical treatment of a truck with a fluid basin see [12].

In the present paper we consider output trajectory tracking for moving water tank systems by funnel control. The concept of funnel control was developed in [13], see also the survey [14]. The funnel controller is an adaptive controller of high-gain type and proved its potential for tracking problems in various applications, such as temperature control of chemical reactor models [15], control of industrial servo-systems [16] and underactuated multibody systems [17], voltage and current control of electrical circuits [18], control of peak inspiratory pressure [19] and adaptive cruise control [20]. We like to emphasize that the funnel controller is a model-free feedback controller, i.e., it does not require specific system parameters for feasibility. This makes it a suitable choice for the application to the water tank system, for which we assume that it contains a non-vanishing friction term as modeled in the Saint-Venant equations e.g. in [21], but the exact shape of this term is unknown and not available to the controller.

It is our aim to show that the funnel controller introduced in [22] is feasible for these systems. While a very large class of functional differential equations with higher relative degree is considered in [22] and funnel control is shown to work for those systems (cf. also Section II), it is not clear exactly which systems containing PDEs are encompassed by this class. It is our main result that the linearized model of the moving water tank, where the above mentioned effect of sloshing appears, belongs to the aforementioned system class.

A. Nomenclature

Throughout this article, we use the following notation: $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{R}_{\geq 0} = [0, \infty)$. We write $\mathbb{C}_\omega = \{ \lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > \omega \}$ for $\omega \in \mathbb{R}$ and $\mathbb{C}_+ = \mathbb{C}_0$. With $L^p(I; \mathbb{K}^n)$ we denote the Lebesgue space of all measurable and pth power integrable functions $f : I \to \mathbb{K}^n$, $I \subseteq \mathbb{R}$ an interval, where $p \in [1, \infty)$ and $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$; $L^\infty(I; \mathbb{K}^n)$ denotes the Lebesgue space of all measurable and essentially bounded functions $f : I \to \mathbb{K}^n$. We write $\| \cdot \|_\infty$ for $\| \cdot \|_{L^\infty(\mathbb{R}_{\geq 0}; \mathbb{K}^n)}$. By $L^\infty_{\mathrm{loc}}(I; \mathbb{K}^n)$ we denote the set of measurable and locally essentially bounded functions $f : I \to \mathbb{K}^n$, by $W^{k,p}(I; \mathbb{K}^n)$, $k \in \mathbb{N}_0$, the Sobolev space of k-times weakly differentiable functions $f : I \to \mathbb{K}^n$ such that $f, \dot{f}, \ldots, f^{(k)} \in L^p(I; \mathbb{K}^n)$, and by $\mathcal{C}^k(I; \mathbb{K}^n)$ the set of k-times continuously differentiable functions $f : I \to \mathbb{K}^n$, $k \in \mathbb{N}_0 \cup \{\infty\}$ where $\mathcal{C}(I; \mathbb{K}^n) := \mathcal{C}^0(I; \mathbb{K}^n)$. We further use the abbreviation $H^k(I; \mathbb{K}^n) := W^{k,2}(I; \mathbb{K}^n)$. For $\omega \in \mathbb{R}$ we use the notation $L^2_\omega(\mathbb{R}_{\geq 0}; \mathbb{K}) := \{ e^{\omega \cdot} f(\cdot) \mid f \in L^2(\mathbb{R}_{\geq 0}; \mathbb{K}) \}$ with norm $\| e^{\omega \cdot} f \|_{L^2_\omega} = \| f \|_{L^2(\mathbb{R}_{\geq 0}; \mathbb{K})}$. By $\mathcal{L}(\mathcal{X}; \mathcal{Y})$, where $\mathcal{X}, \mathcal{Y}$ are Hilbert spaces, we denote the set of all bounded linear operators $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$.

B. Mathematical Model

In the present paper we study the horizontal movement of a water tank as depicted in Fig. 1, where we neglect the wheels' inertia and friction between the wheels and the ground. We assume that there

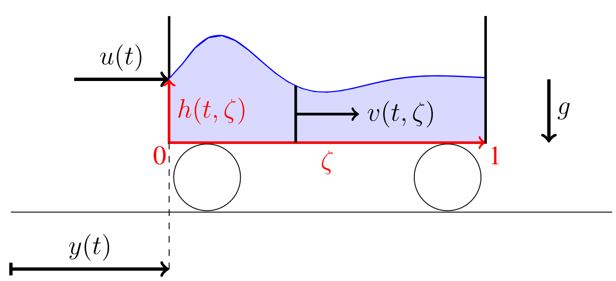

Fig. 1: Horizontal movement of a water tank.

is an external force acting on the water tank, which we denote by $u(t)$ as this will be the control input of the resulting system, cf. also Section I-C. The measurement output is the horizontal position $y(t)$

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of the water tank, and the mass of the empty tank is denoted by m_T. The dynamics of the water under gravity g are described by the *Saint-Venant equations* (first derived in [23]; also called one-dimensional *shallow water equations*)

$$\begin{aligned}\partial_t h + \partial_\zeta(hv) &= 0,\\ \partial_t v + \partial_\zeta\left(\frac{v^2}{2} + gh\right) + hS\left(\frac{v}{h}\right) &= -\ddot{y}\end{aligned} \tag{1}$$

with boundary conditions $v(t,0) = v(t,1) = 0$. Here $h : \mathbb{R}_{\geq 0} \times [0,1] \to \mathbb{R}$ denotes the height profile and $v : \mathbb{R}_{\geq 0} \times [0,1] \to \mathbb{R}$ the (relative) horizontal velocity profile, where the length of the container is normalized to 1. The friction term $S : \mathbb{R} \to \mathbb{R}$ is typically modeled by a high velocity coefficient of the form $C_S v^2/h^2$ and a viscous drag of the form $C_D v/h$ for some positive constants C_S, C_D. In the present paper, we do not specify S, but we do assume that $S(0) = 0$ and $S'(0) > 0$. The condition $S(0) = 0$ means that, whenever the velocity is zero, then there is no friction. The condition $S'(0) > 0$ means that the viscous drag does not vanish; this is the case in most real-world non-ideal situations, but sometimes neglected in the literature, see e.g. [21, Sec. 1.4].

For a derivation of the Saint-Venant equations (1) of a moving water tank we refer to [1], [8], see also the references therein. The friction term in the model is the general version of that used in [21, Sec. 1.4]. Let us emphasize that in our framework the input is the force acting on the water tank, which can be manipulated using an engine for instance. In contrast to this, in [8], [9] the acceleration of the tank is used as input, but this can usually not be influenced directly. Note that — in the presence of sloshing — the applied force does not equal the product of the tanks's mass and acceleration. We also stress that, if the acceleration is used as input, then the input-output relation is given by the simple double integrator $\ddot{y} = u$, and the Saint-Venant equations (1) do not affect this relation.

As shown in [7], [8], the linearization of the Saint-Venant equations is relevant in the context of control since it provides a model which is much simpler to solve (both analytically and numerically) and it can be an insightful approximation for motion planning purposes. Therefore, we restrict ourselves to the linearization of (1). In order to derive the linearization we first consider the general operator differential equation

$$\partial_t x(t) - F\big(x(t)\big) = f(t), \tag{2}$$

where $F : \mathcal{D}(F) \subseteq X \to Y$, $f : \mathbb{R}_{\geq 0} \to Y$, X, Y are suitable Hilbert spaces and $\mathcal{D}(F)$ is the domain of the operator F. We call a point $x^* \in \mathcal{D}(F)$ a *steady-state* of (2), if $F(x^*) = 0$. If F is Fréchet differentiable in x^* with Fréchet derivative $A := D_{x^*}F : X \to Y$, then the *linearization* of (2) around the steady-state x^* is given by

$$\partial_t x(t) - Ax(t) = f(t).$$

In the case of the Saint-Venant equations (1) we have $X = H^1([0,1];\mathbb{R}^2)$, $Y = L^2([0,1];\mathbb{R}^2)$, $f(t) = \left(\begin{smallmatrix} 0 \\ -\ddot{y}(t)\end{smallmatrix}\right)$ and the operator

$$F(x_1,x_2) = -\begin{pmatrix} \partial_\zeta(x_1x_2) \\ \partial_\zeta\left(\frac12 x_2^2 + gx_1\right) + x_1 S\left(\frac{x_2}{x_1}\right)\end{pmatrix}$$

with

$$\mathcal{D}(F) = \left\{ (x_1,x_2) \in X \;\middle|\; \begin{array}{l} x_2(0) = x_2(1) = 0, \\ \forall \zeta \in [0,1] : \; x_1(\zeta) > 0 \end{array} \right\}.$$

A steady-state $x^* = (H,V) \in \mathcal{D}(F)$ is a solution of the boundary-value problem

$$\begin{aligned}\partial_\zeta(HV) &= 0,\\ V\partial_\zeta V + g\partial_\zeta H + HS\left(\frac{V}{H}\right) &= 0, \quad V(0) = V(1) = 0.\end{aligned}$$

Since $S(0) = 0$ and $H(\zeta) > 0$ for all $\zeta \in [0,1]$, we may infer that $V \equiv 0$ and $H \equiv h_0 > 0$. A straightforward computation yields that F is Fréchet differentiable in $x^* = (h_0, 0) \in \mathcal{D}(F)$ with bounded Fréchet derivative $A := D_{x^*}F : X \to Y$ given by

$$Az = -\big(h_0\partial_\zeta z_2, g\partial_\zeta z_1 + S'(0)z_2\big)^\top, \quad z \in X.$$

Note that $A : X \to Y$ is bounded, but $A : (X, \|\cdot\|_Y) \subseteq Y \to Y$ will be unbounded since a weaker norm is used. Define $\mu := \frac12 S'(0)$,

$$P_1 := \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \mathcal{H} := \begin{bmatrix} g & 0 \\ 0 & h_0 \end{bmatrix}, \quad P_0 := \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \quad b := \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Then the linearization of the Saint-Venant equations (1) is given by

$$\partial_t z = Az + b\ddot{y} = P_1\partial_\zeta(\mathcal{H}z) - \mu P_0 z + b\ddot{y} \tag{3}$$

with boundary conditions $z_2(t,0) = z_2(t,1) = 0$. Note that by the first equation in (3) (conservation of mass) and the boundary conditions we have

$$\partial_t \int_0^1 z_1(t,\zeta)\,\mathrm{d}\zeta = -h_0\big(z_2(t,1) - z_2(t,0)\big) = 0,$$

hence $\int_0^1 z_1(t,\zeta)\,\mathrm{d}\zeta = \text{const}$. Furthermore, if (z_1,z_2) is a solution of (3), then also $(z_1,z_2) + (c,0)$ is a solution of (3) for all $c \in \mathbb{R}$. Hence, without loss of generality we may restrict ourselves to solutions which satisfy $\int_0^1 z_1(t,\zeta)\,\mathrm{d}\zeta = 0$ for all $t \geq 0$. This justifies to choose

$$\hat{X} = \left\{ (f_1,f_2) \in L^2([0,1];\mathbb{R}^2) \;\middle|\; \int_0^1 f_1(\zeta)\,\mathrm{d}\zeta = 0 \right\} \tag{4}$$

as new state space and to consider the operator $A : \mathcal{D}(A) \subseteq \hat{X} \to Y$,

$$\mathcal{D}(A) = \left\{ (z_1,z_2) \in \hat{X} \;\middle|\; \begin{array}{l} z_1, z_2 \in H^1([0,1];\mathbb{R}), \\ z_2(0) = z_2(1) = 0 \end{array} \right\}. \tag{5}$$

Note that for any $z \in \mathcal{D}(A)$ we have $\int_0^1 \partial_\zeta z_2(\zeta)\,\mathrm{d}\zeta = 0$, hence $Az \in \hat{X}$. Therefore, $A : \mathcal{D}(A) \subseteq \hat{X} \to \hat{X}$ and we like to stress that A may be unbounded. From now on, with some abuse of notation, we write X instead of $\hat{X}$.

In order to complete the model, we introduce the *momentum*

$$p(t) := m_T\dot{y}(t) + \int_0^1 \big(z_1(t,\zeta) + h_0\big)\big(z_2(t,\zeta) + \dot{y}(t)\big)\,\mathrm{d}\zeta\,,$$

and consider the balance law $\dot{p}(t) = u(t)$. Using (3) we calculate

$$\begin{aligned}\dot{p}(t) = m_T\ddot{y}(t) &- \frac{g}{2}\left(z_1(t,1)^2 - z_1(t,0)^2\right) - h_0 g\big(z_1(t,1) - z_1(t,0)\big)\\ &- 2\mu\int_0^1 \big(z_1(t,\zeta) + h_0\big)z_2(t,\zeta)\,\mathrm{d}\zeta\,.\end{aligned}$$

Altogether the model that we consider in the present paper is described by the following nonlinear equations,

$$\begin{aligned}&\partial_t z = P_1\partial_\zeta(\mathcal{H}z) - \mu P_0 z + b\ddot{y},\\ &\ddot{y}(t) = \frac{g}{2m_T}\left(z_1(t,1) - z_1(t,0)\right)\big(2h_0 + z_1(t,1) + z_1(t,0)\big)\\ &\qquad + \frac{2\mu h_0}{m_T}\int_0^1 z_2(t,\zeta)\,\mathrm{d}\zeta + \frac{2\mu}{m_T}\int_0^1 z_1(t,\zeta)z_2(t,\zeta)\,\mathrm{d}\zeta + \frac{u(t)}{m_T},\\ &0 = z_2(t,0) = z_2(t,1)\end{aligned} \tag{6}$$

on the state space X, with input u, state z and output y.

C. Control objective

The objective is to design an output error feedback $u(t) = F\big(t, e(t), \dot{e}(t)\big)$, where $y_{\text{ref}} \in W^{2,\infty}(\mathbb{R}_{\geq 0};\mathbb{R})$ is a reference signal, which applied to (6) results in a closed-loop system where the tracking error $e(t) = y(t) - y_{\mathrm{ref}}(t)$ evolves within a prescribed performance funnel

$$\mathcal{F}_\varphi := \{ \, (t,e) \in \mathbb{R}_{\ge 0} \times \mathbb{R} \;|\; \varphi(t)|e| < 1 \, \},$$

which is determined by a function φ belonging to

$$\Phi := \left\{ \varphi \in \mathcal{C}^1(\mathbb{R}_{\ge 0} \to \mathbb{R}) \;\middle|\; \begin{array}{l} \varphi, \dot\varphi \text{ are bounded,} \\ \varphi(\tau) > 0 \text{ for all } \tau > 0, \\ \text{and } \liminf_{\tau\to\infty} \varphi(\tau) > 0 \end{array} \right\}.$$

Furthermore, all signals $u, e, \dot e$ should remain bounded.

The funnel boundary is given by the reciprocal of φ, see Fig. 2. The case $\varphi(0) = 0$ is explicitly allowed and puts no restriction on the initial value since $\varphi(0)|e(0)| < 1$; in this case the funnel boundary $1/\varphi$ has a pole at $t = 0$.

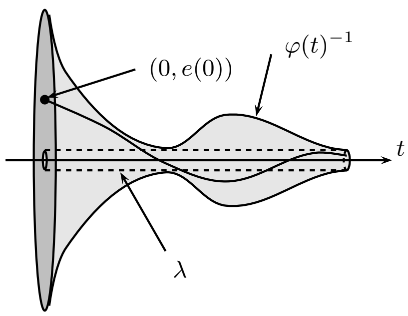

Fig. 2: Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $\varphi(t)^{-1}$.

Note that boundedness of φ implies that there exists $\lambda > 0$ such that $1/\varphi(t) \ge \lambda$ for all $t > 0$. The funnel boundary is not necessarily monotonically decreasing and there are situations, like in the presence of periodic disturbances, where widening the funnel over some later time interval might be beneficial.

It was shown in [22] that for $\varphi_0, \varphi_1 \in \Phi$, the funnel controller

$$\boxed{\begin{aligned} u(t) &= -k_1(t)\big(\dot e(t) + k_0(t)e(t)\big), \\ k_0(t) &= \frac{1}{1 - \varphi_0(t)^2\|e(t)\|^2}, \\ k_1(t) &= \frac{1}{1 - \varphi_1(t)^2\|\dot e(t) + k_0(t)e(t)\|^2}, \end{aligned}} \tag{7}$$

achieves the above control objective for a large class of nonlinear systems with relative degree two. In the present paper we extend this result and show feasibility of (7) for the model described by (6).

D. Organization of the present paper

In Section II we recall a recent result in funnel control from [22]. We show that in order to achieve the control objective formulated in Section I-C it suffices to show that a certain operator is causal, locally Lipschitz continuous and maps bounded functions to bounded functions. To this end, in Section III we consider the linearized Saint-Venant equations in the framework of well-posed systems and show, in particular, that the corresponding impulse response is a measure with bounded total variation. In Section IV we exploit this result to show that the operator associated with the internal dynamics of (6) is well-defined and has the properties mentioned above. The application of the funnel controller to the moving water tank system is illustrated in Section V.

II. Funnel control

In this section we formulate how the funnel controller (7) described in Subsection I-C achieves the control objective for system (6) — this is the main result of this article. The initial conditions for (6) are

$$\begin{aligned} \big(z_1(0,\cdot), z_2(0,\cdot)\big) &= \big(\tilde h_0(\cdot), v_0(\cdot)\big) &&\in \mathcal{D}(A), \\ \big(y(0), \dot y(0)\big) &= \big(y^0, y^1\big) &&\in \mathbb{R}^2, \end{aligned} \tag{8}$$

since the initial value for z needs to belong to the domain of the operator A_μ in (6). In [22] the controller (7) is shown to be feasible for a large class of nonlinear systems of the form

$$\begin{aligned} \ddot y(t) &= f\big(d(t), \mathcal{S}(y,\dot y)(t)\big) + \Gamma\, u(t) \\ \big(y(0), \dot y(0)\big) &= \big(y^0, y^1\big) \in \mathbb{R}^2 \end{aligned} \tag{9}$$

where

(N1) the disturbance satisfies $d \in L^\infty(\mathbb{R}_{\ge 0}; \mathbb{R}^p)$, $p \in \mathbb{N}$;
(N2) $f \in \mathcal{C}(\mathbb{R}^p \times \mathbb{R}^q; \mathbb{R})$, $q \in \mathbb{N}$,
(N3) the high-frequency gain satisfies $\Gamma > 0$, and
(N4) $\mathcal{S} : \mathcal{C}(\mathbb{R}_{\ge 0}; \mathbb{R}^2) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_{\ge 0}; \mathbb{R}^q)$ satisfies the properties:

a) $\mathcal{S}$ maps bounded trajectories to bounded trajectories, i.e., for all $c_1 > 0$, there exists $c_2 > 0$ such that for all $\zeta \in \mathcal{C}(\mathbb{R}_{\ge 0}; \mathbb{R}^2) \cap L^\infty(\mathbb{R}_{\ge 0}; \mathbb{R}^2)$, $\mathcal{S}(\zeta) \in L^\infty(\mathbb{R}_{\ge 0}; \mathbb{R}^q)$ and
$$\|\zeta\|_\infty \le c_1 \implies \|\mathcal{S}(\zeta)\|_\infty \le c_2,$$

b) $\mathcal{S}$ is causal, i.e, for all $t \ge 0$ and all $\zeta, \xi \in \mathcal{C}(\mathbb{R}_{\ge 0}; \mathbb{R}^2)$,
$$\zeta|_{[0,t)} = \xi|_{[0,t)} \implies \mathcal{S}(\zeta)|_{[0,t)} \overset{\text{a.e.}}{=} \mathcal{S}(\xi)|_{[0,t)}.$$

c) $\mathcal{S}$ is locally Lipschitz continuous in the following sense: for all $t \ge 0$ and $\xi \in \mathcal{C}([0,t]; \mathbb{R}^2)$ there exist $\tau, \delta, c > 0$ such that, for all $\zeta_1, \zeta_2 \in \mathcal{C}(\mathbb{R}_{\ge 0}; \mathbb{R}^2)$ with $\zeta_i|_{[0,t]} = \xi$ and $\|\zeta_i(s) - \xi(t)\| < \delta$ for all $s \in [t, t+\tau]$ and $i = 1, 2$, we have
$$\left\| (\mathcal{S}(\zeta_1) - \mathcal{S}(\zeta_2))\,|_{[t,t+\tau]} \right\|_\infty \le c \left\| (\zeta_1 - \zeta_2)|_{[t,t+\tau]} \right\|_\infty.$$

In [13], [25]–[27] it is shown that the class of systems (9) encompasses linear and nonlinear systems with strict relative degree two and input-to-state stable internal dynamics. The operator $\mathcal{S}$ allows for infinite-dimensional (linear) systems, systems with hysteretic effects or (when a slightly more general version of (9) with a memory component is considered) nonlinear delay elements, and combinations thereof. The linear infinite-dimensional systems that are considered in [13], [27] are in a special Byrnes-Isidori form that is discussed in detail in [28]. While the internal dynamics in these systems is allowed to correspond to a strongly continuous semigroup, all other operators are assumed to be bounded. In contrast to this, the equation (6) that we consider here is nonlinear and involves unbounded operators.

In [22], the existence of solutions of the initial value problem resulting from the application of the funnel controller (7) to a system (9) is investigated. By a *solution* of (7), (9) on $[0, \omega)$ we mean a function $y \in \mathcal{C}^1([0,\omega); \mathbb{R})$, $\omega \in (0, \infty]$, such that $\dot y$ is weakly differentiable and satisfies (9) with u defined in (7) for almost all $t \in [0, \omega)$; y is called *maximal*, if it has no right extension that is also a solution. Existence of solutions of functional differential equations has been investigated in [13] for instance.

The following result is from [22]. Note that in [22] a slightly stronger version of condition (N4) c) is used. However, the existence part of the proof there relies on a result from [26] where the version from the present paper is used.

Theorem II.1. *Consider a system* (9) *with properties (N1)–(N4). Let* $y_{\mathrm{ref}} \in W^{2,\infty}(\mathbb{R}_{\ge 0}; \mathbb{R})$, $\varphi_0, \varphi_1 \in \Phi$ *and* $(y^0, y^1) \in \mathbb{R}^2$ *be initial conditions such that*

$$\begin{aligned} &\varphi_0(0)|y^0 - y_{\mathrm{ref}}(0)| < 1 \\ \text{and}\quad &\varphi_1(0)|y^1 - \dot y_{\mathrm{ref}}(0) + k_0(0)\big(y^0 - y_{\mathrm{ref}}(0)\big)| < 1. \end{aligned}$$

Then the funnel controller (7) *applied to* (9) *yields an initial-value problem which has a solution, and every solution can be extended* *to a maximal solution $y : [0,\omega) \to \mathbb{R}$, $\omega \in (0,\infty]$, which has the following properties:*

(i) *The solution is global (i.e., $\omega = \infty$).*
(ii) *The input $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$, the gain functions $k_0, k_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $y, \dot{y} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ are bounded.*
(iii) *The tracking error $e = y - y_{\mathrm{ref}}$ is uniformly bounded away from the funnel boundary in the following sense:*

$$\exists \varepsilon > 0 \; \forall t > 0 : \; |e(t)| \leq \varphi_0(t)^{-1} - \varepsilon. \tag{10}$$

To show feasibility of the funnel controller (7) for (6), (8), we will show that (6), (8) belongs to the class of systems (9). Then feasibility is a consequence of the above Theorem II.1.

By the change of variables $x(t) = z(t) - b\eta(t)$ where we use $\eta(t) := \dot{y}(t) - \dot{y}(0)$, System (6) can be rewritten as

$$\ddot{y}(t) = \mathcal{S}(y, \dot{y})(t) + \frac{u(t)}{m_T}, \tag{11}$$

where $\mathcal{S} : \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^2) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R})$ is given by

$$\mathcal{S}(y_1, y_2) := \mathcal{T}(y_2 - y_2(0)) \tag{12}$$

for the operator $\mathcal{T} : \mathcal{C}_0(\mathbb{R}_{\geq 0}; \mathbb{R}) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R})$, where $\mathcal{C}_0(\mathbb{R}_{\geq 0}; \mathbb{R}) = \{ \, f \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}) \mid f(0) = 0 \, \}$, defined by

$$\begin{aligned}
\mathcal{T}(\eta)(t) = \; & \frac{g}{2m_T} \left(x_1(t,1) - x_1(t,0) \right) \big(2h_0 + x_1(t,1) + x_1(t,0) \big) \\
& + \frac{2\mu h_0}{m_T} \int_0^1 x_2(t,\zeta) \, \mathrm{d}\zeta + \frac{2\mu}{m_T} \int_0^1 x_1(t,\zeta) x_2(t,\zeta) \, \mathrm{d}\zeta \\
& - \frac{2\mu h_0}{m_T} \eta(t),
\end{aligned} \tag{13}$$

$$\begin{aligned}
\dot{x}(t) &= P_1 \partial_\zeta \big(\mathcal{H}(x(t) + b\eta(t)) \big) - \mu P_0 (x(t) + b\eta(t)), \\
x(0) &= x_0 = (\tilde{h}_0, v_0).
\end{aligned} \tag{14}$$

Note that $\mathcal{T}$ depends on $x = x(t,\zeta)$ which in turn is given through η and x_0 as the solution of the linear PDE (14) that is a one-dimensional wave equation. We like to point out that the operator $\mathcal{S}$ essentially models the internal dynamics of system (6).

Theorem II.2. *For $\mu > 0$ the system consisting of* (6), (8) *belongs to the class of systems* (9)*, where the operator $\mathcal{S}$ is given by* (12)*. Thus, the assertions of Theorem II.1 hold for the considered system.*

Proof. First observe that for equation (11) conditions (N1)–(N3) are obviously satisfied, so it remains to show the properties of the operator $\mathcal{S}$ as required in (N4). By Proposition IV.1 the operator $\mathcal{T}$ given by (13), (14) is well-defined, locally Lipschitz continuous and maps bounded functions to bounded functions. As it is easy to see that $\mathcal{S}$ is causal it thus follows that it satisfies (N4). □

Remark II.3. In the case $\mu = 0$ the statement of Theorem II.2 is false in general, since the operator $\mathcal{S}$ does not satisfy condition a) in (N4). To be more precise we need to consider the later results derived in Sections III and IV: If $\mu = 0$, then $\mathfrak{h} = \mathcal{L}^{-1}(\mathbf{H})$ derived in Lemma III.2 does not have bounded total variation, which, upon inspecting the proof of Proposition IV.1, reveals that, for example, $\mathcal{T}(\sin)(\cdot)$ is unbounded.

The remainder of the paper deals with proving Proposition IV.1.

III. Linearized model – abstract framework

In this section we gather preliminary results concerning the operator associated with the linearized Saint-Venant equations (3). This includes admissibility of the involved unbounded control and evaluation operators and the transfer functions of the linear subsystems. Finally we show that the impulse response of these transfer functions defines measures with bounded total variation.

We consider the complexification of the state space from (4),

$$X = \left\{ \, (f_1, f_2) \in L^2([0,1]; \mathbb{C}^2) \; \middle| \; \int_0^1 f_1(\zeta) \, \mathrm{d}\zeta = 0 \, \right\},$$

and the linear operators $A_\mu : \mathcal{D}(A_\mu) \subseteq X \to X$ given by

$$A_\mu z := P_1 \partial_\zeta (\mathcal{H} z) - \mu P_0 z \tag{15}$$

with domain $\mathcal{D}(A_\mu) = \mathcal{D}(A)$ as the complexification of (5).

Clearly, the solution of the linear, one-dimensional damped wave equation $\dot{z} = A_\mu z$ with $z(0) = z_0$ can be derived by a Fourier ansatz. More general, the solution theory for linear PDEs can be discussed using semigroup theory. Let us recall a few basics from semigroup theory and admissible operators in the context of linear systems, which can all be found e.g. in [29]. A *semigroup* $(\mathcal{T}(t))_{t \geq 0}$ on X is a $\mathcal{L}(X;X)$-valued map satisfying $\mathcal{T}(0) = I_X$ and $\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(s)$, $s,t \geq 0$, where I_X denotes the identity operator. Furthermore, we assume that semigroups are strongly continuous, i.e., $t \mapsto \mathcal{T}(t)x$ is continuous for every $x \in X$. Semigroups are characterized by their generator $\mathcal{A}$, which is a possibly unbounded operator on X. For any semigroup there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|\mathcal{T}(t)\| \leq M \mathrm{e}^{\omega t}$ for all $t \geq 0$. The infimum over all ω such that this inequality is valid for some M is called the growth bound $\omega_{\mathcal{A}}$ and $\mathcal{T}$ is *exponentially stable*, if $\omega_{\mathcal{A}} < 0$. If $\|\mathcal{T}(t)\| \leq 1$ for all $t \geq 0$, then $\mathcal{T}$ is called *contraction semigroup*. For β in the resolvent set $\rho(\mathcal{A})$ of the generator $\mathcal{A}$, we denote by $\mathcal{X}_{-1}$ the completion of $\mathcal{X}$ with respect to the norm $\|\cdot\|_{\mathcal{X}_{-1}} = \|(\beta I - \mathcal{A})^{-1} \cdot \|_{\mathcal{X}}$. Note that $\mathcal{X}_{-1}$ is independent of the choice of β and that $(\beta I - \mathcal{A})$ uniquely extends to an surjective isometry $(\beta I - \mathcal{A}_{-1}) \in \mathcal{L}(\mathcal{X}; \mathcal{X}_{-1})$. The semigroup $(\mathcal{T}(t))_{t \geq 0}$ in $\mathcal{X}$ has a unique extension to a semigroup $(\mathcal{T}_{-1}(t))_{t \geq 0}$ in $\mathcal{X}_{-1}$, which is generated by $\mathcal{A}_{-1}$.

The notion of *admissible* operators appears naturally in infinite-dimensional linear systems theory with unbounded control and observation operators, as present in boundary control. Let $\mathcal{U}, \mathcal{Y}$ be Hilbert spaces, $\mathcal{A}$ as above and $p \in [1,\infty]$. We call $B \in \mathcal{L}(\mathcal{U}; \mathcal{X}_{-1})$ an L^p*-admissible* control operator (for $(\mathcal{T}(t))_{t \geq 0}$), if for all $t \geq 0$ and all $u \in L^p([0,t]; \mathcal{U})$ we have

$$\Phi_t u := \int_0^t \mathcal{T}_{-1}(t-s) B u(s) \, \mathrm{d}s \in \mathcal{X},$$

which implies that the operator $\Phi_t \in \mathcal{B}(L^p([0,t]; \mathcal{U}), \mathcal{X})$. An operator $C \in \mathcal{L}(\mathcal{D}(\mathcal{A}); \mathcal{Y})$ is called L^p*-admissible* observation operator (for $(\mathcal{T}(t))_{t \geq 0}$), if for some (and hence all) $t \geq 0$ the mapping

$$\Psi_t : \mathcal{D}(\mathcal{A}) \to L^p([0,t]; \mathcal{Y}), \; x \mapsto CT(\cdot)x$$

can be extended to a bounded operator from $\mathcal{X}$ to $L^p([0,t]; \mathcal{Y})$, again denoted by Ψ_t. We call B or C *infinite-time L^p-admissible*, if $\sup_{t>0} \|\Phi_t\| < \infty$ or $\sup_{t>0} \|\Psi_t\| < \infty$, respectively. Both admissibility notions are combined in the stronger concept of *well-posedness*: Let $(\mathcal{A}, B, C)$ represent a system where $\mathcal{A}$ is the generator of a semigroup, B is an L^2-admissible control operator and C is an L^2-admissible observation operator in the sense described above. If there exists a function $\mathbf{G} : \mathbb{C}_\omega \to \mathcal{L}(\mathcal{U}; \mathcal{Y})$, $\omega > \omega_{\mathcal{A}}$, which satisfies

$$\mathbf{G}(s) - \mathbf{G}(t) = C((sI - \mathcal{A})^{-1} - (tI - \mathcal{A})^{-1}) B \tag{16}$$

for all $s,t \in \mathbb{C}_\omega$ and $\mathbf{G}$ is proper, i.e., $\sup_{s \in \mathbb{C}_\omega} \|\mathbf{G}(s)\| < \infty$, then we say that $(\mathcal{A}, B, C)$ is *well-posed*. Note that $\mathbf{G}$ is uniquely determined up to a constant and that well-posedness can be defined in different, but equivalent ways, see [30], [31]. If $\lim_{\mathrm{Re}\, s \to \infty} \mathbf{G}(s)v$ exists for any $v \in \mathcal{U}$, then $(\mathcal{A}, B, C)$ is called *regular*.

In order to prove Theorem II.2 we study the PDE (14) in combination with two observation operators which appear in the definition of the operator $\mathcal{T}$ in (13), that is we investigate the input-output behaviour of the linear systems

$$\begin{aligned}\dot{x} &= A_\mu x + A_\mu b\eta,\\ v_i &= C_i x := \frac{1}{2}(x_1(1) + (-1)^i x_1(0))\end{aligned} \tag{Σ_i}$$

for $i = 1, 2$, where $C_i : \mathcal{D}(A) \to \mathbb{C}$. Whereas it will be essential to show that the associated input-output map $\eta \mapsto v_i$ is bounded with respect to L^∞-norms, we first restrict ourselves to the classical case of boundedness with respect to L^2-norms. In Proposition III.1 (iv) below we show that (Σ_i) is regular and well-posed. This then implies by definition, cf. [30], [31], that the input-output map

$$\begin{aligned}F_i &: W_0^{1,\infty}(\mathbb{R}_{\geq 0};\mathbb{C}) \cap L^2_\omega(\mathbb{R}_{\geq 0};\mathbb{C}) \to L^2_\omega(\mathbb{R}_{\geq 0};\mathbb{C}),\\ \eta &\mapsto \left(t \mapsto C_i \int_0^t (T_\mu)_{-1}(t-s)B\eta(s)\,\mathrm{d}s\right),\end{aligned} \tag{17}$$

where $W_0^{1,\infty}(\mathbb{R}_{\geq 0};\mathbb{C}) = \left\{ f \in W^{1,\infty}(\mathbb{R}_{\geq 0};\mathbb{C}) \;\middle|\; f(0) = 0 \right\}$, is well-defined for all $\omega > \omega_{A_\mu}$ and can be continuously extended to $L^2_\omega(\mathbb{R}_{\geq 0};\mathbb{C})$. Here, we identify C_i with a suitable extension, see [30, Sec. 5] for details. Therefore, the *transfer function* of (Σ_i) can be defined by representing F_i in terms of the Laplace transform, that is

$$\mathcal{L}(v_i)(s) = \mathcal{L}(F_i\eta)(s) = \mathbf{H}^i(s)\mathcal{L}(\eta)(s), \tag{18}$$

where $\mathbf{H}^i : \mathbb{C}_\omega \to \mathbb{C}$, $i = 1, 2$.

As the proof of the following result rests on a standard routine, details are omitted. For an explicit derivation we refer to [32].

Proposition III.1. *Let $c := \sqrt{gh_0}$, $\mu \geq 0$, $b = \left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right)$ and A_μ be defined by* (15). *Then the following assertions hold for all $i = 1, 2$.*

(i) *A_μ generates a contraction semigroup $(T_\mu(t))_{t\geq 0}$ with growth bound $\omega_{A_\mu} = -\mu + \operatorname{Re}\sqrt{\mu^2 - \pi^2 c^2}$ and the spectrum of A_μ consists of the eigenvalues $\theta_n^\pm = -\mu \pm \mathrm{i}\phi_n$, where*

$$\phi_n = \sqrt{\sigma_n^2 - \mu^2}, \quad \sigma_n = n\pi c, \quad n \in \mathbb{N}. \tag{19}$$

(ii) *$B = A_\mu b \in \mathcal{L}(\mathbb{C}; X_{-1})$ is L^p-admissible for all $p \in [2, \infty]$;*

(iii) *$C_i \in \mathcal{L}(\mathcal{D}(A);\mathbb{C})$ is L^2-admissible;*

(iv) *(A_μ, B, C_i) is well-posed and regular and for $\omega > \omega_{A_\mu}$ the transfer functions $\mathbf{H}^i : \mathbb{C}_\omega \to \mathbb{C}$ are given by*

$$\mathbf{H}(\lambda) := \mathbf{H}^1(\lambda) = -\sqrt{\frac{4h_0}{g}\frac{\lambda}{\lambda + 2\mu}}\tanh\left(\frac{\sqrt{\lambda(\lambda+2\mu)}}{2c}\right) \tag{20}$$

and $\mathbf{H}^2(\lambda) = 0$, for all $\lambda \in \mathbb{C}_\omega$.

If $\mu > 0$, then B, C_1, C_2 are even infinite-time admissible.

Proof. Assertion (i) follows from a standard argument which can e.g. be found in [24]. To show (ii) and (iii) first note that $T_\mu(t)$ is in fact boundedly invertible for any $t \geq 0$ and that A_μ and $B = A_\mu b$ are bounded perturbations of A_0 and $A_0 b$, resp. To show L^2-admissibility of B, C_1, C_2, it thus suffices to consider $\mu = 0$ and to show that

$$\sup_{\operatorname{Re}\lambda = 1} \|(\lambda I - A_0)^{-1}B\|_X < \infty, \tag{21}$$

$$\sup_{\operatorname{Re}\lambda = 1} \|C_i(\lambda I - A_0)^{-1}\|_{\mathcal{L}(X;\mathbb{C})} < \infty, \quad i = 1, 2, \tag{22}$$

see [29, Rem. 2.11.3, Thm. 5.2.2 and Cor. 5.2.4]. Consider

$$x_{\mu,\lambda} := (\lambda I - A_\mu)^{-1}B = -b + \lambda(\lambda I - A_\mu)^{-1}b = -b + \lambda z_{\mu,\lambda},$$

where $z_{\mu,\lambda}(\zeta)$ can be computed to be

$$\frac{h_0}{\theta}\left(\frac{\cosh\left(\frac{\theta}{c^2}\right)-1}{\sinh\left(\frac{\theta}{c^2}\right)}\begin{bmatrix}\cosh\left(\frac{\theta\zeta}{c^2}\right)\\ -\frac{\lambda g}{\theta}\sinh\left(\frac{\theta\zeta}{c^2}\right)\end{bmatrix} + \begin{bmatrix}-\sinh\left(\frac{\theta\zeta}{c^2}\right)\\ \frac{\lambda g}{\theta}\left(\cosh\left(\frac{\theta\zeta}{c^2}\right)-1\right)\end{bmatrix}\right),$$

with $\theta = c\sqrt{\lambda(\lambda+2\mu)}$. Thus, (21) holds as $\sup_{\operatorname{Re}\lambda=1}\|\lambda x_{0,\lambda}\|_X < \infty$. Similarly, (22) can be checked. Thus, B, C_1, C_2 are L^2-admissible control/observation operators for $(T_\mu(t))_{t\geq 0}$ and hence B is also L^p-admissible for all $p \in [2, \infty]$ by the nesting property of L^p spaces on compact intervals. To show (iv) we construct functions $\mathbf{G}_i : \mathbb{C}_\omega \to \mathbb{C}$ which satisfy

$$\begin{aligned}\mathbf{G}_i(\lambda_1) - \mathbf{G}_i(\lambda_2) &= C_i((\lambda_1 I - A_\mu)^{-1} - (\lambda_2 I - A_\mu)^{-1})B\\ &= C_i(x_{\mu,\lambda_1} - x_{\mu,\lambda_2})\end{aligned}$$

for all $\lambda_1, \lambda_2 \in \mathbb{C}_\omega$. Using the explicit formula for $x_{\mu,\lambda}$ from above gives that $\mathbf{G}_i$ can be chosen as $\mathbf{H}^i$ defined in the statement of the proposition. Since $\mathbf{H}^i$ is proper and $\lim_{\operatorname{Re}\lambda\to\infty}\mathbf{H}^i(\lambda)$ exists, the system (A_μ, B, C_i) is well-posed and regular. This also implies (18), which shows that $\mathbf{H}^i$ is the transfer function of the system.
For $\mu > 0$, $(T_\mu(t))_{t\geq 0}$ is exponentially stable, whence admissibility implies infinite-time admissibility, cf. [33, Lem. 2.9]. ☐

In the next step we obtain a series representation for $\mathbf{H}(\lambda)$ and its inverse Laplace transform, which is shown to be a measure of bounded total variation on $\mathbb{R}_{\geq 0}$. The latter set is denoted by $\mathrm{M}(\mathbb{R}_{\geq 0})$ and the total variation by $\|f\|_{\mathrm{M}(\mathbb{R}_{\geq 0})}$ for $f \in \mathrm{M}(\mathbb{R}_{\geq 0})$; we refer to the textbook [34] for more details.

Lemma III.2. *Let $\mu > 0$, $\omega > \omega_{A_\mu}$ and $\sigma_n = n\pi c$ as in* (19). *The transfer function $\mathbf{H} : \mathbb{C}_\omega \to \mathbb{C}$ defined in* (20) *can be represented as*

$$\mathbf{H}(\lambda) = -8h_0\sum_{n\in\mathbb{N}}\mathbf{H}_n(\lambda) = -8h_0\sum_{n\in 2\mathbb{N}_0+1}\frac{\lambda}{\lambda^2 + 2\mu\lambda + \sigma_n^2},$$

with inverse Laplace transform $\mathfrak{h} = \mathcal{L}^{-1}(\mathbf{H}) \in \mathrm{M}(\mathbb{R}_{\geq 0})$. Moreover,

$$\mathfrak{h} = \mathfrak{h}_{L^1} + \frac{1}{4c}\mathfrak{h}_\delta,$$

where $\mathfrak{h}_{L^1}(t) := e^{-\mu t}(t^2\mathfrak{f}_2(t) + t\mathfrak{f}_1(t) + \mathfrak{f}_0(t))$, $t \geq 0$, and

$$\begin{aligned}\mathfrak{h}_\delta &:= \delta_0 - 2e^{-\mu/c}\delta_{1/c}\\ &\quad + 2\sum_{k\in\mathbb{N}}\left(e^{-2k\mu/c}\delta_{2k/c} - e^{-(2k+1)\mu/c}\delta_{(2k+1)/c}\right),\end{aligned}$$

for some $\mathfrak{f}_0, \mathfrak{f}_1, \mathfrak{f}_2 \in L^\infty(\mathbb{R}_{\geq 0};\mathbb{R})$, and δ_t denotes the Dirac delta distribution at $t \in \mathbb{R}$.

Proof. Let us first show the series representation of $\mathbf{H}$. Recall that

$$\tanh(z) = 8z\sum_{k=1}^\infty \frac{1}{\pi^2(2k-1)^2 + 4z^2}, \quad z \notin i\pi(1 + 2\mathbb{Z}).$$

Using this in (20) gives the desired formula for $\mathbf{H}$. Next we study the inverse Laplace transform of $\mathbf{H}$; in particular, $\mathbf{H}_n(\lambda) = 0$ for $n \in 2\mathbb{N}_0$. It is easy to see that $\mathbf{H}$ is also continuous on $\overline{\mathbb{C}_+}$ and that the series converges locally uniformly along the imaginary axis. Thus, the partial sums converge to $\alpha \mapsto \mathbf{H}(i\alpha)$ in the distributional sense when considered as tempered distributions on $i\mathbb{R}$. By continuity of the Fourier transform $\mathcal{F}(\cdot)$, this gives that the series

$$-8h_0\sum_{n\in\mathbb{N}}\mathcal{F}^{-1}(\mathbf{H}_n(i\cdot)) = -8h_0\sum_{n\in\mathbb{N}}\mathcal{L}^{-1}(\mathbf{H}_n)$$

converges to $\mathfrak{h} = \mathcal{F}^{-1}(\mathbf{H}(i\cdot)) = \mathcal{L}^{-1}(\mathbf{H})$ in the distributional sense[1]. It remains to study $\mathcal{L}^{-1}(\mathbf{H}_n)$ and the limit of the corresponding sum. By well-known rules for the Laplace transform we have $\mathcal{L}^{-1}(\mathbf{H}_n)(t) = e^{-\mu t}\mathfrak{g}_n(t)$ for $t \geq 0$, where

$$\mathfrak{g}_n(t) = \cos(\phi_n t) - \mu\phi_n^{-1}\sin(\phi_n t), \quad n \in 2\mathbb{N}_0 + 1.$$

The idea of the proof is to use Fourier series that are related to the frequencies σ_n in contrast to the 'perturbed' harmonics $\sin\phi_n$ and

[1]Here we identify functions on $\mathbb{R}_{\geq 0}$ with their trivial extension to $\mathbb{R}$ and use the relation between Fourier and Laplace transform.

$\cos\phi_n$. We write

$$\mathfrak{g}_n(t) = [\cos(\phi_n t) - \cos(\sigma_n t)] + \frac{\mu}{\phi_n}[\sin(\sigma_n t) - \sin(\phi_n t)] + \cos(\sigma_n t) + \frac{\mu}{\phi_n}\sin(\sigma_n t).$$

By the mean value theorem there exist $\alpha_n, \beta_n \in [\phi_n, \sigma_n]$ and $\omega_n \in [\alpha_n, \sigma_n]$ such that

$$\cos(\phi_n t) - \cos(\sigma_n t) = t(\sigma_n - \phi_n)\sin(\alpha_n t) = \frac{\mu^2 t \sin(\alpha_n t)}{\sigma_n + \phi_n},$$
$$\sin(\alpha_n t) = t(\alpha_n - \sigma_n)\cos(\omega_n t) + \sin(\sigma_n t),$$
$$\sin(\sigma_n t) - \sin(\phi_n t) = t(\sigma_n - \phi_n)\cos(\beta_n t) = \frac{\mu^2 t\cos(\beta_n t)}{\sigma_n + \phi_n},$$

where we used the identity $\sigma_n^2 - \phi_n^2 = \mu^2$ from (19). Hence,

$$\mathfrak{g}_n(t) = t^2\frac{\mu^2(\alpha_n - \sigma_n)}{\sigma_n + \phi_n}\cos(\omega_n t) + \frac{\mu^3 t}{\phi_n(\sigma_n + \phi_n)}\cos(\beta_n t) + \cos(\sigma_n t) + \left[t(\sigma_n - \phi_n) + \frac{\mu}{\phi_n}\right]\sin(\sigma_n t)$$

The coefficient sequences of the first two terms in the sum,

$$a_n := \mu^2\frac{\alpha_n - \sigma_n}{\sigma_n + \phi_n}, \quad b_n := \frac{\mu^3}{\phi_n(\sigma_n + \phi_n)},$$

are absolutely summable sequences since

$$0 > a_n > \mu^2\frac{\phi_n - \sigma_n}{\sigma_n + \phi_n} = \frac{-\mu^4}{(\sigma_n + \phi_n)^2}.$$

Let us further rewrite the coefficient of the last term, recalling that $\sigma_n^2 - \phi_n^2 = \mu^2$ implies that $\frac{1}{\sigma_n+\phi_n} - \frac{1}{2\sigma_n} = \frac{\mu^2}{2\sigma_n(\sigma_n+\phi_n)^2}$, and hence

$$t(\sigma_n - \phi_n) = \frac{\mu^2 t}{\sigma_n + \phi_n} = \frac{\mu^4 t}{2\sigma_n(\sigma_n + \phi_n)^2} + \frac{\mu^2 t}{2\sigma_n},$$
$$\frac{\mu}{\phi_n} = \frac{\mu}{\phi_n} + \frac{\mu}{\sigma_n} - \frac{\mu}{\sigma_n} = \frac{\mu}{\sigma_n} + \frac{\mu^3}{\sigma_n\phi_n(\sigma_n + \phi_n)}.$$

Thus, with $c_n = \frac{\mu^4}{2\sigma_n(\sigma_n+\phi_n)^2}$ and $d_n = \frac{\mu^3}{\sigma_n\phi_n(\sigma_n+\phi_n)}$, which define absolutely summable sequences, we have

$$\mathfrak{g}_n(t) = t^2 a_n\cos(\omega_n t) + t b_n\cos(\beta_n t) + [tc_n + d_n]\sin(\sigma_n t) + \cos(\sigma_n t) + (\mu t + 2)\frac{\mu}{2\sigma_n}\sin(\sigma_n t).$$

Consider the last two terms of the sum: As $\sigma_n = n\pi c$, we have by basics on Fourier series that $4c\sum_{n\in 2\mathbb{N}_0+1}\frac{1}{\sigma_n}\sin(\sigma_n t)$ converges to

$$H_0(t) = \begin{cases} 1, & t \in [2k/c, (2k+1)/c),\ k \in \mathbb{N}_0 \\ -1, & t \in [(2k+1)/c, (2k+2)/c),\ k \in \mathbb{N}_0 \end{cases}$$

for almost all $t \geq 0$. Therefore, for almost all $t \geq 0$ we have

$$\sum_{n\in 2\mathbb{N}_0+1}\frac{\mu}{2\sigma_n}\sin(\sigma_n t) = \frac{\mu}{8c}H_0(t).$$

Since the coefficients $\frac{\mu}{\sigma_n}$ are square summable, the series even converges in L^2 on any bounded interval and thus particularly in the distributional sense on $\mathbb{R}_{\geq 0}$.

Finally, by well-known facts on the Fourier series of delta distributions, $4c\sum_{n\in 2\mathbb{N}_0+1}\cos(\sigma_n\cdot)$ converges to the $\frac{2}{c}$-periodic extension of $(\delta_0 - 2\delta_{\frac{1}{c}} + \delta_{\frac{2}{c}})$ in the distributional sense as we have

$$\lim_{N\to\infty}\left\langle 4c\sum_{\substack{n=1\\ n \text{ odd}}}^N\cos(\sigma_n\cdot), \psi\right\rangle = \lim_{N\to\infty}\int_0^{\frac{2}{c}} 4c\sum_{\substack{n=1\\ n \text{ odd}}}^N\cos(\sigma_n s)\psi(s)\,\mathrm{d}s = \langle\delta_0 - 2\delta_{1/c} + \delta_{2/c}, \psi\rangle$$

for any function $\psi \in C^\infty([0, \frac{2}{c}]; \mathbb{R})$. Altogether, and as multiplying with $e^{-\mu t}$ preserves the distributional convergence, this yields that

$$\sum_{n\in 2\mathbb{N}_0+1}\mathcal{L}^{-1}(\mathbf{H}_n)(\cdot) = \sum_{n\in 2\mathbb{N}_0+1}e^{-\mu\cdot}\mathfrak{g}_n(\cdot) = \mathfrak{h}_{L^1}(\cdot) + \frac{1}{4c}\mathfrak{h}_\delta$$

with $\mathfrak{h}_{L^1}$, $\mathfrak{h}_\delta$ as in the assertion and where the functions

$$\mathfrak{f}_2(t) := \sum_{n\in 2\mathbb{N}_0+1}a_n\cos(\omega_n t)$$
$$\mathfrak{f}_1(t) := \frac{\mu^2}{8c}H_0(t) + \sum_{n\in 2\mathbb{N}_0+1}b_n\cos(\beta_n t) + c_n\sin(\sigma_n t),$$
$$\mathfrak{f}_0(t) := \frac{\mu}{4c}H_0(t) + \sum_{n\in\mathbb{N}}d_n\sin(\sigma_n t), \qquad t \geq 0,$$

are bounded since a_n, b_n, c_n, d_n are absolutely summable sequences. By this representation, $\mathfrak{h}_{L^1} \in L^1(\mathbb{R}_{\geq 0}; \mathbb{R})$ and can thus be identified with an element in $\mathrm{M}(\mathbb{R}_{\geq 0})$, while obviously $\mathfrak{h}_\delta \in \mathrm{M}(\mathbb{R}_{\geq 0})$ as the total variation $\|h_\delta\|_{\mathrm{M}(\mathbb{R}_{\geq 0})} = 1 + 2\sum_{k\in\mathbb{N}}e^{-\mu k/c}$ is finite. ☐

IV. The operator $\mathcal{T}$

In this section we show that the nonlinear operator $\mathcal{T}$ given by (13), (14) is well-defined and maps bounded functions to bounded functions. To this end, we calculate the different parts of the operator $\mathcal{T}$ using the mild solution x of the PDE (14).

Proposition IV.1. *Let $x_0 \in \mathcal{D}(A)$ as defined in (5). Then the operator $\mathcal{T}$ given by (13), (14) is well-defined from $W_0^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R})$ to $L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R})$ and there exist $k_1, k_2, k_3, k_4 > 0$ such that for every $\eta \in W_0^{1,\infty}(\mathbb{R}_{\geq 0})$ we have*

$$\|\mathcal{T}(\eta)\|_\infty \leq k_1(\|x_0\|_X + \|A_\mu x_0\|_X + \|\eta\|_\infty) + k_2(\|x_0\|_X + \|\eta\|_\infty)^2 + k_3(\|x_0\|_X^2 + \|A_\mu x_0\|_X^2) + k_4\|A_\mu x_0\|_X\|\eta\|_\infty.$$

Moreover, $\mathcal{T}$ can be extended to an operator defined from $\mathcal{C}_0(\mathbb{R}_{\geq 0}; \mathbb{R})$ to $L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R})$, which is locally Lipschitz in the sense of condition (N4) c) and the estimate extends to $\eta \in \mathcal{C}_0(\mathbb{R}_{\geq 0}; \mathbb{R}) \cap L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R})$.

Proof. Recall that the (mild) solution to the PDE (14) is given by

$$x(t) = T_\mu(t)x_0 + \int_0^t (T_\mu)_{-1}(t-s)A_\mu b\eta(s)\,\mathrm{d}s, \quad t \geq 0. \tag{23}$$

By Proposition III.1 (ii), $A_\mu b \in \mathcal{L}(\mathbb{C}; X_{-1})$ is infinite-time L^∞-admissible, hence $x \in \mathcal{C}(\mathbb{R}_{\geq 0}; X)$ and there exists $\tilde{k} > 0$ such that

$$\|x(t)\|_X \leq \tilde{k}(\|x_0\|_X + \|\eta\|_{L^\infty([0,t];\mathbb{R})})$$

for all $t \geq 0$, any $x_0 \in X$ and $\eta \in \mathcal{C}_0(\mathbb{R}_{\geq 0}; \mathbb{R})$. Since $x_0(\cdot)$ and $\eta(\cdot)$ are real-valued we have that x, as a function in time and space, is real-valued as well. Let C_i denote the operators from (Σ_i) and define

$$\mathcal{M} : X \to \mathbb{R},\ x \mapsto \frac{2\mu h_0}{m_T}\int_0^1 x_2(\zeta)\,\mathrm{d}\zeta\,,$$
$$\mathcal{N} : X \to \mathbb{R},\ x \mapsto \frac{2\mu}{m_T}\int_0^1 x_1(\zeta)x_2(\zeta)\,\mathrm{d}\zeta\,.$$

Then $\mathcal{T}$ defined in (13) can be written as $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$, where, for $\eta \in \mathcal{C}_0(\mathbb{R}_{\geq 0}; \mathbb{R})$,

$$\mathcal{T}_1(\eta)(t) = \frac{g}{2m_T}\big(C_1(x(t))\big)\big(2h_0 + C_2(x(t))\big)$$
$$\mathcal{T}_2(\eta)(t) = \mathcal{M}(x(t)) + \mathcal{N}(x(t)) - \frac{2\mu h_0}{m_T}\eta(t), \quad t \geq 0,$$

and x is given by (23). While it is obvious that $\mathcal{T}_2$ is well-defined on $\mathcal{C}_0(\mathbb{R}_{\geq 0}; \mathbb{R})$, this is not yet clear for $\mathcal{T}_1$.

In order to estimate $\|\mathcal{T}(\eta)\|_\infty$, we first study the operator $\mathcal{T}_2$. From the definition of $\mathcal{M}$ and $\mathcal{N}$ we readily get for $x \in X$ that

$$|\mathcal{M}(x)| \leq \frac{2\mu h_0}{m_T}\|x\|_X \quad \text{and} \quad |\mathcal{N}(x)| \leq \frac{\mu}{m_T}\|x\|_X^2.$$

Hence, for $\eta \in \mathcal{C}_0(\mathbb{R}_{\geq 0};\mathbb{R}) \cap L^\infty(\mathbb{R}_{\geq 0};\mathbb{R})$ we obtain

$$\begin{aligned}\|\mathcal{T}_2(\eta)\|_\infty \leq\ & \frac{2\mu h_0}{m_T}\tilde{k}(\|x_0\|_X + \|\eta\|_\infty) \\ & + \frac{\mu}{m_T}\tilde{k}^2(\|x_0\|_X + \|\eta\|_\infty)^2.\end{aligned}$$

In the remainder of the proof we consider $\mathcal{T}_1$. Let $\eta \in W_0^{1,\infty}(\mathbb{R}_{\geq 0};\mathbb{R}) \cap L^2(\mathbb{R}_{\geq 0};\mathbb{R})$ in the following. First note that $C_2(x(\cdot))$ only depends on x_0 and is hence constant as a function of η. By Proposition III.1 (iv) we have that $g_2(\lambda) = 0$, which implies that $C_2(x(\cdot)) = C_2 T_\mu(\cdot)x_0$, which is well-defined since $x_0 \in \mathcal{D}(A_\mu)$ and moreover bounded, i.e.,

$$\begin{aligned}|C_2(x(t))| &\leq \|C_2\|_{\mathcal{L}(\mathcal{D}(A);\mathbb{R})}\|T_\mu(t)A_\mu x_0\|_X \\ &\leq \|C_2\|_{\mathcal{L}(\mathcal{D}(A);\mathbb{R})} M\|A_\mu x_0\|_X\end{aligned}$$

with $M = \sup_{t\geq 0}\|T_\mu(t)\|$. Analogously, $C_1 T_\mu(\cdot)x_0$ is bounded by $\|C_1\|_{\mathcal{L}(\mathcal{D}(A);\mathbb{R})} M\|A_\mu x_0\|_X$. Using the input-output map F_1 defined in (17) we may infer from the variation of constants formula that

$$C_1(x(\cdot)) = C_1 T_\mu(\cdot)x_0 + F_1(\eta)(\cdot).$$

It remains to investigate whether the real-valued extension of F_1 to L^2, which we again denote by F_1, that is the map

$$\begin{aligned}F_1 : W_0^{1,\infty}(\mathbb{R}_{\geq 0};\mathbb{R}) \cap L^2(\mathbb{R}_{\geq 0};\mathbb{R}) &\to L^2(\mathbb{R}_{\geq 0};\mathbb{R}), \\ \eta &\mapsto \left(t \mapsto C_1\int_0^t (T_\mu)_{-1}(t-s)B\eta(s)\,\mathrm{d}s\right),\end{aligned}$$

is bounded in the L^∞-norms. By Proposition III.1 (iv), the transfer function $\mathbf{H}$ is an element of $H^\infty(\mathbb{C}_+)$ and thus

$$\mathcal{L}(F_1(\eta))(\lambda) = \mathbf{H}(\lambda)\cdot\mathcal{L}(\eta)(\lambda), \quad \lambda \in \mathbb{C}_+.$$

Hence, there exists a tempered distribution $\mathfrak{h} = \mathcal{L}^{-1}(\mathbf{H})$ such that

$$F_1(\eta) = \mathfrak{h} * \eta \tag{24}$$

for Schwartz-class functions η with support in $\mathbb{R}_{\geq 0}$ — here and in the following we extend functions defined on $\mathbb{R}_{\geq 0}$ to $\mathbb{R}$ by zero. By Lemma III.2, $\mathfrak{h}$ can be identified with a Radon measure on $\mathbb{R}_{\geq 0}$ with bounded total variation $\|\mathfrak{h}\|_{\mathrm{M}(\mathbb{R}_{\geq 0})}$. Hence, by a variant of Young's integral inequality, $F_1(\eta) \in L^\infty(\mathbb{R}_{\geq 0};\mathbb{R})$ and

$$\|F_1(\eta)\|_\infty \leq \|\mathfrak{h}\|_{\mathrm{M}(\mathbb{R}_{\geq 0})}\,\|\eta\|_\infty \tag{25}$$

for all Schwartz functions η supported in $\mathbb{R}_{\geq 0}$; we refer to [34, Sec. 2.5.4] for details on convolution operators with $\mathfrak{h} \in \mathrm{M}(\mathbb{R}_{\geq 0})$. Thus, F_1 (and hence also $\mathcal{T}_1$ and $\mathcal{T}$) can, in the form (24), be extended to $\mathcal{C}_0(\mathbb{R}_{\geq 0};\mathbb{R})$ and we find that for $\eta \in \mathcal{C}_0(\mathbb{R}_{\geq 0};\mathbb{R}) \cap L^\infty(\mathbb{R}_{\geq 0};\mathbb{R})$

$$\begin{aligned}\|\mathcal{T}_1(\eta)\|_\infty \leq\ & \tfrac{g}{2m_T}(\|C_1 T(\cdot)x_0\|_\infty + \|\mathfrak{h}\|_{\mathrm{M}(\mathbb{R}_{\geq 0})}\,\|\eta\|_\infty) \\ & \cdot(2h_0 + \|C_2 T(\cdot)x_0\|_\infty) \\ \leq\ & k_3\|A_\mu x_0\|_X^2 + k_4(\|A_\mu x_0\|_X + 1)\|\eta\|_\infty \\ & + k_5\|A_\mu x_0\|_X\end{aligned}$$

for some $k_3, k_4, k_5 > 0$. Finally, it remains to show that $\mathcal{T}$ satisfies condition (N4) c). To this end, first observe that $\mathcal{T}(\eta) - \mathcal{N}(x)$, where x is as in (23), is linear in η and hence trivially locally Lipschitz. To show (N4) c) for $\mathcal{N}(x)$ fix $t \geq 0$ and $\xi \in \mathcal{C}([0,t];\mathbb{R})$ as well as $\eta_i \in \mathcal{C}_0(\mathbb{R}_{\geq 0};\mathbb{R})$ with $\eta_i|_{[0,t]} = \xi$ and $|\eta_i(s) - \xi(t)| < 1$ for all $s \in [t, t+1]$ and $i = 1, 2$. Let x^i denote the mild solution as in (23) corresponding to $\eta = \eta_i$ for $i = 1, 2$. Then we have

$$\begin{aligned}& x_1^1(s)x_2^1(s) - x_1^2(s)x_2^2(s) \\ &\quad = \big(x_1^1(s) - x_1^2(s)\big)x_2^2(s) + x_1^1(s)\big(x_2^1(s) - x_2^2(s)\big)\end{aligned}$$

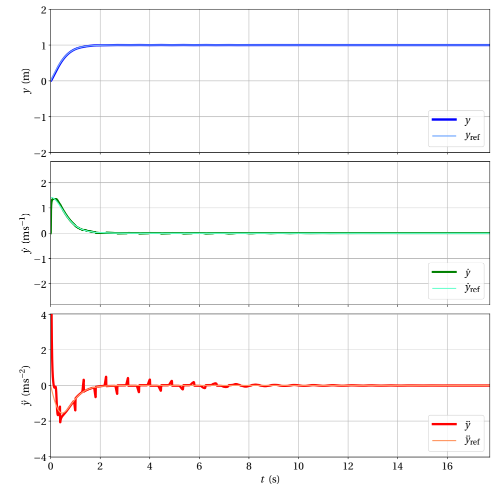

Fig. 3: Output y and reference signal y_{ref} and corresponding first and second derivatives.

for $s \in [t, t+1]$ and hence

$$\begin{aligned}& |\mathcal{N}(x^1)(s) - \mathcal{N}(x^2)(s)| \\ & \leq \frac{\mu}{m_T}\|x^1(s) - x^2(s)\|_X\left(\|x^1(s)\|_X + \|x^2(s)\|_X\right) \\ & \leq \frac{\mu}{m_T}\tilde{k}^2\|\eta_1 - \eta_2\|_\infty\left(2\|x_0\|_X + \|\eta_1|_{[0,t+1]}\|_\infty + \|\eta_2|_{[0,t+1]}\|_\infty\right).\end{aligned}$$

Clearly, $\|\eta_i|_{[0,t+1]}\|_\infty \leq \|\xi\|_\infty + 1$ and thus the assertion is true for $\tau = \delta = 1$ and $c = \frac{2\mu}{m_T}\tilde{k}^2\left(\|x_0\|_X + \|\xi\|_\infty + 1\right)$. $\square$

V. Simulations

In this section we illustrate the application of the funnel controller (7) to the system (6). In the following we present the numerical method used to simulate the corresponding closed-loop system. Using the change of variables $z(\zeta,t) = Q\begin{pmatrix}\eta_1(\zeta,t)\\ \eta_2(\zeta,t)\end{pmatrix}$ with $Q := \begin{bmatrix}1 & 1\\ \frac{g}{c} & -\frac{g}{c}\end{bmatrix}$ in (6) enables us to solve the PDE corresponding to η_1 with an implicit finite difference method and the PDE corresponding to η_2 with an explicit finite difference method, respectively. For the simulation we have used the parameters $m_T = 1\mathrm{kg}$, $h_0 = 0.5\mathrm{m}$, $g = 9.8\mathrm{ms}^{-2}$, $\mu = 0.1\mathrm{s}^{-1}$ and the reference signal $y_{\mathrm{ref}}(t) = \tanh^2(\omega t)$ with $\omega = 2\pi\sqrt{h_0/g}$. The initial values (8) are chosen as $\tilde{h}_0(\zeta), v_0(\zeta)) = (0\mathrm{m}, 0.1\sin^2(4\pi\zeta)\mathrm{ms}^{-1})$ and $(y^0, y^1) = (0\mathrm{m}, 0\mathrm{ms}^{-1})$. For the controller (7) we chose the funnel functions $\varphi_0(t) = \varphi_1(t) = 100\tanh(\omega t)$. Clearly, the initial errors lie within the funnel boundaries as required in Theorem II.1. For the finite differences we have used a grid in t with $M = 4000$ points for the interval $[0, 2\tau]$ with $\tau = f^{-1}$, and a grid in ζ with $N = \lfloor ML/(4c\tau)\rfloor$ points. Furthermore, we have used a tolerance of 10^{-6}. The method has been implemented in Python and the simulation results are shown in Figs. 3 and 4.

It can be seen that even in the presence of sloshing effects a prescribed performance of the tracking error can be achieved with the funnel controller (7), while at the same time the generated input is bounded and shows an acceptable performance. We like to stress that the sloshing effects are theoretically substantiated by the part $\mathfrak{h}_\delta$ of the inverse Laplace transform of the transfer function derived in Lemma III.2. An inspection of the proof of Proposition IV.1 reveals that $\mathfrak{h}_\delta * \eta$ explicitly appears in $\ddot{y}$; the resulting decaying impulses can be seen in Fig. 3.

VI. Conclusion

In the present paper we have shown that the funnel controller (7) is feasible for the moving water tank system (6) which rests on

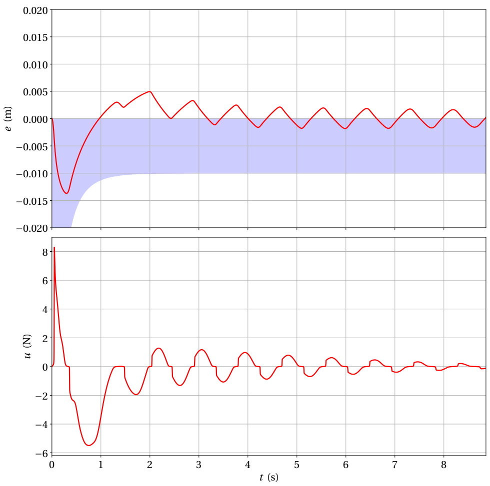

Fig. 4: Performance funnel with tracking error e and input u.

the linearized Saint-Venant equations. We stress that System (6) is nonlinear and that the operators involved are unbounded. Even in the linearized case the motion of the fluid affects the dynamics of the overall system which leads to the effect of sloshing. Nevertheless, the funnel controller is able to handle these effects as shown in Theorems II.1 and II.2 and in the simulations in Section V.

We also like to point out that the controller (7) requires that the derivative of the output is available for control. This may not be true in practice, and it may even be hard to obtain suitable estimates of the output derivative. This drawback may be resolved by combining the controller (7) with a funnel pre-compensator as developed in [35], [36], which results in a pure output feedback.

Several extensions of the moving water tank system (6) may be considered in future research, such as a slope at the bottom of the tank, the interconnection of the tank with a truck as in [12] and, of course, the general nonlinear Saint-Venant equations (1) as well as the two-dimensional case.

References

[1] F. L. Cardoso-Ribeiro, D. Matignon, and V. Pommier-Budinger, “A port-Hamiltonian model of liquid sloshing in moving containers and application to a fluid-structure system,” *Journal of Fluids and Structures*, vol. 69, pp. 402–427, 2017.

[2] E. W. Graham and A. M. Rodriguez, “The characteristics of fuel motion which affect airplane dynamics,” DTIC Document, 1951.

[3] J. T. Feddema, C. R. Dohrmann, G. G. Parker, R. D. Robinett, V. J. Romero, and D. J. Schmitt, “Control for slosh-free motion of an open container,” *IEEE Control Syst. Mag.*, vol. 17, no. 1, pp. 29–36, 1997.

[4] M. Grundelius and B. Bernhardsson, “Control of liquid slosh in an industrial packaging machine,” in *Proc. IEEE Int. Conf. Control Appl.*, Hawai, USA, 1999, pp. 1654–1659.

[5] R. Venugopal and D. Bernstein, “State space modeling and active control of slosh,” in *Proc. IEEE Int. Conf. Control Appl.*, Dearborn, MI, 1996, pp. 1072–1077.

[6] K. Yano, T. Yoshida, M. Hamaguchi, and K. Terashima, “Liquid container transfer considering the suppression of sloshing for the change of liquid level,” in *Proc. 13th IFAC World Congress*, San Francisco, CA, 1996, pp. 701–706.

[7] F. Dubois, N. Petit, and P. Rouchon, “Motion planning and nonlinear simulations for a tank containing a fluid,” in *Proc. 5th European Control Conf. (ECC)*, Karlsruhe, Germany, 1999, pp. 3232–3237.

[8] N. Petit and P. Rouchon, “Dynamics and solutions to some control problems for water-tank systems,” *IEEE Trans. Autom. Control*, vol. 47, no. 4, pp. 594–609, 2002.

[9] J.-M. Coron, “Local controllability of a 1-d tank containing a fluid modeled by the shallow water equations,” *ESAIM: Control Optim. Calc. Var.*, vol. 8, pp. 513–554, 2002.

[10] C. Prieur and J. de Halleux, “Stabilization of a 1-d tank containing a fluid modeled by the shallow water equations,” *Syst. Control Lett.*, vol. 52, pp. 167–178, 2004.

[11] D. Auroux and S. Bonnabel, “Symmetry-based observers for some water-tank problems,” *IEEE Trans. Autom. Control*, vol. 56, no. 5, pp. 1046–1058, 2011.

[12] M. Gerdts and S.-J. Kimmerle, “Numerical optimal control of a coupled ODE-PDE model of a truck with a fluid basin,” in *Proc. 10th AIMS Int. Conf. Dynam. Syst. Diff. Equ. Appl.*, Madrid, Spain, 2015.

[13] A. Ilchmann, E. P. Ryan, and C. J. Sangwin, “Tracking with prescribed transient behaviour,” *ESAIM: Control Optim. Calc. Var.*, vol. 7, pp. 471–493, 2002.

[14] A. Ilchmann and E. P. Ryan, “High-gain control without identification: a survey,” *GAMM Mitt.*, vol. 31, no. 1, pp. 115–125, 2008.

[15] A. Ilchmann and S. Trenn, “Input constrained funnel control with applications to chemical reactor models,” *Syst. Control Lett.*, vol. 53, no. 5, pp. 361–375, 2004.

[16] C. M. Hackl, *Non-identifier Based Adaptive Control in Mechatronics–Theory and Application*, ser. Lecture Notes in Control and Information Sciences. Cham, Switzerland: Springer-Verlag, 2017, vol. 466.

[17] T. Berger, S. Otto, T. Reis, and R. Seifried, “Combined open-loop and funnel control for underactuated multibody systems,” *Nonlinear Dynamics*, vol. 95, no. 3, pp. 1977–1998, 2019.

[18] T. Berger and T. Reis, “Zero dynamics and funnel control for linear electrical circuits,” *J. Franklin Inst.*, vol. 351, no. 11, pp. 5099–5132, 2014.

[19] A. Pomprapa, S. Weyer, S. Leonhardt, M. Walter, and B. Misgeld, “Periodic funnel-based control for peak inspiratory pressure,” in *Proc. 54th IEEE Conf. Decis. Control, Osaka, Japan*, 2015, pp. 5617–5622.

[20] T. Berger and A.-L. Rauert, “A universal model-free and safe adaptive cruise control mechanism,” in *Proceedings of the MTNS 2018*, Hong Kong, 2018, pp. 925–932.

[21] G. Bastin and J.-M. Coron, *Stability and Boundary Stabilization of 1-D Hyperbolic Systems*, ser. Progress in Nonlinear Differential Equations and Their Applications: Subseries in Control. Birkhäuser, 2016, vol. 88.

[22] T. Berger, H. H. Lê, and T. Reis, “Funnel control for nonlinear systems with known strict relative degree,” *Automatica*, vol. 87, pp. 345–357, 2018.

[23] A.J.C.B. de Saint-Venant, , “Théorie du mouvement non permanent des eaux avec applications aux crues des rivières et à l’introduction des marées dans leur lit,” *Comptes Rendus de l’Académie des Sciences de Paris*, vol. 73, pp. 147–154 and 237–240, 1871.

[24] B. Jacob and H. Zwart, *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*, ser. Operator Theory: Advances and Applications. Birkhäuser, 2012, vol. 223.

[25] C. M. Hackl, N. Hopfe, A. Ilchmann, M. Mueller, and S. Trenn, “Funnel control for systems with relative degree two,” *SIAM J. Control Optim.*, vol. 51, no. 2, pp. 965–995, 2013.

[26] A. Ilchmann and E. P. Ryan, “Performance funnels and tracking control,” *Int. J. Control*, vol. 82, no. 10, pp. 1828–1840, 2009.

[27] A. Ilchmann, E. P. Ryan, and P. Townsend, “Tracking with prescribed transient behavior for nonlinear systems of known relative degree,” *SIAM J. Control Optim.*, vol. 46, no. 1, pp. 210–230, 2007.

[28] A. Ilchmann, T. Selig, and C. Trunk, “The Byrnes-Isidori form for infinite-dimensional systems,” *SIAM J. Control Optim.*, vol. 54, no. 3, pp. 1504–1534, 2016.

[29] M. Tucsnak and G. Weiss, *Observation and Control for Operator Semigroups*, ser. Birkhäuser Advanced Texts Basler Lehrbücher. Basel: Birkhäuser, 2009.

[30] ——, “Well-posed systems – the LTI case and beyond,” *Automatica*, vol. 50, pp. 1757–1779, 2007.

[31] O. Staffans, *Well-Posed Linear Systems*, ser. Encyclopedia of Mathematics and its Applications. Cambridge: Cambridge University Press, 2005, vol. 103.

[32] M. Puche, *Adaptive Control for Boundary Control Systems*. PhD thesis, Universität Hamburg, Germany, 2019.

[33] B. Jacob, R. Nabiullin, J. Partington, and F. L. Schwenninger, “Infinite-dimensional input-to-state stability and Orlicz spaces,” *SIAM J. Control Optim.*, vol. 56, no. 2, pp. 868–889, 2018.

[34] L. Grafakos, *Classical Fourier analysis*, 3rd ed., ser. Graduate Texts in Mathematics. New York: Springer-Verlag, 2014, vol. 249.

[35] T. Berger and T. Reis, “The Funnel Pre-Compensator,” *Int. J. Robust & Nonlinear Control*, vol. 28, no. 16, pp. 4747–4771, 2018.

[36] ——, “Funnel control via funnel pre-compensator for minimum phase systems with relative degree two,” *IEEE Trans. Autom. Control*, vol. 63, no. 7, pp. 2264–2271, 2018.